Homework 2

Andrew Hwang

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Section 1.4

0.1 Question 6

Show that the vectors $(1,1,0)$, $(1,0,1)$, and $(0,1,1)$ generate F^3.

First, we must assume that F does not have characteristic two. Let $(a,b,c) \in F^3$ be arbitrary. Let us find coefficients $r,s,t \in F$ such that

$$r(1,1,0) + s(1,0,1) + t(0,1,1) = (a,b,c).$$

The equation can be further expanded as

$$(r+s, r+t, s+t) = (a,b,c).$$

Thus, we are led to a following system of linear equations:

$$r + s = a \tag{1}$$
$$r + t = b \tag{2}$$
$$s + t = c \tag{3}$$

Adding equation (1) and (2) yields $2r + s + t = a + b$. Substituting equation (3) into this equation yields

$$2r + c = a + b,$$

and rearranging this equation yields

$$r = \frac{1}{2}(a + b - c).$$

By the same logic, adding equation (1) and (3), and then substituting equation (2) yields

$$s = \frac{1}{2}(a - b + c).$$

Lastly, adding equation (2) and (3), and then substituting equation (1) yields

$$t = \frac{1}{2}(-a + b + c).$$

Therefore, every $(a,b,c) \in F^3$ can be expressed as a linear combination of vectors $(1,1,0), (1,0,1), (0,1,1)$ with coefficients $r,s,t \in F$. Hence $\{(1,1,0),(1,0,1),(0,1,1)\}$ generates F^3.

0.2 Question 14

Show that if S_1 and S_2 are arbitrary subsets of a vector space V, then $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$.

Proof. If $S_1 = S_2 = \emptyset$, then $S_1 \cup S_2 = \emptyset$. Thus, the condition is satisfied because

$$\text{span}(S_1 \cup S_2) = \text{span}(\emptyset) = \{0\}$$

and

$$\text{span}(S_1) + \text{span}(S_2) = \{0\} + \{0\} = \{0\}.$$

Suppose now that S_1 and S_2 were not empty. Define subsets $S_1 = \{u_1, u_2, \ldots, u_n\} \subseteq V$ and $S_2 = \{v_1, v_2, \ldots, v_m\} \subseteq V$. Let us also define coefficients $a_1, a_2, \ldots, a_n \in F$ and $b_1, b_2, \ldots, b_m \in F$.

We can thus create arbitrary vectors $x, y \in V$, $x \in \text{span}(S_1)$, $y \in \text{span}(S_2)$ expressed as linear combinations such that

$$x = a_1u_1 + a_2u_2 + \ldots + a_nu_n, \quad y = b_1v_1 + b_2v_2 + \cdots + b_mv_m.$$

We can add x and y so that

$$x + y = a_1u_1 + a_2u_2 + \ldots + a_nu_n + b_1v_1 + b_2v_2 + \cdots + b_mv_m.$$

This is the equivalent of a linear combination of vectors in $S_1 \cup S_2$. Hence,

$$x + y \in \text{span}(S_1 \cup S_2).$$

Thus,

$$\text{span}(S_1) + \text{span}(S_2) \subseteq \text{span}(S_1 \cup S_2).$$

Now let $z \in \text{span}(S_1 \cup S_2)$. Then z can be expressed as a linear combination of vectors in $S_1 \cup S_2$. That is,

$$z = c_1w_1 + c_2w_2 + \cdots + c_kw_k$$

where each $w_i \in S_1 \cup S_2$ and each $c_i \in F$. Every w_i is either in S_1 or in S_2. Let x be the sum of all terms c_iw_i where $w_i \in S_1$, and let y be the sum of all terms c_iw_i where $w_i \in S_2$. Then $x \in \text{span}(S_1)$ and $y \in \text{span}(S_2)$, and

$$z = x + y.$$

Therefore, $z \in \text{span}(S_1) + \text{span}(S_2)$, so

$$\text{span}(S_1 \cup S_2) \subseteq \text{span}(S_1) + \text{span}(S_2).$$

Since both inclusions hold, we conclude that

$$\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2).$$

0.3 Question 15

Let S_1 and S_2 be subsets of a vector space V. Prove that $\operatorname{span}(S_1 \cap S_2) \subseteq \operatorname{span}(S_1) \cap \operatorname{span}(S_2)$. Give an example in which $\operatorname{span}(S_1 \cap S_2)$ and $\operatorname{span}(S_1) \cap \operatorname{span}(S_2)$ are equal and one in which they are unequal.

Proof. Given an arbitrary vector $x \in \operatorname{span}(S_1 \cap S_2)$, we can express x as a linear combination of vectors $v_1, v_2, \ldots, v_n \in S_1 \cap S_2$ and scalars $a_1, a_2, \ldots, a_n \in F$ as

$$x = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n.$$

Since $v_i \in S_1 \cap S_2$ for each $i = 1, 2, \ldots, n$, we have $v_i \in S_1$ and $v_i \in S_2$. Thus the same linear combination shows that $x \in \operatorname{span}(S_1)$ and $x \in \operatorname{span}(S_2)$. Therefore,

$$x \in \operatorname{span}(S_1) \cap \operatorname{span}(S_2).$$

Hence,

$$\operatorname{span}(S_1 \cap S_2) \subseteq \operatorname{span}(S_1) \cap \operatorname{span}(S_2).$$

An example of a proof where the $\operatorname{span}(S_1 \cap S_2)$ and $\operatorname{span}(S_1) \cap \operatorname{span}(S_2)$ are equal is

$$S_1 = \{(1,0,0), (0,1,0)\}, \quad S_2 = \{(1,0,0), (0,0,1)\}.$$

The intersection of the two sets is

$$S_1 \cap S_2 = \{(1,0,0)\}.$$

Thus,

$$\operatorname{span}(S_1 \cap S_2) = \operatorname{span}\{(1,0,0)\} = \{(a,0,0) : a \in \mathbb{R}\}.$$

Next, observe that

$$\operatorname{span}(S_1) = \operatorname{span}\{(1,0,0), (0,1,0)\} = \{(a,b,0) : a, b \in \mathbb{R}\},$$

and

$$\operatorname{span}(S_2) = \operatorname{span}\{(1,0,0), (0,0,1)\} = \{(a,0,c) : a, c \in \mathbb{R}\}.$$

Therefore,

$$\operatorname{span}(S_1) \cap \operatorname{span}(S_2) = \{(a,b,0)\} \cap \{(a,0,c)\} = \{(a,0,0) : a \in \mathbb{R}\}.$$

Hence, we conclude that in this example,

$$\operatorname{span}(S_1 \cap S_2) = \operatorname{span}(S_1) \cap \operatorname{span}(S_2) = \{(a,0,0) : a \in \mathbb{R}\}.$$

An example of a proof where the $\operatorname{span}(S_1 \cap S_2)$ and $\operatorname{span}(S_1) \cap \operatorname{span}(S_2)$ are not equal is

$$S_1 = \{(1,0,0),(0,1,0),(0,0,1)\}, \quad S_2 = \{(1,1,1)\}.$$

The intersection of the two sets is

$$S_1 \cap S_2 = \{\emptyset\}.$$

Thus,

$$\operatorname{span}(S_1 \cap S_2) = \operatorname{span}\{\emptyset\} = 0.$$

Next, observe that

$$\operatorname{span}(S_1) = \operatorname{span}\{(1,0,0),(0,1,0),(0,0,1)\} = \{(a,b,c) : a,b,c \in \mathbb{R}\},$$

and

$$\operatorname{span}(S_2) = \operatorname{span}\{(1,1,1)\} = \{(d,d,d) : d \in \mathbb{R}\}.$$

Therefore,

$$\operatorname{span}(S_1) \cap \operatorname{span}(S_2) = \{(a,b,c)\} \cap \{(d,d,d)\} = \{(a,a,a) : a \in \mathbb{R}\}.$$

Hence, we conclude that in this example,

$$\operatorname{span}(S_1 \cap S_2) \neq \operatorname{span}(S_1) \cap \operatorname{span}(S_2).$$

Section 1.5

0.4 Question 8

Let $S = \{(1,1,0),(1,0,1),(0,1,1)\}$ be a subset of the vector space F^3.
(a) Prove that if $F = R$, then S is linearly independent.

Proof: Assume S is linearly dependent. Then there exist scalars $r, s, t \in \mathbb{R}$, not all zero, such that

$$r(1,1,0) + s(1,0,1) + t(0,1,1) = (0,0,0).$$

Expanding gives

$$(r+s,\ r+t,\ s+t) = (0,0,0),$$

so

$$r+s=0, \quad r+t=0, \quad s+t=0.$$

From substituting the first two equations we obtain $s = t$. Substituting into $s + t = 0$ gives $2s = 0$. Since $\mathrm{char}(F) \neq 2$, it follows that $s = 0$. Hence $t = 0$ and then $r = 0$. Therefore $r = s = t = 0$, contradicting the assumption that not all were zero. Thus, S is linearly independent.

(b) Prove that if F has characteristic two, then S is linearly dependent.
Proof: If

$$r(1,1,0) + s(1,0,1) + t(0,1,1) = (0,0,0),$$

then

$$r+s=0, \quad r+t=0, \quad s+t=0.$$

From the first two equations we obtain $s = t$, and substituting into $s + t = 0$ gives $2s = 0$. Since F is characteristic two, we have $2 = 0$, so $2s = 0$ holds for every $s \in F$. Hence, we may choose $s = 1$, which yields $t = 1$ and $r = -1 = 1$. Therefore $r = s = t = 1$ is a nontrivial solution, so S is linearly dependent.

0.5 Question 9

Let u and v be distinct vectors in a vector space V. Show that $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of the other.

Proof. In the first direction, assume $\{u, v\}$ is linearly dependent. Then there exist scalars $a, b \in F$, not both zero, such that

$$au + bv = 0.$$

If $a \neq 0$, then

$$au = -bv \quad \Rightarrow \quad u = -\frac{b}{a}v,$$

so u is a scalar multiple of v.

If $a = 0$, then $bv = 0$. Since $b \neq 0$, it follows that $v = 0_v$, and hence $v = 0 \cdot u$, so v is a multiple of u. Thus, in all cases, u or v is a multiple of the other.

In the other direction, assume one vector is a multiple of the other. Without loss of generality, suppose $u = cv$ for some scalar $c \in F$. Then

$$(1)u + (-c)v = 0.$$

This is a nontrivial linear combination of u and v because the coefficients are 1 and $-c$. Thus, both directions are proven and therefore $\{u, v\}$ is linearly dependent.

0.6 Question 19

Prove that if $\{A_1, A_2, \ldots, A_k\}$ is a linearly independent subset of $M_{n\times n}(F)$, then $\{A_1^t, A_2^t, \ldots, A_k^t\}$ is also linearly independent.

Proof. Assume $\{A_1, \ldots, A_k\}$ is linearly independent in $M_{n\times n}(F)$. To show $\{A_1^t, \ldots, A_k^t\}$ is linearly independent, suppose

$$c_1A_1^t + c_2A_2^t + \cdots + c_kA_k^t = O,$$

where $c_1, \ldots, c_k \in F$.

Let $A_m = (A_m)_{ij}$ for each $m = 1, \ldots, k$. Then

$$A_m^t = (A_m)_{ji},$$

so the (i, j)-entry of the matrix $c_1A_1^t + \cdots + c_kA_k^t$ equals

$$c_1(A_1)_{ji} + c_2(A_2)_{ji} + \cdots + c_k(A_k)_{ji}.$$

Since the sum is the zero matrix, every entry is 0, hence for all i, j,

$$c_1(A_1)_{ji} + c_2(A_2)_{ji} + \cdots + c_k(A_k)_{ji} = 0.$$

Thus, $c_1A_1^t + \cdots + c_kA_k^t = O$ produces a system of equations $c_1(A_1)_{ji} + c_2(A_2)_{ji} + \cdots + c_k(A_k)_{ji} = 0$ for all i, j. This collection is identical to the one for $c_1A_1 + \cdots + c_kA_k = 0$, since swapping i and j only reorders the equations. Therefore, $\{A_1^t, \ldots, A_k^t\}$ is linearly independent.

Section 1.6

0.7 Question 3b

Determine if the set $\{1+2x+x^2, 3+x^2, x+x^2\}$ is a basis for $P_2(\mathbb{R})$.

Since $\dim(P_2(\mathbb{R})) = 3$ and the set

$$S = \{1+2x+x^2,\ 3+x^2,\ x+x^2\}$$

has three polynomials, the set is a basis if S is linearly independent.

Let $a_1, a_2, a_3 \in \mathbb{R}$ and suppose

$$a_1(1+2x+x^2) + a_2(3+x^2) + a_3(x+x^2) = 0.$$

Expanding and collecting like terms yields

$$(a_1+3a_2) + (2a_1+a_3)x + (a_1+a_2+a_3)x^2 = 0.$$

Since the zero polynomial has all coefficients equal to 0, we obtain the system of equations

$$a_1 + 3a_2 = 0 \tag{4}$$
$$2a_1 + a_3 = 0 \tag{5}$$
$$a_1 + a_2 + a_3 = 0. \tag{6}$$

From (5), $a_3 = -2a_1$. Substituting into (6) gives

$$a_1 + a_2 - 2a_1 = 0 \quad \Rightarrow \quad a_2 = a_1.$$

Substituting $a_2 = a_1$ into (1) yields

$$a_1 + 3a_1 = 0 \quad \Rightarrow \quad 4a_1 = 0 \quad \Rightarrow \quad a_1 = 0.$$

Thus $a_2 = 0$ and $a_3 = 0$ as well. Therefore, the only solution is

$$a_1 = a_2 = a_3 = 0,$$

a trivial solution. Hence, S is a linearly independent set of three vectors in $P_2(\mathbb{R})$, so S is a basis for $P_2(\mathbb{R})$.

0.8 Question 11

Let u and v be distinct vectors of a vector space V. Show that if $\{u, v\}$ is a basis for V and a and b are nonzero scalars, then both $\{u + v, au\}$ and $\{au, bv\}$ are also bases for V.

Proof: Since $\{u, v\}$ is a basis for V, $\dim V = 2$. Hence, to show that each of the sets $\{u + v, au\}$ and $\{au, bv\}$ is a basis for V, it suffices to show that each set is linearly independent.

First, we show that $\{u + v, au\}$ is linearly independent. Let $c_1, c_2 \in F$ and suppose

$$c_1(u + v) + c_2(au) = 0.$$

Expanding and grouping by u and v gives

$$(c_1 + ac_2)u + c_1 v = 0.$$

Since $\{u, v\}$ is linearly independent, the coefficients must both be zero:

$$c_1 = 0, \qquad c_1 + ac_2 = 0.$$

Substituting $c_1 = 0$ into the second equation yields $ac_2 = 0$. Because $a \neq 0$, it follows that $c_2 = 0$. Therefore the only solution is $c_1 = c_2 = 0$, so $\{u + v, au\}$ is linearly independent, and hence a basis for V.

Next, we show that $\{au, bv\}$ is linearly independent. Let $d_1, d_2 \in F$ and suppose

$$d_1(au) + d_2(bv) = 0.$$

Grouping terms gives

$$(ad_1)u + (bd_2)v = 0.$$

Again using linear independence of $\{u, v\}$, we conclude

$$ad_1 = 0 \quad \text{and} \quad bd_2 = 0.$$

Since $a \neq 0$ and $b \neq 0$, this implies $d_1 = 0$ and $d_2 = 0$. Thus $\{au, bv\}$ is linearly independent, and therefore it is a basis for V.

Hence both $\{u + v, au\}$ and $\{au, bv\}$ are bases for V.