Real Analysis - P1

Learning Theory and Applications Group

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Contents

Part I
Lebesgue Intergration

Key definitions here:

1 The Real Number: Sets, Sequences, and Functions

1.1 The Field, Positivity, and Completeness Axioms

1.1.1 Excercise

Ex 1. For $a \neq 0$ and $b \neq 0$, show that $(ab)^{-1} = a^{-1}b^{-1}$

$$(a^{-1}b^{-1})(ab) = a^{-1}b^{-1}ba = a^{-1}1a = 1$$

As a result, $a^{-1}b^{-1} = (ab^{-1})$

Ex 2. Verify the following:

- For each real number $a \neq 0$, $a^2 > 0$. In particular, $1 > 0$ since $1 \neq 0$ and $1 = 1^2$
- For each positive number a, its multiplicative inverse a^{-1} also is positive
- If $a > b$, then

$$ac > bc \text{ if } c > 0 \text{ and } ac < bc \text{ if } c < 0$$

For the first point, we first need to prove that, for any a, then $-a = (-1)a$,

$$a + (-a) = 0 = (1 + (-1))a = a + (-1)a$$

Next, for each $a \neq 0$, if a is positive, then a^2 is positive by definition of positiveness. On the other hand, if $a < 0$, then let $a = -b$ with $b > 0$,

$$a^2 = (-b)^2 = (-1)b(-1)b = (-1)(-b)b = (-1)^2 b^2 > 0 \tag{1}$$

For the second point, assuming by contradiction that $a^{-1} < 0$ for any $a > 0$, then let $a^{-1} = -b$ with $b > 0$. then

$$1 = a(a^{-1}) = a(-b) = (-1)ab < 0$$

Here $ab > 0$ since both a and b are positive, and we know from previous point that $0 > -(ab) = (-1)ab$

The last point is straighforward from the definition of $>$.

$$ac - bc = \underbrace{(a-b)}_{>0}\underbrace{c}_{>0} > 0$$

$$ac - bc = \underbrace{(a-b)}_{>0}\underbrace{c}_{<0} = (-1)\underbrace{(a-b)}_{>0}\underbrace{d}_{>0} < 0 \text{ with } d = -c$$

Ex 3. For a nonempty set of real numbers E, show that $\inf E = \sup E$ if and only if E consists of a single point.

If the set E has a single element, then the least upper bound equal to that single element. This similarly applies to lowerbound. In other word, its sup and inf coincides.

On the other direction, if a set E has its sup and inf equal, and assuming by contradiction that E has at least 2 distinct elements, then the gap between these two points $\neq 0$. The difference between sup and inf is lowerbounded by this gap, so they cannot equal.

Ex 4. Let a and b be real numbers.

i Show that if $ab = 0$, then $a = 0$ or $b = 0$

ii Verify that $a^2 - b^2 = (a-b)(a+b)$ and conclude from part (i) that if $a^2 = b^2$, then $a = b$ or $a = -b$.

iii Let c be a positive real number. Define $E = \{x \in \mathbb{R} | x^2 < c\}$ verify that E is nonempty and bounded above. Define $x_0 = \sup E$. Show that $x_0^2 = c$. Use part (ii) to show that there is a unique $x > 0$ for which $x^2 = c$. It is denoted by $\sqrt{c}$

For the first point, suppose that $ab = 0$ and both a and b are not 0, then there exists a^{-1} and b^{-1}, then we have

$$abb^{-1}a^{-1} = 1$$

which means that $b^{-1}a^{-1} = (ab)^{-1}$, but since $ab = 0$, no such number exists.

The second point is a straighforward application of distributive property,

$$(a-b)(a+b) = a(a+b) + (-b)(a+b) = a^2 + ab - ba - b^2 = a^2 - b^2$$

Then from part (i), since $(a-b)(a+b) = 0$, one of the two terms must be 0.

In part (iii), we see that $0^2 = 0 < c$ for all $c > 0$, so E is nonempty. By contradiction, suppose E is not bounded from above, that is, for every $b > 0$, we can always choose some $x \in E$ such that $x > b$, letting $b > c$ lead to a contradiction with the definition of E.

Next, since E is bounded from above, then it has a supremum by completeness axiom. Denote $x_0 = \sup E$. We will show that $x_0^2 \geq c$ and $x_0^2 \leq c$ to conclude that $x_0^2 = c$.

Since $x^2 < c, \forall x \in E$, c is an upperbound of E^2, and because $\sup E$ is the smallest/least upperbound, then $\sup(E)^2 \leq c$. On the otherhand, $x_0 \geq x, \forall x \in E$ and E contains **all** real numbers whose square less than c, so $x_0^2 \geq c$.

Finally, we need to show that x_0 is a unique positive real number such that $x_0^2 = c$. By contradiction, suppose there is some $x > 0$ such that $x \neq x_0$ and $x^2 = c$, then by part (ii), since $x_0^2 = x^2$, we have either $x = x_0$ or $x = -x_0$, but x is positive and $-x_0$ is negetive, so $x = x_0$.

Ex 5. Let a, b, c be real bumbers such that $a \neq 0$ and consider the quadratic equation

$$ax^2 + bx + c = 0, x \in \mathbb{R}$$

i Suppose $b^2 - 4ac > 0$, use the Field Axiom and the preceding problem to complete the square and thereby show that this equation has exactly two solutions given by

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

ii Now suppose $b^2 - 4ac < 0$. Show that the quadratic equation fails to have any solution.

Suppose that $b^2 - 4ac > 0$, then from previous problem, there exists a unique positive number $\sqrt{b^2 - 4ac}$. we can verify that

$$\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = x^2 - x\frac{-b - \sqrt{b^2 - 4ac}}{2a} - x\frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{b^2 - b^2 + 4ac}{4a^2}$$
$$= x^2 + x\frac{b}{a} + \frac{c}{a} = 0.$$

Then also from the previous problem, either one of the two terms equal 0. As a result, the equation has exactly two solutions.

On the other hand, if $b^2 - 4ac < 0$, then the equation can be rewritten as

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{4ac - b^2}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4} > 0,$$

which does not have any solution.

Ex 6. Use the Completeness Axiom to show that every nonempty set of real numbers that is bounded below has an infimum and that

$$\inf E = -\sup\{-x|x \in E\}.$$

The set E is bounded below, which means that the set $E' = \{-x|x \in E\}$ is bounded from above, then its supremum exists by completeness axiom. Denote $x_0 = \sup E'$, then $x_0 \geq -x, \forall x \in E \rightleftarrows -x_0 \leq x, \forall x \in E$. As a result, $-x_0 \leq \inf E$.

Suppose that there exists some x' such that $x' > -x_0$ and $x' \leq x, \forall x \in E$; i.e. x' is a "greater" lowerbound of E than x_0. Then we can show that $-x'$ is a "smaller" upperbound of E', which contradicts with the definition of supremum. As a result, no such x' exists, and $-x_0$ is the infimum of E

Ex 7. For real bumbers a and b, verify the following:

i $|ab| = |a||b|$

ii $|a + b| \leq |a| + |b|$

iii For $\epsilon > 0$,

$$|x - a| < \epsilon \text{ if and only if } a - \epsilon < x < a + \epsilon$$

First we define the sign operator as $\text{sg}(x) \in \{1, -1\}, x \neq 0$. The absolute value can be written as the product with the sign operator

$$|a| = a\text{sg}(a)$$

Then the first claim can be verified as

$$|ab| = ab\text{sg}(ab) = a\text{sg}(a)b\text{sg}(b) = |a||b|$$

by noting $\text{sg}(ab) = \text{sg}(a)\text{sg}(b)$, and

$$|a + b| = (a + b)\text{sg}(a + b) = a\text{sg}(a + b) + b\text{sg}(a + b) \leq a\text{sg}(a) + b\text{sg}(b) = |a| + |b|$$

by noting $a\text{sg}(a) = \max(a, -a) \geq a\text{sg}(c), \forall c$

Final point: if $x - a > 0$, then $|x - a| = x - a$ and $|x - a| < \epsilon \rightleftarrows a < x < a + \epsilon$

Similar, if $x - a < 0$, then $|x - a| < \epsilon \rightleftarrows a > x > a - \epsilon$, combining the both cases and with the zero case yield the desired claim.

1.2 The Natural and Rational Numbers

Hoang Anh:

1.2.1 Excercise

1.3 The Countable and Uncountable Sets

Quan:

1.3.1 Excercise

1.4 Open Sets, Closed Sets, and Borel Sets of Real Numbers

1.4.1 Excercise

1.5 Sequences of Real Numbers

1.5.1 Summary

A sequence is a function $f : \mathbb{N} \to \mathbb{R}$ with customary notation $\{a_n\}$ where n is called the index, the number a_n is the nth term.

A sequence $\{a_n\}$ is said to be

- bounded if $\exists c \geq 0$ s.t. $|a_n| \leq c \forall n$
- increasing if $a_n < a_{n+1} \forall n$
- decreasing if the sequence $\{-a_n\}$ is increasing
- monotone if it's either increasing or decreasing

For any sequence $\{a_n\}$ and a strictly increase sequence $\{n_k\} \in \mathbb{N}$, call the sequence $\{a_{n_k}\}$ a subsequence of $\{a_n\}$.

Definition 1. *A sequence $\{a_n\}$ converges to it's limit a (write $\lim_{n\to\infty} a_n = a$ or $\{a_n\} \to a$) if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t.*

$$n \geq N \implies |a - a_n| < \epsilon.$$

Proposition 1. *If $\{a_n\} \to a$, then the limit is unique, the sequence is bounded, and, $\forall c \in \mathbb{R}$,*

$$a_n \leq c \forall n \implies a \leq c.$$

Theorem 1. *A monotone sequence of real numbers converges if and only if it is bounded.*

Theorem 2 (Bolzano-Weierstrass)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Definition 2. *A sequence of real numbers $\{a_n\}$ is Cauchy if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t.*

$$n, m \geq N \implies |a_m - a_n| < \epsilon.$$

Theorem 3. *A sequence of real numbers converges if and only if it is Cauchy.*

Theorem 4. *Convergent real sequences are linear and monotonic.*

Definition 3. *A sequence $\{a_n\}$ converges to infinity (write $\lim_{n\to\infty} a_n = \infty$ or $\{a_n\} \to \infty$) if $\forall c \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t.*

$$n \geq N \implies a_n \geq c.$$

Similar definitions are made at $-\infty$.

Definition 4. *The limit superior and limit inferior of a sequence $\{a_n\}$ is defined as,*

$$\limsup \{a_n\} = \lim_{n\to\infty} \left[\sup \{a_k | k \geq n\}\right]$$
$$\liminf \{a_n\} = \lim_{n\to\infty} \left[\inf \{a_k | k \geq n\}\right]$$

Proposition 2. *Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers* *(i)* $\limsup \{a_n\} = \ell \in \mathbb{R}$ *if and only if for each* $\epsilon > 0$*, there are infinitely many indices* n *for which* $a_n > \ell - \epsilon$ *and only finitely many indices* n *for which* $a_n > \ell + \epsilon$.

(ii) $\limsup \{a_n\} = \infty$ *if and only if* $\{a_n\}$ *is not bounded above.*

(iii) $\limsup \{a_n\} = -\liminf \{-a_n\}$.

(iv) A sequence of real numbers $\{a_n\}$ *converges to* $a \in \mathbb{R}$ *if and only if* $\liminf \{a_n\} = \limsup \{a_n\} = a$.

(v) $a_n \leq b_n \forall n \implies \limsup \{a_n\} \leq \liminf \{b_n\}$.

Proof ***Ex 39.***

Definition 5. *For every sequence* $\{a_k\}$ *of real numbers, define a sequence of partial sums* $\{s_n\}$ *where* $s_n = \sum_{k=1}^{n}$. *The series* $\sum_{k=1}^{\infty} a_k$ *is summable to* $s \in \mathbb{R}$ *when* $\{s_n\} \to s$.

Proposition 3. *Let* $\{a_n\}$ *be a sequence of real numbers.*

(i) The series $\sum_{k=1}^{\infty} a_k$ *is summable if and only if for each* $\epsilon > 0, \exists N \in \mathbb{N}$ *s.t.*

$$\sum_{k=n}^{n+m} a_k < \epsilon \forall n \geq N, m \in \mathbb{N}.$$

(ii) If the series $\sum_{k=1}^{\infty} |a_k|$ *is summable, then* $\sum_{k=1}^{\infty} a_k$ *also is summable.*

(iii) If each term a_k *is nonnegative, then the series* $\sum_{k=1}^{\infty} a_k$ *is summable if and only if the sequence of partial sums is bounded.*

Proof ***Ex 45.***

1.5.2 Excercise

Problems done: 38, 39, 40, 41, 45.

Ex 38.

Lemma 1. *For any set* $X \subseteq \mathbb{R}$, $\forall d > 0 \in \mathbb{R}, \exists x \in X$ *s.t.* $x < \inf X + d$.

Proof We prove by contradiction. Assume there exists $d > 0 \in \mathbb{R}$ s.t. $\forall x \in X, \inf X + d \leq x$. There is now a greater lower bound $\inf X + d$, which contradicts the definition of infimum.

We use the above lemma to solve this excercise.
Let $\liminf \{a_n\} = L$.

- $\liminf \{a_n\}$ is a cluster point.

  By the above lemma, for every n, we can pick the smallest index $k_n \geq n$ satisfying $a_{k_n} \leq \inf \{a_k | k \geq n\} + \frac{1}{n}$
  Now, $\forall \epsilon > 0, \exists N \in \mathbb{N}, N \geq 1/\epsilon$ s.t. $n \geq N \implies a_{k_n} - L < 1/N < \epsilon$. The subsequence $\{a_{k_n}\}$ converges to L by defintion.

- There does not exist a cluster point M satisfying $M < \liminf \{a_n\}$.

  We argue by contradiction. Assume there exists such a cluster point, this means there also exists a subsequence $\{a_{m_j}\}$ that converges to M.
  Let $\epsilon = \frac{M-L}{2}$, by definition, $\exists J \in \mathbb{N}$ s.t.

  $$j \geq J \implies a_{m_j} - M < \epsilon \iff a_{m_j} < M + \epsilon = \frac{L+M}{2}.$$

  Also, by definition, $L = \liminf \{a_n\} = \lim_{n\to\infty} \{\inf \{a_k | k \geq n\}\}$, as such $\exists N \in \mathbb{N}, N > J$ s.t.

  $$n \geq N \implies L - \inf \{a_k | k \geq n\} < \epsilon \iff \inf \{a_k | k \geq n\} > L - \epsilon = \frac{L+M}{2}.$$

  This is a contradiction, as there exists $N \in \mathbb{N}$ satisfying

  $$n \geq N \implies \begin{cases} a_{m_n} < \frac{L+M}{2} \\ \inf \{a_k | k > n\} \geq \frac{L+M}{2} \end{cases}.$$

Proof is similar for $\limsup \{a_n\}$

Ex 39. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers

(i) $\limsup \{a_n\} = \ell \in \mathbb{R}$ if and only if for each $\epsilon > 0$, there are infinitely many indices n for which $a_n > \ell - \epsilon$ and only finitely many indices n for which $a_n > \ell + \epsilon$.

Trivial. Use definition of suprimum and the fact that the collection of sequences $\{\{a_k | k \geq n\}\}_{n=1}^{\infty}$ is decending.

(ii) $\limsup \{a_n\} = \infty$ if and only if $\{a_n\}$ is not bounded above.

We prove the above through showing that $\limsup \{a_n\} < \infty$ if and only if $\{a_n\}$ is bounded above. Note that $\limsup$ of a sequence always exists.

- If $\{a_n\}$ is bounded above, then $\exists M < \infty \in \mathbb{R}$ s.t. $a_n \leq M \forall n$. As a result, $\sup \{a_k | k \geq n\} \leq M$.
- If $\limsup \{a_n\} < \infty$, then $\sup \{a_k \ k \geq n\}$ is bounded.
  Because there exists $c > 0$ satisfying $a_n \leq \sup \{a_k | k \geq 1\} \leq c$ for all n, the sequence $\{a_n\}$ is also bounded above.

(iii) $\limsup \{a_n\} = -\liminf \{-a_n\}$.

$$\limsup \{a_n\} = \lim_{n\to\infty} \sup \{a_k | k \geq n\} = -\lim_{n\to\infty} \inf \{-a_k | k \geq n\} = -\liminf \{-a_n\}.$$

(iv) A sequence of real numbers $\{a_n\}$ converges to $a \in \mathbb{R}$ if and only if $\liminf \{a_n\} = \limsup \{a_n\} = a$.

- $\liminf \{a_n\} = \limsup \{a_n\} = a \implies \{a_n\} \to a$
  For any $\epsilon > 0$, there exists $N, M \in \mathbb{N}$ s.t.

$$\begin{cases} n \geq N \implies -\epsilon < a - \sup \{a_k | k \geq n\} \leq a - a_n \\ n \geq M \implies a - a_n \leq a - \inf \{a_k | k \geq n\} < \epsilon \end{cases}$$

  So $\exists L = \max N, M \in \mathbb{N}$ s.t.

$$n \geq L \implies \begin{cases} -\epsilon < a - a_n \\ a - a_n < \epsilon \end{cases} \implies |a - a_n| < \epsilon.$$

  By definition, $\{a_n\} \to a$
- $\{a_n\} \to a \implies \liminf \{a_n\} = \limsup \{a_n\} = a$
  For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ s.t.

$$\forall n \geq N, \begin{cases} -\epsilon < a - a_n \\ a - a_n < \epsilon \end{cases} \implies \forall n \geq N, \begin{cases} \inf \{a_k | k \geq n\} \leq a_n < a + \epsilon \\ a - \epsilon < a_n \implies a - \epsilon < \inf \{a_k | k \geq n\} \end{cases}.$$

  which is equivalent to $n \geq N \implies |a - \inf \{a_k | k \geq n\}| < \epsilon$ and so $\liminf \{a_n\} = a$ by definition.
  Similar proof is done for $\limsup \{a_n\} = a$.

(v) $a_n \leq b_n \forall n \implies \limsup \{a_n\} \leq \liminf \{b_n\}$. (similar to book)

Consider a new sequence $c_n = \inf \{b_k | k \geq n\} - \sup \{a_k | k \geq n\}$ for all n.

By linearity of convergent sequences, $\{c_n\} \to c = \liminf \{b_n\} - \limsup \{a_n\}$. This means, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t.

$$n \geq N \implies -\epsilon < c - c_n < \epsilon.$$

In particular, $0 \leq c_N < c + \epsilon$. Since $c \geq -\epsilon$ for any positive number ϵ, $c \geq 0$.

Ex 40.

Prove above in **<u>Ex. 38</u>**, $\liminf \{a_n\}$ and $\limsup \{a_n\}$ are the smallest and largest cluster points of $\{a_n\}$.
Shown above in **<u>Ex. 39</u>**, $\{a_n\} \to a \iff \liminf \{a_n\} = \limsup \{a_n\} = a$.
The proof is now trivial.
The sequence $\{a_n\}$ has only one cluster point if and only if $\liminf \{a_n\} = \limsup \{a_n\} = a$, which is equivalent to $\{a_n\} \to a$.

Ex 41. At every index n,

$$\inf\{a_k | k \geq n\} \leq \sup\{a_k | k \geq n\}$$

And so, by the linearity property of convergent sequences, $\lim_{n\to\infty} \inf\{a_k|k \geq n\} \leq \lim_{n\to\infty} \sup\{a_k|k \geq n\}$ or $\liminf\{a_n\} \leq \limsup\{a_n\}$.

Q.E.D

Ex 45. Let $\{a_n\}$ be a sequence of real numbers.

(i) The series $\sum_{k=1}^{\infty} a_k$ is summable if and only if for each $\epsilon > 0, \exists N \in \mathbb{N}$ s.t.

$$\left|\sum_{k=n}^{n+m} a_k\right| < \epsilon \forall n \geq N, m \in \mathbb{N}.$$

The series $\sum_{k=1}^{\infty} a_k$ is summable if and only if $\{s_n\}$ converges.

As such, for each $\epsilon > 0, \exists N \in \mathbb{N}$ s.t.

$$i \geq n \geq N \implies \epsilon > \left|\sum_{k=n}^{i} a_k\right|$$
$$\iff n \geq N, m \in \mathbb{N} \implies \epsilon > \left|\sum_{k=n}^{n+m} a_k\right|$$

(ii) If the series $\sum_{k=1}^{\infty} |a_k|$ is summable, then $\sum_{k=1}^{\infty} a_k$ also is summable.

If the series $\sum_{k=1}^{\infty} |a_k|$ is summable, then the partial sum sequence $\{\sum_{k=1}^{n} |a_k|\}$ converges.

As such, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t.

$$n > m \geq N \implies \epsilon > \left|\sum_{k=m}^{n} |a_k|\right| \geq \left|\sum_{k=m}^{n} a_k\right|.$$

The partial sum sequence $\{\sum_{k=1}^{n} |a_k|\}$ converges because it is Cauchy. As a result, the series $\sum_{k=1}^{\infty} a_k$ also is summable.

(iii) If each term a_k is nonnegative, then the series $\sum_{k=1}^{\infty} a_k$ is summable if and only if the sequence of partial sums is bounded.

Since $a_k > 0 \forall k \in \mathbb{N}$, $s_n = \sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n+1} a_k = s_{n+1}$ for all n. In other words, the partial sum sequence is nondecreasing.

The series $\sum_{k=1}^{\infty} a_k$ is summable if and only if $\{s_n\}$ converges.

- If $\{s_n\}$ converges then it is bounded
- If $\{s_n\}$ is bounded, then it converges to $s = \sup\{s_n | n \in \mathbb{N}\}$ (note that the suprimum exists thanks to the Completeness Axiom)

  For any $\epsilon > 0$,

  +) $s_n \leq s < s + \epsilon$ for all n.

  +) Because $s - \epsilon$ is not an upperbound of $\{s_n | n \in \mathbb{N}\}$, $\exists N \in \mathbb{N}$ s.t $s_N > s - \epsilon$.
  And since the sequence $\{s_n\}$ is nondecreasing, $n \geq N \implies s_n > s - \epsilon$.

  By definition, $\{s_n\}$ converges to s.