CS 3110

Balanced Trees

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Spring 2020

Today's scene: Libe Slope

Review

Previously in 3110:

- Efficiency
- Hash tables: imperative constant-time maps

Today:

- Balanced trees: functional maps

Running example: Sets

```
module type Set = sig
  type 'a t
  val empty : 'a t
  val insert : 'a -> 'a t -> 'a t
  val mem : 'a -> 'a t -> bool
  ...
end
```

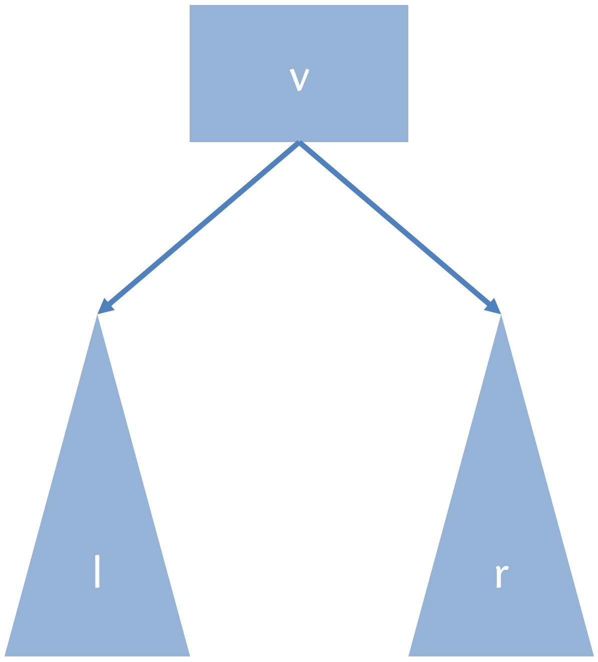
LIST SET

Demo

BST SET

Binary search tree (BST)

- Binary tree: every node has two subtrees
- BST invariant:
  - all values in l are less than v
  - all values in r are greater than v



Demo

Set implementations: performance

	Workload 1	
	`insert`	`mem`
`ListSet`	35s	106s

MacBook, 1.3 GHz Intel Core m7, 8 GB RAM, median of three runs

Set implementations: performance

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Set implementations: performance

	Workload 1		Workload 2	
	`insert`	`mem`	`insert`	`mem`
`ListSet`	35s	106s	35s	106s
`BstSet`	130s	149s	0.07s	0.07s

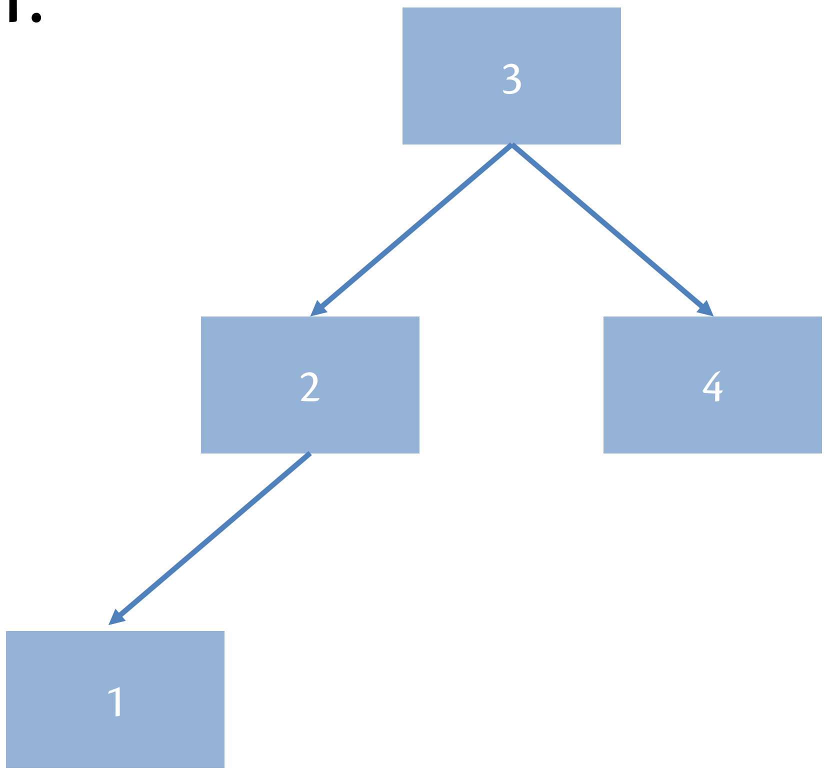
MacBook, 1.3 GHz Intel Core m7, 8 GB RAM, median of three runs

Workloads

- Workload 1:
  - `insert`: 50,000 elements in ascending order
  - `mem`: 100,000 elements, half of which not in set
- Workload 2:
  - `insert`: 50,000 elements in random order
  - `mem`: 100,000 elements, half of which not in set

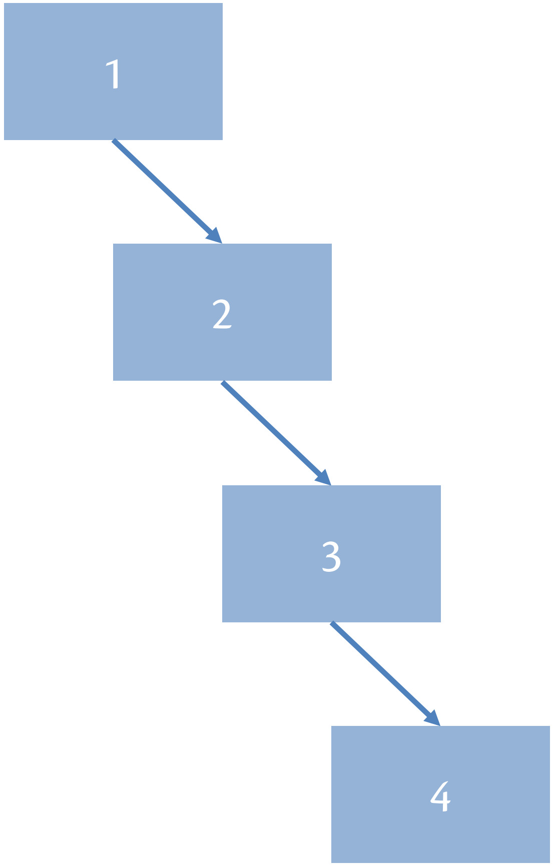
Insert in random order

- Resulting tree depends on exact order
- One possibility for inserting 1..4 in random order 3, 2, 4, 1:



Insert in linear order

Only one possibility for inserting 1..4 in linear order 1, 2, 3, 4:

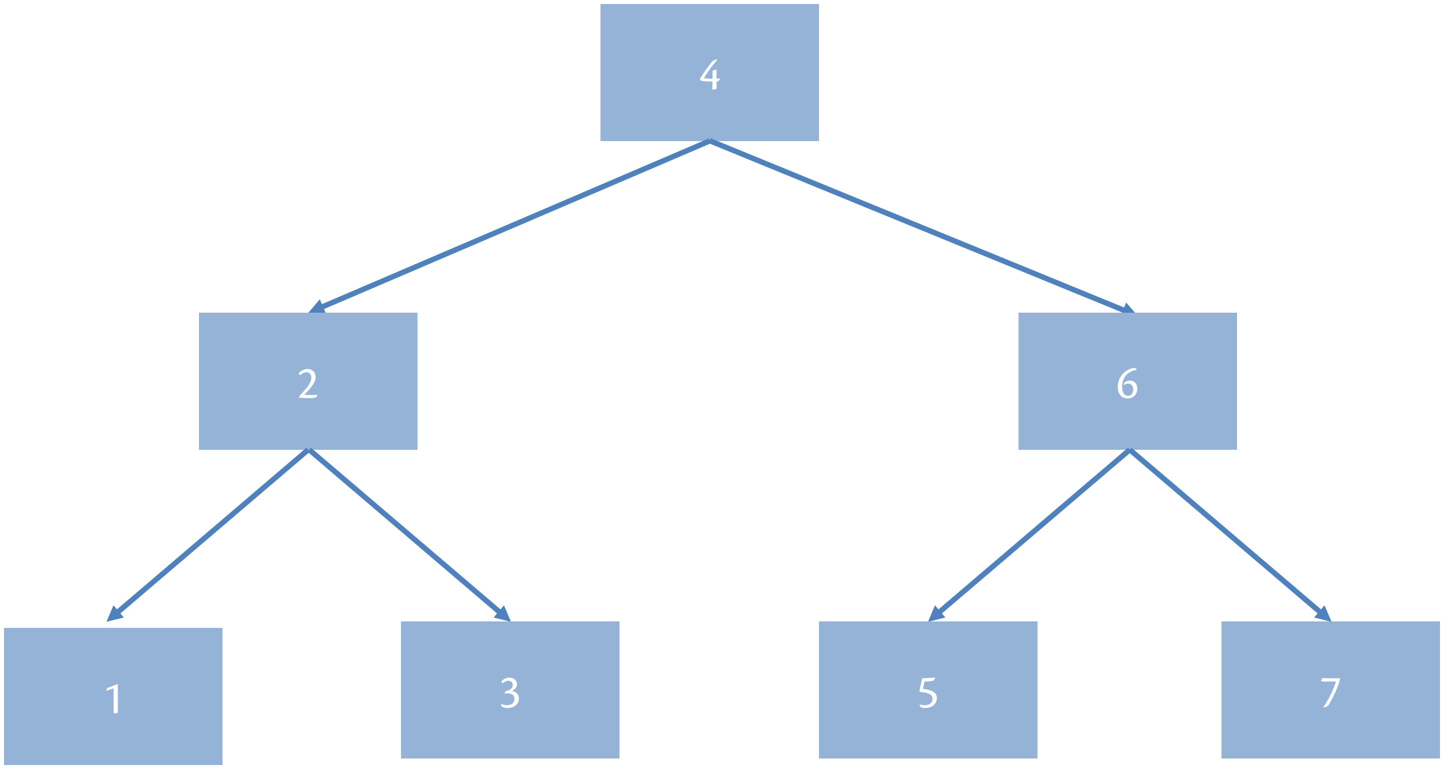


unbalanced: leaning toward the right

When trees get big

- Inserting next element in linear tree always takes *n* operations where *n* is number of elements in tree already
- Inserting next element in randomly-built tree might take far fewer…

Best case tree



all paths through *perfect binary tree* have same length: $log_2\ (n+1)$,
where n is the number of nodes,
recalling there are implicitly leafs below each node at bottom level

Performance of BST

- `insert` and `mem` are both O(n)
- But if trees always had short paths instead of long paths, could be better: O(log n)
- How could we ensure short paths?
  i.e., *balance* trees so they don't lean

BALANCED TREES

Strategies for achieving balance

- In general:
  - Strengthen the RI to require balance
  - And modify insert to guarantee it
- Well-known data structures:
  - 2-3 trees: all paths have same length
  - AVL trees: length of shortest and longest path from any node differ at most by one
  - Red-black trees: length of shortest and longest path from any node differ at most by factor of two
- All of these achieve O(log n) `insert` and `mem`

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RED-BLACK TREES

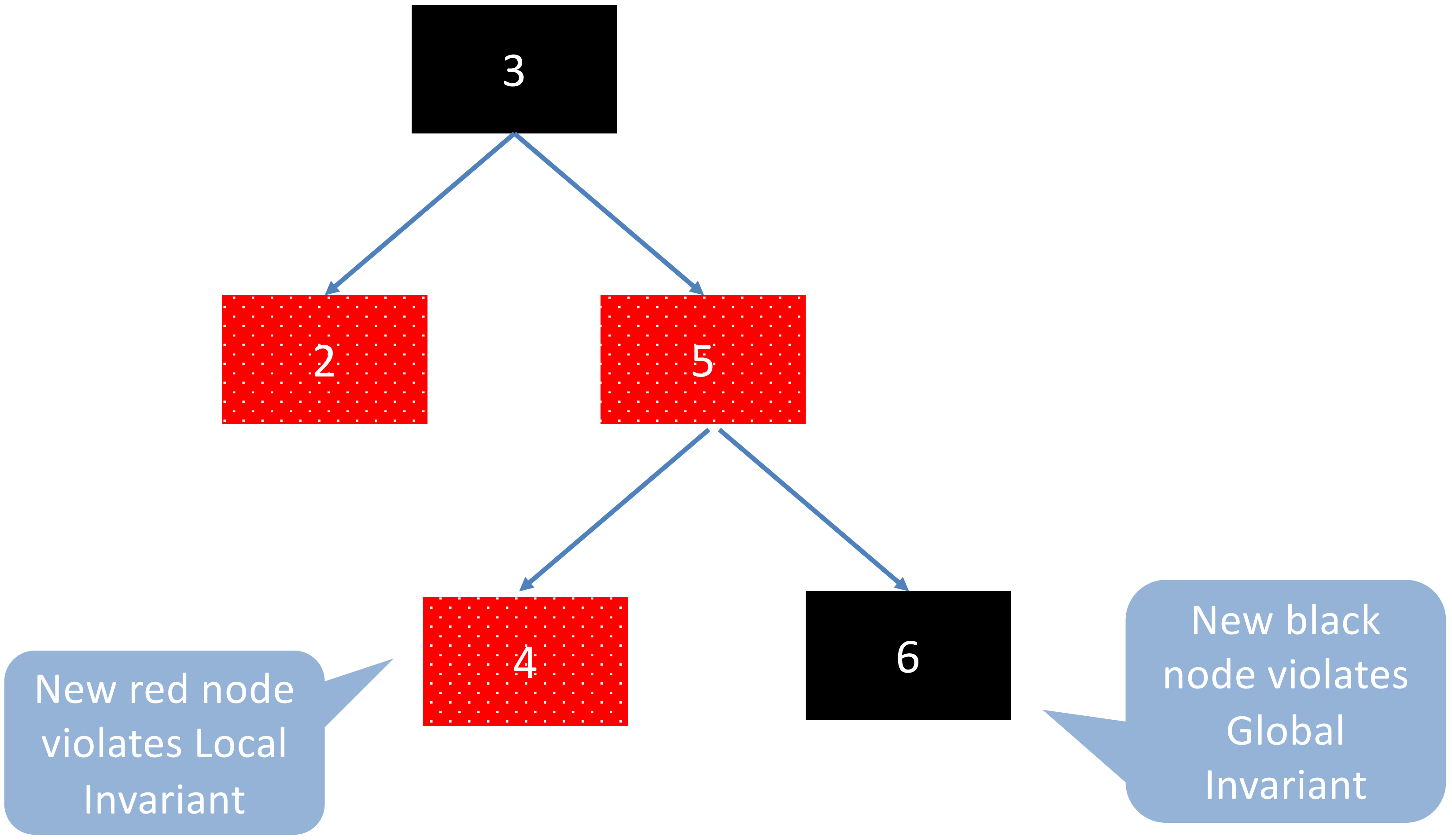
Red-black trees

- [Guibas and Sedgewick 1978], [Okasaki 1998]
- Binary search tree with:
  - Each node colored red or black
  - Leaves and root colored black
- RI: BST +
  - Local invariant: No red node has a red child
  - Global invariant: Every path from the root to a leaf has the same number of black nodes

Path length

- Invariants:
  - No red node has a red child
  - Every path from the root to a leaf has the same number of black nodes
- Together imply: length of longest path is at most twice length of shortest path
  - e.g., B-R-B-R-B-R-B vs. B-B-B-B

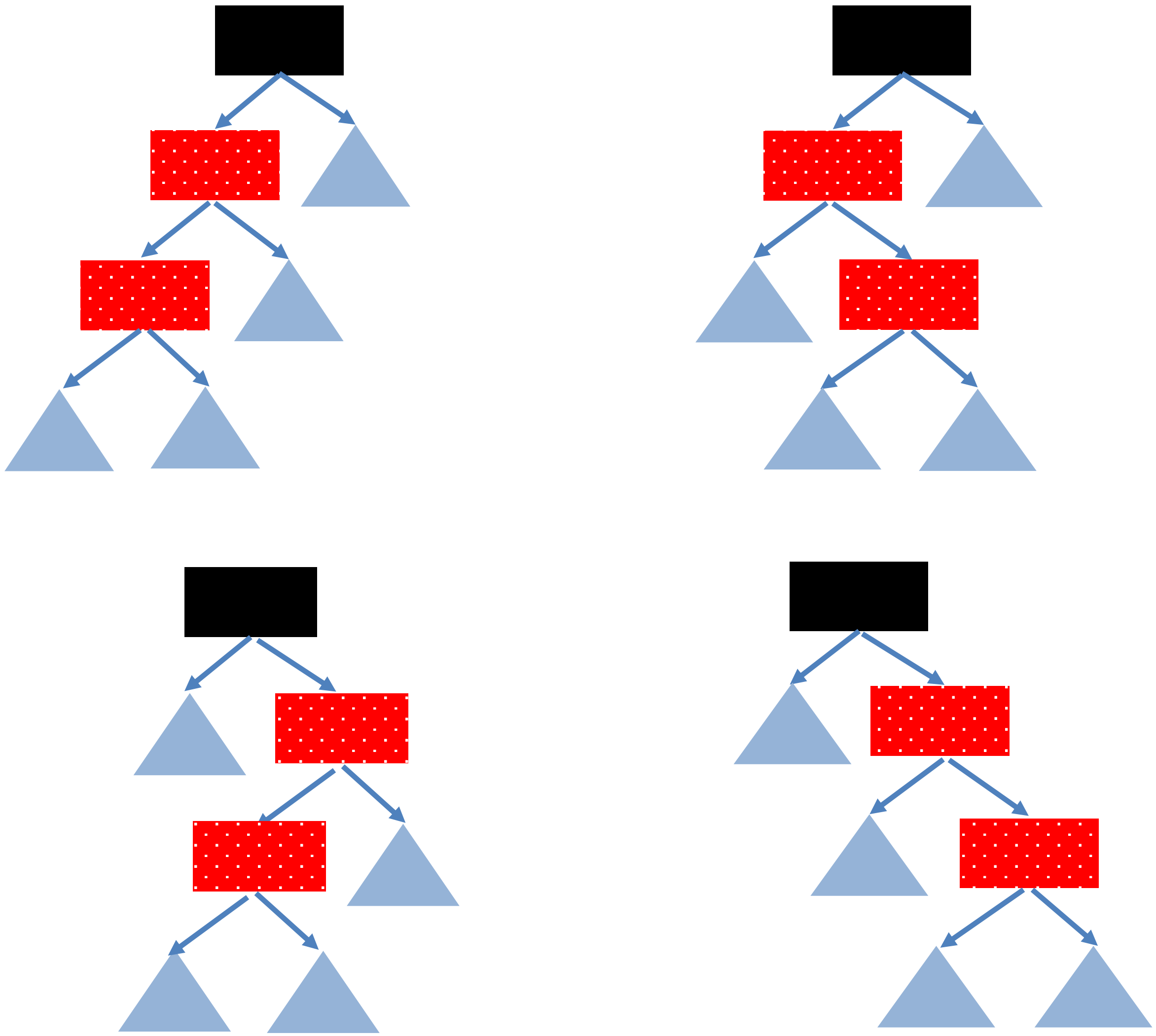
Examples of new nodes



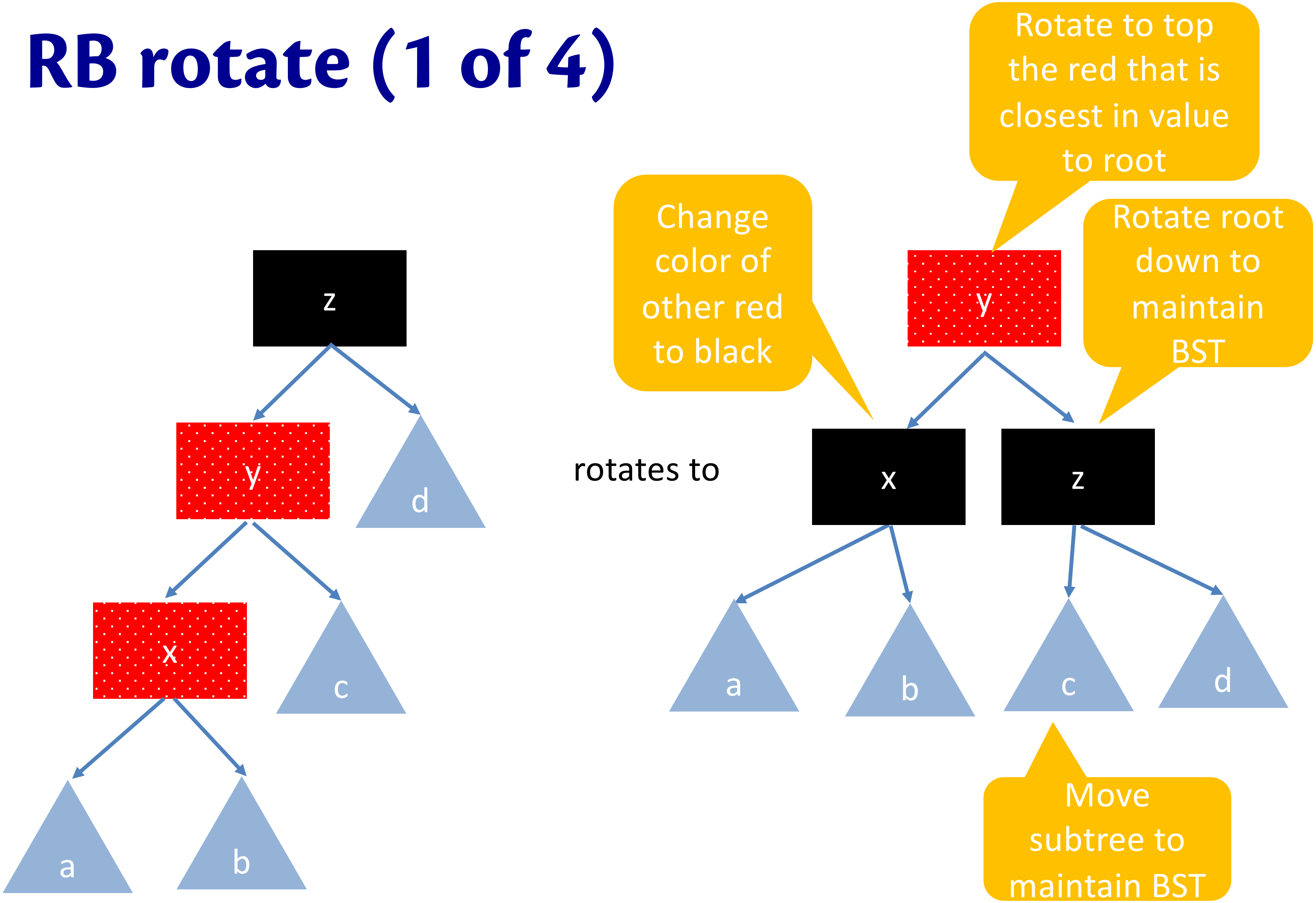
Okasaki's algorithm

- [Okasaki 1998]: functional RB tree
- Always maintain BST + Global Invariant
- Make new node red
- Recurse back up tree
  - Look at the two nodes immediately beneath current node
  - Fix any violations of Local Invariant with a rotation that balances tree

Only four possible violations



RB rotate (1 of 4)



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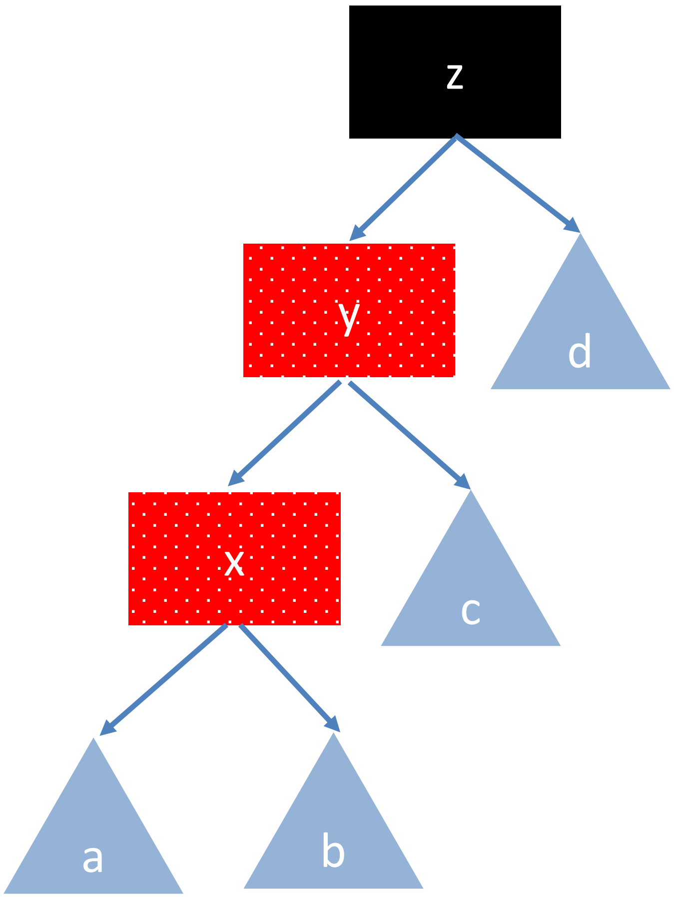
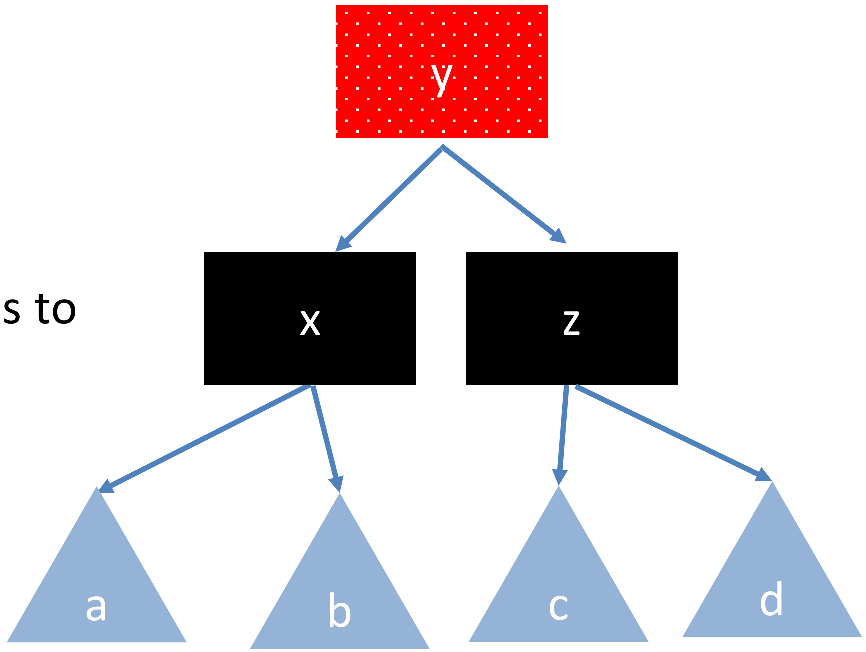
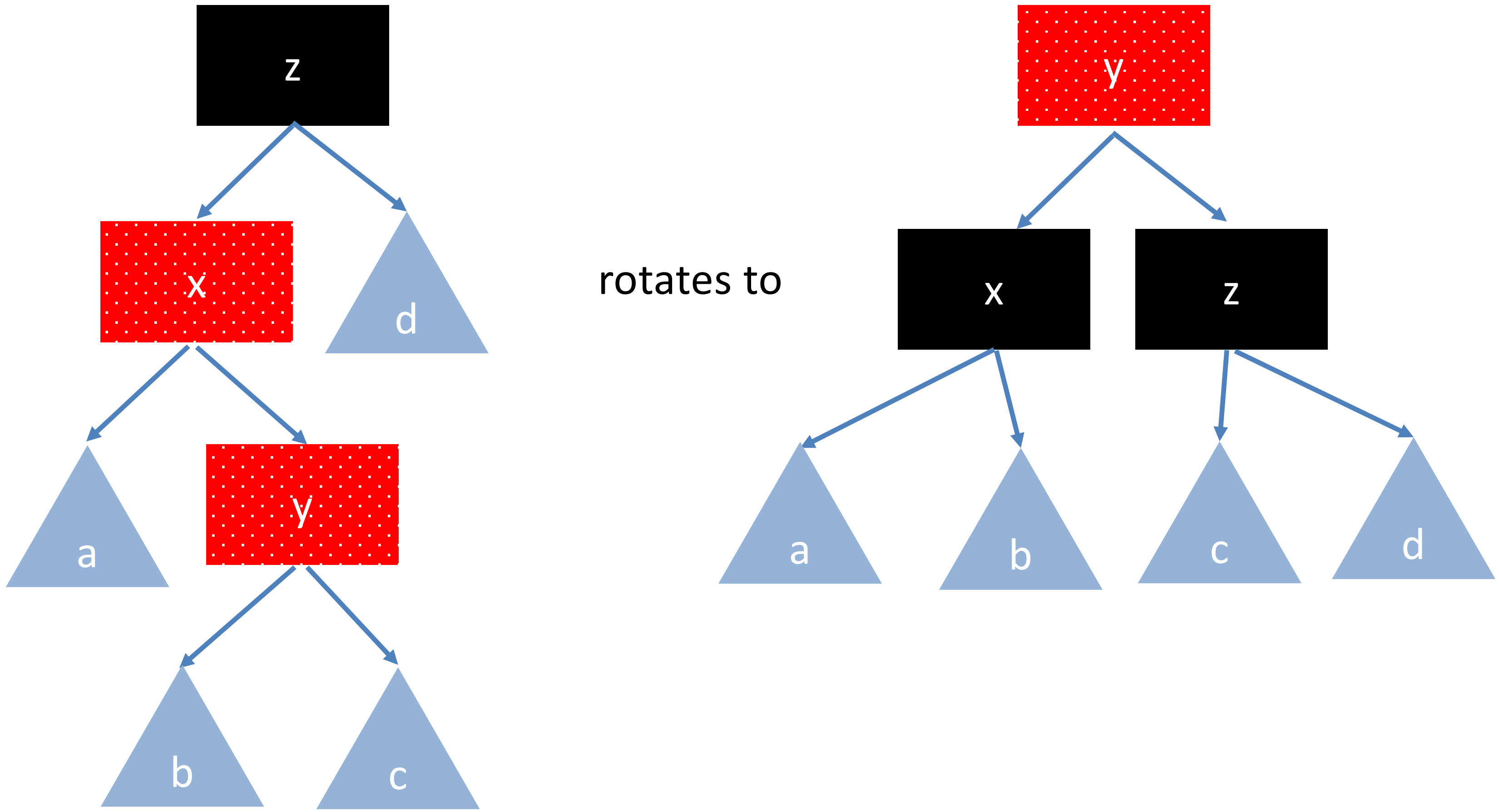


rotates to

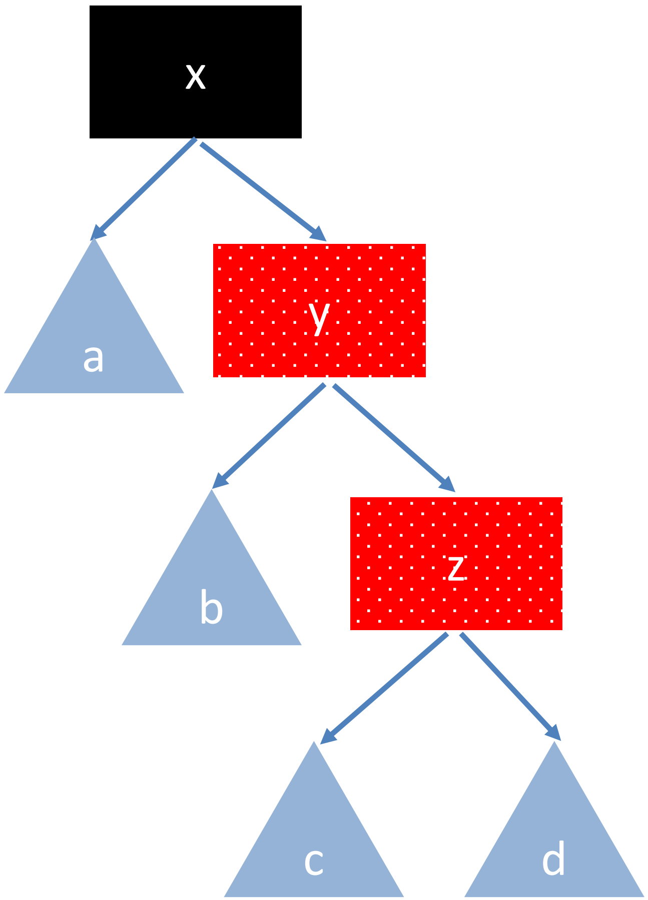


BST + Global Invariant: maintained
Local Invariant: fixed for x & y
…but maybe broken for y and z's original parent!
…so keep recursing up

RB rotate (2 of 4)



RB rotate (3 of 4)



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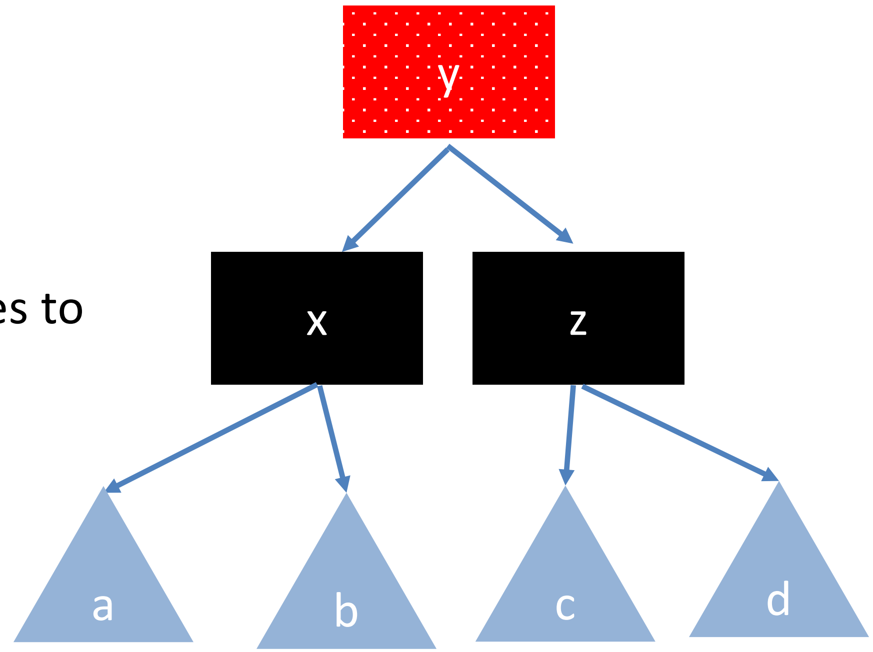
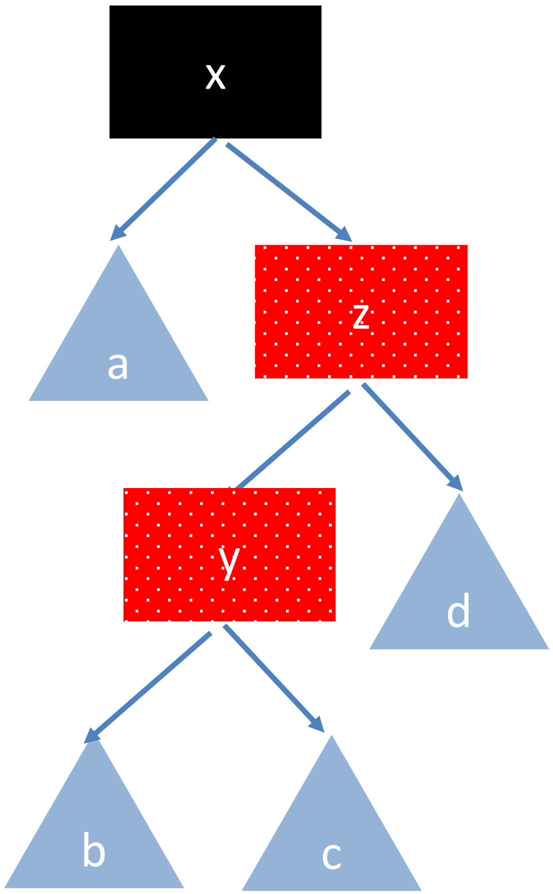


RB rotate (4 of 4)



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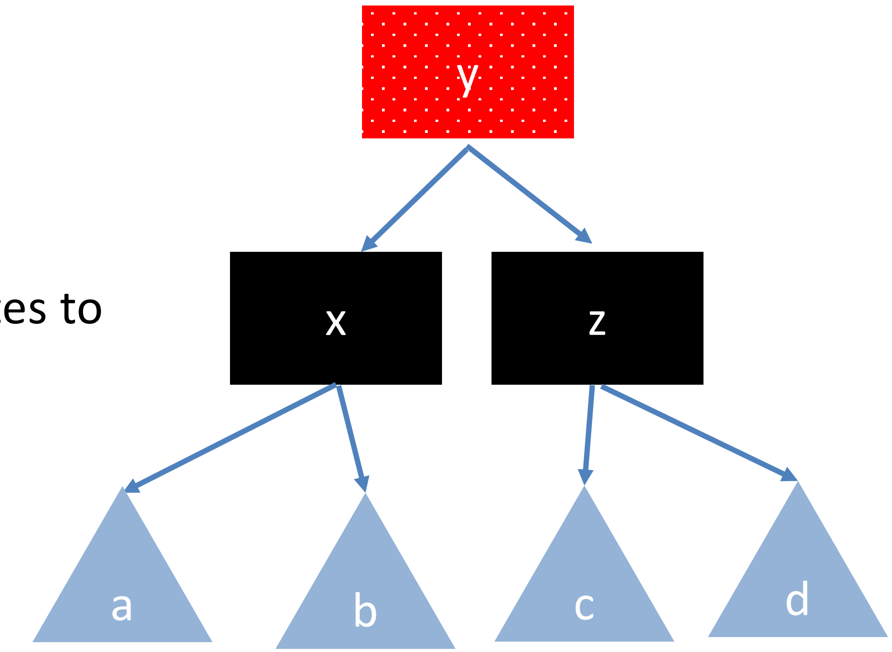


OCaml implementation

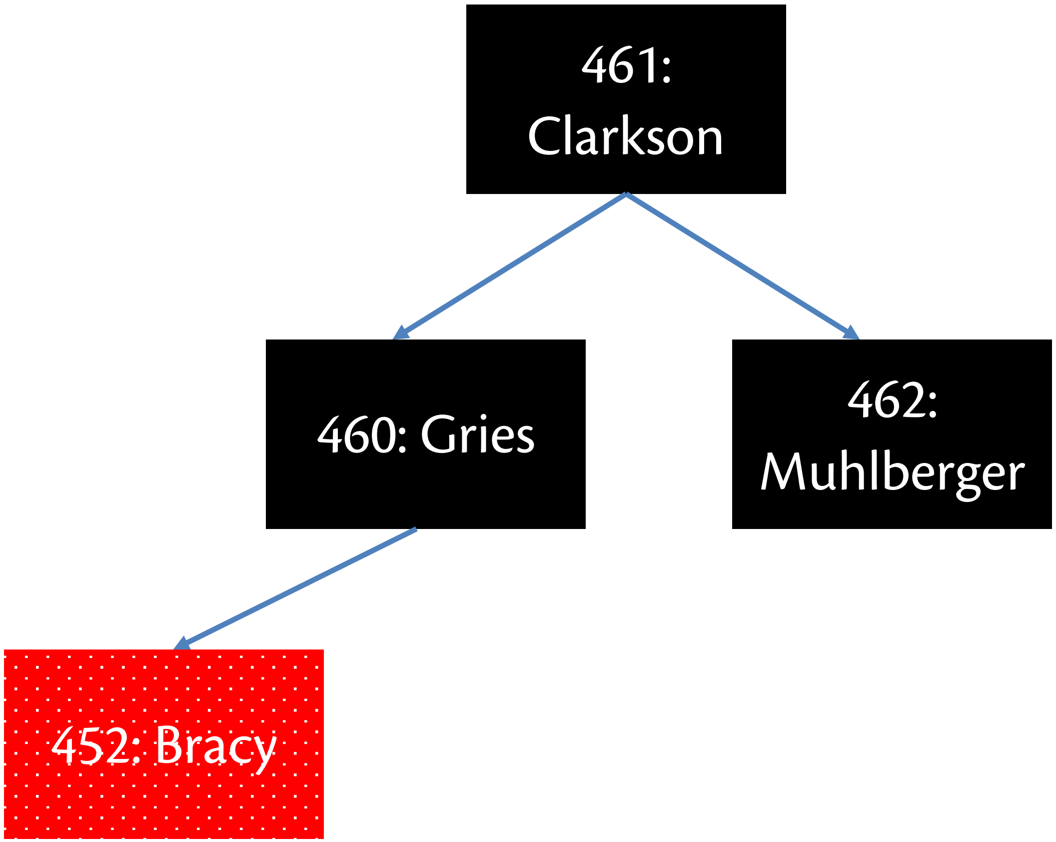
```
let balance = function
  | (Blk, Node (Red, Node (Red, a, x, b), y, c), z, d) (* 1 *)
  | (Blk, Node (Red, a, x, Node (Red, b, y, c)), z, d) (* 2 *)
  | (Blk, a, x, Node (Red, Node (Red, b, y, c), z, d)) (* 4 *)
  | (Blk, a, x, Node (Red, b, y, Node (Red, c, z, d))) (* 3 *)
    -> Node (Red, Node (Blk, a, x, b), y, Node (Blk, c, z, d))
  | t -> Node t
```

Efficiency of red-black set

- mem: O(log n)
  - Worst case: walk down one entire path
  - Length of every path is the same as a perfect tree, or at most twice that long
- insert: O(log n)
  - You will see algorithm in recitation (and textbook)
  - Worst case: walk down then up one entire path

Red-black dictionary

- Store (key, value) pair at each node
- Order as a BST by keys



Map implementations

	insert	find	remove
Arrays	O(1)	O(1)	O(1)
Association lists	O(1)	O(n)	O(n)
Hash tables	O(1)	O(1)	O(1)
Red-black trees	O(log n)	O(log n)	O(log n)

- **Arrays:** fast, but keys must be integers
- **Association lists:** allow any keys, but slower
- **Hash tables:** fast, but requires good hash function; and worst-case performance relaxed to expected & amortized performance
- **Red-black trees:** almost as fast, and immutable

Set implementations: performance

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`RbSet`	0.12s	0.07s	0.15s	0.08s

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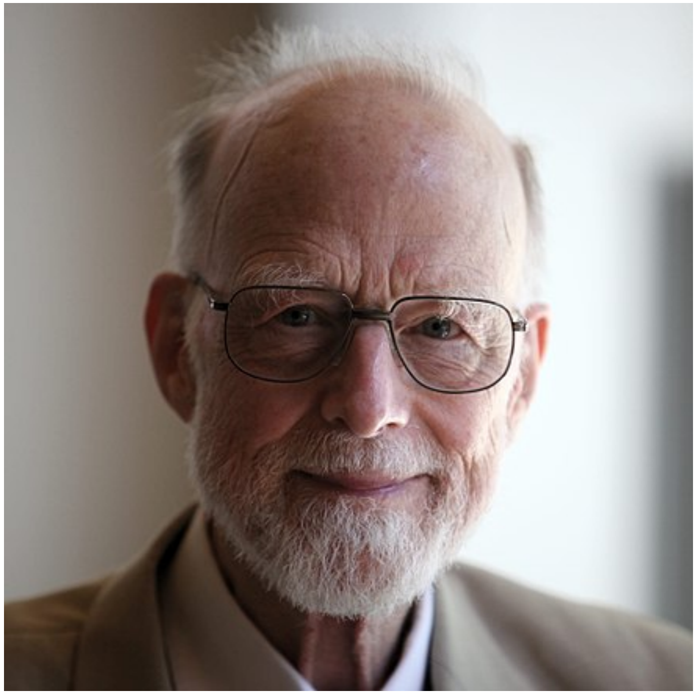
Sir Tony Hoare



b. 1934

Turing Award Winner 1980

For his fundamental contributions to the definition and design of programming languages.

"We should forget about small efficiencies, say about 97% of the time: premature optimization is the root of all evil."

Upcoming events

- [Monday] A4 released
- [Tuesday] Discussion sections start
- [Wednesday] Form partners on CMS
- [Friday] MS0 due

This is blissfully balanced.

THIS IS 3110