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1.1 a

We know that, for a basic feasible solution x associated with basis matrix B, that $\bar{c}_i > 0$, for all indices i within the set of nonbasic indices N. We must show that x is a unique optimal solution.

Consider any arbitrary feasible solution y, and the vector $y - x$. Since both x and y are feasible, we have $Ax = Ay = b$, meaning that $Ad = Ax - Ay = b - b = 0$.

Ad is equivalent to the form

$$Bd_B + \sum_{i \in N} A_i d_i = 0$$

Since B is invertible, we have

$$d_B = -\sum_{i \in N} B^{-1} A_i d_i$$

and

$$c'd = c'_B d_B + \sum_{i \in N} c_i d_i = \sum_{i \in N} (c_i - c' B B^{-1} A_i) d_i = \sum_{i \in N} \bar{c}_i d_i$$

For all nonbasic indices $i \in N$, we have $x_i = 0$, and since y is a feasible solution, we have $y_i \geq 0$. Therefore $d_i \geq 0$. We also know that $c_i > 0$ for all $i \in N$. Therefore $c'd \geq 0$.

Furthermore, since all $c_i > 0$, we know that $c'd = 0$ only if $d_i = 0$ for all $i \in N$. If this is the case, then we have

$$d_B = -\sum_{i \in N} B^{-1} A_i d_i d_B = -\sum_{i \in N} B^{-1} A_i (0) d_B = 0$$

Thus $d = 0$, and $y = x$. This means that for any $y \neq x$, $c'd > 0$, meaning $c'y > c'x$ for any feasible y. Thus, by definition, x is a unique optimal solution.

1.2 b

We know that x is a unique optimal nondegenerate solution, and we must show that $\bar{c} > 0$.

Suppose that x is uniquely optimal, nondegenerate basic feasible solution, and that $\bar{c}_j \leq 0$ for some index j.