Real Analysis - P1

Learning Theory and Applications Group

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Contents

Part I
Lebesgue Intergration

Key definitions here:

1 The Real Number: Sets, Sequences, and Functions

1.1 The Field, Positivity, and Completeness Axioms

1.1.1 Excercise

Ex 1. For $a \neq 0$ and $b \neq 0$, show that $(ab)^{-1} = a^{-1}b^{-1}$

$$(a^{-1}b^{-1})(ab) = a^{-1}b^{-1}ba = a^{-1}1a = 1$$

As a result, $a^{-1}b^{-1} = (ab^{-1})$
Ex 2. Verify the following:

- For each real number $a \neq 0$, $a^2 > 0$. In particular, $1 > 0$ since $1 \neq 0$ and $1 = 1^2$
- For each positive number a, its multiplicative inverse a^{-1} also is positive
- If $a > b$, then
$$ac > bc \text{ if } c > 0 \text{ and } ac < bc \text{ if } c < 0$$

For the first point, we first need to prove that, for any a, then $-a = (-1)a$,

$$a + (-a) = 0 = (1 + (-1))a = a + (-1)a$$

Next, for each $a \neq 0$, if a is positive, then a^2 is positive by definition of positiveness. On the other hand, if $a < 0$, then let $a = -b$ with $b > 0$,

$$a^2 = (-b)^2 = (-1)b(-1)b = (-1)(-b)b = (-1)^2 b^2 > 0 \tag{1}$$

For the second point, assuming by contradiction that $a^{-1} < 0$ for any $a > 0$, then let $a^{-1} = -b$ with $b > 0$. then

$$1 = a(a^{-1}) = a(-b) = (-1)ab < 0$$

Here $ab > 0$ since both a and b are positive, and we know from previous point that $0 > -(ab) = (-1)ab$

The last point is straighforward from the definition of $>$.

$$ac - bc = \underbrace{(a-b)}_{>0}\underbrace{c}_{>0} > 0$$

$$ac - bc = \underbrace{(a-b)}_{>0}\underbrace{c}_{<0} = (-1)\underbrace{(a-b)}_{>0}\underbrace{d}_{>0} < 0 \text{ with } d = -c$$

Ex 3. For a nonempty set of real numbers E, show that $\inf E = \sup E$ if and only if E consists of a single point.

If the set E has a single element, then the least upper bound equal to that single element. This similarly applies to lowerbound. In other word, its sup and inf coincides.

On the other direction, if a set E has its sup and inf equal, and assuming by contradiction that E has at least 2 distinct elements, then the gap between these two points $\neq 0$. The difference between sup and inf is lowerbounded by this gap, so they cannot equal.

Ex 4. Let a and b be real numbers.

i Show that if $ab = 0$, then $a = 0$ or $b = 0$

ii Verify that $a^2 - b^2 = (a-b)(a+b)$ and conclude from part (i) that if $a^2 = b^2$, then $a = b$ or $a = -b$.

iii Let c be a positive real number. Define $E = \{x \in \mathbb{R} | x^2 < c\}$ verify that E is nonempty and bounded above. Define $x_0 = \sup E$. Show that $x_0^2 = c$. Use part (ii) to show that there is a unique $x > 0$ for which $x^2 = c$. It is denoted by $\sqrt{c}$

For the first point, suppose that $ab = 0$ and both a and b are not 0, then there exists a^{-1} and b^{-1}, then we have

$$abb^{-1}a^{-1} = 1$$

which means that $b^{-1}a^{-1} = (ab)^{-1}$, but since $ab = 0$, no such number exists.

The second point is a straighforward application of distributive property,

$$(a-b)(a+b) = a(a+b) + (-b)(a+b) = a^2 + ab - ba - b^2 = a^2 - b^2$$

Then from part (i), since $(a-b)(a+b) = 0$, one of the two terms must be 0.

In part (iii), we see that $0^2 = 0 < c$ for all $c > 0$, so E is nonempty. By contradiction, suppose E is not bounded from above, that is, for every $b > 0$, we can always choose some $x \in E$ such that $x > b$, letting $b > c$ lead to a contradiction with the definition of E.

Next, since E is bounded from above, then it has a supremum by completeness axiom. Denote $x_0 = \sup E$. We will show that $x_0^2 \geq c$ and $x_0^2 \leq c$ to conclude that $x_0^2 = c$.

Since $x^2 < c, \forall x \in E$, c is an upperbound of E^2, and because $\sup E$ is the smallest/least upperbound, then $\sup(E)^2 \leq c$. On the otherhand, $x_0 \geq x, \forall x \in E$ and E contains **all** real numbers whose square less than c, so $x_0^2 \geq c$.

Finally, we need to show that x_0 is a unique positive real number such that $x_0^2 = c$. By contradiction, suppose there is some $x > 0$ such that $x \neq x_0$ and $x^2 = c$, then by part (ii), since $x_0^2 = x^2$, we have either $x = x_0$ or $x = -x_0$, but x is positive and $-x_0$ is negetive, so $x = x_0$.

Ex 5. Let a, b, c be real bumbers such that $a \neq 0$ and consider the quadratic equation

$$ax^2 + bx + c = 0, x \in \mathbb{R}$$

i Suppose $b^2 - 4ac > 0$, use the Field Axiom and the preceding problem to complete the square and thereby show that this equation has exactly two solutions given by

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

ii Now suppose $b^2 - 4ac < 0$. Show that the quadratic equation fails to have any solution.

Suppose that $b^2 - 4ac > 0$, then from previous problem, there exists a unique positive number $\sqrt{b^2 - 4ac}$. we can verify that

$$\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = x^2 - x\frac{-b - \sqrt{b^2 - 4ac}}{2a} - x\frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{b^2 - b^2 + 4ac}{4a^2}$$
$$= x^2 + x\frac{b}{a} + \frac{c}{a} = 0.$$

Then also from the previous problem, either one of the two terms equal 0. As a result, the equation has exactly two solutions.

On the other hand, if $b^2 - 4ac < 0$, then the equation can be rewritten as

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{4ac - b^2}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4} > 0,$$

which does not have any solution.

Ex 6. Use the Completeness Axiom to show that every nonempty set of real numbers that is bounded below has an infimum and that

$$\inf E = -\sup\{-x|x \in E\}.$$

The set E is bounded below, which means that the set $E' = \{-x|x \in E\}$ is bounded from above, then its supremum exists by completeness axiom. Denote $x_0 = \sup E'$, then $x_0 \geq -x, \forall x \in E \rightleftarrows -x_0 \leq x, \forall x \in E$. As a result, $-x_0 \leq \inf E$.

Suppose that there exists some x' such that $x' > -x_0$ and $x' \leq x, \forall x \in E$; i.e. x' is a "greater" lowerbound of E than x_0. Then we can show that $-x'$ is a "smaller" upperbound of E', which contradicts with the definition of supremum. As a result, no such x' exists, and $-x_0$ is the infimum of E

Ex 7. For real bumbers a and b, verify the following:

i $|ab| = |a||b|$

ii $|a + b| \leq |a| + |b|$

iii For $\epsilon > 0$,

$$|x - a| < \epsilon \text{ if and only if } a - \epsilon < x < a + \epsilon$$

First we define the sign operator as $\text{sg}(x) \in \{1, -1\}, x \neq 0$. The absolute value can be written as the product with the sign operator

$$|a| = a\text{sg}(a)$$

Then the first claim can be verified as

$$|ab| = ab\text{sg}(ab) = a\text{sg}(a)b\text{sg}(b) = |a||b|$$

by noting $\text{sg}(ab) = \text{sg}(a)\text{sg}(b)$, and

$$|a + b| = (a + b)\text{sg}(a + b) = a\text{sg}(a + b) + b\text{sg}(a + b) \leq a\text{sg}(a) + b\text{sg}(b) = |a| + |b|$$

by noting $a\text{sg}(a) = \max(a, -a) \geq a\text{sg}(c), \forall c$

Final point: if $x - a > 0$, then $|x - a| = x - a$ and $|x - a| < \epsilon \rightleftarrows a < x < a + \epsilon$

Similar, if $x - a < 0$, then $|x - a| < \epsilon \rightleftarrows a > x > a - \epsilon$, combining the both cases and with the zero case yield the desired claim.

1.2 The Natural and Rational Numbers

Hoang Anh:

1.2.1 Excercise

1.3 The Countable and Uncountable Sets

Quan:

1.3.1 Excercise