CS 3110

Proofs about Programs

Nate Foster
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Today's scene: Taughannock Falls

Review

Previously in 3110:
- Functional programming
- Modular programming
- Efficiency
- Interpreters

Final unit of course: proofs about programs

Today:
- Equational reasoning
- Proving correctness of recursive functions

Approaches to validation [lec 10]

- Social
  - Code reviews
  - Extreme/Pair programming
- Methodological
  - Design patterns
  - Test-driven development
  - Version control
  - Bug tracking
- Technological
  - Static analysis
    (“lint” tools, FindBugs, …)
  - Fuzzers
- Mathematical
  - Sound type systems
  - “Formal” verification

Less formal: Techniques may miss problems in programs

All of these methods should be used!

Even the most formal can still have holes:
- did you prove the right thing?
- do your assumptions match reality?

More formal: eliminate *with certainty* as many problems as possible.

Slide credit: Benjamin C. Pierce (UPenn)

Verification

- In the 1970s, scaled to about tens of LOC
- Now, research projects scale to real software:
  - CompCert: verified C compiler
  - seL4: verified microkernel OS
  - Ynot: verified DBMS, web services
  - Four color theorem
  - Project Everest: verified HTTPS stack [in progress]
  - Etc.
- In another 40 years?

Our goals

- Write small, pure functional programs
  - no side effects, mutability, I/O; always terminating
  - integers, lists, options, trees
- Prove correctness theorems
  - CS 2800 mathematics: induction, logic
- Be rigorous but not completely formal

CORRECTNESS

Specifications

```
(** [fact n] is [n] factorial,
    i.e., [n!].
    Requires: [n >= 0]. *)
let rec fact n = ...
```

Postcondition gives an equality between function output and an English/mathematical description involving input

Correctness proofs

- Based on equality between expressions
- When does e = e'?
  - Not asking about OCaml Boolean equality
  - Asking whether two pieces of code are equal…

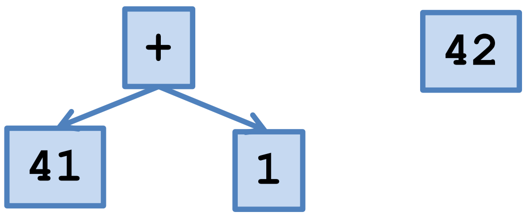
Equality of expressions

$$41 + 1 \stackrel{?}{=} 42$$

Semantically: yes

$$41 + 1 \longrightarrow^* 42$$
$$42 \longrightarrow^* 42$$

Syntactically: no



Equality of expressions

`fun x -> x` $\stackrel{?}{=}$ **`fun y -> y`**

Extensionality

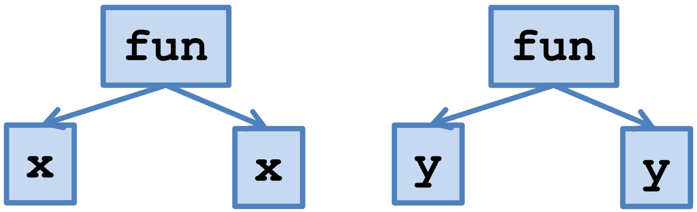
Semantically: yes

for all v,

(**fun** x -> x) v $\longrightarrow^*$ v

(**fun** y -> y) v $\longrightarrow^*$ v

Syntactically: no



Equality of expressions

$$e = e'$$

if e and e’ evaluate to the same value

must be well typed, pure, total

PART II: EQUATIONAL REASONING

Example 1

```
let twice f x = f (f x)
let compose f g x = f (g x)
```

twice h x = h (h x) (by evaluation)

compose h h x = h (h x) (by evaluation)

so

twice h x = compose h h x (by transitivity)

Example 1

```
let twice f x = f (f x)
let compose f g x = f (g x)
```

  twice h x
= {evaluation}
  h (h x)
= {evaluation}
  compose h h x

Please use this proof format

Example 2

```
let (<<) = compose
```

Theorem: composition is associative.

(f << g) << h = f << (g << h)

Proof: by extensionality, we need to show:

forall x, ((f << g) << h) x = (f << (g << h)) x

```
let compose f1 f2 x = f1 (f2 x)
```

Example 2

((f << g) << h) x   =   (f << (g << h)) x

((f << g) << h) x
= { evaluation }
(f << g) (h x)
= { evaluation }
f (g (h x))

(f << (g << h)) x
= { evaluation }
f ((g << h) x)
= { evaluation }
f (g (h x))

QED

PART III: PROOFS WITH RECURSION

Example 1: Summation

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

```
let rec sumto n =
  if n = 0 then 0
  else n + sumto (n - 1)
```

$\texttt{sumto n} \stackrel{?}{=} \texttt{n * (n + 1) / 2}$

Induction on natural numbers

Theorem: forall n, P(n).

Proof: by induction on n

Base case: n = 0
Show: P(0)

Inductive case: n = k+1
IH: P(k)
Show: P(k+1)

QED

Please use this proof format

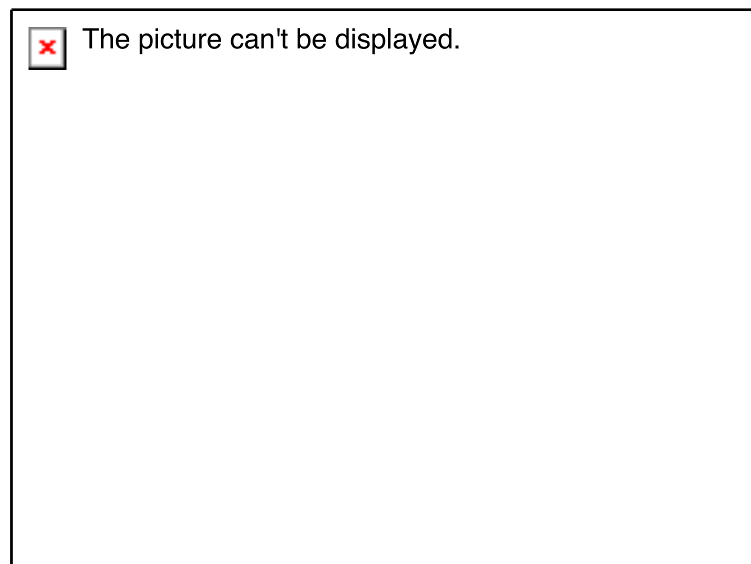
Induction principle on naturals

forall properties P,

if P(0)

and (forall k, P(k) implies P(k+1))

then (forall n, P(n))



Summation: proof structure

Claim: forall n, sumto n = n * (n + 1) / 2

Proof: by induction on n.

P(n) = sumto n = n * (n + 1) / 2

Base case: n = 0

Show: sumto 0 = 0 * (0 + 1) / 2

Inductive case: n = k + 1

IH: ???

Summation: proof structure

Claim: forall n, sumto n = n * (n + 1) / 2

Proof: by induction on n.
P(n) = sumto n = n * (n + 1) / 2

Base case: n = 0
Show: sumto 0 = 0 * (0 + 1) / 2

Inductive case: n = k + 1
IH: sumto k = k * (k + 1) / 2
Show: sumto (k + 1) = (k + 1) * ((k + 1) + 1) / 2

Summation: base case

Base case: n = 0

Show: sumto 0 = 0 * (0 + 1) / 2

sumto 0

= { evaluation }

0

= { algebra (or evaluation) }

0 * (0 + 1) / 2

```
let rec sumto n =
  if n = 0 then 0
  else n + sumto (n - 1)
```

Summation: inductive case

Inductive case: n = k + 1
IH: sumto k = k * (k + 1) / 2
Show: sumto (k + 1) = (k + 1) * ((k + 1) + 1) / 2

  sumto (k + 1)
= { evaluation }
  k + 1 + sumto k
= { IH }
  k + 1 + k * (k + 1) / 2
= { algebra }
  (k + 1) * ((k + 1) + 1) / 2

QED

```
let rec sumto n =
  if n = 0 then 0
  else n + sumto (n - 1)
```

Example 2: Factorial

```
let rec fact n =
  if n = 0 then 1
  else n * fact (n - 1)


let rec facti acc n =
  if n = 0 then acc
  else facti (acc * n) (n - 1)


let fact_tr n = facti 1 n
```

“i” suggests
iterative

Factorial: correctness

Claim: forall n, fact n = facti 1 n

Proof: by induction on n.
P(n)   =   fact n = facti 1 n

Base case: n = 0
Show: fact 0 = facti 1 0

Inductive case: n = k + 1
IH: fact k = facti 1 k
Show: fact (k + 1) = facti 1 (k + 1)

Factorial: inductive case

Inductive case: n = k + 1

IH: fact k = facti 1 k

Show: fact (k + 1) = facti 1 (k + 1)

fact (k + 1)

= { evaluation }

(k + 1) * fact k

= { IH }

(k + 1) * facti 1 k

STUCK

facti 1 (k + 1)

= { evaluation }

facti (k + 1) k

want to move (k + 1) into accumulator:

facti (k + 1) k

but how?

```
let rec fact n =
  if n = 0 then 1
  else n * fact (n - 1)

let rec facti acc n =
  if n = 0 then acc
  else facti (acc * n) (n - 1)
```

Strengthened IH

What we have from IH:

(k + 1) * fact k = (k + 1) * facti 1 k

What we want:

(k + 1) * fact k = facti (k + 1) k

want to multiply
(k + 1) into acc

So, strengthen P(n) to give us what we want:

forall p, p * fact n = facti p n

can multiply
anything into acc

Factorial: correctness, take 2

Lemma: forall n, forall p, p * fact n = facti p n

Proof: by induction on n.
P(n)   =   forall p, p * fact n = facti p n

Base case: n = 0
Show: forall p, p * fact 0 = facti p 0

Inductive case: n = k + 1
IH: forall p, p * fact k = facti p k
Show: forall p, p * fact (k + 1) = facti p (k + 1)

Factorial: strengthened ind. case

Inductive case: n = k + 1

IH: forall p, p * fact k = facti p k

Show: forall p, p * fact (k + 1) = facti p (k + 1)

p * fact (k + 1)

= { evaluation }

(p * (k + 1)) * fact k

= { IH with p := p * (k + 1) }

facti (p * (k + 1)) k

facti p (k + 1)

= { evaluation }

facti (p * (k + 1)) k

QED

```
let rec fact n =
  if n = 0 then 1
  else n * fact (n - 1)

let rec facti acc n =
  if n = 0 then acc
  else facti (acc * n) (n - 1)
```

Factorial: correctness

Claim: forall n, fact n = fact_tr n

Proof:

```
  fact_tr n
=  { evaluation }
  facti 1 n
= { previous lemma with p := 1 }
  fact n
```

QED

Upcoming events

- [Friday] Project MS2 due

This is rigorous.

THIS IS 3110