微积分

zhx

常用符号

- ∀：对于任意
- ∃：存在
- s.t.：使得

上界、下界、无界、有界

- $\exists M \in R, s.t. \forall x \in A, x \leq M$，则$M$为$A$的上界
- $\exists M \in R, s.t. \forall x \in A, x \geq M$，则$M$为$A$的下界

- 若A有上下界，则称为有界
- 否则称为无界

- 上确界sup，下确界inf（最小的上界、最大的下界）

极限的定义

- 对于数列a_n，$\exists A \in R, \forall \epsilon > 0, \exists N = f(\epsilon), s.t.$ 当$n > N$时，有$|a_n - A| < \epsilon$，则称$\lim\limits_{n \to \infty} A$

- 若a_n没有极限，则称a_n发散

- 极限能够进行加减乘除

单调数列的极限

- 有上确界的单调递增或者单调非减数列的极限即为上确界
- 有下确界的单调递增或者单调非减数列的极限即为下确界

函数的极限

- $N(x_0, \delta) = (x_0 - \delta, x_0 + \delta)$
- $U(x_0, \delta) = N(x_0, \delta) \setminus \{x_0\}$

- 若$\forall \epsilon > 0, \exists \delta \in (0, \rho), s.t, |f(x) - A| < \epsilon, \forall x \in U(x_0, \delta)$
- 则称$\lim_{x \to x_0} f(x) = A$

- $\lim_{x \to 0} \left(1 + \frac{1}{x}\right)^x = e$

Question 1

- $\lim_{x \to 0} x \sin \frac{1}{x}$
- $\lim_{x \to 1} \frac{x^2 - 3x + 2}{x^2 - x}$

高阶无穷

- 若$\lim\limits_{x\to x_0}\frac{f(x)}{g(x)}=0$，则称$x\to x_0$时，$f(x)$是$g(x)$的高阶无穷小，记做$f(x)\to o\big(g(x)\big)(x\to x_0)$
- 若$\lim\limits_{x\to x_0}\frac{f(x)}{g(x)}=\infty$，则称$x\to x_0$时，$f(x)$是$g(x)$的高阶无穷大
- 若$\lim\limits_{x\to x_0}\frac{f(x)}{g(x)}=c$，则称$x\to x_0$时，$f(x),g(x)$同阶，记做$f(x)\sim g(x)$
- 若$\lim\limits_{x\to x_0}\frac{f(x)}{(x-x_0)^k}=c\neq 0$，则称$x\to x_0$时，$f(x)$是$k$阶无穷小量

常见无穷小

- 当$x \to 0$时
- $\sin x \sim x$ $\tan x \sim x$ $\arcsin x \sim x$ $\arctan x \sim x$
- $1 - \cos x \sim \frac{1}{2}x^2$ $\ln(1+x) \sim x$
- $e^x - 1 \sim x$
- $a^x - 1 \sim x \ln a$
- $(1+x)^a - 1 \sim ax$

Question 2

- $\lim_{x \to 0^+} (e^x + 2x)^{\frac{1}{x}}$

Question 2

- $\lim\limits_{x\to 0^+}(e^x + 2x)^{\frac{1}{x}} = \lim\limits_{x\to 0^+}(1 + e^x + 2x - 1)^{\frac{1}{e^x+2x-1}\times\frac{e^x+2x-1}{x}}$

- $\lim\limits_{x\to 0^+}\frac{e^x+2x-1}{x} = 2 + \lim\limits_{x\to 0^+}\frac{e^x-1}{x} = 3$

导数

- 导数：$f'(x_0)=\lim\limits_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$
- 左导数：$f'(x_0)=\lim\limits_{\Delta x\to 0^-}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$
- 右导数：$f'(x_0)=\lim\limits_{\Delta x\to 0^+}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$

- 导数的几何意义？
- 可导的几何意义？

Question 3

- $(\sin x)'$
- $(\cos x)'$

Question 3

- $(\sin x)' = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \to 0} \frac{2 \sin\frac{h}{2} \cos\left(x + \frac{h}{2}\right)}{h} = \cos x$
- $(\cos x)' = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \to 0} \frac{-2\sin\frac{h}{2} \sin\left(x + \frac{h}{2}\right)}{h} = -\sin x$

Question 4

- $(a^x)'$
- $(x^a)'$

Question 4

- $(a^x)' = \lim_{h\to 0} \frac{a^{x+h}-a^x}{h} = a^x \lim_{h\to 0} \frac{a^h-1}{h} = a^x \ln a$
- $(x^a)' = \lim_{h\to 0} \frac{(x+h)^a-x^a}{h} = x^a \lim_{h\to 0} \frac{\left(1+\frac{h}{x}\right)^a-1}{h} = x^a \lim_{h\to 0} \frac{\frac{ah}{x}}{h} = ax^{a-1}$

求导法则

- $(f+g)'(x_0) = f'(x_0) + g'(x_0)$
- $(cf)'(x_0) = cf'(x_0)$
- $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$
- $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}$

- 证明乘法和除法

求导法则

- $(fg)'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} = \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}$
- $\left(\frac{f}{g}\right)'(x) = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h} = \lim_{h\to 0} \frac{f(x+h)g(x)-f(x)g(x+h)}{hg(x)g(x+h)}$
- $= \frac{1}{g(x)^2} \lim_{h\to 0} \frac{f(x+h)g(x)-f(x)g(x+h)}{h}$
- $= \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$

复合函数求导法则

- $h(x) = f\left(g(x)\right)$
- $h'(x) = f'\left(g(x)\right) \cdot g'(x)$

- 设$y = g(x)$
- $h'(x) = \frac{dh}{dx} = \frac{dh}{dy} \cdot \frac{dy}{dx} = f'\left(g(x)\right) \cdot g'(x)$

Question 5

- $f(x) = \left(\frac{x+1}{x-1}\right)^{\frac{3}{2}}$
- 求$f'(x)$

Question 5

- 令$g(u) = u^{\frac{3}{2}}, h(x) = \frac{x+1}{x-1}$，则$f(x) = g\big(h(x)\big)$
- $g'(u) = \frac{3}{2}u^{\frac{1}{2}}$
- $h'(x) = \frac{-2}{(x-1)^2}$
- $f'(x) = g'\big(h(x)\big)h'(x) = \frac{3}{2}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}} \cdot \frac{-2}{(x-1)^2} = \frac{-3(x+1)^{\frac{1}{2}}}{(x-1)^{\frac{5}{2}}}$

Question 6

- $f(x) = f_1(x)f_2(x)\cdots f_n(x)$，求$f'(x)$

Question 6

- 取对数转换成加法
- 再求导

常见导数列表

$c' = 0,$ $\qquad (x^{\alpha})' = \alpha x^{\alpha - 1},$

$(\sin x)' = \cos x,$ $\qquad (\cos x)' = -\sin x,$

$(\tan x)' = \sec^2 x,$ $\qquad (\cot x)' = -\csc^2 x,$

$(\sec x)' = \sec x \tan x,$ $\qquad (\csc x)' = -\csc x \cot x$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, \qquad \arctan x = \frac{1}{1+x^2}$$

$$(\arccos x)' = \frac{-1}{\sqrt{1-x^2}}, \qquad \operatorname{arc}\cot x = \frac{-1}{1+x^2}$$

$(a^x)' = a^x \ln a,$ $\qquad (e^x)' = e^x$

$$(\log_a x)' = \frac{1}{x \ln a}, \qquad (\ln x)' = \frac{1}{x}$$

$$\left(\ln\left|x + \sqrt{x^2 \pm a^2}\right|\right)' = \frac{1}{\sqrt{x^2 \pm a^2}}$$

牛顿迭代法

- 求解函数零点的方法
- $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$

- HDU 2899

高阶导数

- $y''(x) = \frac{d^2y}{dx^2} = y^{(2)}(x)$

- $(f+g)^{(n)}(x) = f^{(n)}(x) + g^{(n)}(x)$
- $(cf)^{(n)}(x) = cf^{(n)}(x)$
- $(f \cdot g)^{(n)}(x) = \sum_{k=0}^{n} C_n^k f^{(k)}(x) g^{(n-k)}(x)$

Question 7(may be hard)

- $x^2 + xy + y^2 = 1$确定了隐函数$y = y(x)$，求$y''(x)$。

Question 7

- $x^2 + xy + y^2 = 1$
- $2x + y + xy' + 2yy' = 0 \Rightarrow y' = -\frac{2x+y}{x+2y}$
- $y'' = -\frac{(2x+y)'(x+2y)-(2x+y)(x+2y)'}{(x+2y)^2}$
- $= \frac{3\left(xy'-y\right)}{(x+2y)^2} = \frac{-6}{(x+2y)^3}$

洛必达法则

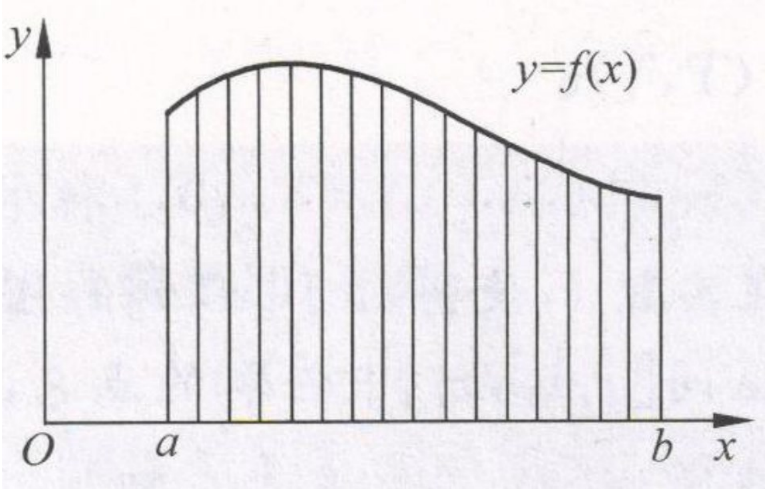
- 如果$\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} g(x) = 0$或者$\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} g(x) = \infty$时
- 有
- $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = \lim\limits_{x \to x_0} \frac{f'(x)}{g'(x)}$

Question 8

- $\lim_{x \to x_0} \frac{\sin x}{x}$

积分（黎曼积分）

- 积分：曲边梯形的面积



- 积分是某种意义上求导的逆运算
- 定积分：给定上下界
- 不定积分：不给定上下界

常见积分表

- $\int f'(x)\,dx = f(x) + C$
- $\int \cos x\,dx = \sin x + C$
- $\int \sin x\,dx = -\cos x + C$
- $\int x^a\,dx = \frac{x^{a+1}}{a+1} + C$
- $\int \frac{1}{x}\,dx = \ln x + C$
- $\int a^x\,dx = \frac{a^x}{\ln a} + C$

不定积分换元法

- $\frac{df\left(\phi(x)\right)}{dx} = f'\left(\phi(x)\right) \cdot \phi'(x) \Rightarrow \int f'\left(\phi(x)\right)\phi'(x)dx = f\left(\phi(x)\right) + C$

- 第一换元法：
- $\int f'\left(\phi(x)\right)\phi'(x)dx = \int f'\left(\phi(x)\right)d\phi(x) = f\left(\phi(x)\right) + C$
- 第二换元法：
- $\int f'(u)du = \int f'\left(\phi(x)\right)\phi'(x)dx = g\left(\phi^{-1}(u)\right) + C$

Question 9

- $\int 2xe^{x^2}dx$

Question 10

- $\int \frac{dx}{1+\sqrt[3]{1+x}}$

Question 10

- 令$t = \sqrt[3]{1+x}$
- $\int \frac{dx}{1+\sqrt[3]{1+x}} = \int \frac{d(t^3-1)}{1+t} = \int \frac{3t^2}{1+t} dt = \int \left(t - 1 + \frac{1}{1+t}\right) dt$
- $= 3\left(\frac{1}{2}t^2 - t + \ln|1+t|\right) + C$

分部积分法

- $\big(u(x)v(x)\big)' = u'(x)v(x) + u(x)v'(x)$
- $\Rightarrow \int u'(x)v(x)dx + \int u(x)v'(x)dx = \int \big(u(x)v(x)\big)' dx$
- $\Rightarrow \int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$
- 也记做
- $\int u(x)dv(x) = u(x)v(x) - \int v(x)du(x)$

Question 11

- $\int x \cos x \, dx$
- $\int \ln x \, dx$

Question 11

- $\int x \cos x \, dx = \int x \, d \sin x = x \sin x - \int \sin x \, dx = x \sin x - \cos x + C$
- $\int \ln x \, dx = x \ln x - \int x d \ln x = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - x + C$

Problem 1

- 圆上任取N个点
- 问这N个点组成的凸多边形内角有锐角的概率
- 有标号
- $N \le 10^9$

Problem 1

- 存在两个点A,B分别为锐角和钝角
- 过圆心找对点A',B'
- 则B的下一个点必须在BA'上
- 其余点必须在$BA'B'$上
- 设$A'B'$的弧度是$x\cdot\pi$
- 则答案为$\int_0^1 \left(\frac{1}{2}\right)^{n-2} - \left(\frac{x}{2}\right)^{n-2} dx = \frac{n-2}{n-1}\left(\frac{1}{2}\right)^{n-2}$
- Bzoj 5146

Problem 2

- 求一次函数和二次函数交出的图形的面积

Problem 2

- HDU 1071

Problem 3

- 一个人以v_1做匀速圆周运动，半径为R
- 一个人从圆心出发速度是v_2，去追另外一个人
- 要求两个人加圆心一定三点共线
- 问一定时间内能否追上

Problem 3

- 三点共线推出角速度相等
- $\frac{v_1}{R} = \frac{v_x}{r} \Rightarrow v_y = \sqrt{v_2^2 - \frac{v_1^2 \times r^2}{R^2}} = \frac{dr}{dt} \Rightarrow dt = \frac{dr}{\sqrt{v_2^2 - \frac{v_1^2 \times r^2}{R^2}}}$
- 两边积分可得$\int_0^R \frac{dr}{\sqrt{v_2^2 - \frac{v_1^2 \times r^2}{R^2}}} = t$
- 令$x = \frac{v_1 \times r}{v_2 \times R} \Rightarrow \int_0^{\frac{v_1}{v_2}} \frac{1}{\sqrt{1-x^2}} \times \frac{1}{v_2} d\frac{R \times v_2 \times x}{v_1} = \frac{R}{v_1} \times \operatorname{asin}\frac{v_1}{v_2}$
- HDU 4969

Problem 4

- 给定$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$的椭圆
- 求内接矩形的周长的期望

Problem 4

- $C = 4(x + y) = 4y + 4a\sqrt{1 - \frac{y^2}{b^2}}$，设$y = b\sin\theta$，则
- $\int_0^b 4y + 4a\sqrt{1 - \frac{y^2}{b^2}}\,dy = \int_0^{\frac{\pi}{2}}(4b\sin\theta + 4a\cos\theta)db\sin\theta$
- $= \int_0^{\frac{\pi}{2}} 4b^2\sin\theta\cos\theta + 4ab\cos^2\theta\,d\theta$
- $= \int_0^{\frac{\pi}{2}} 2b^2\sin 2\theta + 2ab(\cos 2\theta + 1)d\theta$
- $= 2b^2 + \pi ab$
- 这是期望吗?
- HDU 6362

辛普森积分

- $\int_l^r f(x)dx \approx \frac{r-l}{6}[f(l) + 4f\left(\frac{r-l}{2}\right) + f(r)]$

- 当左右计算值与整体计算值相等时停止
- 不可用于非连续函数的积分