CO2035

4. Signal and System in Frequency Domain

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- Frequency Analysis of Continuous-Time Signals
- Frequency Analysis of Discrete-Time Signals
- Properties of the Fourier Transform for Discrete-Time Signals
- LTI Systems in Frequency Domain
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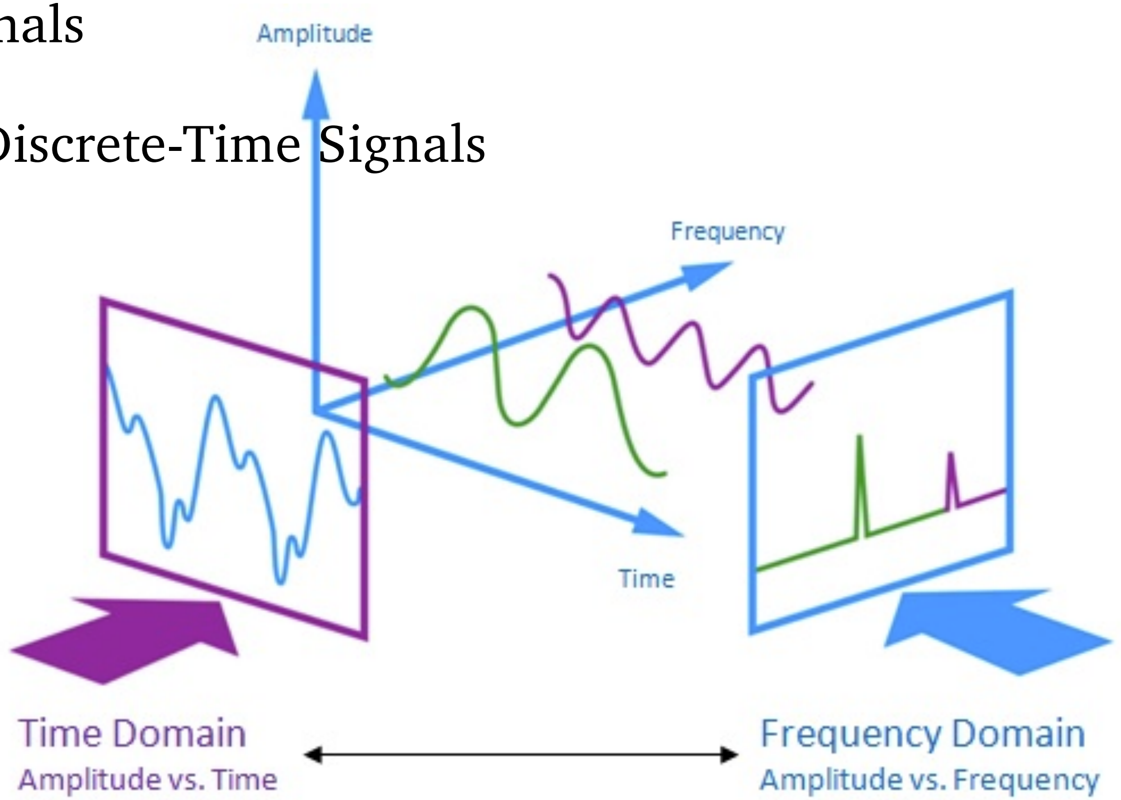
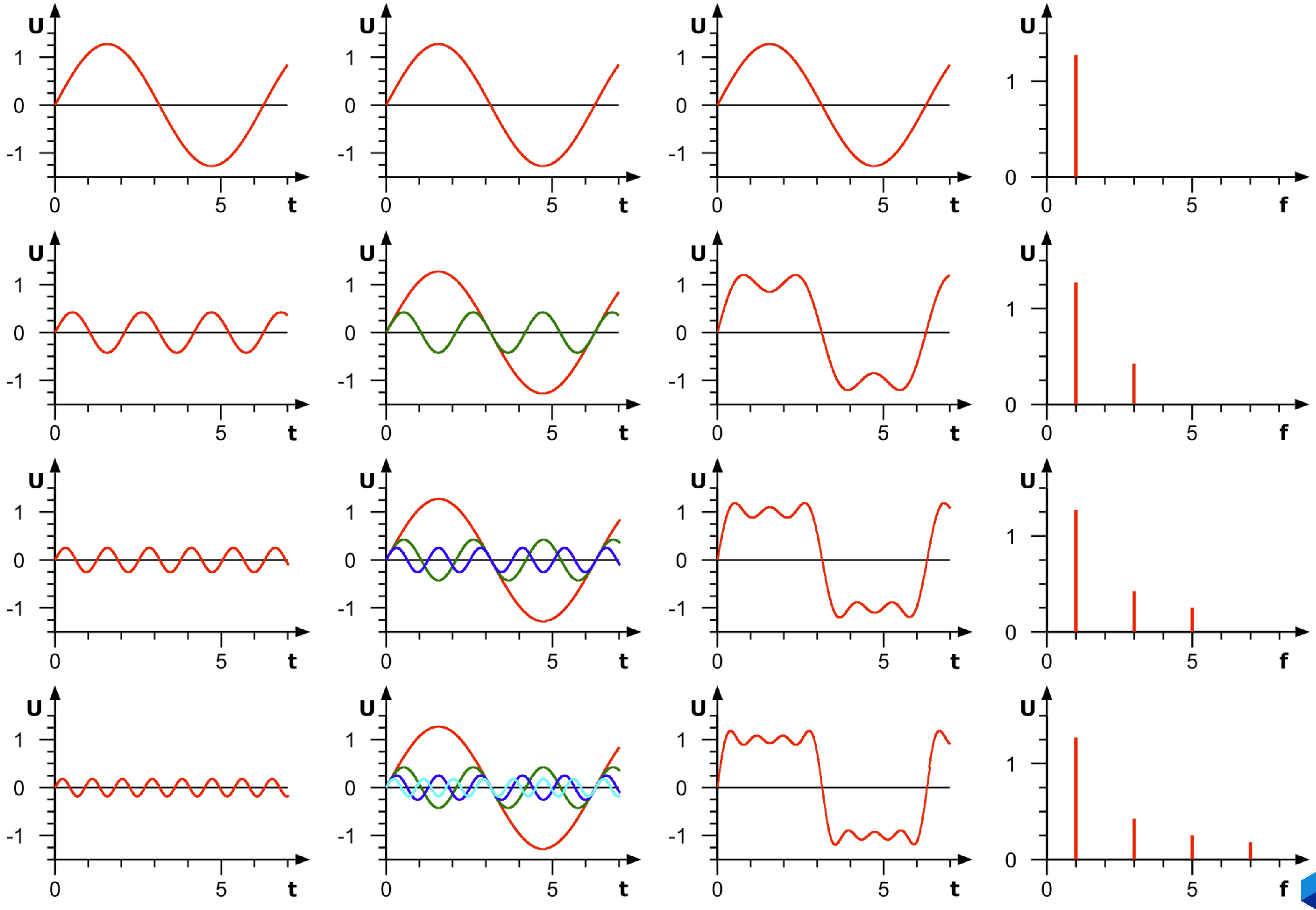
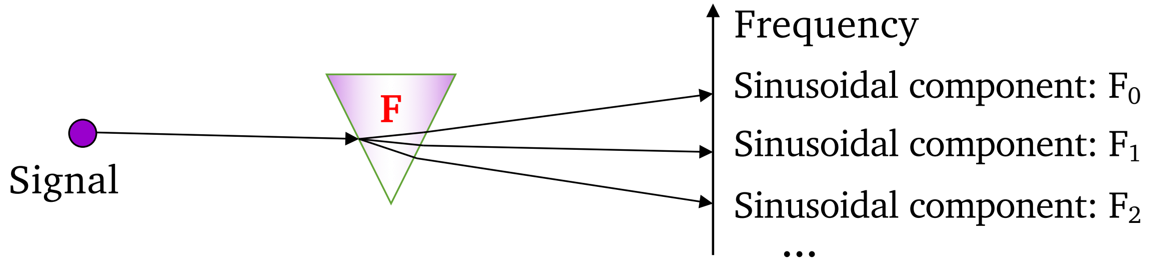


Time Domain
vs.
Frequency Domain



Frequency Analysis



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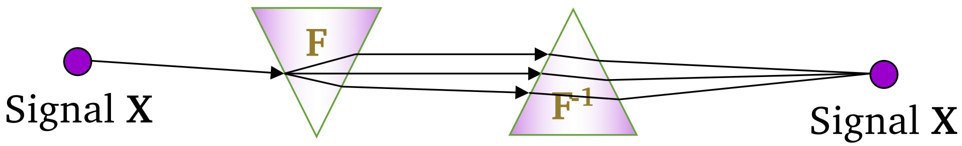
Frequency Analysis

- Fourier series – Periodic Signals
- Fourier Transform – Energy signals, aperiodic signals

(J.B.J. Fourier: 1768 - 1830)



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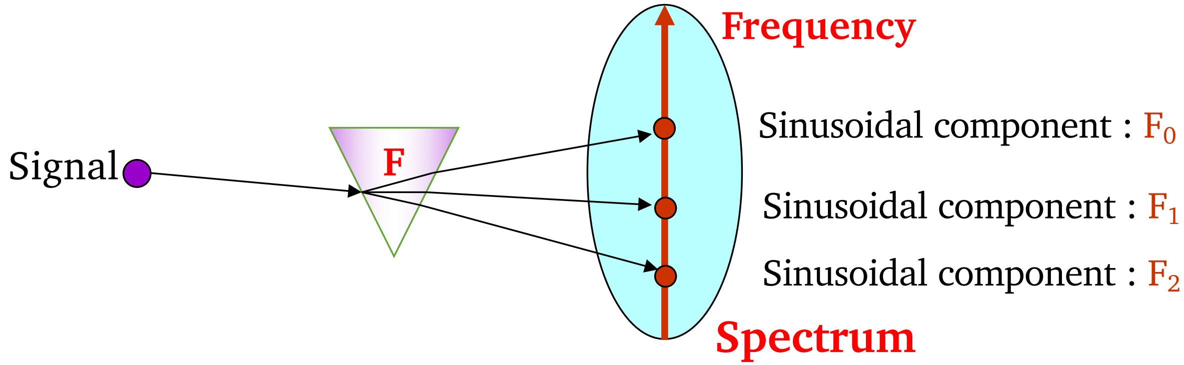
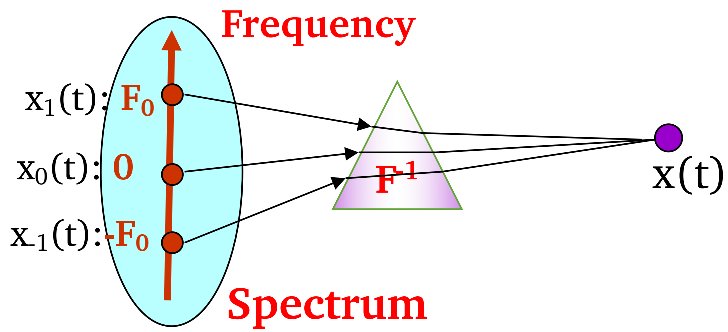
Frequency Synthesis

- Inverse Fourier Series – Periodic signals
- Inverse Fourier Transform – Energy signals, aperiodic signals

Frequency Analysis



Spectrum: Frequency components of a signal
Spectrum Analysis: Determine the spectrum of a signal by the proper mathematical tools
Spectrum Estimation: Determine the spectrum of a signal based on actual measurements of the signals



Frequency Analysis: Determine a given signal from its frequency components

Continuous-Time Periodic Signals

- Fourier series
  - x(t): continuous-time, periodic with fundamental period $T_p = 1/F_0$

$$\mathbf{x(t)} = \sum_{k=-\infty}^{+\infty} \mathbf{c_k e^{j2\pi k F_0 t}}$$

Synthesis Equation

  - Suppose that $\mathbf{x_k(t)} = \mathbf{c_k e^{j2\pi k F_0 t}}$
    - $x_k(t)$ is periodic with a period $\mathbf{T_k} = \frac{\mathbf{T_p}}{\mathbf{k}}$ (kF_0: frequency)
    - $x_k(t)$ (the frequency kF_0) contributes to x(t) a value c_k

$$\mathbf{x(t)} = \sum_{k=-\infty}^{+\infty} \mathbf{x_k(t)}$$

  - Cofficients of Fourier series

$$\mathbf{c_k} = \frac{\mathbf{1}}{\mathbf{T_p}} \int_{\mathbf{T_p}} \mathbf{x(t) e^{-j2\pi k F_0 t} dt}$$

Analysis Equation

Amplitude

$$c_k = |c_k| e^{j\theta_k}$$

Phase

Continuous-Time Periodic Signals

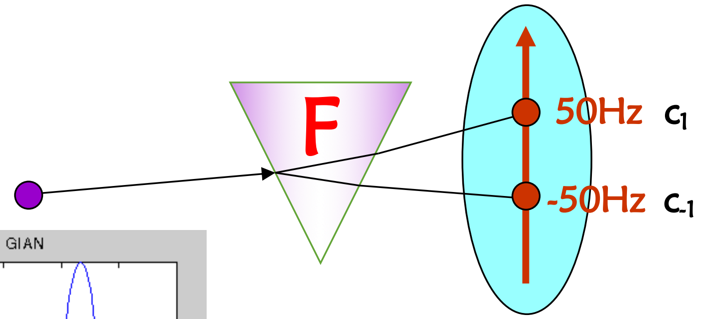
- Example: determine the Fourier series of the following signal $x(t) = 3\cos(100\pi t - \frac{\pi}{3})$

$$x(t) = \frac{3}{2}e^{j(100\pi t - \frac{\pi}{3})} + \frac{3}{2}e^{-j(100\pi t - \frac{\pi}{3})} = \frac{3}{2}e^{-j\frac{\pi}{3}}e^{j100\pi t} + \frac{3}{2}e^{j\frac{\pi}{3}}e^{-j100\pi t}$$

Compare with the systhesis equation

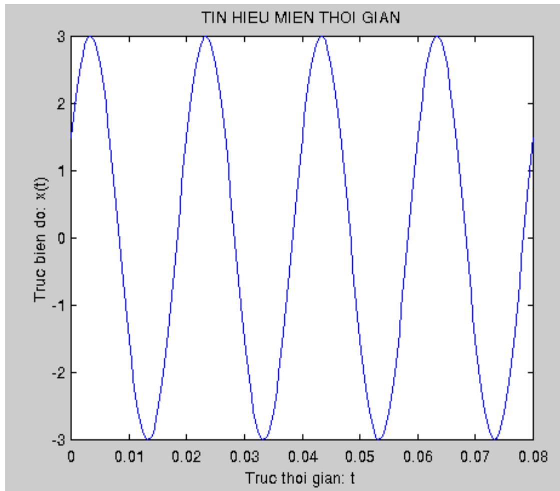
$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{j2\pi k F_0 t}$$

$$\Rightarrow \begin{cases} c_1 = \frac{3}{2}e^{-j\frac{\pi}{3}} \\ c_{-1} = \frac{3}{2}e^{j\frac{\pi}{3}} \end{cases}$$

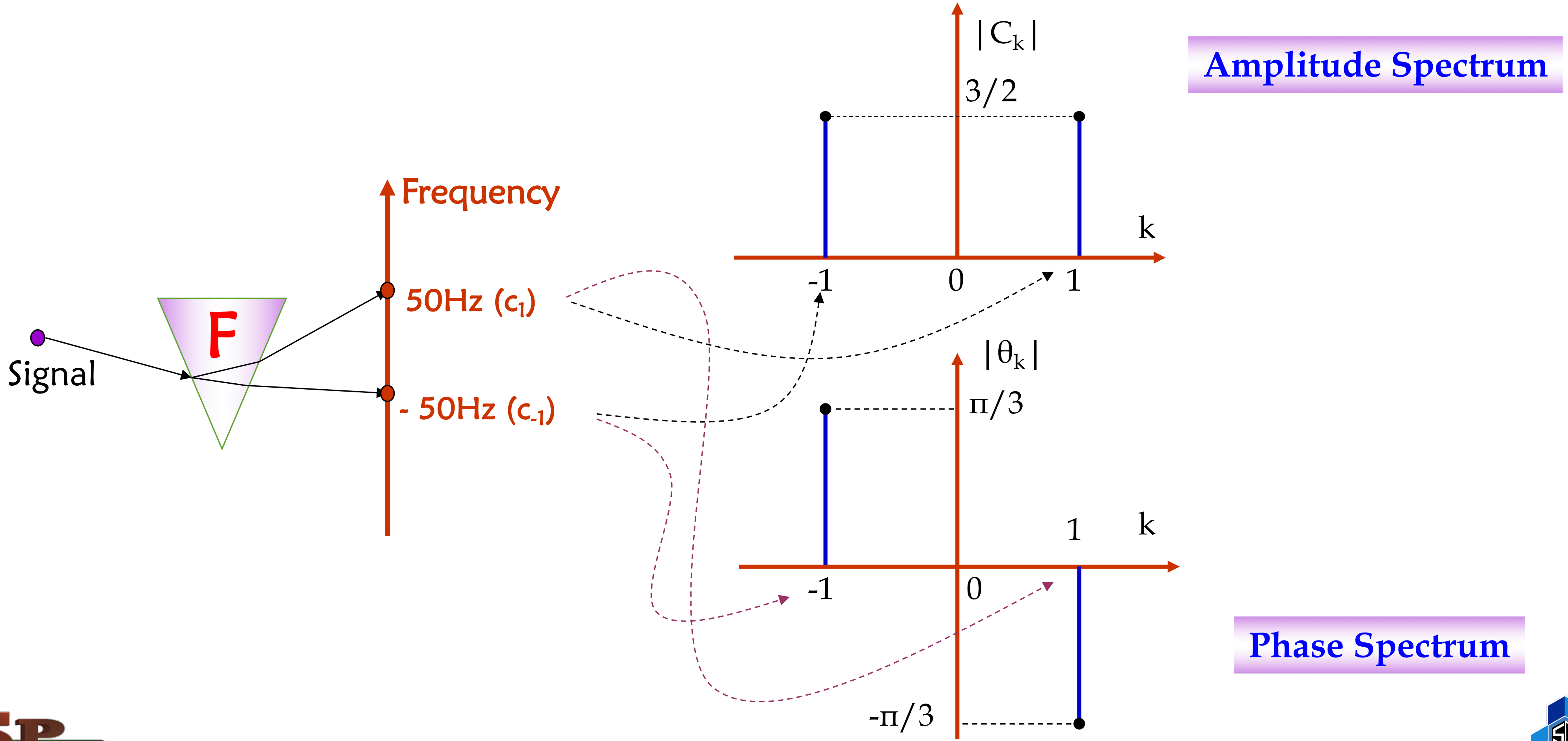


Frequency Spectrum

The signal in time domain



Continuous-Time Periodic Signals



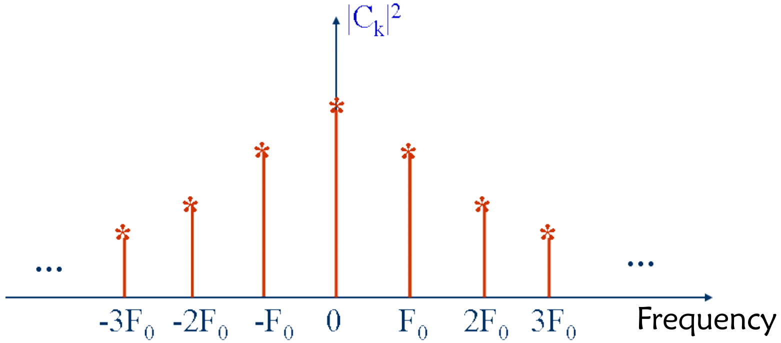
Continuous-Time Periodic Signals

- Average Power
$$P_x = \frac{1}{T_p}\int_{T_p} |x(t)|^2 dt = \frac{1}{T_p}\int_{T_p} x(t)x^*(t)dt \qquad \text{where } x^*(t) = \sum_{k=-\infty}^{+\infty} c_k^* e^{-j2\pi kF_0 t}$$

$$\Longrightarrow P_x = \frac{1}{T_p}\int_{T_p}\left[x(t)\sum_{k=-\infty}^{+\infty} c_k^* e^{-j2\pi F_0 t}\right]dt = \sum_{k=-\infty}^{+\infty} c_k^*\left[\frac{1}{T_p}\int_{T_p} x(t)e^{-j2\pi F_0 t}\,dt\right] = \sum_{k=-\infty}^{+\infty} c_k^* c_k$$

  - Therefore
$$P_x = \frac{1}{T_p}\int_{T_p} |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |c_k|^2$$
Parseval's Relation

- Power Density Spectrum



Continuous-Time Periodic Signals

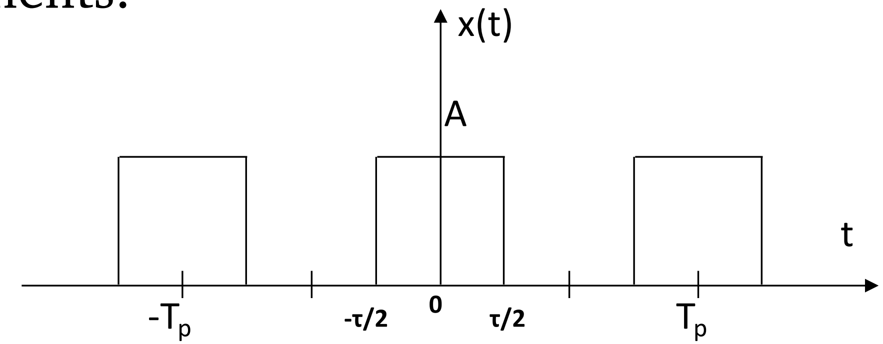
- **Example 1**: determine the average power of $x(t) = 3\cos(100\pi t - \pi/3)$
  - Frequency analysis, $\mathbf{c_1 = \frac{3}{2} e^{-j\frac{\pi}{3}}}$ and $\mathbf{c_{-1} = \frac{3}{2} e^{j\frac{\pi}{3}}}$
  - Apply Parseval, $\mathbf{P_x = |c_{-1}|^2 + |c_1|^2 = 4.5}$

Continuous-Time Periodic Signals

- **Example 2**: Given continuous-time signal $x(t)$ that is periodic with a period T_p. Decompose $x(t)$ into frequency components.

Time Domain

$$\mathbf{x(t)} = \begin{cases} \mathbf{A} & |\mathbf{t}| \le \dfrac{\boldsymbol{\tau}}{\mathbf{2}} \\ \mathbf{0} & |\mathbf{t}| > \dfrac{\boldsymbol{\tau}}{\mathbf{2}} \end{cases}$$



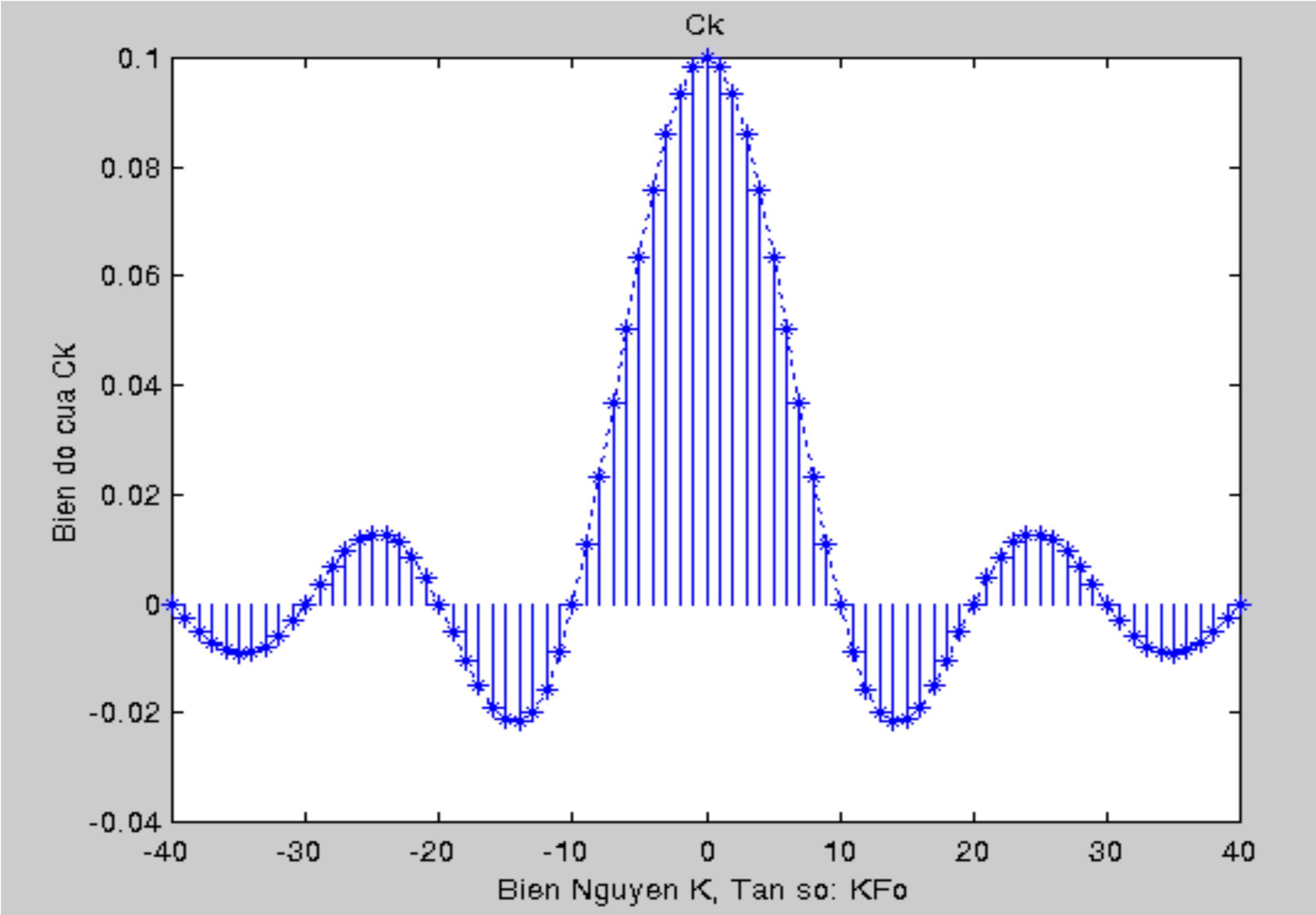
Frequency Domain

$$c_k = \frac{1}{T_p}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} Ae^{-j2\pi kF_0 t}dt = \frac{A}{T_p}\left[\frac{e^{-j2\pi kF_0 t}}{-j2\pi kF_0}\right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = \frac{A}{T_p\pi kF_0}\frac{e^{j\pi kF_0\tau} - e^{j\pi kF_0\tau}}{2j} = \frac{\mathbf{A\tau}}{\mathbf{T_p}}\frac{\sin(\pi kF_0\tau)}{\pi kF_0\tau}$$

$$c_0 = \frac{1}{T_p}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} Adt = \frac{\mathbf{A\tau}}{\mathbf{T_p}}$$

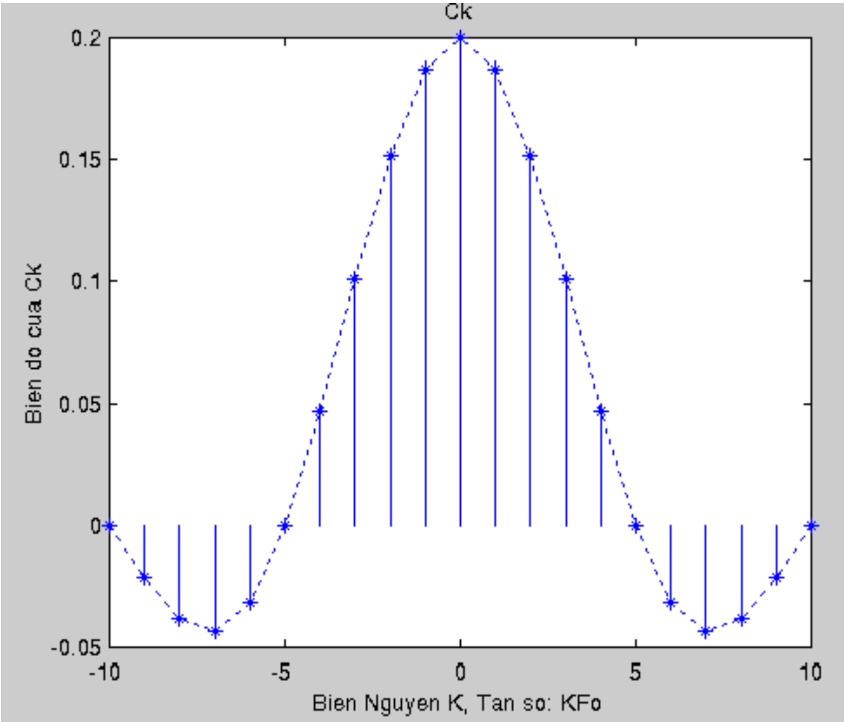
Continuous-Time Periodic Signals

$$c_k = \frac{A\tau}{T_p}\frac{\sin(\pi k F_0 \tau)}{\pi k F_0 \tau}$$



Continuous-Time Periodic Signals

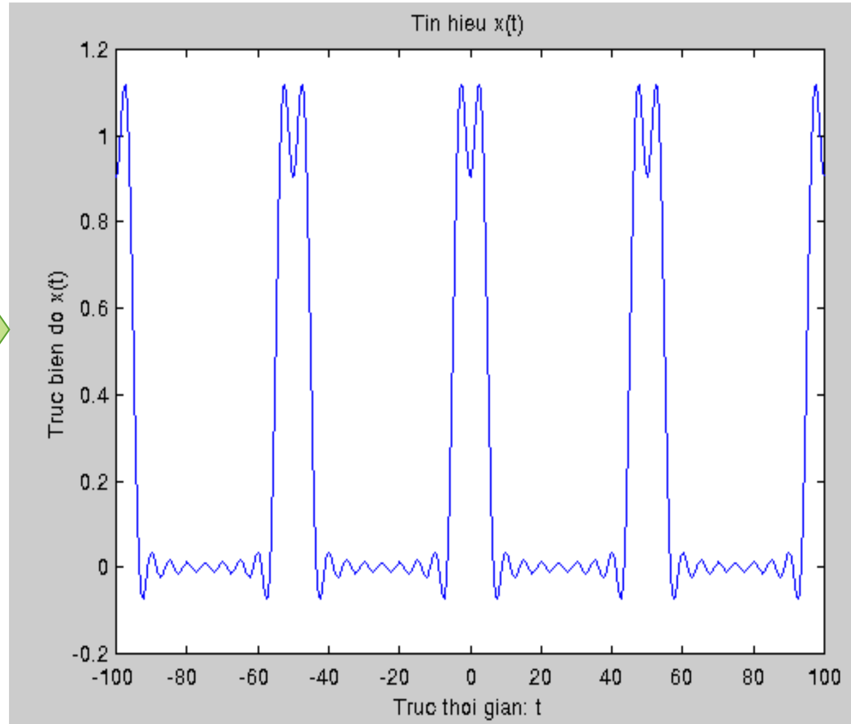
Signal x(t) is synthesized from sinusoidal components



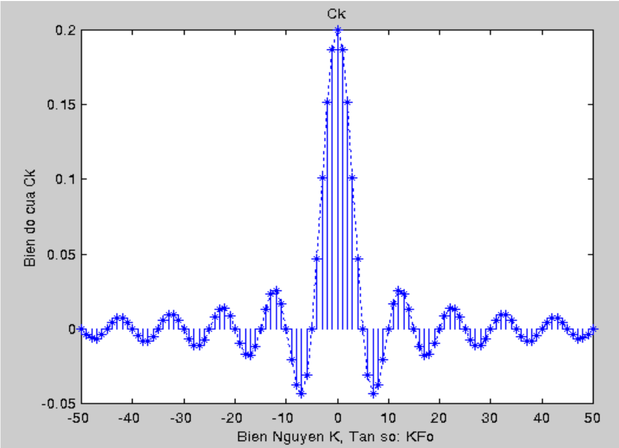
Parameters:

$T_p = 50s$
$\tau = 0.2T_p$
$A = 1$

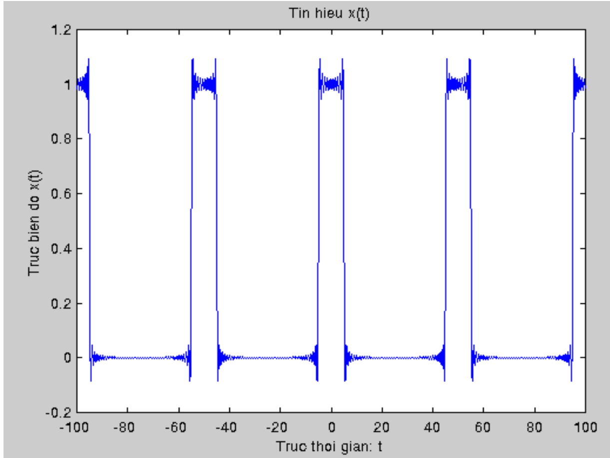
Synthesis from 21 components

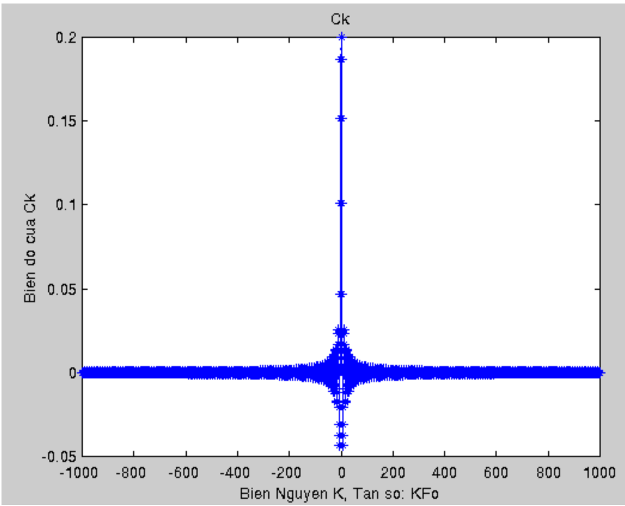


Continuous-Time Periodic Signals

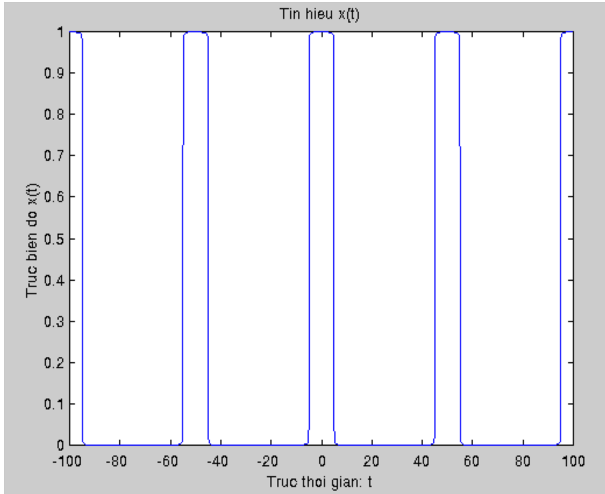


Synthesis from
101 components

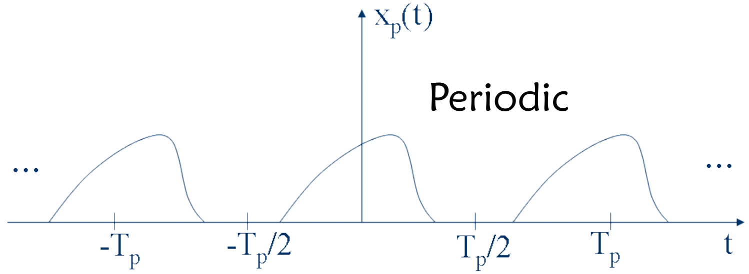




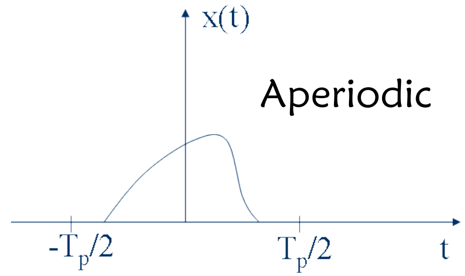
Synthesis from
2001 components



Continuous-Time Aperiodic Signals



- Periodic signal $x_p(t)$
  - Signal is repeated x(t)
  - Fundamental period T_p
  - **The spectrum of a periodic signal is discrete** ($F_0=1/T_p$)



- Aperiodic signal x(t)
  - To be $x_p(t)$ where $T_p \rightarrow \infty$
  - $F_0 = 1/T_p \rightarrow 0$

$\Rightarrow$ **The spectrum of an aperiodic signal is continuous.**

Continuous-Time Aperiodic Signals

- Fourier transform
  - x(t): continuous-time aperiodic

$$X(F) = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi Ft}dt$$

Analysis Equation
(Fourier Transform)

- Fourier coefficients $c_k = \frac{1}{T_p}X(kF_0) = F_0X(kF_0)$

$$x(t) = \int_{-\infty}^{+\infty} X(F)e^{j2\pi Ft}dF$$

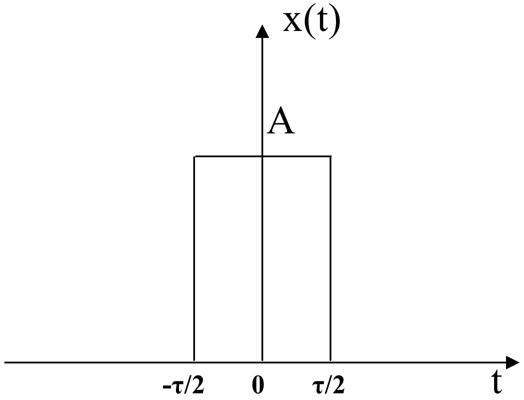
Synthesis Equation
(Inverse Fourier Transform)

Continuous-Time Aperiodic Signals

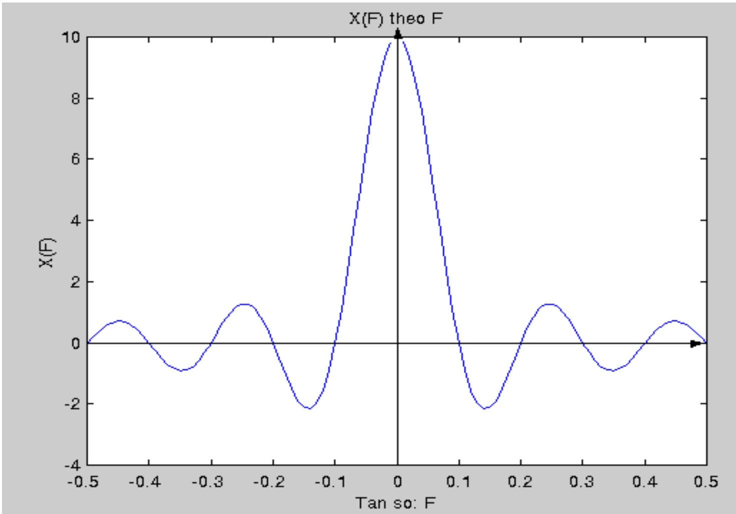
- Example: given x(t) is aperiodic. Decompose x(t) into frequency components.

$$\mathbf{x(t)} = \begin{cases} \mathbf{A} & |\mathbf{t}| \leq \frac{\tau}{2} \\ \mathbf{0} & |\mathbf{t}| > \frac{\tau}{2} \end{cases} \quad \xrightarrow{\mathbf{F}} \quad \mathbf{X(F)} = \int_{-\infty}^{+\infty} \mathbf{A}e^{-j2\pi Ft}\mathbf{dt} = \mathbf{A}\tau\frac{\sin(\pi Ft)}{\pi Ft}$$

Time Domain



Frequency Domain



Continuous-Time Aperiodic Signals

- **Energy** $$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} x(t)x^*(t)dt \quad \text{where } x^*(t) = \int_{-\infty}^{+\infty} X^*(F)e^{-j2\pi Ft}dF$$

$$\Rightarrow E_x = \int_{-\infty}^{+\infty} x(t)\left[\int_{-\infty}^{+\infty} X^*(F)e^{-j2\pi Ft}dF\right]dt = \int_{-\infty}^{+\infty} X^*(F)dF\left[\int_{-\infty}^{+\infty} x(t)e^{-j2\pi Ft}dt\right] = \int_{-\infty}^{+\infty} X^*(F)X(F)dF$$

Therefore

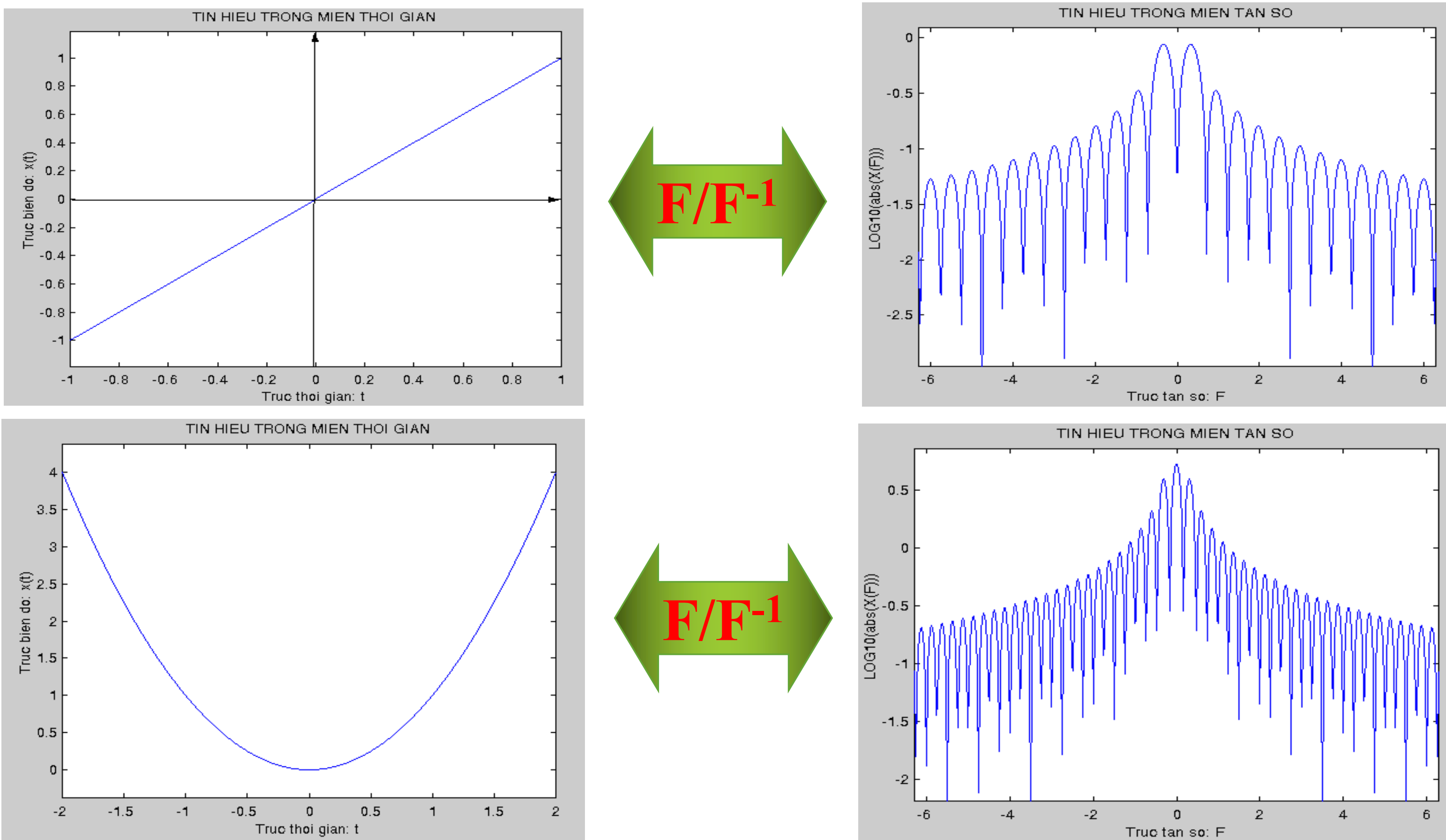
$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |X(F)|^2 dF$$

Parseval's Relation

- Energy Density Spectrum $S_{xx}(F) = |X(F)|^2$
- If x(t) is real signal

$$\left.\begin{aligned} |X(-F)| &= |X(F)| \\ \angle X(-F) &= -\angle X(-F) \end{aligned}\right\} \quad S_{xx}(F) = S_{xx}(-F)$$

Continuous-Time Aperiodic Signals



Discrete-Time Periodic Signals

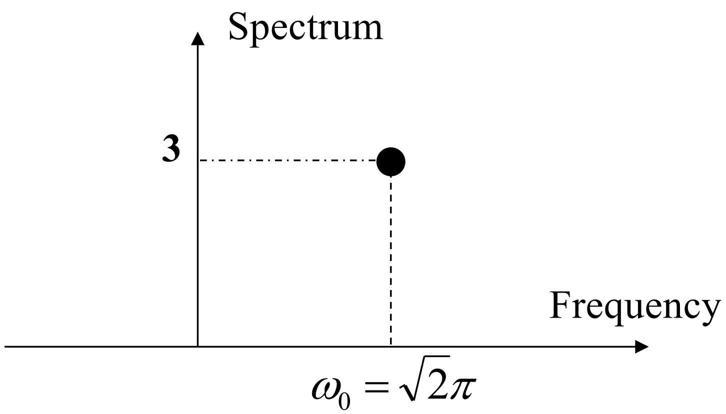
- Given x(n) is periodic signal with a period N (i.e. $x(n+N)=x(n)\ ,\ \forall n$)
- Fourier series of a discrete-time signal has maximum N frequency components (in range [0, 2π] or [-π, π])
- Discrete-time Fourier Series (DTFS) $$\mathbf{x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi \frac{k}{N} n}}$$ Synthesis Equation
- Fourier coefficients $$\mathbf{c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi \frac{k}{N} n}}$$ Analysis Equation
  - Express x(n) in frequency domain (c_k represents amplitude and phase of frequency component $s_k(n) = e^{j2\pi kn/N}$)
  - $c_{k+N} = c_k \Rightarrow$ Spectrum of a periodic signal x(n) with a period N is a periodic series with a period N.

Discrete-Time Periodic Signals

- **Example**: determine and draw the spectrum of the following signals
- $\mathbf{x(n) = 3cos(\sqrt{2}\pi n)}$

$$\omega_0 = \sqrt{2}\pi \Leftrightarrow f_0 = \frac{1}{\sqrt{2}}$$

f_0 : is not fractional
→ x(n) is aperiodic
→ Spectrum consists of a single f_0



Discrete-Time Periodic Signals

- **Example:** determine and draw the spectrum of the following signals
- $\mathbf{x(n) = 3cos(\frac{\pi}{3}n)}$

$$x(n) = 3\cos\left(\frac{2\pi n}{6}\right) \Rightarrow f_0 = \frac{1}{6} \Leftrightarrow N = 6$$

$\Rightarrow$ x(n) is periodic with a period N=6

Coefficients

$$c_k = \frac{1}{6}\sum_{n=0}^{5} x(n)e^{-j2\pi\frac{k}{6}n} \qquad k = 0 \dots 5$$

Otherwise,

$$x(n) = 3\cos\left(2\pi\frac{1}{6}n\right) = \frac{3}{2}e^{j2\pi\frac{1}{6}n} + \frac{3}{2}e^{-j2\pi\frac{1}{6}n} = \frac{3}{2}e^{-j2\pi\frac{-1}{6}n} + \frac{3}{2}e^{-j2\pi\frac{1}{6}n}$$

Combine with the systhesis equation, we have

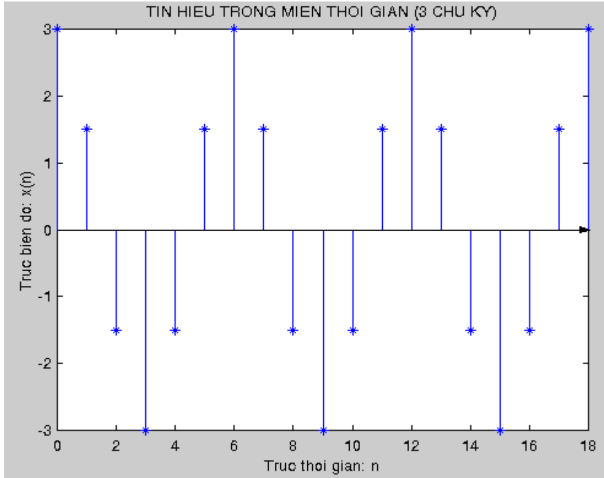
$$c_0 = c_2 = c_3 = c_4 = 0 \text{ and } c_1 = c_5 = \frac{3}{2}$$

$$c_5 = c_{-1}$$

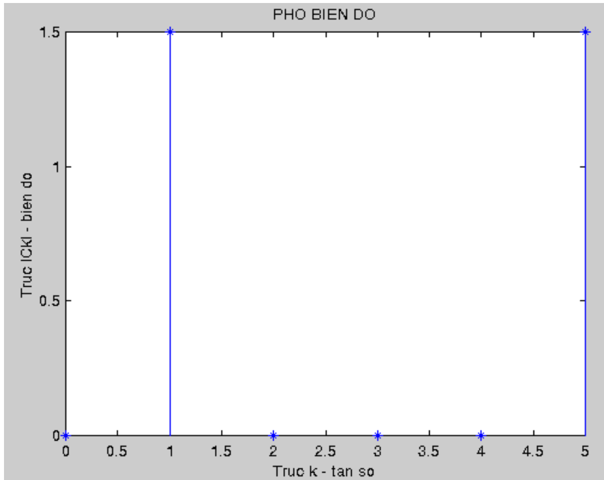
Discrete-Time Periodic Signals

$$x(n) = 3\cos\left(\frac{\pi}{3}n\right)$$

Signal in Time-domain (3 periods)



Signal in Frequency-domain



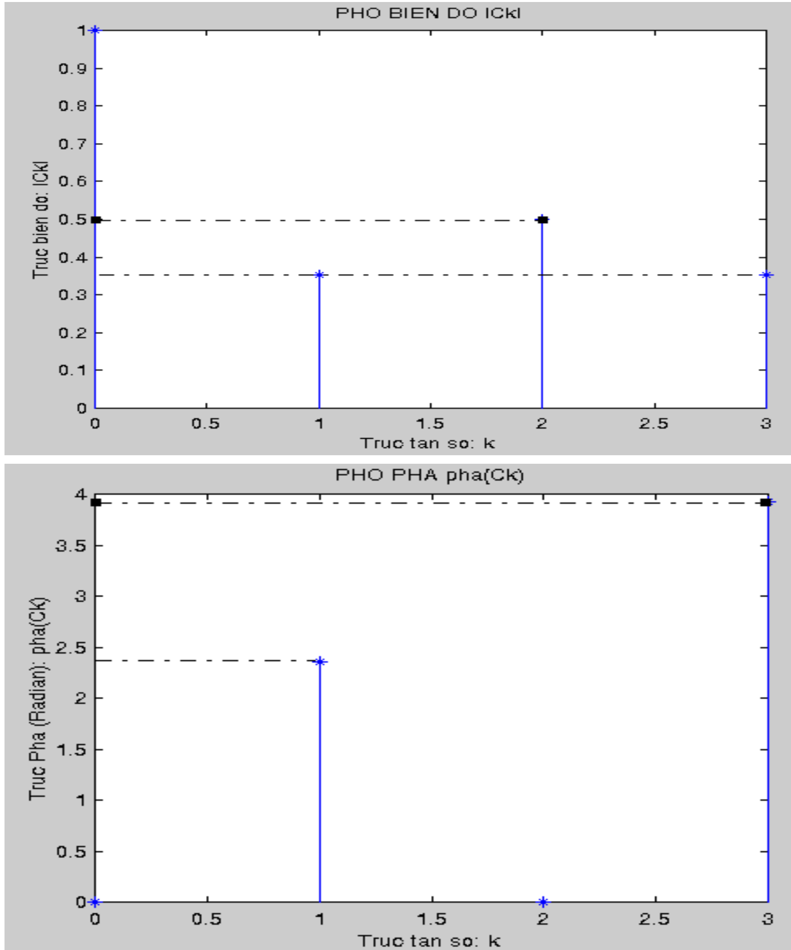
$$c_0 = c_2 = c_3 = c_4 = 0 \text{ and } c_1 = c_5 = \frac{3}{2}$$

Discrete-Time Periodic Signals

$\mathbf{x(n)\ is\ periodic\ with\ \{\overset{\uparrow}{1}\ \ 0\ \ 2\ \ 1\}}$

$$c_k = \frac{1}{4}\sum_{n=0}^{3} x(n)e^{-j2\pi\frac{k}{4}n} = \frac{1}{4}\left(1 + 2e^{-j\pi k} + e^{-j\frac{3}{2}\pi k}\right) \quad k = 0 \dots 3$$

$$\Rightarrow \begin{cases} c_0 = \frac{1}{4}(1+2+1) = 1 \\ c_1 = \frac{1}{4}(1-2+j) = \frac{j-1}{4} = \frac{\sqrt{2}}{4}e^{j\frac{3\pi}{4}} \\ c_2 = \frac{1}{4}(1+2-1) = \frac{1}{2} \\ c_3 = \frac{1}{4}(1-2-j) = \frac{-1-j}{4} = \frac{\sqrt{2}}{4}e^{j\frac{5\pi}{4}} \end{cases}$$



Discrete-Time Periodic Signals

- **Average Power** $P_x = \frac{1}{N}\sum_{n=0}^{N-1}|x(n)|^2 = \frac{1}{N}\sum_{n=0}^{N-1}x(n)x^*(n)$ where $x^*(n) = \sum_{k=0}^{N-1}c_k^* e^{-j2\pi\frac{k}{N}n}$

$$\Rightarrow P_x = \frac{1}{N}\sum_{n=0}^{N-1}x(n)\left(\sum_{k=0}^{N-1}c_k^* e^{-j2\pi\frac{k}{N}n}\right) = \sum_{k=0}^{N-1}c_k^*\left(\frac{1}{N}\sum_{n=0}^{N-1}x(n)e^{-j2\pi\frac{k}{N}n}\right) = \sum_{k=0}^{N-1}c_k^* c_k$$

  - Therefore

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1}|x(n)|^2 = \sum_{k=0}^{N-1}|c_k|^2$$

Parseval's Relation

  - Series $|c_k|^2$: the power density spectrum of a periodic signal.

- Energy of the signal in a period $E_N = \sum_{n=0}^{N-1}|x(n)|^2 = N\sum_{k=0}^{N-1}|c_k|^2$

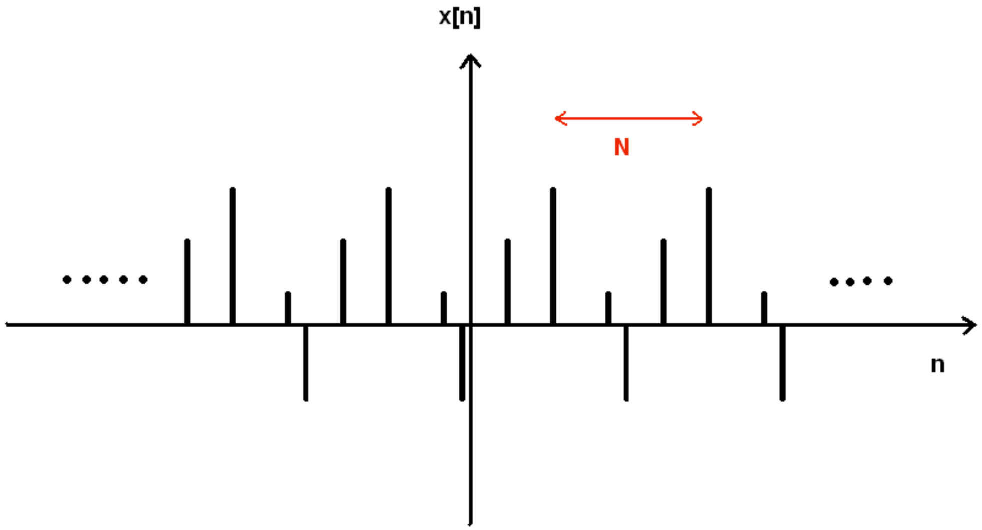
Discrete-Time Periodic Signals

- If x(n) is real signal $[x^*(n) = x(n)]$, $\Rightarrow c_k^* = c_{-k}$
  - This means

$$\begin{cases} |c_{-k}| = |c_k| \\ -\angle c_{-k} = \angle c_k \end{cases}$$

  - Beside, $c_{N+k} = c_k$, we have

$$\begin{cases} |c_k| = |c_{N-k}| \\ \angle c_k = -\angle c_{N-k} \end{cases}$$



  - $\Rightarrow$ Amplitude spectrum is **even symmetric**
  - $\Rightarrow$ Phase spectrum is **odd symmetric**

Discrete-Time Aperiodic Signals

- Only consider energy signal x(n) $E_x = \sum_{-\infty}^{+\infty} |x(n)|^2 < \infty$
- **Fourier Transform**

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

Analysis Equation

  - $X(\omega)$: represents the frequency content of the signal x(n)

$$x(n) = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}d\omega$$

Systhesis Equation

Discrete-Time Aperiodic Signals

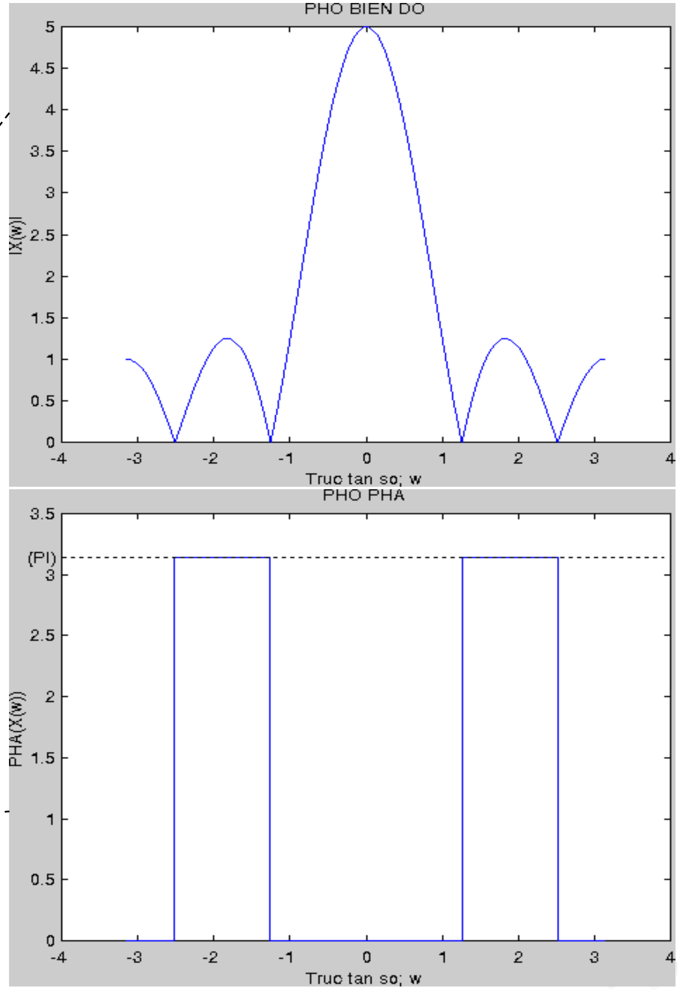
Frequency Domain

- **Example**: determine the X(ω) of the following signal
- $\mathbf{x(n) = \{\ldots 0 \quad 1 \quad 1 \quad \overset{\uparrow}{1} \quad 1 \quad 1 \quad 0 \quad \ldots\}}$

$$\mathbf{X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}}$$

$$\mathbf{X(\omega) = e^{j2\omega} + e^{j\omega} + 1 + e^{-j\omega} + e^{-j2\omega}}$$

$$\mathbf{X(\omega) = 1 + 2\cos\omega + 2\cos(2\omega)}$$



Discrete-Time Aperiodic Signals

- **Energy** $\quad \mathbf{E_X} = \sum_{n=-\infty}^{+\infty} |\mathbf{x(n)}|^2 = \sum_{n=-\infty}^{+\infty} \mathbf{x(n)x^*(n)} \qquad \text{where } x^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\omega)e^{-j\omega n}d\omega$

$$\Longrightarrow \mathbf{E_X} = \sum_{n=-\infty}^{+\infty} \mathbf{x(n)}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{X^*(\omega)e^{-j\omega n}d\omega}\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{X^*(\omega)}\left[\sum_{n=-\infty}^{+\infty} \mathbf{x(n)e^{-j\omega n}}\right]\mathbf{d\omega} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{X^*(\omega)X(\omega)d\omega}$$

  - Then

$$\mathbf{E_x} = \sum_{n=-\infty}^{+\infty} |\mathbf{x(n)}|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\mathbf{X(\omega)}|^2\mathbf{d\omega} \qquad \text{Parseval's Relation}$$

  - $X(\omega)$ is complex $\quad \mathbf{X(\omega) = |X(\omega)|e^{j\Theta(\omega)}}$
    - Amplitude $\quad |\mathbf{X(\omega)}| \quad$ and $\quad$ Phase $\quad \mathbf{\Theta(\omega)}$
    - Energy density spectrum $\quad \mathbf{S_{xx}(\omega) = |X(\omega)|^2 = X(\omega)X^*(\omega)}$

Discrete-Time Aperiodic Signals – Example

- Given signal $x(n) = a^n u(n)$, $-1<a<1$, let's determine
  - Representation of signal in frequency domain
  - Representation of amplitude,phase and energy spectral
  - Draw the spectrum
  - Check if the frequency ($\pi/2$) whether it contributes to $x(n)$ or not? If yes, compute the corresponding amplitude and phase.

$X(\omega) = ?$

$$X(\omega) = \sum_{n=-\infty}^{+\infty} a^n u(n) e^{-j\omega n} = \sum_{n=0}^{+\infty} a^n e^{-j\omega n} = \sum_{n=0}^{+\infty} (ae^{-j\omega})^n$$

$$X(\omega) = \frac{1}{1 - ae^{-j\omega}}$$

Discrete-Time Aperiodic Signals

$|X(\omega)|, \Theta(\omega), S_{xx}(\omega) = ?$

$$X(\omega) = \frac{1}{1 - ae^{-j\omega}} = \frac{1 - ae^{j\omega}}{(1 - ae^{-j\omega})(1 - ae^{j\omega})} = \frac{(1 - a\cos\omega) - j(a\sin\omega)}{1 - 2a\cos\omega + a^2}$$

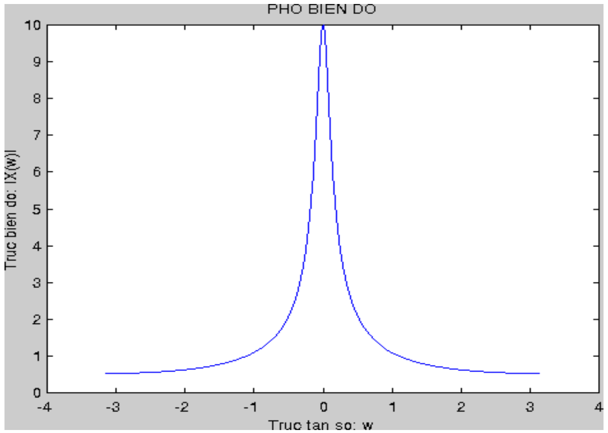
$$X_R(\omega) = \frac{(1 - a\cos\omega)}{1 - 2a\cos\omega + a^2} \qquad X_I(\omega) = \frac{-a\sin\omega}{1 - 2a\cos\omega + a^2}$$

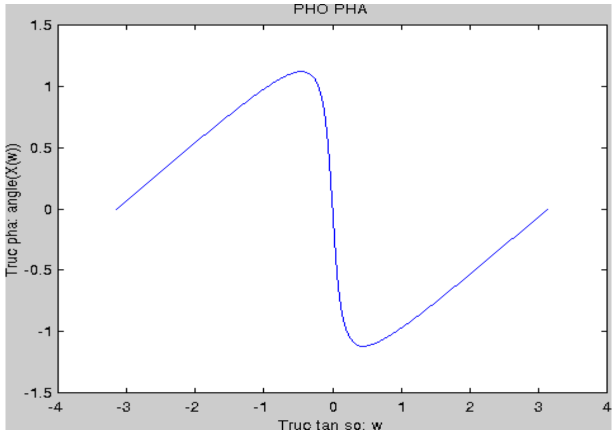
$$|X(\omega)| = \sqrt{(X_R(\omega))^2 + (X_I(\omega))^2} \qquad \Theta(\omega) = \tan^{-1}\left(\frac{X_I(\omega)}{X_R(\omega)}\right)$$

$$S_{XX}(\omega) = X(\omega)X^*(\omega) = \frac{1}{(1 - ae^{-j\omega})(1 - ae^{j\omega})} = \frac{1}{1 - 2a\cos\omega + a^2}$$

Discrete-Time Aperiodic Signals

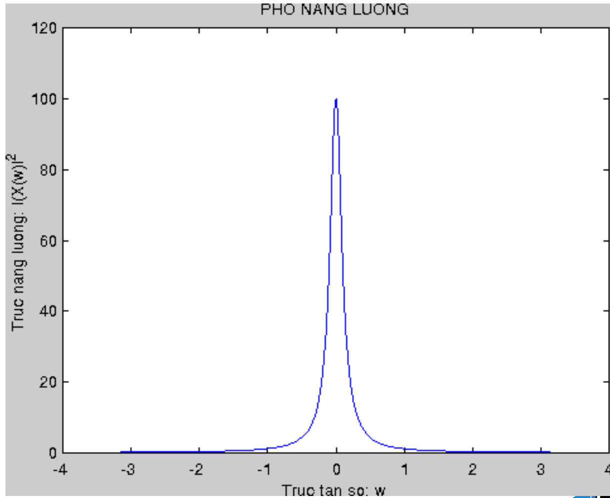
Draw spectrum





ω=π/2

$$X\left(\frac{\pi}{2}\right)=\frac{1}{1-ae^{-j\frac{\pi}{2}}}=\frac{1}{1+ja}=\frac{1-ja}{1+a^2}=\frac{1}{1+a^2}+j\frac{-a}{1+a^2}$$



$$\left|X\left(\frac{\pi}{2}\right)\right|=\sqrt{\left(\frac{1}{1+a^2}\right)^2+\left(\frac{-a}{1+a^2}\right)^2}=\frac{1}{\sqrt{1+a^2}}$$

$$\Theta(\omega)=\tan^{-1}(-a)$$

$|X(\pi/2)| \neq 0 \Rightarrow$ The frequency $\pi/2$ contributes to the signal

Discrete-Time Aperiodic Signals

- **If x(n) is real**
  - $X^*(\omega) = X(-\omega)$

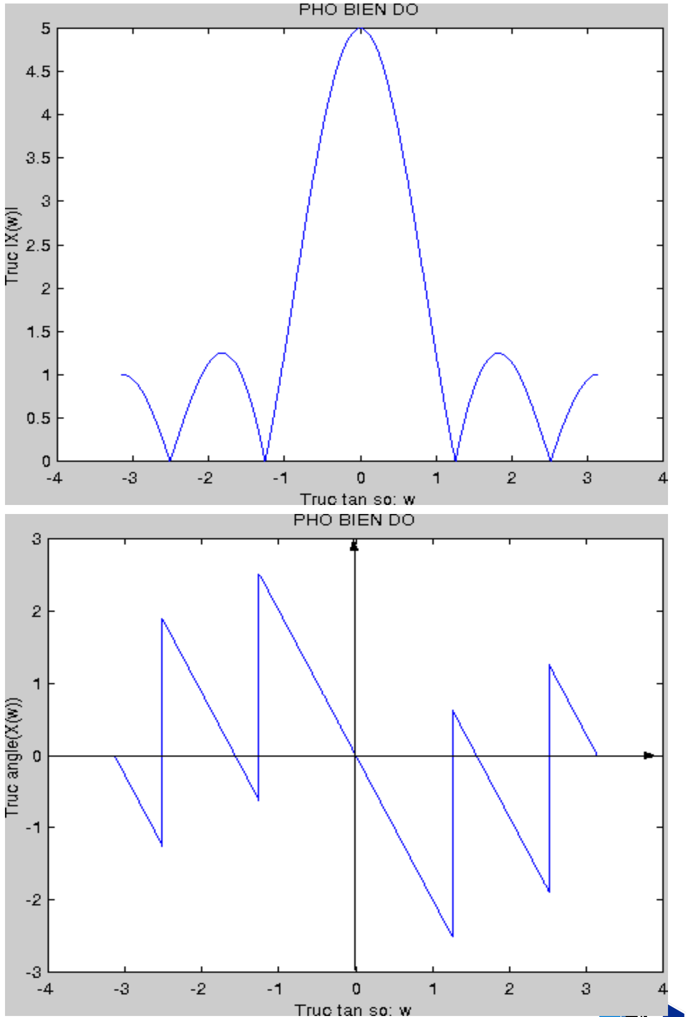
$$\begin{cases} |X(-\omega)| = |X(\omega)| \\ \angle X(-\omega) = \angle X(\omega) \end{cases}$$

  - $S_{xx}(-\omega) = S_{xx}(\omega)$
- **Example**

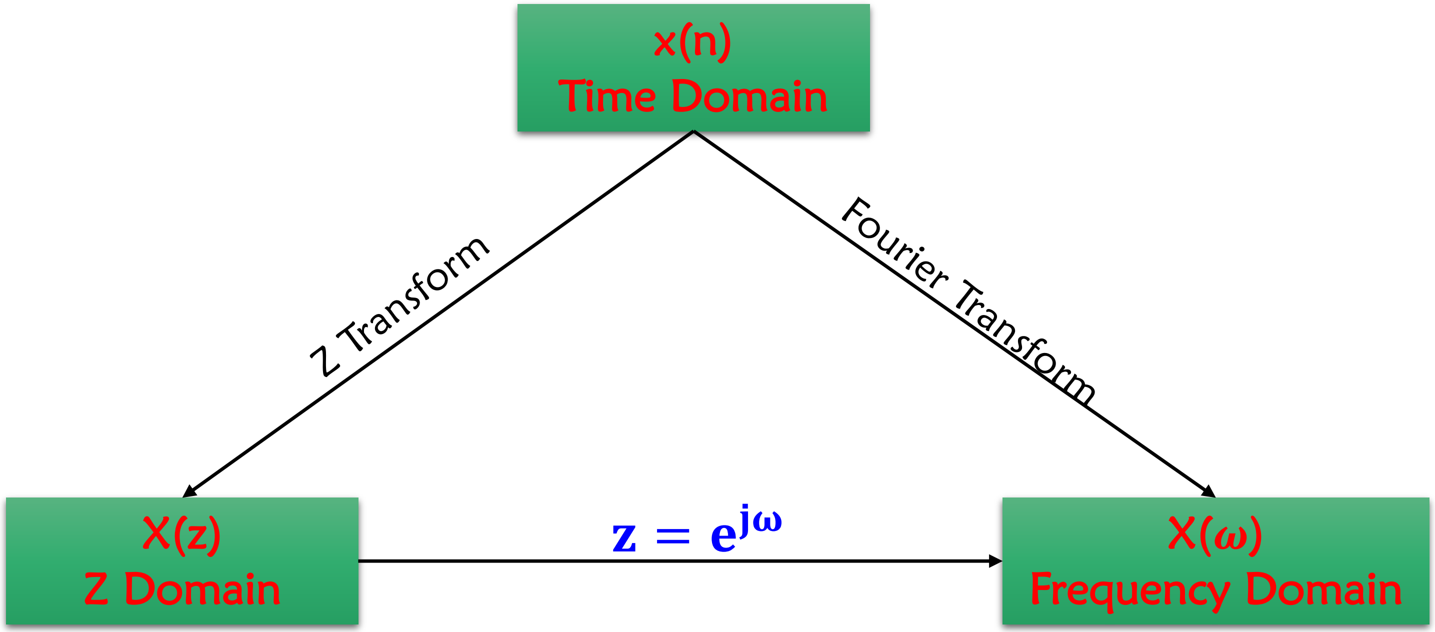
$$x(n) = \begin{cases} A & 0 \le n \le L-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{X(\omega) = Ae^{-j\frac{\omega}{2}(L-1)} \frac{\sin\frac{\omega L}{2}}{\sin\frac{\omega}{2}}}$$

L=5
A=1



Fourier Transform and Z-Transform



$$X(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n}$$

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n}$$

Properties of Fourier Transform

- Linearity

$$\begin{cases} x_1(n) \xleftrightarrow{F} X_1(\omega) \\ x_2(n) \xleftrightarrow{F} X_2(\omega) \end{cases} \Longrightarrow \quad a_1x_1(n) + a_2x_2(n) \xleftrightarrow{F} a_1X_1(\omega) + a_2X_2(\omega)$$

- Time shifting

$$x(n) \xleftrightarrow{F} X(\omega) \quad \Longrightarrow \quad x(n-k) \xleftrightarrow{F} e^{-j\omega k}X(\omega)$$

- Inverse in time domain

$$x(n) \xleftrightarrow{F} X(\omega) \quad \Longrightarrow \quad x(-n) \xleftrightarrow{F} X(-\omega)$$

- Convolution

$$\begin{cases} x_1(n) \xleftrightarrow{F} X_1(\omega) \\ x_2(n) \xleftrightarrow{F} X_2(\omega) \end{cases} \Longrightarrow \quad x(n) = x_1(n) * x_2(n) \xleftrightarrow{F} X(\omega) = X_1(\omega)X_2(\omega)$$

Exercises

- Determine the Fourier transform of the following signals

- a) $\mathbf{x(n) = x_1(n) + x_2(n)}$

$$\mathbf{x_1(n) = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases} \quad \text{and} \quad x_2(n) = \begin{cases} 0 & n \geq 0 \\ a^{-n} & n < 0 \end{cases} \quad \text{where} \; -1 < a < 1}$$

- b) $\mathbf{x(n) = 3\left(\frac{1}{2}\right)^{n-3} u(n-2)}$

Properties of Fourier Transform

- Frequency shifting

$$x(n) \xleftrightarrow{F} X(\omega) \quad \Longrightarrow \quad e^{j\omega_0 n}\, x(n) \xleftrightarrow{F} X(\omega - \omega_0)$$

- Modulation

$$x(n) \xleftrightarrow{F} X(\omega) \quad \Longrightarrow \quad x(n)\cos(\omega_0 n) \xleftrightarrow{F} \frac{1}{2}\left[X(\omega + \omega_0) + X(\omega - \omega_0)\right]$$

- Parseval

$$\begin{cases} x_1(n) \xleftrightarrow{F} X_1(\omega) \\ x_2(n) \xleftrightarrow{F} X_2(\omega) \end{cases} \Longrightarrow \sum_{n=-\infty}^{+\infty} x_1(n) x_2^*(n) \xleftrightarrow{F} \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\omega) X_2^*(\omega)\, d\omega$$

- Differentiation in frequency domain

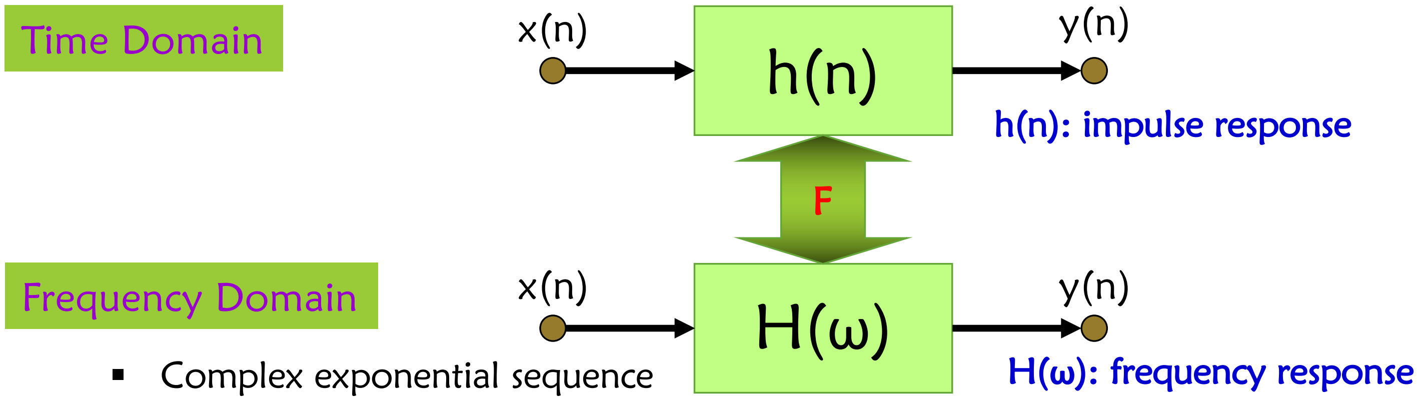
$$x(n) \xleftrightarrow{F} X(\omega) \quad \Longrightarrow \quad n x(n) \xleftrightarrow{F} j\frac{dX(\omega)}{d\omega}$$

Exercises

- Determine the Fourier Transform x*(n)

$$x(n) \xleftrightarrow{F} X(\omega) \quad \Longrightarrow \; F = \{x^*(n)\} = ?$$

LTI Systems in Frequency Domain



- Complex exponential sequence
- Sinusoid

- Example: frequency response of a complex exponential sequence $\mathbf{x(n) = Ae^{j\omega n}}$ $-\infty < n < \infty$

$$\mathbf{y(n) = x(n) * h(n) = \sum_{k=-\infty}^{+\infty} h(k)x(n-k) = \sum_{k=-\infty}^{+\infty} h(k)Ae^{j\omega(n-k)} = Ae^{j\omega n}\sum_{k=-\infty}^{+\infty} h(k)e^{-j\omega k} = AH(\omega)e^{j\omega n}}$$

LTI Systems in Frequency Domain

- $H(\omega)$ can be expressed in polar form as $\mathbf{H(\omega) = |H(\omega)|e^{j\Theta(\omega)}}$
- We have

$$\mathbf{H(\omega) = \sum_{k=-\infty}^{+\infty} h(k)e^{-j\omega k} = \sum_{k=-\infty}^{+\infty} h(k)\cos(\omega k) - j\sum_{k=-\infty}^{+\infty} h(k)\sin(\omega k)}$$

$$\mathbf{H(\omega) = H_R(\omega) + jH_I(\omega) = \sqrt{H_R^2(\omega) + H_I^2(\omega)}\; e^{j\tan^{-1}\left(\frac{H_I(\omega)}{H_R(\omega)}\right)}}$$

where

$$\mathbf{H_R(\omega) = \sum_{k=-\infty}^{+\infty} h(k)\cos(\omega k)} \qquad \textbf{Even Function}$$

$$\mathbf{H_I(\omega) = -\sum_{k=-\infty}^{+\infty} h(k)\sin(\omega k)} \qquad \textbf{Odd Function}$$

$$\mathbf{|H(\omega)| = \sqrt{H_R^2(\omega) + H_I^2(\omega)}} \qquad \textbf{Even}$$

$$\mathbf{\Theta(\omega) = e^{j\tan^{-1}\left(\frac{H_I(\omega)}{H_R(\omega)}\right)}} \qquad \textbf{Odd}$$

- Therefore, if we know $|H(\omega)|$ and $\Theta(\omega)$ for $0 \le \omega \le \pi$, we also know these functions for $-\pi \le \omega \le 0$

LTI Systems in Frequency Domain

- Response of sinusoid signal
  - We have

$$x_1(n) = Ae^{j\omega n} \rightarrow y_1(n) = A|H(\omega)|e^{j\Theta(\omega)}e^{j\omega n}$$

$$x_2(n) = Ae^{-j\omega n} \rightarrow y_2(n) = A|H(-\omega)|e^{j\Theta(-\omega)}e^{-j\omega n} = A|H(\omega)|e^{-j\Theta(\omega)}e^{-j\omega n}$$

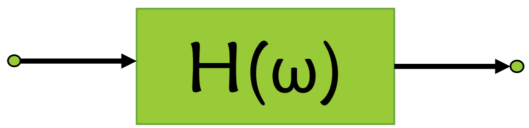
$$\mathbf{x(n) = A\cos(\omega n) = \frac{1}{2}[x_1(n) + x_2(n)] \rightarrow y(n) = \frac{1}{2}[y_1(n) + y_2(n)] = A|H(\omega)|\cos[\omega n + \Theta(\omega)]}$$

$$\mathbf{x(n) = A\sin(\omega n) = \frac{1}{2j}[x_1(n) - x_2(n)] \rightarrow y(n) = \frac{1}{2j}[y_1(n) - y_2(n)] = A|H(\omega)|\sin[\omega n + \Theta(\omega)]}$$

LTI System in Frequency Domain

- The response to **periodic** input signal

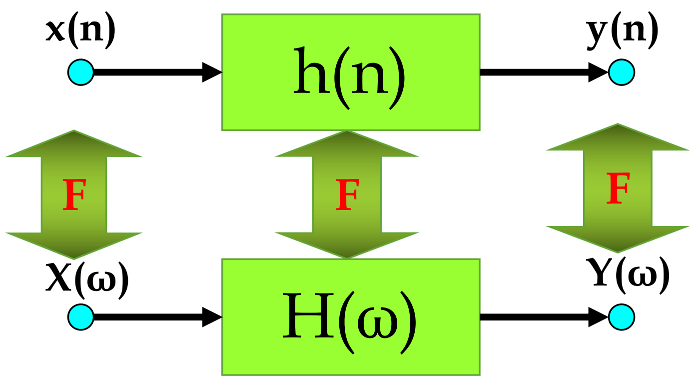
$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi \frac{k}{N} n}$$



$$y(n) = \sum_{k=0}^{N-1} c_k H\left(\frac{2\pi k}{N}\right) e^{j2\pi \frac{k}{N} n}$$

  - **The response of the system to the periodic input signal is also periodic with the same period N**

- The response to **aperiodic** input signal



y(n) = x(n)*h(n)

Y(ω) = X(ω)H(ω)

$Y(\omega_0) = X(\omega_0)H(\omega_0) = |H(\omega_0)|\, e^{j\Theta(\omega_0)} X(\omega_0)$

➔ The response to a specific frequency (ω_0):

- Amplitude: scaling $|H(\omega_0)|$
- Phase: shift $\Theta(\omega_0)$

LTI Systems in Frequency Domain

- System function H(z) vs. Frequency response function $H(\omega)$

$$\mathbf{H(\omega) = H(z)\Big|_{z\,=\,e^{j\omega}} = \sum_{n=-\infty}^{+\infty} h(n)e^{-j\omega n}}$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \xrightarrow{\text{The system is stable}} H(\omega) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} a_k e^{-j\omega k}}$$

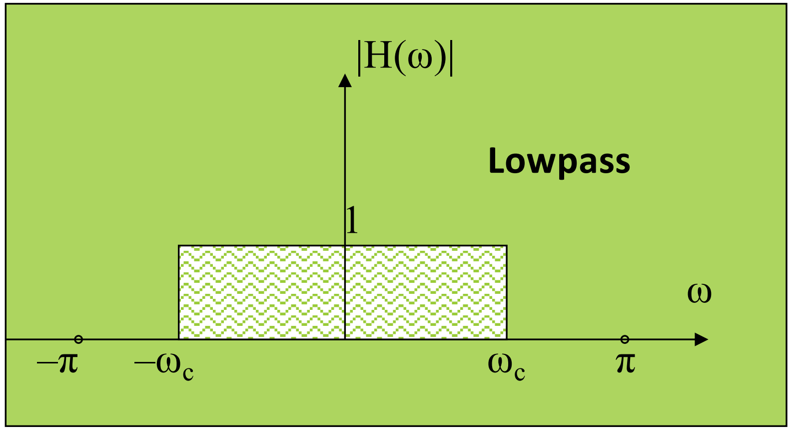
$$\mathbf{|H(\omega)|^2 = H(\omega)H^*(\omega) = H(\omega)H(-\omega) = H(z)H(z^{-1})}$$

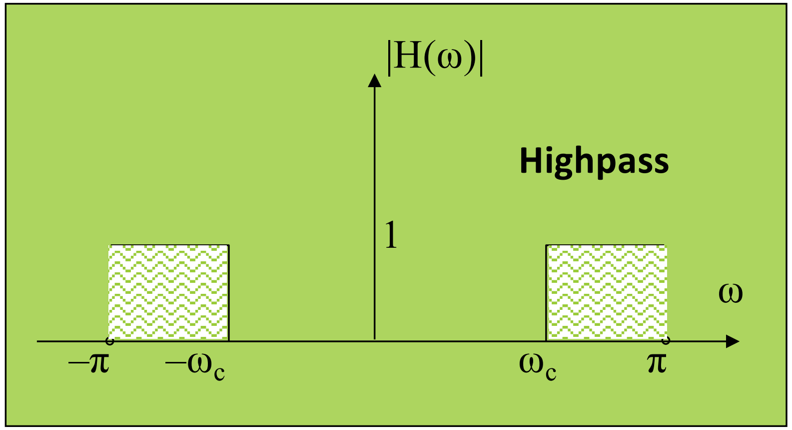
LTI Systems as Frequency-Selective Filters

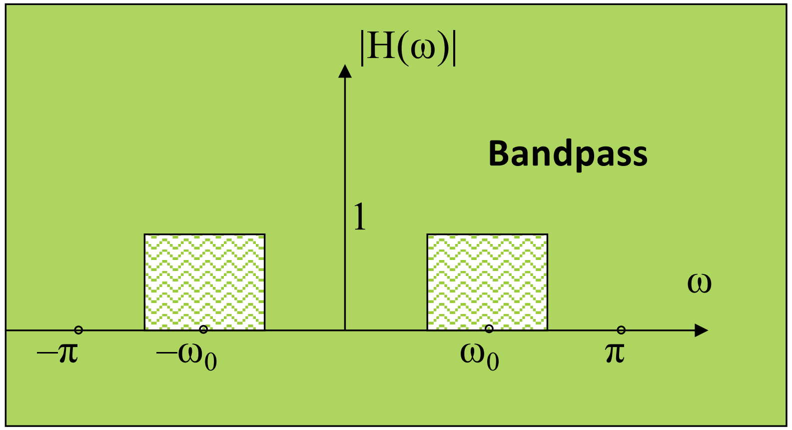
- Filters
  - It is used to describe a device that discriminates, according to some attribute of the objects applied as its input.
  - For example: an air filter, an oil filter, an ultraviolet filter.

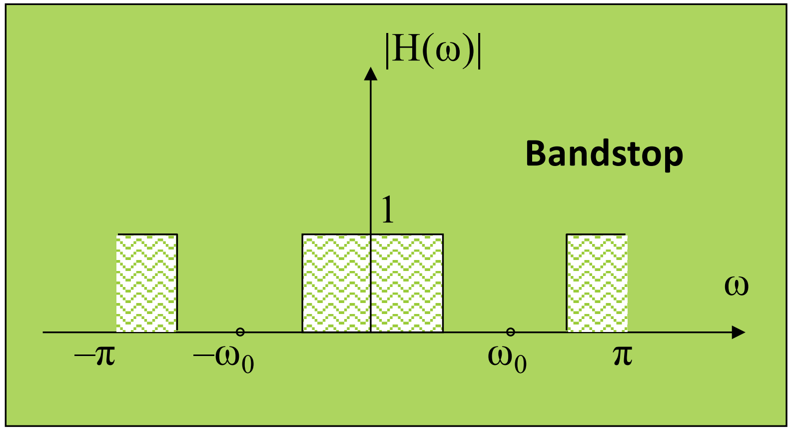
- LTI Systems
  - Modify the input signal spectrum $X(\omega)$ according to its frequency response $H(\omega)$ to yield an output signal with spectrum $Y(\omega) = H(\omega)X(\omega)$.
  - Therefore, LTI systems are considered as frequency filters where $H(\omega)$ acts as a weighting function or a spectral shaping function.
  - Frequency-selective filters
    - Removal of undesirable noise
    - Spectral shaping such as equalization of communication channels
    - Spectral analysis of signals
    - Signal detection in Radar, Sonar, …

Filters


|H(ω)|
Lowpass
1
ω
−π
−ωc
ωc
π


|H(ω)|
Highpass
1
ω
−π
−ωc
ωc
π


|H(ω)|
Bandpass
1
ω
−π
−ω0
ω0
π


|H(ω)|
Bandstop
1
ω
−π
−ω0
ω0
π

Exercise