Homework 5

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Question 8. Let A be an $m \times n$ matrix. Prove that if c is any nonzero scalar, then $\text{rank}(cA) = \text{rank}(A)$.

Proof. Observe that $R(L_{cA}) = R(cL_A) = cL_A(F^n) = L_A(cF^n)$. Since F^n is closed under multiplication, this implies that $L_A(cF^n) = R(L_A)$. Hence, $R(L_{cA}) = R(L_A)$, and therefore $\text{rank}(cA) = \text{rank}(A)$. $\square$

Question 14. Let $T, U : V \to W$ be linear transformations.

(a) Prove that $R(T+U) \subseteq R(T) + R(U)$.

(b) Prove that if W is finite-dimensional, then $\text{rank}(T+U) \leq \text{rank}(T) + \text{rank}(U)$.

(c) Deduce from (b) that $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$ for any matrices A and B.

Proof. (a) Let $w \in R(T+U)$. By definition, there exists a vector $v \in V$ such that

$$w = (T+U)(v) = T(v) + U(v).$$

Since $T(v) \in R(T)$ and $U(v) \in R(U)$, we have expressed w as a sum of an element from $R(T)$ and an element from $R(U)$. By the definition of the sum of subspaces, $w \in R(T) + R(U)$. Thus, $\boxed{R(T+U) \subseteq R(T) + R(U).}$

(b) From part (a), we know $R(T+U)$ is a subspace of $R(T) + R(U)$. It follows from the properties of finite-dimensional vector spaces that

$$\dim(R(T+U)) \leq \dim(R(T) + R(U)).$$

Recall the dimension formula for the sum of two subspaces S_1 and S_2 is

$$\dim(S_1 + S_2) = \dim(S_1) + \dim(S_2) - \dim(S_1 \cap S_2).$$

Since $\dim(S_1 \cap S_2) \geq 0$, we can arrange the inequality as $\dim(S_1 + S_2) \leq \dim(S_1) + \dim(S_2)$. Applying this formula to the set of ranges gives us

$$\boxed{\text{rank}(T+U) \leq \text{rank}(T) + \text{rank}(U).}$$

(c) Let A and B be $m \times n$ matrices. Consider the associated left multiplication transformations L_A, L_B. Since F^m is finite-dimensional, we can apply the result from part (b) to get

$$\text{rank}(L_A + L_B) = \text{rank}(L_{A+B}) \leq \text{rank}(L_A) + \text{rank}(L_B)$$

Since the rank of a matrix is equal to the rank of its corresponding linear transformation, it is hence proven that

$$\boxed{\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B).}$$

□

Question 17. Prove that if B is a 3×1 matrix and C is a 1×3 matrix, then the 3×3 matrix BC has rank at most 1. Conversely, show that if A is any 3×3 matrix having rank 1, then there exists a 3×1 matrix B and a 1×3 matrix C such that $A = BC$.

Proof. To prove the first part, let B be a 3×1 matrix and C be a 1×3 matrix such that

$$B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad C = \begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix}.$$

The product BC is the 3×3 matrix

$$BC = \begin{pmatrix} b_1c_1 & b_1c_2 & b_1c_3 \\ b_2c_1 & b_2c_2 & b_2c_3 \\ b_3c_1 & b_3c_2 & b_3c_3 \end{pmatrix}$$

Observe that each column j is simply the vector B scaled by the scalar c_j. Thus, the column space of BC is spanned by the single vector B (unless $B = O$ or $C = O$, in which case the rank is 0). Therefore, $\operatorname{rank}(BC) \leq 1$.

Conversely, suppose A is a 3×3 matrix with $\operatorname{rank}(A) = 1$. This implies that the column space of A is one-dimensional. Let B be a non-zero vector (a 3×1 matrix) that forms a basis for $\operatorname{Col}(A)$. Then, every column a_j of A (for $1 \leq j \leq 3$) must be a scalar multiple of B. That is, $a_j = c_jB$ for some scalars $c_j \in F$. If we define C as the 1×3 matrix $\begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix}$, then by the definition of matrix multiplication,

$$A = [c_1B, c_2B, c_3B] = B \begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix} = BC.$$

Thus, A can be expressed as the product of a 3×1 matrix and a 1×3 matrix. $\square$

Question 21. Let A be an $m \times n$ matrix with rank m. Prove that there exists a $n \times m$ matrix B such that $AB = I_m$.

Proof. Since A has rank m, we know that the left multiplication $L_A : \mathbb{F}^n \to \mathbb{F}^m$ is onto. This means that the range of L_A is the entire space $\mathbb{F}^m$. Consequently, every vector in $\mathbb{F}^m$ is the image of at least one vector in $\mathbb{F}^n$. Then, it must be true that there exists a set of a vectors $v_1, \ldots, v_j \in \mathbb{F}^n$ such that

$$L_A(v_j) = Av_j = e_j,$$

where $1 \leq j \leq m$. Now, let B be the $n \times m$ matrix whose columns are these vectors $v_1, v_2, \ldots, v_m$:

$$B = [v_1, \ldots, v_m]$$

By the definition of matrix multiplication, the j-th column of AB is the product of A and the j-th column of B. Hence, $(AB)_j = A(v_j) = e_j$. Since the j-th column of AB is e_j for all $1 \leq j \leq m$,

$$AB = [e_1, \ldots e_m] = I_m.$$

Thus, there exists a matrix B such that $AB = I_m$. $\square$

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Question 5. Give an example of a system of n linear equations in n unknowns with infinitely many solutions.

Solution. Define a system of n equations as follows:

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= 1 \\ 2x_1 + 2x_2 + \cdots + 2x_n &= 2 \\ &\vdots \\ nx_1 + nx_2 + \cdots + nx_n &= n \end{aligned}$$

Because each equation is a scalar multiple of the first equation ($x_1 + x_2 + \cdots + x_n = 1$), any solution to the first equation is necessarily a solution to the entire system. Since a single linear equation with $n > 1$ unknowns has infinitely many solutions, the entire system has infinitely many solutions. □

Question 8. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(a, b, c) = (a + b, b - 2c, a + 2c)$. For each vector v in $\mathbb{R}^3$, determine whether $v \in R(T)$.

(a) $v = (1, 3, -2)$

(b) $v = (2, 1, 1)$

Solution. In order for $v \in R(T)$, the system $T(a, b, c) = v$ must be consistent. This can be expressed as the matrix equation $Ax = v$:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The system is consistent if and only if $\mathrm{rank}(A) = \mathrm{rank}(A|v)$. We first determine $\mathrm{rank}(A)$ through row reduction:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus, $\mathrm{rank}(A) = 2$. To find the condition for consistency, we apply the same row operations to the augmented matrix $(A|v)$:

$$\begin{pmatrix} 1 & 1 & 0 & | & x \\ 0 & 1 & -2 & | & y \\ 1 & 0 & 2 & | & z \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & | & x \\ 0 & 1 & -2 & | & y \\ 0 & -1 & 2 & | & z - x \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & | & x \\ 0 & 1 & -2 & | & y \\ 0 & 0 & 0 & | & z - x + y \end{pmatrix}$$

Hence, the system is consistent if and only if $z - x + y = 0$.

(a) For $v = (1, 3, -2)$, we have $z - x + y = (-2) - (1) + (3) = 0$. Since the condition is satisfied, $\mathrm{rank}(A|v) = 2 = \mathrm{rank}(A)$, and $\boxed{(1, 3, -2) \in R(T).}$

(b) For $v = (2, 1, 1)$, we have $z - x + y = (1) - (2) + (1) = 0$. Since the condition is satisfied, $\mathrm{rank}(A|v) = 2 = \mathrm{rank}(A)$, and $\boxed{(2, 1, 1) \in R(T).}$

$\square$