1 Problem 1

1.1 a

We know that, for a basic feasible solution x associated with basis matrix B, that $\bar{c}_i > 0$, for all indices i within the set of nonbasic indices N. We must show that x is a unique optimal solution.

Consider any arbitrary feasible solution y, and the vector $y - x$. Since both x and y are feasible, we have $Ax = Ay = b$, meaning that $Ad = Ax - Ay = b - b = 0$.

Ad is equivalent to the form

$$Bd_B + \sum_{i \in N} A_i d_i = 0$$

Since B is invertible, we have

$$d_B = -\sum_{i \in N} B^{-1} A_i d_i$$

and

$$c'd = c'_B d_B + \sum_{i \in N} c_i d_i = \sum_{i \in N} (c_i - c'_B B^{-1} A_i) d_i = \sum_{i \in N} \bar{c}_i d_i$$

For all nonbasic indices $i \in N$, we have $x_i = 0$, and since y is a feasible solution, we have $y_i \geq 0$. Therefore $d_i \geq 0$. We also know that $c_i > 0$ for all $i \in N$. Therefore $c'd \geq 0$.

Furthermore, since all $c_i > 0$, we know that $c'd = 0$ only if $d_i = 0$ for all $i \in N$. If this is the case, then we have

$$\begin{aligned} d_B &= -\sum_{i \in N} B^{-1} A_i d_i \\ d_B &= -\sum_{i \in N} B^{-1} A_i (0) \\ d_B &= 0 \end{aligned} \tag{1}$$

Thus $d = 0$, and $y = x$. This means that for any $y \neq x$, $c'd > 0$, meaning $c'y > c'x$ for any feasible y. Thus, by definition, x is a unique optimal solution.

1.2 b

We know that x is a unique optimal nondegenerate solution, and we must show that the reduced cost of every nonbasic variable is positive. We provide a proof by contradiction.

Suppose that x is a uniquely optimal, nondegenerate basic feasible solution, and that $\bar{c}_j \leq 0$ for some nonbasic variable x_j. Since x is a nondegenerate basic feasible solution, the j^{th} basic direction d_j is a feasible direction and by definition there exists some feasible $y = x + \theta d_j$. Since the reduced cost $\bar{c}_j$ is non-positive, $c'y \leq c'x$.

If $c'y < c'x$, then x is not optimal, and if $c'y = c'x$, then x is optimal, but not uniquely so. Either way, we have reached a contradiction, and the reduced cost of **all** nonbasic variables are positive.

Q.E.D.

2 Problem 3

2.1 a

We know that x^* is an optimal basic feasible solution with a corresponding optimal basis B^*. We also know that I is empty, meaning **all nonbasic indices** have corresponding reduced costs that are nonzero.

We must show that x^* is the only optimal solution. We provide a proof by contradiction, showing that the existence of another distinct optimal solution y^* violates the optimality of the basis B^*.

Suppose that, there is another optimal solution $y^* \neq x^*$. Consider the feasible direction $d = y - x \neq 0$. It follows that

$$c'd = c'y - c'x = 0$$

Since y^* is feasible, we know that

$$y_N^* = x_N^* + d_N \geq 0$$

But since x is a basic feasible solution $x_N^* = 0$ and

$$d_i \geq 0$$

Furthermore, since y^* and x^* are both feasible, the equality conditions require $Ad = 0$. Since $Ad = Bd_B + \sum_{i \in N} A_i d_i$=0, we can see that if $d_i = 0$ for all i, then $d = 0$. This would be a contradiction, so there must exist some $j \in N$ such that $d_j > 0$.

We then rewrite $c'd$:

$$\begin{aligned} c'd &= c_B' d_B + \sum_{i \in N} c_i d_i \\ c'd &= \sum_{i \in N} (c_i - c_B' B^{-1} A_i) d_i \\ c'd &= \sum_{i \in N} \bar{c}_i d_i = 0 \end{aligned} \tag{2}$$

Thus we have $c'd = 0 = \sum_{i \in N} \bar{c}_i d_i$. Since B^* is an optimal basis, we know that $\bar{c} \geq 0$. Since I is empty, $\bar{c}_i \neq 0$ for all $i \in N$. Therefore, for all $i \in N$, $\bar{c}_i > 0$.

However, we know that $d_j > 0$. Thus, $\bar{c}_j d_j > 0$, and we have

$$\begin{aligned} -\bar{c}_j d_j &= \sum_{i \in N | i \neq j} \bar{c}_i d_i \\ 0 &> \sum_{i \in N | i \neq j} \bar{c}_i d_i \end{aligned} \tag{3}$$

Since all $\bar{c}_i, d_i \geq 0$, this summation cannot be negative, and we have reached our contradiction. Thus, if all of the problem constraints hold, x^* must be the only optimal solution.

Q.E.D.

2.2 b