1 Problem 1

1.1 a

We know that, for a basic feasible solution x associated with basis matrix B, that $\bar{c}_i > 0$, for all indices i within the set of nonbasic indices N. We must show that x is a unique optimal solution.

Consider any arbitrary feasible solution y, and the vector $y - x$. Since both x and y are feasible, we have $Ax = Ay = b$, meaning that $Ad = Ax - Ay = b - b = 0$.

Ad is equivalent to the form

$$Bd_B + \sum_{i \in N} A_i d_i = 0$$

Since B is invertible, we have

$$d_B = -\sum_{i \in N} B^{-1} A_i d_i$$

and

$$c'd = c'_B d_B + \sum_{i \in N} c_i d_i = \sum_{i \in N} (c_i - c'_B B^{-1} A_i) d_i = \sum_{i \in N} \bar{c}_i d_i$$

For all nonbasic indices $i \in N$, we have $x_i = 0$, and since y is a feasible solution, we have $y_i \geq 0$. Therefore $d_i \geq 0$. We also know that $c_i > 0$ for all $i \in N$. Therefore $c'd \geq 0$.

Furthermore, since all $c_i > 0$, we know that $c'd = 0$ only if $d_i = 0$ for all $i \in N$. If this is the case, then we have

$$\begin{aligned} d_B &= -\sum_{i \in N} B^{-1} A_i d_i \\ d_B &= -\sum_{i \in N} B^{-1} A_i (0) \\ d_B &= 0 \end{aligned} \tag{1}$$

Thus $d = 0$, and $y = x$. This means that for any $y \neq x$, $c'd > 0$, meaning $c'y > c'x$ for any feasible y. Thus, by definition, x is a unique optimal solution.

1.2 b

We know that x is a unique optimal nondegenerate solution, and we must show that the reduced cost of every nonbasic variable is positive. We provide a proof by contradiction.

Suppose that x is a uniquely optimal, nondegenerate basic feasible solution, and that $\bar{c}_j \leq 0$ for some nonbasic variable x_j. Since x is a nondegenerate basic feasible solution, the j^{th} basic direction d_j is a feasible direction and by definition there exists some feasible $y = x + \theta d_j$. Since the reduced cost $\bar{c}_j$ is non-positive, $c'y \leq c'x$.

If $c'y < c'x$, then x is not optimal, and if $c'y = c'x$, then x is optimal, but not uniquely so. Either way, we have reached a contradiction, and the reduced cost of **all** nonbasic variables are positive.

Q.E.D.

2 Problem 3

We know that x^* is an optimal basic feasible solution with a corresponding optimal basis B^*.

2.1 a

We also know that I is empty, meaning **all nonbasic indices** have corresponding reduced costs that are nonzero.

We must show that x^* is the only optimal solution. We provide a proof by contradiction, showing that the existence of another distinct optimal solution y^* violates the optimality of the basis B^*.

Suppose that, there is another optimal solution $y^* \neq x^*$. Consider the feasible direction $d = y - x \neq 0$. It follows that

$$c'd = c'y - c'x = 0$$

Since y^* is feasible, we know that

$$y^*_N = x^*_N + d_N \geq 0$$

But since x is a basic feasible solution $x^*_N = 0$ and

$$d_i \geq 0$$

Furthermore, since y^* and x^* are both feasible, the equality conditions require $Ad = 0$. Since $Ad = Bd_B + \sum_{i \in N} A_i d_i$=0, we can see that if $d_i = 0$ for all i, then $d = 0$. This would be a contradiction, so there must exist some $j \in N$ such that $d_j > 0$.

We then rewrite $c'd$:

$$
\begin{aligned}
c'd &= c_B' d_B + \sum_{i \in N} c_i d_i \\
c'd &= \sum_{i \in N} (c_i - c_B' B^{-1} A_i) d_i \\
c'd &= \sum_{i \in N} \bar{c}_i d_i = 0
\end{aligned} \tag{2}
$$

Thus we have $c'd = 0 = \sum_{i \in N} \bar{c}_i d_i$. Since B^* is an optimal basis, Definition 3.3 tells us that $\bar{c} \geq 0$. Since I is empty, $\bar{c}_i \neq 0$ for all $i \in N$. Therefore, for all $i \in N$, $\bar{c}_i > 0$.

However, we know that $d_j > 0$. Thus, $\bar{c}_j d_j > 0$, and we have

$$
\begin{aligned}
-\bar{c}_j d_j &= \sum_{i \in N | i \neq j} \bar{c}_i d_i \\
0 &> \sum_{i \in N | i \neq j} \bar{c}_i d_i
\end{aligned} \tag{3}
$$

Since all $\bar{c}_i, d_i \geq 0$, this summation cannot be negative, and we have reached our contradiction. Thus, if all of the problem constraints hold, x^* must be the only optimal solution.

Q.E.D.

2.2 b

We prove both directions, referring to the "following linear programming problem" as q:

x^* is the unique optimal solution $\Rightarrow$ q has an optimal value of zero

We first show that x^* is a feasible solution to q with cost 0. Since x^* is feasible, we satisfy $Ax = b$. By definition, $x_i^* = 0$ for all $i \in N$, meaning we satisfy $x_i^* = 0, i \in N \backslash I$. Since x^* is feasible, we know that $x^* \geq 0$, meaning that we satisfy $x_i^* \geq 0$ for all $i \in B \cup I$. Finally, we note that since $I \subseteq N$, $x_i^* = 0$ for all $i \in I$ and thus the cost of x^* is 0.

We now show that x^* is the only solution to q, as the existence of any other solution would violate the unique optimality of x^*.

Consider any other solution y to q. We know that $Ay = b$, and we also can combine the second and third constraints to show that $y \geq 0$; this means that y is a feasible solution in our polyhedron.

Consider $d = y - x^*$. We note that

$$
c'd = \sum_{i \in N} \bar{c}_i d_i = \sum_{i \in N \backslash I} \bar{c}_i d_i + \sum_{i \in I} \bar{c}_i d_i
$$

Since both x^* and y are feasible solutions to q, we know that $x_i^* = y_i = 0$, meaning $d_i = 0$ for all $i \in N \backslash I$. Thus we have

$$c'd = \sum_{i \in I} \bar{c}_i d_i$$

But, by the definition of I, $\bar{c}_i = 0$ for all $i \in I$, leading to

$$c'd = 0$$

Therefore $c'y = c'(x^* + d) = c'x^* + 0 = c'x^*$, and y is also an optimal solution in our minimization problem. Thus, we have reached a contradiction, and for x^* to be uniquely optimal there can be no other solutions to q.

Since x^* is the only solution to q, the optimal cost of q is its' optimal cost, i.e. 0.

Q.E.D.

q has an optimal value of zero $\Rightarrow$ x^* is the unique optimal solution

We provide a proof by contrapositive. Suppose that x^* is optimal, but not uniquely so, and that there exists some distinct optimal solution y.

Consider $d = y - x \neq 0$. Since y and x^* are feasible,

$$Ad = Ay - Ax = b - b = 0$$

We can rewrite this to yield the equation

$$B^{*-1} d_B + \sum_{i \in N} A_i d_i = 0$$

Which, rearranging to solve for d_B yields

$$d_B = -\sum_{i \in N} B^{*-1} A_i d_i$$

Considering this equation, it is clear that for $d \neq 0$ there must be some $j \in N$ such that $d_j \neq 0$. Furthermore, y's feasibility requires $y_j \geq 0$, and $x_j = 0$, meaning $d_j > 0$.

Since x^* and y are both optimal, they share the same optimal cost, and $c'd = 0$. We can rewrite this as

$$0 = \sum_{i \in N} \bar{c}_i d_i$$

Since B^* is an optimal basis, by definition we have $\bar{c}_i \geq 0$. Since y is feasible and $x_i = 0$ for all $i \in N$, $d_i \geq 0$ for all $i \in N$. Since all terms are non-negative, the above equality only holds if all terms $\bar{c}_i d_i = 0$ for all $i \in N$. If $d_j > 0$, then $\bar{c}_j = 0$, which means $j \in I$.

We now consider y in terms of the maximization problem q. Since y is feasible in our minimization problem, we know that $Ay = b$ and $y_i \geq 0, i \in$

$B \cup I$. Furthermore, since we know that $\bar{c}_i d_i = 0$ for all $i \in N$ and that $\bar{c}_i \neq 0$ for all $i \in N \backslash I$, $d_i = 0$ for all $i \in N \backslash I$ and thus $y_i = d_i + x_i = 0$ for all $i \in N \backslash I$. Thus y is also a feasible solution to our maximization problem.

We finally consider the cost of y, which is

$$\begin{aligned} \sum_{i \in I} y_i = \sum_{i \in I} x_i + d_i = \sum_{i \in I} d_i \\ \sum_{i \in I} y_i = d_j + \sum_{i \in I | i \neq j} d_i \\ \sum_{i \in I} y_i > 0 \end{aligned} \tag{4}$$

Thus, y has positive cost, and the optimal cost for q must also be positive. We have proven the contrapositive, and thus the original claim holds. Q.E.D.

3 Problem 4

3.1 a

We can easily convert this problem into standard form by adding slack variables to the two non-negativity constraints of the problem:

$$\begin{array}{ll} \text{minimize} & -2x_1 - x_2 \\ \text{subject to} & x_1 - x_2 + x_3 = 2 \\ & x_1 + x_2 + x_4 = 6 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

To construct a basic feasible solution x, we choose the linearly independent columns $A_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, with $B = \begin{bmatrix} A_3 & A_4 \end{bmatrix}$. Thus we have $x_1 = x_2 = 0$, and solve the remaining system of equations

$$\begin{aligned} x_3 = 2 \\ x_4 = 6 \end{aligned}$$

to determine that $x_3 = 2, x_4 = 6$. Thus we have the initial basic feasible solution

$$x = (0, 0, 2, 6)$$

3.2 b

We start by constructing the initial tableau:

		x_1	x_2	x_3	x_4
	0	-2	-1	0	0
x_3	2	1	-1	1	0
x_4	6	1	1	0	1

Since $\bar{c}_1 \leq 0$, we let x_1 enter the basis. The pivot column is $u = \begin{pmatrix} 1 & 1 \end{pmatrix}$, and we compute θ_i, for $i = 3, 4$:

- $\theta_3 = \frac{2}{1} = 2$
- $\theta_4 = \frac{6}{1} = 6$

We have $\theta^* = \min\{\theta_3, \theta_4\} = 2$, and we have $i = 3, l = 1$, and the first basic variable x_3 exits the basis. The new basis will be $\bar{B}(1) = 1, \bar{B}(2) = 4$.

After performing the row operations (described to the right), we obtain the following new tableau

		x_1	x_2	x_3	x_4	
	4	0	-3	2	0	$R_0 = R_0 + 2R_1$
x_1	2	1	-1	1	0	$R1 = R1$
x_4	4	0	2	-1	1	$R2 = R2 - R1$

The corresponding basic feasible solution is $x = \begin{pmatrix} 2 & 0 & 0 & 4 \end{pmatrix}$

Now, we have $\bar{c}_2 = -3$, so we let x_2 enter the basis. The pivot column is $u = \begin{pmatrix} -1 & 2 \end{pmatrix}$, and we select $i = 4, l = 2$ since it is the only nonzero element. x_4 exits the basis, and the new basis will be $\bar{B}(1) = 1, \bar{B}(2) = 2$.

After performing the row operations (described to the right), we obtain the following new tableau

		x_1	x_2	x_3	x_4	
	10	0	0	0.5	0.75	$R_0 = R_0 + \frac{3R_2}{2}$
x_1	4	1	0	0.5	0.5	$R1 = R1 + \frac{R2}{2}$
x_4	2	0	1	-0.5	0.5	$R2 = \frac{R2}{2}$

The corresponding basic feasible solution is $x = \begin{pmatrix} 4 & 2 & 0 & 0 \end{pmatrix}$

At this point, all reduced costs are non-negative and the algorithm terminates. Our optimal basic feasible solution is $x = \begin{pmatrix} 4 & 2 & 0 & 0 \end{pmatrix}$.

3.3 c

We can see the path taken in the algorithm above in Figure 1.

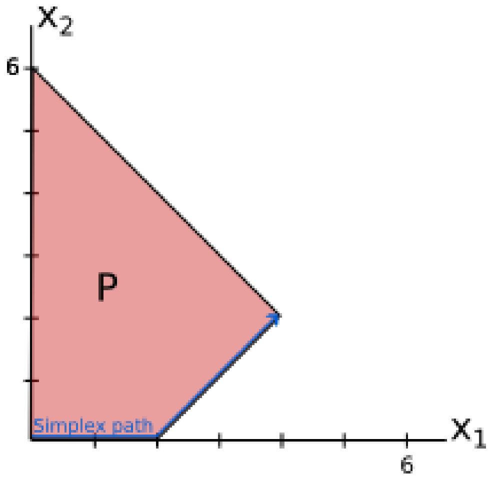

Figure 1: The geometry of the minimization problem (red) and the path taken by the simplex algorithm (blue)

4 Problem 6

Consider the basic feasible solution y. Since it is a basic feasible solution, it is also a vertex, and there exists some cost function c which is uniquely minimized at y.

We consider the linear programming problem of minimizing c over the polyhedron P (containing the basic feasible solutions x and y), starting from the basic feasible solution x, and the intuition behind the simplex method can deliver us from x to y in a finite number of iterations.

Suppose that any iteration, we are at basic feasible solution $z \neq y$ (to start, suppose $z = x$). Consider $d = y - z$, and the the difference in cost $c'd$. Since $c'y < c'z$, $c'd < 0$. Since we are at a basic feasible solution, if we define N_z as the set of nonbasic indices at z, we have

$$c'd = \sum_{i \in N_z} \bar{c}_i d_i < 0$$

For the summation to be negative, there must exist some index $j \in N_z$ such that $\bar{c}_j d_j < 0$. Since $j \in N_z$, $z_j = 0$, for y to be feasible we must have $d_j > 0$, meaning $\bar{c}_j < 0$. There may be many different values of j yielding $\bar{c}_j < 0$, however there must exist one such that d_j is a feasible direction at z (if there wasn't, then the simplex algorithm would tell us that z is optimal, which it is not because we know $z \neq y$. Therefore, by traveling along d_j, we will decrease the cost until we violate another constraint (which we must do, as otherwise the optimal cost would approach $-\infty$, which is impossible as the optimal cost is $c'y$). Theorem 3.2 tells us that when we hit that new constraint, we have reached a new basic feasible solution $\bar{z}$ **that is adjacent**

to z. If $\bar{z} = y$, we have finished; otherwise, we iterate again from $\bar{z}$.

We have thus shown that each iteration strictly decreases our cost. To complete this proof, we argue that we will reach y within a finite number of iterations. We note that there are a finite number of basic feasible solutions, since any polyhedron must have a finite number of constraints. Since every iteration produces a basic feasible solution whose cost is less than the one before, we cannot visit a basic feasible solution more than once during our iteration. Since there are a finite number of basic feasible solutions, we must terminate before we have exhausted all of the basic feasible solutions, and we will eventually reach y.

We have shown that we