Real Analysis - P1

Learning Theory and Applications Group

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Contents

Part I
Lebesgue Intergration

Key definitions here:

1 The Real Number: Sets, Sequences, and Functions

1.1 The Field, Positivity, and Completeness Axioms

1.1.1 Excercise

Ex 1. For $a \neq 0$ and $b \neq 0$, show that $(ab)^{-1} = a^{-1}b^{-1}$

$$(a^{-1}b^{-1})(ab) = a^{-1}b^{-1}ba = a^{-1}1a = 1$$

As a result, $a^{-1}b^{-1} = (ab^{-1})$
Ex 2. Verify the following:

- For each real number $a \neq 0$, $a^2 > 0$. In particular, $1 > 0$ since $1 \neq 0$ and $1 = 1^2$
- For each positive number a, its multiplicative inverse a^{-1} also is positive

- If $a > b$, then
$$ac > bc \text{ if } c > 0 \text{ and } ac < bc \text{ if } c < 0$$

For the first point, we first need to prove that, for any a, then $-a = (-1)a$,

$$a + (-a) = 0 = (1 + (-1))a = a + (-1)a$$

Next, for each $a \neq 0$, if a is positive, then a^2 is positive by definition of positiveness. On the other hand, if $a < 0$, then let $a = -b$ with $b > 0$,

$$a^2 = (-b)^2 = (-1)b(-1)b = (-1)(-b)b = (-1)^2b^2 > 0 \tag{1}$$

For the second point, assuming by contradiction that $a^{-1} < 0$ for any $a > 0$, then let $a^{-1} = -b$ with $b > 0$. then

$$1 = a(a^{-1}) = a(-b) = (-1)ab < 0$$

Here $ab > 0$ since both a and b are positive, and we know from previous point that $0 > -(ab) = (-1)ab$

The last point is straighforward from the definition of $>$.

$$ac - bc = \underbrace{(a-b)}_{>0}\underbrace{c}_{>0} > 0$$

$$ac - bc = \underbrace{(a-b)}_{>0}\underbrace{c}_{<0} = (-1)\underbrace{(a-b)}_{>0}\underbrace{d}_{>0} < 0 \text{ with } d = -c$$

Ex 3. For a nonempty set of real numbers E, show that $\inf E = \sup E$ if and only if E consists of a single point.

If the set E has a single element, then the least upper bound equal to that single element. This similarly applies to lowerbound. In other word, its sup and inf coincides.

On the other direction, if a set E has its sup and inf equal, and assuming by contradiction that E has at least 2 distinct elements, then the gap between these two points $\neq 0$. The difference between sup and inf is lowerbounded by this gap, so they cannot equal.

Ex 4. Let a and b be real numbers.

i Show that if $ab = 0$, then $a = 0$ or $b = 0$

ii Verify that $a^2 - b^2 = (a-b)(a+b)$ and conclude from part (i) that if $a^2 = b^2$, then $a = b$ or $a = -b$.

iii Let c be a positive real number. Define $E = \{x \in \mathbb{R} | x^2 < c\}$ verify that E is nonempty and bounded above. Define $x_0 = \sup E$. Show that $x_0^2 = c$. Use part (ii) to show that there is a unique $x > 0$ for which $x^2 = c$. It is denoted by $\sqrt{c}$

For the first point, suppose that $ab = 0$ and both a and b are not 0, then there exists a^{-1} and b^{-1}, then we have

$$abb^{-1}a^{-1} = 1$$

which means that $b^{-1}a^{-1} = (ab)^{-1}$, but since $ab = 0$, no such number exists.

The second point is a straighforward application of distributive property,

$$(a-b)(a+b) = a(a+b) + (-b)(a+b) = a^2 + ab - ba - b^2 = a^2 - b^2$$

Then from part (i), since $(a-b)(a+b) = 0$, one of the two terms must be 0.

In part (iii), we see that $0^2 = 0 < c$ for all $c > 0$, so E is nonempty. By contradiction, suppose E is not bounded from above, that is, for every $b > 0$, we can always choose some $x \in E$ such that $x > b$, letting $b > c$ lead to a contradiction with the definition of E.

Next, since E is bounded from above, then it has a supremum by completeness axiom. Denote $x_0 = \sup E$. We will show that $x_0^2 \geq c$ and $x_0^2 \leq c$ to conclude that $x_0^2 = c$.

Since $x^2 < c, \forall x \in E$, c is an upperbound of E^2, and because $\sup E$ is the smallest/least upperbound, then $\sup(E)^2 \leq c$. On the otherhand, $x_0 \geq x, \forall x \in E$ and E contains **all** real numbers whose square less than c, so $x_0^2 \geq c$.

Finally, we need to show that x_0 is a unique positive real number such that $x_0^2 = c$. By contradiction, suppose there is some $x > 0$ such that $x \neq x_0$ and $x^2 = c$, then by part (ii), since $x_0^2 = x^2$, we have either $x = x_0$ or $x = -x_0$, but x is positive and $-x_0$ is negetive, so $x = x_0$.

Ex 5. Let a, b, c be real bumbers such that $a \neq 0$ and consider the quadratic equation

$$ax^2 + bx + c = 0, x \in \mathbb{R}$$

i Suppose $b^2 - 4ac > 0$, use the Field Axiom and the preceding problem to complete the square and thereby show that this equation has exactly two solutions given by

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

ii Now suppose $b^2 - 4ac < 0$. Show that the quadratic equation fails to have any solution.

Suppose that $b^2 - 4ac > 0$, then from previous problem, there exists a unique positive number $\sqrt{b^2 - 4ac}$. we can verify that

$$\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = x^2 - x\frac{-b - \sqrt{b^2 - 4ac}}{2a} - x\frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{b^2 - b^2 + 4ac}{4a^2}$$
$$= x^2 + x\frac{b}{a} + \frac{c}{a} = 0.$$

Then also from the previous problem, either one of the two terms equal 0. As a result, the equation has exactly two solutions.

On the other hand, if $b^2 - 4ac < 0$, then the equation can be rewritten as

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{4ac - b^2}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4} > 0,$$

which does not have any solution.

Ex 6. Use the Completeness Axiom to show that every nonempty set of real numbers that is bounded below has an infimum and that

$$\inf E = -\sup\{-x|x \in E\}.$$

The set E is bounded below, which means that the set $E' = \{-x|x \in E\}$ is bounded from above, then its supremum exists by completeness axiom. Denote $x_0 = \sup E'$, then $x_0 \geq -x, \forall x \in E \rightleftarrows -x_0 \leq x, \forall x \in E$. As a result, $-x_0 \leq \inf E$.

Suppose that there exists some x' such that $x' > -x_0$ and $x' \leq x, \forall x \in E$; i.e. x' is a "greater" lowerbound of E than x_0. Then we can show that $-x'$ is a "smaller" upperbound of E', which contradicts with the definition of supremum. As a result, no such x' exists, and $-x_0$ is the infimum of E

Ex 7. For real bumbers a and b, verify the following:

i $|ab| = |a||b|$

ii $|a + b| \leq |a| + |b|$

iii For $\epsilon > 0$,

$$|x - a| < \epsilon \text{ if and only if } a - \epsilon < x < a + \epsilon$$

First we define the sign operator as $\text{sg}(x) \in \{1, -1\}, x \neq 0$. The absolute value can be written as the product with the sign operator

$$|a| = a\text{sg}(a)$$

Then the first claim can be verified as

$$|ab| = ab\text{sg}(ab) = a\text{sg}(a)b\text{sg}(b) = |a||b|$$

by noting $\text{sg}(ab) = \text{sg}(a)\text{sg}(b)$, and

$$|a + b| = (a + b)\text{sg}(a + b) = a\text{sg}(a + b) + b\text{sg}(a + b) \leq a\text{sg}(a) + b\text{sg}(b) = |a| + |b|$$

by noting $a\text{sg}(a) = \max(a, -a) \geq a\text{sg}(c), \forall c$

Final point: if $x - a > 0$, then $|x - a| = x - a$ and $|x - a| < \epsilon \rightleftarrows a < x < a + \epsilon$

Similar, if $x - a < 0$, then $|x - a| < \epsilon \rightleftarrows a > x > a - \epsilon$, combining the both cases and with the zero case yield the desired claim.

1.2 The Natural and Rational Numbers

1.2.1 Excercise

Ex 13. Show that each real number is the supremum of a set of rational numbers and also supremum of a set of irrational numbers.

Let x be any real number. We want to show that x is the supremum of both a set of rational numbers and a set of irrational numbers.

Define a set of rational numbers as: $S = \{q \in \mathbb{Q} : q < x\}$. According to Theorem 2, rational numbers are dense in $\mathbb{R}$, therefore there are rational numbers arbitrarily close to x, meaning S is nonempty. The upper bound of S is x, since every rational number $q \in S$ must satisfies $q < r$. To prove x is the least upper bound of S, we use The density of the rational (and irrational) numbers in R, which guarantees that between any number s that is less than a given real number x, there exists a rational number. This means there is a number $q \in S$ that satisfies $s < q < x$. Thus, no number smaller than x can be an upper bound of S, which confirms that $x = \sup(S)$ is indeed the least upper bound.

Similarly, for irrational numbers, we define a set $T = \{t \in \mathbb{R}/\mathbb{Q} : t < x\}$. We have to prove T is dense in $\mathbb{R}$, and the proof for rational numbers can be applied for irrational numbers. We can prove T is dense in $\mathbb{R}$ through irrational numbers are dense in $\mathbb{R}$. Since Q are dense in R, therefore $Q + \sqrt{2}$ are dense in $R + \sqrt{2}$. We know that $Q + \sqrt{2}$ is a subset in of the irrational numbers, therefore irrational numbers are dense in R. From this, we can prove there exists an irrational number t satisfies $s < t < x$. This mean $x = \sup(T)$ is indeed the least upper bound.

Ex 14. Show that if $r > 0$, then for each natural number n, $(1 + r)^n \geq 1 + n.r$.

For $n = 1$, we have $(1 + r)^1 = 1 + 1.r$. Assume that for $n = k$, the statement is true. Let $n = k + 1$, we have

$$(1 + r)^(k + 1) \geq (1 + k.r)(1 + r) = 1 + (k + 1).r + k.r^2 > 1 + (k + 1).r.$$

Hence, the statement is true for $n = k + 1$. By the induction, we finish the proof. The equation happends when $n = 1$.

1.3 The Countable and Uncountable Sets

Quan:

1.3.1 Excercise

1.4 Open Sets, Closed Sets, and Borel Sets of Real Numbers

1.4.1 Excercise

1.5 Sequences of Real Numbers

1.5.1 Summary

A sequence is a function $f : \mathbb{N} \to \mathbb{R}$ with customary notation $\{a_n\}$ where n is called the index, the number a_n is the nth term.
A sequence $\{a_n\}$ is said to be

- bounded if $\exists c \geq 0$ s.t. $|a_n| \leq c \forall n$
- increasing if $a_n < a_{n+1} \forall n$
- decreasing if the sequence $\{-a_n\}$ is increasing
- monotone if it's either increasing or decreasing

For any sequence $\{a_n\}$ and a strictly increase sequence $\{n_k\} \in \mathbb{N}$, call the sequence $\{a_{n_k}\}$ a subsequence of $\{a_n\}$.

Definition 1. *A sequence $\{a_n\}$ converges to it's limit a (write $\lim_{n\to\infty} a_n = a$ or $\{a_n\} \to a$) if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t.*

$$n \geq N \implies |a - a_n| < \epsilon.$$

Proposition 1. *If $\{a_n\} \to a$, then the limit is unique, the sequence is bounded, and, $\forall c \in \mathbb{R}$,*

$$a_n \leq c \forall n \implies a \leq c.$$

Proof ***Ex Extra.***

Theorem 1. *A monotone sequence of real numbers converges if and only if it is bounded.*

Theorem 2 (Bolzano-Weierstrass)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Definition 2. *A sequence of real numbers $\{a_n\}$ is Cauchy if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t.*

$$n, m \geq N \implies |a_m - a_n| < \epsilon.$$

Theorem 3. *A sequence of real numbers converges if and only if it is Cauchy.*

Theorem 4. *Convergent real sequences are linear and monotonic.*

Definition 3. *A sequence $\{a_n\}$ converges to infinity (write $\lim_{n\to\infty} a_n = \infty$ or $\{a_n\} \to \infty$) if $\forall c \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t.*

$$n \geq N \implies a_n \geq c.$$

Similar definitions are made at $-\infty$.

Definition 4. *The limit superior and limit inferior of a sequence $\{a_n\}$ is defined as,*

$$\limsup \{a_n\} = \lim_{n\to\infty} [\sup \{a_k | k \geq n\}]$$
$$\liminf \{a_n\} = \lim_{n\to\infty} [\inf \{a_k | k \geq n\}]$$

Proposition 2. *Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers*

(i) $\limsup \{a_n\} = \ell \in \mathbb{R}$ *if and only if for each $\epsilon > 0$, there are infinitely many indices n for which $a_n > \ell - \epsilon$ and only finitely many indices n for which $a_n > \ell + \epsilon$.*

(ii) $\limsup \{a_n\} = \infty$ *if and only if $\{a_n\}$ is not bounded above.*

(iii) $\limsup \{a_n\} = -\liminf \{-a_n\}$.

(iv) A sequence of real numbers $\{a_n\}$ converges to $a \in \mathbb{R}$ if and only if $\liminf \{a_n\} = \limsup \{a_n\} = a$.

(v) $a_n \leq b_n \forall n \implies \limsup \{a_n\} \leq \liminf \{b_n\}$.

Proof ***Ex 39.***

Definition 5. *For every sequence $\{a_k\}$ of real numbers, define a sequence of partial sums $\{s_n\}$ where $s_n = \sum_{k=1}^{n}$. The series $\sum_{k=1}^{\infty} a_k$ is summable to $s \in \mathbb{R}$ when $\{s_n\} \to s$.*

Proposition 3. *Let $\{a_n\}$ be a sequence of real numbers.*

(i) The series $\sum_{k=1}^{\infty} a_k$ is summable if and only if for each $\epsilon > 0, \exists N \in \mathbb{N}$ s.t.

$$\sum_{k=n}^{n+m} a_k < \epsilon \forall n \geq N, m \in \mathbb{N}.$$

(ii) If the series $\sum_{k=1}^{\infty} |a_k|$ is summable, then $\sum_{k=1}^{\infty} a_k$ also is summable.

(iii) If each term a_k is nonnegative, then the series $\sum_{k=1}^{\infty} a_k$ is summable if and only if the sequence of partial sums is bounded.

Proof ***Ex 45.***

1.5.2 Excercise

Problems done: 38, 39, 40, 41, 45. and proved the first Proposition (i.e. Ex Extra.).

Ex 38.

Lemma 1. *For any set $X \subseteq \mathbb{R}$, $\forall d > 0 \in \mathbb{R}, \exists x \in X$ s.t. $x < \inf X + d$.*

Proof We prove by contradiction. Assume there exists $d > 0 \in \mathbb{R}$ s.t. $\forall x \in X, \inf X + d \leq x$. There is now a greater lower bound $\inf X + d$, which contradicts the definition of infimum.

We use the above lemma to solve this excercise.
Let $\liminf \{a_n\} = L$.

- $\liminf \{a_n\}$ is a cluster point.

  By the above lemma, for every n, we can pick the smallest index $k_n \geq n$ satisfying $a_{k_n} \leq \inf \{a_k | k \geq n\} + \frac{1}{n}$
  Now, $\forall \epsilon > 0, \exists N \in \mathbb{N}, N \geq 1/\epsilon$ s.t. $n \geq N \implies a_{k_n} - L < 1/N \leq \epsilon$. The subsequence $\{a_{k_n}\}$ converges to L by defintion.

- There does not exist a cluster point M satisfying $M < \liminf \{a_n\}$.

  We argue by contradiction. Assume there exists such a cluster point, this means there also exists a subsequence $\{a_{m_j}\}$ that converges to M.
  Let $\epsilon = \frac{M-L}{2}$, by definition, $\exists J \in \mathbb{N}$ s.t.

  $$j \geq J \implies a_{m_j} - M < \epsilon \iff a_{m_j} < M + \epsilon = \frac{L+M}{2}.$$

  Also, by definition, $L = \liminf \{a_n\} = \lim_{n\to\infty} \{\inf \{a_k | k \geq n\}\}$, as such $\exists N \in \mathbb{N}, N > J$ s.t.

  $$n \geq N \implies L - \inf \{a_k | k \geq n\} < \epsilon \iff \inf \{a_k | k \geq n\} > L - \epsilon = \frac{L+M}{2}.$$

  This is a contradiction, as there exists $N \in \mathbb{N}$ satisfying

  $$n \geq N \implies \begin{cases} a_{m_n} < \frac{L+M}{2} \\ \inf \{a_k | k > n\} \geq \frac{L+M}{2} \end{cases}.$$

Proof is similar for $\limsup \{a_n\}$
Ex 39. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers

(i) $\limsup \{a_n\} = \ell \in \mathbb{R}$ if and only if for each $\epsilon > 0$, there are infinitely many indices n for which $a_n > \ell - \epsilon$ and only finitely many indices n for which $a_n > \ell + \epsilon$.

Trivial. Use definition of suprimum and the fact that the collection of sequences $\{\{a_k | k \geq n\}\}_{n=1}^{\infty}$ is decending.

(ii) $\limsup \{a_n\} = \infty$ if and only if $\{a_n\}$ is not bounded above.

We prove the above through showing that $\limsup \{a_n\} < \infty$ if and only if $\{a_n\}$ is bounded above. Note that the limit superior of a sequence always exists.

- If $\{a_n\}$ is bounded above, then $\exists M < \infty \in \mathbb{R}$ s.t. $a_n \leq M \forall n$. As a result, $\sup \{a_k | k \geq n\} \leq M$.
- If $\limsup \{a_n\} < \infty$, then $\sup \{a_k \; k \geq n\}$ is bounded.
  And because there exists $c > 0$ satisfying $a_n \leq \sup \{a_k | k \geq 1\} \leq c$ for all n, the sequence $\{a_n\}$ is also bounded above.

(iii) $\limsup \{a_n\} = -\liminf \{-a_n\}$.

$$\limsup \{a_n\} = \lim_{n\to\infty} \sup \{a_k | k \geq n\} = -\lim_{n\to\infty} \inf \{-a_k | k \geq n\} = -\liminf \{-a_n\}.$$

(I ommitted the proof to $\lim_{n\to\infty} \{a_n\} = -\lim_{n\to\infty} \{-a_n\}$. It is trivial and uses the definition.)

(iv) A sequence of real numbers $\{a_n\}$ converges to $a \in \mathbb{R}$ if and only if $\liminf \{a_n\} = \limsup \{a_n\} = a$.

- $\liminf \{a_n\} = \limsup \{a_n\} = a \implies \{a_n\} \to a$
  For any $\epsilon > 0$, there exists $N, M \in \mathbb{N}$ s.t.

$$\begin{cases} n \geq N \implies -\epsilon < a - \sup \{a_k | k \geq n\} \leq a - a_n \\ n \geq M \implies a - a_n \leq a - \inf \{a_k | k \geq n\} < \epsilon \end{cases}$$

  So $\exists L = \max(N, M) \in \mathbb{N}$ s.t.

$$n \geq L \implies \begin{cases} -\epsilon < a - a_n \\ a - a_n < \epsilon \end{cases} \implies |a - a_n| < \epsilon.$$

  By definition, $\{a_n\} \to a$
- $\{a_n\} \to a \implies \liminf \{a_n\} = \limsup \{a_n\} = a$
  For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ s.t.

$$\forall n \geq N, \begin{cases} -\epsilon < a - a_n \\ a - a_n < \epsilon \end{cases} \implies \forall n \geq N, \begin{cases} \inf \{a_k | k \geq n\} \leq a_n < a + \epsilon \\ a - \epsilon < a_n \implies a - \epsilon < \inf \{a_k | k \geq n\} \end{cases}.$$

  which is equivalent to $|a - \inf \{a_k | k \geq n\}| < \epsilon$ for all $n \geq N$ and so $\liminf \{a_n\} = a$ by definition.
  Similar proof is done for $\limsup \{a_n\} = a$.

(v) $a_n \leq b_n \forall n \implies \limsup \{a_n\} \leq \liminf \{b_n\}$. (similar to book)
Consider the sequence $\{c_n\}$, where $c_n = \inf \{b_k | k \geq n\} - \sup \{a_k | k \geq n\}$ for all n.
By linearity of convergent sequences, $\{c_n\} \to c = \liminf \{b_n\} - \limsup \{a_n\}$. This means, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t.

$$n \geq N \implies -\epsilon < c - c_n < \epsilon.$$

In particular, $0 \leq c_N < c + \epsilon$. Since $c \geq -\epsilon$ for any positive number ϵ, $c \geq 0$.

Ex 40.
Proven above in **<u>Ex. 38</u>**, $\liminf \{a_n\}$ and $\limsup \{a_n\}$ are the smallest and largest cluster points of $\{a_n\}$.
Shown above in **<u>Ex. 39</u>**, $\{a_n\} \to a \iff \liminf \{a_n\} = \limsup \{a_n\} = a$.
The proof is now trivial.
The sequence $\{a_n\}$ has only one cluster point if and only if $\liminf \{a_n\} = \limsup \{a_n\} = a$, which is equivalent to $\{a_n\} \to a$.

Ex 41. At every index n,

$$\inf \{a_k | k \geq n\} \leq \sup \{a_k | k \geq n\}$$

And so, by the linearity property of convergent sequences, $\lim_{n \to \infty} \inf \{a_k | k \geq n\} \leq \lim_{n \to \infty} \sup \{a_k | k \geq n\}$ or $\liminf \{a_n\} \leq \limsup \{a_n\}$.

Ex 45. Let $\{a_n\}$ be a sequence of real numbers.

(i) The series $\sum_{k=1}^{\infty} a_k$ is summable if and only if for each $\epsilon > 0, \exists N \in \mathbb{N}$ s.t.

$$\left| \sum_{k=n}^{n+m} a_k \right| < \epsilon \forall n \geq N, m \in \mathbb{N}.$$

The series $\sum_{k=1}^{\infty} a_k$ is summable if and only if $\{s_n\}$ converges.
As such, for each $\epsilon > 0, \exists N \in \mathbb{N}$ s.t.

$$\begin{aligned} j > i - 1 \geq N &\implies \epsilon > \left| \sum_{k=i}^{j} a_k \right| \\ \iff n \geq N, m \in \mathbb{N} &\implies \epsilon > \left| \sum_{k=n}^{n+m} a_k \right| \quad (i - 1 = n, j = n + m) \end{aligned}$$

(ii) If the series $\sum_{k=1}^{\infty} |a_k|$ is summable, then $\sum_{k=1}^{\infty} a_k$ also is summable.

If the series $\sum_{k=1}^{\infty} |a_k|$ is summable, then the partial sum sequence $\{\sum_{k=1}^{n} |a_k|\}$ converges.

As such, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t.

$$n, m \geq N \implies \epsilon > \left| \sum_{k=\min(m,n)}^{\max(m,n)} |a_k| \right| \geq \left| \sum_{k=\min(m,n)}^{\max(m,n)} a_k \right|.$$

The partial sum sequence $\{\sum_{k=1}^{n} a_k\}$ converges because it is Cauchy. As a result, the series $\sum_{k=1}^{\infty} a_k$ also is summable.

(iii) If each term a_k is nonnegative, then the series $\sum_{k=1}^{\infty} a_k$ is summable if and only if the sequence of partial sums is bounded.

Since $a_k > 0 \forall k \in \mathbb{N}$, $s_n = \sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n+1} a_k = s_{n+1}$ for all n. In other words, the partial sum sequence is nondecreasing.

The series $\sum_{k=1}^{\infty} a_k$ is summable if and only if $\{s_n\}$ converges.

- If $\{s_n\}$ converges then it is bounded.
- If $\{s_n\}$ is bounded, then it converges to $s = \sup\{s_n | n \in \mathbb{N}\}$ (note that the suprimum exists thanks to the Completeness Axiom)

  For any $\epsilon > 0$, we have:

  +) $s_n \leq s < s + \epsilon$ for all n.

  +) Because $s - \epsilon$ is not an upperbound of $\{s_n | n \in \mathbb{N}\}$, $\exists N \in \mathbb{N}$ s.t $s_N > s - \epsilon$.
  And since the sequence $\{s_n\}$ is nondecreasing, $n \geq N \implies s_n > s - \epsilon$.

  By definition, $\{s_n\}$ converges to s.

Ex Extra. If $\{a_n\} \to a$, then:

- The limit is unique.

  We prove by contradiction. Assume $\{a_n\} \to a, \{a_n\} \to b$ and $a \neq b$.

  Let $d = |a - b|$ and $\epsilon = \frac{d}{2}$. By definition, there exists $N, M \in \mathbb{N}$ s.t.

$$\begin{cases} n \geq N \implies |a - a_n| < \epsilon \\ n \geq M \implies |b - a_n| < \epsilon \end{cases}$$

  So $\exists L = \max(N, M) \in \mathbb{N}$ s.t. $n \geq L$ implies both $|a - a_n|$ and $|b - a_n|$ are less than ϵ.

  By the triangle inequality, $d = |a - b| \leq |a - a_n| + |b - a_n| < 2\epsilon = 2 \times \frac{d}{2} = d$. In other words, $d < d$, which is a contradiction.

- The sequence is bounded.

  Choose any $\epsilon > 0$.

  By definition, $\exists N \in \mathbb{N}$ s.t.

$$n \geq N \implies -\epsilon < a - a_n < \epsilon \iff a - \epsilon < a_n < a + \epsilon \implies |a_n| < |a| + \epsilon$$

  Denote $M_1 = \max[\{a_n | n \in \mathbb{N}, n < N\}]$, note that we can always find M_1 because this sequence is finite.

  We conclude that $\{a_n\}$ is bounded by $\max(|a| + \epsilon, M_1)$.

- $\forall c \in \mathbb{R}$, if $a_n \leq c \forall n$ then $a \leq c$.

Approach 1) Using only the definition.
For any $\epsilon > 0, \exists N \in \mathbb{N}$ s.t

$$n \geq N \implies |a - a_n| < \epsilon \implies a - \epsilon < a_n \leq c$$

Since $a - \epsilon < c$ is true for all $\epsilon > 0$, we conclude that $a \leq c$.

Approach 2) Using only the definition.
Prove by contradiction. Assume $a > c$, then set $\epsilon = a - c > 0$...

Approach 3) Consider the sequence $\{c_n\}$, where $c_n = c \forall n$ and use the monotonic property of convergent sequences. (Trivial)

1.6 Continuous Real-Valued Functions of Real Variable

1.6.1 Excercise

Ex 50. Show that a Lipschitz function is uniformly continuous but there are uniformly continuous functions that are not Lipschitz

We can prove that not all functions that are uniformly continuous are a Lipschitz function by using contradiction. Suppose we have function $f = \sqrt{x}$ uniformly continuous on $\{0, 1\}$ and is a Lipschitz function. Based on definition, there is a $c > 0$ for which

$$|\sqrt{x'} - \sqrt{x}| \leq c|x' - x|. \tag{2}$$

If we take $x = 0$, the equation become $|\sqrt{x'}| \leq c|x'|$. We can rewrite this as $|\sqrt{x'}|/|x'| \leq c$. However, if $x' \to 0$, we have $|\sqrt{x'}|/|x'| \to \infty$ which contradicts the inequality. Hence, $f = \sqrt{x}$ is not a Lipschitz function.

Ex 53. Show that a set E of real numbers is closed and bounded if and only if every open cover of E has a finite subcover.

(i) ($\Rightarrow$) According to Heine-Borel theorem, if a set E of real numbers is closed and bounded, every open cover of E has a finite subcover.

(ii) ($\Leftarrow$) We first prove that if every open cover of E has a finite subcover, then E is bounded. We form an open cover of E by defining a set $O_x = (x - 1, x + 1)$ for every $x \in E$. The collection $\{O_x : x \in E\}$ is an open cover for E. This collection must have a finite subcover $\{O_{x_1}, O_{x_2}, \ldots, O_{x_n}\}$. Since E is contained in a finite union of bounded sets, E must be bounded.

We now prove that E must be closed. Suppose E is not closed. Let $y \notin E$ be a point of closure of E. We form an open cover of E by defining a set $O_x = (x - r_x, x + r_x)$ where $r_x = |y - x|/2$ for every $x \in E$. The collection $\{O_x : x \in E\}$ must have a finite subcover $\{O_{x_1}, O_{x_2}, \ldots, O_{x_n}\}$. Let $r_{\min} = \min\{r_{x_1}, r_{x_2}, \ldots, r_{x_n}\}$. Since y is a point of closure of E, the open interval $(y - r_{\min}, y + r_{\min})$ must contain a point $x' \in E$. This means $|x' - y| < r_{\min}$. We now show that x' is not in the subcover.

$$\forall i : 1 \leq i \leq n, |x_i - x'| > |x_i - y| - |x' - y| > |x_i - y| - r_{\min} > 2r_{x_i} - r_{\min} > r_{x_i}$$

$\forall i : |x_i - x'| > r_{x_i} \Rightarrow \forall i : x' \notin O_{x_i} \Rightarrow x' \notin \bigcup_{1 \leq i \leq n} O_{x_i}$. This means the finite subcover fails to cover E. This contradiction implies that E is closed.