FEM - 1D

V = virtual displacement, a bit movement from u but need to satisfy $V(0)=0$ at fixed end

1) Strong form:

$$-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x) \qquad (1)$$

Multiplying 2 sides of (1) with a test function $v(x)$

$$\int_\Omega -\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) v(x)\, dx = \int_\Omega f(x)\, dx \qquad (**)$$

This is the weak form. If (**) is true for every $v(x)$ then we can get back to the strong form

2 Integration by part (**)

For any $v(x)$ with $v=0$ at fixed end

$$\int_\Omega c(x)\frac{du}{dx}\frac{dv}{dx}\, dx - \left[c(x)\frac{du}{dx}\, v(x)\right]_0^1\Big|_\Omega = f(x)v(x)\,dx$$

Assume $du/dx = 0$ at free end

3 Galerkin method:

- Start with a weak form.
- Change that continuous weak form to discrete one

$$KU = F$$

(Instead of a function unknown, I want to have n-unknown. I will give a discrete equation, which will eventually be $KU = F$

- Choose trial function (basis / shape) function: $\phi_1(x) \ldots \phi_N(x)$
- Approximate solution:

$$u = \underbrace{u_1}_{} \cdot \phi_1(x) + \ldots + \underbrace{u_N}_{}\phi_N(x)$$

N unknown numbers

- Choose test function $v_1(x), v_2(x), \ldots, v_N(x)$. Each v_i gives 1 equation $\Rightarrow$ get N equation -

$\Rightarrow$ A square matrix, a linear system $Ku = F$

NOTE:

Galerkin only applied weak form to trial & test function, not to the real (continuous) weak form for a whole a lot of v

FEM - 1D

Weak Form → Galerkin → Choose $\begin{cases} \phi_1, \dots, \phi_N \\ v_1, \dots, v_N \end{cases} \Rightarrow KU = F$

very often they are the same

$$-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x) \Rightarrow \int c \frac{du}{dx}\frac{dv}{dx}\,dx = \int f(x)\,v(x)\,dx$$

STRONG WEAK

Constraint: If $u(1) = 0$ then $v(1) = 0$

What choice will we make for ϕ_i? How do we get from all that preparation to the equation that we actually solve $KU = F$

① Example of $\phi(x)$ as hat function

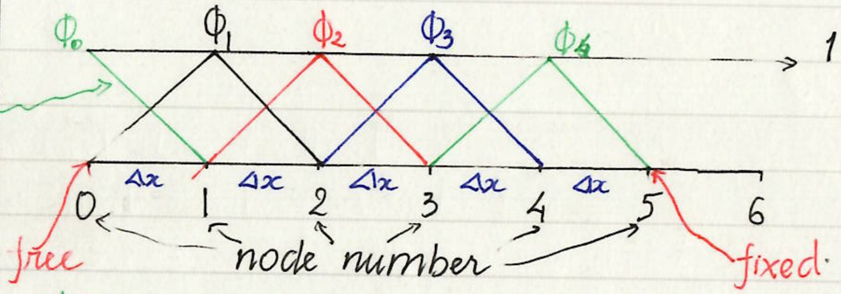
Keep ϕ_i simple

half of hat func

$\phi_0 = -\frac{x}{\Delta x} + 1$

$\left(x = 0 \Rightarrow \phi_0 = 1\right.$

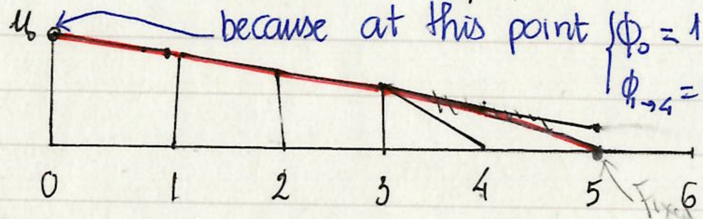
$\left.x = \Delta x \Rightarrow \phi_0 = 0\right)$



$\begin{cases} \text{basis func} \\ \text{shape func} \\ \text{trial func} \end{cases}$

Approximation

$u(x) = u_0\phi_0(x) + u_1\phi_1(x) + \dots + u_4\phi_4(x) \Rightarrow u(\text{node } 5) = 0$



because at this point $\begin{cases} \phi_0 = 1 \\ \phi_{1 \to 4} = 0 \end{cases}$; basis funcs are "local"

. FEM: We just decide ϕ_i, Galerkin give us equations.

. Where does the equation come from?

Weak form,

$$\underbrace{\int_0^1 c(x)\frac{du}{dx}\frac{dv}{dx}\,dx}_{} = \int_0^1 f(x)\,v_i(x)\,dx \rightarrow F = \begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{bmatrix}$$

$(u_0\phi_0 + u_1\phi_1 + \dots + u_4\phi_4)$

Assume: $f(x) = 1$, $c(x) = 0$ $\quad (\Rightarrow u'' = 1 \Rightarrow u = x^2 - 1)$

$$\Rightarrow F = \Delta x \begin{bmatrix} 1/2 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Equation 0: FEM 1D

$$\int_0^1 -\left(u_0\phi_0' + \dots + u_4\phi_4'\right)\frac{dv_0}{dx}\,dx = F_0 = \frac{\Delta x}{2}$$

$$\frac{1}{\Delta x}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \Delta x \begin{bmatrix} 1/2 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- $K_{00} = \int_0^1 -\phi_0' \frac{dv_0}{dx}\,dx = -\int_0^{\Delta x} (\phi_0')^2\,dx = \frac{1}{\Delta x}$

  $\phi_0' = v_0' = \left(1 - \frac{x}{\Delta x}\right)' = -\frac{1}{\Delta x}$

- $K_{01} = \int_0^1 -\phi_1 \frac{dv_0}{dx}\,dx = \int_0^{\Delta x} -\phi_1'\phi_0'\,dx = -\int_0^{\Delta x} \frac{1}{\Delta x}\left(-\frac{1}{\Delta x}\right)dx = -\frac{1}{\Delta x}$

- $K_{02} = \int_0^1 \phi_2' v_0'\,dx = 0$

  only $\neq 0$ from node $1 \rightarrow 3$

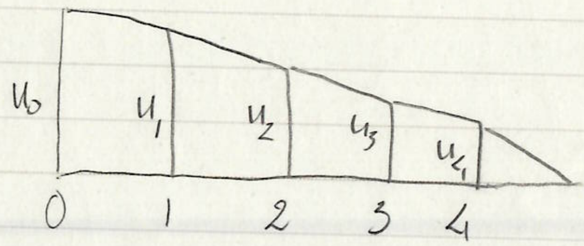
  only $\neq 0$ from node $0 \rightarrow 1$

- $K_{03} = K_{04} = 0$

- $K_{10} = K_{01} = -1/\Delta x$

- $K_{11} = \int_0^1 -(\phi_1')^2\,dx = 2/\Delta x$, $\phi_1' = v_1' = \begin{cases} 1/\Delta x & 0 \rightarrow \Delta x \\ -1/\Delta x & \Delta x \rightarrow 2\Delta x \end{cases}$

- $K_{12} = \int_0^1 \phi_1'\phi_2'\,dx = K_{01} = -1/\Delta x$



$u_0 = 0.5$

$u_1 = 0.48$

$u_2 = 0.42$

$u_3 = 0.32$

$u_4 = 0.18$

$u_5 = 0$ ← automatic, not calculated