Automata, Logics, and Infinite Games

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Master2 RI 2010

Model-Checking

- The Model-checking Problem: Given a system *Sys* and a specification *Spec*, decide whether *Sys* satisfies *Spec*.
- Example: Mutual exclusion protocol

```
Process 1: repeat                 Process 2:  repeat
00: non-critical section 1        00: non-critical section 2
01: wait unless turn = 0          01: wait unless turn = 1
10: critical section 1            10: critical section 2
11: turn := 1                     11: turn := 0
```

- A state is a bit vector

  (line no. of process 1,line no. of process 2, value of turn)

  Start from `(00000)`.
- *Spec* = "a state `(1010b)` is never reached", and "always when a state of the form `(01bcd)` is reached, then later a state `(10b'c'd')` is reached". Similarly for Process 2, i.e. with states `(bc01d)` and `(b'c'10d')`.

The Formal Approach

- Models of systems = Kripke Structures
- Specifications = Temporal Logic formulas

Kripke Structures

Fix $Prop = p_1, \ldots, p_n$ a set of atomic propositions (local properties).

Definition

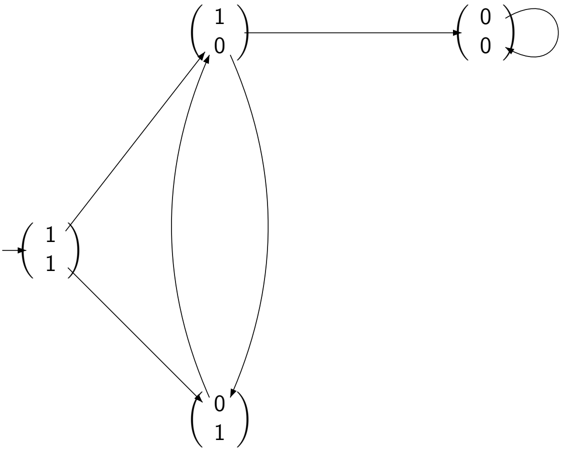
Kripke Structure over *Prop* A *Kripke Structure* is $\mathcal{S} = (S, R, \lambda)$

- S is a set of states (worlds)
- $R \subseteq S \times S$ is a transition relation
- $\lambda : S \rightarrow 2^{Prop}$ associates those p_i which are assumed true in s. Write $\lambda(s)$ as a bit vector $(b_1, \ldots, b_n)$ with $b_i = 1$ iff $p_i \in \lambda(s)$

A rooted Kripke Structure is a pair $(\mathcal{S}, s)$ where s is a distinguished state, called the *initial state*.

A Toy System

Over two propositions p_1, p_2



Example : Mutual Exclusion Protocol

- Use p_1 (resp. p_2) for "being in wait instruction before critical section of Process 1 (resp. Process 2)"
- Use p_3 (resp. p_2) for "being in critical section of Process 1 (resp. Process 2)"
- Example of label function $\lambda(01101) = \{p_1, p_4\}$ (encoded by (1001))
- The relation R is as defined by the transitions of the protocol.

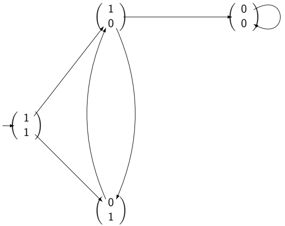
Exercise: Draw the KS.

Paths and Words

Let $\mathcal{S} = (S, R, \lambda)$ be Kripke Structure over *Prop*

- A path from s in $\mathcal{S}$ (i.e. through $(\mathcal{S}, s)$) is a sequence $s_0, s_1, s_2, \ldots$ where $s_0 = s$ and $(s_i, s_{i+1}) \in R$ for $i \geq 0$
- A word from s in $\mathcal{S}$ is a sequence $\alpha = \lambda(s_0)\lambda(s_1)\lambda(s_2)\ldots \in (\mathbb{B}^n)^\omega$ where $s_0 s_1 s_2 \ldots$ is a path from s in $\mathcal{S}$
  Use $\text{Words}_s(\mathcal{S})$ for the set of words from s in $\mathcal{S}$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}^\omega$ in



- If $\alpha \in \text{Words}_{\alpha(0)}(\mathcal{S})$,
  1. α^i stands for $\alpha(i)\alpha(i+1)\ldots$ So $\alpha = \alpha^0$
  2. $(\alpha(i))_j$ is the jth component of $\alpha(i)$; the value of proposition p_j

Linear Time Logic for Properties of Words

[Eme90] We use modalities

G	denotes	"*Always*"
F	denotes	"*Eventually*"
X	denotes	"*Next*"
U	denotes	"*Until*"

The syntax of the logic LTL is:

$$\varphi_1, \varphi_2 (\ni LTL) ::= p \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi_1 \mid \mathbf{X}\, \varphi_1 \mid \varphi_1 \,\mathbf{U}\, \varphi_2$$

where $p \in Prop$
Use standard abbreviaitons $\mathtt{true}, \mathtt{false}, \varphi_1 \wedge \varphi_2$, $\varphi_1 \Rightarrow \varphi_2$, and $\varphi_1 \Leftrightarrow \varphi_2$, etc.

Semantics of LTL

Given a KS $\mathcal{S}$, $\alpha \in \text{Words}_{\alpha(0)}(\mathcal{S})$, and $\varphi \in$ LTL, we now define the expression $\alpha \models \varphi$, to mean "α satisfies φ"
Rather $\alpha^i \models \varphi$ (for $i \geq 0$), by induction over φ:

- $\alpha^i \models p_j$ iff $(\alpha(i))_j = 1$
- $\alpha^i \models \varphi_1 \vee \varphi_2$ iff ...
- $\alpha^i \models \neg\varphi_1$ iff
- $\alpha^i \models \mathbf{X}\,\varphi_1$ iff $\alpha^{i+1} \models \varphi_1$
- $\alpha^i \models \varphi_1 \,\mathbf{U}\, \varphi_2$ iff for some $j \geq i$, $\alpha^j \models \varphi_2$, and
  for all $k = i, \ldots, j-1$, $\alpha^k \models \varphi_1$

$$\text{Let } \begin{cases} \mathbf{F}\varphi \stackrel{\text{def}}{=} \mathtt{true}\,\mathbf{U}\,\varphi, \text{ hence } \alpha^i \models \mathbf{F}\varphi \text{ iff } \alpha^j \models \varphi \text{ for some } j \geq i. \\ \mathbf{G}\varphi \stackrel{\text{def}}{=} \neg\mathbf{F}\neg\varphi, \text{ hence } \alpha^i \models \mathbf{G}\varphi_1 \text{ iff } \alpha^j \models \varphi_1 \text{ for every } j \geq i. \end{cases}$$

Examples

Formulas over p_1 and p_2:

1. $\alpha \models \mathbf{GF}p_1$ iff "in α, infinitely often 1 appears in the first component".
2. $\alpha \models \mathbf{X}\,\mathbf{X}\,(p_2 \Rightarrow \mathbf{F}p_1)$ iff "if the second component of $\alpha(2)$ is 1, so will be the first component of $\alpha(j)$ for some $j \geq 2$".
3. $\alpha \models \mathbf{F}(p_1 \wedge \mathbf{X}\,(\neg p_2\,\mathbf{U}\,p_1))$ iff "α has two letters of the form $\begin{pmatrix} 1 \\ \star \end{pmatrix}$ such that only letters of the form $\begin{pmatrix} \star \\ 0 \end{pmatrix}$ occur in between".

Augmenting LTL: the logic CTL^*

We want to specify that every word of $(\mathcal{S}, s)$ satisfies an LTL specification φ, or that there exists a word in the Kripke Structure such that something holds. We use CTL^* [EH83] which extends LTL with quantfications over words:

$$\psi_1, \psi_2 (\ni CTL^*) ::= \mathbf{E}\,\psi \mid p \mid \psi_1 \vee \psi_2 \mid \neg\psi_1 \mid \mathbf{X}\,\psi_1 \mid \psi_1\,\mathbf{U}\,\psi_2$$

Semantics: for a word α, a position i, and a rooted Kripke Structure $(\mathcal{S}, s)$:

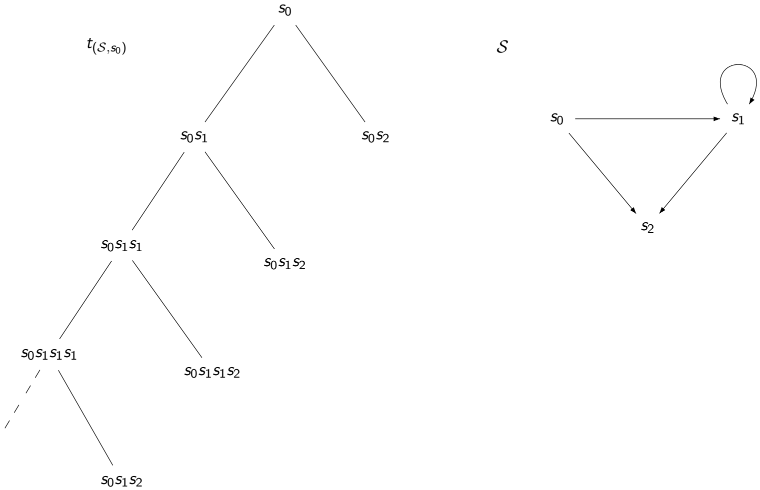
$$\alpha^i \models \mathbf{E}\,\psi \text{ iff } \beta \models \psi \text{ for some } \beta \text{ in } \mathrm{Words}_{\alpha^i(0)}(\mathcal{S})$$

Let $\mathbf{A}\,\psi \stackrel{\text{def}}{=} \neg\mathbf{E}\,\neg\psi$

CTL* is more expressive than LTL: $\mathbf{A}\,[\mathbf{G}\mathrm{life} \;\Rightarrow\; \mathbf{GEX}\,\mathrm{death}]$

Interpretation over Trees

- We unravel $\mathcal{S} = (S, R, \lambda)$ from s as a tree $t_{(\mathcal{S},s)}$.
- Paths of $\mathcal{S}$ are retrieved in the tree $t_{(\mathcal{S},s)}$ as branches.



Σ-Labeled Full Binary Trees

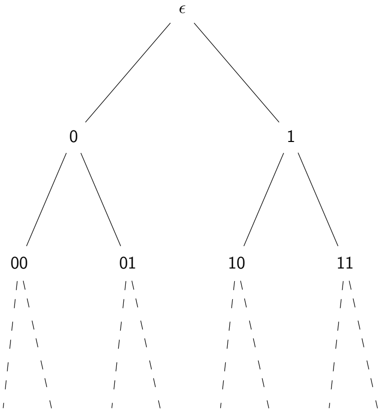
For simplicity we assume that states have exactly two successors $\Rightarrow$ we consider (only) binary trees

- The full binary tree T^ω is the set $\{0,1\}^*$ of finite words over a two element alphabet.
- The root is the empty word ϵ
- A node $w \in \{0,1\}^*$ has left son $w0$ and right son $w1$.
- A Σ-labeled full binary tree is a function $t : \{0,1\}^* \to \Sigma$
- *Trees*(Σ) is the set of Σ-labeled full binary trees.

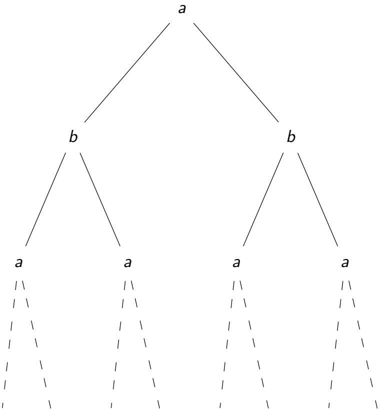
If the formulas are over the set *Prop* of propositions, then take $\Sigma = 2^{Prop}$ (or equivalently $\mathbb{B}^{|Prop|}$)

Example: alternating levels of a's and b's

The full binary tree T^ω



A tree t



Model-checking and Satisfiabilty

- The Model-checking Problem: does a tree t satisfy the specification *Spec*?
- The Satisfiability Problem: Is there a tree model of the specification *Spec*?

$$\begin{aligned} \text{Model-checking} &= \text{Program Verification} \\ \text{Satisfiability} &= \text{Program Synthesis} \end{aligned}$$

More specifically in this course

- Tree Automata: devices which recognize models of formulas

$$\Phi \rightsquigarrow \mathcal{A}_\Phi \text{ such that } L(\mathcal{A}_\Phi) = \{t \in \mathit{Trees}(\Sigma) \mid t \models \Phi\}$$

The Model-checking Problem $\rightsquigarrow$ The Membership Problem

The Satisfiability Problem $\rightsquigarrow$ The Emptiness Problem

- Games are fundamental to solve those

- Mu-calculus is a unifying logical formalism

Games

- Two-person games on directed graphs.
- How they are played?
- What is a strategy? What does it mean to say that a player wins the game?
- Determinacy, forgetful strategies, memoryless strategies

Arena

An arena (or a game graph) is

- $G = (V_0, V_1, E)$
- V_0 Player 0 positions, and V_1 Player 1 positions (partition of V)
- $E \subseteq V \times V$ is the edged-relation
- write $\sigma \in \{0, 1\}$ to designate a player, and $\overline{\sigma} = 1 - \sigma$

Plays

- A token is placed on some initial vertex $v \in V$
- When v is a σ-vertex, the Player σ moves the token from v to some successor position $v' \in vE$.
- This is repeated infinitely often or until a vertex $\bar{v}$ without successor is reached ($\bar{v}E = \emptyset$)
- Formally, a play in the arena G is either
  - an infinite path $\pi = v_0 v_1 v_2 \ldots \in V^\omega$ with $v_{i+1} \in v_i E$ for all $i \in \omega$, or
  - a finite path $\pi = v_0 v_1 v_2 \ldots v_l \in V^+$ with $v_{i+1} \in v_i E$ for all $i < l$, but $v_l E = \emptyset$.

Games and Winning sets

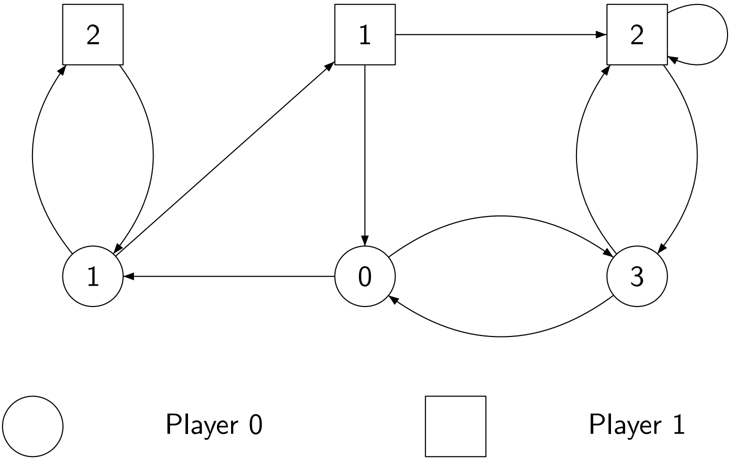
- Let be G an arena and $Win \subseteq V^\omega$ be the winning condition
- The pair $\mathcal{G} = (G, Win)$ is called a game
- Player 0 is declared the winner of a play π in the game $\mathcal{G}$ if
  - π is finite and $last(\pi) \in V_1$ and $last(\pi)E = \emptyset$, or
  - π is infinite and $\pi \in Win$.
- Player 1 wins π if Player 0 does not win π.
- Initialized game $(\mathcal{G}, v_I)$.

Parity Winning Conditions

- We color vertices of the arena by $\chi : V \rightarrow C$ where C is a finite set of so-called colors; it extends to plays $\chi(\pi) = \chi(v_0)\chi(v_1)\chi(v_2)\ldots$.
- C is a finite set of integers called priorities (or colors)
- Let $Inf_\chi(\pi)$ be the set of colors that occurs infinitely often in $\chi(\pi)$.

  Win is the set of infinite paths π such that $min(Inf_C(\pi))$ is even.

Parity Game Example



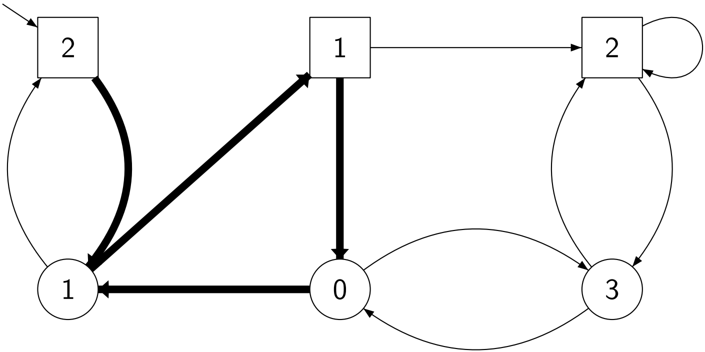
Exercises

- What are the links between parity games, and *reachability games* in which Player 0 wins whenever she reaches/hits a given set of "good positions"?
- What are the links between parity games, and *safety games* in which Player 0 wins whenever she never hits a given set of "bad positions"? Comment on previous question.
- Can the winning condition "visiting infinitely often a position of some given set" expressible with parity condition? Justify.

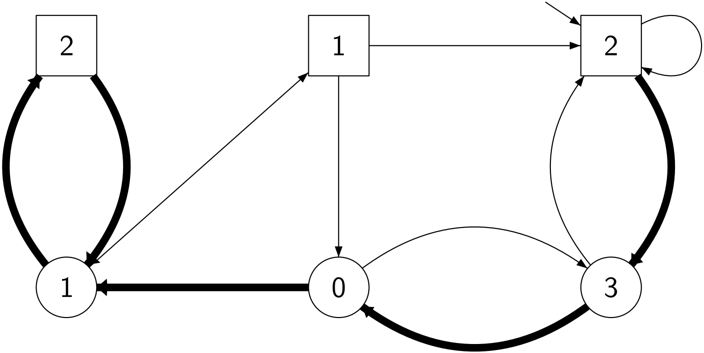
Strategies

- A strategy for Player σ is a function $f_\sigma \colon V^* V_\sigma \to V$
- A prefix play $\pi = v_0 v_1 v_2 \dots v_l$ is conform with f_σ if for every i with $0 \leq i < l$ and $v_i \in V_\sigma$ the function f_σ is defined and we have $v_{i+1} = f_\sigma(v_0 \dots v_i)$.
- A play is conform with f_σ if each of its prefix is conform with f_σ.
- f_σ is a strategy for Player σ on $U \subseteq V$ if it is defined for every prefix of a play which is conform with it, starts in a vertex in U, and does not end in a dead end of Player σ.
- A strategy f_σ is a winning strategy for Player σ on U if all plays which are conform with f_σ and start from a vertex in U are wins for Player σ.
- Player σ wins a game $\mathcal{G}$ on $U \subseteq V$ if he has a winning strategy on U.

Winning Play for Player 0



Winning Play for Player 1



Winning Regions

- The winning region for Player σ is the set $W_\sigma(\mathcal{G}) \subseteq V$ of all vertices such that Player σ wins $(\mathcal{G}, v)$, i.e.Player 0 wins $\mathcal{G}$ on $\{v\}$.
- Hence, for any $\mathcal{G}$, Player σ wins $\mathcal{G}$ on $W_\sigma(\mathcal{G})$.

Determinacy of Parity Games

- A game $\mathcal{G} = ((V, E), Win)$ is determined when the sets $W_\sigma(\mathcal{G})$ and $W_{\overline{\sigma}}(\mathcal{G})$ form a partition of V.

Theorem

Every parity game is determined.

- A strategy f_σ is positional (or memoryless) strategy whenever when defined for πv and $\pi' v$, we have $f_\sigma(\pi v) = f_\sigma(\pi' v)$.

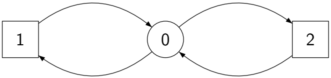
Theorem

[EJ91, Mos91] In every parity game, both players win memoryless.

See [GTW02, Chaps. 6 and 7]

Games that are not Memoryless

In Muller games, a set $\mathcal{F} \subseteq 2^C$ is given and $Win = \{\pi \in V^\omega | Inf_\chi(\pi) \in \mathcal{F}\}$
Here every color must occur infinitely often; Player 0 must remember something (but the strategy is finite memory = forgetful strategy)



Forgetful Determinacy of Regular Games

Muller games (and any other regular games, Rabin, Streett, Rabin Chain, Buchi, ...) can be simulated by larger parity games. They are also determined (also see determinacy result from [Mar75] for every game with Borel type). As a corollary of previous results, we have the very general following result for

Corollary

Regular games are forgetful determined.

Complexity Results

Theorem

$\textsc{Wins} =$
$\{(\mathcal{G}, v) \mid \mathcal{G}$ *a finite parity game and* v *a winning position of Player 0*$\}$
is in NP $\cap$ *co-NP*

1. Guess a memoryless strategy f of Player 0
2. Check whether f is memoryless winning strategy

Step 2. can be carried out in polynomial time: $\mathcal{G}_f$ is a subgraph of $\mathcal{G}$ where all edges $(v, v")$ where $v" \neq f(v)$ have been eliminated. Given $\mathcal{G}_f$, check existence of a vertex v' reachable from v such that 1) $\chi(v')$ is odd and 2) v' lies on cycle in $\mathcal{G}_f$ containing only priorities greater than equal to $\chi(v')$. Such v' does not exist iff Player 0 has a winning strategy. Hence, $\textsc{Wins} \in$ NP. By determinacy, deciding $(\mathcal{G}, v) \notin \textsc{Wins}$ means to decide whether v is a winning position for Player 1 (as above but 1') $\chi(v')$ is even), or use algorithm above on the dual game. Hence, $\textsc{Wins} \in$ co-NP.

Algorithms for Parity Games

- Compute the winning region: look at "Algorithms for Parity Games", Chapter 7 of Automata, Logics, and Infinite Games A Guide to Current Research. Series: Lecture Notes in Computer Science , Vol. 2500 Grädel, Erich; Thomas, Wolfgang; Wilke, Thomas (Eds.) 2002, XI, 385 p.
- Reduction to Safety Games
  We will show an algorithm in $O(n(n/d)^{\lceil d/2 \rceil})$ (the known bound until now)

Winning conditions

Let $\mathcal{G} = ((V, E), Win)$ be a game. Its winning condition Win is of type

- Reachability if $Win = V^* R V^\omega$ for some $R \subseteq V$
- Safety if $Win = S^\omega$ for some $S \subseteq V$
- Buchi if $Win = (V^* B)^\omega$ for some $B \subseteq V$
- co-Buchi if $Win = V^* C^\omega$ for some $C \subseteq V$

EXERCISE: show that the above are expressible with a parity condition

The controlled predecessor operator

We define a predicate transformer *Cpre* : $2^V \longrightarrow 2^V$ which given a subset $V' \subseteq V$ of positions in a game, returns the set of positions from which Player 0 can force the move towards un position in V'. Formally,

$$
\begin{gathered}
v \in Cpre(V') \\
\text{iff} \\
\text{either } v \in V_0, \text{ then } \exists v' \in V',\ v \longrightarrow v' \\
\text{or } v \in V_1 \text{ and } \forall v' \in V',\ v \longrightarrow v'
\end{gathered}
$$

Notice that *Cpre* is monotonic in the lattice $(2^V, \subseteq)$.

Algorithm for Safety

$\mathcal{G} = ((V, E), S^\omega)$ a safety game, with $S \subseteq V$

- To solve the game, we must compute the set of positions $W := W_0(\mathcal{G}) \subseteq V$ from which Player 0 can maintain the game within S for any number of rounds.
- Clearly $W \subseteq S$
- If W_i is the set of positions from which Player 0 can keep the game within S for i steps, then $W_{i+1} \subseteq W_i$and $W_{i+1} := W_i \cap Cpre(W_i)$.
- The predicate transformer $f : V' \rightarrow V' \cap Cpre(V')$ is montononic ($Cpre$ and $\cap$), hence (Tarski-Knaster 1955) it has fix-points.
- Notice that the set W is such that $W = f(W)$ and it is the greatest such one.
- It can be computed as the limit of the sequence $W_0 := V$ (the top element of the lattice) ; $W_{i+1} \subseteq S \cap W_i \cap Cpre(W_i)$
- Notice that the sequence stabilizes in at most $|S|$ steps.

Example Safety

Algorithms for Reachability

$\mathcal{G} = ((V, E), V^*RV^\omega)$ a reachability game, with $R \subseteq V$

- A direct algorithm where you compute the limit of

$$\begin{cases} W_0 := R \\ W_{i+1} := R \cup W_i \cup Cpre(W_i) \end{cases}$$

  It is the least fix-point of $V' \rightarrow R \cup V' \cup Cpre(V')$

- Other solution : Safety and Reachbility are dual. Use algorithm for safety but for Player 1 (define $\widetilde{Cpre}$)

From Parity to Safety

- A simple reduction to safety games (similar to [BJW02] but simpler, see also [BD08] - FSTTCS08 paper)
- $\mathcal{G} = (V, E, \chi : V \rightarrow \{0, d\})$ a parity game, and write n_c the number of positions with priority c. item We associate to each odd priority c a (bounded) counter $k(c)$ wich take values in $\{0, \dots, n_c\} \cup \{\infty\}$.
- Initially, all counters have value 0. The counter $k(c)$ is incremented when a position with priority c is visited, and it is reset when a position with an even priority $c' < c$ is visited.

Remark. As Buchi and co-Buchi are special cases of parity, they can be handled by this reduction too.

From Parity to Safety (2)

- Let $[n] = \{0, 1, \ldots, n\} \cup \{\infty\}$ and for $x \in [n]$, define $x \oplus 1 := \infty$ if $x \in \{n, \infty\}$, and $x + 1$ otherwise.
- Consider the safety game

$$\mathcal{H} = (V \times [n_1] \times [n_3] \times \ldots \times [n_d], H, V \times \{0, 1, \ldots, n\}^{\lceil d/2 \rceil})$$

where

$$H = \{((v, k), (v, \mathtt{update}(k, c))) \mid (v, v') \in E \text{ and } c = c(v')\}$$

and $\mathtt{update}((k_1, k_3, \ldots, k_d), c) :=$

$$\begin{cases} (k_1, \ldots, k_{c-1}, 0, 0, \ldots, 0) & \text{if } c \text{ is even} \\ (k_1, \ldots, k_{c-2}, k_c \oplus 1, k_{c+2}, \ldots, k_d) & \text{if } c \text{ is odd} \end{cases}$$

- Note that the size of $\mathcal{H}$ is in $O(n(n/d)^{\lceil d/2 \rceil})$

Automata on Infinite Objects

- We refer to [Tho90]
- Connection with Logic LTL, CTL* - membership and emptiness -
- Connection with Games
- Automata on words, trees, and graphs.

ω-automata

We refer to [GTW02, Chap. 1]

- Inputs are infinite words.
- Acceptance conditions: Buchi, Muller, Rabin and Streett, Parity
- All coincide with ω-regular languages ($L = \bigcup_i K_i R_i^\omega$)
- *LTL* corresponds to star-free languages

Automata on Infinite Trees

- Acceptance conditions: Buchi, Muller, Rabin and Streett, Parity on each branch of the input tree.
- Buchi tree automata are weaker [Rab70].
  [KSV96] L is recognizable by a nondeterministic Buchi word automaton but not by a deterministic Buchi word automaton iff $trees(L)$ is recognizable by a Rabin tree automaton and not by a Buchi tree automaton.
- Here we restrict to labeled full binary trees and to parity acceptance conditions, but the results generalize.

Non-deterministic Parity Tree Automata

- A (Σ-labeled full binary) tree t is an input to an automaton.
- In a current node in the tree, the automaton has to decide which state to assume in each of the two successor nodes.
- $\mathcal{A} = (Q, \Sigma, q^0, \delta, c)$ where
  - $Q(\ni q^0)$ is a finite set of states (q^0 the initial state)
  - $\delta \subseteq Q \times \Sigma \times Q \times Q$ is the transition relation
  - $c : Q \to \{0, \ldots, k\}$, $k \in \mathbb{N}$ is the coloring function which assigns the index values (colors) to each states of $\mathcal{A}$

Runs

- A run of $\mathcal{A}$ on input tree $t \in Trees(\Sigma)$ is a tree $\rho \in Trees(Q)$ satisfying
  - $\rho(\epsilon) = q^0$, and
  - for every node $w \in \{0,1\}^*$ of t (and its sons $w0$ and $w1$), we have

$$(\rho(w0), \rho(w1)) \in \delta(\rho(w), t(w))$$

- A run ρ is accepting (successful) iff every path $\pi \in \{0,1\}^\omega$ of the tree ρ satisfies the acceptance condition (often written $\pi \in Acc$) is satisfied.
  Here we consider parity, where $\pi \in Acc$ whenever among the colors that occur infinitely often along π path, the ninimal one is even.
- A tree t is accepted by $\mathcal{A}$ iff there exists an accepting run of $\mathcal{A}$ on t.
- The tree language recognized by $\mathcal{A}$ is $L(\mathcal{A}) = \{t \mid t \text{ is accepted by } \mathcal{A}\}$

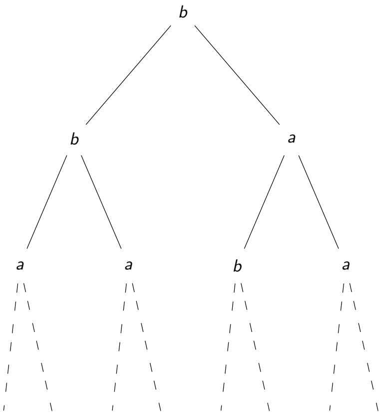
Example of an parity automaton

- Let $L_0 = \{t \in Trees(\{a, b\}) | t \models \mathbf{A}\,\mathbf{F}a\}$ be the set of trees which have an a-labeled node on each branch.
- Consider the automaton whose states are q (initial and for "awaiting an a") and $\top$ (for "I have seen an a"), with $c(q) = 1$ and $c(\top) = 0$, and whose transitions are defined by:

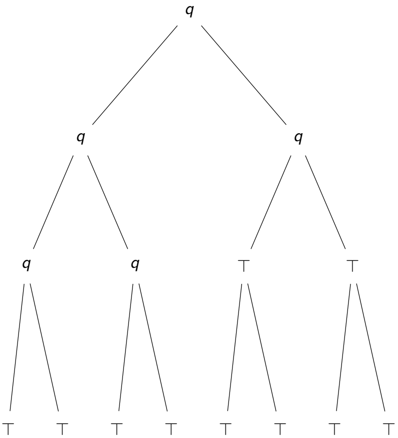
$$
\begin{array}{rclcrcl}
\delta(q, a) & = & \{(\top, \top)\} & \qquad & \delta(q, b) & = & \{(q, q)\}(\text{where } b \neq a) \\
\delta(\top, a) & = & \{(\top, \top)\} & \qquad & \delta(\top, b) & = & \{(\top, \top)\}
\end{array}
$$

Example of a run

A tree t



A run ρ on t



Other Acceptance Conditions

- Buchi is specified by a set $F \subset Q$

$$Acc = \{\pi \mid Inf(\pi) \cap F \neq \emptyset\}$$

- Muller is specified by a set $\mathcal{F} \subseteq \mathcal{P}(Q)$,

$$Acc = \{\pi \mid Inf(\pi) \in \mathcal{F}\}$$

- Rabin is specified by a set $\{(R_1, G_1), \ldots, (R_k, G_k)\}$ where $R_i, G_j \subseteq Q$,

$$Acc = \{\pi \mid \forall i, Inf(\pi) \cap R_i = \emptyset \text{ and } Inf(\pi) \cap G_i \neq \emptyset\}$$

- Streett is specified by a set $\{(R_1, G_1), \ldots, (R_k, G_k)\}$ where $R_i, G_j \subseteq Q$,

$$Acc = \{\pi \mid \forall i, Inf(\pi) \cap R_i = \emptyset \text{ or } Inf(\pi) \cap G_i \neq \emptyset\}$$

For the relationship between these conditions see [GTW02].
In the following, when the definition and results apply to any acceptance conditions presented so far (including parity condition), we simply denote by *Acc* this condition.

Example 2

- Let $L_a^\infty = \{t \in \mathit{Trees}(\{a,b\} | t \models \mathbf{E} \overset{\infty}{\mathbf{F}} a\}$ be the set of trees having a path with infinitely many a's
- Consider the automaton with states $q_a, q_b, \top$ and transitions ($*$ stands for either a or b)

$$\begin{array}{rcl} \delta(q_*, a) & = & \{(q_a, \top), (\top, q_a)\} \\ \delta(q_*, b) & = & \{(q_b, \top), (\top, q_b)\} \\ \delta(\top, *) & = & \{(\top, \top)\} \end{array}$$

and coloring $c(q_b) = 1$ and $c(q_a) = c(\top) = 0$
(this a Buchi condition, only 0 and 1 colors)

Example 2 (Cont.)

$\delta(q_*, a) = \{(q_a, \top), (\top, q_a)\}, \delta(q_*, b) = \{(q_b, \top), (\top, q_b)\}, \delta(\top, *) = \{(\top, \top)\}$

- From state $\top$, $\mathcal{A}$ accepts any tree.
- Any run from q_a consists of a single path labeled with states q_a, q_b (whereas the rest of the run tree is labeled with $\top$). There are infinitely many states q_a on this path iff there are infinitely many vertices labeled by a.

Exercises

- Find a (parity) tree automaton which recognizes the set of trees (over $\{a, b\}$) such that each branch belongs to $a^\omega \cup (a+b)^* b^\omega$.
- Find a Muller automaton which recognizes the complement of L_a^∞ (finitely many a's on each branch).

Regular Tree Languages and Properties

Theorem

A tree language $L \subseteq Trees(\Sigma)$ is regular iff there exists a parity (Muller, Rabin, Streett) tree automaton which recognizes L.

There exists effective transformations between the different kinds of tree automata. Take you favorite one as for a definition of a "regular" tree language. ⚠ Buchi tree automata are weaker ($\neq$ word automata): The complement of L_a^∞ (finitely many a's on each branch) is not recognizable by any Buchi tree automaton.

Theorem

Parity tree automata are closed under union, projection, and complementation.

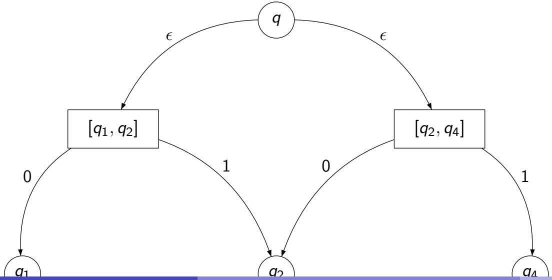
Union, intersection and projection is easy on Muller tree automata. The proof for complementation uses the determinization result for word automata. It is a difficult proof [Rab70], but using games [GTW02, Chap. 8] things get much faster. ⚠ Tree automata cannot be determinized:

Alternating Tree Automata: Motivaations

- Design an automaton for the language $\{t \in \mathit{Trees}(\{a,b,c\}) \mid t \models \mathbf{A}\,\mathbf{F}a \wedge \mathbf{A}\,\mathbf{F}b \wedge \mathbf{A}\,\mathbf{F}c\}$
- Quite difficult to design with a non-deterministic tree automaton (combinatorics between the occurrences of a and b and c) but easy to write as
$$\delta(q,*) = (q_a,\epsilon) \wedge (q_b,\epsilon) \wedge (q_c,\epsilon)$$
where q_a (resp. q_b, q_c) is the initial states of the automaton for $\mathbf{A}\,\mathbf{F}a$ (resp. $\mathbf{A}\,\mathbf{F}b$, $\mathbf{A}\,\mathbf{F}c$).
- The automaton splits into three "copies" checking in parallel $\mathbf{A}\,\mathbf{F}a$, $\mathbf{A}\,\mathbf{F}b$, and $\mathbf{A}\,\mathbf{F}c$.

What is Alternation?

- Recall for ND tree automata that *e.g.* $\delta(q, a) = \{(q_1, q_2), (q_2, q_4)\}$ to mean that from state q and node w in the input tree (with $t(w) = a$): the automaton (1) non-deterministically chooses between the two "disjuncts" $[q_1, q_2]$ and $[q_2, q_4]$, and (2) proceeds accordingly to the left and right sons of w in t.
- We extend the non-deterministic tree automaton with a notion of universal moves (similar to alternating Turing machines extend non-deterministic Turing machines).



What is Alternation (con't)?

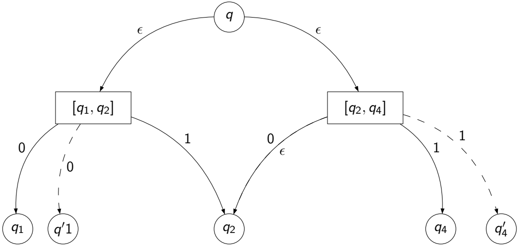
- In the transitions relation, we allow positive Boolean combinations of terms (q, d), $d \in \{0, 1, \epsilon\}$:
  - For non-deterministic automata, we had
    $\delta(q, a) = (q_1, 0) \wedge (q_2, 1) \vee (q_2, 0) \wedge (q_4, 1)$
  - Now we can write things like
    $\delta(q, a) = (q_1, 0) \wedge (q_1', 0) \wedge (q_2, 1) \vee (q_2, 0) \wedge (q_4, 1) \wedge (q_5, \epsilon)$

  Notice that different "copies" of he automaton can proceed along the same subtree, e.g. $\mathcal{A}, q_1$ and $\mathcal{A}, q_1'$ on the left subtree of nodes labeled by a.

Clearly, Alternating Tree Automata extend Non-deterministic Tree Automata

Example

$\delta(q, a) = (q_1, 0) \wedge (q_1', 0) \wedge (q_2, 1) \vee (q_2, 0) \wedge$



$(q_4, 1)$

Decision problems on ATA

- We use parity games to define the semantics of ATA
- Parity games provide a straightforward construction to complement ATA (parity accpetance). Determinacy of games gives the correction of this construction.
- We use parity games to show the decidability of the membership problem (for emptiness see [GTW02, Chap. 9]).
- We will see that ATA have a logical counter part: the μ-calculus, an extension of modal logic with fix-points.

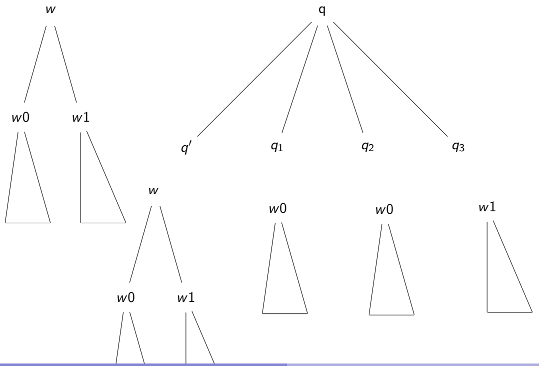
Formal Definition of ATA

An alternating tree automaton is $\mathcal{A} = (Q, Q^{\exists}, Q^{\forall}, \Sigma, q^0, \delta, Acc)$

- $\{Q^{\exists}, Q^{\forall}\}$ is a partition of Q
- $\delta : Q \times \Sigma \to \mathcal{P}(Q \times \{0, 1, \epsilon\})$ is a function and ϵ-transitions are allowed.

We can write $\delta(q, a) = (q', \epsilon) \wedge (q_1, 0) \wedge (q_2, 0) \wedge (q_3, 1) \vee ...$

We could give the semantics in terms of runs, as before, but the runs are tree with possibly a degree > 2



Semantics of ATA

Runs and Acceptance of the automaton are formalized in terms of two-player games.

- Given a tree $t \in \mathit{Trees}(\Sigma)$, we define the acceptance game $\mathcal{G}(\mathcal{A}, t)$ by:
- $V_0 = \{0,1\}^* \times Q^{\exists}$
- $V_1 = \{0,1\}^* \times Q^{\forall}$
- From each position (w, q) and $(q', d) \in \delta(q, t(w))$, there is an edged to (wd, q')
- The acceptance condition *Acc* consists of the sequences $(w_0, q_0)(w_1, q_1)\dots$ such that the sequence $q_0 q_1 \dots$ is in *Acc*
- $\mathcal{A}$ accepts a tree t iff Player 0 has a winning strategy in $\mathcal{G}(\mathcal{A}, t)$

Alternating Tree Automata over Kripke Structures

Follow the same lines

- Consider a rooted Kripke Structure $(\mathcal{S}, s^0)$ (which unfolds as a tree)
- Define $\mathcal{G}(\mathcal{A}, (\mathcal{S}, s^0))$ as for trees, but notice that if $\mathcal{S}$ is finite so is $\mathcal{G}(\mathcal{A}, (\mathcal{S}, s^0))$
- $\mathcal{A}$ accepts $(\mathcal{S}, s^0)$ iff Player 0 has a winning strategy in $\mathcal{G}(\mathcal{A}, (\mathcal{S}, s^0))$

Properties of ATA

- Closed under disjunction and conjunction
- Closed under negation (complementation), see proof next slide
- Unfortunately, it is difficult to show that alternating automata are closed under projection. Muller and Schupp showed that

Theorem

(Simulation Theorem) [MS95]
Any alternating tree automaton is equivalent to a non-deterministic tree automaton (with an exponential blow up in the number of states).

Complementation of Alternating Parity Tree Automata

Lemma

For every alternating parity tree automaton $\mathcal{A}$ there is a dual parity tree automaton $\bar{\mathcal{A}}$ such that $L(\bar{\mathcal{A}}) = Trees(\Sigma) \setminus L(\mathcal{A})$. Moreover, regarding size, $|\bar{\mathcal{A}}| = |\mathcal{A}|$

Proof $\mathcal{A} = (Q, Q^\exists, Q^\forall, \Sigma, q^0, \delta, Acc) \rightsquigarrow \bar{\mathcal{A}} = (Q, Q^\forall, Q^\exists, \Sigma, q^0, \delta, \bar{c})$ where $\bar{c}(q) = c(q) + 1$ for every $q \in Q$. Now, compare $\mathcal{G}(\mathcal{A}, t)$ and $\mathcal{G}(\bar{\mathcal{A}}, t)$:

- Same graph but positions of Player 0 become positions of Player 1, and vice versa.
- For every infinite play π, π is winning for Player 0 in $\mathcal{G}(\mathcal{A}, t)$ iff π is winning for Player 1 in $\mathcal{G}(\bar{\mathcal{A}}, t)$.
  Hence Player 0 has a winning strategy in $\mathcal{G}(\mathcal{A}, t)$ iff Player 1 has a winning strategy in $\mathcal{G}(\bar{\mathcal{A}}, t)$ (same strategy).
- So, $t \in L(\mathcal{A})$ iff $t \notin L(\bar{\mathcal{A}})$

Decision Problems

- the Membership Problem: given an ATA $\mathcal{A}$ and a tree t, does $t \in L(\mathcal{A})$? (see next slide)
- the Emptiness Problem: given an ATA $\mathcal{A}$, is $L(\mathcal{A}) = \emptyset$?

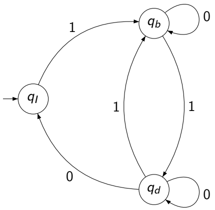
the Membership Problem

$\mathcal{A} = (Q, Q^{\exists}, Q^{\forall}, \Sigma, q^0, \delta, c)$, k colors, and $t \in Trees(\Sigma)$, does $t \in L(\mathcal{A})$?

- t is regular, as the unravelling of some finite Kripke Structure $(\mathcal{S}, s^0)$.
- Build the finite parity game $\mathcal{G}(\mathcal{A}, (\mathcal{S}, s^0))$ and solve it (decidable).
- The size of $\mathcal{G}(\mathcal{A}, (\mathcal{S}, s^0))$: $|Q| \times |S|$ positions and k priorities
- Complexity in NP $\cap$ co-NP (as for parity games)

Exercise

- Remember the automaton for the property $\mathbf{E} \overset{\infty}{\mathbf{F}} b$ on Slide 57
- Design an ATA $\mathcal{A}$ which accepts
  $\{t \in \mathit{Trees}(\{a,b,c\}) \mid t \models \mathbf{E} \overset{\infty}{\mathbf{F}} b \wedge \mathbf{E} \overset{\infty}{\mathbf{F}} d\}$
- Consider the following regular model $\mathcal{T}$, where q_I satisfies neither b nor d, q_b (resp. q_d) satisfies b (resp. d) and not d (resp. b)



- Build $\mathcal{G}(\mathcal{A}, (\mathcal{T}, q_I))$ and decide whether $\mathcal{T}, s^0 \models \mathbf{E} \overset{\infty}{\mathbf{F}} b \wedge \mathbf{E} \overset{\infty}{\mathbf{F}} d$

the Emptiness Problem

Theorem

The Emptiness Problem for parity ATA (i.e. "Do we have $L(\mathcal{A}) = \emptyset$?") is EXPTIME-complete

- First method: Convert the ATA into a NTA (Simulation Theorem) and solve the Emptiness Problem for non-deterministic tree automata.
- Second method: Based on Parity Games on Tiles (see [GTW02, Chap. 9]).

We look at a polytime algorithm to solve the Emptiness of Non-deterministic Tree Automaton

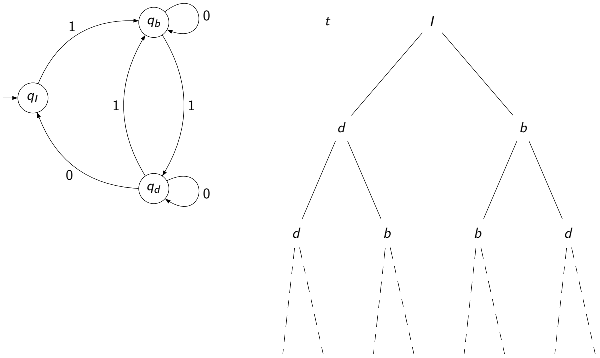
Emptiness of Non-deterministic Tree Automaton

We need consider the method used in [GTW02, Chap. 8]

- Regular trees generated by deterministic finite-state automata with an input function
- Input-free automata
- Relate deterministic finite-state automata and input-free automata
- Relate input-free automata and parity games (runs and strategies)

Regular Trees generated by deterministic finite-state Automata with an input function

- $A = (Q, \{0,1\}, \Delta, q_I, f)$ a finite automaton
- $f : Q \to \Sigma'$ an output function
- It generates the tree such that $t(w) = f(\Delta(q_I, w))$



Input-free Automata

- An input-free (IF) automaton is $\mathcal{B} = (Q, \delta, q_I, Acc)$ where $\delta \subseteq Q \times Q \times Q$; we may remove Acc.
- Runs are defined as usual: they are binary trees
- By determinism, the run is unique. Moreover it is regular since

Recall

t is regular iff $\{t^u \mid u \in \{0,1\}^*\}$ is finite, where $t^u(v) = t(uv)$

Deterministic Finite Automata on $\{0,1\}$ and Deterministic IF Automata (without Acc)

- Let $A = (Q, \{0,1\}, \Delta, q_I, f : Q \to \Sigma')$ be a deterministic Finite Automata on $\{0,1\}$.
- Define the IF automaton $\mathcal{B}_A = (Q \times \Sigma', \delta, (q_I, f(q_I)))$ by: $\forall q \in Q$,

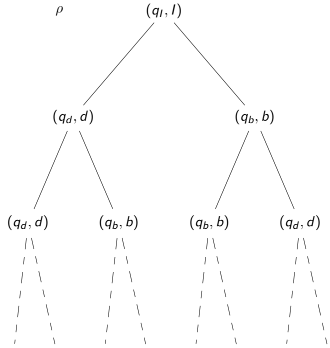
$$((q, f(q)), (\Delta(q,0), f(\Delta(q,0))), (\Delta(q,1), f(\Delta(q,1))) \in \delta$$

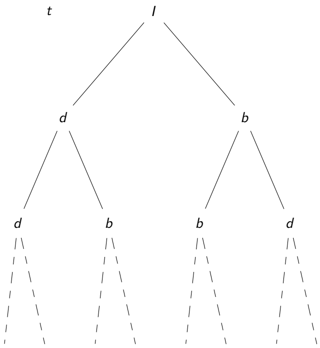
Lemma

$\mathcal{B}_A$ is deterministic and a run of $\mathcal{B}_A$ generates (in the second component of its nodes' label) the very tree that A generates.

Example

Consider the deterministic finite-state automaton A of Slide 80. Then, $\mathcal{B}_A$ has states $\{(q_I, I), (q_b, b), (q_d, d)\}$ and transitions $((q_I, I), (q_d, d), (q_d, d))$, $((q_d, d), (q_d, d), (q_d, d))$, $((q_d, d), (q_d, d), (q_d, d))$. (q_I, I) is intial





Lemma

For each parity automaton $\mathcal{A}$ there exists an IF automaton $\mathcal{A}'$ such that $L(\mathcal{A}) \neq \emptyset$ iff $\mathcal{A}'$ admits a successful run.

Proof.

$\mathcal{A} = (Q, \Sigma, q^0, \delta, c)$ and define $\mathcal{A}' = (Q \times \Sigma, \{q_I\} \times \Sigma, \delta', c')$.
$\mathcal{A}'$ will guess non-deterministically the second component of its states (i.e. the labeling of a model).
Formally,

- for each $(q, a, q', q") \in \delta$, we generate $((q, a), (q', x), (q", y)) \in \delta'$, if $(q', x, p, p'), (q", y, r, r') \in \delta$ for some $p, p', q, q' \in Q$
- $c'(q, a) = c(q)$

It is easy to verify that the lemma holds for this construction (Exercise). $\square$

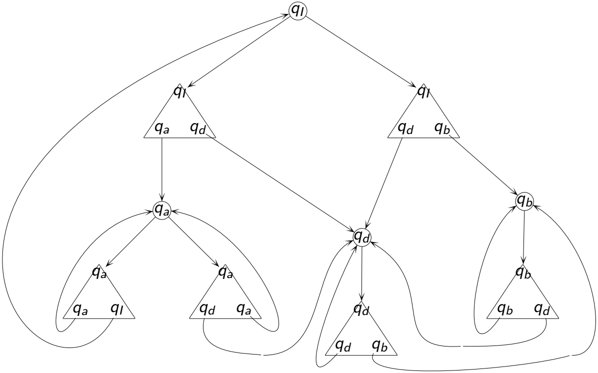
From IF Automata to Parity Games

Given $\mathcal{A} = (Q, \Sigma, q^0, \delta, c)$ an IF automaton, we build a parity game $\mathcal{G}_{\mathcal{A}} = ((V, E), \chi)$ as follows.

- Positions $V_0 = Q$ and $V_1 = \delta$
- Moves for all $(q, q', q'') \in \delta$
  - $(q, (q, q', q"")) \in E$
  - $((q, q', q""), q'), ((q, q', q""), q"") \in E$
- Priorities $\chi(q) = c(q) = \chi((q, q', q""))$

Example of $\mathcal{G}_{\mathcal{A}}$

Consider the IF automaton $\mathcal{A}$ with states $Q = \{q_I, q_a, q_b, q_d\}$, initial state q_I and transition relation
$\Delta = \{(q_I, a_a, q_b, q_d), (q_I, q_d, q_b), (q_a, q_d, q_a), (q_d, q_d, q_b), (q_b, q_b, q_d))\}$



From IF Automata to Parity Games

Given $\mathcal{A} = (Q, \Sigma, q^0, \delta, c)$ an IF automaton, we build a parity game $\mathcal{G}_\mathcal{A} = ((V, E), \chi)$ as follows.

- Positions $V_0 = Q$ and $V_1 = \delta$
- Moves for all $(q, q', q'') \in \delta$
  - $(q, (q, q', q''))) \in E$
  - $((q, q', q''), q'), ((q, q', q''), q'') \in E$
- Priorities $\chi(q) = c(q) = \chi((q, q', q''))$

Notice that $\mathcal{G}_\mathcal{A}$ has a finite number of positions.

Lemma

(Winning) Strategies of Player 0 and (successful) runs of $\mathcal{A}$ correspond.

The Emptiness Problem for Nondeterministic Tree Automata

Theorem

For parity tree automata it is decidable whether their recognized language is empty or not.

Proof.

$\mathcal{A} \rightsquigarrow \mathcal{A}'$ an IF automaton $\rightsquigarrow \mathcal{G}_{\mathcal{A}'}$, and solve $\mathcal{G}_{\mathcal{A}'}$. $\square$

Finite Model Property

Corollary

If the language of a parity tree automaton is not empty, then it contains a regular tree.

Proof.

Take $\mathcal{A}$ and its corresponding IF automatan $\mathcal{A}'$. Assume a successful run of $\mathcal{A}'$, that is a memoryless strategy f for Player 0 in $\mathcal{G}_{\mathcal{A}'}$ from some position (q_I, a).
The subgraph $\mathcal{G}_{\mathcal{A}'_f}$ induces a deteministic IF automaton $\mathcal{A}$" (without Acc) by extracting the transitions out of $\mathcal{G}_{\mathcal{A}_f}$ from positions in V_1.
$\mathcal{A}$" is clearly a subautomaton of $\mathcal{A}'$. It $\mathcal{A}$" generates a regular tree t in the second component of its states. Now, $t \in L(\mathcal{A})$ because $\mathcal{A}'$ behaves like $\mathcal{A}$. See details in [GTW02, Chap. 8]. □

Complexity results

Corollary

The Emptiness Problem for parity non-deterministic tree automata is in $NP \cap co\text{-}NP$.

Proof.

Immediate since the size of $\mathcal{G}_{\mathcal{A}'}$ is polynomial in the size of $\mathcal{A}$ (see [GTW02, p. 150, Chap. 8]) $\square$

Important remark: the Universality problem is EXPTIME-complete (already for finite trees).

The Mu-calculus

Bibliography

- invented by Dana Scott and Jaco de Bakker, and further developed by Dexter Kozen [Koz83]
- D. Kozen.
  Results on the propositional μ-calculus. Theoretical Computer Science, 27(3):333-354, 1983.
- A. Arnold and D. Niwinski.
  Rudiments of mu-calculus. North-Holland, 2001.
- E. A. Emerson and C. S. Jutla.
  Tree automata, mu-calculus and determinacy. In Proceedings 32nd Annual IEEE Symp. on Foundations of Computer Science, FOCS'91, San Jose, Puerto Rico, 1-4 Oct 1991, pages 368-377. IEEE Computer Society Press, Los Alamitos, California, 1991.

Syntax of L_μ

- Alphabet Σ and Propositions $Prop = \{P_a\}_{a \in \Sigma}$
- Variables $Var = \{Z, Z', \dots\}$
- Formulas

$$\beta, \beta' \in L_\mu ::= P_a \mid Z \mid \neg\beta \mid \beta \wedge \beta' \mid \langle 0 \rangle \beta \mid \langle 1 \rangle \beta \mid \mu Z.\beta$$

  where $Z \in Var$.
- Use usual abbreviations $\mathtt{false} = P_a \wedge \neg P_a$ (for some P_a), $\vee$, $\mathtt{true} = P_a \vee \neg P_a$, $\Rightarrow$, etc.
- Define $\langle\ \rangle\beta = \langle 0 \rangle \beta \vee \langle 1 \rangle \beta$ and $[\ ]\beta = \neg\langle\ \rangle\neg\beta$
- Well-formed formulas: for every formula $\mu Z.\beta$, Z appears only under the scope of an even number of $\neg$ symbols in β.
- β is a sentence if all variables in β are bounded by a μ operator.
- Write $\beta' \leq \beta$ when β' is a subformula of β.

Semantics of L_μ

- Assume given a tree $t \in \mathit{Trees}(\Sigma)$ and a valuation $\mathit{val} : \mathit{Var} \to 2^{\{0,1\}^*}$ of the variables.
- For every $N \subseteq \{0,1\}^*$, we write $\mathit{val}[N/Z]$ for val' defined as val except that $\mathit{val}'(Z) = N$
- We define $[\![\beta]\!]^t_{val} \subseteq \{0,1\}^*$ by:

$$
\begin{array}{lcl}
[\![Z]\!]^t_{val} & = & \mathit{val}(Z) \\
[\![P_a]\!]^t_{val} & = & t^{-1}(a) \qquad \rightsquigarrow \text{ we may write } a \text{ instead of } P_a \\
[\![\beta \wedge \beta']\!]^t_{val} & = & [\![\beta]\!]^t_{val} \cap [\![\beta']\!]^t_{val} \\
[\![\langle 0 \rangle \beta]\!]^t_{val} & = & \{w \in \{0,1\}^* \mid w0 \in [\![\beta]\!]^t_{val}\} = \mathit{pre}_0([\![\beta]\!]^t_{val}) \\
[\![\langle 1 \rangle \beta]\!]^t_{val} & = & \{w \in \{0,1\}^* \mid w1 \in [\![\beta]\!]^t_{val}\} = \mathit{pre}_1([\![\beta]\!]^t_{val}) \\
[\![\mu Z.\beta]\!]^t_{val} & = & \bigcap\{S' \subseteq S \mid [\![\beta]\!]^t_{val[S'/Z]} \subseteq S'\}
\end{array}
$$

As a consequence $[\![\mathtt{false}]\!]^t_{val} = \emptyset$ and $[\![\mathtt{true}]\!]^t_{val} = \{0,1\}^*$

Also $[\![[\,]\beta]\!]^t_{val}$ is the set of nodes which both children belong to $[\![\beta]\!]^t_{val}$

About Fix-points

- $\mu Z.\beta$ denotes the least fix-point of

$$\tau : 2^{\{0,1\}^*} \longrightarrow 2^{\{0,1\}^*}$$
$$\tau(N) = [\![\, \beta \,]\!]^t_{val[N/Z]}$$

  By the assumption on "positive" occurrences of Z in β, τ is monotonic: $N' \subseteq N$ implies $\tau(N') \subseteq \tau(N)$ (prove it). Henceforth, since $(2^{\{0,1\}^*}, \emptyset, \{0,1\}^*, \subseteq)$ is a complete lattice, by [Tar55], the least fix-point (and the greatest fix-point) exists.

- Let $\nu Z.\beta \stackrel{\text{def}}{=} \neg\mu Z.\neg\beta[\neg Z/Z]$. It is a greatest fix-point.

Tarski-Knaster Theorem

Theorem

(Tarski-Knaster) Assume a set D. Let $\tau : 2^D \to 2^D$ be monotonic, then

$$\mu z.\tau(z) = \cap\{z \mid \tau(z) = z\} = \cap\{z \mid \tau(z) \subseteq z\}$$
$$\nu z.\tau(z) = \cup\{z \mid \tau(z) = z\} = \cup\{z \mid \tau(z) \supseteq z\}$$

$\mu z.\tau(z) = \cup_i \tau^i(\emptyset)$, *where i ranges over all ordinals of cardinality at most the state space D; when D is finite, $\mu z.\tau(z)$ is the union of the following ascending chain $\emptyset \subseteq \tau(\emptyset) \subseteq \tau^2(\emptyset)\ldots$*

$\nu z.\tau(z) = \cap_i \tau^i(D)$, *where i ranges over all ordinals of cardinality at most the state space D; when D is finite, $\nu z.\tau(z)$ is the intersection of the following descending chain $D \supseteq \tau(D) \supseteq \tau^2(D)\ldots$*

Therefore, if t is regular, i.e. representing the unravelling of a finite rooted KS $(\mathcal{S}, s)$, the fix-points can be effectively computed.

"Easy" L_μ formulas

- Evaluate $\mu Z.Z$ in a tree t. Iterate the predicate transformer (from the bottom element $\emptyset$ of the lattice $2^{\{0,1\}}$):

$$id : 2^{\{0,1\}^*} \longrightarrow 2^{\{0,1\}^*}$$
$$id(N) = [\![Z]\!]^t_{val[N/Z]}$$

  We get the fixed point immediately since $id(\emptyset) = \emptyset$. Therefore $\mu Z.Z$ is equivalent to `false`

- Similarly $\nu Z.Z$ is equivalent to `true` (iterate *id* from the top element $\{0,1\}^*$ of the lattice)
- $\mu Z.P$ and $\nu Z.P$ are equivalent to P
- $\mu Z.\langle 0\rangle Z$ is the least fixed point of $\tau : 2^{\{0,1\}^*} \longrightarrow 2^{\{0,1\}^*}$ with $\tau(N) = pre_0(N)$. That is $\emptyset$.

Other L_μ formulas

- $\mu Z.b \vee \langle\,\rangle Z$ is the least fixed point of $\tau : 2^{\{0,1\}^*} \rightarrow 2^{\{0,1\}^*}$ with $\tau(N)$ is the set of nodes that are labeled by b (write it b) or which have a child that belongs to N. Write $pre(N)$ for the set $\{w \mid w0 \in N \vee w1 \in N\}$.
  We iterate from the least element $\emptyset$ of the lattice:

iteration 0	$\emptyset$
iteration 1	$\tau(\emptyset) = b$
iteration 2	$\tau^2(\emptyset) = b \cup pre(b)$
iteration 3	$\tau^3(\emptyset) = b \cup pre(b \cup pre(b)) = b \cup pre(b) \cup pre^2(2)$
	$\ldots$

  iteration $n+1$

  $$\tau^n(\emptyset) = b \cup pre(b) \cup pre^2(2) \cup pre^3(3) \cup \ldots \cup pre^{n-1}(n-1)$$

  At the end, we get the set $\bigcup_{n \geq 0} pre^n(n)$, that is the nodes which have a finite path to a node labeled by b. In CTL this is written $\mathbf{E}\,\mathbf{F}b$.
- $\nu Z.\langle\,\rangle Z$ is the greatest fixed point of $\tau : 2^{\{0,1\}^*} \rightarrow 2^{\{0,1\}^*}$, where $\tau(N) = pre(N)$. Since very node has a child, $\nu Z.\langle\,\rangle Z$ is the set of all nodes, that is $\{0,1\}^*$, henceforth $\nu Z.\langle\,\rangle Z$ is equivalent to

Intuitively, μ (resp. ν) correspond to finite (resp. infinite) computations.

- Exercises:
  - We have seen that the CTL formula **E F** b expresses as $\mu Z.P_b \vee \langle\ \rangle Z$.
  - Find out how you can express the CTL formula **E** a **U** b. Justify your answer.
  - Same for **AG** a. Justify your answer.
- How can you write "infinitely often a along the branch 0^ω?
  Answer: $\nu Z.\mu Z'.(Z \wedge a \vee \langle 0\rangle Z')$
  This is not expressible in CTL.
- How do you write "all along any branch there is a possibility to reach a node labeled by a"?
  Answer: $\nu Z.(\mu Z'.(a \vee \langle\ \rangle Z') \wedge [\ ]Z)$
  Note that this is not expressible in LTL.

Miscellaneous

- We can push negation inside a formula (notice that $\neg\langle d\rangle\beta = \langle d\rangle\neg\beta$, for $d \in \{0,1\}$) to get a formula in positive normal form.

- Write $t \models \beta$ whenever $\epsilon \in [\![\beta]\!]^t_{val}$.
- Notice that when β is a sentence (no free variables), $[\![\beta]\!]^t_{val}$ is independent of *val*, and we may write $[\![\beta]\!]^t$

From the Mu-calculus to Alternating Tree Automata

Theorem

Given a sentence $\beta \in L_\mu$ (in positive normal form), we can construct in polynomial time an ATA $\mathcal{A}_\beta$ such that

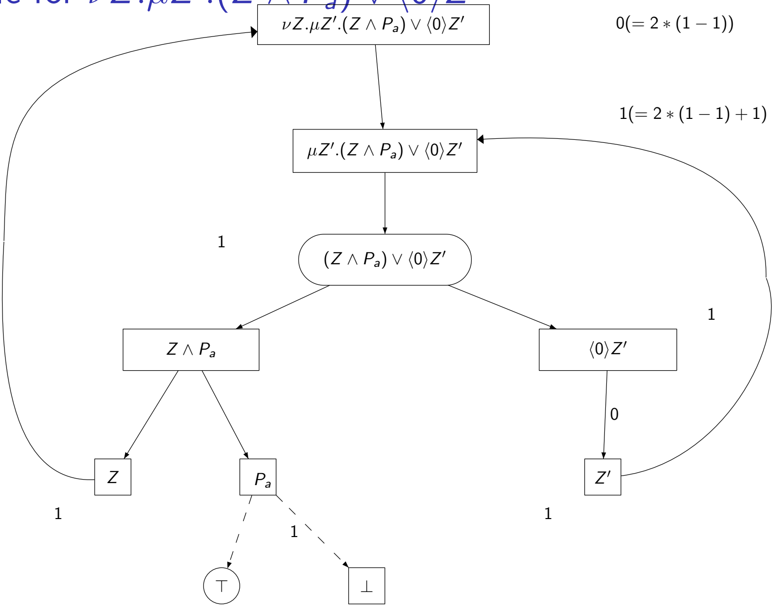
$$L(\mathcal{A}_\beta) = \{t \in \mathit{Tree}(\Sigma) \mid t \models \beta)\}$$

Moreover, $|\mathcal{A}_\beta| = O(|\beta|)$.

Corollary

The Model-checking problem and the Satisfiability problem for the Mu-calculus reduce to the membership and emptiness problems for ATA respectively.

Example for $\nu Z.\mu Z'.(Z \wedge P_a) \vee \langle 0 \rangle Z'$



Alternation Depth

Let $\beta \in L_\mu$ be in postive normal form. We define $ad(\beta)$, the alternation depth of β inductively by:

- $ad(P_a) = ad(\neg P_a) = 0$
- $ad(\beta \wedge \beta') = ad(\beta \vee \beta') = max\{ad(\beta), ad(\beta')\}$
- $ad(\langle d \rangle \beta) = ad(\beta)$, for $d \in \{0, 1\}$
- $ad(\mu Z.\beta) = max(\{0, ad(\beta)\} \cup \{ad(\nu Z'.\beta') + 1 \,|\, \nu Z'.\beta' \leq \beta, Z \in free(\nu Z'.\beta')\})$
- $ad(\nu Z.\beta) = max(\{0, ad(\beta)\} \cup \{ad(\mu Z'.\beta') + 1 \,|\, \nu Z'.\beta' \leq \beta, Z \in free(\mu Z'.\beta')\})$

For example, $ad(\nu Z.\mu Z'.(Z \wedge a) \vee \langle 0 \rangle Z') = 1$

The construction of $\mathcal{A}_\beta$

$\mathcal{A}_\beta \stackrel{\text{def}}{=} (Q, \Sigma, q^0, \delta, c)$ where

- $Q = \{\alpha \,|\, \alpha \leq \beta\} \cup \{\top, \bot\}$ and $q_I = \beta$
- $Q^\exists$ is composed of all subformulas of the form $\alpha \vee \alpha'$, $Q^\forall$ contains the rest.
- $\delta : Q \times \Sigma \to \mathcal{P}(Q \times \{0, 1, \epsilon\})$ is defined by induction over $\alpha \in Q$:
  - $\delta(P_a, a) = \{(\top, \epsilon)\}$ and $\delta(P_a, b) = \{(\bot, \epsilon)\}$ for all $b \neq a$
  - $\delta(\neg P_a, a) = \{(\bot, \epsilon)\}$ and $\delta(\neg P_a, b) = \{(\top, \epsilon)\}$ for all $b \neq a$
  - $\delta(Z, a) = \{\beta_Z, \epsilon)\}$
  - $\delta(\alpha \wedge \alpha') = \{(\alpha, \epsilon), (\alpha', \epsilon)\}$ and the same for $\delta(\alpha \vee \alpha')$
  - $\delta(\langle d \rangle \alpha) = (\alpha, d)$, for $d \in \{0, 1\}$
  - $\delta(\theta Z.\alpha) = (\alpha, \epsilon)$, for $\theta \in \{\mu, \nu\}$
- The coloring function c is defined by (let $M = ad(\beta)$)
  - $c(\alpha) = 2 * (M - ad(\alpha))$ if α is a ν-formula
  - $c(\alpha) = 2 * (M - ad(\alpha)) + 1$ if α is a μ-formula
  - $c(\alpha) = M$ if α is not a fix-point formula.

For the correctness of the construction see [GTW02, Chap. 10].

From Alternating Tree Automata to the Mu-calculus

Theorem

Given a an ATA $\mathcal{A}$, there exist a formula $\beta_\mathcal{A} \in L_\mu$ such that

$$\{t \in \mathit{Tree}(\Sigma) \mid t \models \beta_\mathcal{A}\}) = L(\mathcal{A})$$

Moreover, $|\beta_\mathcal{A}| = O(|\mathcal{A}|)$.

The translation from Alternating Parity Tree Automata to the Mu-calculus uses vectorial Mu-calulus, see [AN01]. We omit the proof here.

Summary

- The Mu-calculus and Alternating Parity Tree Automata have the same expressive power.
- Complexity results:
  - The satisfiability problem for the Mu-calculus is EXPTIME-complete ([SE89, EJ88]).
  - The model-checking for the Mu-calculus is NP $\cap$ co-NP; it is open whether it is in P.
- The Mu-calculus subsumes every temporal logics.
  - CTL translates into the alternation free fragment of the Mu-calculus. It has a polynomial time model-checking procedure (retrieve why according to previous results).
  - CTL* can be translated into the Mu-calculus [Dam94], but there is an exponential blow-up. Notice that you cannot express $\nu Z.a \wedge [\,][\,]Z \in L_\mu$ saying that "*a* is true at every even moment of all computations".

Mu-calculus and Parity Games

We have seen a reduction from the Model-checking Problem of the Mu-calculus to Parity Games (via Automata), but there is a reduction in the reverse direction.

A parity game $\mathcal{G}, V_0, V_1, E)$ with a priority function $\chi : V \to \{0, \dots, k-1\}$ (k priorities) can be seen as a Kripke Structure (V, E, λ) where λ maps states onto the set of propositions $\{V_0, V_1, P_0, \dots, P_k\}$ where $P_i = \{v \mid \chi(v) = i\}$.
The formula

$$Win_k \stackrel{\text{def}}{=} \nu Z_0.\mu Z_1.\dots.\theta Z_{k-1} \bigvee_{j=0}^{k-1} ((V_0 \wedge P_j \wedge (\langle . \rangle Z_j) \vee (V_1 \wedge P_j \wedge ([.] Z_j))$$

(where $\theta = \nu$ if k is odd, and $\theta = \mu$ if k is even) defines the winning region of Player 0 in any parity game with priorities $0, \dots, k-1$.

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