[STATS 531] HW2

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Question 2.1

A.

For $h > 0$:

$$
\begin{aligned}
\gamma_h &= \mathrm{Cov}(X_n, X_{n+h}) \\
&= \mathrm{Cov}(X_n, \phi X_{n+h-1} + \epsilon_{n+h}) \\
&= \phi \cdot \mathrm{Cov}(X_n, X_{n+h-1}) + \mathrm{Cov}(X_n, \epsilon_{n+h}) \\
&= \phi \gamma_{h-1} \quad (\epsilon_{n+h} \text{ independent of } X_n \text{ for } h > 0)
\end{aligned}
$$

This gives a first-order linear homogeneous recurrence, so the solutions are in the form $\gamma_h = A\lambda^h$,

$$
\begin{aligned}
A\lambda^h &= \phi A \lambda^{h-1} \\
\implies \lambda &= \phi \quad (\text{dividing by } A\lambda^{h-1}, \text{ assume } A \text{ and } \lambda \text{ are not } 0)
\end{aligned}
$$

Thus $\gamma_h = A\phi^h$ for $h > 0$. To find A, we first find γ_0. We have

$$
\begin{aligned}
\gamma_0 &= \mathrm{Var}(X_n) \\
&= \mathrm{Var}(\phi X_{n-1} + \epsilon_n) \\
&= \phi^2 \mathrm{Var}(X_{n-1}) + \mathrm{Var}(\epsilon_n) \quad (X_{n-1} \perp \epsilon_n) \\
&= \phi^2 \gamma_0 + \sigma^2 \\
\implies \gamma_0 - \phi^2 \gamma_0 &= \sigma^2 \\
\gamma_0 (1 - \phi^2) &= \sigma^2 \\
\gamma_0 &= \frac{\sigma^2}{1 - \phi^2}
\end{aligned}
$$

Evaluate $\gamma_h = A\phi^h$ at $h = 0$, we have

$$
\gamma_0 = A \implies A = \frac{\sigma^2}{1 - \phi^2}
$$

Therefore,

$$
\gamma_h = \frac{\sigma^2}{1 - \phi^2} \phi^h, \quad h \geq 0
$$

B.

$$
\begin{aligned}
g(x) &= \frac{1}{1-\phi x} \\
&= 1 + \phi x + (\phi x)^2 + (\phi x)^3 + \cdots \quad \text{(geometric series, } |\phi x| < 1) \\
&= 1 + \phi x + \phi^2 x^2 + \phi^3 x^3 + \cdots
\end{aligned}
$$

Thus $g_0 = 1$, $g_1 = \phi$, $g_2 = \phi^2$, $g_3 = \phi^3$, and in general $g_j = \phi^j$ for $j \geq 0$.

MA(∞) representation of AR(1):

$$
\begin{aligned}
X_n &= \phi X_{n-1} + \epsilon_n \\
&= \phi(\phi X_{n-2} + \epsilon_{n-1}) + \epsilon_n \\
&= \phi^2 X_{n-2} + \phi \epsilon_{n-1} + \epsilon_n \\
&= \phi^2(\phi X_{n-3} + \epsilon_{n-2}) + \phi \epsilon_{n-1} + \epsilon_n \\
&= \phi^3 X_{n-3} + \phi^2 \epsilon_{n-2} + \phi \epsilon_{n-1} + \epsilon_n \\
&\vdots \\
X_n &= \sum_{j=0}^{\infty} \phi^j \epsilon_{n-j}.
\end{aligned}
$$

This is an MA(∞) process with coefficients $\psi_j = \phi^j$ for $j \geq 0$.

Hence, the general formula for the autocovariance function is

$$
\gamma_h = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h} \quad [1]
$$

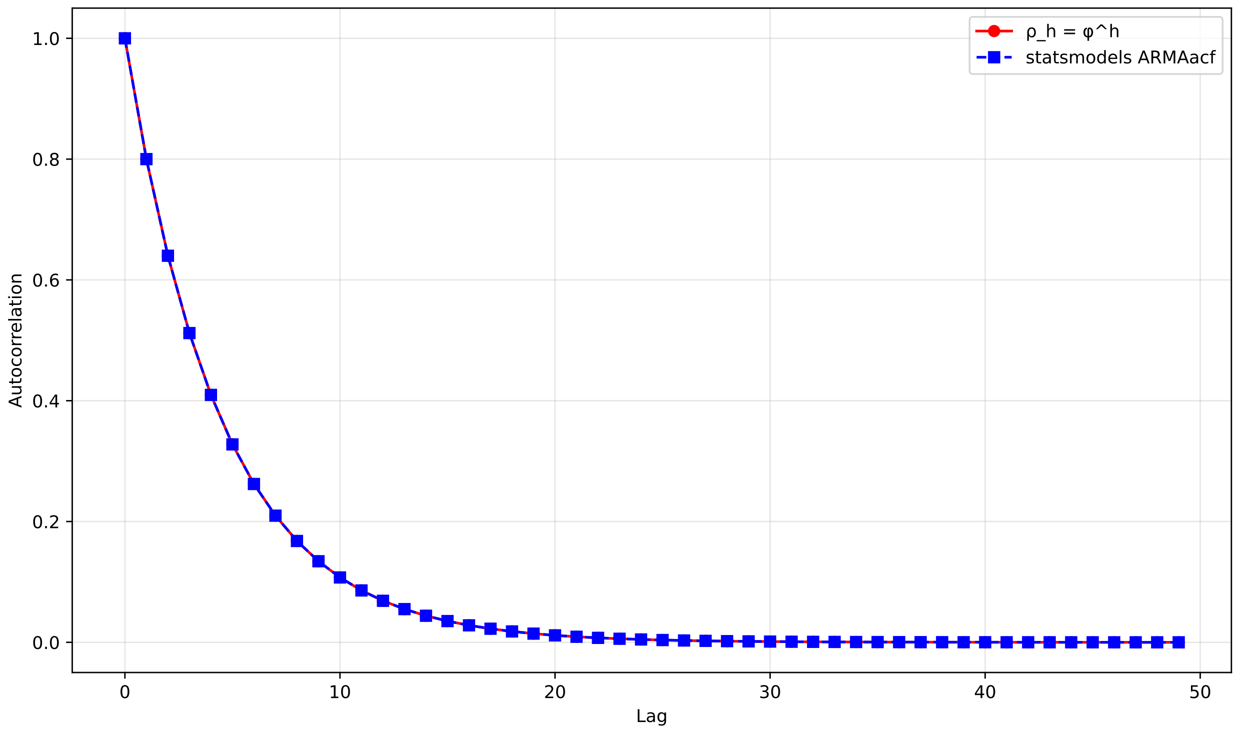
Applying this with $\psi_j = \phi^j$, we have

$$
\begin{aligned}
\gamma_h &= \sigma^2 \sum_{j=0}^{\infty} \phi^j \phi^{j+h} \\
&= \sigma^2 \sum_{j=0}^{\infty} \phi^{2j+h} \\
&= \sigma^2 \phi^h \sum_{j=0}^{\infty} \phi^{2j} \\
&= \frac{\sigma^2}{1-\phi^2} \phi^h.
\end{aligned}
$$

which yields the same answer in A.

C.

`True`



The result returned by `statsmodels.tsa.arima_process.arma_acf` is similar to the formula given from A and B.

Question 2.2

We know the solution of stochastic difference equation of the random walk model is $X_n = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n = \sum_{i=1}^{n} \epsilon_i$.
Without loss of generality, assume $m \leq n$. Then we have

$$\begin{aligned}
\gamma_{mn} &= \mathrm{Cov}(X_m, X_n) \\
&= \mathrm{Cov}\left(\sum_{i=1}^{m} \epsilon_i, \sum_{j=1}^{n} \epsilon_j\right) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} \mathrm{Cov}(\epsilon_i, \epsilon_j) \\
&= \sum_{i=1}^{m} \mathrm{Cov}(\epsilon_i, \epsilon_i) \quad \text{(only non-zero when } i = j \text{ and } i \leq m) \\
&= \sum_{i=1}^{m} \sigma^2 \\
&= m\sigma^2
\end{aligned}$$

Therefore, the autocovariance function for the random walk is

$$\gamma_{mn} = \mathrm{Cov}(X_m, X_n) = \min(m, n)\sigma^2$$

References

1. https://ionides.github.io/531w26/04/notes.pdf (Chapter 4, Page 2)