Homework 1

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Section 1.1

0.1 Question 2a

Find the equation of the line through the following pair of points in space.
$(3, -2, 4)$ **and** $(-5, 7, 1)$

Let A and B represent points with coordinates $(3, -2, 4)$ and $(-5, 7, 1)$, respectively. Define the point C as the endpoint of a vector emanating from the origin and having the same direction as the vector from A to B. C has coordinates

$$(-5, 7, 1) - (3, -2, 4) = (-8, 9, -3).$$

Thus, the equation of the line through A and B is

$$x(t) = (-5, 7, 1) + t(-8, 9, -3), \quad t \in \mathbb{R}.$$

0.2 Question 3a

Find the equation of the plane containing the following points in space. $(2, -5, -1)$**,** $(0, 4, 6)$**, and** $(-3, 7, 1)$

Let A, B, and C represent the points $(2, -5, -1)$, $(0, 4, 6)$, and $(-3, 7, 1)$, respectively. The endpoint of the other vector emanating from the origin equal to the vector beginning at A and terminating at B is

$$(0, 4, 6) - (2, -5, -1) = (-2, 9, 7).$$

Similarly, the endpoint of the other vector emanating from the origin equal to the vector beginning at A and terminating at C is

$$(-3, 7, 1) - (2, -5, -1) = (-5, 12, 2).$$

Thus, the equation of the plane containing the three given points is

$$x(t) = (2, -5, -1) + s(-2, 9, 7) + t(-5, 12, 2), \quad s, t \in \mathbb{R}.$$

0.3 Question 7

Prove that the diagonals of a parallelogram bisect each other.

Proof. Let V be a vector space over $\mathbb{R}$. Additionally, let x, $y \in V$ be nonzero vectors and adjacent side vectors of a parallelogram with a vertex at the origin. Let us also assume that a vertex of the parallelogram lies on the origin.

The vertices of the parallelogram are

$$(0_{\mathrm{v}}, x, x + y, y).$$

A midpoint of a line segment that starts from the endpoint of one vector u to the endpoint of another vector v in the textbook is defined to be

$$\frac{u + v}{2}.$$

The midpoint of the diagonal that starts from the endpoint of x to the endpoint of y is

$$\frac{x + y}{2}.$$

Additionally, the midpoint of the other diagonal that starts from the endpoint of 0_{v} to the endpoint of $x + y$ is

$$\frac{0_{\mathrm{v}} + (x + y)}{2} = \frac{x + y}{2}.$$

Because the midpoints of the two diagonals coincide, it is thus proven that the diagonals of a parallelogram bisect each other.

Section 1.2

0.4 Question 15

Let $V = \{(a_1, A_2, \ldots, a_n) : a_i \in \mathbb{R} \text{ for } i = 1, 2, \ldots, n\}$; **so** V **is a vector space over** $\mathbb{R}$ **by Example 1. Is** V **a vector space over the field of complex numbers with the operations of coordinatewise addition and multiplication?**

Proof. Let V be a vector space such that

$$V = \{(a_1, a_2, \ldots, a_n) : a_i \in \mathbb{R}\}.$$

To be a vector space over $\mathbb{C}$, V must be closed under scalar multiplication by all scalars in $\mathbb{C}$.

Let $v = (1, 0, \ldots, 0)$ and $c = i \in \mathbb{C}$. Then

$$cv = i(1, 0, \ldots, 0) = (i, 0, \ldots, 0).$$

But $(i, 0, \ldots, 0) \notin V$, because $a_1 = i \notin \mathbb{R}$. Thus V is not closed under scalar multiplication by complex scalars, so V cannot be a vector space over $\mathbb{C}$.

0.5 Question 16

Let V denote the set of all $m \times n$ matrices with real entries; so V is a vector space over $\mathbb{R}$ by Example 2. Let F be the field of rational numbers. Is V a vector space over F with the usual definitions of matrix addition and scalar multiplication?

Proof. Let V be the set of all $m \times n$ matrices with real entries. Since $\mathbb{Q} \subseteq \mathbb{R}$, we may restrict the scalar multiplication of the $\mathbb{R}$-vector space V to scalars in $\mathbb{Q}$.

Closure holds: if $A, B \in V$ then $A + B \in V$, and if $q \in \mathbb{Q}$ and $A \in V$ then $qA \in V$ because $q \in \mathbb{R}$.

All vector space axioms over $\mathbb{Q}$ are the same identities that hold over $\mathbb{R}$, and therefore remain true when scalars are restricted to $\mathbb{Q}$.

Hence V is a vector space over $\mathbb{Q}$.

0.6 Question 18

Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$**. For** $(a_1, a_2), (b_1, b_2)$ **and** $c \in \mathbb{R}$**, define**

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, ca_2).$$

Is V **a vector space over** $\mathbb{R}$ **with these operations? Justify your answer.**

First, for (x_1, x_2) and $a, b \in \mathbb{R}$, let us attempt to fulfill the axiom

$$(a + b)(x_1, x_2) = a(x_1, x_2) + b(x_1, x_2).$$

Using the defined rule of multiplication, the left side of the axiom equation can be expressed as

$$(a + b)(x_1, x_2) = ((a + b)x_1, (a + b)x_2) = (ax_1 + bx_1, ax_2, +bx_2).$$

Similarly, using the defined rules of addition and multiplication, the right side of the axiom equation is expressed as

$$a(x_1, x_2) + b(x_1, x_2) = (ax_1, ax_2) + (bx_1, bx_2) = (ax_1 + 2bx_1, ax_2 + 3bx_2).$$

Because $(ax_1 + bx_1, ax_2, +bx_2) \neq (ax_1 + 2bx_1, ax_2 + 3bx_2)$, the axiom does not hold. Therefore, V is not a vector space over $\mathbb{R}$.

Section 1.3

0.7 Question 3

Prove that $(aA + bB)^t = aA^t + bB^t$ **for any** $A, B \in M_{m\times n}(F)$ **and any** $a, b \in F$.

Proof. Let $A, B \in M_{m\times n}(F)$, and let $a, b \in F$. Based on the rules of matrix addition and scalar multiplication, the (i, j)-entry of $(aA + bB)^t$ can be expressed as

$$((aA + bB)^t)_{ij} = (aA + bB)_{ji} = aA_{ji} + bB_{ji}.$$

Similarly, the (i, j)-entry of $aA^t + bB^t$ can be expressed as

$$(aA^t + bB^t)_{ij} = a(A^t)_{ij} + b(B^t)_{ij} = aA_{ji} + bB_{ji}.$$

Thus, $((aA+bB)^t)_{ij} = (aA^t+bB^t)_{ij} = aA_{ji}+bB_{ji}$, and therefore makes the two matrices $(aA + bB)^t$ and $aA^t + bB^t$ equal.

0.8 Question 19

Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Proof. In the first direction, let us assume $W_1 \subseteq W_2$. Then, this simply makes $W_1 \cup W_2 = W_2$. Because W_2 is already defined to be a subspace of V, $W_1 \cup W_2$ is therefore a subspace of V.

In the other direction, let us assume $W_1 \cup W_2 \subseteq V$. The proof is sufficient if $W_1 \nsubseteq W_2$ and $W_2 \subseteq W_1$.

Define two arbitrary vectors $w_1 \in W_1, w_1 \notin W_2$ and $w_2 \in W_2$. Based on these definitions, it is true that

$$w_1 + w_2 \in W_1 \cup W_2.$$

If $w_1 + w_2 \in W_2$, then $w_1 = (w_1 + w_2) + (-w_2) \in W_2$ because W_2 is closed under addition. However, this statement contradicts, therefore confirming $w_1 + w_2 \in W_1$. Because W_1 is closed under addition, we can express

$$w_2 = (w_1 + w_2) + (-w_1) \in W_1.$$

Since w_2 is an arbitrary vector, $W_2 \subseteq W_1$, and thus both directions are proven.

0.9 Question 23

Let W_1 and W_2 be subspaces of a vector space V.
(a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

Proof. First, let us prove that $W_1 + W_2$ is a subspace. Since it is true that $0_{\text{v}} \in W_1$ and $0_{\text{v}} \in W_2$, $0_{\text{v}} + 0_{\text{v}} = 0_{\text{v}} \in W_1 + W_2$.

Now, let us define two vectors $x, y \in W_1 + W_2$ and scalar $a \in F$. By the definition of the sum of two subsets, there must exist vectors x_1 and x_2 where $x = x_1 + x_2, x_1 \in W_1, x_2 \in W_2$. Similarly, there must be vectors y_1 and y_2 where $y = y_1 + y_2, y_1 \in W_1, y_2 \in W_2$. When expanding $ax + y$, we get

$$ax + y = a(x_1 + x_2) + (y_1 + y_2) = (ax_1 + y_1) + (ax_2 + y_2)$$

Since W_1 is a subspace of V, it is closed under addition and scalar multiplication. Therefore, $ax_1 + y_1 \in W_1$. By the same reasoning in W_2, $ax_2 + y_2 \in W_2$. Thus, $(ax_1 + y_1) + (ax_2 + y_2) \in W_1 + W_2$. These conditions satisfy the variation of TSST, and makes $W_1 + W_2$ a subspace of V.

To show $W_1 + W_2$ contains both subspaces, let $w \in W_1$. Since $0_V \in W_2$, we can write $w = w + 0_V$, so $w \in W_1 + W_2$, proving $W_1 \subseteq W_1 + W_2$. Likewise, if $w \in W_2$, then $w = 0_V + w$ with $0_V \in W_1$, so $w \in W_1 + W_2$, proving $W_2 \subseteq W_1 + W_2$.

(b) Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

Proof. Let U be a subspace of V such that $W_1 \subseteq U$ and $W_2 \subseteq U$. Then there exists an arbitrary $z \in W_1 + W_2$. By definition of the sum of subspaces, there also exist vectors $w_1 \in W_1$ and $w_2 \in W_2$ such that

$$z = w_1 + w_2.$$

Since $W_1 \subseteq U$ and $W_2 \subseteq U$, it follows that $w_1, w_2 \in U$. Because U is a subspace, it is closed under addition, so $w_1 + w_2 \in U$. Hence $z \in U$, and therefore $W_1 + W_2 \subseteq U$.

0.10 Question 28

A matrix M is skew-symmetric if $M^t = -M$. Clearly, a skew-symmetric matrix is square. Let F be a field. Prove that set W_1 of all skew-symmetric $n \times n$ matrices with form F is a subspace of $M_{n\times n}(F)$. Now assume that F is not of characteristic two, and let W_2 be the subspace of $M_{n\times n}$ consisting of all symmetric $n\times n$ matrices. Prove that $M_{n\times n}(F) = W_1 \bigoplus W_2$.

First, I will prove that $W_1 = \{M \in M_{n\times n}(F) : M^t = -M\}$ is a subspace of $M_{n\times n}(F)$.

Proof: We know that the zero matrix is in W_1 because $0^t = -0 = 0$. Next, let us define matrices $A, B \in W_1$ and scalar $a \in F$. We can express $(aA + B)^t$ as

$$(aA + B)^t = (aA)^t + B^t = aA^t + B^t = a(-A) + (-B) = -(aA + B).$$

Thus, $(aA+B)^t \in W_1$. These statements satisfy the conditions for the variation of TSST, and thus we have proved that that W_1 is a subspace of $M_{n\times n}(F)$.

Next I will prove that $M_{n\times n}(F) = W_1 \bigoplus W_2$.

Proof: Given $W_1 = \{M \in M_{n\times n}(F) : M^t = -M\}$ and $W_2 = \{M \in M_{n\times n}(F) : M^t = M\}$, let us define an arbitrary matrix $M \in W_1 \cap W_2$. Then,

$$M^t = -M = M.$$

The only possible matrix to satisfy this condition is the zero matrix assuming F is not of characteristic two. Thus, $W_1 \cap W_2 = \{0\}$. Next, let us define matrices $A = \frac{1}{2}(M + M^t), B = \frac{1}{2}(M - M^t)$. Note that $\frac{1}{2}$ exists in F because it is not of characteristic two.

It is true that $A \in W_2$ because

$$A^t = (\frac{1}{2}(M + M^t))^t = \frac{1}{2}(M + M^t)^t = \frac{1}{2}(M^t + M) = \frac{1}{2}(M + M^t) = A$$

Similarly, it is true that $B \in W_1$ because

$$B^t = (\frac{1}{2}(M - M^t))^t = \frac{1}{2}(M - M^t)^t = \frac{1}{2}(M^t - M) = -\frac{1}{2}(M - M^t) = -B$$

When adding matrices A and B, we get

$$A + B = \frac{1}{2}(M + M^t) + \frac{1}{2}(M - M^t) = \frac{1}{2}(2M) = M.$$

Thus, for any matrix M, $W_1 + W_2 = M_{n\times n}(F)$. These conditions satisfy the necessary claim that $M_{n\times n}(F) = W_1 \bigoplus W_2$.

0.11 Question 30

Let W_1 and W_2 be subspaces of a vector space V. Prove that V is the direct sum of W_1 and W_2 if and only if each vector V can be uniquely written as x_1+x_2 where $x_1 \in W_1$ and $x_2 \in W_2$.

Proof: In the first direction, let us assume V is a direct sum of W_1 and W_2. This means that there exists a vector $v = x_1 + x_2$ where $x_1 \in W_1$ and $x_2 \in W_2$. Furthermore, $W_1 \cap W_2 = \{0\}$. Let us introduce a second set of vectors where $y_1 \in W_1$ and $y_2 \in W_2$ such that $v = y_1 + y_2$.

This makes

$$v = x_1 + x_2 = y_1 + y_2.$$

Rearranging this equation yields $x_1 - y_1 = y_2 - x_2$. Since $x_1 - y_1 \in W_1$ and $y_2 - x_2 \in W_2$, this proves that $x_1 - y_1 \in W_1 \cap W_2 = \{0\}$. Moreover, this also shows that $x_1 = y_1$ and $x_2 = y_2$, therefore proving that each vector in V can be uniquely written as $x_1 + x_2$.

In the other direction, let us assume each vector V can be uniquely written as $x_1 + x_2$ where $x_1 \in W_1$ and $x_2 \in W_2$. For any arbitrary vector $v \in V$ such that $v = x_1 + x_2$, this also means that $V = W_1 + W_2$.

Now we show that $W_1 \cap W_2 = \{0\}$ by contradiction. Suppose instead that $W_1 \cap W_2 \neq \{0\}$. Then there exists some nonzero vector $a \in W_1 \cap W_2$. Since $a \in W_1$, we may write

$$a = a + 0,$$

where $a \in W_1$ and $0 \in W_2$. Since $a \in W_2$, we may also write

$$a = 0 + a,$$

where $0 \in W_1$ and $a \in W_2$. These are two different representations of a as x_1+x_2 with $x_1 \in W_1$ and $x_2 \in W_2$, because $a \neq 0$. This contradicts the assumption that such a representation is unique. Therefore no nonzero vector can lie in $W_1 \cap W_2$, and hence

$$W_1 \cap W_2 = \{0\}.$$

These conditions satisfy the statement that $V = W_1 \bigoplus W_2$.