CO2029

1. Introduction of Signals and Systems

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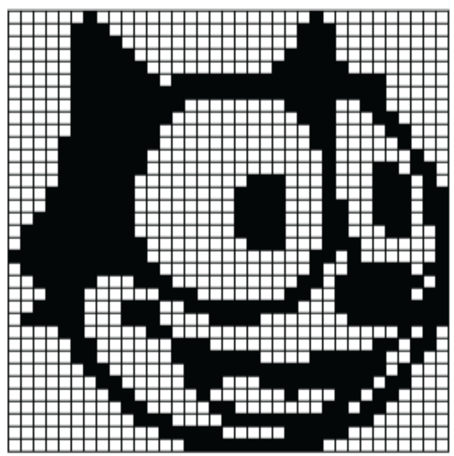
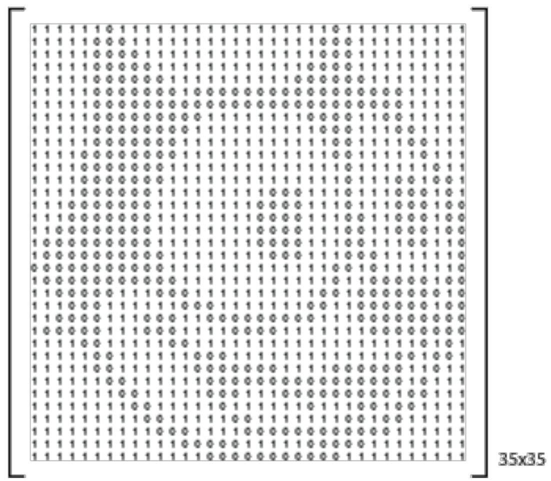
What is a Signal?

- Any physical quantity that varies with time, space, or any other independent variable or variables.
- Examples: pressure as a function of altitude, sound as a function of time, color as a function of space, etc.
- Representation
  - $x(t) = \cos(2\pi t)$, $x(t) = 4pt + t^3$, $x(m;n) = (m+n)^3$





What is a System?

- A physical device or program that performs an operation on a signal such as information transform and extraction.
  - Performing an operation on a signal is called **signal processing**

- Examples
  - Analog amplifier
  - Noise canceler
  - Communication channel
  - Transistor
  - etc.

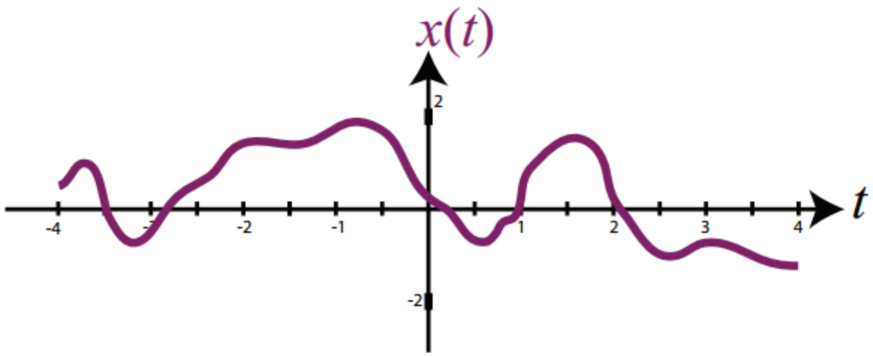
- Representation

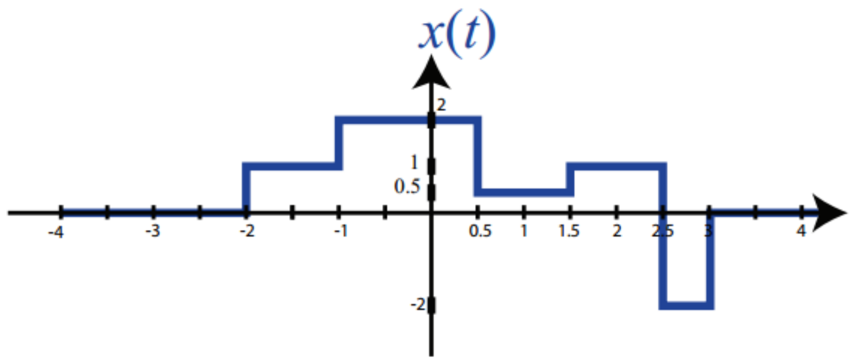
$$y(t) = -4x(t),\ \frac{dy(t)}{dt} + 3y(t) = -\frac{dx(t)}{dt} + 6x(t),$$
$$y(n) - \frac{1}{2}y(n-2) = 3x(n) + x(n-2)$$

Continuous-Time vs. Discrete-Time Signals

- **Continuous-Time Signals**: signal is defined for every value of time in a given interval (a, b) where $a \geq -\infty$ and $b \leq -\infty$

- Examples
  - Voltages as a function of time
  - Height as a function of pressure
  - Number of positron emissions as a function of time

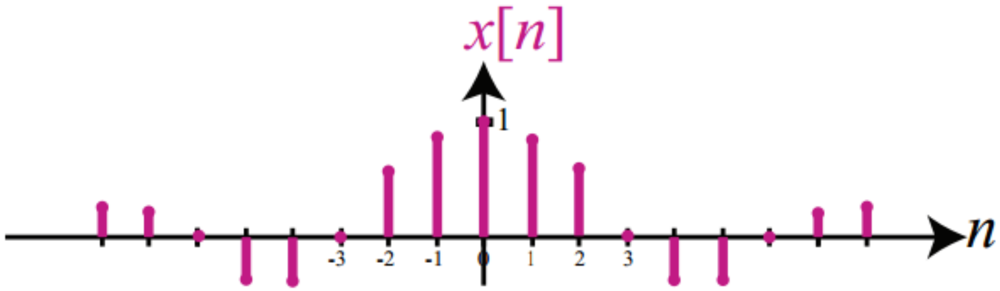


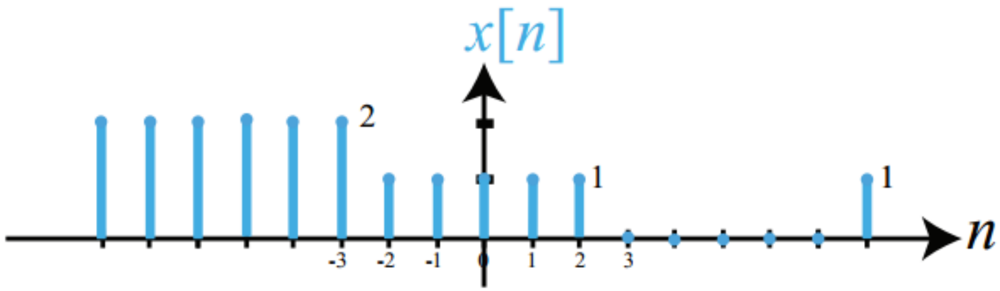


Continuous-Time vs. Discrete-Time Signals

- **Discrete-Time Signals**: signal is defined only for certain specific values of time; typically taken to be equally spaced points in an interval.

- Examples
  - Number of stocks traded per day
  - Average income per province

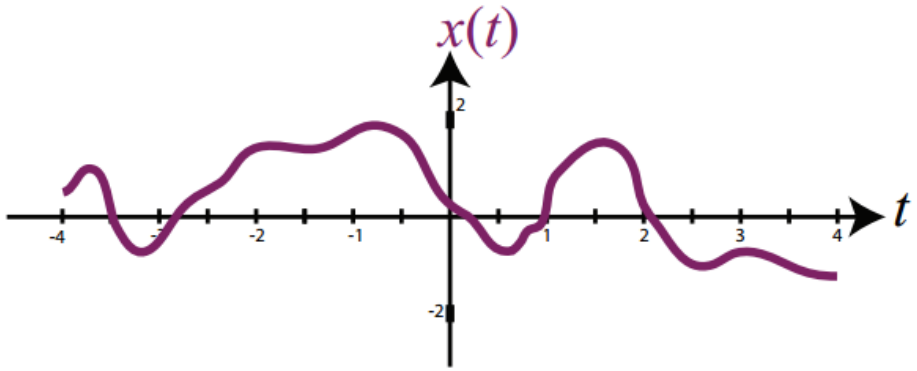


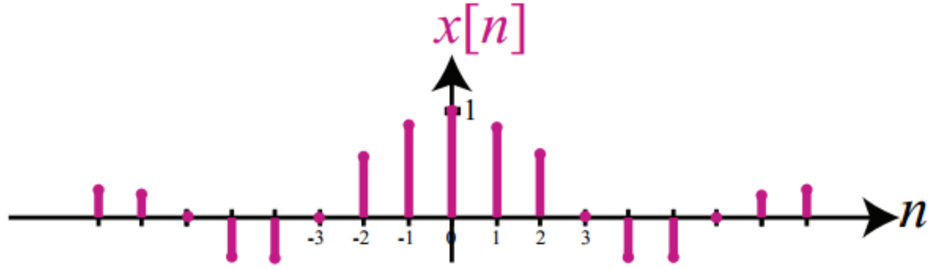


Continuous-Amplitude vs. Discrete-Amplitude Signals

- **Continuous-Amplitude Signals**: signal amplitude takes on a spectrum of values within one or more intervals.

- Examples
  - Color
  - Temperature
  - Pain-level

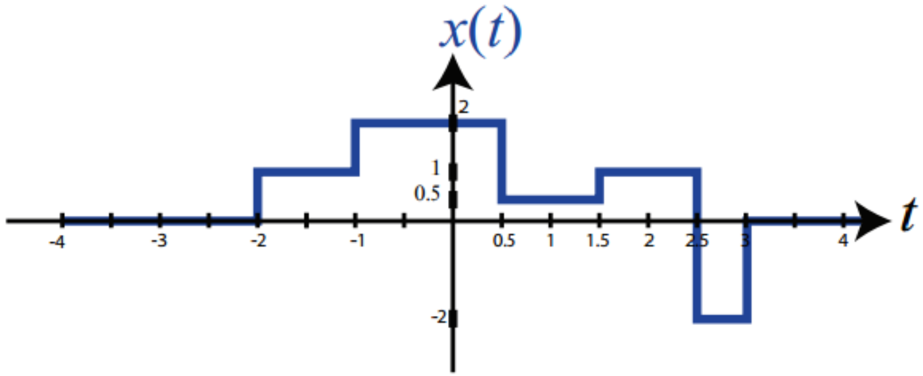


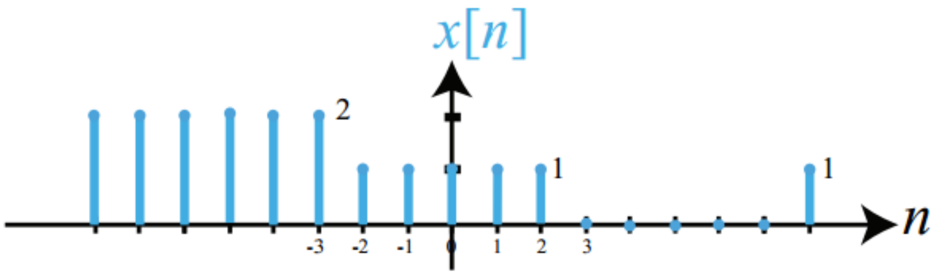


Continuous-Amplitude vs. Discrete-Amplitude Signals

- **Discrete-Amplitude Signals**: signal amplitude takes on values from a finite set.

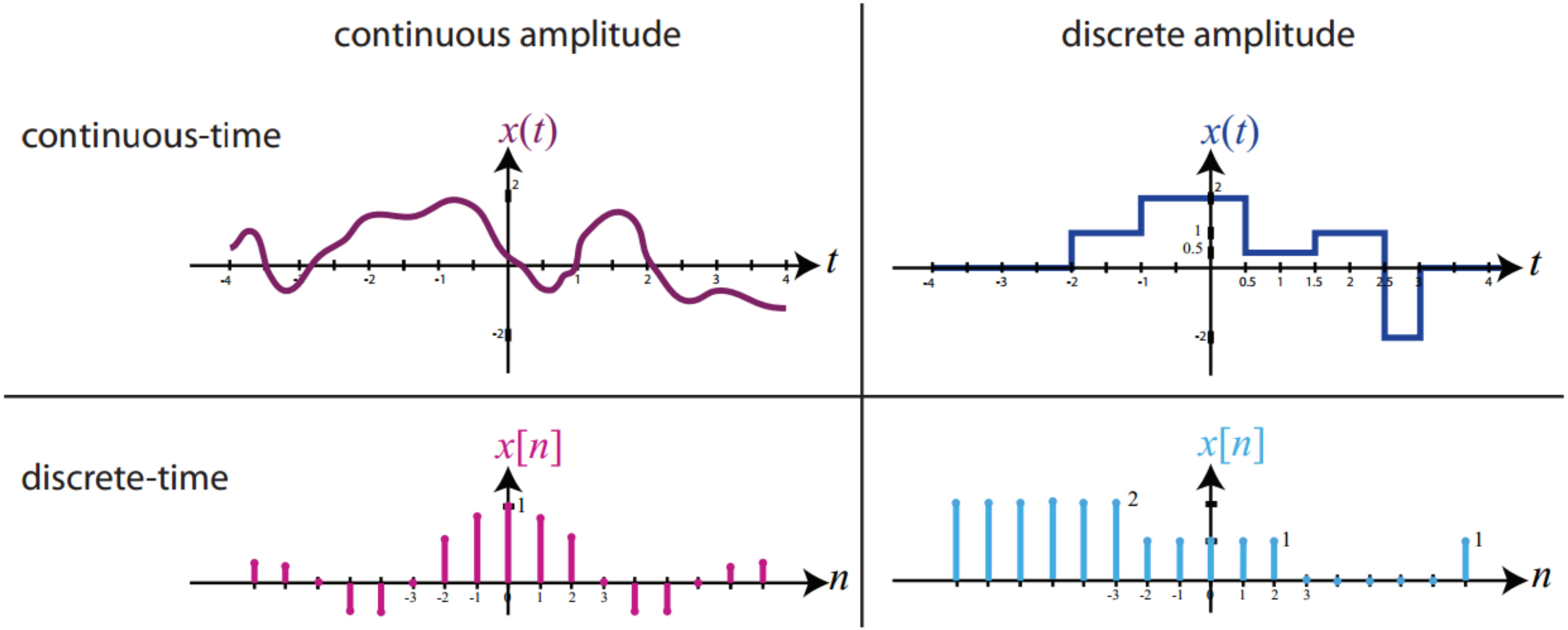
- Examples
  - Digital image
  - Population of a country





Analog and Digital Signals

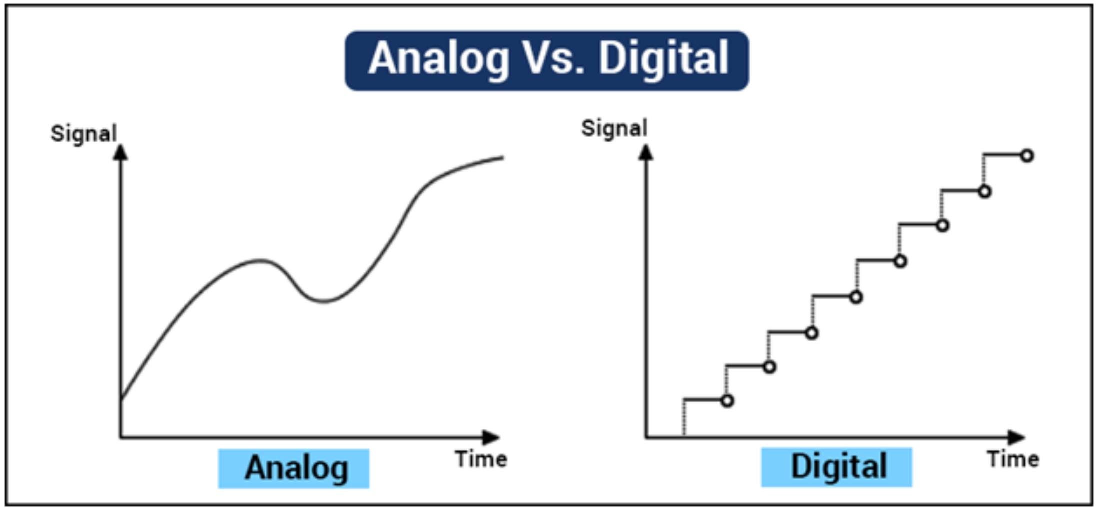
- Analog Signal = Continuous-Time + Continuous-Amplitude
- Digital Signal = Discrete-Time + Discrete-Amplitude



Analog and Digital Signals

- Analog signals are fundamentally significant because we must interface with the real world which is analog by nature.

- Digital signals are important because they facilitate the use of digital signal processing (DSP) systems, which have practical and performance advantages for several applications.



Analog and Digital Systems

- **Analog system** =

  analog signal input + analog signal output

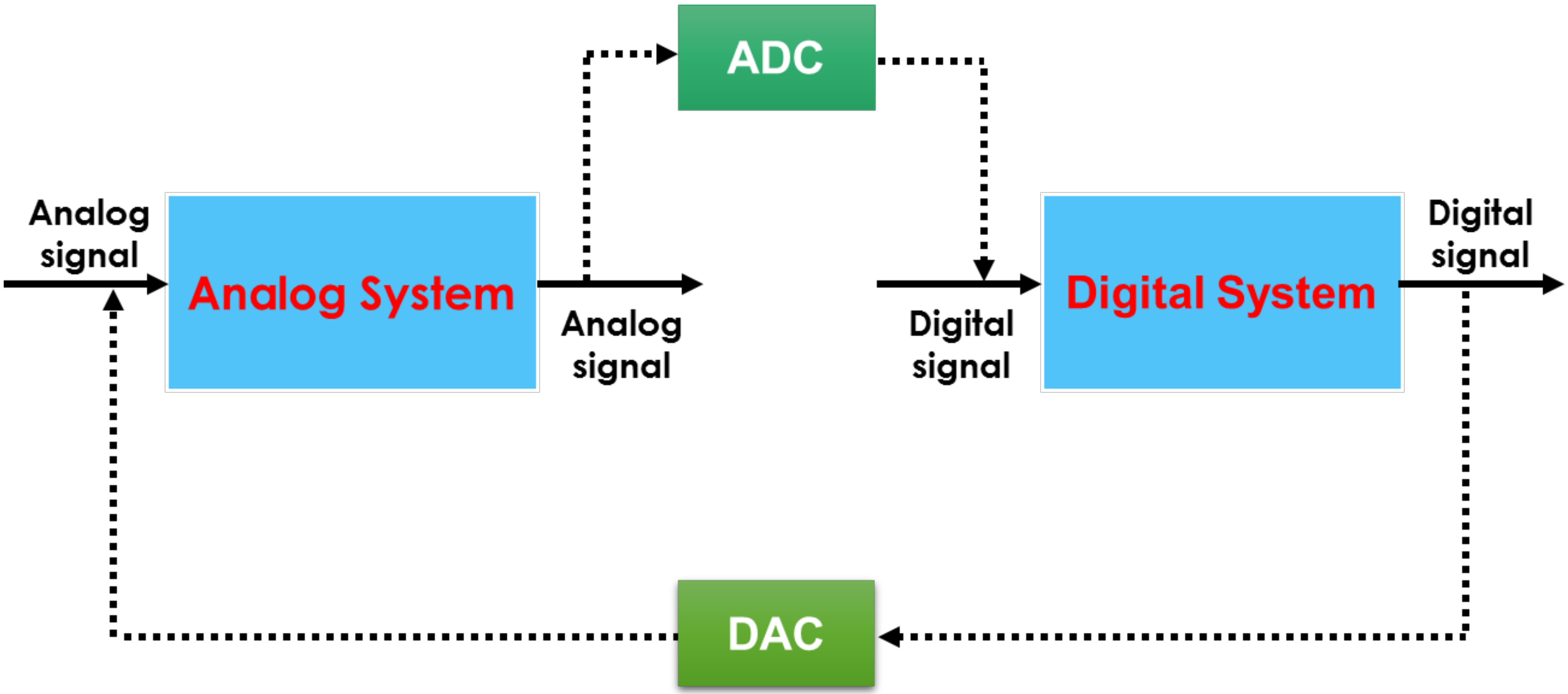
  - **Advantages**: easy to interface to real world, do not need A/D or D/A converters, speed not dependent on clock rate.

- **Digital system** =

  digital signal input + digital signal output

  - Advantages: re-configurability using software, greater control over accuracy/resolution, predictable and reproducible behavior.

Analog and Digital Systems



Multichannel and Multidimensional Signals

- **Multichannel Signals**
  - Signal is generated by multiple sources and usually represented in vector form.
  - Example
    - ECG – ElectroCardioGram
    - EEG – ElectroEncephaloGram
    - Color Image - RGB

- **Multidimensional Signal**
  - Signal is a function of M independent variables ($M > 1$).
  - Example
    - Image: $\sim (x, y)$
    - Black/White TV Image: $\sim (x, y, t)$

- **Signal is multichannel and multidimensional**
  - Color TV Image

Deterministic vs. Random Signals

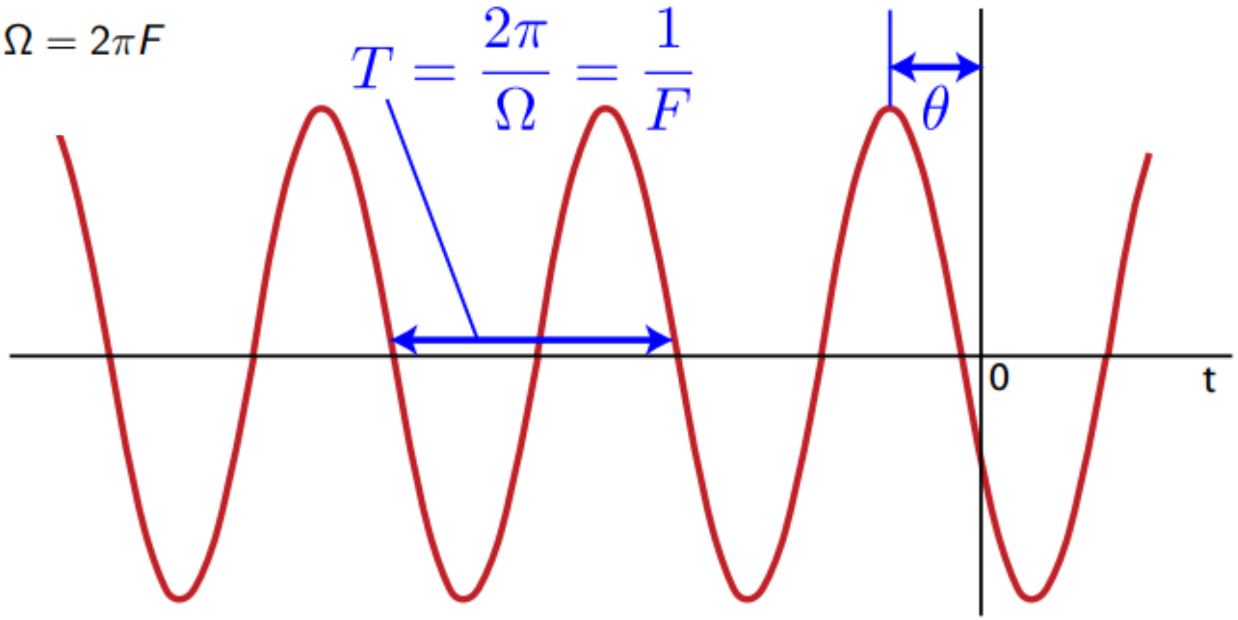
- **Deterministic signal**
  - Any signal that can be uniquely described by an explicit mathematical expression, a table of data, or a well-defined rule.
  - past, present and future values of the signal are known precisely without any uncertainty.

- **Random signal**
  - Any signal that lacks a unique and explicit mathematical expression and thus evolves in time in an unpredictable manner.
  - It may not be possible to accurately describe the signal.
  - The deterministic model of the signal may be too complicated to be of use.

What is a “pure frequency” signal?

$$x_a(t) = A\cos(\Omega t + \theta) = A\cos(2\pi F t + \theta), \quad t \in \mathbb{R}$$

- analog signal, $\because -A \leq x_a(t) \leq A$ and $-\infty < t < \infty$
- A = amplitude
- Ω = frequency in rad/s
- F = frequency in Hz (or cycles/s); note: $\Omega = 2\pi F$
- θ = phase in rad



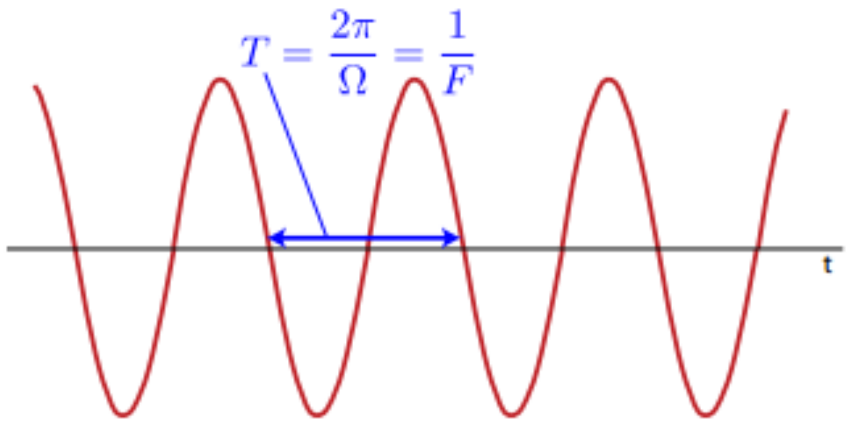
Continuous-time Sinusoids

$$x_a(t) = A\cos(\Omega t + \theta) = A\cos(2\pi F t + \theta), \quad t \in \mathbb{R}$$

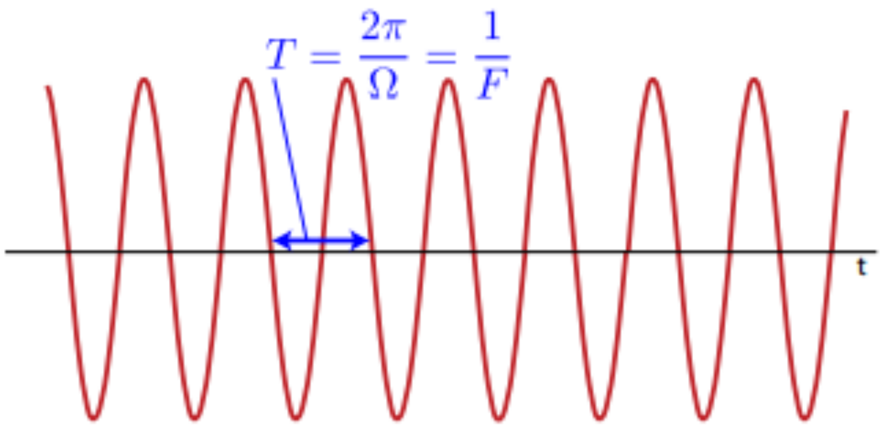
1. for $F \in \mathbb{R}$, $x_a(t)$ is periodic
   - i.e., there exists $T_p \in \mathbb{R}^+$ such that $x_a(t) = x_a(t + T_p)$
2. distinct frequencies result in distinct sinusoids
   - i.e., for $F_1 \neq F_2$, $A\cos(2\pi F_1 t + \theta) \neq A\cos(2\pi F_2 t + \theta)$
3. increasing frequency results in an increase in the rate of oscillation of the sinusoid
   - i.e., for $|F_1| < |F_2|$, $A\cos(2\pi F_1 t + \theta)$ has a lower rate of oscillation than $A\cos(2\pi F_2 t + \theta)$

Continuous-time Sinusoids: Frequency

- Smaller F, larger T



- Larger F, smaller T

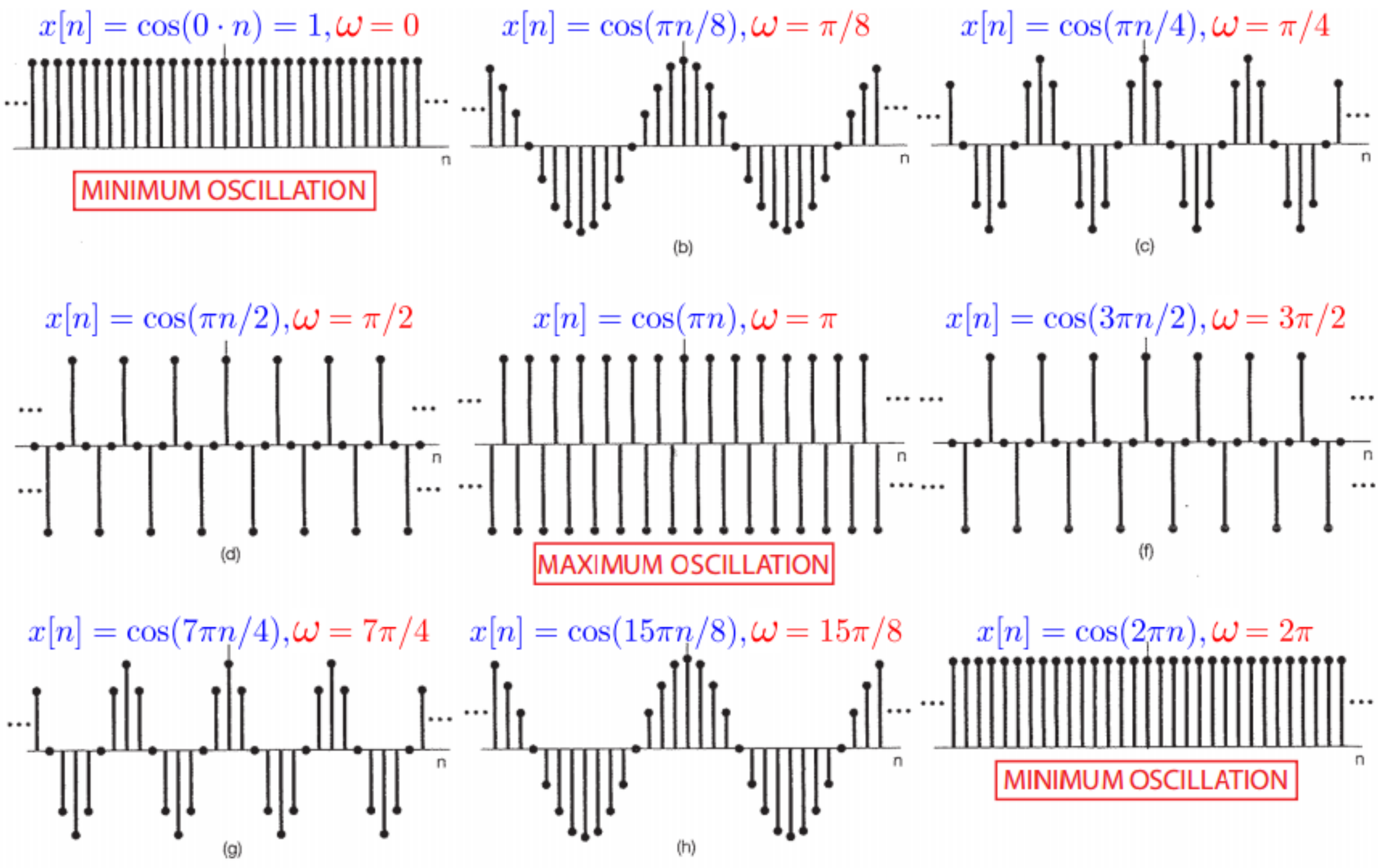


Discrete-time Sinusoids

$$x(n) = A\cos(\omega n + \theta) = A\cos(2\pi fn + \theta), \quad n \in \mathbb{Z}$$

- discrete-time signal (not digital), $\because -A \leq x_a(t) \leq A$ and $n \in \mathbb{Z}$
- $A =$ amplitude
- $\omega =$ frequency in rad/sample
- $f =$ frequency in cycles/sample; note: $\omega = 2\pi f$
- $\theta =$ phase in rad

- x(n) is periodic only if its frequency f is a rational number.
  - Note: rational number is of the form $\frac{k_1}{k_2}$ for $k_1, k_2 \in \mathbb{Z}$
  - periodic discrete-time sinusoids: $x(n) = 2\cos(\frac{4}{7}\pi n), x(n) = \sin(-\frac{\pi}{5}n + \sqrt{3})$
  - aperiodic discrete-time sinusoids: $x(n) = 2\cos(\frac{4}{7}n), x(n) = \sin(\sqrt{2}\pi n + \sqrt{3})$
- Radian frequencies separated by an integer multiple of 2π are identical.
- Lowest rate of oscillation is achieved for $\omega = 2k\pi$ and highest rate of oscillation is achieved for $\omega = (2k+1)\pi$, for $k \in Z$.


$x[n] = \cos(0 \cdot n) = 1, \omega = 0$
MINIMUM OSCILLATION
$x[n] = \cos(\pi n/8), \omega = \pi/8$
(b)
$x[n] = \cos(\pi n/4), \omega = \pi/4$
(c)
$x[n] = \cos(\pi n/2), \omega = \pi/2$
(d)
$x[n] = \cos(\pi n), \omega = \pi$
MAXIMUM OSCILLATION
$x[n] = \cos(3\pi n/2), \omega = 3\pi/2$
(f)
$x[n] = \cos(7\pi n/4), \omega = 7\pi/4$
(g)
$x[n] = \cos(15\pi n/8), \omega = 15\pi/8$
(h)
$x[n] = \cos(2\pi n), \omega = 2\pi$
MINIMUM OSCILLATION

Complex Exponentials

$$e^{j\phi} = \cos(\phi) + j\sin(\phi) \quad \text{Euler's relation}$$

$$\cos(\phi) = \frac{e^{j\phi}+e^{-j\phi}}{2}$$

$$\sin(\phi) = \frac{e^{j\phi}-e^{-j\phi}}{2j}$$

where $j \triangleq \sqrt{-1}$

- Continuous-time

$$A\ e^{j(\Omega t+\theta)} = A\ e^{j(2\pi F t+\theta)}$$

- Discrete-time:

$$A\ e^{j(\omega n+\theta)} = A\ e^{j(2\pi f n+\theta)}$$

Periodicity: Continuous-time

$$
\begin{aligned}
x(t) &= x(t+T),\ T \in \mathbb{R}^{+} \\
A\ e^{j(2\pi F t+\theta)} &= A\ e^{j(2\pi F(t+T)+\theta)} \\
e^{j2\pi F t} \cdot e^{j\theta} &= e^{j2\pi F t} \cdot e^{j2\pi F T} \cdot e^{j\theta} \\
1 &= e^{j2\pi F T} \\
e^{j2\pi k} = 1 &= e^{j2\pi F T},\quad k \in \mathbb{Z} \\
T &= \frac{k}{F}\ k \in \mathbb{Z} \\
T_0 &= \frac{1}{|F|},\quad k = \operatorname{sgn}(F)
\end{aligned}
$$

Periodicity: Discrete-time

$$
\begin{aligned}
x(n) &= x(n+N),\ N \in \mathbb{Z}^+ \\
A\, e^{j(2\pi f n+\theta)} &= A\, e^{j(2\pi f(n+N)+\theta)} \\
e^{j2\pi f n} \cdot e^{j\theta} &= e^{j2\pi f n} \cdot e^{j2\pi f N} \cdot e^{j\theta} \\
1 &= e^{j2\pi f N} \\
e^{j2\pi k} = 1 &= e^{j2\pi f N},\quad k \in \mathbb{Z} \\
f &= \frac{k}{N}\ \ k \in \mathbb{Z} \\
N_0 &= \frac{k'}{f}, \text{min } |k'| \in \mathbb{Z} \text{ such that } \frac{k'}{f} \in \mathbb{Z}^+
\end{aligned}
$$

Example 1

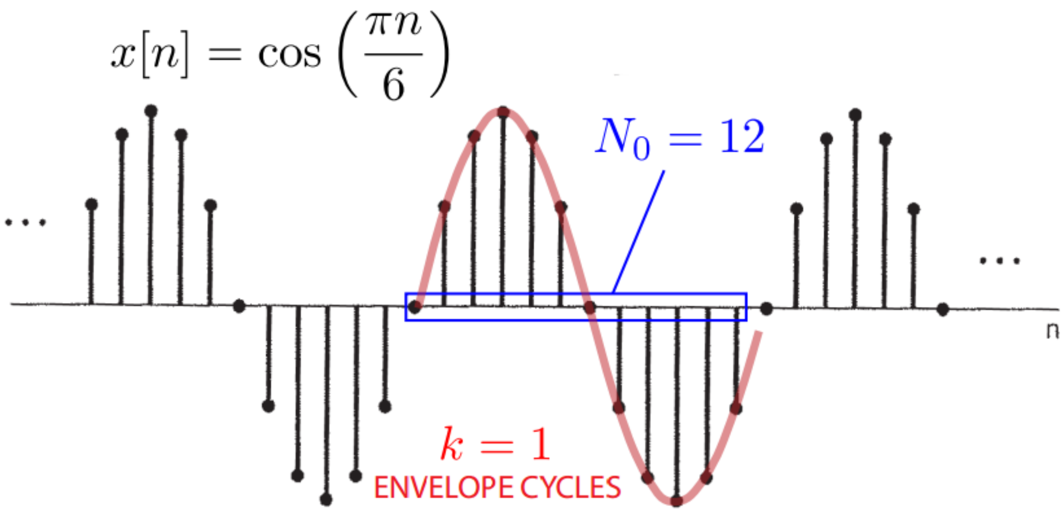
$$x[n] = \cos\left(\frac{\pi n}{6}\right)$$

$$\omega = \pi/6 = \pi \cdot \boxed{\frac{1}{6}}$$

$$N = \frac{2\pi k}{\Omega} = \frac{2\pi k}{\pi \frac{1}{6}} = 12k$$

$$N_0 = 12 \quad \text{for } k = 1$$

- The fundamental period is 12 which corresponds to k = 1 envelope cycles.



Example 2

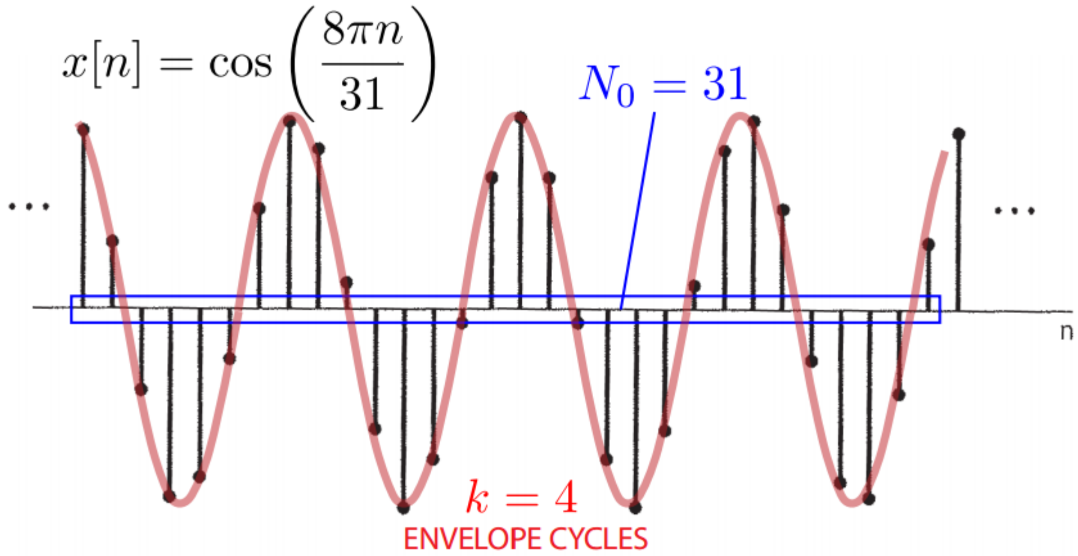
$$x[n] = \cos\left(\frac{8\pi n}{31}\right)$$

$$\omega = 8\pi/31 = \pi \cdot \boxed{\frac{8}{31}}$$

$$N = \frac{2\pi k}{\Omega} = \frac{2\pi k}{\pi \frac{8}{31}} = \frac{31}{4}k$$

$$N_0 = 31 \quad \text{for } k = 4$$

- The fundamental period is 31 which corresponds to k = 4 envelope cycles.



Example 3

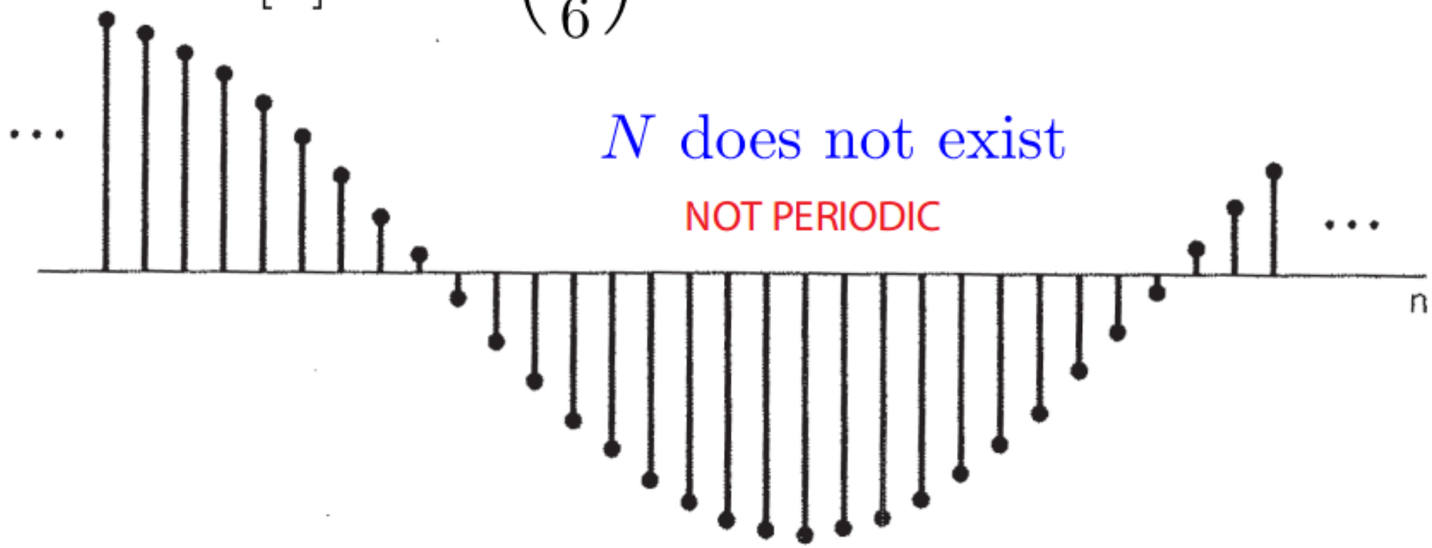
$$x[n] = \cos\left(\frac{n}{6}\right)$$

$$\omega = 1/6 = \pi \cdot \boxed{\frac{1}{6\pi}}$$

$$N = \frac{2\pi k}{\Omega} = \frac{2\pi k}{\frac{1}{6}} = 12\pi k$$

$N \in \mathbb{Z}^+$ does not exist for any $k \in \mathbb{Z}$; $x[n]$ is non-periodic.

$$x[n] = \cos\left(\frac{n}{6}\right)$$



Uniqueness: Continuous-time

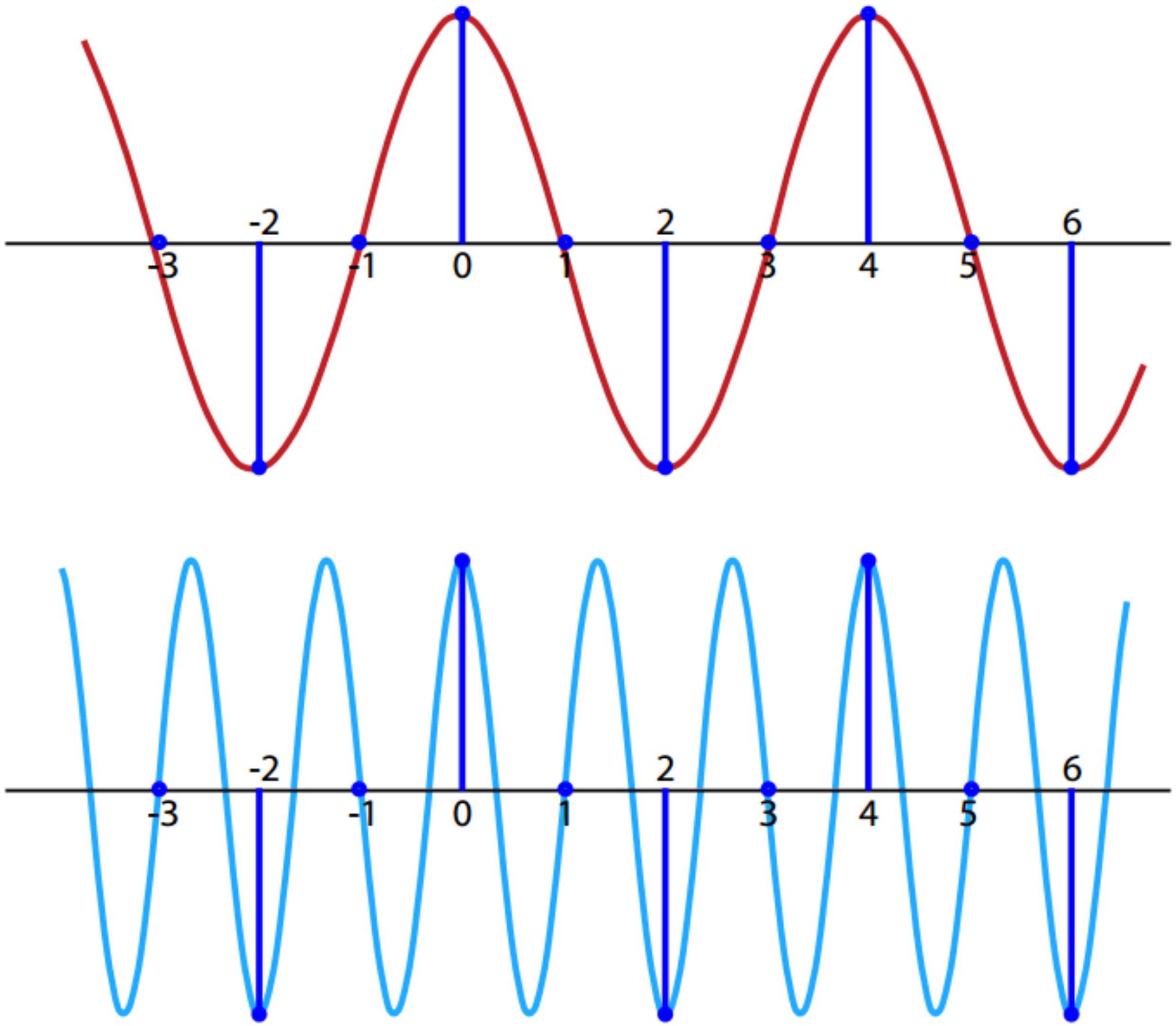
For $F_1 \neq F_2$,

$$A\cos(2\pi F_1 t + \theta) \neq A\cos(2\pi F_2 t + \theta)$$

except at discrete points in time.

Let $f_1 = f_0 + k$ where $k \in \mathbb{Z}$,

$$\begin{aligned} x_1(n) &= A\, e^{j(2\pi f_1 n + \theta)} \\ &= A\, e^{j(2\pi (f_0 + k) n + \theta)} \\ &= A\, e^{j(2\pi f_0 n + \theta)} \cdot e^{j(2\pi k n)} \\ &= x_0(n) \cdot 1 = x_0(n) \end{aligned}$$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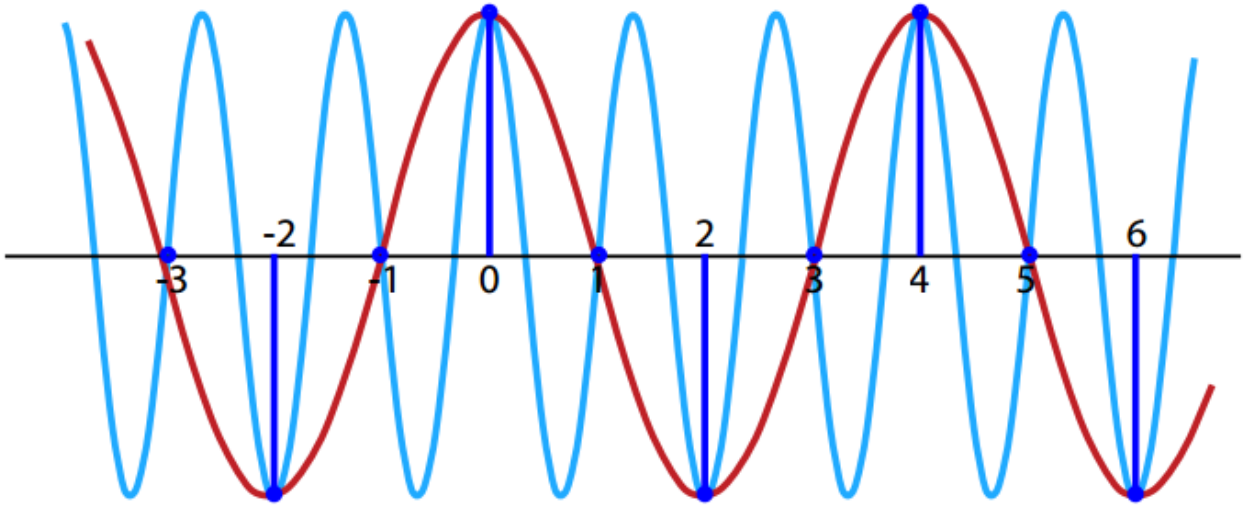
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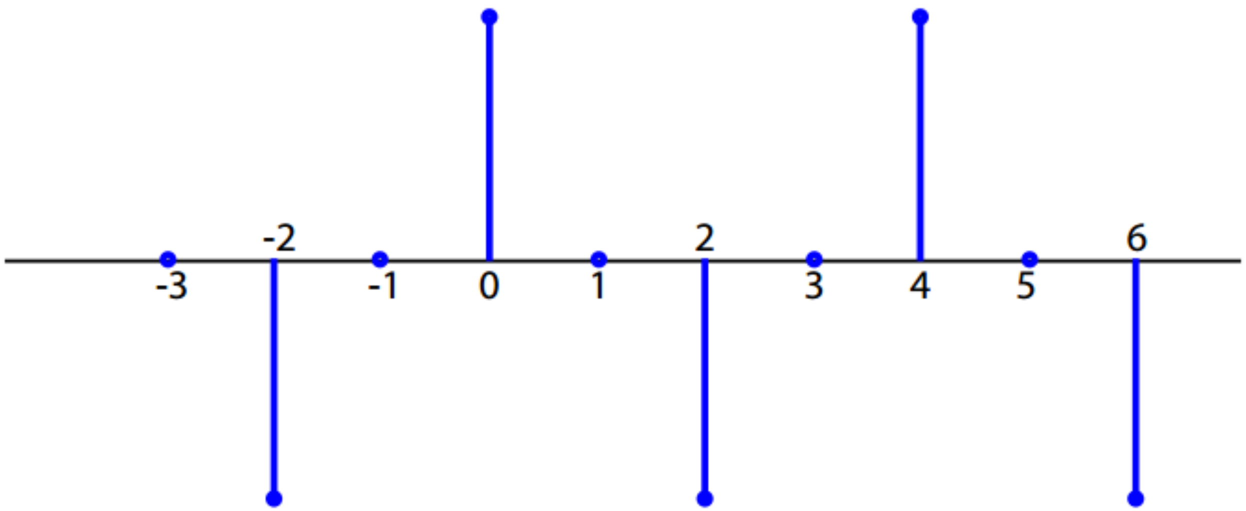
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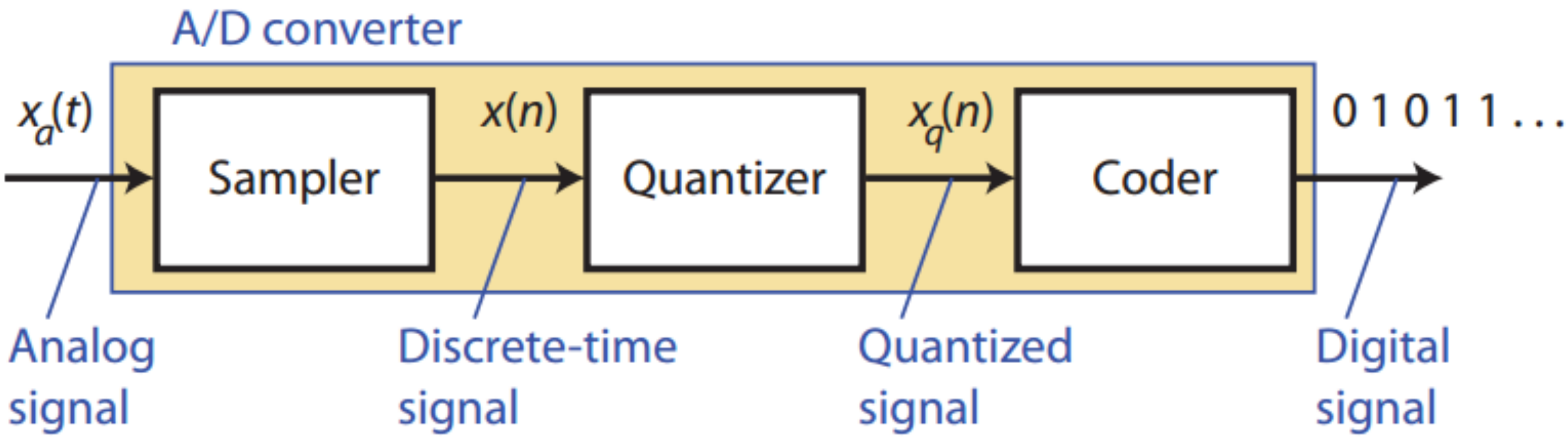
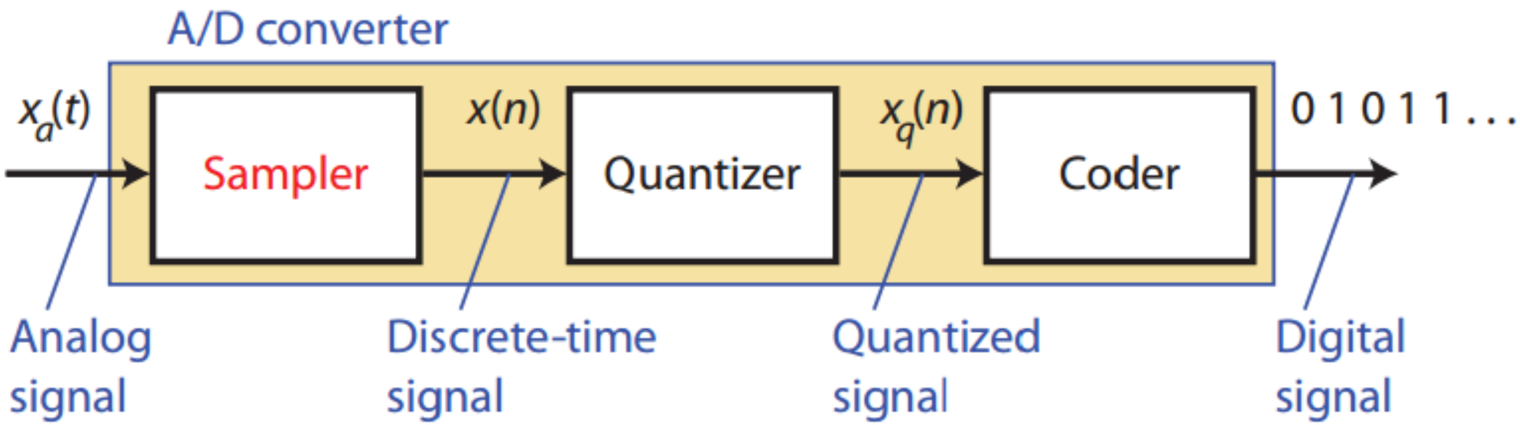
Analog-to-Digital Conversion



- Sampler
  - Sampling
- Quantizer
  - Quantization
- Coder
  - Coding

Analog-to-Digital Conversion: Sampling



- **Sampling**
  - Conversion from continuous-time to discrete-time by taking "samples" at discrete time instants.
  - E.g., uniform sampling: $x(n) = x_a(nT)$ where T is the sampling period and n € Z.

Sampling Theorem

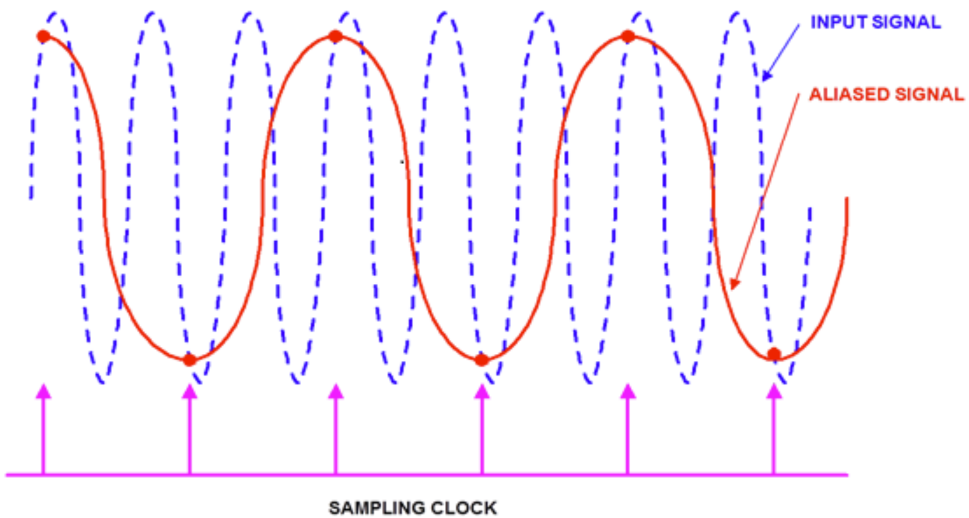
If the highest frequency contained in an analog signal $x_a(t)$ is $F_{max} = B$ and the signal is sampled at a rate

$$F_s > 2F_{max} = 2B$$

then $x_a(t)$ can be exactly recovered from its sample values using the interpolation function

$$g(t) = \frac{\sin(2\pi Bt)}{2\pi Bt}$$

Note: $F_N = 2B = 2F_{max}$ is called the Nyquist rate.



Example

- Do sampling $x_1(t)$ and $x_2(t)$ with sampling frequency $F_s=100Hz$
  - $x_1(t) = 3\cos(20\pi t)$
  - $x_2(t) = 3\cos(220\pi t)$

$x_1(t)$ $x_2(t)$

Sampling

$$x_1(n) = 3\cos([20/100]\pi n) = 3\cos(\pi n/5)$$

$$x_2(n) = 3\cos([220/100]\pi n) = 3\cos([11/5]\pi n) = 3\cos([(10+1)/5]\pi n)$$

$$x_1(n) = 3\cos(\pi n/5)$$

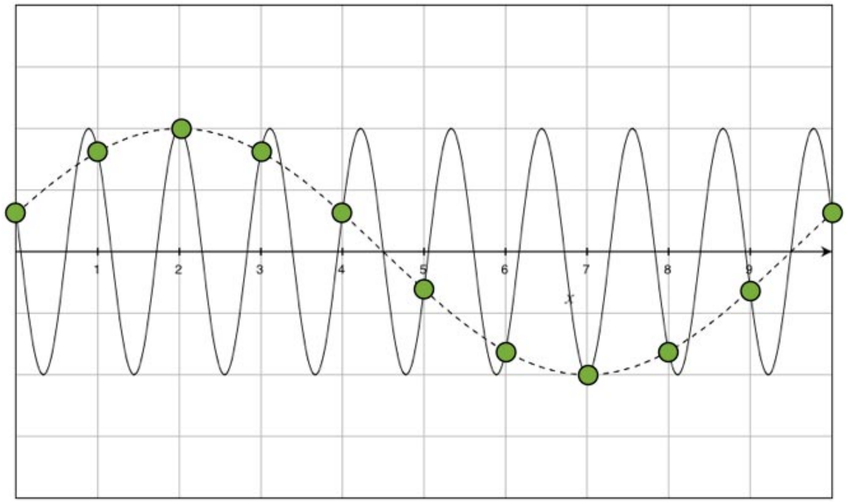
Aliasing

- **What is aliasing?**

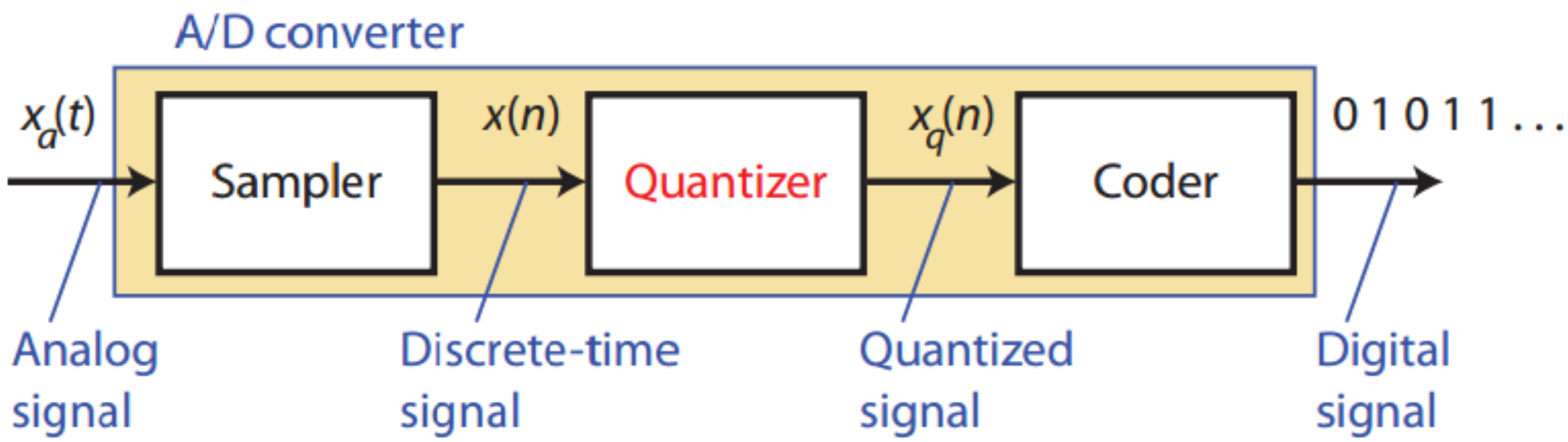
$x_0(t) = ACos(2\pi F_0 t + \theta)$

$x_k(t) = ACos(2\pi F_k t + \theta)$ where $F_k = F_0 + kF_s$ $(k \in Z)$

If $x_k(t)$ is sampled by F_s, the sampling result will be same as $x_0(t)$



Analog-to-Digital Conversion: Quantization



- **Quantization**
  - Conversion from discrete-time continuous-amplitude signal to a discrete-time discrete-amplitude signal.
  - Quantization error: $e_q(n) = x_q(n) - x(n)$ for all n € Z.

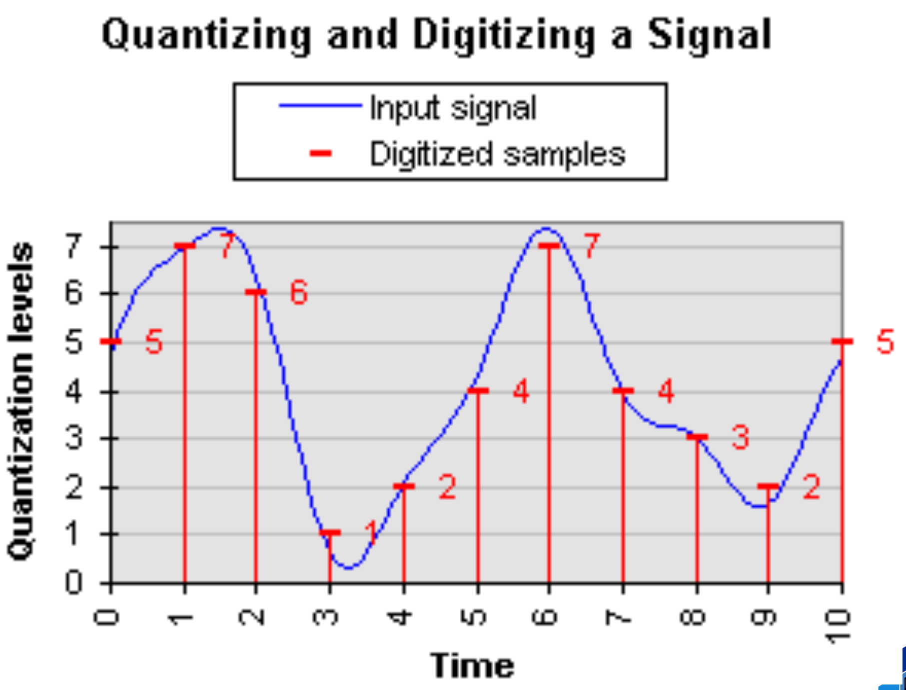
Analog-to-Digital Conversion: Quantization

- **Quantization**
  - Conversion from discrete-time continuous-amplitude signal to a discrete-time discrete-amplitude signal.
  - Methods: rounding or truncated.
  - Notes:
    - L the number of quantization levels
    - Y_{max}, Y_{min}: the max and min value of the signal
    - Δ: quantization step

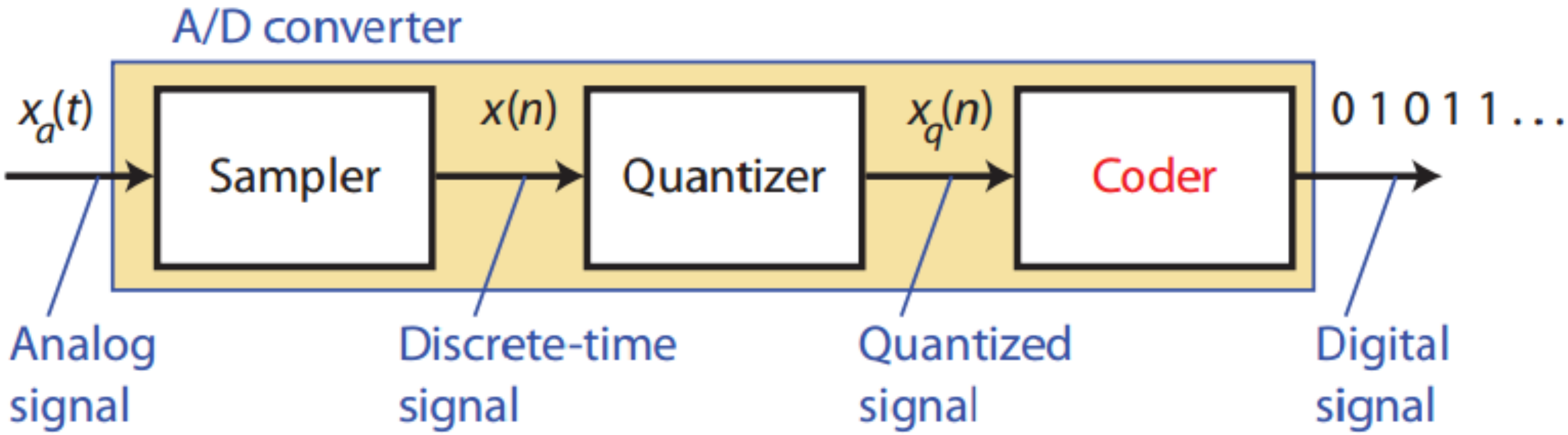
$\Delta = (Y_{max} - Y_{min})/(L-1)$

Quantization error:

- Rounding: $| e_q(n) | <= \Delta/2$
- Truncated: $| e_q(n) | < \Delta$

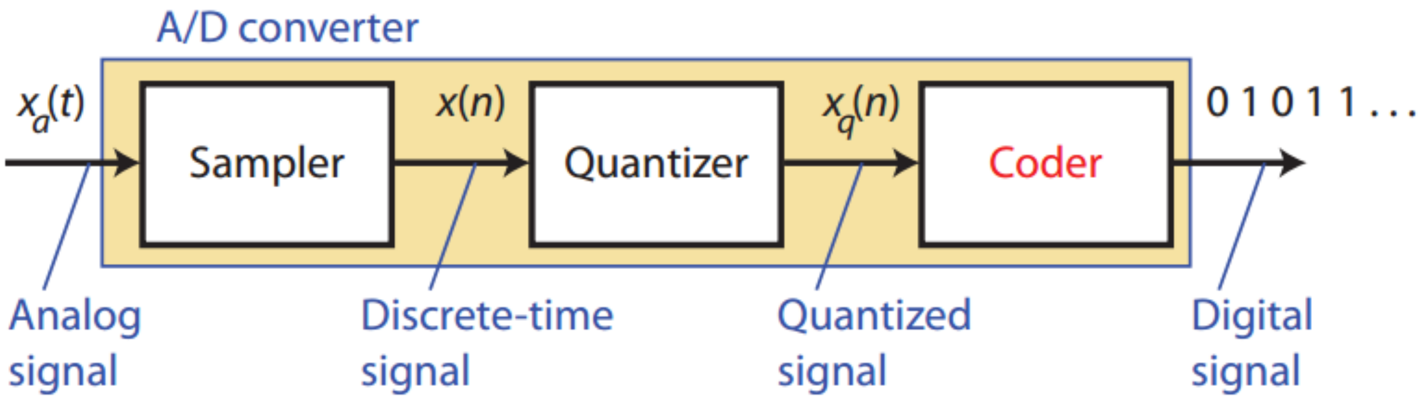


Analog-to-Digital Conversion: Coding

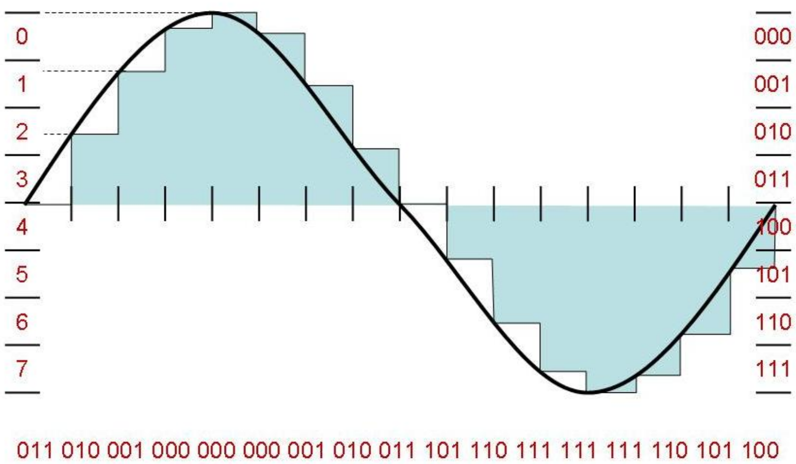


- **Coding**
  - Representation of each discrete-amplitude $x_q(n)$ by a b-bit binary sequence.
    - $2^b \geq L \Rightarrow b \geq \mathrm{ceil}(\log_2 L)$
  - E.g., if for any n, $x_q(n)$ € {0; 1;…; 6; 7}, then the coder may use the following mapping to code the quantized amplitude.

Analog-to-Digital Conversion: Coding



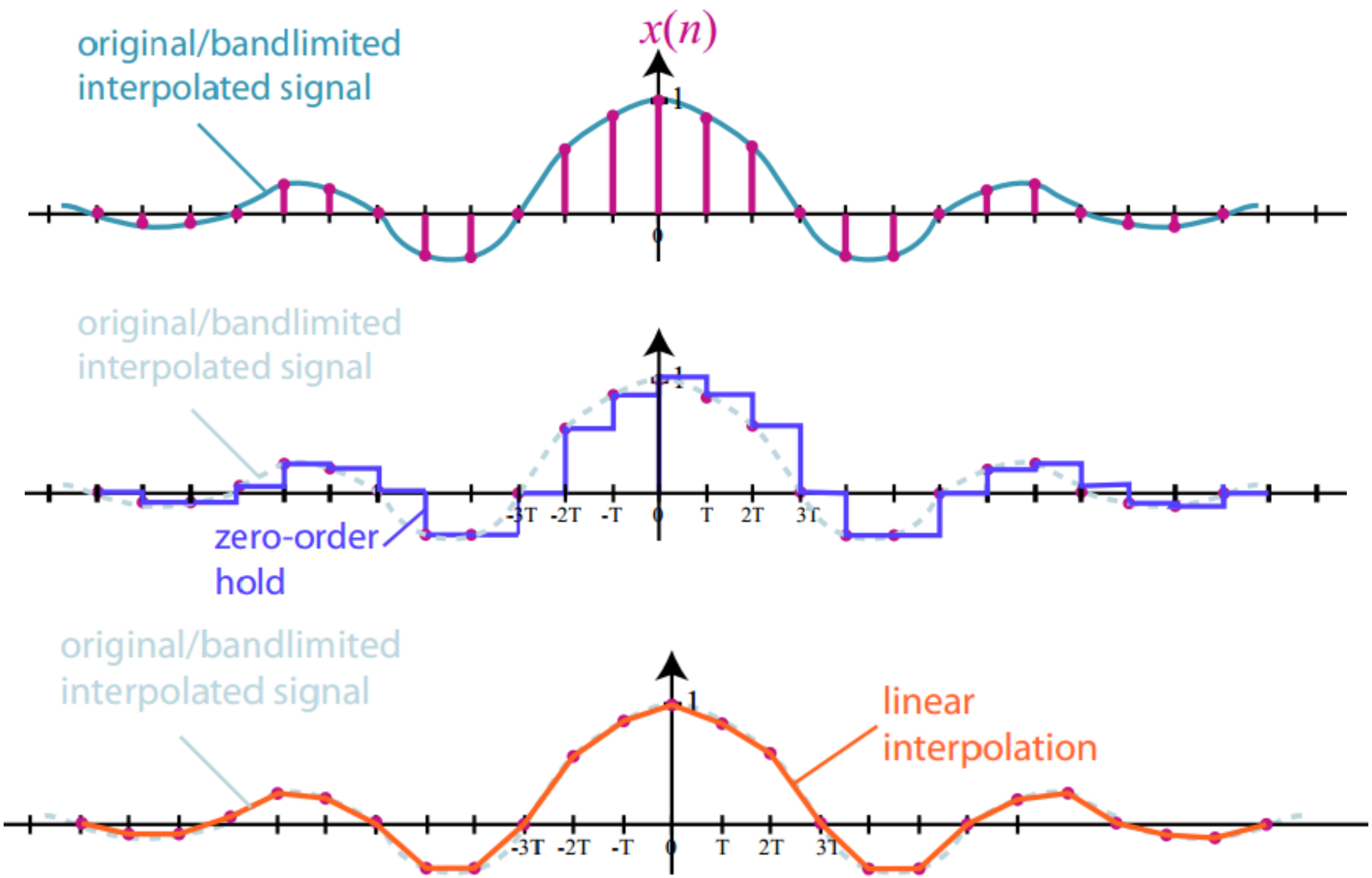
Example coder:



Digital-to-Analog Conversion

- To convert digital signal to analog signal.

- Common interpolation approaches
  - Bandlimited interpolation
  - Zero-order hold
  - Linear interpolation
  - Higher-order interpolation techniques


original/bandlimited
interpolated signal
x(n)
1
0
original/bandlimited
interpolated signal
-3T -2T -T 0 T 2T 3T
zero-order
hold
original/bandlimited
interpolated signal
-3T -2T -T 0 T 2T 3T
linear
interpolation

Exercise

1

A given signal $x(t) = \cos(\pi t/2) - sin(\pi t/8) + 3\cos\left(\frac{\pi t}{4} + \frac{\pi}{3}\right)$, determine

a. Sampling frequency F_s that satisfies the sampling theorem.

b. x(n) using F_s determined in (a)

c. The number of quantization levels L of x(n) with Δ=0.1

d. The binary sequence corresponding to each quantized value of x(n). (Using truncated method for quantization)

2

- A given signal $x(t) = 3\cos(600\pi t) + 2\,sin(1800\pi t)$, determine

a. Sampling frequency F_s that satisfies the sampling theorem.

b. x(n) using F_s determined in (a)

c. Quantization error if using 1024 quantization levels

d. The binary sequence corresponding to each quantized value of x(n). (Using rounding method for quantization)

Exercise

3 Determine which of the following sinusoids are periodic and compute their fundamental period.

- $\cos(0.01\pi n)$ $\cos\left(\pi\frac{30n}{105}\right)$ $\cos(3\pi n)$ $\sin(3n)$ $\sin(\pi\frac{62n}{10})$

4 Consider the following analog signal

$$x_a(t) = 3\sin(100\pi t)$$

- The signal $x_a(t)$ is sampled with a sampling rate $\mathbf{F_s = 300\ samples/s}$. Determine the discrete signal x(n) and determine the periodic property of x(n). If x(n) is periodic signal, determine the frequency and period of x(n). Then, compute the sample values in one period of x(n).