A Bayesian Approach to Problems in Stochastic Estimation and Control

Y. C. HO, SENIOR MEMBER, IEEE AND R. C. K. LEE

Summary—In this paper, a general class of stochastic estimation and control problems is formulated from the Bayesian Decision-Theoretic viewpoint. A discussion as to how these problems can be solved step by step in principle and practice from this approach is presented. As a specific example, the closed form Wiener-Kalman solution for linear estimation in Gaussian noise is derived. The purpose of the paper is to show that the Bayesian approach provides; 1) a general unifying framework within which to pursue further researches in stochastic estimation and control problems, and 2) the necessary computations and difficulties that must be overcome for these problems. An example of a nonlinear, non-Gaussian estimation problem is also solved.

Single Stage Estimation Problem

FOR THE PURPOSE of illustrating the concepts involved, the single-stage estimation problem will be discussed first. Once this is accomplished, the multistage problem can be treated straightforwardly.

Problem Statement

The following information is assumed, given;

1) A set of measurements $z_1, z_2, \cdots, z_k$ which are denoted by the vector z.
2) The physical relationship between the state of nature which is to be estimated and the measurements. This is given by

$$z = g(x, v), \tag{1}$$

where
z is the measurement vector $(k\times 1)$,
x is the state (signal) vector $(n\times 1)$, and
v is the noise vector $(q\times 1)$.

3) The joint density function $p(x, v)$. From this the respective marginal density functions, $p(x)$ and $p(v)$, are readily obtained.

It is assumed that information for 3) is available in analytical form or can be approximated by analytical distributions. Item 2) can be either in closed form or merely computable. The problem is to obtain an estimate $\hat{x}$ of x upon which to base the best measurements which will be defined later.

The Bayesian Solution

The Bayesian solution to the above problem now proceeds via the following steps:

1) Evaluate $p(z)$—This can be done analytically, at least in principle, or experimentally by Monte Carlo methods since $z=g(x, v)$ and $p(x, v)$ are given. In the latter case, it is assumed possible to fit the experimental distribution again by a member of a family of distributions.
2) At this point, two alternatives are possible, one may be superior to the other depending on the nature of the problem.
   a) Evaluate $p(x, z)$. This is possible analytically if v is of the same dimension as z and one can obtain the functional relationship $v=g^*(x, z)$ from (1). Then, using $p(x, v)$ and the theory of derived distributions, one obtains

$$p(x, z) = p(x, v = g^*(x, z))J \tag{2}$$

where

$$J = \det\left[\frac{\partial g^*(x, z)}{\partial z}\right].$$

   b) Evaluate $p(z/x)$. This conditional density function can always be obtained either analytically, whenever possible, or experimentally from the $z=g(x, v)$ and $p(x, v)$.

Note that 2a) may be difficult to obtain in general since g^* may not exist either because of the nonlinear nature of g or the fact that z, v are of different dimensions. Nevertheless, 2b) can always be carried out. This fact will be demonstrated in the nonlinear example in the sequel.

3) Evaluate $p(x/z)$ using the following relationships:
   a) Following 2a)

$$p(x/z) = \frac{p(x, z)}{p(z)}\cdot \tag{3}$$

   b) Following 2b), use the Bayes' rule

$$p(x/z) = \frac{p(z/x)p(x)}{p(z)}\cdot \tag{4}$$

Depending on the class of distributions one has assumed or obtained for $p(x, v)$, $p(z)$, $p(z/x)$, this key step may be easy or difficult to carry out. Several classes of distribution which have nice properties for this purpose can be found in Raiffa and Schlaifer [1]. The density function $p(x/z)$ is known as the *a posteriori* density function of x. It is the knowledge about the state of nature *after* the measurements z. By definition, it contains all the information necessary for estimation.

Manuscript received October 31, 1963; revised July 2, 1964.
Y. C. Ho is with Harvard University, Cambridge, Mass., and is also a Consultant at the Minneapolis-Honeywell Regulator Co., Boston, Mass.
R. C. K. Lee was formerly with the Minneapolis-Honeywell Regulator Co., Minneapolis, Minn.

4) Depending on the criterion function for estimation, one can compute estimate $\hat{x}$ from $p(x/z)$. Some typical examples are
   a) Criterion: Maximize the Probability ($\hat{x}=x$).
   Solution: $\hat{x}=$ Mode of $p(x/z)$. (5)
   This is defined as the most probable estimate. When the *a priori* density function $p(x)$ is uniform, this estimate is identical to the classical maximum likelihood estimate.
   b) Criterion: Minimize $\int \|x-x\|^2 p(x/z)dx$.
   Solution: $\hat{x}=E(x/z)$[1] (6)
   This is the conditional mean estimate.
   c) Criterion: Minimize Maximum $|x-\hat{x}|$.
   Solution: $\hat{x}=$ Medium of $p(x/z)$. (7)
   This can be defined as the minimax estimate.

Pictorially, the three estimates are shown in Fig. 1 for a general $p(x/z)$ for a scalar case.

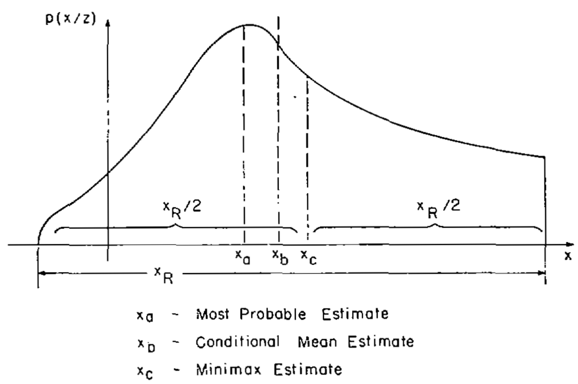

Fig. 1—Estimates based on *a posteriori* density.

Clearly, other estimates, as well as confidence intervals can be derived from $p(x/z)$ directly.

Special Case of the Wiener-Kalman Filter (single stage)

Now a special case of the above estimation problem will be considered. Let there be given;

1) A set of measurements $z=(z_1, z_2, \cdots, z_\kappa)$.
2) The physical relationship

$$z = Hx + v. \tag{8}$$

3) The independent noise and state density functions

$$p(x, v) = p(x)p(v) \tag{9}$$

$p(x)$ be Gaussian with

$$\left.\begin{aligned} E(x) &= \bar{x} \\ \mathrm{Cov}\,(x) &= P_0 \end{aligned}\right\} \tag{10}$$

$p(v)$ be Gaussian with

$$\left.\begin{aligned} E(v) &= 0 \\ \mathrm{Cov}\,(v) &= R \end{aligned}\right\}. \tag{11}$$

Now, following the steps for the Bayesian solution,

1) Evaluate $p(z)$.
   Since $z=Hx+v$ and x, v are Gaussian and independent, one immediately gets

$$\left.\begin{aligned} &p(z) \text{ is Gaussian} \\ &E(z) = H\bar{x} \\ &\mathrm{Cov}\,(z) = HP_0H^T + R \end{aligned}\right\}. \tag{12}$$

2a) Evaluate $p(x, z)$. Since $(\partial g^*/\partial z) =$ Identify matrix, it follows

$$\begin{aligned} p(x, z) &= p(x, v = z - Hx) \\ &= p(x)p_v(z - Hx).^2 \end{aligned} \tag{13}$$

2b) Evaluate $p(z/x)$.[3]

$$p(z/x) = \frac{p(x, z)}{p(x)} = p(v) = p_v(z - Hx). \tag{14}$$

3) Evaluate $p(x/z)$. One gets from Bayes' rule,

$$p(x/z) = \frac{p(x)p(v)}{p(z)}. \tag{15}$$

By direct substitution of (10), (11), and (12), one obtains

$$\begin{aligned} p(x/z) = {} & \frac{|HP_0H^T + R|^{1/2}}{(2\pi)^{n/2}|P_0|^{1/2}|R|^{1/2}} \\ & \cdot \exp\{-1/2[(x - \bar{x})^T P_0^{-1}(x - \bar{x}) \\ & + (z - Hx)^T R^{-1}(z - Hx) \\ & - (z - H\bar{x})^T(HP_0H^T + R)^{-1}(z - H\bar{x})]\}. \end{aligned} \tag{16}$$

Now completing squares in the { }, (16) simplifies to

$$\begin{aligned} p(x/z) = {} & \frac{|HP_0H^T + R|^{1/2}}{(2\pi)^{n/2}|P_0|^{1/2}|R|^{1/2}} \\ & \cdot \exp\{-1/2(x - \hat{x})^T P^{-1}(x - \hat{x})\} \end{aligned} \tag{17}$$

where

$$P^{-1} = P_0^{-1} + H^T R^{-1} H \tag{18}$$

or equivalently,

$$P = P_0 - P_0H^T(HP_0H^T + R)^{-1}HP_0 \tag{19}$$

and

$$\hat{x} = \bar{x} + PH^T R^{-1}(z - H\bar{x}). \tag{20}$$

4) Now, since $p(x/z)$ is Gaussian, the most probable, conditional mean and minimax estimate all coincide and is given by $\hat{x}$.

This is the derivation of the single stage Weiner-Kalman filter [2], [3]. The pair $(P, \hat{x})$ is called a *suffi-*

[1] It is assumed that $p(x/z)$ has finite second moment.

[2] $p_v(z-Hx)$ means substituting $(z-Hx)$ for v in $p(v)$.
[3] Note: step (2b) is redundant.

cient statistic for the problem in the sense that $p(x/z) = p(x/P, \hat{x})$.

Multistage Estimation Problem

The problem formulation and the solution in this case is basically similar to the single-stage problem. The only additional complication is that now the state is changing from stage to stage according to some dynamic relationship, and that the *a posteriori* density function is to be computed recursively.

Problem Statement

It is assumed that at any stage $k+1$, the following data is given as a result of previous computation or as part of the problem statement.

1) The system equations governing the evolution of the state.

$$\left.\begin{aligned} x_{k+1} &= f(x_k, w_k) \\ z_{k+1} &= h(x_{k+1}, v_{k+1}) \end{aligned}\right\} \tag{21}$$

where

x_{k+1} is the state vector at $k+1$,
v_{k+1} is the measurement noise at $k+1$,
z_{k+1} is the additional measurement available at $k+1$,
w_k is the disturbance vector at k.

2) The complete set of measurements $Z_{k+1} \triangleq (z_1, \cdots, z_{k+1})$.
3) The density functions[4]
$p(x_k/z_1, \cdots, z_k) \triangleq p(x_k/Z_k)$
$p(w_k, v_{k+1}/x_k)$—statistics of a vector random sequence with components w_k and v_{k+1} which depends on x_k.

Now it is required to estimate x_{k+1} based on measurements $z_1, \cdots, z_{k+1}$.

The Bayesian Solution

The procedure is analogous to the single-stage case.

1) Evaluate $p(x_{k+1}/x_k)$. This can be accomplished either experimentally or analytically from knowledge of $p(w_k, v_{k+1}/x_k)$, $p(x_k/Z_k)$ and (21).
2) Evaluate $p(z_{k+1}/x_k, x_{k+1})$. This is derived from $p(w_k, v_{k+1}/x_k)$ and (21).
3) Evaluate

$$p(x_{k+1}, z_{k+1}/Z_k) = \int p(z_{k+1}/Z_k, x_{k+1})p(x_{k+1}/x_k) \cdot p(x_k/Z_k)dx_k. \tag{22}$$

From this the marginal density functions $p(x_{k+1}/Z_k)$ and $p(z_{k+1}/Z_k)$ can be directly evaluated.

4) Evaluate

$$p(x_{x+1}/Z_{k+1}) = \frac{p(x_{k+1}, z_{k+1}/Z_k)}{p(z_{k+1}/Z_k)} \tag{23}$$

from (22)

$$= \frac{\int p(z_{+1}/Z_k, x_{k+1})p(x_{k+1}/x_k)p(x_k/Z_k)dx_k}{\int\int p(z_{k+1}/Z_k, x_{k+1})p(x_{k+1}/x_k)p(x_k/Z_k)dx_{k+1}dx_k}. \tag{24}$$

Eq. (24) is a functional-integral-difference equation governing the evolution of the *a posteriori* density function of the state of (21).

5) Estimates for x_{k+1} can now be obtained from $p(x_{k+1}/Z_{k+1})$ exactly as in the single-stage case.

Special Case of the Wiener-Kalman Filter[5]

The given data at $k+1$ is specified as follows:

The physical model is given by

$$\begin{aligned} x_{k+1} &= \Phi x_k + \Gamma w_k \\ z_k &= Hx_k + v_k \end{aligned} \tag{25}$$

where w and v are independent, white, Gaussian random sequences with

$$p(x_k/Z_k) \text{ is Gaussian}$$

$$\left.\begin{aligned} E(x_k/Z_k) &\triangleq \hat{x}_k \\ \text{Cov }(x_k/Z_k) &= P_k \end{aligned}\right\} \tag{26}$$

$$p(w_k, v_{k+1}/x_k, Z_k) = p(w_k)p(v_{k+1})$$

$$\left.\begin{aligned} E(v_k) = E(v_{k+1}) &= 0 \\ \text{Cov }(w_k) = Q;\ \text{Cov }(v_{k+1}) &= R \end{aligned}\right\}. \tag{27}$$

[4] The product of the two density functions yields $p(w_k, v_{k+1}, x_k/Z_k)$ by the Markov property of (21). It is also assumed that if $p(w\ v/x) = p(w, v)$ then w, v are white random sequences.

[5] This development of the multistage Wiener-Kalman filtering method is very similar to a paper by H. Rauch, F. Tung, and C. C. Striebel entitled "On The Maximum Likelihood Estimate for Linear Dynamic Systems" presented at the SIAM Conference on System Optimization, 1964, Monterey, Calif. The only difference between the two developments is that the Rauch-Tung-Striebel paper does not explicitly compute $p(x/z)$ but simply computes its maximum and uses it as the estimate. In the author's approach, the computation of the maximum plays a secondary role. The explicit calculation of the *a posteriori* probability is emphasized as the Bayesian viewpoint. The authors are indebted to Prof. A. E. Bryson for bringing this reference to their attention. Similar development of the Wiener-Kalman filter is also presented in NASA TR-R-135 1962 by G. L. Smith, S. Schmidt, and L. A. Megee. This was brought to the authors' attention after the publication of the JACC preprint.

Since in this case, the noise w_k, v_{k+1} is not dependent on the state, (24) simplifies to

$$p(x_{k+1}/Z_{k+1}) = \frac{p(z_{k+1}/x_{k+1})}{p(z_{k+1}/Z_k)} p(x_{k+1}/Z_k). \qquad (24)'$$

Hence, the solution only involved the evaluation of the three density functions on the rhs of (24) given the data (25–27). This is carried out below. From (27), it is noted that $p(x_{k+1}/Z_k)$ is Gaussian and independent of v_{k+1}

$$\left.\begin{aligned} E(x_{k+1}/Z_k) &= \Phi\hat{x}_k \\ \text{Cov}\,(x_{k+1}/Z_k) &= \Phi P_k \Phi^T + \Gamma Q \Gamma^T \triangleq P_{k+1} \end{aligned}\right\}. \qquad (28)$$

Similarly, $p(z_{k+1}/Z_k)$ is Gaussian and

$$\left.\begin{aligned} E(z_{k+1}/Z_k) &= H\Phi\hat{x}_k \\ \text{Cov}\,(z_{k+1}/Z_k) &= HP_{k+1}H^T + R \end{aligned}\right\}. \qquad (29)$$

Finally $p(z_{k+1}/x_{k+1})$ is also Gaussian with

$$\left.\begin{aligned} P(z_{k+1}/x_{k+1}) &= Hx_{k+1} \\ \text{Cov}\,(z_{k+1}/x_{k+1}) &= R \end{aligned}\right\}. \qquad (30)$$

Combining (28–30) using (24), one gets

$$\begin{aligned} p(x_{k+1}/Z_{k+1}) = {} & \frac{|HM_{k+1}H^T + R|^{1/2}}{(2\pi)^{n/2}|R|^{1/2}|M_{k+1}|^{1/2}} \\ & \cdot \exp\left[-1/2[(x_{k+1} - \Phi\hat{x}_k)^T M_{k+1}{}^{-1}\right. \\ & \cdot (x_{k+1} - \Phi\hat{x}_k) \\ & + (z_{k+1} - Hx_{k+1})^T R^{-1}(z_{k+1} - Hx_{k+1}) \\ & - (z_{k+1} - H\Phi\hat{x}_k)^T(HM_{k+1}H^T + R)^{-1} \\ & \cdot (z_{k+1} - H\Phi\hat{x}_k)]\}. \end{aligned} \qquad (31)$$

Now completing squares in $\{\ \}$ one gets,

$$\begin{aligned} & p(x_{k+1}/Z_{k+1}) \\ & = \frac{|HM_{k+1}H^T + R|^{1/2}}{(2\pi)^{n/2}|R|^{1/2}|M_{k+1}|^{1/2}} \\ & \quad \cdot \exp\{-1/2(x_{k+1} - \hat{x}_{k+1})^T P_{k+1}{}^{-1}(x_{k+1} - \hat{x}_{k+1})\} \end{aligned} \qquad (32)$$

where

$$\begin{aligned} \hat{x}_{k+1} = {} & \Phi\hat{x}_k + M_{k+1}H^T(HM_{k+1}H^T + R)^{-1} \\ & \cdot (z_{k+1} - H\Phi\hat{x}_k) \end{aligned} \qquad (33)$$

$$P_{k+1}{}^{-1} = M_{k+1}{}^{-1} + H^T R^{-1} H \qquad (34)$$

or equivalently,

$$P_{k+1} = M_{k+1} - M_{k+1}H^T(HM_{k+1}H^T + R)^{-1}HM_{k+1} \qquad (35)$$

and

$$M_{k+1} = \Phi P_k \Phi^T + \Gamma Q \Gamma^T. \qquad (36)$$

Eqs. (33–36) are exactly the discrete Wiener-Kalman filter in the multistage case [3], [4].

A Simple Nonlinear Non-Gaussian Estimation Problem

The discussions in the above sections have been carried out in terms of continuous density functions. However, it is obvious that the same process can be applied to problems involving discrete density function and discontinuous functional relationships. It is worthwhile, at this point, to carry out one such solution for a simple *contrived* example which nevertheless illustrates the application of the basic approach.

The problems can be visualized as an abstraction of the following physical estimation problem. An infrared detector followed by a threshold device is used in a satellite to detect hot targets on the ground. However, extraneous signals, particularly reflection from clouds, obscure the measurements. The problem is to design a multistage estimation process to estimate the presence of hot targets through measurement of the output of the threshold detector.

Let s_k (target) be scalar independent Bernoulli process with,

$$p(s_k) = (1 - q)\delta(s_k) + q\delta(1 - s_k), \qquad (37)^6$$

n_k (cloud noise) be a scalar Markov process with,

$$p(n_1) = (1 - a)\delta(n_1) + a\delta(1 - n_1) \qquad (38)$$

$$\begin{aligned} p(n_{k+1}/n_k) = {} & \left(1 - a - \frac{n_k}{2}\right)\delta(n_{k+1}) \\ & + \left(a + \frac{n_k}{2}\right)\delta(1 - n_{k+1}) \end{aligned} \qquad (39)$$

and the scalar measurement,

$$z_k = s_k \oplus n_k \qquad (40)$$

where $\oplus$ indicates the logical "OR" operation.

Essentially (37–40) indicate the fact that as the detector sweeps across the field of view, cloud reflection tends to appear in groups while targets appear in isolated dots.

Now to proceed to the Bayesian solution. First, there is,

n_1	0	0	1	1
s_1	0	1	0	1
z_1	0	1	1	1
Probability of z_1	$(1-a)(1-q)$	$q(1-a)$	$a(1-q)$	aq

$$p(z_1) = (1 - a)(1 - q)\delta(z_1) + (a + q - aq)\delta(z_1 - 1). \qquad (41)$$

[6] The notation

$$\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

is used here. Also, $p(x)$ is to be interpreted as mass functions.

Also,

$$p(z_1/n_1) = \delta(z_1 - 1)n_1 + (1 - q)\delta(z_1) + q\delta(z_1 - 1)](1 - n_1). \quad (42)$$

Then by direct calculation,

$$p(n_1/z_1) = \frac{p(z_1/n_1)p(n_1)}{p(z_1)} = (1 - a'(z_1))\delta(n_1) + a'(z_1)\delta(n_1 - 1), \quad (43)$$

where

$$a'(z_1) = \frac{a\delta(z_1 - 1)}{(1 - a)(1 - q)\delta(z_1) + (a + q - aq)\delta(z_1 - 1)}; \quad (44)$$

which, after straightforward but somewhat laborious manipulations, becomes

$$= \left(1 - a - \frac{a'(z_1)}{2}\right)\delta(n_2) + \left(\frac{a'(z_1)}{2} + a\right)\delta(n_2 - 1)$$
$$\triangleq (1 - a(z_1))\delta(n_2) + a(z_1)\delta(n_2 - 1).$$

Furthermore,

$$p(s_2/z_1) \triangleq p(s_2) = (1 - q)\delta(s_2) + q\delta(s_2 - 1). \quad (50)$$

Eqs. (49) and (50) now take the place of (37) and (38) and by the same process, one can get, in general,

$$p(n_k/Z_k) \triangleq p(n_k/z_k, z_{k-1}, \cdots) = (1 - a'(Z_k))\delta(n_k) + a'(Z_k)\delta(n_{k-1}) \quad (51)$$

$$a'(Z_k) \triangleq a'(z_k, z_{k-1}, \cdots) = \frac{a(Z_{k-1})\delta(z_k - 1)}{[1 - a(Z_{k-1})](1 - q)\delta(z_k) + (a(Z_{k-1}) + q - a(Z_{k-1})q)\delta(z_k - 1)} \quad (52)$$

$$a(Z_{k-1}) \triangleq a(z_{k-1}z_{k-2}, \cdots) = a + \frac{a'(Z_{k-1})}{2} \quad (53)$$

$$p(s_k/Z_k) \triangleq p(s_k/z_k, z_{k-1}, \cdots) = 1 - q'(Z_k)\delta(s_k) + q'(Z_k)\delta(s_{k-1}) \quad (54)$$

$$q'(Z_k) \triangleq q'(z_k, z_{k-1}, \cdots) = \frac{q\delta(z_{k-1})}{[1 - a(Z_{k-1})](1 - q)\delta(z_k) + [a(Z_{k-1}) + q - a(Z_{k-1})q]\delta(z_k - 1)} \quad (55)$$

$$\begin{aligned} p(n_{k+1}/Z_k) &= (1 - a(Z_k))\delta(n_{k+1}) + a(Z_k)\delta(n_{k+1} - 1) \\ p(s_{k+1}/Z_k) &= p(s_{k+1}). \end{aligned} \quad (56)$$

similarly,

$$p(z_1/s_1) = \delta(z_1 - 1)s_1 + [(1 - a)\delta(z_1) + a\delta(z_1 - 1)](1 - s_1), \quad (45)$$

and

$$p(s_1/z_1) = \frac{p(z_1/s_1)p(s_1)}{p(z_1)} = (1 - q'(z_1))\delta(s_1) + q'(z_1)\delta(s_1 - 1) \quad (46)$$

where

$$q'(z_1) = \frac{q\delta(z_1 - 1)}{(1 - a)(1 - q)\delta(z_1) + (a + q - aq)\delta(z_1 - 1)}, \quad (47)$$

and a reasonable estimate is

$$\hat{s}_1 = \begin{cases} 1 & \text{if } q'(z_1) > \epsilon \text{ (Given constant)} \\ 0 & \text{if } q'(z_1) < \epsilon \end{cases} \quad (48)$$

where $\hat{s}_1 = 1$ may be interpreted as an alarm.

Now consider a second measurement z_2 has been made. One has

$$p(n_2/z_1) = \int_{-\infty}^{\infty} p(n_2/n_1)p(n_1/z_1)dn_1, \quad (49)$$

Eqs. (51–57) now represent the general recursion solution for the multistage estimation process.

As a check, two possible observed sequences for z, namely (0, 1) and (1, 1) are considered. With $a = 1/4$ and $q = 1/4$ it is found that $p(s_2/z_2, z_1) = 0.571$ and 0.337, respectively. This agrees with intuition since the sequence (1, 1) has a higher probability of being cloud reflections. On the other hand, the numbers also showed that under the circumstances, it is very difficult to detect targets with accuracy using the system contrived here.

Oftentimes, one is actually interested in $p(s_k/Z_{k+\tau})$ with $\tau > 0$ in order to obtain the so-called "smoothed" estimate for s_k. The desired density function can be computed from $p(s_k/Z_k)$ by further manipulations. However, the calculation becomes involved and will not be done here.

Relationship to General Bayesian Statistical Decision Theory

It is worthwhile to point out the relationship of the above formulation and solution of the estimation problem to and its difference from the general statistical decision problem. For simplicity, the single-stage case is considered again. In the general statistical decision problem, the input data is somewhat different. One typical form is,[7]

$p(x)$—*a priori* density of x
$\{e\}$—a set of choices of experiments from which we can derive measurements z with
$p(z/x, e)$—conditional density of z for given x and e.
$\{u\}$—a set of choices of decisions
$J(e, z, u, x)$—a criterion function which is a possible function of e, z, u and x.

The problem is then stated as the determination of e and u so that $E(J)$ is optimized. The optimal J is given by

$$J_{\text{opt}} = \underset{e}{\text{Max}}\,(\text{Min}) \int \left\{ \underset{u}{\text{Max}}\,(\text{Min}) \cdot \left[\int J(e, z, u, x) \cdot p(x/z, e) dx \right] \right\} p(z/e) dz. \qquad (58)^{8}$$

Thus, the main differences between the estimation problem and the general decision problem are as follows:

1) In the estimation problem there is no choice of experiment. One always makes the same type of measurement z given by $g(x, v)$. To generalize the estimation problem, one can specify,

$$z_e = g_e(x, v);\ \{e\} = 1, 2, \cdots = \text{possible sets of measurements} \qquad (59)$$

and then require that

$$\hat{x} = \underset{e}{\text{Opt}} \{(\hat{x})_e\ e = 1, 2, \cdots\}.$$

2) In the general decision problem, the function $z = g(x, v)$ is implicit in $p(z/x, e)$. Hence step 2a) and 2b) for the estimation solution is not required. This is often a tremendous simplification.
3) In the estimation problem the criterion function J is always a simple function of x only. There is, furthermore, no choice of action (one has to make an estimate by definition). On the other hand, the general decision problem is more analogous to a combined estimation and control problem where one has a further choice of action after determining $p(x/z)$, and like a control problem, the criteria function is generally more complex.
4) It is, however, to be noted that the key step is the computation of $p(x/z)$ for both problems. The choice of action is determined only *after* the computation of $p(x/z)$. Thus, a general decision problem can be composed into two problems, namely, determination of $p(x/z)$ (estimation problem) and choice of action (control problem). In control-theoretic technology, this fact is called the Generalized Decomposition Axiom.

As an example, consider the single stage Wiener-Kalman problem and the added requirement that,

$$J(e, z, u, x) = J(u, x) = E\|Bx + u\|^2 = \int \|Bx + u\|^2 p(x/z) dx \qquad (60)$$

be a minimum. Expanding (60), one gets

$$J = E\|x\|^2_{B^TB} + 2u^T B\hat{x} + u^T u; \qquad (61)$$

clearly,

$$u_{\text{opt}} = u(\hat{x}) \triangleq u(x(z)) \triangleq u(z) = -B\hat{x}(z), \qquad (62)$$

which is one of the fundamental results of linear stochastic control. Thus, the control action u is only a function of the criterion J and the *a posteriori* density function $p(x/z)$. In fact, in this case only $\hat{x}$ of $p(x/z)$ is needed. We call $\hat{x}$ as the *minimal sufficient statistic for the control problem.*

In the more general multistage case, the decomposition property clearly still holds, the only difference being that $p(x_{k+1}/Z_{k+1})$ is now dependent on u_k. However, this dependence is entirely *deterministic* since, in a given situation, one always knows what u_k's are. In fact, in the Wiener-Kalman control problem, it is known that u_k is a linear function of $\hat{x}_k$ only.

Conclusion

In the above sections, the problem of estimation from the Bayesian viewpoint is discussed. It is the author's thesis that this approach offers a unifying methodology, at least conceptually, to the general problems of estimation and control.

The *a posteriori* conditional density function $p(x/z)$ is seen to be the key to the solution of the general problem. Difficulties associated with the solution of the general problem now appear more specifically as difficulties in steps leading to the computation of $p(x/z)$. From the above discussions, it is relatively obvious that these difficulties are

1) Computation of $p(z/x)$. In both the single-stage or multistage case, this problem is complicated by the nonlinear functional relationships between z and x. Except in the case when z and x are linearly related or when z and x are scalars, very little can be done in general, analytically or experimentally. As was mentioned earlier, this difficulty does not appear in the usual decision problem, since there it is assumed that $p(z/x)$ is given as part of the problem.
2) Requirement that $p(x/z)$ be in analytical form.

[7] For other equivalent forms, see Raiffa and Schlaifer [1].
[8] See [1].

This is an obvious requirement if we intend to use the solution in real-time applications.

3) Requirements that $p(x)$, $p(z)$, $p(x/z)$ be conjugate distributions [1]. This is simply the requirement that $p(x)$ and $p(x/z)$ be density functions from the same family. Note that all the examples discussed in this paper possess this desirable property. This is precisely the reason that multistage computation can be done efficiently. This imposed a further restriction on the functions g, f and h.

The difficulties (1–3) listed above are formidable ones. It is not likely that they can be easily circumvented except for special classes of problems such as those discussed. However, it is worthwhile first to pinpoint these difficulties. Research toward their solution can then be effectively initiated. Finally, it is felt that the Bayesian approach offers a unified and intuitive viewpoint particularly adaptable to handling modern-day control problems where the *State* and the *Markov* assumptions play a fundamental role.

References

[1] H. Raiffa and R. Schlaifer, "Applied Statistical Decision Theory" Harvard University Press, Cambridge, Mass., chs. 1–3; 1961.

[2] A. E. Bryson and M. Frazler, "Smoothing in linear and nonlinear systems," *Proc. of Optimal System Synthesis Symp.*, Wright Field, Ohio; 1962.

[3] Y. C. Ho, "On stochastic approximation and optimal filtering method," *J. Math. Analysis and Applications*, vol. 6, pp. 152–154; February, 1963.

[4] R. E. Kalman, "New Methods and Results in Linear Prediction and Filtering Theory," Research Institute for Advanced Studies, Baltimore, Md., No. 61-1.

The Minimization of Measurement Error in a General Perturbation-Correlation Process Identification System

H. J. PERLIS, SENIOR MEMBER, IEEE

Summary—**A general identification system is studied for an important class of realistic, time-varying processes. This class consists of those in which the process is nominally known, and the statistical characteristics of its varying parameters and of the environment are also known. The expression for identification error in terms of the spectral properties of the parameter variations and of the output transducer noise is developed. Optimization procedures are given to minimize the perturbation-correlation system's mean-square identification error.**

Introduction

With the increase in automation and the demand for more self-contained, sophisticated systems, a more general approach to system measurement must be taken. This is especially true in the field of self-adaptive control [1]–[13]. Here, before the system can perform self-adjustment it first must identify itself. Then, this measurement must be evaluated in accordance with a predetermined criterion and mathematical model of the dynamical system. Thus, one sees that the basic functions for self-adaption are: identification, evaluation, and adjustment.

Manuscript received October 15, 1963; revised July 13, 1964.
The author is with the Dept. of Electrical Engineering, Rutgers, The State University, New Brunswick, N. J.

Although only one of the basic functions, the self-adaptive subsystem is dominated by the identification technique employed. It has been shown that regardless of its particular form, the identification process performs its function by measuring one of the following:

1) the coefficients of the differential equation,
2) the impulse or step response,
3) the frequency response.

Similarly, in order to identify a system, it is necessary to have available, either directly or indirectly, both the input and the output signals. Either normal input or special test signals are employed. Usually, the latter is preferred, since it guarantees system measurement regardless of the level of normal plant activity.

The basic function of identification has been treated extensively in the literature. However, the combined consideration of measurement noise, *a priori* knowledge, and time-varying system parameters has not been given sufficient treatment. The major block of identification techniques deals with the learning process associated with abstract systems. Cooper and Lindenlaub [14] pointed out that no investigation had been made which incorporated the effects of noise and *a priori* knowledge.