1 Problem 1

1.1 a

We know that, for a basic feasible solution x associated with basis matrix B, that $\bar{c}_i > 0$, for all indices i within the set of nonbasic indices N. We must show that x is a unique optimal solution.

Consider any arbitrary feasible solution y, and the vector $y - x$. Since both x and y are feasible, we have $Ax = Ay = b$, meaning that $Ad = Ax - Ay = b - b = 0$.

Ad is equivalent to the form

$$Bd_B + \sum_{i \in N} A_i d_i = 0$$

Since B is invertible, we have

$$d_B = -\sum_{i \in N} B^{-1} A_i d_i$$

and

$$c'd = c'_B d_B + \sum_{i \in N} c_i d_i = \sum_{i \in N} (c_i - c'_B B^{-1} A_i) d_i = \sum_{i \in N} \bar{c}_i d_i$$

For all nonbasic indices $i \in N$, we have $x_i = 0$, and since y is a feasible solution, we have $y_i \geq 0$. Therefore $d_i \geq 0$. We also know that $c_i > 0$ for all $i \in N$. Therefore $c'd \geq 0$.

Furthermore, since all $c_i > 0$, we know that $c'd = 0$ only if $d_i = 0$ for all $i \in N$. If this is the case, then we have

$$\begin{aligned} d_B &= -\sum_{i \in N} B^{-1} A_i d_i \\ d_B &= -\sum_{i \in N} B^{-1} A_i (0) \\ d_B &= 0 \end{aligned} \tag{1}$$

Thus $d = 0$, and $y = x$. This means that for any $y \neq x$, $c'd > 0$, meaning $c'y > c'x$ for any feasible y. Thus, by definition, x is a unique optimal solution.

1.2 b

We know that x is a unique optimal nondegenerate solution, and we must show that the reduced cost of every nonbasic variable is positive. We provide a proof by contradiction.

Suppose that x is a uniquely optimal, nondegenerate basic feasible solution, and that $\bar{c}_j \leq 0$ for some nonbasic variable x_j. Since x is a nondegenerate basic feasible solution, the j^{th} basic direction d_j is a feasible direction and by definition there exists some feasible $y = x + \theta d_j$. Since the reduced cost $\bar{c}_j$ is non-positive, $c'y \leq c'x$.

If $c'y < c'x$, then x is not optimal, and if $c'y = c'x$, then x is optimal, but not uniquely so. Either way, we have reached a contradiction, and the reduced cost of **all** nonbasic variables are positive.

Q.E.D.

2 Problem 3

We know that x^* is an optimal basic feasible solution with a corresponding optimal basis B^*.

2.1 a

We also know that I is empty, meaning **all nonbasic indices** have corresponding reduced costs that are nonzero.

We must show that x^* is the only optimal solution. We provide a proof by contradiction, showing that the existence of another distinct optimal solution y^* violates the optimality of the basis B^*.

Suppose that, there is another optimal solution $y^* \neq x^*$. Consider the feasible direction $d = y - x \neq 0$. It follows that

$$c'd = c'y - c'x = 0$$

Since y^* is feasible, we know that

$$y^*_N = x^*_N + d_N \geq 0$$

But since x is a basic feasible solution $x^*_N = 0$ and

$$d_i \geq 0$$

Furthermore, since y^* and x^* are both feasible, the equality conditions require $Ad = 0$. Since $Ad = Bd_B + \sum_{i \in N} A_i d_i$=0, we can see that if $d_i = 0$ for all i, then $d = 0$. This would be a contradiction, so there must exist some $j \in N$ such that $d_j > 0$.

We then rewrite $c'd$:

$$
\begin{aligned}
c'd &= c'_B d_B + \sum_{i \in N} c_i d_i \\
c'd &= \sum_{i \in N} (c_i - c'_B B^{-1} A_i) d_i \\
c'd &= \sum_{i \in N} \bar{c}_i d_i = 0
\end{aligned} \tag{2}
$$

Thus we have $c'd = 0 = \sum_{i \in N} \bar{c}_i d_i$. Since B^* is an optimal basis, Definition 3.3 tells us that $\bar{c} \geq 0$. Since I is empty, $\bar{c}_i \neq 0$ for all $i \in N$. Therefore, for all $i \in N$, $\bar{c}_i > 0$.

However, we know that $d_j > 0$. Thus, $\bar{c}_j d_j > 0$, and we have

$$
\begin{aligned}
-\bar{c}_j d_j &= \sum_{i \in N | i \neq j} \bar{c}_i d_i \\
0 &> \sum_{i \in N | i \neq j} \bar{c}_i d_i
\end{aligned} \tag{3}
$$

Since all $\bar{c}_i, d_i \geq 0$, this summation cannot be negative, and we have reached our contradiction. Thus, if all of the problem constraints hold, x^* must be the only optimal solution.

Q.E.D.

2.2 b

We prove both directions, referring to the "following linear programming problem" as q:

x^* is the unique optimal solution $\Rightarrow$ q has an optimal value of zero

We first show that x^* is a feasible solution to q with cost 0. Since x^* is feasible, we satisfy $Ax = b$. By definition, $x_i^* = 0$ for all $i \in N$, meaning we satisfy $x_i^* = 0, i \in N \backslash I$. Since x^* is feasible, we know that $x^* \geq 0$, meaning that we satisfy $x_i^* \geq 0$ for all $i \in B \cup I$. Finally, we note that since $I \subseteq N$, $x_i^* = 0$ for all $i \in I$ and thus the cost of x^* is 0.

We now show that x^* is the only solution to q, as the existence of any other solution would violate the unique optimality of x^*.

Consider any other solution y to q. We know that $Ay = b$, and we also can combine the second and third constraints to show that $y \geq 0$; this means that y is a feasible solution in our polyhedron.

Consider $d = y - x^*$. We note that

$$
c'd = \sum_{i \in N} \bar{c}_i d_i = \sum_{i \in N \backslash I} \bar{c}_i d_i + \sum_{i \in I} \bar{c}_i d_i
$$

Since both x^* and y are feasible solutions to q, we know that $x_i^* = y_i = 0$, meaning $d_i = 0$ for all $i \in N \backslash I$. Thus we have

$$c'd = \sum_{i \in I} \bar{c}_i d_i$$

But, by the definition of I, $\bar{c}_i = 0$ for all $i \in I$, leading to

$$c'd = 0$$

Therefore $c'y = c'(x^* + d) = c'x^* + 0 = c'x^*$, and y is also an optimal solution in our minimization problem. Thus, we have reached a contradiction, and for x^* to be uniquely optimal there can be no other solutions to q.

Since x^* is the only solution to q, the optimal cost of q is its' optimal cost, i.e. 0.

Q.E.D.

q has an optimal value of zero $\Rightarrow$ x^* is the unique optimal solution

We provide a proof by contrapositive. Suppose that x^* is optimal, but not uniquely so, and that there exists some distinct optimal solution y.

Consider $d = y - x \neq 0$. Since y and x^* are feasible,

$$Ad = Ay - Ax = b - b = 0$$

We can rewrite this to yield the equation

$$B^{*-1} d_B + \sum_{i \in N} A_i d_i = 0$$

Which, rearranging to solve for d_B yields

$$d_B = -\sum_{i \in N} B^{*-1} A_i d_i$$

Considering this equation, it is clear that for $d \neq 0$ there must be some $j \in N$ such that $d_j \neq 0$. Furthermore, y's feasibility requires $y_j \geq 0$, and $x_j = 0$, meaning $d_j > 0$.

Since x^* and y are both optimal, they share the same optimal cost, and $c'd = 0$. We can rewrite this as

$$0 = \sum_{i \in N} \bar{c}_i d_i$$

Since B^* is an optimal basis, by definition we have $\bar{c}_i \geq 0$. Since y is feasible and $x_i = 0$ for all $i \in N$, $d_i \geq 0$ for all $i \in N$. Since all terms are non-negative, the above equality only holds if all terms $\bar{c}_i d_i = 0$ for all $i \in N$. If $d_j > 0$, then $\bar{c}_j = 0$, which means $j \in I$.

We now consider y in terms of the maximization problem q. Since y is feasible in our minimization problem, we know that $Ay = b$ and $y_i \geq 0, i \in$

$B \cup I$. Furthermore, since we know that $\bar{c}_i d_i = 0$ for all $i \in N$ and that $\bar{c}_i \neq 0$ for all $i \in N \backslash I$, $d_i = 0$ for all $i \in N \backslash I$ and thus $y_i = d_i + x_i = 0$ for all $i \in N \backslash I$. Thus y is also a feasible solution to our maximization problem.

We finally consider the cost of y, which is

$$\begin{aligned}\sum_{i \in I} y_i = \sum_{i \in I} x_i + d_i = \sum_{i \in I} d_i \\ \sum_{i \in I} y_i = d_j + \sum_{i \in I | i \neq j} d_i \\ \sum_{i \in I} y_i > 0\end{aligned} \tag{4}$$

Thus, y has positive cost, and the optimal cost for q must also be positive.

We have proven the contrapositive, and thus the original claim holds.

Q.E.D.