Homework 3

Andrew Hwang
Math 24
Professor Williams

February 3, 2026

1 1.6

Question 14. Find bases for the following subspaces of F^5:

$$W_1 = \{(a_1, a_2, a_3, a_4, a_5) \in F^5 : a_1 - a_3 - a_4 = 0)\}$$

and

$$W_2 = \{(a_1, a_2, a_3, a_4, a_5) \in F^5 : a_2 = a_3 = a_4, a_1 + a_5 = 0)\}.$$

What are the dimensions of W_1 and W_2?

Solution. Define F^5 as the vector space of 5-tuples over the field F. First, let us evaluate W_1. Let $(a_1, a_2, a_3, a_4, a_5) \in W_1$. Then the defining condition

$$a_1 - a_3 - a_4 = 0$$

implies

$$a_1 = a_3 + a_4.$$

Hence, every vector in W_1 has the form

$$(a_3 + a_4,\ a_2,\ a_3,\ a_4,\ a_5).$$

This 5-tuple can be rearranged as a linear combination as such:

$$(a_3 + a_4, a_2, a_3, a_4, a_5) = a_2(0, 1, 0, 0, 0) + a_3(1, 0, 1, 0, 0) + a_4(1, 0, 0, 1, 0) + a_5(0, 0, 0, 0, 1)$$

Let us define set $\beta_1 = \{(0, 1, 0, 0, 0), (1, 0, 1, 0, 0), (1, 0, 0, 1, 0), (0, 0, 0, 0, 1)\}$. It is true that β_1 spans W_1. Now, let us show that β_1 is linearly independent. Suppose

$$c_1(1, 0, 0, 0, 0) + c_2(0, 1, 0, 0, 0) + c_3(1, 0, 1, 0, 0) + c_4(0, 0, 0, 0, 1) = (0, 0, 0, 0, 0).$$

Then, we get a system of equations

$$\begin{aligned} c_1 + c_3 &= 0, \\ c_2 &= 0, \\ c_3 &= 0, \\ c_4 &= 0. \end{aligned}$$

From $c_3 = 0$ we get $c_1 = 0$, and thus $c_1 = c_2 = c_3 = c_4 = 0$. Hence the vectors in β_1 are linearly independent. Since β_1 spans W_1, and is linearly independent, it is a basis set for W_1. Therefore, $\dim W_1 = 4$.

Similarly, let $(a_1, a_2, a_3, a_4, a_5) \in W_2$. The defining conditions

$$a_2 = a_3 = a_4 \quad \text{and} \quad a_1 + a_5 = 0$$

imply

$$a_1 = -a_5.$$

Hence, every vector in W_2 has the form

$$(-a_5,\ a_2,\ a_2,\ a_2,\ a_5).$$

Repeating the procedure from evaluating W_1, we express an arbitrary vector in W_2 as a linear combination:

$$(-a_5, a_2, a_2, a_2, a_5) = a_2(0, 1, 1, 1, 0) + a_5(-1, 0, 0, 0, 1).$$

Thus, $\beta_2 = \{(0, 1, 1, 1, 0), (-1, 0, 0, 0, 1)\}$ spans W_2. To show that β_2 is linearly independent, assume

$$c_1(0, 1, 1, 1, 0) + c_2(-1, 0, 0, 0, 1) = (0, 0, 0, 0, 0).$$

This gives us a system of equations

$$\begin{aligned} -c_2 &= 0, \\ c_1 &= 0, \\ c_1 &= 0, \\ c_1 &= 0, \\ c_2 &= 0. \end{aligned}$$

Hence $c_1 = c_2 = 0$, so the vectors in β_2 are linearly independent. Therefore, β_2 spans W_2 and is linearly independent, so β_2 is a basis for W_2. Thus, $\dim W_2 = 2$.

$\square$

Question 17. The set of all skew-symmetric $n \times n$ matrices is a subspace W of $M_{n\times n}(F)$. Find a basis for W. What is the dimension of W?

Solution. Let

$$W = \{A \in M_{n\times n}(F) : A^t = -A\}.$$

Comparing each entry in the matrix, the condition $A^t = -A$ is equivalent to

$$A_{ji} = -A_{ij} \quad \text{for all } i, j.$$

In particular, when $i = j$ we get $A_{ii} = -A_{ii}$, so $2A_{ii} = 0$. Hence if F is not of characteristic 2, then $A_{ii} = 0$ for all i.

For each pair $1 \le i < j \le n$, define

$$K_{ij} = E_{ij} - E_{ji}.$$

Then $K_{ij}^t = E_{ji} - E_{ij} = -K_{ij}$, so $K_{ij} \in W$. Let $A = (A_{ij}) \in W$ with $i < j$. Since A is skew-symmetric, $A_{ji} = -A_{ij}$ and $A_{ii} = 0$. Define

$$B = \sum_{1\le i<j\le n} A_{ij}K_{ij}.$$

Then for $i < j$, the (i, j) entry of B is A_{ij}, and the (j, i) entry is $-A_{ij} = A_{ji}$. All diagonal entries of B are zero. Hence B has the same entries as A, so $A = B$. Therefore, every $A \in W$ is a linear combination of the matrices K_{ij}, and

$$W = \operatorname{span}(\{K_{ij} : 1 \le i < j \le n\}).$$

Next, we prove that $\{K_{ij} : 1 \le i < j \le n\}$ is linearly independent. Suppose

$$\sum_{1\le i<j\le n} c_{ij}K_{ij} = 0.$$

Fix $i < j$. The matrix K_{ij} is the only basis element with a nonzero entry in the (i, j) position, where it equals 1. Hence, the (i, j) entry of the sum equals c_{ij}. Since the zero matrix has a zero in this position, we obtain $c_{ij} = 0$ for all $i < j$. Therefore, the set is linearly independent, and also a basis set.

To compute $\dim(W)$, note that an $n \times n$ matrix has n^2 entries. In a skew-symmetric matrix $A^t = -A$, the diagonal entries satisfy $A_{ii} = -A_{ii}$, so $a_{ii} = 0$ for all i, giving n forced zero entries on the diagonal. Among the remaining $n^2 - n$ off-diagonal entries, the skew-symmetry condition pairs each entry (i, j) with its mirror (j, i), so each pair is determined by a single variable. Thus, the number of free parameters is

$$\frac{n^2 - n}{2} = \frac{n(n-1)}{2}.$$

Hence,

$$\dim(W) = \frac{n(n-1)}{2}.$$

□

Question 23. Let $v_1, v_2, \ldots, v_k, v$ be vectors in a vector space V, and define

$$W_1 = \operatorname{span}(\{v_1, v_2, \ldots, v_k\}), \qquad W_2 = \operatorname{span}(\{v_1, v_2, \ldots, v_k, v\}).$$

(a) Find necessary and sufficient conditions on v such that $\dim(W_1) = \dim(W_2)$.

Claim. Dimensions $\dim(W_1) = \dim(W_2)$ if and only if $v \in W_1$.

Proof. In the first direction, assume $\dim(W_1) = \dim(W_2)$. Since $W_1 \subseteq W_2$, if $v \notin W_1$ then $W_1 \subset W_2$. But proper containment of subspaces forces a strict inequality of dimensions, i.e. $\dim(W_1) < \dim(W_2)$, contradicting the assumption. Hence $v \in W_1$.

In the other direction, assume $v \in W_1$. Then there exist scalars $a_1, \ldots, a_k \in F$ such that

$$v = a_1 v_1 + \cdots + a_k v_k.$$

Let $x \in W_2$ be arbitrary. Then for some scalars $b_1, \ldots, b_k, b \in F$,

$$x = b_1 v_1 + \cdots + b_k v_k + bv.$$

Substituting the expression for v gives

$$x = b_1 v_1 + \cdots + b_k v_k + b(a_1 v_1 + \cdots + a_k v_k) = \sum_{i=1}^{k} (b_i + b a_i)\, v_i \in W_1.$$

Thus $W_2 \subseteq W_1$. Since $W_1 \subseteq W_2$, we have $W_1 = W_2$, and therefore $\dim(W_1) = \dim(W_2)$. Hence, both directions are proven.

□

(b) State and prove a relationship involving $\dim(W_1)$ and $\dim(W_2)$ in the case that $\dim(W_1) \neq \dim(W_2)$.

Claim. If $\dim(W_1) \neq \dim(W_2)$, then $\dim(W_2) = \dim(W_1) + 1$.

Proof. Assume $\dim(W_1) \neq \dim(W_2)$. Since $W_1 \subseteq W_2$, we must have

$$\dim(W_1) < \dim(W_2).$$

By part (a), $\dim(W_1) = \dim(W_2)$ holds exactly when $v \in W_1$. Hence $\dim(W_1) \neq \dim(W_2)$ implies $v \notin W_1$.

Let $B = \{w_1, \ldots, w_m\}$ be a basis of W_1, so $m = \dim(W_1)$. We show that $B \cup \{v\}$ is a basis for W_2.

First, $B \cup \{v\}$ spans W_2 because any $x \in W_2$ can be written as

$$x = y + cv$$

for some $y \in W_1$ and $c \in F$ (since $W_2 = \text{span}(W_1 \cup \{v\})$). Because B spans W_1, we can write $y = \sum_{i=1}^{m} d_i w_i$, so

$$x = \sum_{i=1}^{m} d_i w_i + cv \in \text{span}(B \cup \{v\}).$$

Thus $W_2 \subseteq \text{span}(B \cup \{v\})$, and the reverse inclusion is immediate from $B \subseteq W_1 \subseteq W_2$ and $v \in W_2$. Hence

$$W_2 = \text{span}(B \cup \{v\}).$$

Next, $B \cup \{v\}$ is linearly independent. Suppose

$$cv + \sum_{i=1}^{m} c_i w_i = 0.$$

If $c \neq 0$, then

$$v = -\frac{1}{c} \sum_{i=1}^{m} c_i w_i \in \text{span}(B) = W_1,$$

contradicting $v \notin W_1$. Therefore $c = 0$, and then $\sum_{i=1}^{m} c_i w_i = 0$ implies $c_1 = \cdots = c_m = 0$ since B is linearly independent. Hence $B \cup \{v\}$ is linearly independent.

Basis set B contains $\dim(W_1)$ vectors. Adding the vector v produces a basis for W_2 with one more vector. Hence,

$$\dim(W_2) = \dim(W_1) + 1.$$

$\square$

Question 33a. Let W_1 and W_2 be subspaces of a vector space V such that $V = W_1 \oplus W_2$. If β_1 and β_2 are bases for W_1 and W_2, respectively, show that $\beta_1 \cap \beta_2 = \emptyset$ and $\beta_1 \cup \beta_2$ is a basis for V.

Proof. First, we will prove $\beta_1 \cap \beta_2 = \emptyset$. Assume $\beta_1 \cap \beta_2 \neq \emptyset$. This means that there is at least one vector u such that $u \in \beta_1$ and $u \in \beta_2$. Since β_1 is a basis for W_1 and β_2 is a basis for W_2, we have $u \in W_1$ and $u \in W_2$, hence $u \in W_1 \cap W_2$. Because $V = W_1 \oplus W_2$, we know $W_1 \cap W_2 = \{0\}$, so $u = 0$. This is a contradiction since the zero vector cannot be a basis vector. Therefore $\beta_1 \cap \beta_2 = \emptyset$.

Next, we will prove $\beta_1 \cup \beta_2$ is a basis for V. To prove $\beta_1 \cup \beta_2$ is a generating set, let $v \in V$ be arbitrary. Since $V = W_1 \oplus W_2$, we can write

$$v = v_1 + v_2$$

where $v_1 \in W_1$ and $v_2 \in W_2$. Let $\beta_1 = \{x_1, \dots, x_n\}$ and $\beta_2 = \{y_1, \dots, y_m\}$. Since β_1 and β_2 are bases, there exist scalars $a_1, \dots, a_n$ and $b_1, \dots, b_m$ such that

$$v_1 = \sum_{i=1}^{n} a_i x_i, \qquad v_2 = \sum_{j=1}^{m} b_j y_j.$$

Adding these gives

$$v = v_1 + v_2 = \sum_{i=1}^{n} a_i x_i + \sum_{j=1}^{m} b_j y_j,$$

which is a linear combination of vectors from $\beta_1 \cup \beta_2$. Hence $\beta_1 \cup \beta_2$ spans V.

To prove $\beta_1 \cup \beta_2$ is linearly independent, suppose

$$\sum_{i=1}^{n} a_i x_i + \sum_{j=1}^{m} b_j y_j = 0.$$

Rearranging,

$$\sum_{i=1}^{n} a_i x_i = -\sum_{j=1}^{m} b_j y_j.$$

The left-hand side is in W_1, and the right-hand side is in W_2. Therefore the common vector lies in $W_1 \cap W_2$. Since $W_1 \cap W_2 = \{0\}$, it follows that

$$\sum_{i=1}^{n} a_i x_i = 0 \quad \text{and} \quad \sum_{j=1}^{m} b_j y_j = 0.$$

Because β_1 is linearly independent, we must have $a_1 = \cdots = a_n = 0$. Because β_2 is linearly independent, we must have $b_1 = \cdots = b_m = 0$. Hence $\beta_1 \cup \beta_2$ is linearly independent. Since $\beta_1 \cup \beta_2$ spans V and is linearly independent, it is a basis for V. $\square$

2 2.1

Question 9. In this exercise, $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a function. For each of the following parts, state why T is not linear.

(a) $T(a_1 a_2) = (1, a_2)$

Solution. Transformation T does not satisfy the condition $T(c(a_1, a_2)) = cT(a_1, a_2)$. Expanding the two sides gives us the following:

$$T(c(a_1, a_2)) = T(ca_1, ca_2) = (1, ca_2),$$

$$cT(a_1, a_2) = c(1, a_2) = (c, ca_2)$$

Since $(1, ca_2) \neq (c, ca_2)$, T is not linear. $\square$

(b) $T(a_1 a_2) = (a_1, a_1^2)$

Solution. Similarly to part (a), T does not satisfy the condition $T(c(a_1, a_2)) = cT(a_1, a_2)$. Expanding the two sides gives us the following:

$$T(c(a_1, a_2)) = T(ca_1, ca_2) = (ca_1, (ca_1)^2) = (ca_1, c^2 a_1^2),$$

$$cT(a_1, a_2) = c(a_1, a_1^2) = (ca_1, ca_1^2)$$

Since $(ca_1, c^2 a_1^2) \neq (ca_1, ca_1^2)$, T is not linear. $\square$

(c) $T(a_1 a_2) = (\sin a_1, 0)$

Solution. Transformation T does not satisfy the condition $T(a_1 + a_2) = T(a_1) + T(a_2)$, where $a_1 = (x_1, y_1)$ and $a_2 = (x_2, y_2) \in \mathbb{R}^2$. Expanding both sides gives

$$T(a_1 + a_2) = T(x_1 + x_2, y_1 + y_2) = (\sin(x_1 + x_2), 0)$$

and

$$T(a_1) + T(a_2) = (\sin x_1, 0) + (\sin x_2, 0) = (\sin x_1 + \sin x_2, 0).$$

In general, $\sin(x_1 + x_2) \neq \sin x_1 + \sin x_2$ (for example, take $x_1 = x_2 = \frac{\pi}{2}$). Hence

$$T(a_1 + a_2) \neq T(a_1) + T(a_2),$$

so T is not linear. $\square$

Question 11. Prove that there exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$. What is $T(8,11)$?

Proof. A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ is uniquely determined by its values on a basis of $\mathbb{R}^2$. Thus it suffices to show that $\{(1,1),(2,3)\}$ is a basis. The set has two vectors. A linearly independent set of two vectors in $\mathbb{R}^2$ is automatically a basis, so let us prove the set is linearly independent. Given two scalars $a, b \in \mathbb{R}$, evaluate the equation

$$a(1,1) + b(2,3) = 0.$$

Expanding the equation as a system of equations gives us

$$a + 2b = 0 \tag{1}$$
$$a + 3b = 0 \tag{2}$$

Subtracting (1) from (2) gives $b = 0$, which implies $a = 0$. Since the only solution is the trivial one, set $\{(1,1),(2,3)\}$ is linearly independent and hence, a basis set. Therefore, there exists a unique linear transformation T with such values.

To find $T(8,11)$, note that $(8,11)$ can be decomposed into $2(1,1) + 3(2,3)$. We can use the properties of linear transformations where

$$T(8,11) = 2T(1,1) + 3T(2,3) = 2(1,0,2) + 3(1,-1,4) = (5,-3,16).$$

$\square$

Question 17. Let V and W be finite-dimensional vector spaces and $T : V \to W$ be linear.

(a) Prove that if $\dim(V) < \dim(W)$, then T cannot be onto.

(b) Prove that if $\dim(V) > \dim(W)$, then T cannot be one-to-one.

Proof. **(a)** Assume $\dim(V) < \dim(W)$. Suppose, for contradiction, that $T : V \to W$ is onto. If T is onto, then $R(T) = W$, so

$$\operatorname{rank}(T) = \dim(W).$$

By the Dimension Theorem,

$$\dim(V) = \operatorname{nullity}(T) + \operatorname{rank}(T) = \operatorname{nullity}(T) + \dim(W)$$

However, since $\operatorname{nullity}(T)$ cannot be negative, this contradicts the assumption $\dim(V) < \dim(W)$. Therefore, T cannot be onto.

(b) Assume $\dim(V) > \dim(W)$. Suppose, for contradiction, that $T : V \to W$ is one-to-one. If T is one-to-one, then $\ker(T) = \{0\}$, so

$$\operatorname{nullity}(T) = 0.$$

Again by the Dimension Theorem,

$$\dim(V) = \operatorname{nullity}(T) + \operatorname{rank}(T) = \operatorname{rank}(T).$$

But $R(T) \subseteq W$, so

$$\operatorname{rank}(T) = \dim(R(T)) \leq \dim(W).$$

This contradicts assumption $\dim(V) > \dim(W)$. Therefore, T cannot be one-to-one. $\square$

Problem 27. Using the notation in the definition above, assume that $T : V \to V$ is the projection on W_1 to W_2.

(a) Prove that T is linear and $W_1 = \{x \in V : T(x) = x\}$

(b) Prove that $W_1 = R(T)$ and $W_2 = N(T)$.

Proof. Assume $V = W_1 \oplus W_2$. Then for each $x \in V$ there exist unique vectors $x_1 \in W_1$ and $x_2 \in W_2$ such that $x = x_1 + x_2$. Define $T : V \to V$ by

$$T(x) = x_1,$$

i.e., T is the projection onto W_1 along W_2.

(a) We first prove that T is linear. Let $x, y \in V$ and $c \in F$. Write $x = x_1 + x_2$ and $y = y_1 + y_2$ with $x_1, y_1 \in W_1$ and $x_2, y_2 \in W_2$. Then

$$cx + y = (cx_1 + y_1) + (cx_2 + y_2),$$

where $cx_1 + y_1 \in W_1$ and $cx_2 + y_2 \in W_2$ since W_1 and W_2 are subspaces. By the definition of T,

$$T(cx + y) = cx_1 + y_1.$$

On the other hand,

$$cT(x) + T(y) = cx_1 + y_1.$$

Thus $T(cx + y) = cT(x) + T(y)$, so T is linear.

Next we show $W_1 = \{x \in V : T(x) = x\}$. If $x \in W_1$, then $x = x + 0$ with $0 \in W_2$, so $T(x) = x$. Hence $W_1 \subseteq \{x \in V : T(x) = x\}$.

Conversely, if $T(x) = x$, write $x = x_1 + x_2$ with $x_1 \in W_1$, $x_2 \in W_2$. Then $T(x) = x_1$, so $x_1 = x$. Hence $x \in W_1$. Therefore,

$$W_1 = \{x \in V : T(x) = x\}.$$

(b) We prove $R(T) = W_1$ and $N(T) = W_2$.

First, $R(T) \subseteq W_1$ because for any $x \in V$, $T(x) = x_1 \in W_1$. For the reverse inclusion, let $w \in W_1$. Then $T(w) = w$ by part (a), so $w \in R(T)$. Hence $R(T) = W_1$.

For the null space, let $x \in N(T)$. Write $x = x_1 + x_2$ with $x_1 \in W_1$, $x_2 \in W_2$. Since $0 = T(x) = x_1$, we get $x = x_2 \in W_2$, so $N(T) \subseteq W_2$. Conversely, if $x_2 \in W_2$, then $x_2 = 0 + x_2$ so $T(x_2) = 0$, meaning $x_2 \in N(T)$. Thus $W_2 \subseteq N(T)$, and therefore $N(T) = W_2$. □

3 2.2

Question 4. Define

$$T : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R}) \quad \text{by} \quad T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)x + bx^2.$$

Let

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \right\} \quad \text{and} \quad \gamma = \{1, x, x^2\}.$$

Compute $[T]_\beta^\gamma$.

Solution. The matrix

$$[T]_\beta^\gamma = \left[\left[T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_\gamma, \left[T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]_\gamma, \left[T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right]_\gamma, \left[T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]_\gamma \right],$$

where β is a basis of $M_{2\times 2}(\mathbb{R})$ and γ is a basis of $P_2(\mathbb{R})$. Since

$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (1+0) + (2(0))x + (0)x^2 = 1,$$

$$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = (0+1) + (2(0))x + (1)x^2 = 1 + x^2,$$

$$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = (0+0) + (2(0))x + (0)x^2 = 0,$$

and

$$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = (0+0) + (2(1))x + (0)x^2 = 2,$$

we evaluate the coordinate vectors

$$\left[T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_\gamma = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\left[T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]_\gamma = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

$$\left[T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right]_\gamma = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and

$$\left[T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]_\gamma = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}.$$

Therefore the matrix

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

$\square$

Question 8. Let V be an n-dimensional vector space with an ordered basis β. Define $T : V \to F^n$ by $T(x) = [x]_\beta$. Prove that T is linear.

Proof. Transformation T is linear if

$$T(cx + y) = cT(x) + T(y),$$

where $x, y \in V$. Define an ordered basis $\beta = \{v_1, v_2, \ldots, v_n\}$. Vectors x and y can be expressed as a linear combination of the vectors in the β and scalar $a_1, a_2, \ldots, a_n$ and $b_1, b_2, b \ldots, b_n$, respectively, as

$$x = a_1v_1 + a_2v_2 + \ldots + a_nv_n, \quad y = b_1v_1 + b_2v_2 + \ldots + b_nv_n.$$

Hence, the transformation is equal to

$$T(x) = [x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad T(y) = [y]_\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

On the left side, $cx + y$ is equal to

$$cx + y = c(a_1v_1 + a_2v_2 + \ldots + a_nv_n) + b_1v_1 + b_2v_2 + \ldots + b_nv_n.$$

Factoring results in

$$= (ca_1 + b_1)v_1 + (ca_2 + b_2) + \ldots + (ca_n + b_n)v_n.$$

Hence,

$$T(cx + y) = [cx + y]_\beta = \begin{pmatrix} ca_1 + b_1 \\ ca_2 + b_2 \\ \vdots \\ ca_n + b_n \end{pmatrix}.$$

On the right side, $cT(x) + T(y)$ expands to

$$cT(x) + T(y) = c[x]_\beta + [y]_\beta = c\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} ca_1 + b_1 \\ ca_2 + b_2 \\ \vdots \\ ca_n + b_n \end{pmatrix}.$$

Since both sides are equal, the transformation T is linear. $\square$

Question 10. Let V be a vector space with the ordered basis $\beta = \{v_1, v_2, \ldots, v_n\}$. Define $v_0 = 0$. By Theorem 2.6, there exist a linear transformation $T : V \to V$ such that $T(v_j) = v_j + v_{j-1}$ for $j = 1, 2, \ldots, n$. Compute $[T]_\beta$.

Solution. Let us use the ordered basis is the standard ordered basis $\beta = \{e_1, e_2, \ldots, e_n\}$ to calculate the matrix. Recall that

$$[T]_\beta = \left[[T(v_1)]_\beta, [T(v_2)]_\beta, \ldots, [T(v_n)]_\beta\right].$$

Next, let us find $T(v_n)$ for each index $j \in [1, n]$:

$$\begin{aligned}
T(v_1) &= v_1 + v_0 = e_1, \\
T(v_2) &= v_2 + v_1 = e_2 + e_1, \\
T(v_3) &= v_3 + v_2 = e_3 + e_2, \\
&\vdots \\
T(v_j) &= v_j + v_{j-1} = e_j + e_{j-1}.
\end{aligned}$$

Then, the respective column vector $[T(v_n)]_\beta \in F^n$ is

$$\begin{aligned}
[T(v_1)]_\beta &= (1, 0, 0, \ldots, 0), \\
[T(v_2)]_\beta &= (1, 1, 0, \ldots, 0), \\
[T(v_3)]_\beta &= (0, 1, 1, 0, 0, 0 \ldots, 0), \\
&\vdots \\
[T(v_j)]_\beta &= (0, 0, 0, \ldots, 1, 1).
\end{aligned}$$

Hence, the matrix $[T]_\beta$ is

$$[T]_\beta = (a_{ij}), \quad a_{ij} = \delta_{ij} + \delta_{i,j-1}.$$

for $1 \le i, j \le n$. $\square$