

Math 717 Homework #2

Due date: Monday, March 4th, 23:59pm.

1. Consider a process with state space $S = \{1, 2, 3\}$, initial distribution $\alpha = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and transition matrix

$$\begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

- a) Construct a Markov Chain with length $N = 10000$ and plot it. Attach your code.
b) Use the above transition matrix to compute the equilibrium distribution.
c) Use the Markov Chain obtained from Part a) to validate your result in Part b).

Solution:

- a) The code is attached. The plot for 10000 is shown below.

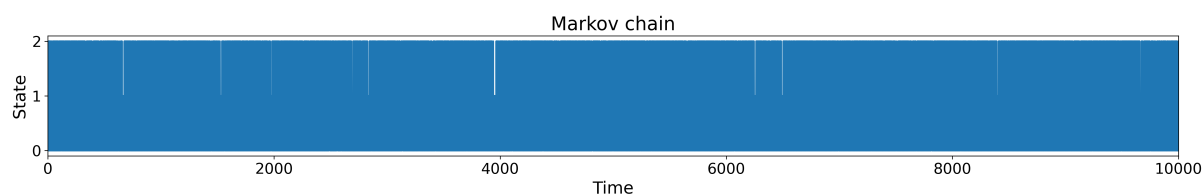


Figure 0.1: Markov Chain with length 10000

This plot however does not say much, I thus also plotted for $N = 1000$. See below

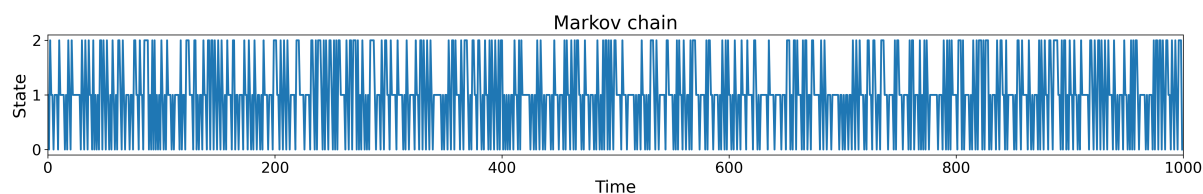


Figure 0.2: Markov Chain with length 1000

- b) The equilibrium distribution is the eigenvector of the transition matrix corresponding to the eigenvalue 1. Computing this we get the equilibrium distribution as:

$$\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

- c) The code for comparing the equilibrium distribution is attached. Given below is the absolute difference error between the computed equilibrium distribution and the one obtained from the Markov Chain:

Simulated equilibrium distribution: [0.2886 0.4487 0.2627]

Surprisingly, I find that the simulated equilibrium distribution is not very close to the actual equilibrium distribution. I even increase the number of samples to $N = 1000000$ and still find that the simulated equilibrium distribution is not very close to the actual equilibrium distribution. I am not sure why this is the case.

- Solution:

$$\begin{aligned} P\{T_1 > s+t \mid T_1 > s\} &= P\{X(m) = 1 \ \forall m \in [0, s+t] \mid X(m) = 1 \ \forall m \in [0, s]\} \\ &= P\{X(m) = 1 \ \forall m \in [s, s+t] \mid X(s) = 1\} \quad (\text{Markov property}) \\ &= P\{X(m) = 1 \ \forall m \in [0, t] \mid X(0) = 1\} \quad (\text{Time Homogeneity}) \\ &= P\{T_1 > t\}. \end{aligned}$$
$$\begin{aligned} P\{T_1 \leq t\} &= 1 - e^{-\nu t}, \\ P\{T_2 \leq t\} &= 1 - e^{-\mu t}. \end{aligned}$$

Now to prove the formula for the expectation $E[X]$, notice that since T_1 which is the time spent in state 1 is exponentially distributed with rate ν , the expected time spent in state 1 is given by $E[T_1] = \frac{1}{\nu}$. Similarly, $E[T_2] = \frac{1}{\mu}$. Therefore, the “probability” of being in state 1 is given by $\frac{E[T_1]}{E[T_1] + E[T_2]} = \frac{\mu}{\mu + \nu}$. Similarly, the “probability” of being in state 2 is given by $\frac{\nu}{\mu + \nu}$. Therefore, the expectation is given by:

b) See code attached. This is the output it produces

Let state 1 be 1 and state 2 be 2
Let $\nu = 0.2$ be the rate of switching from state 1 to state 2 and
 $\mu = 0.5$ be the rate of switching from state 2 to state 1
Emperical probability of being in state 1: 0.7129576497361925
Analytical probability of being in state 1: 0.7142857142857143
Emperical probability of being in state 2: 0.2870423502638063
Analytical probability of being in state 2: 0.28571428571428575
Emperical expected value: 1.287042350263805
Analytical expected value: 1.2857142857142858

3. Write down the time evolution equations of the mean and variance of the complex OU process

$$dx_t = ((-a + i\omega)x_t + f)dt + \sigma dW_t.$$

Start from an arbitrary initial condition, plot the time evolutions of the mean and variance (you choose the model parameters).

Solution:

The time evolution differential equations for the mean and variance of the complex OU process are given by:

$$\begin{aligned} d\langle x_t \rangle &= ((-a + i\omega)\langle x_t \rangle + f)dt, \\ d\langle x_t'^2 \rangle &= (-2(a + i\omega)\langle x_t'^2 \rangle + \sigma^2)dt \end{aligned}$$

Solving these ODEs, we get the time evolution of the mean and variance as:

$$\begin{aligned} \langle x_t \rangle &= \langle x_0 \rangle e^{(-a+i\omega)t} + \frac{f}{a-i\omega}(1 - e^{(-a+i\omega)t}), \\ \langle x_t'^2 \rangle &= \langle x_0'^2 \rangle e^{-2(a+i\omega)t} + \frac{\sigma^2}{2(a+i\omega)}(1 - e^{-2(a+i\omega)t}). \end{aligned}$$

The plot for the time evolution of the mean and variance is shown below. The parameters chosen are $a = 1.$, $\omega = 0.5$, $f = 0.1 + i0.1$, $x_0 = 1 + i1$ and $x_0'^2 = 2 + i2$. The time interval is $[0, 5]$.

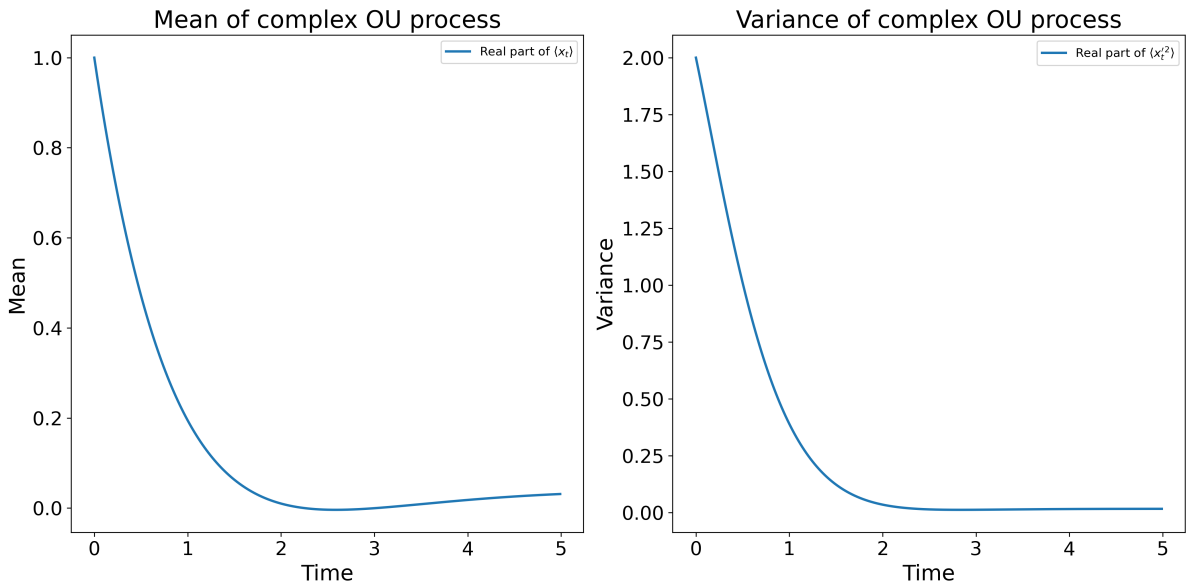


Figure 0.3: Time evolution of the mean and variance