Homework 6

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1 3.4

Question 3. Suppose that the augmented matrix of a system $Ax = b$ is transformed into a matrix $(A'|b')$ in reduced row echelon form by a finite sequence of elementary row operations.

(a) Prove that $\text{rank}(A') \neq \text{rank}(A'|b')$ if and only if $(A'|b')$ contains a row in which the only nonzero entry lies in the last column.

(b) Deduce that $Ax = b$ is consistent if and only if $(A'|b')$ contains no row in which the only nonzero entry lies in the last column.

Proof. For part (a), suppose that $\text{rank}(A') \neq \text{rank}(A'|b')$. Because the columns of A' are a subset of the columns of the augmented matrix $(A'|b')$, it must be that $\text{rank}(A'|b') > \text{rank}(A')$. This difference in rank implies that there is at least one row i which is a non-zero row in $(A'|b')$ but a zero row in A'. For row i to be a zero row in A', we must have $a'_{ij} = 0$ for all $j = 1, \ldots, n$. However, since row i is non-zero in the augmented matrix, the entry in the last column must satisfy $b'_i \neq 0$. Thus, $(A'|b')$ contains a row in which the only non-zero entry lies in the last column.

Conversely, suppose that $(A'|b')$ contains a row i in which the only non-zero entry lies in the last column. This means that row i of the augmented matrix takes the form $[0\ 0\ \ldots\ 0 \mid b'_i]$ where $b'_i \neq 0$. In the reduced row echelon form, this non-zero entry b'_i must be a leading entry for $(A'|b')$. Since this entry is in the last column, the corresponding row in the coefficient matrix A' contains only zeros and thus does not have a leading entry. Because the rank of a matrix in reduced row echelon form is determined by the number of its leading entries, the augmented matrix $(A'|b')$ has a higher rank than A'. Therefore, it follows that $\text{rank}(A') \neq \text{rank}(A'|b')$.

For part (b), suppose the system $Ax = b$ is consistent. Then $\text{rank}(A') = \text{rank}(A'|b')$. If $(A'|b')$ contained a row where the only non-zero entry was in the last column, then by the result in part (a), we would have $\text{rank}(A') \neq \text{rank}(A'|b')$. This contradicts our assumption of consistency. Therefore, $(A'|b')$ must contain no such row.

In the other direction, suppose $(A'|b')$ contains no row in which the only non-zero entry lies in the last column. This implies that every non-zero row in $(A'|b')$ must also have at least one non-zero entry in the first n columns (the A' portion). Consequently, every leading entry of the augmented matrix $(A'|b')$ is also a leading entry of A'. Since the rank of a matrix in reduced row echelon form is equal to its number of leading entries, it follows that $\operatorname{rank}(A') = \operatorname{rank}(A'|b')$. Thus, the system is consistent. $\square$

Question 10. Let

$$V = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 - 2x_2 + 3x_3 - x_4 + 2x_5 = 0\}.$$

(a) Show that $S = \{(0, 1, 1, 1, 0)\}$ is a linearly independent subset of V.

(b) Extend S to a basis for V.

Proof. For part (a), the set S is clearly linearly independent since there is one nonzero vector. Additionally, we know that the vector $(0, 1, 1, 1, 0) \in V$ because $0-2(1)+3(1)-(1)+2(0) = 0$.

For part (b), the dimension of V is

$$\dim(V) = n - \operatorname{rank}(A) = 5 - 1 = 4.$$

To extend S to a basis for V, we must find three additional vectors $v_1, v_2, v_3 \in V$ such that the set $\{(0, 1, 1, 1, 0), v_1, v_2, v_3\}$ is a basis set. We can rearrange the condition such that

$$x_1 = 2x_2 - 3x_3 + x_4 - 2x_5.$$

Therefore, for arbitrary scalars $x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}$, we get

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_5) &= (2x_2 - 3x_3 + x_4 - 2x_5, x_2, x_3, x_4, x_5)\\ &= x_2(2, 1, 0, 0, 0) + x_3(-3, 0, 1, 0, 0) + x_4(1, 0, 0, 1, 0) + x_5(-2, 0, 0, 0, 1).\end{aligned}$$

Hence, the set $\{(2, 1, 0, 0, 0), (-3, 0, 1, 0, 0), (1, 0, 0, 1, 0), (-2, 0, 0, 0, 1)\}$ is a basis set for V. However, the set must extend S and hence include the vector $(0, 1, 1, 1, 0)$. Vector $(0, 1, 1, 1, 0)$ is a linear combination where $(0, 1, 1, 1, 0) = (2, 1, 0, 0, 0)+(-3, 0, 1, 0, 0)+(1, 0, 0, 1, 0)$. Since $(0, 1, 1, 1, 0)$ is a linear combination of the basis vectors with a non-zero coefficient for v_1, v_2, and v_3, we can replace any of those three vectors to form a new basis for V that extends S. By replacing $v_3 = (1, 0, 0, 1, 0)$, we obtain the basis

$$\boxed{\{(0, 1, 1, 1, 0), (2, 1, 0, 0, 0), (-3, 0, 1, 0, 0), (-2, 0, 0, 0, 1)\}.}$$

2 4.1

Question 5. Prove that if B is the matrix obtained by interchanging the rows of a 2×2 matrix A, then $\det(A) = -\det(B)$.

Proof. Let A be a 2×2 matrix such that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

By the definition of the problem, the matrix B is hence

$$B = \begin{pmatrix} c & d \\ a & b \end{pmatrix}.$$

Therefore,

$$\boxed{\det(A) = ad - bc = -(bc - ad) = -\det(B).}$$

$\square$

Problem 7. Prove that $\det(A^t) = \det(A)$ for any $A \in M_{2\times 2}(F)$.

Proof. Let A be a 2×2 matrix such that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This implies that

$$A^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Therefore,

$$\boxed{\det(A^t) = ad - bc = \det(A).}$$

□

3 4.2

Question 6. Evaluate the determinant of the following matrix by cofactor expansion along the first row.

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 3 & 0 \end{pmatrix}$$

Solution. Given

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij})$$

and $i = 1$, we get the following cofactor expansion:

$$\begin{aligned} \det(A) &= (1)\det\begin{pmatrix} 1 & 5 \\ 3 & 0 \end{pmatrix} - 0 + (2)\det\begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix} \\ &= -15 - 0 + 2(1) = \boxed{-13.} \end{aligned}$$

□

Question 12. Evaluate the determinant of the following matrix by cofactor expansion along the fourth row.

$$\begin{pmatrix} 1 & -1 & 2 & -1 \\ -3 & 4 & 1 & -1 \\ 2 & -5 & -3 & 8 \\ -2 & 6 & -4 & 1 \end{pmatrix}$$

Solution. Given

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij})$$

and $i = 4$, we get the following cofactor expansion:

$$\begin{aligned}
\det(A) &= (2)\det\begin{pmatrix} -1 & 2 & -1 \\ 4 & 1 & -1 \\ -5 & -3 & 8 \end{pmatrix} + (6)\det\begin{pmatrix} 1 & 2 & -1 \\ -3 & 1 & -1 \\ 2 & -3 & 8 \end{pmatrix} \\
&+ (4)\det\begin{pmatrix} 1 & -1 & -1 \\ -3 & 4 & -1 \\ 2 & -5 & 8 \end{pmatrix} + (1)\det\begin{pmatrix} 1 & -1 & 2 \\ -3 & 4 & 1 \\ 2 & -5 & -3 \end{pmatrix} \\
&= 2\left[-1\det\begin{pmatrix} 1 & -1 \\ -3 & 8 \end{pmatrix} - (2)\det\begin{pmatrix} 4 & -1 \\ -5 & 8 \end{pmatrix} - (1)\det\begin{pmatrix} 4 & 1 \\ -5 & -3 \end{pmatrix}\right] \\
&+ 6\left[\det\begin{pmatrix} 1 & -1 \\ -3 & 8 \end{pmatrix} - (2)\det\begin{pmatrix} -3 & -1 \\ 2 & 8 \end{pmatrix} - (1)\det\begin{pmatrix} -3 & 1 \\ 2 & -3 \end{pmatrix}\right] \\
&+ 4\left[\det\begin{pmatrix} 4 & -1 \\ -5 & 8 \end{pmatrix} + \det\begin{pmatrix} -3 & -1 \\ 2 & 8 \end{pmatrix} - (1)\det\begin{pmatrix} -3 & 4 \\ 2 & -5 \end{pmatrix}\right] \\
&+ \left[\det\begin{pmatrix} 4 & 1 \\ -5 & -3 \end{pmatrix} + \det\begin{pmatrix} -3 & 1 \\ 2 & -3 \end{pmatrix} + (2)\det\begin{pmatrix} -3 & 4 \\ 2 & -5 \end{pmatrix}\right] \\
&= 2[-(8-3) - 2(32-5) - (-12+5)] + 6[(8-3) - 2(-24+2) - (9-2)] \\
&+ 4[(32-5) + (-24+2) - (15-8)] + [(-12+5) + (9-2) + 2(15-8)] \\
&= 2(-5-54+7) + 6(5+44-7) + 4(27-22-7) + (-7+7+14) \\
&= \boxed{154.}
\end{aligned}$$

□

Question 23. Prove that the determinant of an upper triangular matrix is the product of its diagonal entries.

Proof. Let us solve this by proof by induction. In a 2×2 upper triangular matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$,

$$\det(A) = a_{11}(a_{22}) - 0(a_{12}) = a_{11}a_{22}.$$

Now, let us assume the case holds for $(n-1) \times (n-1)$ upper triangular matrices where $n \geq 2$. In the last row n of the a $n \times n$ upper triangular matrix A, all entries are zero except the last entry, which is some constant a_{nn}. By cofactor expansion along the last row,

$$\begin{aligned}\det(A) &= 0 + 0 + \ldots + 0 + (-1)^{n+n}(a_{nn}) \det(\tilde{A}_{nn}) \\ &= (-1)^{2n}(a_{nn}) \det(\tilde{A}_{nn}).\end{aligned}$$

However, we know that $\tilde{A}_{nn}$ is also an upper triangular matrix and we assumed that the case holds for the $n-1 \times n-1$ matrices. Hence, we can say that $\det(\tilde{A}_{nn}) = a_{11}a_{22} \ldots a_{(n-1)(n-1)}$. Additionally, $(-1)^{2n} = (-1)^{2^n} = 1^n = 1$. Therefore,

$$\boxed{\det(A) = (a_{nn}) \det(\tilde{A}_{nn}) = (a_{nn})(a_{11}a_{22} \ldots a_{(n-1)(n-1)}) = a_{11}a_{22} \cdots a_{nn}.}$$

$\square$

Problem 25. Prove that $\det(kA) = k^n \det(A)$ for any $A \in \mathrm{M}_{n\times n}(F)$.

Proof. Let A be a $n \times n$ matrix defined as such

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix},$$

where a_i are n-tuple row vectors. Notice that multiplying A by a scalar $k \in F$, all of the n row vectors are mutiplied by the same scalar. In terms of elementary matrix operations, this is the equivalent of a type 1 row operation n times on each row vector.

Therefore,

$$kA = E_n E_{n-1} \dots E_1 A,$$

where E_i are elementary matrices corresponding to multiplying the ith row by k. Then,

$$\det(kA) = \det(E_n)\det(E_{n-1})\dots\det(E_1)\det(A).$$

Recall that in an elementary matrix corresponding to multiplying a row by k, $\det(E_i) = k$. Hence,

$$\boxed{\det(kA) = \underbrace{k(k)\dots(k)}_{n} \cdot \det(A) = k^n \det(A).}$$

□

Question 26. Let $A \in \mathrm{M}_{n\times n}(F)$. Under what conditions is $\det(-A) = \det(A)$?

Solution. It is obvious that the trivial solution $A = O$ exists. However, utilizing our solution from problem 25, by setting $k = -1$, we get that

$$\det(-A) = \underbrace{-1(-1)\dots(-1)}_{n} \cdot \det(A) = (-1)^n \det(A).$$

In order for $\det(-A) = \det(A)$, $(-1)^n = 1$. Hence, n must be even. Therefore, for the condition to be true, **the number of rows and columns in A must be even.** $\square$

4 4.3

Question 12. A matrix $Q \in \mathrm{M}_{n\times n}(R)$ is called orthogonal if $QQ^t = I$. Prove that if Q is orthogonal, then $\det(Q) = \pm 1$.

Proof. If Q is orthogonal, then it is true that $QQ^t = I$. For this proof, we will define $a \in R$ such that $a = \det(A)$. By definition, $\det(I) = 1$ and $\det(Q^t) = \det(Q) = a$. Therefore, we are given an equation

$$a^2 = 1.$$

Hence, $\boxed{a = \det(A) = \pm 1.}$ □

Question 14. Let $\beta = \{u_1, u_2, \ldots u_n\}$ be a subset of $\mathbb{F}^n$ containing n distinct vectors, and let B be the matrix in $\mathrm{M}_{n\times n}(\mathbb{F})$ having u_j as column j. Prove that β is a basis for $\mathbb{F}^n$ if and only if $\det(B) \neq 0$.

Proof. In the first direction, let us assume that β is a basis for $\mathbb{F}^n$. This means that β is linearly independent. Since every vector in this linearly independent set is a column vector in B, this in turn implies that B has full rank, which proves that B is invertible. Therefore, $\det(B) \neq 0$.

In the other direction, let us assume $\det(B) \neq 0$. This implies that B is invertible, which therefore implies that it is rank full. Since we also know that β has distinct vectors, this implies that β is linearly independent. However, β has n vectors and $\dim(\mathbb{F}^n) = n$. Therefore, β is a basis for $\mathbb{F}^n$. $\square$

Question 15. Prove that if $A, B \in \mathrm{M}_{n\times n}(F)$ are similar, then $\det(A) = \det(B)$.

Proof. If matrices A and B are similar, then it is implied that

$$A = Q^{-1}BQ,$$

where Q is a change of basis matrix. By definition of inverses and by composition of matrices we get that

$$\begin{aligned}\det(A) &= \det(Q^{-1}BQ) = \det(Q^{-1})\det(B)\det(Q)\\ &= \frac{1}{\det(Q)}\det(B)\det(Q).\end{aligned}$$

Since determinants are scalars, we can move around the first and third terms to cancel them out. Therefore, $\boxed{\det(A) = \det(B).}$

$\square$