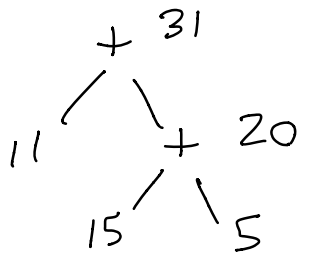
Recitation 17: The Substitution Model

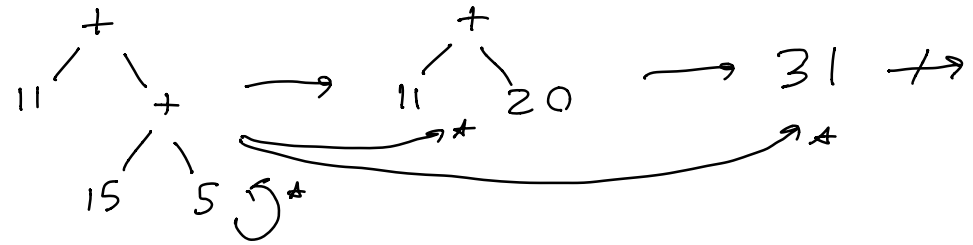
11 + (15 + 5)



No element

$(31, e) \in \rightarrow$

Single-step relation $\rightarrow$ $\left(\begin{array}{c} + \\ 11 \quad 20 \end{array}, 31\right) \in \rightarrow$



Multi-step relation $\rightarrow^*$

Single-step semantics for +

$e_1 + e_2 \quad e_1 * e_2$

if condition
LHS $\rightarrow$ RHS

if $e \rightarrow e'$
$e + e_1 \rightarrow e' + e_1$

if $e \rightarrow e'$
$v + e \rightarrow v + e'$

if "$v_1 + v_2$" $= n$
$v_1 + v_2 \rightarrow n$

Let bindings

let $y = \underline{3+4}$ in $y + 5$

let $x = \underline{e_1}$ in e_2

if $e_1 \to e_1'$

$\mathtt{let}\ x = e_1\ \mathtt{in}\ e_2 \to \mathtt{let}\ x = e_1'\ \mathtt{in}\ e_2$

(no condition)

$\mathtt{let}\ x = v\ \mathtt{in}\ e_2 \to e_2\{v/x\}$

Substitution

$x\{v/x\} = v$

$(e_1 + e_2)\{v/x\} = e_1\{v/x\} + e_2\{v/x\}$

$i\{v/x\} = i \quad y\{v/x\} = y$

$(\mathtt{let}\ y = e_1\ \mathtt{in}\ e_2)\{v/x\} \to$

$\mathtt{let}\ y = e_1\{v/x\}\ \mathtt{in}\ e_2\{v/x\}$

$(\mathtt{let}\ x = e_1\ \mathtt{in}\ e_2)\{v/x\}$

$\to \mathtt{let}\ x = e_1\{v/x\}\ \mathtt{in}\ e_2$

Exercises

(1) $\mathtt{let}\ x = \underline{2+2}\ \mathtt{in}\ x + x$

$\longrightarrow \mathtt{let}\ x = 4\ \mathtt{in}\ x + x$

$\longrightarrow (x+x)\{4/x\}$

$= 4 + 4$

$\longrightarrow 8$

(2) $\mathtt{let}\ x = 5\ \mathtt{in}\ ((\mathtt{let}\ x = 6\ \mathtt{in}\ x) + x)$

$\longrightarrow ((\mathtt{let}\ x = 6\ \mathtt{in}\ x) + x)\{5/x\}$

$= (\text{let } x = 6 \text{ in } x)\{5/x\} + 5$

$= (\text{let } x = 6 \text{ in } x) + 5$

$\longrightarrow x\{6/x\} + 5$

$= 6 + 5$

$\longrightarrow 11$

(3) $\text{let } x = 1 \text{ in } (\text{let } x = x + x \text{ in } x + x)$

$\longrightarrow (\text{let } x = x + x \text{ in } x + x)\{1/x\}$

$= \text{let } x = 1 + 1 \text{ in } x + x$

$\longrightarrow \text{let } x = 2 \text{ in } x + x$

$\longrightarrow (x + x)\{2/x\}$

$= 2 + 2$

$\longrightarrow 4$

Core OCaml

$i ::= \langle \text{integers} \rangle$

$x ::= \langle \text{identifiers} \rangle$

$bop ::= + \mid *$

$b ::= \text{true} \mid \text{false}$

$e ::= i \mid x \mid e_1 \; bop \; e_2 \mid$
$\text{let } x = e_1 \text{ in } e_2 \mid$
$(e_1, e_2) \mid \text{fst } e \mid \text{snd } e \mid$
$\text{Left } e \mid \text{Right } e \mid$

match e with Left $x_1 \to e_1$; Right $x_2 \to e_2$ |
if e_1 then e_2 else e_3 |
fun $x \to e$ | e_1 e_2

$v ::= i \mid b \mid (v_1, v_2) \mid$ Left $v \mid$ Right $v \mid$ fun $x \to e$

Semantics of Core OCaml

Pairs

if $e \to e'$

$(e, e_1) \to (e', e_1)$

$(v, e) \to (v, e')$

$\text{fst}\,(v_1, v_2) \to v_1$

$\text{snd}\,(v_1, v_2) \to v_2$

Variants

if $e \to e'$

Left $e \to$ Left e'

Right $e \to$ Right e'

match e with ... $\longrightarrow$ match e' with ...

match Left v with Left $x_1 \to e_1$; Right $x_2 \to e_2$

$\longrightarrow e_1\{v/x_1\}$

Conditionals

if $e \to e'$

if e then e_2 else $e_3 \to$ if e' then e_2 else e_3

if true then e_2 else $e_3 \to e_2$

Functions

Functions

if $e \to e'$

$e\ e_1 \to e'\ e_1$

$(\text{fun } x \to e_2)\ e \to (\text{fun } x \to e_2)\ e'$

$(\text{fun } x \to e_2)\ v \to e_2\{v/x\}$

Substitution in Core OCaml

Functions

$(\text{fun } x \to e)\{v/x\} = \text{fun } x \to e$

$(\text{fun } y \to e)\{v/x\} = \text{fun } y \to (e\{v/x\})$

---

let x = 3 in ...

fun y → x

fun z → x

let x = z in (fun y → x)

→ (fun y → x){z/x}

= fun y → z

let x = z in (fun z → x)

→ (fun z → x){z/x}

= ~~fun z → z~~

= (fun z_1 → x){z/x}

=

$= \lim_{z_1 \to z}$