Chapter 3: Solutions to Exercises

Exercise 1

For $\theta \in \Theta = (0, 1)$, the Bernoulli($\theta$) p.m.f. is

$$\begin{aligned}
p(x|\theta) &= \theta^x (1-\theta)^{1-x} \mathbb{1}(x \in \{0,1\}) \\
&= (1-\theta)\big(\theta/(1-\theta)\big)^x \mathbb{1}(x \in \{0,1\}) \\
&= \exp\Big(x \log\big(\theta/(1-\theta)\big) + \log(1-\theta)\Big) \mathbb{1}(x \in \{0,1\}) \\
&= \exp\big(\varphi(\theta) t(x) - \kappa(\theta)\big) h(x)
\end{aligned}$$

with

$$\begin{aligned}
t(x) &= x \\
\varphi(\theta) &= \log\big(\theta/(1-\theta)\big) \\
\kappa(\theta) &= -\log(1-\theta) \\
h(x) &= \mathbb{1}(x \in \{0,1\}).
\end{aligned}$$

Exercise 2

For $\theta \in \Theta = (0, 1)$, the Binomial($n, \theta$) p.m.f. is

$$\begin{aligned}
p(x|\theta) &= \binom{n}{x} \theta^x (1-\theta)^{n-x} \mathbb{1}(x \in S) \\
&= \exp\Big(x \log\big(\theta/(1-\theta)\big) + n \log(1-\theta)\Big) \binom{n}{x} \mathbb{1}(x \in S) \\
&= \exp\big(\varphi(\theta) t(x) - \kappa(\theta)\big) h(x)
\end{aligned}$$

where $S = \{0, 1, \ldots, n\}$, with

$$\begin{aligned}
t(x) &= x \\
\varphi(\theta) &= \log\big(\theta/(1-\theta)\big) \\
\kappa(\theta) &= -n \log(1-\theta) \\
h(x) &= \binom{n}{x} \mathbb{1}(x \in S).
\end{aligned}$$

Exercise 3

Fix $a > 0$. For $b > 0$ and $x > 0$, the Gamma p.d.f. is

$$\mathrm{Gamma}(x|a,b) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} \underset{b}{\propto} b^a e^{-bx}.$$

The form of this as a function of b suggests that a Gamma prior would be conjugate. If we try choosing $p(b) = \mathrm{Gamma}(b|\alpha,\beta)$, the resulting posterior on b is

$$\begin{aligned}
p(b|x_{1:n}) &\underset{b}{\propto} p(x_{1:n}|b)p(b) \\
&= \Big(\frac{b^a}{\Gamma(a)}\Big)^n (x_1 \cdots x_n)^{a-1} e^{-b\sum x_i} \frac{\beta^\alpha}{\Gamma(\alpha)} b^{\alpha-1} e^{-\beta b} \mathbb{1}(b > 0) \\
&\underset{b}{\propto} b^{\alpha+na-1} \exp\big(-(\beta + \textstyle\sum x_i)b\big)\mathbb{1}(b > 0) \\
&\propto \mathrm{Gamma}\big(b \mid \alpha + na,\ \beta + \textstyle\sum x_i\big)
\end{aligned}$$

for $x_1, \ldots, x_n > 0$. Since $\alpha + na > 0$ and $\beta + \sum x_i > 0$, this confirms that the Gamma distribution is a conjugate prior family for b (with a fixed).

Exercise 4

For $a, b > 0$,

$$\begin{aligned}
\mathrm{Gamma}(x|a,b) &= \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} \mathbb{1}(x > 0) \\
&= \exp\Big(a \log x - bx + \log \frac{b^a}{\Gamma(a)}\Big) x^{-1} \mathbb{1}(x > 0) \\
&= \exp\big(\theta^{\mathsf{T}} t(x) - \kappa(\theta)\big) h(x)
\end{aligned}$$

with

$$\theta = \begin{pmatrix} a \\ b \end{pmatrix} \qquad t(x) = \begin{pmatrix} \log x \\ -x \end{pmatrix} \qquad h(x) = x^{-1}\mathbb{1}(x > 0).$$

Exercise 5

Consider a one-parameter exponential family indexed by $\theta \in \Theta$, with p.d.f./p.m.f.

$$p(x|\theta) = \exp\big(\varphi(\theta) t(x) - \kappa(\theta)\big) h(x)$$

and consider the prior with p.d.f.

$$p_{n_0,t_0}(\theta) \propto \exp\big(n_0 t_0 \varphi(\theta) - n_0 \kappa(\theta)\big) \mathbb{1}(\theta \in \Theta).$$

Then the likelihood is

$$p(x_{1:n}|\theta) = \exp\Big(\varphi(\theta) \sum_{i=1}^n t(x_i) - n\kappa(\theta)\Big) h(x_1) \cdots h(x_n)$$

and the resulting posterior is

$$\begin{aligned}
p(\theta|x_{1:n}) &\propto p(x_{1:n}|\theta)p_{n_0,t_0}(\theta) \\
&\propto \exp\Big(\big(n_0 t_0 + \textstyle\sum t(x_i)\big)\varphi(\theta) - (n_0+n)\kappa(\theta)\Big)\mathbb{1}(\theta\in\Theta) \\
&= \exp\big(n't'\varphi(\theta) - n'\kappa(\theta)\big)\mathbb{1}(\theta\in\Theta) \\
&\propto p_{n',t'}(\theta)
\end{aligned}$$

where $n' = n_0 + n$ and $t' = \big(n_0 t_0 + \sum t(x_i)\big)/(n_0+n)$.

Exercise 6

For $\alpha \in H$, define the p.d.f.

$$\pi_\alpha(\theta) = \frac{p_\alpha(\theta)g(\theta)}{z(\alpha)}.$$

Consider data $x_1,\ldots,x_n$. Since $\{p_\alpha : \alpha\in H\}$ is a conjugate prior family, then for any $\alpha\in H$, there is an $\alpha'\in H$ such that $p(x_{1:n}|\theta)p_\alpha(\theta)\propto p_{\alpha'}(\theta)$. Thus, using $\pi_\alpha(\theta)$ as the prior results in the posterior

$$\begin{aligned}
p(\theta|x_{1:n}) &\propto p(x_{1:n}|\theta)\pi_\alpha(\theta) \\
&= p(x_{1:n}|\theta)p_\alpha(\theta)\frac{g(\theta)}{z(\alpha)} \\
&\propto p_{\alpha'}(\theta)\frac{g(\theta)}{z(\alpha)} \\
&\propto p_{\alpha'}(\theta)\frac{g(\theta)}{z(\alpha')} \\
&= \pi_{\alpha'}(\theta).
\end{aligned}$$

Therefore, $\{\pi_\alpha : \alpha\in H\}$ is a conjugate prior family.

Exercise 7

Suppose the prior p.d.f. is

$$p(\theta) = \sum_{i=1}^{k}\pi_i p_{\alpha_i}(\theta)$$

for some $\alpha_1,\ldots,\alpha_k\in H$, $\pi\in\Delta_k$. Consider data $x_1,\ldots,x_n$. Conjugacy of $\{p_\alpha : \alpha\in H\}$ implies that for each $i=1,\ldots,k$, there is some $\alpha_i'\in H$ and some $c_i>0$ such that

$$p(x_{1:n}|\theta)p_{\alpha_i}(\theta) = c_i p_{\alpha_i'}(\theta)$$

for all $\theta \in \Theta$. Therefore, the posterior is

$$\begin{aligned}
p(\theta|x_{1:n}) &\propto p(x_{1:n}|\theta)p(\theta) \\
&= p(x_{1:n}|\theta)\sum_{i=1}^{k}\pi_i p_{\alpha_i}(\theta) \\
&= \sum_{i=1}^{k}\pi_i p(x_{1:n}|\theta)p_{\alpha_i}(\theta) \\
&= \sum_{i=1}^{k}\pi_i c_i\, p_{\alpha_i'}(\theta) \\
&\propto \sum_{i=1}^{k}\pi_i' p_{\alpha_i'}(\theta)
\end{aligned}$$

where

$$\pi_i' = \frac{\pi_i c_i}{\sum_{j=1}^{k}\pi_j c_j}.$$