Theoretical Assignment

Tanguy Kerdoncuff

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1 Problem 1

Goal : Prove that for all $n \in \mathbb{N} \quad P(\eta = n) = \frac{(p\lambda)^n}{n!}e^{-p\lambda}$
Let $n \in \mathbb{N}$

$$P(\eta = n) = \sum_{k \in \mathbb{N}} P(\eta = n | \xi = k) P(\xi = k) \quad \text{Law of total probability}$$

We notice that for all $k \in \mathbb{N}$ $P(\eta = n | \xi = k) = \binom{k}{n} p^n (1-p)^{k-n}$ if $k \geq n$ and 0 if not (we can't have more success than the number of trial). This is due to the fact that for all $k \in \mathbb{N} \quad \eta_{\xi=k} \hookrightarrow B(k,p)$ as a sum of k independents (this is not specify clearly in the question) Bernoulli trials of parameter p. Then we have :

$$\begin{aligned}
P(\eta = n) &= \sum_{k \geq n} \binom{k}{n} p^n (1-p)^{k-n} \frac{(\lambda)^k}{k!} e^{-\lambda} \\
&= \sum_{k \geq n} \frac{k!}{n!(k-n)!} p^n (1-p)^{k-n} \frac{\lambda^k}{k!} e^{-\lambda} \\
&= \frac{p^n}{n!} e^{-\lambda} \sum_{k \geq n} \frac{1}{(k-n)!} (1-p)^{k-n} \lambda^k \\
&= \frac{p^n}{n!} e^{-\lambda} \sum_{l \geq 0} \frac{1}{l!} (1-p)^l \lambda^{l+n} \\
&= \frac{p^n \lambda^n}{n!} e^{-\lambda} \sum_{l \geq 0} \frac{((1-p)\lambda)^l}{l!} \\
&= \frac{(p\lambda)^n}{n!} e^{-\lambda} \sum_{l \geq 0} \frac{((1-p)\lambda)^l}{l!} e^{-(1-p)\lambda} e^{(1-p)\lambda} \\
&= \frac{(p\lambda)^n}{n!} e^{-\lambda + (1-p)\lambda} \left(\sum_{l \geq 0} \frac{((1-p)\lambda)^l}{l!} e^{-(1-p)\lambda} \right) \\
&= \frac{(p\lambda)^n}{n!} e^{-\lambda + \lambda - \lambda p} \times 1 \quad \text{Sum of Poisson distribution with parameter } (1-p)\lambda \\
&= \frac{(p\lambda)^n}{n!} e^{-\lambda p}
\end{aligned}$$

We have proved that η follows a Poisson distribution with the parameter $p\lambda$ ($p\lambda > 0$ so the distribution is well defined).

2 Problem 2

Let K be the discret random variable which is 2 if the application was checked by a kind reviewer and 1 if not. Let t be the continuous random variable which is the time that has been taken to check assigned application.

Goal : We want to compute $p_{K|t}(2|10)$.

This is not a classical probability (because the probability $P(K=2|t=10)$ is not define for $P(t=10)=0$) but this is a probability mass function. We use the Bayes rule in a discrete/continuous case.

$$p_{K|t}(2|10) = \frac{f_{t|K}(10|2)p_K(2)}{f_t(10)} = \frac{f_{t|K}(10|2)p_K(2)}{f_{t|K}(10|2)p_K(2) + f_{t|K}(10|1)p_K(1)}$$

$f_{t|K}(10|2)$ is the density function of $t2$, $f_{t|K}(10|1)$ is the density function of $t1$. Moreover $p_K(2) = \frac{1}{2} = p_K(1)$.

$$p_{K|t}(2|10) = \frac{\frac{1}{\sqrt{2\pi}\sigma_2}e^{-\frac{(10-\mu_2)^2}{2\sigma_2^2}}}{\frac{1}{\sqrt{2\pi}\sigma_2}e^{-\frac{(10-\mu_2)^2}{2\sigma_2^2}} + \frac{1}{\sqrt{2\pi}\sigma_1}e^{-\frac{(10-\mu_1)^2}{2\sigma_1^2}}} = \frac{\frac{1}{5}e^{-2}}{\frac{1}{5}e^{-2} + \frac{1}{10}e^{-2}} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$$

We can conclude that the probability that the application was checked by a kind reviewer conditionally to the time of review t = 10 is $\frac{2}{3}$.