ChE 597 Computational Optimization

Homework 1

January 21st 11:59 pm

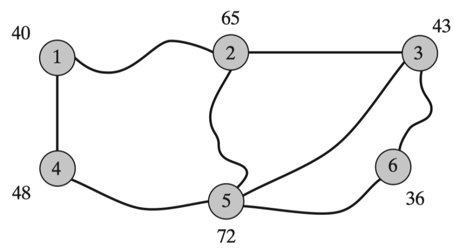
1. A small engineering consulting firm has 3 senior designers available to work on the firm's 4 current projects over the next 2 weeks. Each designer has 80 hours to split among the projects, and the following table shows the manager's scoring ($0 =$ nil to $100 =$ perfect) of the capability of each designer to contribute to each project, along with his estimate of the hours that each project will require.

	Project			
Designer	**1**	**2**	**3**	**4**
1	90	80	10	50
2	60	70	50	65
3	70	40	80	85
Required	70	50	85	35

   (a) Formulate an allocation LP to choose an optimal work assignment.

   (b) Solve your model using pyomo.

2. The following map shows the 8 intersections at which automatic traffic monitoring devices might be installed. A station at any particular node can monitor all the road links meeting that intersection. Numbers next to nodes reflect the monthly cost (in thousands of dollars) of operating a station at that location. This problem is known as the set covering problem in combinatorial optimization.



   (a) Formulate the problem of providing full coverage at minimum total cost as a set covering integer program.
   (b) Use pyomo to solve part (a)
   (c) Revise your formulation of part (a) to obtain an ILP minimizing the number of uncovered road links while using at most 2 stations.
   (d) Use pyomo to solve part (c)

3. Prove the following results covered in class. Hint: Use the definition of convex functions, the first and second order characterizations.

   (a) Univariate functions:
      - Exponential function: e^{ax} is convex for any a over $\mathbb{R}$
      - Power function: x^a is convex for $a \geq 1$ or $a \leq 0$ over $\mathbb{R}_+$ (nonnegative reals)
      - Power function: x^a is concave for $0 \leq a \leq 1$ over $\mathbb{R}_+$
      - Logarithmic function: $\log x$ is concave over $\mathbb{R}_{++}$

   (b) Affine function: $a^T x + b$ is both convex and concave

   (c) Quadratic function: $\frac{1}{2}x^T Qx + b^T x + c$ is convex provided that $Q \succeq 0$ (positive semidefinite)

   (d) Least squares loss: $\|y - Ax\|_2^2$ is always convex (since $A^T A$ is always positive semidefinite)

   (e) **Norm:** $\|x\|$ is convex for any norm; e.g., ℓ_p norms,

   $$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \text{ for } p \geq 1, \quad \|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$$

   (f) **Support function:** for any set C (convex or not), its support function

   $$I_C^*(x) = \max_{y \in C} x^T y$$

   is convex.

   (g) **Max function:** $f(x) = \max\{x_1, \ldots, x_n\}$ is convex.

4. Prove the following properties of convex sets.

   (a) **Intersection**: the intersection of convex sets is convex.

   (b) **Scaling and translation**: if C is convex, then

   $$aC + b = \{ax + b : x \in C\}$$

   is convex for any a, b.

   (c) **Affine images and preimages**: if $f(x) = Ax + b$ and C is convex then

   $$f(C) = \{f(x) : x \in C\}$$

   is convex, and if D is convex then

   $$f^{-1}(D) = \{x : f(x) \in D\}$$

   is convex.

   (d) **Perspective images and preimages:** the perspective function is $\mathbf{P} : \mathbb{R}^n \times \mathbb{R}_{++} \to \mathbb{R}^n$ (where $\mathbb{R}_{++}$ denotes positive reals),

   $$\mathbf{P}(x, z) = \frac{x}{z}$$

   for $z > 0$. If $C \subseteq \mathrm{dom}(\mathbf{P})$ is convex then so is $\mathbf{P}(C)$, and if D is convex then so is $\mathbf{P}^{-1}(D)$

   (e) **Linear-fractional images and preimages:** the perspective map composed with an affine function,

   $$f(x) = \frac{Ax + b}{c^T x + d}$$

   is called a **linear-fractional function**, defined on $c^T x + d > 0$. If $C \subseteq \mathrm{dom}(f)$ is convex then so if $f(C)$, and if D is convex then so is $f^{-1}(D)$

5. Prove the following properties of convex functions

   (a) **Nonnegative linear combination:** $f_1, \ldots, f_m$ convex implies $a_1 f_1 + \ldots + a_m f_m$ convex for any $a_1, \ldots, a_m \geq 0$

   (b) **Pointwise maximization:** if f_s is convex for any $s \in S$, then $f(x) = \max_{s \in S} f_s(x)$ is convex. Note that the set S here (number of functions f_s) can be infinite

   (c) **Partial minimization:** if $g(x,y)$ is convex in x,y, and C is convex, then $f(x) = \min_{y \in C} g(x,y)$ is convex