Daniel's class notes

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1 The PHD filter prediction

Consider a multi-object Chapman-Kolmogorov equation of the form:

$$G_{k+1|k}(h) = G_\gamma(h) \cdot G_k(G_{s,k+1|k}(h|\cdot)),$$

with prior

$$G_k(h) = \sum_{n\geq 0} \frac{1}{n!} \int \prod_{i=1}^{n} h(x_i) P_k(x_1, \ldots, x_n) \mathrm{d}x_1 \ldots \mathrm{d}x_n,$$

and first moment $D_k(x)$, Poisson birth model

$$G_\gamma(h) = \exp\left(\int \gamma(x)(h(x) - 1)\mathrm{d}x \right),$$

and Markov model

$$G_{s,k+1|k}(h|y) = 1 - p_s(y) + p_s(y) \int h(x) \underbrace{f_{k+1|k}(x|y)}_{\text{Markov kernel}} \mathrm{d}x.$$

Theorem 1. *Suppose that we have the model above. Then, the first-order moment is given by*

$$D_{k+1|k}(x) = \gamma(x) + \int p_s(y) D_k(y) f_{k+1|k}(x|y)\mathrm{d}y.$$

Proof. Find $D_{k+1|k}(x) = \delta G_{k+1|k}(h; \varphi)\big|_{h=1,\varphi=\delta_x}$. □

2 Gaussian mixture prediction

Suppose that $\gamma(x)$ and $D_k(x)$ are mixture of Gaussian densities, and that $f_{k+1|k}(x|y)$ is a linear Gaussian kernel, and $p_s(y) = p_s$ is a scalar constant.

Exercise 2. Determine the form of $D_{k+1|k}(x)$.

Solution. Given the assumptions

$$\begin{aligned}
\gamma(x) &= \sum_{i=1}^{n_\gamma} w_\gamma^{(i)} \mathcal{N}(x|\mu_\gamma^{(i)}, \Sigma_\gamma^{(i)}), \\
D_k(x) &= \sum_{i=1}^{n_k} w_k^{(i)} \mathcal{N}(x|\mu_k^{(i)}, \Sigma_k^{(i)}), \\
f_{k+1|k}(x|y) &= \mathcal{N}(x|\mathrm{F}y, \mathrm{Q}),
\end{aligned}$$

then

$$
\begin{aligned}
D_{k+1|k}(x) &= \gamma(x) + \int p_s(y) D_k(y) f_{k+1|k}(x|y)\mathrm{d}y \\
&= \sum_{i=1}^{n_\gamma} w_\gamma^{(i)} \mathcal{N}(x|\mu_\gamma^{(i)}, \Sigma_\gamma^{(i)}) + \int p_s \mathcal{N}(x|\mathrm{F}y, \mathrm{Q}) \sum_{j=1}^{n_k} w_k^{(j)} \mathcal{N}(y|\mu_k^{(j)}, \Sigma_k^{(j)})\mathrm{d}y \\
&= \sum_{i=1}^{n_\gamma} w_\gamma^{(i)} \mathcal{N}(x|\mu_\gamma^{(i)}, \Sigma_\gamma^{(i)}) + \sum_{j=1}^{n_k} p_s w_k^{(j)} \int \mathcal{N}(x|\mathrm{F}y, \mathrm{Q}) \mathcal{N}(y|\mu_k^{(j)}, \Sigma_k^{(j)})\mathrm{d}y \\
&= \sum_{i=1}^{n_\gamma} w_\gamma^{(i)} \mathcal{N}(x|\mu_\gamma^{(i)}, \Sigma_\gamma^{(i)}) + \sum_{j=1}^{n_k} p_s w_k^{(j)} \mathcal{N}(x|\mathrm{F}\mu_k^{(j)}, \mathrm{F}\Sigma_k^{(j)}\mathrm{F}^{\mathrm{T}} + \mathrm{Q}) \\
&\equiv \sum_{\ell=1}^{n_{k+1|k}} w_{k+1|k}^{(\ell)} \mathcal{N}(x|\mu_{k+1|k}^{(\ell)}, \Sigma_{k+1|k}^{(\ell)}),
\end{aligned}
$$

where $n_{k+1|k} = n_\gamma + n_k$,

$$
w_{k+1|k}^{(\ell)} = \begin{cases} w_\gamma^{(i)}, & \ell = i, \quad i \in [1..n_\gamma], \\ p_s w_k^{(j)}, & \ell = n_\gamma + j, \quad j \in [1..n_k]. \end{cases}
$$

$$
\mu_{k+1|k}^{(\ell)} = \begin{cases} \mu_\gamma^{(i)}, & \ell = i, \quad i \in [1..n_\gamma], \\ \mathrm{F}\mu_k^{(j)}, & \ell = n_\gamma + j, \quad j \in [1..n_k]. \end{cases}
$$

$$
\Sigma_{k+1|k}^{(\ell)} = \begin{cases} \Sigma_\gamma^{(i)}, & \ell = i, \quad i \in [1..n_\gamma], \\ \mathrm{F}\Sigma_k^{(j)}\mathrm{F}^{\mathrm{T}} + \mathrm{Q}, & \ell = n_\gamma + j, \quad j \in [1..n_k]. \end{cases}
$$

3 PHD filter update

Consider the bivariate p.g.fl.

$$
F(g,h) = \overbrace{G_c(g)}^{\text{false alarm process (Poisson)}} \cdot \underbrace{G_{k+1|k}}_{\text{prior (Poisson)}} (h \overbrace{G_d(g|\cdot)}^{\text{Bernoulli likelihood}}),
$$

where $G_d(g|x) = 1 - p_d(x) + p_d(x) \int \ell_{k+1}(z|x) g(z)\mathrm{d}z$, p_d is the probability of detection, $\ell_{k+1}(\cdot|\cdot)$ is the likelihood, the false alarm process is a Poisson process with p.g.fl.

$$
G_c(g) = \exp\left(\int \lambda(z)(g(z)-1)\mathrm{d}z\right),
$$

and the prior is a Poisson point process with p.g.fl.

$$
G_{k+1|k}(h) = \exp\left(\int D_{k+1|k}(x)(h(x)-1)\mathrm{d}x\right).
$$

Theorem 3. *Under this model, the PHD update is given by*

$$
D_{k+1}(x) = (1-p_d(x))D_{k+1|k}(x) + \sum_{z \in Z_k} \frac{p_d(x)\ell_{k+1}(z|x)}{\lambda(z) + \int p_d(y)\ell_{k+1}(z|y)D_{k+1|k}(y)\mathrm{d}y} D_{k+1|k}(x).
$$

Proof. Find $D_{k+1}(x) = \delta\left(\delta^{|Z_k|} F(g,h; (\delta_z)_{z\in Z_k}); \delta_x\right)\Big|_{g=0,h=1}$. □

4 Gaussian mixture PHD filter update

Suppose that:

- $D_{k+1|k}(x)$ is a mixture of Gaussian curves,
- $\ell_{k+1}(z|x)$ is a Gaussian likelihood,
- $p_d(x) = p_d$ is a constant,

$$D_{k+1}(x) = (1 - p_d(x))D_{k+1|k}(x) + \sum_{z \in Z_k} \frac{p_d(x)\ell_{k+1}(z|x)}{\lambda(z) + \int p_d(y)\ell_{k+1}(z|y)D_{k+1|k}(y)\mathrm{d}y} D_{k+1|k}(x).$$

- $\lambda(z)$ is a constant.

Exercise 4. Determine the form of the Gaussian mixture PHD filte update.

Solution 5. Given the assumptions

$$\begin{aligned} D_{k+1|k}(x) &= \sum_{\ell=1}^{n_{k+1|k}} w_{k+1|k}^{(\ell)} \mathcal{N}(x|\mu_{k+1|k}^{(\ell)}, \Sigma_{k+1|k}^{(\ell)}), \\ \ell_{k+1}(z|x) &= \mathcal{N}(z|\mathrm{H}x, \mathrm{R}), \end{aligned}$$

and defining $q_d \triangleq 1 - p_d$, then

$$\begin{aligned} D_{k+1}(x) &= q_d D_{k+1|k}(x) + \sum_{z \in Z_k} \frac{p_d \ell_{k+1}(z|x)}{\lambda(z) + \int p_d \ell_{k+1}(z|y) D_{k+1|k}(y)\mathrm{d}y} D_{k+1|k}(x) \\ &= q_d \sum_{\ell=1}^{n_{k+1|k}} w_{k+1|k}^{(\ell)} \mathcal{N}(x|\mu_{k+1|k}^{(\ell)}, \Sigma_{k+1|k}^{(\ell)}) \\ &+ \sum_{z \in Z_k} \frac{p_d \mathcal{N}(z|\mathrm{H}x, \mathrm{R})}{\lambda(z) + \int p_d \ell_{k+1}(z|y) D_{k+1|k}(y)\mathrm{d}y} \sum_{\ell=1}^{n_{k+1|k}} w_{k+1|k}^{(\ell)} \mathcal{N}(x|\mu_{k+1|k}^{(\ell)}, \Sigma_{k+1|k}^{(\ell)}) \\ &= \sum_{\ell=1}^{n_{k+1|k}} q_d w_{k+1|k}^{(\ell)} \mathcal{N}(x|\mu_{k+1|k}^{(\ell)}, \Sigma_{k+1|k}^{(\ell)}) \\ &+ \sum_{z \in Z_k} \frac{\sum_{\ell=1}^{n_{k+1|k}} p_d w_{k+1|k}^{(\ell)} \mathcal{N}(z|\mathrm{H}x, \mathrm{R}) \mathcal{N}(x|\mu_{k+1|k}^{(\ell)}, \Sigma_{k+1|k}^{(\ell)})}{\lambda(z) + \sum_{\ell'=1}^{n_{k+1|k}} p_d w_{k+1|k}^{(\ell')} \int \mathcal{N}(z|\mathrm{H}y, \mathrm{R}) \mathcal{N}(y|\mu_{k+1|k}^{(\ell')}, \Sigma_{k+1|k}^{(\ell')})\mathrm{d}y} \\ &= \sum_{\ell=1}^{n_{k+1|k}} q_d w_{k+1|k}^{(\ell)} \mathcal{N}(x|\mu_{k+1|k}^{(\ell)}, \Sigma_{k+1|k}^{(\ell)}) \\ &+ \sum_{z \in Z_k} \frac{\sum_{\ell=1}^{n_{k+1|k}} p_d w_{k+1|k}^{(\ell)} \mathcal{N}(z|\mathrm{H}\mu_{k+1|k}^{(\ell)}, \mathrm{H}\Sigma_{k+1|k}^{(\ell)}\mathrm{H}^{\mathrm{T}} + \mathrm{R}) \overbrace{\mathcal{N}(x|\mu_{k+1}^{(\ell)}, \Sigma_{k+1}^{(\ell)})}^{\text{Kalman filter update}}}{\lambda(z) + \sum_{\ell'=1}^{n_{k+1|k}} p_d w_{k+1|k}^{(\ell')} \mathcal{N}(z|\mathrm{H}\mu_{k+1|k}^{(\ell')}, \mathrm{H}\Sigma_{k+1|k}^{(\ell')}\mathrm{H}^{\mathrm{T}} + \mathrm{R})} \\ &= \sum_{\ell=1}^{n_{k+1|k}} q_d w_{k+1|k}^{(\ell)} \mathcal{N}(x|\mu_{k+1|k}^{(\ell)}, \Sigma_{k+1|k}^{(\ell)}) \\ &+ \sum_{\ell=1}^{n_{k+1|k}} \sum_{j=1}^{|Z_k|} \frac{p_d w_{k+1|k}^{(\ell)} \mathcal{N}(z^{(j)}|\mathrm{H}\mu_{k+1|k}^{(\ell)}, \mathrm{H}\Sigma_{k+1|k}^{(\ell)}\mathrm{H}^{\mathrm{T}} + \mathrm{R})}{\lambda(z^{(j)}) + \sum_{\ell'=1}^{n_{k+1|k}} p_d w_{k+1|k}^{(\ell')} \mathcal{N}(z^{(j)}|\mathrm{H}\mu_{k+1|k}^{(\ell')}, \mathrm{H}\Sigma_{k+1|k}^{(\ell')}\mathrm{H}^{\mathrm{T}} + \mathrm{R})} \mathcal{N}(x|\mu_{k+1}^{(\ell)}, \Sigma_{k+1}^{(\ell)}) \\ &\equiv \sum_{i=1}^{n_{k+1}} w_{k+1}^{(i)} \mathcal{N}(x|\mu_{k+1}^{(i)}, \Sigma_{k+1}^{(i)}), \end{aligned}$$

where $n_{k+1} = (|Z_k| + 1) \cdot n_{k+1|k}$ and where

$$w_{k+1}^{(i)} = \begin{cases} q_d w_{k+1|k}^{(\ell)}, & i = \ell, \quad \ell \in [1..n_{k+1|k}], \\ \frac{p_d w_{k+1|k}^{(\ell)} \mathcal{N}(z^{(j)} | \mathrm{H}\mu_{k+1|k}^{(\ell)}, \mathrm{H}\Sigma_{k+1|k}^{(\ell)}\mathrm{H}^{\mathrm{T}} + \mathrm{R})}{\lambda(z^{(j)}) + \sum_{\ell'=1}^{n_{k+1|k}} p_d w_{k+1|k}^{(\ell')} \mathcal{N}(z^{(j)} | \mathrm{H}\mu_{k+1|k}^{(\ell')}, \mathrm{H}\Sigma_{k+1|k}^{(\ell')}\mathrm{H}^{\mathrm{T}} + \mathrm{R})}, & i = \ell \cdot n_{k+1|k} + j, \quad j \in [1..|Z_k|], \quad \ell \in [1..n_{k+1|k}]. \end{cases}$$

$$\mu_{k+1}^{(i)} = \begin{cases} \mu_{k+1|k}^{(\ell)}, & i = \ell, \quad \ell \in [1..n_{k+1|k}], \\ \mu_{k+1|k}^{(\ell)} + \Sigma_{k+1|k}^{(\ell)}\mathrm{H}^{\mathrm{T}}(\mathrm{H}\Sigma_{k+1|k}^{(\ell)}\mathrm{H}^{\mathrm{T}} + \mathrm{R}^{\mathrm{T}})^{-1}(z^{(j)} - \mathrm{H}\mu_{k+1|k}^{(\ell)}), & i = \ell \cdot n_{k+1|k} + j, \quad j \in [1..|Z_k|], \quad \ell \in [1..n_{k+1|k}]. \end{cases}$$

$$\Sigma_{k+1}^{(i)} = \begin{cases} \Sigma_{k+1|k}^{(\ell)}, & i = \ell, \quad \ell \in [1..n_{k+1|k}], \\ (\mathbb{I}_{\dim(x)} - \Sigma_{k+1|k}^{(\ell)}\mathrm{H}^{\mathrm{T}}(\mathrm{H}\Sigma_{k+1|k}^{(\ell)}\mathrm{H}^{\mathrm{T}} + \mathrm{R}^{\mathrm{T}})^{-1}\mathrm{H})\Sigma_{k+1|k}^{(\ell)}, & i = \ell \cdot n_{k+1|k} + j, \quad j \in [1..|Z_k|], \quad \ell \in [1..n_{k+1|k}]. \end{cases}$$