Problem

Given $x_1, \ldots, x_n \in \mathbb{R}^p$, consider the following linear regression model:

$$Y_i \sim N(\beta^{\mathtt{T}} x_i, \lambda^{-1})$$

independently for $i = 1, \ldots, n$, where $\beta \in \mathbb{R}^p$ and $\lambda > 0$. (Here, $\beta^{\mathtt{T}}$ denotes the transpose of β.)

1. Define conditionally-conjugate (a.k.a. semi-conjugate) priors for β and λ.

2. Derive the full conditional distributions (of β and λ) required for Gibbs sampling.

Solution

1. Conditionally-conjugate priors: $\beta \sim N(\beta_0, \Sigma_0)$ and $\lambda \sim \text{Gamma}(\nu_0/2, \nu_0\sigma_0^2/2)$ independently.

2. Full conditionals: First, letting $y = (y_1, \ldots, y_n)^{\mathtt{T}}$ and $A = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^{\mathtt{T}}$,

$$\sum_{i=1}^n (y_i - \beta^{\mathtt{T}} x_i)^2 = (y - A\beta)^{\mathtt{T}}(y - A\beta) = y^{\mathtt{T}} y - 2\beta^{\mathtt{T}} A^{\mathtt{T}} y + \beta^{\mathtt{T}} A^{\mathtt{T}} A\beta,$$

thus,

$$\begin{aligned}
p(\beta|y, x, \lambda) &\propto p(y|x, \beta, \lambda)p(\beta) \\
&\propto \exp\big(-\tfrac{1}{2}\lambda \textstyle\sum_{i=1}^n (y_i - \beta^{\mathtt{T}} x_i)^2\big) \exp\big(-\tfrac{1}{2}(\beta - \beta_0)^{\mathtt{T}} \Sigma_0^{-1}(\beta - \beta_0)\big) \\
&\propto \exp\big(-\tfrac{1}{2}\lambda\beta^{\mathtt{T}} A^{\mathtt{T}} A\beta + \lambda\beta^{\mathtt{T}} A^{\mathtt{T}} y - \tfrac{1}{2}\beta^{\mathtt{T}}\Sigma_0^{-1}\beta + \beta^{\mathtt{T}}\Sigma_0^{-1}\beta_0\big) \\
&= \exp\big(-\tfrac{1}{2}\beta^{\mathtt{T}}(\lambda A^{\mathtt{T}} A + \Sigma_0^{-1})\beta + \beta^{\mathtt{T}}(\lambda A^{\mathtt{T}} y + \Sigma_0^{-1}\beta_0)\big) \\
&\propto N(\beta \mid \mu_n, \Sigma_n)
\end{aligned}$$

where

$$\begin{aligned}
\Sigma_n &= (\lambda A^{\mathtt{T}} A + \Sigma_0^{-1})^{-1} \\
\mu_n &= \Sigma_n(\lambda A^{\mathtt{T}} y + \Sigma_0^{-1}\beta_0).
\end{aligned}$$

Defining $S(\beta) = \sum_{i=1}^n (y_i - \beta^{\mathtt{T}} x_i)^2$,

$$\begin{aligned}
p(\lambda|y, x, \beta) &\propto p(y|x, \beta, \lambda)p(\lambda) \\
&\propto \lambda^{n/2} \exp\big(-\tfrac{1}{2}\lambda\, S(\beta)\big)\lambda^{\nu_0/2-1} \exp(-\tfrac{1}{2}\nu_0\sigma_0^2\lambda) \\
&= \lambda^{(n+\nu_0)/2-1} \exp\big(-(\tfrac{1}{2}S(\beta) + \tfrac{1}{2}\nu_0\sigma_0^2)\lambda\big) \\
&\propto \text{Gamma}\big(\tfrac{1}{2}(n + \nu_0),\ \tfrac{1}{2}S(\beta) + \tfrac{1}{2}\nu_0\sigma_0^2\big).
\end{aligned}$$