Nonlinear Bayesian Estimation Using Gaussian Sum Approximations

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Abstract—**Knowledge of the probability density function of the state conditioned on all available measurement data provides the most complete possible description of the state, and from this density any of the common types of estimates (e.g., minimum variance or maximum *a posteriori*) can be determined. Except in the linear Gaussian case, it is extremely difficult to determine this density function. In this paper an approximation that permits the explicit calculation of the *a posteriori* density from the Bayesian recursion relations is discussed and applied to the solution of the nonlinear filtering problem. In particular, it is noted that a weighted sum of Gaussian probability density functions can be used to approximate arbitrarily closely another density function. This representation provides the basis for procedure that is developed and discussed.**

I. Introduction

THE PROBLEM of estimating the state of a nonlinear stochastic system from noisy measurement data has been the subject of considerable research interest during the past few years but, although a great deal has been published on the subject, the basic objective of obtaining a solution that can be implemented in a straightforward manner for specific applications has not been satisfactorily realized. This is manifested by the fact that the Kalman filter equations [1], [2], that were derived for linear Gaussian systems, continue to be widely used for nonlinear non-Gaussian systems. Of course, continued application has resulted in the development of ad hoc techniques [3], [4] that have improved the performance of the Kalman filter and that give it some of the characteristics of nonlinear filters.

Central to the nonlinear estimation problem is the determination of the probability density function or the state conditioned on the available measurement data. If this *a posteriori* density function were known, an estimate of the state for any performance criterion could be determined. Unfortunately, although the manner in which the density evolves with time and additional measurement data can be described in terms of difference (or differential) equations, these relations are generally very difficult to solve, either in closed form or numerically, so that it is usually impossible to determine the *a posteriori* density for specific applications. Because of this difficulty, it is natural to investigate the possibility of approximating the density with some tractable form. This approximation problem and one approach to its solution are the subject of this paper.

Since the objective of the work was to determine computational algorithms, attention is directed toward the development of policies that explicitly assume that events occur at discrete instants of time. The approximations discussed below are introduced in order to allow a solution of the estimation problem for the following system.

Suppose that the n-dimensional state vector $\boldsymbol{x}$ of the system evolves according to the nonlinear stochastic difference equation

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}_{k+1}(\boldsymbol{x}_k) + \boldsymbol{w}_k, \qquad k = 0,1,\cdots N-1 \quad (1.1)$$

where $\boldsymbol{w}_k$ represents a Gaussian white-noise sequence with zero mean and covariance Q_k. The system model could be modified to eliminate the additive Gaussian character of the noise, but the generalization serves primarily to complicate the notation and to confuse the discussion. The initial state $\boldsymbol{x}_0$ is assumed to be described by a known density function $p(\boldsymbol{x}_0)$.

The behavior of the plant is observed imperfectly through m measurement quantities $\boldsymbol{z}_k$ that are related in a prescribed fashion to the state but which contain random errors.

$$\boldsymbol{z}_k = \boldsymbol{h}_k(\boldsymbol{x}_k) + \boldsymbol{v}_k, \qquad k = 0,1,\cdots N \quad (1.2)$$

where $\boldsymbol{v}_k$ is a Gaussian white-noise sequence with mean zero and covariance R_k. The noise sequences[1] $\boldsymbol{w}_k$ and $\boldsymbol{v}_k$ are assumed to be mutually independent and independent of the initial state $\boldsymbol{x}_0$.

The state vector $\boldsymbol{x}_k$ is a random variable. In the probabilistic context of this discussion, the *a posteriori* density $p(\boldsymbol{x}_k|\boldsymbol{z}^k)$ provides the most complete description possible of $\boldsymbol{x}_k$. This density is determined recursively from the following relations:

$$p(\boldsymbol{x}_k|\boldsymbol{z}^k) = c_k p(\boldsymbol{x}_k|\boldsymbol{z}^{k-1}) p(\boldsymbol{z}_k|\boldsymbol{x}_k) \quad (1.3)$$

$$p(\boldsymbol{x}_k|\boldsymbol{z}^{k-1}) = \int p(\boldsymbol{x}_{k-1}|\boldsymbol{z}^{k-1}) p(\boldsymbol{x}_k|\boldsymbol{x}_{k-1})\, d\boldsymbol{x}_{k-1} \quad (1.4)$$

where the normalizing constant c_k is given by

$$1/c_k \triangleq p(\boldsymbol{z}_k|\boldsymbol{z}^{k-1}) = \int p(\boldsymbol{x}_k|\boldsymbol{z}^{k-1}) p(\boldsymbol{z}_k|\boldsymbol{x}_k)\, d\boldsymbol{x}_k$$

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[1] Throughout this discussion the use of a superscript $(\ )^k$ will denote the sequence up to and including k. For example, $\boldsymbol{w}^k = \{\boldsymbol{w}_0, \boldsymbol{w}_1, \cdots, \boldsymbol{w}_k\}$.

and the initial condition for (1.3) is

$$p(x_0|z^{-1}) \triangleq p(x_0).$$

The densities $p(z_k|x_k)$ and $p(x_k|x_{k-1})$ are determined from (1.1) and (1.2) and the *a priori* distribution for v_k and w_{k-1}.

In this context the filtering problem can be regarded as having been solved when the density $p(x_k|z^k)$ can be determined for all k. No more complete description of the state is possible. The introduction of an estimation criterion (e.g., minimum error variance, most probable) reduces the available information concerning the state to a finite collection of numbers. As such it is important to recognize that, in general, an estimate obtained using some arbitrary criterion will provide an incomplete and therefore a possibly inadequate description of the state. The principal exception to this conclusion occurs when the system is linear and Gaussian. Then $p(x_k|z^k)$ is Gaussian so it is completely defined in terms of a finite set of parameters, the conditional mean $E[x_k|z^k]$ and covariance $E[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^T|z^k]$.

Except when the system (1.1), (1.2) is linear and the *a priori* distributions are Gaussian, it is generally impossible to determine $p(x_k|z^k)$ in a closed form using (1.3) and (1.4). It is this fact that has led to the dearth of practical nonlinear filter schemes. In the linear Gaussian case the relations describing the conditional mean and covariance of $p(x_k|z^k)$ (and, therefore, $p(x_k|z^k)$) have become famous as the Kalman filter equations [1], [2]. The difficulties associated with the explicit determination of the *a posteriori* density have led to the development of approximate procedures for estimating the state of nonlinear stochastic systems. The most commonly used approximation involves the assumptions that the *a priori* distributions are Gaussian and that the nonlinear system can be linearized relative to prescribed reference values. This approximation produces a system that is linear and Gaussian so that the Kalman filter equations can be utilized. This procedure has come to be known as the "extended" Kalman filter and has seen widespread application to nonlinear systems [5], [6].

The extended Kalman filter has performed satisfactorily in many applications, but there are numerous examples in which unsatisfactory results have been obtained. This has spurred the development of other procedures. Most of these (e.g., [3]–[9]) either implicitly or explicitly retain the assumption that $p(x_k|z^k)$ is Gaussian and essentially provide a means for modifying the mean and covariance of the density. However, the Gaussian assumption greatly reduces the amount of information that is contained in the true density, particularly when it is multimodal. In fact, for multimodal systems the extended Kalman filter operates more as a maximum likelihood estimator than as a minimum variance estimator and the mean follows (hopefully) one of the peaks of the density function. In the following discussion the Gaussian assumption is discarded entirely and the problem of approximating the *a posteriori* density function for nonlinear systems (1.1), (1.2) is attacked directly.

II. Gaussian Sum Approximations

Since it appears to be impossible to obtain exact representations of the *a posteriori* density for nonlinear systems, it is reasonable to seek approximations that will make (1.3) and (1.4) more tractable. The Gram–Charlier and Edgeworth expansions have been proposed and investigated [7], [10]. While this approach has several advantages and its usefulness has been demonstrated in some applications, it has the distinct disadvantage that, when truncated, the resulting series approximation is not itself a valid density function. To avoid or at least reduce the nonpositivity of the approximation, it is sometimes necessary to retain a large number of terms in the series, thereby marring the computational attractiveness of the procedure. Thus it appears to be desirable to seek other approximations.

Many approximations other than the Gram–Charlier series could be considered. For example, Aoki [11] has suggested that the Pearson-type density function be used as an approximation. This type of approximation has the disadvantage that the recursion relations (1.3), (1.4) still cannot be solved explicitly so that it is difficult to determine the parameters of the distribution. As has been noted, (1.3) and (1.4) can be solved when the densities are Gaussian. Thus the attractiveness of the Gram–Charlier series comes from the presence of the Gaussian weighting function. This permits to some extent the properties of the Gaussian to be used in the calculation of $p(x_k|z^k)$ even though non-Gaussian densities are being considered. The Gaussian sum approximation that is discussed below has the advantage provided by the presence of Gaussian terms. Unlike the Gram–Charlier series, any truncation of the Gaussian sum is a valid density function. However, this series is nonorthogonalizable so that it is extremely difficult to obtain meaningful error bounds. Nonetheless, the practical utility of the approach is sufficiently clear that the approximation merits serious consideration for nonlinear filtering problems.

The idea of using a weighted sum of Gaussian density functions for approximation of the *a posteriori* density function has been suggested before. Aoki [12] mentioned this possibility briefly in 1965. Cameron [13] and Lo [14] as well as the present authors [15], [16] have recently used an approximation of this type for linear systems with Gaussian noise but a non-Gaussian distribution for the initial state. In all of these $p(x_0)$ was approximated by a Gaussian sum and then recursion relations were derived for the parameters of the sum for all subsequent stages. In addition, the application of this approximation to linear systems with non-Gaussian plant and measurement noise has been investigated by Alspach and Soren-

son [15], [16]. The succeeding discussion relates to the manner in which the Gaussian sum approximation can be used for the nonlinear system with nonlinear measurements (1.1), (1.2).

The Gaussian sum representation p_A of a density function p associated with a vector-valued random variable $\boldsymbol{x}$ is defined as[2]

$$p_A(\boldsymbol{x}) \triangleq \sum_{i=1}^{l} \alpha_i N(\boldsymbol{x} - \boldsymbol{a}_i, B_i) \tag{2.1}$$

where

$$\sum_{i=1}^{l} \alpha_i = 1, \qquad \alpha_i \geq 0 \text{ for all } i.$$

It can be shown [14]–[16] that p_A converges uniformly to any density function of practical concern as the number of terms l increase and the covariance B_i approaches the zero matrix.

The parameters α_i, $\boldsymbol{a}_i$, B_i of the sum (2.1) can be selected in various ways [15] but it is convenient to regard them in the following manner. The mean values $\boldsymbol{a}_i$ are used to establish a grid in the region of state space that contains the probability mass (or at least the significant part of it). The α_i are chosen as the normalized values $p(\boldsymbol{a}_i)$ of the density p that is to be approximated (i.e., the $p(\boldsymbol{a}_i)$ are normalized so that $\sum_{i=1}^{n} \alpha_i = 1$). Finally, it is convenient to set all of the covariance matrices B_i equal to bI where b is a positive scalar. The value of b is determined so that the error in the approximation $(p - p_A)$ is minimized in some prescribed sense. For example, b can be chosen to minimize the L^1 error $\int |p - p_A| \, d\boldsymbol{x}$. Note that as b tends to zero, the Gaussian terms each approach a unit impulse function located at the mean values $\boldsymbol{a}_i$. Thus, for small but nonzero values of b, it is important to realize that a term of the Gaussian sum is effectively equal to zero everywhere except in a small neighborhood of $\boldsymbol{a}_i$. This type of procedure can be used to obtain an approximation of the *a priori* densities in a problem. Then, the *a posteriori* density is determined in the manner described in the next section.

III. Gaussian Sum Filters for Nonlinear Systems

The most frequently used approach for the estimation of the state of a nonlinear system involves the linearization of the system relative to prescribed reference values and the subsequent application of the Kalman filter equations. This approach results in approximations of two types. First, the nonlinear physical system is replaced by a linear model, and, second, the non-Gaussian *a posteriori* density function $p(\boldsymbol{x}_k|\boldsymbol{z}^k)$ is approximated by a Gaussian density. While this procedure must be regarded as having been singularly successful, it has by no means been satisfactory in all applications. Certainly, in those cases in which the extended Kalman filter exhibits unsatisfactory performance, the inadequacy of the linear, Gaussian approximation of the nonlinear, non-Gaussian system must be blamed. Many methods have been proposed to compensate for these approximation errors (e.g., divergence control methods [3], [4]), second-order methods [17], [19], and iterative procedures [17], [18] and most retain the assumption that $p(\boldsymbol{x}_k|\boldsymbol{z}^k)$ is Gaussian. While effectively eliminating divergence in many cases, it would appear that the optimality of the data processing is often degraded substantially because of the inappropriateness of the Gaussian assumption.

In the following paragraphs a filtering procedure is defined that is based on the utilization of the Gaussian sum representation of the *a posteriori* density function in conjunction with the linearization procedure that has proven so effective in Kalman filter applications. The procedure greatly extends the range of validity of the approximation of the physical system and eliminates the assumption that $p(\boldsymbol{x}_k|\boldsymbol{z}^k)$ is Gaussian.

In the Gaussian sum suppose that a sufficient number of terms are included that the covariance B_i (or the parameter b where $B_i = bI$ for all i) can be chosen small enough that each term is effectively zero outside a neighborhood of $\boldsymbol{a}_i$. Suppose that

$$p(\boldsymbol{x}_k|\boldsymbol{z}^{k-1}) = \sum_{i=1}^{\xi_k'} \alpha_{ki}' N(\boldsymbol{x}_k - \boldsymbol{a}_{ki}, P_{ki}') \tag{3.1}$$

where P_{ki}' is small in some sense (e.g., the maximum eigenvalue of P_{ki}' is less than a parameter s for all i). Then, from (1.3), one can write

$$p(\boldsymbol{x}_k|\boldsymbol{z}^k) = c_k \left[\sum_{i=1}^{\xi_k'} \alpha_{ki}' N(\boldsymbol{x}_k - \boldsymbol{a}_{ki}, P_{ki}') p(\boldsymbol{z}_k - \boldsymbol{h}_k(\boldsymbol{x}_k)) \right]. \tag{3.2}$$

To put (3.2) into a more tractable form, consider the general term

$$[N(\boldsymbol{x}_k - \boldsymbol{a}_{kl}, P_{kl}') p(\boldsymbol{z}_k - \boldsymbol{h}_k(\boldsymbol{x}_k))].$$

As in the extended Kalman filter, linearize $\boldsymbol{h}_k$ relative to $\boldsymbol{a}_{kl}$ so that $p(\boldsymbol{z}_k - \boldsymbol{h}_k(\boldsymbol{x}_k))$ can be approximated by a Gaussian in the region around $\boldsymbol{a}_{kl}$. Since the covariance P_{kl}' is small, the region in which the approximation must be valid is correspondingly small (obviously much smaller than the region that would result if $p(\boldsymbol{x}_k|\boldsymbol{z}^{k-1})$ were represented by a single Gaussian). With this approximation (3.2) can be rewritten as

$$p(\boldsymbol{x}_k|\boldsymbol{z}^k) = c_k \left[\sum_{i=1}^{\xi_k'} \alpha_{ki}' N(\boldsymbol{x}_k - \boldsymbol{a}_{ki}, P_{ki}') N(\boldsymbol{\zeta}_i, R_k) \right]$$

where

$$\boldsymbol{\zeta}_i \triangleq \boldsymbol{z}_k - \boldsymbol{h}_k(\boldsymbol{a}_{ki}) - H_{ki}[\boldsymbol{x}_k - \boldsymbol{a}_{ki}]$$

and

$$H_{ki} \triangleq \frac{\partial \boldsymbol{h}_k}{\partial \boldsymbol{x}_k}(\boldsymbol{a}_i).$$

[2] $N(\boldsymbol{a}, B) \triangleq \exp\{-\frac{1}{2}\boldsymbol{a}^T B^{-1} \boldsymbol{a}\}/(2\pi)^{n/2}|B|^{\frac{1}{2}}$.

But this reduces to

$$p(x_k|z^k) = \sum_{i=1}^{\xi_k} \alpha_{ki} N(x_k - \mu_{ki}, P_{ki}) \tag{3.3}$$

where

$$\mu_{ki} = a_{ki} + K_{ki}(z_k - h_k(a_{ki}))$$
$$P_{ki} = P_{ki}' - K_{ki}H_{ki}P_{ki}'$$
$$K_{ki} = P_{ki}'H_{ki}^T[H_{ki}P_{ki}'H_{ki}^T + R_k]^{-1}$$

and

$$\alpha_{ki} = \frac{\alpha_{ki}'\beta_{ki}}{\sum_{j=1}^{\xi_k'} \alpha_{kj}'\beta_{kj}}$$

where

$$\beta_{ki} \triangleq N(z_k - h_k(a_{ki}), H_{ki}P_{ki}'H_{ki}^T + R_k).$$

The number of terms in the sum ξ_k are the same as the previous sum ξ_k' except for circumstances discussed later.

The preceding discussion has described a method whereby the Gaussian sum representation for $p(x_k|z^k)$ is derived from $p(x_k|z^{k-1})$. One sees that $p(x_k|z^k)$ is represented by a finite number of parameters and that the parameters are obtained using the extended Kalman filter equations. Thus, the Gaussian sum representation is formed as the convex combination of the output of several Kalman filters operating in parallel.

Consider the problem of obtaining the Gaussian sum representation of $p(x_{k+1}|z^k)$. Two cases must be treated. First, when there is little or no plant noise, one proceeds essentially as described above. However, when there is a significant amount of plant noise, it may be necessary to introduce a Gaussian sum representation for $p(w_k)$.

Suppose the covariance of the plant noise is comparable to that of the covariance of the terms of the Gaussian sum P_{ki}. Then, linearize the plant equation f_{k+1} relative to the mean values of the Gaussian sum μ_{ki}. Using (1.4), it follows that

$$p(x_{k+1}|z^k) = \sum_{i=1}^{\xi_{k+1}'} \alpha'_{(k+1)i} N(x_{k+1} - a_{(k+1)i}, P_{(k+1)i}) \tag{3.4}$$

where

$$a_{(k+1)i} = f_{k+1}(\mu_{ki})$$
$$P_{(k+1)i}' = F_{(k+1)i}P_{ki}F_{(k+1)i}^T + Q_k$$
$$\alpha_{(k+1)i}' = \alpha_{ki}$$
$$\xi_{k+1}' = \xi_k$$

and

$$F_{(k+1)i} \triangleq \frac{\partial f_{k+1}}{\partial x_k}(\mu_{ki}).$$

Note in (3.4) that $p(x_{k+1}|z^k)$ has the same weighting coefficients as $p(x_k|z^k)$. The effect of the plant nonlinearity is to modify the parameters of the Gaussian sum terms. The mean values μ_{ki} become equal to $a_{(k+1)i}$ and the covariances P_{ki} are modified to $P_{(k+1)i}'$.

If the plant noise covariance Q_k is large compared to the Gaussian sum covariance P_{ki}, then it may be necessary to alter the calculation of $p(x_{k+1}|z^k)$. This is necessary since the large plant noise will increase the variance of each term in the Gaussian sum, thereby creating a large overlap of the individual terms. Under these conditions the various linearizations are not as valid, but more importantly, the next measurement will cause the various terms to have nearly the same mean and they will then be combined into one term. Under these conditions the methods discussed here will reduce to the extended Kalman filter. To circumvent this difficulty associated with large plant noise, $p(w_k)$ is itself approximated by a Gaussian sum (even though it is Gaussian)

$$p(w_k) = \sum_{l=1}^{q_k} \gamma_{kl} N(w_k - \omega_{kl}, Q_{kl}). \tag{3.5}$$

Then, using (3.5) and linearizing relative to μ_{ki},

$$p(x_{k+1}|x_k) = \sum_{l=1}^{q_k} N(x_{k+1} - f_{k+1}(\mu_{ki}) - F_{(k+1)i}(x_k - \mu_{ki}) - \omega_{kl}, Q_{kl}). \tag{3.6}$$

Inserting (3.3) and (3.6) into (1.4), one obtains

$$p(x_{k+1}|z^k) = \sum_{i=1}^{\xi_{k+1}'} \alpha_{(k+1)i}' N(x_{k+1} - a_{(k+1)i}, P_{(k+1)i}') \tag{3.7}$$

where

$$\xi_{k+1}' = \xi_k q_k$$
$$\alpha_{(k+1)i}' = \alpha_{kj}\gamma_{kl}$$
$$a_{(k+1)i}' = f_{k+1}(\mu_{kj}) + \omega_{kl}$$
$$P_{(k+1)i}' = F_{(k+1)i}P_{kj}F_{(k+1)i}^T + Q_{kl}.$$

Observe in this case that the number of terms in the Gaussian sum have increased. The growth in the number of terms could seriously reduce the practical utility of this approximation if there were no alleviating circumstances. However, as discussed in [15], it is possible to combine many terms into a single term without seriously affecting the approximation. Also, the weighting factors $\alpha_{(k+1)i}'$ of some terms become sufficiently small that these terms can be neglected and the number of terms reduced. As a consequence of these mechanisms, the total number of terms in the Gaussian sum generally do not increase significantly.

To start the procedure, it is necessary that the probability density function prescribed for the initial state $p(x_0)$ be represented as a Gaussian sum

$$p(x_0) = \sum_{i=1}^{\xi_0'} \alpha_0' N(x_0 - a_0', P_0'). \tag{3.8}$$

Thereafter, one uses (3.3) and either (3.4) or (3.5) to recursively compute $p(x_k|z^k)$ and $p(x_{k+1}|z^k)$. To reduce the computational requirements, it is necessary at each

stage to combine and to neglect terms whenever possible.

It is possible to state [16] theoretical criteria to obtain a measure of the error introduced at each stage by the linearizations. However, these bounds are difficult to implement for a filtering application. Instead, the performance of the Gaussian sum filter is judged by the behavior of the measurement residual.

$$r_k = z_k - E[h_k(x_k)|z^{k-1}]. \tag{3.9}$$

If the observed behavior of residual is inconsistent with its theoretical properties, then it must be concluded that the filter is inadequate (i.e., divergence is occurring) and it is necessary to increase the number of terms in the Gaussian sum in order to reduce approximation errors. This requires a reinitialization of the entire procedure. In the results that are presented in the following section, no attempt was made to use the smallest possible number of terms in the Gaussian sum. Consequently, the approximations appear to introduce negligible error and the residual is always consistent.

IV. Numerical Examples

The Gaussian sum filter is applied to two specific systems and some typical results of this study are presented in this section.

A. Quadratic Scalar Example

Consider the scalar system with the plant described by

$$x_{k+1} = x_k + \eta x_k^2 + w_k. \tag{4.1}$$

The state x_k is to be estimated from the measurement data z^k where

$$z_k = x_k^2 + v_k, \qquad k = 0,1,\cdots. \tag{4.2}$$

The initial state and the plant and measurement noise sequences are independent, white, and Gaussian with

$$E(\hat{x}_0') = 1; \qquad E[(x_0 - \hat{x}_0')^2] = 1;$$
$$E(w_k) = E(v_k) = 0;$$
$$E(w_k^2) = \sigma_w^2; \qquad E(v_k^2) = \sigma_v^2.$$

The *a priori* mean and variance of the initial state are held at these values for all examples presented here, although others have been investigated. The basic parameters of the system in the present study are the variances of the plant and measurement noise and the relative effect of the plant nonlinearity η. These variances have been chosen to be independent of k for clarity of presentation only. The value of each of these parameters will be specified for each case presented.

Results for four different filters are presented and discussed, although not all results are included in the attached figures. When a filter performs very badly, it may fall off the scale of the charts and thus not be shown. The first three are filters that have been considered previously in the literature and in which the *a posteriori* and predicted density functions are assumed to be Gaussian at each stage. The first of these is the extended Kalman filter. This is the filter most often used in practice. The second filter uses one iteration to improve the reference values used in the linearization. The third filter is the Gaussian filter of [19], where second-order terms are used to modify the mean and variance of the next stage predicted and *a posteriori* density functions. The fourth is the Gaussian sum filter of Section III.

The characteristics of the filtering problem depend heavily on the position of the state variable x_k with respect to the point of symmetry of the measurement nonlinearity. When x_k is near that point (zero in this case), the ratio x_k^2/σ_v is small and the Gaussian filters tend to diverge. As the state moves away from this point, the measurement nonlinearity becomes increasingly more negligible and the Gaussian filters tend to perform well. This is particularly clear when there is no plant nonlinearity $\eta = 0$ and no plant noise $\sigma_w = 0$. In this case the relative performance of the different filters depends most strongly on the value of the state variable and less on the particular measurement realization under consideration. For this reason it was found best with a limited number of realizations to choose the true initial value of state as a parameter and only select the measurement and plant noise from a random number generator. This was particularly useful in the Monte-Carlo averages, but was done in all the cases presented below. Two types of sample averages were computed to summarize the results of the limited Monte-Carlo studies that were conducted for each of the four filters.

$$\epsilon_k = \frac{1}{N}\sum_{i=1}^{N} |x_k^{(i)} - \hat{x}_k^{(i)}|$$

$$A_k^2 = \frac{1}{N}\sum_{i=1}^{N} \left(\frac{x_k^{(i)} - \hat{x}_k^{(i)}}{\sigma_k^{(i)}}\right)^2$$

where N = number of realizations. Note that if both the filter estimate $\hat{x}_k$ and the covariance σ_k^2 are correct, the parameter A_k^2 will asymptotically approach a value of one. When A_k^2 is greater than one, it implies that the filter has diverged to some extent. On the other hand, if A_k^2 is less than one, the filter is predicting too large a variance and is therefore conservative.

When there is no plant nonlinearity [$\eta = 0$ in (4.1)], it is impossible from the available measurement data to discriminate between the true value of the state and the negative of that value. Thus $p(x_k|z^k)$ should become bimodal if the value of the state is nonzero. This is, of course, not possible for any of the Gaussian filters. When there is no plant noise or nonlinearity, the *a posteriori* density can be computed exactly. Under these conditions it is (except for a normalization constant) simply given by

$$P(x_k|z^k) = cP_{x_0}(x_0)P_{v_0}(z_0 - h(x_0)) \cdots P_{v_k}(z_k - h(x_k)).$$

The density function of a specific realization is depicted in Fig. 1. The values of the system parameters are stated in the figure. The Gaussian sum filter provided an ap-

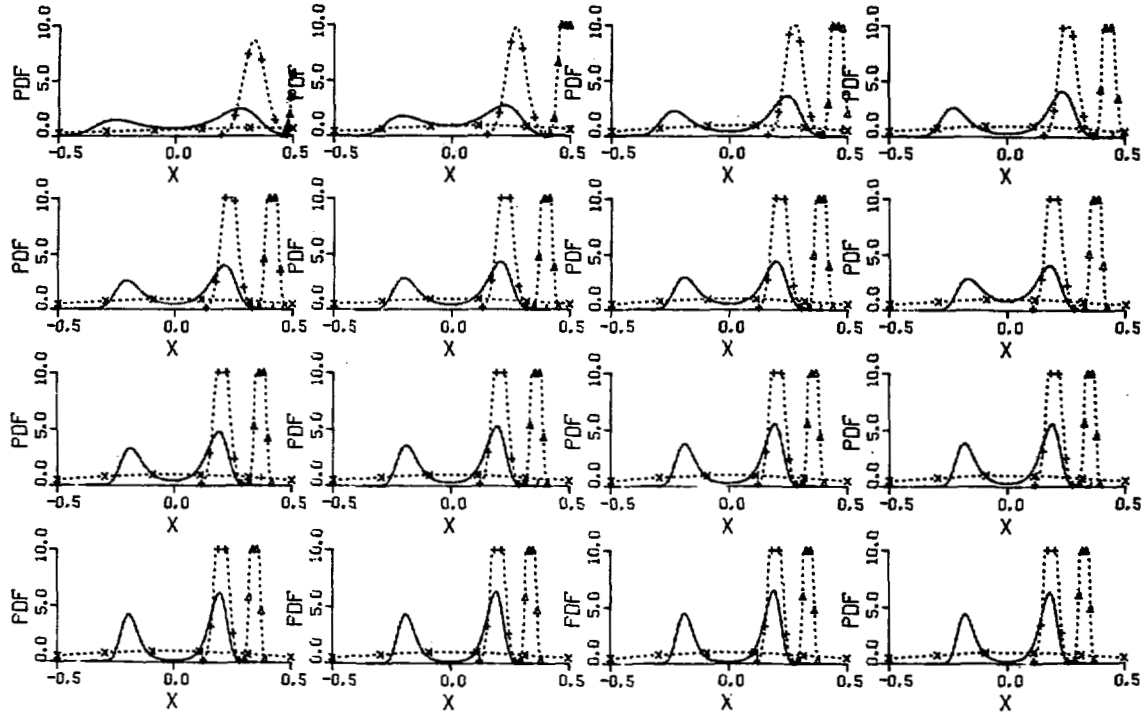

Fig. 1. Filtering density and approximations. Solid line is true PDF. Broken line is Gaussian sum. $x \cdots x$ is second order. $+ \cdots +$ is iterated.

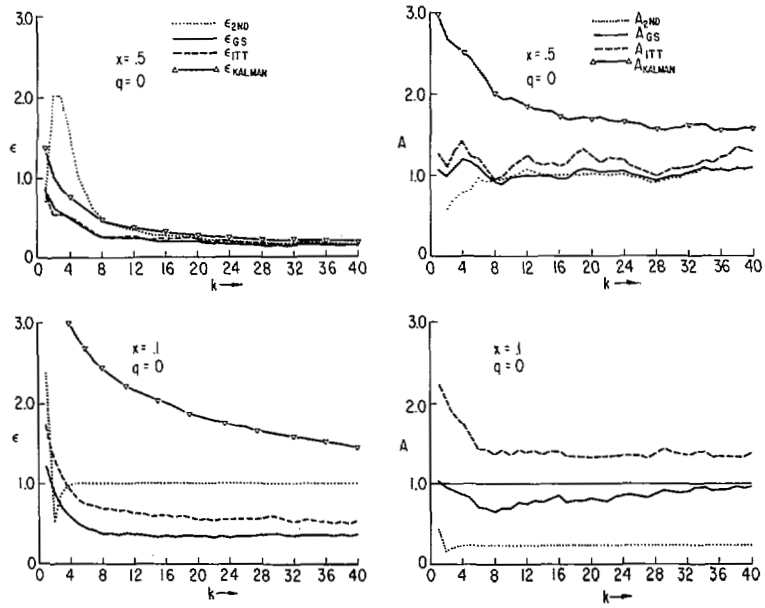

Fig. 2. Monte-Carlo averages. $\eta = 0$, $\sigma_v = 0.1$, $q = 0$, and $N = 50$.

proximation that is indistinguishable from the true *a posteriori* density for the example. In this case the *a priori* density $p(x_0)$ was approximated by a sum of 40 Gaussians. Observe that the second-order filter provides an extremely conservative result and estimates the state to be zero instead of ± 0.2. The extended Kalman filter tends to diverge. Only the iterated filter performs at all satisfactorily and provides an estimate of approximately 0.2.

It is interesting that the minimum variance estimate that one would obtain from $p(x_k|z^k)$ provides an estimate that is between the two peaks (i.e., since the conditional mean is the minimum variance estimate). Clearly, this estimate is very conservative and, consequently, may be unsatisfactory. A maximum likelihood estimate would yield a value close to the true value or its negative.

Sample averages for two cases are depicted in Fig. 2 and corroborate that the behavior shown in Fig. 1 is typical. In the first case, all four filters perform reasonably well. Even in this case the extended Kalman filter tends to diverge since A_k^2 is greater than one. Fig. 2(b) depicts a case in which only the Gaussian sum filter performs satisfactorily. It should be mentioned that for comparative

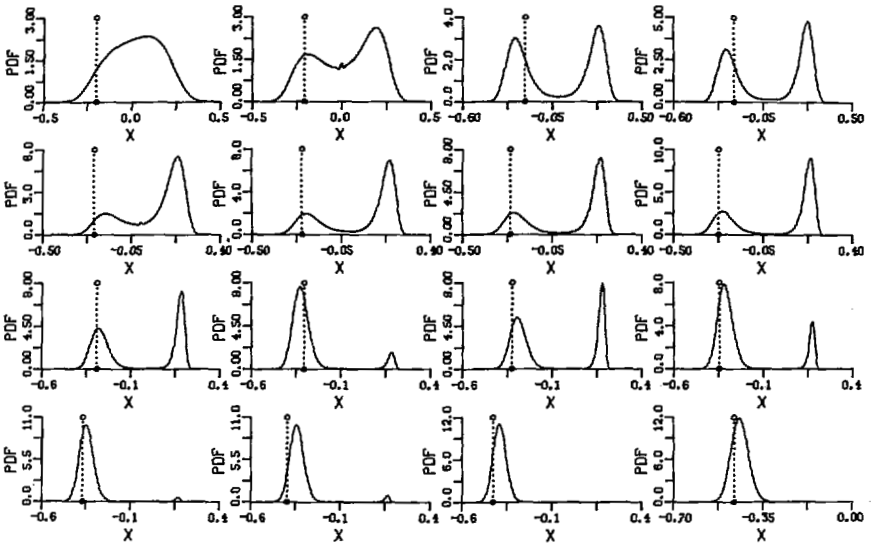

Fig. 3. Gaussian sum approximation to filtering density for nonlinear plant and measurement. Solid line is Gaussian sum PDF. o · · · ● is true value of state. $x_0 = -0.2$, $\eta = -0.2$, $\sigma_w = 0$, and $\sigma_v = 0.05$.

purposes the right-hand peak of the *a posteriori* density is renormalized and treated as the entire density function.

When a plant nonlinearity from (4.2) is included, it is possible to distinguish between the two values and the Gaussian sum filter quickly selects the proper peak. This is shown in Fig. 3 where the value of η is -0.2. Since the state has a negative value, the Gaussian filters all perform unsatisfactorily, so only the results of the Gaussian sum filter are shown. This example demonstrates the difficulty that a maximum likelihood estimator might encounter. It is observed that the maximum value of $p(x_k|z^k)$ switches back and forth from positive to negative. Without complete knowledge of the density function, it is unlikely that a procedure could be devised that would reflect this behavior.

B. Vector Tracking Example

The example discussed here was taken from the paper by Bucy *et al.* [20] on passive receiver design. This is a simple example of a "practical, interesting, and computationally challenging two-dimensional problem." The state vector propagates according to the linear plant

$$x_{k+1} = x_k + w_k \qquad x_k = \begin{pmatrix} x_k \\ y_k \end{pmatrix}$$

and the state is observed by the scalar nonlinear measurement function

$$z_k = h_k(x_k) + v_k$$

where

$$h_k(x_k) = \tan^{-1}\left[(y_k - \sin\beta_k)/(x_k - \cos\beta_k)\right]$$

$$\beta_k = \beta_0 + \dot{\beta}(k-1)$$

where β_0 and $\dot{\beta}$ are given constants. The statistics of the *a priori* random variables x_0, v_k, and w_k that are white, independent, Gaussian random variables are

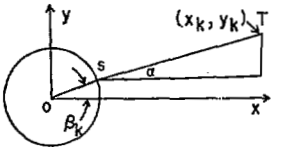

Fig. 4. Geometrical definition of vector tracking example.

$$E(v_k) = 0 \qquad E(w_k) = 0 \qquad E(x_0) = \hat{x}_0'$$

$$E(v_k v_j) = \sigma_v^2 \delta_{kj} \qquad E(w_k w_j^T) = q^2 I \delta_{kj}$$

$$E[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T] = P_0'.$$

The preceding model arises in connection with the tracking geometry of Fig. 4 where target T at the position defined $x_k^T = (x_k\ y_k)$ is undergoing a random walk in the two-dimensional state space.

The observer S is passively measuring the line-of-sight α as it travels in a deterministic orbit around the unit circle.

Results obtained from the application of the Gaussian sum filter to a specific example are shown in Fig. 5. The position of the observer is shown by the cross on the unit orbit and the cross on the density function shows the true position of the target. The *a priori* estimate for the initial state was taken to be

$$\hat{x}_0' = \begin{pmatrix} 2 \\ -0.2 \end{pmatrix} \qquad P_0' = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix},$$

while the true value of the initial state (and all subsequent values since there is no plant noise) was taken to be

$$x_k = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}.$$

The measurement noise has a one sigma value of 0.01 rad or about one-half degree. The non-Gaussian *a posteriori* filtering-density function is seen to propagate from stage 1 to stage 9 in this figure where a measurement is taken every 10°.

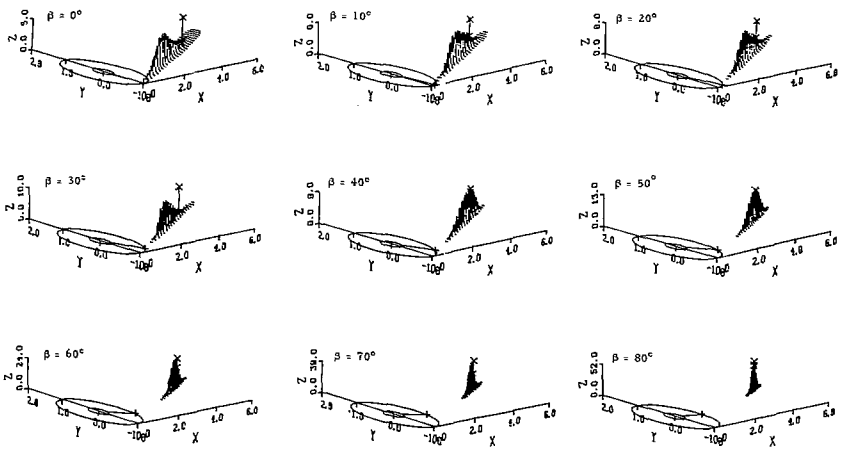

Fig. 5. Filtering density for vector tracking example.

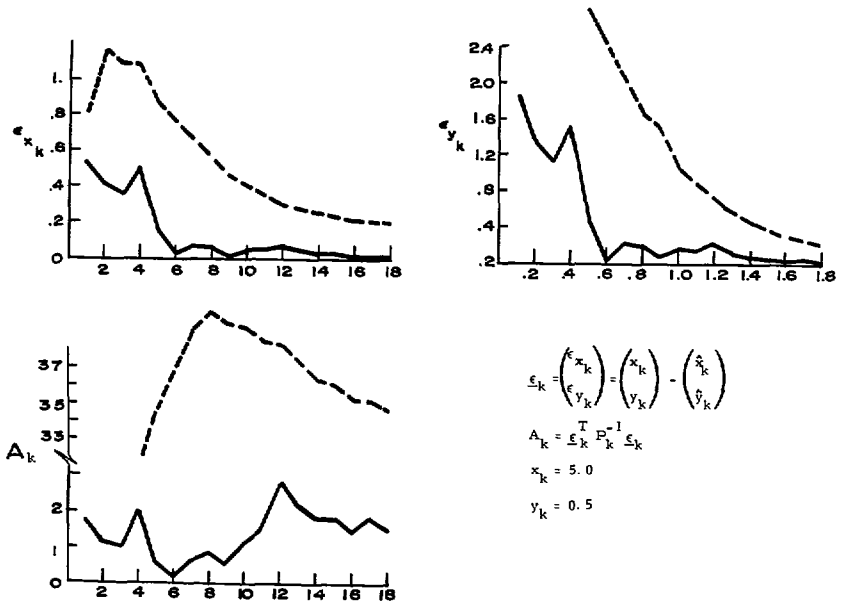

Fig. 6. Relative performance of extended Kalman and Gaussian sum filters for tracking problem. Broken line denotes Kalman. Solid line denotes Gaussian sum.

In Fig. 6 results obtained using the extended Kalman filter and the Gaussian sum filter of Fig. 5 are compared. The parameters ϵ_{xk}, ϵ_{yk}, and A_k for a single realization are presented. The improvement provided by the Gaussian sum filter is striking.

Last, plant noise was added to the system and the initial conditions and *a priori* statistics were changed to be consistent with the Monte-Carlo results presented in [20]. The plant and measurement noise were again white Gaussian sequences with $\sigma_v^2 = 0.1$, $\beta_0 = 0°$, $\beta = 1$ rad/stage, $P_0 = I$, and

$$\Phi = \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix} \quad \hat{x}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad Q_k = \begin{pmatrix} 0.1 & 0.05 \\ 0.05 & 0.1 \end{pmatrix}.$$

A Monte-Carlo average of 100 runs was performed filtering the same sample paths for the Gaussian sum and extended Kalman filters. The results are presented in Table I. With the plant noise added, the increased measurement noise, and the higher rate of rotation of the observer in its orbit, the extended Kalman filter performance with respect to the Gaussian sum filter was greatly improved. Table I shows the results of the filtered estimate for 16 stages. The average error, average covariance, and average divergence parameter are presented. It should be noted from the average covariance that the optimal filter continues to give superior performance even at the later stages. The divergence parameter, which should be such that $\bar{A} = E(A) = 2$ for an infinite number of runs, is consistently larger for the extended Kalman filter. The large values exhibited by this parameter for some stages result from sample paths for which the covariance of the extended Kalman filter becomes nearly singular. It is worth noting that the results reported here correspond closely with results obtained by Bucy and Senne using the method described in [20]. This has been established in private discussions in which both methods were applied to

TABLE I
MONTE-CARLO RESULTS FOR FILTERING DENSITY $p(x_n|z^n)$

	Gaussian Sum		Extended Kalman		Gaussian Sum			Extended Kalman			Gaussian Sum	Extended Kalman
n	ϵ_1	ϵ_2	ϵ_1	ϵ_2	ϵ_1^2	$\epsilon_1\epsilon_2$	ϵ_2^2	ϵ_1^2	$\epsilon_1\epsilon_2$	ϵ_2^2	A	A
0	−0.134	0.074	0.007	0.172	0.499	−0.061	0.319	1.055	0.054	0.485	2.28	6.39
1	−0.036	−0.073	0.091	0.069	0.161	0.072	0.238	0.334	0.227	0.516	2.13	6.97
2	0.055	−0.043	−0.008	0.081	0.070	0.027	0.249	0.134	0.089	0.448	2.33	8.57
3	0.090	0.003	−0.005	0.065	0.393	0.057	0.262	0.100	0.035	0.251	2.55	2.70
4	0.035	0.045	−0.044	0.010	0.145	0.087	0.263	0.155	0.078	0.257	3.12	3.16
5	−0.047	0.046	−0.044	0.009	0.106	0.083	0.271	0.109	0.076	0.302	2.76	3.56
6	0.036	0.058	0.083	0.052	0.100	0.015	0.178	0.113	0.019	0.215	2.25	2.28
7	−0.053	−0.011	0.004	−0.019	0.095	0.041	0.211	0.121	0.067	0.243	2.72	3.53
8	−0.036	−0.120	−0.041	−0.168	0.074	0.054	0.251	0.087	0.072	0.331	2.87	11.73
9	0.033	−0.048	−0.031	−0.073	0.090	0.045	0.274	0.093	0.062	0.316	2.52	2.33
10	−0.012	−0.095	−0.090	−0.156	0.081	0.053	0.238	0.088	0.064	0.286	2.30	2.27
11	0.026	0.000	0.020	−0.075	0.085	0.070	0.327	0.121	0.095	0.420	2.68	12.40
12	−0.132	−0.050	−0.097	−0.126	0.133	0.069	0.350	0.147	0.093	0.445	3.02	3.13
13	−0.089	−0.003	−0.001	−0.024	0.090	0.020	0.279	0.109	0.045	0.377	2.30	2.38
14	−0.065	−0.025	−0.049	0.015	0.077	0.027	0.257	0.091	0.040	3.369	2.44	2.87
15	−0.022	−0.081	−0.083	−0.086	0.076	−0.022	0.337	0.104	0.015	0.370	2.89	3.19

the same measurement realizations and with all other assumptions identical.

V. SUMMARY AND CONCLUSIONS

The Gaussian sum approximation has been introduced and proposed as the means whereby practical nonlinear Bayesian filtering can be accomplished. A procedure for recursively forming the Gaussian sum representation of $p(x_k|z^k)$ and $p(x_{k+1}|z^k)$ has been outlined. As pointed out in the text, the procedure essentially results in the parallel operation of several Kalman filters. In fact there are as many individual filters as there are terms in the Gaussian sum. The convex combination of these filters is formed to obtain the Gaussian sum representation of the *a posteriori* density. Having this density, estimates based on any error criterion can be obtained in a relatively straightforward manner.

As seems to be characteristic of nonlinear filters, the computational requirements of the Gaussian sum filter can be significantly greater than those of the extended Kalman filter. In fact, the observation that the Gaussian sum filter is essentially obtained as the convex combination of several Kalman filters provides an immediate indication of the relative requirements. Nonetheless, the fact that the equations defining the parameters of each term of the sum are the same provides a very easy computer implementation of this nonlinear filter.

Some theoretical bounds on the errors in the approximation of the *a posteriori* density function have been discussed elsewhere [16]. These results are difficult to apply in a specific problem. As in Kalman filter applications, the behavior of the measurement residual provides a practical measure of the performance of the filter. As long as the residual is consistent with its theoretical properties, one can be satisfied that the Gaussian sum filter is performing satisfactorily. If an inconsistency occurs, it may be necessary to reinitialize the filter (i.e., decrease the magnitude of the covariance and add more terms to the sum). In the examples presented here, a sufficiently large number of terms were included so that no divergence problems were encountered.

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On Optimum Distributed-Parameter Filtering and Fixed-Interval Smoothing for Colored Noise

SPYROS G. TZAFESTAS

Abstract—**The optimum filtering problem for a general class of linear distributed-parameter systems with colored observation noise is studied. The input stochastic disturbance is assumed to be white in time, but it may have correlation in space of any type. The optimum filter is derived through a learning theorem which gives the mean value and covariance matrix of a conditional distributed-parameter random variable $X_1(D)$ given $X_2(D)$ where $X_1(D) = \{X_1(x); x \in D\}$ and $X_2(D) = \{X_2(x); x \in D\}$ are Gaussian variables with known mean values and covariance matrices. The fixed-interval smoothing problem for the same class of systems is then considered and solved with the aid of a distributed-parameter theorem concerning the combination of two independent estimates of the state based on different data records. A numerical filtering example is included to illustrate the theory. The results of the paper may find applications in all fields where the information to be processed is distributed in space and depends either on continuous or on discrete time.**

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I. Introduction

THE OPTIMUM filter design problem is of primary importance in both the communication and control fields [1], [2]. In recent years a generalized filtering theory was developed for lumped-parameter stochastic signals and systems. The present paper is a continuation of an attempt to develop an analogous theory for the distributed-parameter case [3], [4]. This theory should be useful not only in the communication [5] or the adaptive control areas [6] but also, generally, in all fields where the information to be processed is distributed in space (e.g., meteorology, radioastronomy, etc.) or depends on some other parameters, besides time, which may be considered spatial.

The contribution of the paper is threefold. First, a number of important lumped-parameter filtering results [7]–[11] are extended for dealing with distributed-parameter systems. Second, a simpler derivation of the filter is presented employing conditional generalized variable techniques [12]. Third, the fixed-time smoothing filter is derived by using the filtering results together with a