Real Analysis - P1

Learning Theory and Applications Group

Academic Year 2024-2025

Contents

Part I
Lebesgue Intergration

Key definitions here:

1 The Real Number: Sets, Sequences, and Functions

1.1 The Field, Positivity, and Completeness Axioms

1.1.1 Excercise

Ex 1. For $a \neq 0$ and $b \neq 0$, show that $(ab)^{-1} = a^{-1}b^{-1}$

$$(a^{-1}b^{-1})(ab) = a^{-1}b^{-1}ba = a^{-1}1a = 1$$

As a result, $a^{-1}b^{-1} = (ab^{-1})$
Ex 2. Verify the following:

- For each real number $a \neq 0$, $a^2 > 0$. In particular, $1 > 0$ since $1 \neq 0$ and $1 = 1^2$

- For each positive number a, its multiplicative inverse a^{-1} also is positive
- If $a > b$, then

$$ac > bc \text{ if } c > 0 \text{ and } ac < bc \text{ if } c < 0$$

For the first point, we first need to prove that, for any a, then $-a = (-1)a$,

$$a + (-a) = 0 = (1 + (-1))a = a + (-1)a$$

Next, for each $a \neq 0$, if a is positive, then a^2 is positive by definition of positiveness. On the other hand, if $a < 0$, then let $a = -b$ with $b > 0$,

$$a^2 = (-b)^2 = (-1)b(-1)b = (-1)(-b)b = (-1)^2 b^2 > 0 \tag{1}$$

For the second point, assuming by contradiction that $a^{-1} < 0$ for any $a > 0$, then let $a^{-1} = -b$ with $b > 0$. then

$$1 = a(a^{-1}) = a(-b) = (-1)ab < 0$$

Here $ab > 0$ since both a and b are positive, and we know from previous point that $0 > -(ab) = (-1)ab$

The last point is straighforward from the definition of $>$.

$$ac - bc = \underbrace{(a-b)}_{>0}\underbrace{c}_{>0} > 0$$

$$ac - bc = \underbrace{(a-b)}_{>0}\underbrace{c}_{<0} = (-1)\underbrace{(a-b)}_{>0}\underbrace{d}_{>0} < 0 \text{ with } d = -c$$

Ex 3. For a nonempty set of real numbers E, show that $\inf E = \sup E$ if and only if E consists of a single point.

If the set E has a single element, then the least upper bound equal to that single element. This similarly applies to lowerbound. In other word, its sup and inf coincides.

On the other direction, if a set E has its sup and inf equal, and assuming by contradiction that E has at least 2 distinct elements, then the gap between these two points $\neq 0$. The difference between sup and inf is lowerbounded by this gap, so they cannot equal.

Ex 4. Let a and b be real numbers.

i Show that if $ab = 0$, then $a = 0$ or $b = 0$

ii Verify that $a^2 - b^2 = (a-b)(a+b)$ and conclude from part (i) that if $a^2 = b^2$, then $a = b$ or $a = -b$.

iii Let c be a positive real number. Define $E = \{x \in \mathbb{R} | x^2 < c\}$ verify that E is nonempty and bounded above. Define $x_0 = \sup E$. Show that $x_0^2 = c$. Use part (ii) to show that there is a unique $x > 0$ for which $x^2 = c$. It is denoted by $\sqrt{c}$

For the first point, suppose that $ab = 0$ and both a and b are not 0, then there exists a^{-1} and b^{-1}, then we have

$$abb^{-1}a^{-1} = 1$$

which means that $b^{-1}a^{-1} = (ab)^{-1}$, but since $ab = 0$, no such number exists.

The second point is a straighforward application of distributive property,

$$(a-b)(a+b) = a(a+b) + (-b)(a+b) = a^2 + ab - ba - b^2 = a^2 - b^2$$

Then from part (i), since $(a-b)(a+b) = 0$, one of the two terms must be 0.

In part (iii), we see that $0^2 = 0 < c$ for all $c > 0$, so E is nonempty. By contradiction, suppose E is not bounded from above, that is, for every $b > 0$, we can always choose some $x \in E$ such that $x > b$, letting $b > c$ lead to a contradiction with the definition of E.

Next, since E is bounded from above, then it has a supremum by completeness axiom. Denote $x_0 = \sup E$. We will show that $x_0^2 \geq c$ and $x_0^2 \leq c$ to conclude that $x_0^2 = c$.

Since $x^2 < c, \forall x \in E$, c is an upperbound of E^2, and because $\sup E$ is the smallest/least upperbound, then $\sup(E)^2 \leq c$. On the otherhand, $x_0 \geq x, \forall x \in E$ and E contains **all** real numbers whose square less than c, so $x_0^2 \geq c$.

Finally, we need to show that x_0 is a unique positive real number such that $x_0^2 = c$. By contradiction, suppose there is some $x > 0$ such that $x \neq x_0$ and $x^2 = c$, then by part (ii), since $x_0^2 = x^2$, we have either $x = x_0$ or $x = -x_0$, but x is positive and $-x_0$ is negetive, so $x = x_0$.

Ex 5. Let a, b, c be real bumbers such that $a \neq 0$ and consider the quadratic equation

$$ax^2 + bx + c = 0, x \in \mathbb{R}$$

i Suppose $b^2 - 4ac > 0$, use the Field Axiom and the preceding problem to complete the square and thereby show that this equation has exactly two solutions given by

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

ii Now suppose $b^2 - 4ac < 0$. Show that the quadratic equation fails to have any solution.

Suppose that $b^2 - 4ac > 0$, then from previous problem, there exists a unique positive number $\sqrt{b^2 - 4ac}$. we can verify that

$$\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = x^2 - x\frac{-b - \sqrt{b^2 - 4ac}}{2a} - x\frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{b^2 - b^2 + 4ac}{4a^2}$$
$$= x^2 + x\frac{b}{a} + \frac{c}{a} = 0.$$

Then also from the previous problem, either one of the two terms equal 0. As a result, the equation has exactly two solutions.

On the other hand, if $b^2 - 4ac < 0$, then the equation can be rewritten as

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{4ac - b^2}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4} > 0,$$

which does not have any solution.

Ex 6. Use the Completeness Axiom to show that every nonempty set of real numbers that is bounded below has an infimum and that

$$\inf E = -\sup\{-x|x \in E\}.$$

The set E is bounded below, which means that the set $E' = \{-x|x \in E\}$ is bounded from above, then its supremum exists by completeness axiom. Denote $x_0 = \sup E'$, then $x_0 \geq -x, \forall x \in E \rightleftarrows -x_0 \leq x, \forall x \in E$. As a result, $-x_0 \leq \inf E$.

Suppose that there exists some x' such that $x' > -x_0$ and $x' \leq x, \forall x \in E$; i.e. x' is a "greater" lowerbound of E than x_0. Then we can show that $-x'$ is a "smaller" upperbound of E', which contradicts with the definition of supremum. As a result, no such x' exists, and $-x_0$ is the infimum of E

Ex 7. For real bumbers a and b, verify the following:

i $|ab| = |a||b|$

ii $|a + b| \leq |a| + |b|$

iii For $\epsilon > 0$,

$$|x - a| < \epsilon \text{ if and only if } a - \epsilon < x < a + \epsilon$$

First we define the sign operator as $\text{sg}(x) \in \{1, -1\}, x \neq 0$. The absolute value can be written as the product with the sign operator

$$|a| = a\text{sg}(a)$$

Then the first claim can be verified as

$$|ab| = ab\text{sg}(ab) = a\text{sg}(a)b\text{sg}(b) = |a||b|$$

by noting $\text{sg}(ab) = \text{sg}(a)\text{sg}(b)$, and

$$|a + b| = (a + b)\text{sg}(a + b) = a\text{sg}(a + b) + b\text{sg}(a + b) \leq a\text{sg}(a) + b\text{sg}(b) = |a| + |b|$$

by noting $a\text{sg}(a) = \max(a, -a) \geq a\text{sg}(c), \forall c$

Final point: if $x - a > 0$, then $|x - a| = x - a$ and $|x - a| < \epsilon \rightleftarrows a < x < a + \epsilon$

Similar, if $x - a < 0$, then $|x - a| < \epsilon \rightleftarrows a > x > a - \epsilon$, combining the both cases and with the zero case yield the desired claim.

1.2 The Natural and Rational Numbers

1.2.1 Excercise

1.2.2 Excercise

Exercise 9:
a) We need to prove that If $n > 1$ is a natural number, then $n - 1$ is also a natural number.
Let P(n) be the assertion that $n \in \mathbb{N}$ and $n > 1 => n - 1 \in \mathbb{N}$
Base Case: Let $n = 2$. Then:

$$n - 1 = 2 - 1 = 1 \in \mathbb{N}.$$

Thus, the base case holds.
Inductive Step: Assume that $P(k)$ is true for some natural number $k \geq 2$, i.e., assume that:

$$k - 1 \in \mathbb{N}.$$

We need to show that $P(k + 1)$ is also true, meaning:

$$(k + 1) - 1 \in \mathbb{N}.$$

Since:

$$(k + 1) - 1 = k,$$

and by our inductive hypothesis, $k \in \mathbb{N}$, it follows that $P(k + 1)$ is true.
By the principle of mathematical induction, for all $n > 1$, we conclude that $n - 1$ is a natural number.
b) We prove that the given statement is true for a fixed n.
Let $P(m)$ be the assertion that for a given natural number n and $m < n$, then $n - m$ is a natural number.
Base case: $P(1)$ is true since $n - 1$ is a natural number, according to part a).
Inductive step: Assume that $P(k)$ is true for some natural number $k \geq 2$ and $k < n$, i.e $n - k \in \mathbb{N}$. We need to show that $P(k + 1)$ is also true, meaning that

$$n - (k + 1) \in \mathbb{N}$$

Since

$$n - (k + 1) = n - k - 1 = (n - k) - 1$$

and given our assumption, $n - k \in \mathbb{N}$, it follows that $(n - k) - 1 \in \mathbb{N}$ i.e. $P(k + 1)$ is true.
By the principle of mathematical induction, for a fixed $n \in \mathbb{N}$ and $m < n$, $n - m$ is a natural number. The same can be proven given a fixed m instead of n.

Ex 13. Show that each real number is the supremum of a set of rational numbers and also supremum of a set of irrational numbers.

Let x be any real number. We want to show that x is the supremum of both a set of rational numbers and a set of irrational numbers.

Define a set of rational numbers as: $S = \{q \in \mathbb{Q} : q < x\}$. According to Theorem 2, rational numbers are dense in $\mathbb{R}$, therefore there are rational numbers arbitrarily close to x, meaning S is nonempty. The upper bound of S is x, since every rational number $q \in S$ must satisfies $q < r$. To prove x is the least upper bound of S, we use The density of the rational (and irrational) numbers in R, which guarantees that between any number s that is less than a given real number x, there exists a rational number. This means there is a number $q \in S$ that satisfies $s < q < x$. Thus, no number smaller than x can be an upper bound of S, which confirms that $x = \sup(S)$ is indeed the least upper bound.

Similarly, for irrational numbers, we define a set $T = \{t \in \mathbb{R}/\mathbb{Q} : t < x\}$. We have to prove T is dense in $\mathbb{R}$, and the proof for rational numbers can be applied for irrational numbers. We can prove T is dense in $\mathbb{R}$ through irrational numbers are dense in $\mathbb{R}$. Since Q are dense in R, therefore $Q + \sqrt{2}$ are dense in $R + \sqrt{2}$. We know that $Q + \sqrt{2}$ is a subset in of the irrational numbers, therefore irrational numbers are dense in R. From this, we can prove there exists an irrational number t satisfies $s < t < x$. This mean $x = \sup(T)$ is indeed the least upper bound.

1.3 The Countable and Uncountable Sets

Hoang Anh

Exercise 16: Consider the mapping from **N** to **Z** defined by

$$f(n) = \begin{cases} 0 & \text{if } n = 1 \\ \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd and } n > 1 \end{cases}$$

If n is a natural number, then $f(2n) = n$ and $f(2n-1) = -n$. We also have $f(1) = 0$. Therefore f is onto.

Now suppose $f(n) = f(n')$. If $f(n)$ equals 0, then $n = n' = 1$. If $f(n)$ is positive, then $\frac{n}{2} = \frac{n'}{2} \implies n = n'$. If $f(n)$ is negative, then $-\frac{n+1}{2} = -\frac{n'+1}{2} \implies n = n'$. Therefore f is one-to-one.

Exercise 18: As a preliminary result, I rst show that every nite set of numbers contains a maximal element. **S(n):** Let $S \subset \mathbb{R}$ be a non-empty set. If there exists a one-to-one correspondence between $\{1, \cdots, n\}$ and S, then S contains a maximal element.

Suppose there exists a one-to-one correspondence f between $\{1\}$ and S. Then $S = \{f(1)\}$, so $s \leq f(1)$ for all $s \in S$. Thus $S(1)$ is true.

Now assume $S(k)$ is true and suppose there exists a one-to-one correspondence between $\{1, \cdots, k+1\}$ and S. Then $S = \{f(i)|1 \leq i \leq k\} \cup \{f(k+1)\}$. By the induction hypothesis, $\{f(i)|1 \leq i \leq k\}$ has a maximal element $\hat{s}$. If $\hat{s} \geq f(k+1)$, then $\hat{s}$ is a maximal element of S. If $\hat{s} < f(k+1)$, then $f(k+1)$ is a maximal element of S. We conclude that $S(k+1)$ must be true.

S(n): The Cartesian product $\underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{n \text{ times}}$ is countably infinite.

The identity function establishes a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{N}$, so $\mathbb{N}$ is countable. Now suppose $\mathbb{N}$ were finite. Then by the preliminary result, there would exist a maximal element m of $\mathbb{N}$. But $m+1$ would then be a natural number larger than m, a contradiction. We conclude that $\mathbb{N}$ is countably infinite, so $S(1)$ is true.

Suppose $S(k)$ is true. Then there exists a one-to-one mapping f of $\mathbb{N}$ onto $\underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{k \text{ times}}$. Consider the mapping from $\underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{k+1 \text{ times}}$ to $\mathbb{N}$ defined by

$$g(n_1, \cdots, n_k, n_{k+1}) = (f^{-1}(n_1, \cdots, n_k) + n_{k+1})^2 + n_{k+1}$$

It is straightforward to check that g is one-to-one using the argument in the text. Thus $\underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{k+1 \text{ times}}$ is equipotent to $g(\underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{k+1 \text{ times}})$, a subset of the countable set $\mathbb{N}$. We infer from Theorem 3 that $\underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{k+1 \text{ times}}$ is countable.

Now suppose $\underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{k+1 \text{ times}}$ is finite. Then there exists a one-to-one mapping f from $\{1, \cdots, n\}$ onto $\underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{k+1 \text{ times}}$ for some $n \in \mathbb{N}$. Consider the mapping from $\underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{k \text{ times}}$ to $\{1, \cdots, n\}$ defined by

$$g(n_1, \cdots, n_k) = f^{-1}(n_1, \cdots, n_k, 1)$$

This establishes a one-to-one correspondence between $\underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{k \text{ times}}$ and a subset of $\{1, \cdots, n\}$, implying that $\underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{k \text{ times}}$ is finite. This contradicts the assumption that $\underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{k \text{ times}}$ is countably infinite. We conclude that $\underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{k+1 \text{ times}}$ is countably infinite, so $S(k+1)$ is true. **Exercise 20:**

Suppose $g(f(a)) = g(f(a'))$. Since g is one-to-one, we must have $f(a) = f(a')$. Since f is one-to-one, we must also have $a = a'$. But this means $g \circ f$ is one-to-one. Now fix $c \in C$. Since g is onto, there exists $b \in B$ such that $g(b) = c$. Since f is onto, there also exists $a \in A$ such that $f(a) = b$. But this means $g(f(a)) = c$, so $g \circ f$ is onto.

Suppose $f^{-1}(b) = f^{-1}(b')$. Then $b = f(f^{-1}(b)) = f(f^{-1}(b')) = b'$, so f^{-1} must be one-to-one. Now suppose $a \in A$. Then $a = f^{-1}(f(a))$, so f^{-1} is onto.

Exercise 22: Suppose $2^{\mathbb{N}}$ is countable. Let $\{X_n|n \in \mathbb{N}\}$ denote an enumeration of $2^{\mathbb{N}}$ and define

$$D = \{n \in \mathbb{N}|n \text{ is not in } X_n\}$$

Then $D \in 2^{\mathbb{N}}$, so $D = X_d$ for some $d \in \mathbb{N}$. If d is not in D, then we would have a contradiction because d would have to be in D by construction. Likewise if d is in D, then we have a contradiction because d could not be in D

by construction. We can conclude that no enumeration can exist, so $2^{\mathbb{N}}$ is uncountable. **Exercise 26:** Let G denote the set of irrational numbers in $(0,1)$ and let $\{q_n | n \in \mathbb{N}\}$ denote an enumeration of the rationals in $(0,1)$. Define

$$i_n = \frac{\sqrt{2}}{2^n}$$

and construct the mapping $f : (0,1) \to G$ as

$$f(x) = \begin{cases} i_{2n} & \text{if } x = q_n \\ i_{2n-1} & \text{if } x = i_n \\ x & \text{otherwise} \end{cases}$$

f defines a one-to-one correspondence between $(0,1)$ and G, so $|(0,1)| = |G|$.

In Problem 25 we showed that $|\mathbb{R}| = |(0,1)|$, so the above result implies $|\mathbb{R}| = |G|$. This means we can find a one-to-one mapping g from $\mathbb{R}$ onto G. Now consider the mapping $h : \mathbb{R} \times \mathbb{R} \to G \times G$ defined by

$$h(x,y) = (g(x), g(y))$$

h defines a one-to-one mapping from $\mathbb{R} \times \mathbb{R}$ onto $G \times G$, so $|\mathbb{R} \times \mathbb{R}| = |G \times G|$.

Recall that if x is an irrational number in $(0,1)$, it can be uniquely written as

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}} = [a_1, a_2, a_3, \cdots]$$

where $a_1, a_2, a_3, \cdots$ is an infinite sequence of natural numbers. (This representation is called the continued fraction expansion of x.) Let $x = [a_1, a_2, \cdots]$ and $y = [b_1, b_2, \cdots]$ denote two elements of G and consider the mapping $m : G \times G \to G$ defined by

$$m(x,y) = [a_1, b_1, a_2, b_2, \cdots]$$

Then m defines a one-to-one correspondence between $G \times G$ and G, so $|G \times G| = |G|$. Combining the above results, we have $|\mathbb{R} \times \mathbb{R}| = |G \times G| = |G| = |\mathbb{R}|$.

1.3.1 Excercise

1.4 Open Sets, Closed Sets, and Borel Sets of Real Numbers

Exercise 28: Suppose A is a non-empty, proper subset of $\mathbf{R}$ that is both open and closed. Then there exists $x \in A$ and $y \in \mathbf{R} \setminus A$. Suppose without loss of generality that $x < y$ and define

$$E = \{x \in A : x < y\}$$

Then E is non-empty ($x \in E$) and bounded above (by y). The completeness axiom implies that there exists a least upper bound of E. Let $x^* = \sup E$ and suppose $x^* \in A$. Since $y \notin A$ and y is an upper bound of E, we must have $x^* < y$. Therefore there exists $r > 0$ such that $x^* + r < y$. But since A is open, we can also find $r^* \in (0,r)$ such that $(x^* - r^*, x^* + r^*) \subset A$. But this implies $x^* + \frac{r}{2} \in E$, so x^* is not an upper bound for E. This contradicts the definition of x^*. Now suppose $x^* \in \mathbf{R} \setminus A$. Since A is closed, $\mathbf{R} \setminus A$ is open. Therefore there exists $r > 0$ such that $(x^* - r, x^* + r) \subset \mathbf{R} \setminus A$. Thus if $x \in A$, $x \le x^* - r$. But this means $x^* - r$ is an upper bound of E, contradicting the assumption that x^* is the least upper bound.

The above argument shows that E cannot have a least upper bound, a contradiction of the completeness axiom. We conclude that no non-empty, proper subset of $\mathbf{R}$ that is both open and closed can exist. **Exercise 31:** Suppose E is a set containing only isolated points. For each $x \in E$, define $f(x) = (p,q)$ where p and q are rational numbers such that $p < x < q$ and $(p,q) \cap E = \{x\}$. f defines a one-to-one mapping from E to $\mathbf{Q} \times \mathbf{Q}$. By Corollary 4 and Problem 23, $\mathbf{Q} \times \mathbf{Q}$ is a countable set. This means there exists a one-to-one mapping g from $\mathbf{Q} \times \mathbf{Q}$ onto $\mathbf{N}$. The composition $g \circ f$ defines a one-to-one mapping from E to $\mathbf{N}$ (see Problem 20), which implies E is countable (see Problem 17). **Exercise 32:** (i) Suppose E is open and $x \in E$. Then there exists an $r > 0$ such that the interval $(x-r, x+r)$ is contained in E. But this means $x \in \text{int } E$, so $E \subseteq \text{int } E$. Since $\text{int } E \subseteq E$ by definition, $E = \text{int } E$.

Conversely, suppose $E = \text{int } E$. If x is a point in E, then $x \in \text{int } E$. But this means there exists an $r > 0$ such that the interval $(x-r, x+r)$ is contained in E, so E is open.

(ii) Let E be dense in $\mathbf{R}$ and suppose $x \in \text{int}(\mathbf{R} \setminus E)$. Then there exists $r > 0$ such that $(x-r, x+r) \subseteq \mathbf{R} \setminus E$. But this means there does not exist an element of E between any two numbers in $(x-r, x+r)$, contradicting the assumption that E is a dense set. We conclude that no such x can be found, so $\text{int}(\mathbf{R} \setminus E) = \emptyset$.

Conversely, suppose $\mathrm{int}(\mathbf{R}\setminus E) = \emptyset$. Let x and y be two real numbers satisfying $x < y$ and suppose $(x, y) \subset \mathbf{R}\setminus E$. Let $z \in (x, y)$ and choose $r \in (0, \min(z - x, y - z))$. Then $(z - r, z + r) \subset (x, y)$, so $(z - r, z + r) \subset \mathbf{R} \setminus E$. But this means $z \in \mathrm{int}(\mathbf{R} \setminus E)$, contradicting the assumption that $\mathrm{int}(\mathbf{R} \setminus E) = \emptyset$. Therefore $(x, y) \not\subset \mathbf{R} \setminus E$, which means there must be an element of E between z and y. But since x and y were arbitrary, this means E is dense in $\mathbf{R}$.

1.4.1 Excercise

1.5 Sequences of Real Numbers

1.5.1 Summary

A sequence is a function $f : \mathbb{N} \to \mathbb{R}$ with customary notation $\{a_n\}$ where n is called the index, the number a_n is the nth term.
A sequence $\{a_n\}$ is said to be

- bounded if $\exists c \geq 0$ s.t. $|a_n| \leq c \forall n$
- increasing if $a_n < a_{n+1} \forall n$
- decreasing if the sequence $\{-a_n\}$ is increasing
- monotone if it's either increasing or decreasing

For any sequence $\{a_n\}$ and a strictly increase sequence $\{n_k\} \in \mathbb{N}$, call the sequence $\{a_{n_k}\}$ a subsequence of $\{a_n\}$.

Definition 1. *A sequence $\{a_n\}$ converges to it's limit a (write $\lim_{n\to\infty} a_n = a$ or $\{a_n\} \to a$) if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t.*

$$n \geq N \implies |a - a_n| < \epsilon.$$

Proposition 1. *If $\{a_n\} \to a$, then the limit is unique, the sequence is bounded, and, $\forall c \in \mathbb{R}$,*

$$a_n \leq c \forall n \implies a \leq c.$$

Proof ***<u>Ex Extra.</u>***

Theorem 1. *A monotone sequence of real numbers converges if and only if it is bounded.*

Theorem 2 (Bolzano-Weierstrass)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Definition 2. *A sequence of real numbers $\{a_n\}$ is Cauchy if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t.*

$$n, m \geq N \implies |a_m - a_n| < \epsilon.$$

Theorem 3. *A sequence of real numbers converges if and only if it is Cauchy.*

Theorem 4. *Convergent real sequences are linear and monotonic.*

Definition 3. *A sequence $\{a_n\}$ converges to infinity (write $\lim_{n\to\infty} a_n = \infty$ or $\{a_n\} \to \infty$) if $\forall c \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t.*

$$n \geq N \implies a_n \geq c.$$

Similar definitions are made at $-\infty$.

Definition 4. *The limit superior and limit inferior of a sequence $\{a_n\}$ is defined as,*

$$\limsup \{a_n\} = \lim_{n\to\infty} [\sup \{a_k | k \geq n\}]$$
$$\liminf \{a_n\} = \lim_{n\to\infty} [\inf \{a_k | k \geq n\}]$$

Proposition 2. *Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers*

(i) $\limsup \{a_n\} = \ell \in \mathbb{R}$ *if and only if for each $\epsilon > 0$, there are infinitely many indices n for which $a_n > \ell - \epsilon$ and only finitely many indices n for which $a_n > \ell + \epsilon$.*

(ii) $\limsup \{a_n\} = \infty$ *if and only if* $\{a_n\}$ *is not bounded above.*

(iii) $\limsup \{a_n\} = -\liminf \{-a_n\}$.

(iv) A sequence of real numbers $\{a_n\}$ *converges to* $a \in \mathbb{R}$ *if and only if* $\liminf \{a_n\} = \limsup \{a_n\} = a$.

(v) $a_n \leq b_n \forall n \implies \limsup \{a_n\} \leq \liminf \{b_n\}$.

Proof **Ex 39.**

Definition 5. *For every sequence* $\{a_k\}$ *of real numbers, define a sequence of partial sums* $\{s_n\}$ *where* $s_n = \sum_{k=1}^{n}$. *The series* $\sum_{k=1}^{\infty} a_k$ *is summable to* $s \in \mathbb{R}$ *when* $\{s_n\} \to s$.

Proposition 3. *Let* $\{a_n\}$ *be a sequence of real numbers.*

(i) The series $\sum_{k=1}^{\infty} a_k$ *is summable if and only if for each* $\epsilon > 0, \exists N \in \mathbb{N}$ *s.t.*

$$\sum_{k=n}^{n+m} a_k < \epsilon \forall n \geq N, m \in \mathbb{N}.$$

(ii) If the series $\sum_{k=1}^{\infty} |a_k|$ *is summable, then* $\sum_{k=1}^{\infty} a_k$ *also is summable.*

(iii) If each term a_k *is nonnegative, then the series* $\sum_{k=1}^{\infty} a_k$ *is summable if and only if the sequence of partial sums is bounded.*

Proof **Ex 45.**

1.5.2 Excercise

Problems done: 38, 39, 40, 41, 45. and proved the first Proposition (i.e. Ex Extra.).

Ex 38.

Lemma 1. *For any set* $X \subseteq \mathbb{R}, \forall d > 0 \in \mathbb{R}, \exists x \in X$ *s.t.* $x < \inf X + d$.

Proof We prove by contradiction. Assume there exists $d > 0 \in \mathbb{R}$ s.t. $\forall x \in X, \inf X + d \leq x$. There is now a greater lower bound $\inf X + d$, which contradicts the definition of infimum.

We use the above lemma to solve this excercise.
Let $\liminf \{a_n\} = L$.

- $\liminf \{a_n\}$ is a cluster point.

  By the above lemma, for every n, we can pick the smallest index $k_n \geq n$ satisfying $a_{k_n} \leq \inf \{a_k | k \geq n\} + \frac{1}{n}$
  Now, $\forall \epsilon > 0, \exists N \in \mathbb{N}, N \geq 1/\epsilon$ s.t. $n \geq N \implies a_{k_n} - L < 1/N \leq \epsilon$. The subsequence $\{a_{k_n}\}$ converges to L by defintion.

- There does not exist a cluster point M satisfying $M < \liminf \{a_n\}$.

  We argue by contradiction. Assume there exists such a cluster point, this means there also exists a subsequence $\{a_{m_j}\}$ that converges to M.

  Let $\epsilon = \frac{M-L}{2}$, by definition, $\exists J \in \mathbb{N}$ s.t.

  $$j \geq J \implies a_{m_j} - M < \epsilon \iff a_{m_j} < M + \epsilon = \frac{L+M}{2}.$$

  Also, by definition, $L = \liminf \{a_n\} = \lim_{n \to \infty} \{\inf \{a_k | k \geq n\}\}$, as such $\exists N \in \mathbb{N}, N > J$ s.t.

  $$n \geq N \implies L - \inf \{a_k | k \geq n\} < \epsilon \iff \inf \{a_k | k \geq n\} > L - \epsilon = \frac{L+M}{2}.$$

  This is a contradiction, as there exists $N \in \mathbb{N}$ satisfying

  $$n \geq N \implies \begin{cases} a_{m_n} < \frac{L+M}{2} \\ \inf \{a_k | k > n\} \geq \frac{L+M}{2} \end{cases}.$$

Proof is similar for $\limsup\{a_n\}$

Ex 39. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers

(i) $\limsup\{a_n\} = \ell \in \mathbb{R}$ if and only if for each $\epsilon > 0$, there are infinitely many indices n for which $a_n > \ell - \epsilon$ and only finitely many indices n for which $a_n > \ell + \epsilon$.

Trivial. Use definition of suprimum and the fact that the collection of sequences $\{\{a_k | k \geq n\}\}_{n=1}^{\infty}$ is decending.

(ii) $\limsup\{a_n\} = \infty$ if and only if $\{a_n\}$ is not bounded above.

We prove the above through showing that $\limsup\{a_n\} < \infty$ if and only if $\{a_n\}$ is bounded above. Note that the limit superior of a sequence always exists.

- If $\{a_n\}$ is bounded above, then $\exists M < \infty \in \mathbb{R}$ s.t. $a_n \leq M \forall n$. As a result, $\sup\{a_k | k \geq n\} \leq M$.
- If $\limsup\{a_n\} < \infty$, then $\sup\{a_k\ k \geq n\}$ is bounded.
  And because there exists $c > 0$ satisfying $a_n \leq \sup\{a_k | k \geq 1\} \leq c$ for all n, the sequence $\{a_n\}$ is also bounded above.

(iii) $\limsup\{a_n\} = -\liminf\{-a_n\}$.

$$\limsup\{a_n\} = \lim_{n\to\infty} \sup\{a_k | k \geq n\} = -\lim_{n\to\infty} \inf\{-a_k | k \geq n\} = -\liminf\{-a_n\}.$$

(I ommitted the proof to $\lim_{n\to\infty}\{a_n\} = -\lim_{n\to\infty}\{-a_n\}$. It is trivial and uses the definition.)

(iv) A sequence of real numbers $\{a_n\}$ converges to $a \in \mathbb{R}$ if and only if $\liminf\{a_n\} = \limsup\{a_n\} = a$.

- $\liminf\{a_n\} = \limsup\{a_n\} = a \implies \{a_n\} \to a$
  For any $\epsilon > 0$, there exists $N, M \in \mathbb{N}$ s.t.
  $$\begin{cases} n \geq N \implies -\epsilon < a - \sup\{a_k | k \geq n\} \leq a - a_n \\ n \geq M \implies a - a_n \leq a - \inf\{a_k | k \geq n\} < \epsilon \end{cases}$$
  So $\exists L = \max(N, M) \in \mathbb{N}$ s.t.
  $$n \geq L \implies \begin{cases} -\epsilon < a - a_n \\ a - a_n < \epsilon \end{cases} \implies |a - a_n| < \epsilon.$$
  By definition, $\{a_n\} \to a$
- $\{a_n\} \to a \implies \liminf\{a_n\} = \limsup\{a_n\} = a$
  For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ s.t.
  $$\forall n \geq N, \begin{cases} -\epsilon < a - a_n \\ a - a_n < \epsilon \end{cases} \implies \forall n \geq N, \begin{cases} \inf\{a_k | k \geq n\} \leq a_n < a + \epsilon \\ a - \epsilon < a_n \implies a - \epsilon < \inf\{a_k | k \geq n\} \end{cases}.$$
  which is equivalent to $|a - \inf\{a_k | k \geq n\}| < \epsilon$ for all $n \geq N$ and so $\liminf\{a_n\} = a$ by definition.
  Similar proof is done for $\limsup\{a_n\} = a$.

(v) $a_n \leq b_n \forall n \implies \limsup\{a_n\} \leq \liminf\{b_n\}$. (similar to book)

Consider the sequence $\{c_n\}$, where $c_n = \inf\{b_k | k \geq n\} - \sup\{a_k | k \geq n\}$ for all n.

By linearity of convergent sequences, $\{c_n\} \to c = \liminf\{b_n\} - \limsup\{a_n\}$. This means, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t.

$$n \geq N \implies -\epsilon < c - c_n < \epsilon.$$

In particular, $0 \leq c_N < c + \epsilon$. Since $c \geq -\epsilon$ for any positive number ϵ, $c \geq 0$.

Ex 40.

Proven above in **Ex. 38**, $\liminf\{a_n\}$ and $\limsup\{a_n\}$ are the smallest and largest cluster points of $\{a_n\}$.

Shown above in **Ex. 39**, $\{a_n\} \to a \iff \liminf\{a_n\} = \limsup\{a_n\} = a$.

The proof is now trivial.

The sequence $\{a_n\}$ has only one cluster point if and only if $\liminf\{a_n\} = \limsup\{a_n\} = a$, which is equivalent to $\{a_n\} \to a$.

Ex 41. At every index n,

$$\inf\{a_k | k \geq n\} \leq \sup\{a_k | k \geq n\}$$

And so, by the linearity property of convergent sequences, $\lim_{n\to\infty} \inf\{a_k|k \geq n\} \leq \lim_{n\to\infty} \sup\{a_k|k \geq n\}$ or $\liminf\{a_n\} \leq \limsup\{a_n\}$.

Ex 45. Let $\{a_n\}$ be a sequence of real numbers.

(i) The series $\sum_{k=1}^{\infty} a_k$ is summable if and only if for each $\epsilon > 0, \exists N \in \mathbb{N}$ s.t.

$$\left|\sum_{k=n}^{n+m} a_k\right| < \epsilon \forall n \geq N, m \in \mathbb{N}.$$

The series $\sum_{k=1}^{\infty} a_k$ is summable if and only if $\{s_n\}$ converges.

As such, for each $\epsilon > 0, \exists N \in \mathbb{N}$ s.t.

$$j > i - 1 \geq N \implies \epsilon > \left|\sum_{k=i}^{j} a_k\right|$$
$$\iff n \geq N, m \in \mathbb{N} \implies \epsilon > \left|\sum_{k=n}^{n+m} a_k\right| \quad (i - 1 = n, j = n + m)$$

(ii) If the series $\sum_{k=1}^{\infty} |a_k|$ is summable, then $\sum_{k=1}^{\infty} a_k$ also is summable.

If the series $\sum_{k=1}^{\infty} |a_k|$ is summable, then the partial sum sequence $\{\sum_{k=1}^{n} |a_k|\}$ converges.

As such, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t.

$$n, m \geq N \implies \epsilon > \left|\sum_{k=\min(m,n)}^{\max(m,n)} |a_k|\right| \geq \left|\sum_{k=\min(m,n)}^{\max(m,n)} a_k\right|.$$

The partial sum sequence $\{\sum_{k=1}^{n} a_k\}$ converges because it is Cauchy. As a result, the series $\sum_{k=1}^{\infty} a_k$ also is summable.

(iii) If each term a_k is nonnegative, then the series $\sum_{k=1}^{\infty} a_k$ is summable if and only if the sequence of partial sums is bounded.

Since $a_k > 0 \forall k \in \mathbb{N}$, $s_n = \sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n+1} a_k = s_{n+1}$ for all n. In other words, the partial sum sequence is nondecreasing.

The series $\sum_{k=1}^{\infty} a_k$ is summable if and only if $\{s_n\}$ converges.

- If $\{s_n\}$ converges then it is bounded.
- If $\{s_n\}$ is bounded, then it converges to $s = \sup\{s_n | n \in \mathbb{N}\}$ (note that the suprimum exists thanks to the Completeness Axiom)

  For any $\epsilon > 0$, we have:

  +) $s_n \leq s < s + \epsilon$ for all n.

  +) Because $s - \epsilon$ is not an upperbound of $\{s_n | n \in \mathbb{N}\}$, $\exists N \in \mathbb{N}$ s.t $s_N > s - \epsilon$.
  And since the sequence $\{s_n\}$ is nondecreasing, $n \geq N \implies s_n > s - \epsilon$.

  By definition, $\{s_n\}$ converges to s.

Ex Extra. If $\{a_n\} \to a$, then:

- The limit is unique.

  We prove by contradiction. Assume $\{a_n\} \to a, \{a_n\} \to b$ and $a \neq b$.

  Let $d = |a-b|$ and $\epsilon = \frac{d}{2}$. By definition, there exists $N, M \in \mathbb{N}$ s.t.

$$\begin{cases} n \geq N \implies |a-a_n| < \epsilon \\ n \geq M \implies |b-a_n| < \epsilon \end{cases}$$

  So $\exists L = \max(N, M) \in \mathbb{N}$ s.t. $n \geq L$ implies both $|a-a_n|$ and $|b-a_n|$ are less than ϵ.

  By the triangle inequality, $d = |a-b| \leq |a-a_n| + |b-a_n| < 2\epsilon = 2 \times \frac{d}{2} = d$. In other words, $d < d$, which is a contradiction.

- The sequence is bounded.

  Choose any $\epsilon > 0$.

  By definition, $\exists N \in \mathbb{N}$ s.t.

$$n \geq N \implies -\epsilon < a - a_n < \epsilon \iff a - \epsilon < a_n < a + \epsilon \implies |a_n| < |a| + \epsilon$$

  Denote $M_1 = \max\left[\{a_n | n \in \mathbb{N}, n < N\}\right]$, note that we can always find M_1 because this sequence is finite.

  We conclude that $\{a_n\}$ is bounded by $\max\left(|a| + \epsilon, M_1\right)$.

- $\forall c \in \mathbb{R}$, if $a_n \leq c \forall n$ then $a \leq c$.

Approach 1) Using only the definition.

For any $\epsilon > 0, \exists N \in \mathbb{N}$ s.t

$$n \geq N \implies |a - a_n| < \epsilon \implies a - \epsilon < a_n \leq c$$

Since $a - \epsilon < c$ is true for all $\epsilon > 0$, we conclude that $a \leq c$.

Approach 2) Using only the definition.

Prove by contradiction. Assume $a > c$, then set $\epsilon = a - c > 0$...

Approach 3) Consider the sequence $\{c_n\}$, where $c_n = c \forall n$ and use the monotonic property of convergent sequences. (Trivial)

1.6 Continuous Real-Valued Functions of Real Variable

1.6.1 Excercise

Ex 50. Show that a Lipschitz function is uniformly continuous but there are uniformly continuous functions that are not Lipschitz

We can prove that not all functions that are uniformly continuous are a Lipschitz function by using contradiction. Suppose we have function $f = \sqrt{x}$ uniformly continuous on $\{0, 1\}$ and is a Lipschitz function. Based on definition, there is a $c > 0$ for which

$$|\sqrt{x'} - \sqrt{x}| \leq c|x' - x|. \tag{2}$$

If we take $x = 0$, the equation become $|\sqrt{x'}| \leq c|x'|$. We can rewrite this as $|\sqrt{x'}|/|x'| \leq c$. However, if $x' \to 0$, we have $|\sqrt{x'}|/|x'| \to \infty$ which contradicts the inequality. Hence, $f = \sqrt{x}$ is not a Lipschitz function.

Ex 53. Show that a set E of real numbers is closed and bounded if and only if every open cover of E has a finite subcover.

(i) ($\Rightarrow$) According to Heine-Borel theorem, if a set E of real numbers is closed and bounded, every open cover of E has a finite subcover.

(ii) ($\Leftarrow$) We first prove that if every open cover of E has a finite subcover, then E is bounded. We form an open cover of E by defining a set $O_x = (x-1, x+1)$ for every $x \in E$. The collection $\{O_x : x \in E\}$ is an open cover for E. This collection must have a finite subcover $\{O_{x_1}, O_{x_2}, \ldots, O_{x_n}\}$. Since E is contained in a finite union of bounded sets, E must be bounded.

We now prove that E must be closed. Suppose E is not closed. Let $y \notin E$ be a point of closure of E. We form an open cover of E by defining a set $O_x = (x - r_x, x + r_x)$ where $r_x = |y - x|/2$ for every $x \in E$. The collection $\{O_x : x \in E\}$ must have a finite subcover $\{O_{x_1}, O_{x_2}, \dots, O_{x_n}\}$. Let $r_{\min} = \min\{r_{x_1}, r_{x_2}, \dots, r_{x_n}\}$. Since y is a point of closure of E, the open interval $(y - r_{\min}, y + r_{\min})$ must contain a point $x' \in E$. This means $|x' - y| < r_{\min}$. We now show that x' is not in the subcover.

$$\forall i : 1 \leq i \leq n, |x_i - x'| > |x_i - y| - |x' - y| > |x_i - y| - r_{\min} > 2r_{x_i} - r_{\min} > r_{x_i}$$

$\forall i : |x_i - x'| > r_{x_i} \Rightarrow \forall i : x' \notin O_{x_i} \Rightarrow x' \notin \bigcup_{1 \leq i \leq n} O_{x_i}$. This means the finite subcover fails to cover E. This contradiction implies that E is closed.