Homework 3

Andrew Hwang
Physics 15
Professor Rimberg

February 3, 2026

Question 1. Similar to K&K 12.10. Two spaceships approach each other. They are each viewed from Earth as having a speed half that of light. What is their speed relative to each other?

Solution. Given the equation for relativistic velocity addition being:

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}},$$

Since both spaceships are moving in opposite directions, set $u = 0.5c$ and $v = -0.5c$. Then,

$$\begin{aligned} u' &= \frac{0.5c - (-0.5c)}{1 - \frac{(-0.5c)(0.5c)}{c^2}} \\ &= \frac{c}{1 + 0.25} \\ &= \boxed{\frac{4}{5}c}. \end{aligned}$$

□

Question 2. K&K 12.15. One of the most prominent spectral lines of hydrogen is the H_α line, a bright red line with a wavelength of 656.1×10^{-9} m.

(a) What is the expected wavelength of the H_α line from a star receding with a speed of 3000 km/s?

(b) The H_α line measured on Earth from opposite ends of the Sun's equator differ in wavelength by 9×10^{-12} m. Assuming that the effect is caused by rotation of the Sun, find the period of rotation. The diameter of the Sun is 1.4×10^6 km.

Solution. For part (a), since the star is *receding*, the expected frequency should be lower than expected, hence making expected wavelength *greater than expected.* Recall that in relativistic Doppler shifts,

$$f_{\text{observer}} = f_{\text{source}} \sqrt{\frac{1-\beta}{1+\beta}}.$$

However, by subtituting $f = c/\lambda$ and rearranging the equation, we can express it as such:

$$\lambda_{\text{observer}} = \lambda_{\text{source}} \sqrt{\frac{1+\beta}{1-\beta}}.$$

We know that $\beta = \frac{3{,}000 \text{ km/s}}{300{,}000\text{km/s}} = 0.01$. Therefore,

$$\lambda_{\text{observer}} = 656.1 \times 10^{-9}\text{m} \sqrt{\frac{1+0.01}{1-0.01}} \approx \boxed{662.7 \times 10^{-9}\text{m or } 662.7 \text{ nm}}.$$

For part (b), let the Sun rotate tangentially with velocity v. On one end of the equator, the source is moving toward Earth with velocity v, and on the opposite end, it is moving away with velocity v. The observed wavelength from the side moving away is

$$\lambda_1 = \lambda_{\text{source}} \sqrt{\frac{1+\beta}{1-\beta}} \approx \lambda_{\text{source}}(1+\beta). \tag{1}$$

The observed wavelength from the side moving toward us is

$$\lambda_2 = \lambda_{\text{source}} \sqrt{\frac{1-\beta}{1+\beta}} \approx \lambda_{\text{source}}(1-\beta). \tag{2}$$

We can obtain the difference in wavelength by subtracting (2) from (1):

$$\lambda_1 - \lambda_2 \approx 2\lambda_{\text{source}}\beta = 2\lambda_{\text{source}}\frac{v}{c}.$$

Then,

$$v = \frac{(\Delta\lambda)c}{2\lambda_{\text{source}}} = \frac{(9 \times 10^{-12} \text{ m})(3 \times 10^8 \text{ m/s})}{2(656.1 \times 10^{-9} \text{ m})} \approx 2057.6 \text{ m/s}.$$

Substituting this into the period of rotation $T = \frac{\pi D}{v}$ gives us

$$T = \frac{\pi(1.4 \times 10^9 \text{ m})}{2057.6 \text{ m/s}} \approx 2.137 \times 10^6 \text{ seconds} \times \frac{1 \text{ day}}{86400 \text{ seconds}} \approx \boxed{24.7 \text{ days}}$$

$\square$

Question 3. Variation on K&K 13.4

(a) Two particles of rest mass m approach each other with equal and opposite velocity, u, as measured in the lab frame. What is the total energy of one particle as measured in the rest frame of the other? Express your answer in terms of u and m.

(b) Express your answer from part (a) in terms of the initial total energy of each particle.

(c) Suppose the two particles are protons ($mc^2 \approx 1$ GeV), each with an initial total energy of 30 GeV. What is the energy of one proton as measured in the rest frame of the other?

Solution. For part (a), the adding relativistic velocities gives us

$$v' = \frac{u-(-u)}{1-\frac{u(-u)}{c^2}} = \frac{2u}{1+\frac{u^2}{c^2}} = \frac{2uc^2}{c^2+u^2}$$

Substituting this into the Lorentz factor equation gives us

$$\gamma' = \frac{1}{\sqrt{1-\left(\frac{2uc}{c^2+u^2}\right)^2}} = \frac{c^2+u^2}{c^2-u^2} = \frac{1+\frac{u^2}{c^2}}{1-\frac{u^2}{c^2}}.$$

Therefore, the total energy $E' = \gamma' mc^2$ is

$$E' = \boxed{\left(\frac{1+\frac{u^2}{c^2}}{1-\frac{u^2}{c^2}}\right) mc^2}.$$

For part (b), the initial total energy is equal to $E_0 = \gamma_0 mc^2$. The Lorentz factor γ_0 is equal to

$$\gamma_0 = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}.$$

To do a trick, we realize that $\gamma' = \gamma_0^2\left(1+\frac{u^2}{c^2}\right)$. We can simply this further since we can set $\frac{u^2}{c^2} = 1 + \frac{1}{\gamma_0^2}$ from the Lorentz factor equation. As a result,

$$\gamma' = \gamma_0^2\left(1+1+\frac{1}{\gamma_0^2}\right) = 2\gamma_0^2 + 1.$$

Substituting this back into the original equation, as well as setting $\gamma_0 = \frac{E_0}{mc^2}$ gives us

$$E' = \gamma' mc^2 = (2\gamma_0^2+1)mc^2 = \left[2\left(\frac{E_0^2}{(mc^2)^2}\right)+1\right]mc^2 = \boxed{2\frac{E_0^2}{mc^2} - mc^2.}$$

For part (c), let us directly use this equation to calculate the energy of the p^+ in the rest frame:

$$E_{p^+} = 2\frac{E_0^2}{mc^2} - mc^2 \approx 2\frac{(30\text{ GeV})^2}{(1\text{ GeV})} - (1\text{ GeV}) = \boxed{1799\text{ GeV, or }1.799\text{ TeV.}}$$

□

Question 4. (K&K 13.1) A cosmic ray proton can have energy up to 10^{13} MeV (almost 10^8 greater than the highest energy achieved with a particle accelerator). Our galaxy has a diameter of about 10^5 light-years. How long does it take the proton to traverse the Galaxy, in its own rest frame?

Solution. To find the time it takes for the proton to traverse the Galaxy in its own rest frame, we must first determine the Lorentz factor γ associated with its extreme energy. We are given the total energy $E = 10^{13}$ MeV and the diameter of the galaxy $D = 10^5$ light-years. The total energy is related to the rest mass energy by:

$$E = \gamma mc^2$$

For a proton, the rest energy $mc^2 \approx 938$ MeV, which we can approximate as 10^3 MeV. Thus,

$$\gamma = \frac{10^{13}\text{ MeV}}{10^3\text{ MeV}} = 10^{10}.$$

In the frame of the Galaxy, the time t required to traverse the distance D at a velocity $v \approx c$ is:

$$t = \frac{D}{c} = \frac{10^5\text{ light-years}}{c} = 10^5\text{ years.}$$

The time in the proton's rest frame (the proper time t') is shorter due to time dilation, expressed as

$$t' = \frac{t}{\gamma} = \frac{10^5\text{ years}}{10^{10}} = 10^{-5}\text{ years.}$$

By converting units, we get

$$10^{-5}\text{ years} \times 365.25\frac{\text{days}}{\text{year}} \times 24\frac{\text{hours}}{\text{day}} \times 60\frac{\text{minutes}}{\text{hour}} \approx \boxed{5.26\text{ minutes.}}$$

□

Question 5.

(a) Find the kinetic energy of a particle with a mass of one gram moving with half the speed of light. Compare your answer with the one you would find using the non-relativistic formula.

(b) What is the total energy of a particle with a rest mass of one gram moving with half the speed of light?

Solution. To compare the kinetic energy using relativistic and non-relativistic mechanics, we consider a particle with mass $m = 1$ g moving at $v = 0.5c$. We first calculate the Lorentz factor γ:

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{1-0.5^2}} = \frac{1}{\sqrt{0.75}} \approx 1.2$$

For part (a), we find the kinetic energy (K):

$$K = (\gamma - 1)mc^2 \approx (1.2 - 1)mc^2 = (0.2)mc^2$$

Substituting $m = 10^{-3}$ kg and $c = 3 \times 10^8$ m/s:

$$K \approx 0.2 \times 10^{-3} \text{ kg} \times (9 \times 10^{16} \text{ m}^2/\text{s}^2) = \boxed{1.8 \times 10^{13} \text{ J.}}$$

In the non-relativistic case,

$$\begin{aligned} K' &= \frac{1}{2}mv^2 = \frac{1}{2}m(0.5c)^2 = 0.125mc^2 \\ &= 0.125 \times 10^{-3} \text{ kg} \times (9 \times 10^{16} \text{ m}^2/\text{s}^2) \\ &= \boxed{1.125 \times 10^{13} \text{ J.}} \end{aligned}$$

The relativistic kinetic energy is greater than the classical prediction.

For part (b), the total energy E is

$$E = \gamma mc^2 \approx 1.2 \times 10^{-3} \text{ kg} \times (9 \times 10^{16} \text{ m}^2/\text{s}^2) = \boxed{1.08 \times 10^{14} \text{ J.}}$$