Homework 4

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Question 14. Let V and W be vector spaces, and let T and U be nonzero linear transformations from V into W. If $R(T) \cap R(U) = \{0\}$, prove that $\{T, U\}$ is a linearly independent subset of $\mathcal{L}(V, W)$.

Proof. To show that $\{T, U\}$ is linearly independent in $\mathcal{L}(V, W)$, we must prove that if

$$aT + bU = T_0$$

(the zero transformation), then $a = b = 0$.

Assume that $aT + bU = 0$. Then for every $x \in V$,

$$(aT + bU)(x) = aT(x) + bU(x) = 0,$$

so

$$aT(x) = -bU(x).$$

Now note that $aT(x) \in R(T)$ for all x, and $-bU(x) \in R(U)$ for all x because both T and U are linear. Therefore, the common value $aT(x) = -bU(x)$ lies in $R(T) \cap R(U)$ for every $x \in V$. But $R(T) \cap R(U) = \{0\}$, so we must have

$$aT(x) = 0 \text{ and } bU(x) = 0 \quad \text{for all } x \in V.$$

Thus, aT is the zero transformation. Since T is nonzero, there exists some $x_0 \in V$ such that $T(x_0) \neq 0$. Therefore, $a = 0$. With the same logic applied to transformation bU, $b = 0$. Hence $a = b = 0$, and therefore $\{T, U\}$ is a linearly independent subset of $\mathcal{L}(V, W)$. $\square$

Question 17. Let V and W be vector spaces such that $\dim(V) = \dim(W)$, and let $T : V \to W$ be linear. Show that there exist ordered bases β and γ for V and W, respectively, such that $[T]_\beta^\gamma$ is a diagonal matrix.

Proof. Let $\dim(V) = \dim(W) = n$. Let $\text{rank}(T) = k$, where $0 \leq k \leq n$. By the Dimension Theorem, we have $\text{nullity}(T) = n - k$. First, we choose a basis $\{v_{k+1}, v_{k+2}, \ldots, v_n\}$ for $N(T)$. We then extend this set to an ordered basis $\beta = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for V.

For the first k vectors in β, we define $w_i = T(v_i)$ for $1 \leq i \leq k$. Since the set $\{v_1, \ldots, v_k\}$ is a basis for a subspace of V complementary to $N(T)$, the image set $\{w_1, w_2, \ldots, w_k\}$ forms a basis for the range $R(T)$.

Because $\{w_1, \ldots, w_k\}$ is a linearly independent set in W, we can extend it to an ordered basis $\gamma = \{w_1, w_2, \ldots, w_k, w_{k+1}, \ldots, w_n\}$ for W.

We now examine the action of T on each element of the basis β to determine the columns of the matrix representation. For $1 \leq i \leq k$, $T(v_i) = e_i$. For $k + 1 \leq i \leq n$, $T(v_i) = 0_W$. Hence, resulting matrix representation $[T]_\beta^\gamma$ takes the form:

$$\begin{aligned} [T]_\beta^\gamma &= [[T(v_1)]_\beta, [T(v_2)]_\beta, \ldots, [T(v_n)]_\beta] \\ &= [e_1, e_2, \ldots, e_k, 0_W, 0_W, \ldots] \\ &= \begin{pmatrix} I_k & O \\ O & O \end{pmatrix} \end{aligned}$$

where I_k is the $k \times k$ identity matrix and O represents zero blocks. This is a diagonal matrix with k ones and $n - k$ zeros on the main diagonal. $\square$

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Problem 11. Let V be a vector space, and let $T : V \to V$ be linear. Prove that $T^2 = T_0$ if and only if $R(T) \subseteq N(T)$.

Proof. In the first direction, suppose $T^2 = T_0$. Expanding this for a vector $x \in V$ gives us $T(T(x)) = 0_V$, implying $T(x) \in N(T)$. Additionally, $R(T) = \{T(x) : x \in V\}$, implying $T(x) \in R(T)$. Hence, $R(T) \subseteq N(T)$.

In the other direction, suppose $R(T) \subseteq N(T)$. For a vector $v \in V$, there is a vector $w = T(v) \in R(T)$. However, the condition $R(T) \subseteq N(T)$ implies that $w \in N(T)$. This means that $T(w) = 0_V$. Substituting w back gives us $T(T(v)) = 0_V$ for any vector $v \in V$. Hence, $T^2 = T_0$. $\square$

Problem 12. Let V, W, and Z be vector spaces, and let $T : V \to W$ and $U : W \to Z$ be linear.

(a) Prove that if UT is one-to-one, then T is one-to-one. Must U also be one-to-one?

(b) Prove that if UT is onto, then U is onto. Must T also be onto?

(c) Prove that if U and T are one-to-one and onto, then UT is also.

Proof. Let V, W, and Z be vector spaces, and let $T : V \to W$ and $U : W \to Z$ be linear transformations.

(a) Suppose UT is one-to-one. To show T is one-to-one, let $v_1, v_2 \in V$ such that $T(v_1) = T(v_2)$. Applying U to both sides gives $U(T(v_1)) = U(T(v_2))$, which is $(UT)(v_1) = (UT)(v_2)$. Since UT is one-to-one, $v_1 = v_2$. Thus, T is one-to-one.

However, U does not need to be one-to-one. For example, let $T : \mathbb{R} \to \mathbb{R}^2$ be defined by $T(x) = (x, 0)$ and $U : \mathbb{R}^2 \to \mathbb{R}$ be defined by $U(x, y) = x$. The composition $(UT)(x) = U(x, 0) = x$ is the identity map on $\mathbb{R}$, which is one-to-one. Yet, U is not one-to-one because $U(0, 1) = U(0, 0) = 0$.

(b) Suppose UT is onto. For any $z \in Z$, there exists $v \in V$ such that $(UT)(v) = z$. This can be written as $U(T(v)) = z$. Let $w = T(v) \in W$. Then $U(w) = z$. Since for every $z \in Z$ we found a $w \in W$ such that $U(w) = z$, U is onto.

T does not necessarily have to be onto. Using the same example as in part (a), let $T : \mathbb{R} \to \mathbb{R}^2$ be $T(x) = (x, 0)$ and $U : \mathbb{R}^2 \to \mathbb{R}$ be $U(x, y) = x$. The composition $(UT)(x) = x$ is onto $\mathbb{R}$. However, T is not onto because there is no $x \in \mathbb{R}$ such that $T(x) = (0, 1)$.

(c) Suppose U and T are both one-to-one and onto. First, let us prove that UT is one-to-one: let $(UT)(v_1) = (UT)(v_2)$. Then $U(T(v_1)) = U(T(v_2))$. Since U is one-to-one, $T(v_1) = T(v_2)$. Since T is one-to-one, $v_1 = v_2$.

Next, let us prove that UT is onto. Let there be a vector $z \in Z$. Since U is onto, there exists $w \in W$ such that $U(w) = z$. Since T is onto, there exists $v \in V$ such that $T(v) = w$. Then $(UT)(v) = U(T(v)) = U(w) = z$. Thus, UT is one-to-one and onto. $\square$

Problem 13. Let A and B be $n \times n$ matrices. Recall that the trace of A is defined by

$$\operatorname{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

Prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ and $\operatorname{tr}(A) = \operatorname{tr}(A^t)$.

Proof. Let A and B be $n \times n$ matrices.

To prove $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, we look at the diagonal entries of the products. The i-th diagonal entry of AB is $(AB)_{ii} = \sum_{j=1}^{n} A_{ij}B_{ji}$. The trace is the sum of these entries:

$$\operatorname{tr}(AB) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}B_{ji}$$

Similarly, the j-th diagonal entry of BA is $(BA)_{jj} = \sum_{i=1}^{n} B_{ji}A_{ij}$. Summing these gives:

$$\operatorname{tr}(BA) = \sum_{j=1}^{n}\sum_{i=1}^{n} B_{ji}A_{ij}$$

Since the terms $A_{ij}B_{ji}$ are scalars and the order of finite summation can be interchanged, $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

To prove $\operatorname{tr}(A) = \operatorname{tr}(A^t)$, recall that the transpose operation swaps indices such that $(A^t)_{ii} = A_{ii}$. Therefore:

$$\operatorname{tr}(A^t) = \sum_{i=1}^{n}(A^t)_{ii} = \sum_{i=1}^{n} A_{ii} = \operatorname{tr}(A)$$

□

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Problem 4. Let A and B be $n \times n$ invertible matrices. Prove that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof. Let A and B be $n \times n$ invertible matrices. To show that AB is invertible, we must find a matrix C such that $(AB)C = I$ and $C(AB) = I$. Consider the matrix $C = B^{-1}A^{-1}$.

First, we check the right inverse:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = AA^{-1} = I.$$

Next, we check the left inverse:

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}(I)B = B^{-1}B = I.$$

Since both products result in the identity matrix, AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. $\square$

Problem 6. Prove that if A is invertible and $AB = O$, then $B = O$.

Proof. Suppose A is an invertible $n \times n$ matrix and $AB = O$, where O is the zero matrix. Since A is invertible, its inverse A^{-1} exists. We multiply the equation $AB = O$ on the left by A^{-1}. By the associativity of matrix multiplication,

$$A^{-1}(AB) = A^{-1}O$$
$$(A^{-1}A)B = O$$
$$IB = O$$
$$B = O.$$

Thus, if A is invertible and $AB = O$, it must be that $B = O$. $\square$

Problem 15. Let V and W be n-dimensional vector spaces, and let $T : V \to W$ be a linear transformation. Suppose that β is a basis for V. Prove that T is an isomorphism if and only if $T(\beta)$ is a basis for W.

Proof. Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be a basis for V.

In the first direction, assume T is an isomorphism. To show $T(\beta) = \{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for W, it suffices to just show that $T(\beta)$ is linearly independent because $\dim(W) = n$. Suppose $\sum_{i=1}^{n} a_i T(v_i) = 0_W$. By linearity, $T(\sum a_i v_i) = 0_W$. Since T is an isomorphism, it is one-to-one, so $N(T) = \{0_W\}$. Thus, $\sum a_i v_i = 0_W$. Since β is a basis, $a_i = 0$ for all i. Hence, $T(\beta)$ is linearly independent. Therefore, $T(\beta)$ is a basis for W.

In the other direction, assume $T(\beta)$ is a basis for W. Since $T(\beta)$ spans W, $R(T) = W$, so T is onto. Since $\dim(V) = \dim(W) = n$ and T is onto, by the Dimension Theorem, $\dim(N(T)) = \dim(V) - \dim(R(T)) = n - n = 0$. Thus, T is one-to-one. Since T is one-to-one and onto, it is an isomorphism. $\square$

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Problem 7a. In $\mathbb{R}^2$, let L be the line $y = mx$, where $m \neq 0$. Find an expression for $T(x, y)$, where T is the reflection of R^2 about L.

Solution. We solve this by changing to a basis β that is aligned with the line L. Let $v_1 = (1, m)$ be a vector along the line L, and let $v_2 = (-m, 1)$ be a vector perpendicular to L. Thus, $\beta = \{v_1, v_2\}$ is an ordered basis for $\mathbb{R}^2$.

In this coordinate system, the reflection T leaves v_1 unchanged and flips the direction of v_2:

$$T(v_1) = v_1 \quad \text{and} \quad T(v_2) = -v_2$$

The matrix representation of T relative to β is therefore:

$$[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let Q be the change of coordinate matrix that changes β'-coordinates (the standard basis) into β-coordinates. The columns of the inverse matrix Q^{-1} are the vectors of β expressed in the standard basis:

$$Q^{-1} = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}$$

We find Q by inverting Q^{-1}:

$$Q = \frac{1}{1(1) - (-m)(m)} \begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix} = \frac{1}{1 + m^2} \begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix}$$

Using the formula $[T]_{\beta'} = Q^{-1}[T]_\beta Q$, we compute the product:

$$[T]_{\beta'} = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\frac{1}{1 + m^2} \begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix} \right)$$

$$[T]_{\beta'} = \frac{1}{1 + m^2} \begin{pmatrix} 1 & m \\ m & -1 \end{pmatrix} \begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix}$$

$$[T]_{\beta'} = \frac{1}{1 + m^2} \begin{pmatrix} 1 - m^2 & 2m \\ 2m & m^2 - 1 \end{pmatrix}$$

Hence,

$$T(x, y) = \frac{1}{1 + m^2} \begin{pmatrix} 1 - m^2 & 2m \\ 2m & m^2 - 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$T(x, y) = \left(\frac{(1 - m^2)x + 2my}{1 + m^2}, \frac{2mx + (m^2 - 1)y}{1 + m^2} \right).$$

□

Problem 10a. Prove that if A and B are similar $n \times n$ matrices, then $\mathrm{tr}(A) = \mathrm{tr}(B)$.

Proof. Let A and B be similar $n \times n$ matrices. By definition, there exists an invertible $n \times n$ matrix P such that $B = P^{-1}AP$. We take the trace of both sides of this equation:

$$\mathrm{tr}(B) = \mathrm{tr}(P^{-1}AP)$$

From 2.3.13 we found that $\mathrm{tr}(XY) = \mathrm{tr}(YX)$ f for any matrices X, Y where the products are square. Thus, let $X = P^{-1}A$ and $Y = P$. Then,

$$\mathrm{tr}((P^{-1}A)P) = \mathrm{tr}(P(P^{-1}A)).$$

By the associativity of matrix multiplication,

$$\mathrm{tr}(P(P^{-1}A)) = \mathrm{tr}((PP^{-1})A) = \mathrm{tr}(IA) = \mathrm{tr}(A).$$

Thus, we conclude that $\mathrm{tr}(B) = \mathrm{tr}(A)$. $\square$