Homework 7

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1 5.1

Question 4b. For the following matrix $A \in \mathrm{M}_{n\times n}(F)$,

(i) Determine all the eigenvalues of A.

(ii) For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ.

(iii) If possible, find a basis for F^n consisting of eigenvectors of A.

(iv) If successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

$$A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix} \qquad \text{for } F = R.$$

Solution. Expanding along the first row for cofactor expansion gives us

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} -\lambda & -2 & -3 \\ -1 & 1-\lambda & -1 \\ 2 & 2 & 5-\lambda \end{pmatrix} \\ &= -\lambda[(\lambda-1)(\lambda-5)+2] + 2(\lambda-5+2) - 3[-2+2(\lambda-1)] \\ &= -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0. \end{aligned}$$

Negating and then factoring the polynomial gives us

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda-1)(\lambda-2)(\lambda-3) = 0.$$

Therefore, our eigenvalues of A is $\boxed{\lambda = 1, 2, 3.}$ For the eigenvectors, we will consider the three different scenarios. First, when $\lambda = 1$:

$$A - I = \begin{pmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{pmatrix}$$

$$A - I = \begin{pmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \implies \boxed{v_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.}$$

For $\lambda = 2$:

$$A - 2I = \begin{pmatrix} -2 & -2 & -3 \\ -1 & -1 & -1 \\ 2 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \implies \boxed{v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.}$$

For $\lambda = 3$:

$$A - 3I = \begin{pmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \implies \boxed{v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.}$$

Therefore, the basis for β for $A \in \mathbb{R}^3$ is

$$\boxed{\beta = \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.}$$

Now, let us find Q^{-1} using an augmented matrix $[Q \mid I] \sim [I \mid Q^{-1}]$:

$$[Q \mid I] = \left(\begin{array}{ccc|ccc} -1 & -1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \sim \cdots \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{array} \right)$$

$$Q^{-1} = \begin{pmatrix} -1 & -1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 2 \end{pmatrix}$$

Therefore,

$$\begin{aligned} D = Q^{-1}AQ &= \begin{pmatrix} -1 & -1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix} \begin{pmatrix} -1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -1 & -1 \\ -2 & 0 & -2 \\ 3 & 3 & 6 \end{pmatrix} \begin{pmatrix} -1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\ &= \boxed{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.} \end{aligned}$$

$\square$

Question 4d. For the following matrix $A \in \mathrm{M}_{n\times n}(F)$,

(i) Determine all the eigenvalues of A.

(ii) For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ.

(iii) If possible, find a basis for F^n consisting of eigenvectors of A.

(iv) If successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix} \quad \text{for } F = R.$$

Solution. Expanding along the second column for the characteristic polynomial:

$$\begin{aligned} \det(A - \lambda I) &= \begin{pmatrix} 2-\lambda & 0 & -1 \\ 4 & 1-\lambda & -4 \\ 2 & 0 & -1-\lambda \end{pmatrix} \\ &= (1-\lambda)[(2-\lambda)(-1-\lambda) - (-2)] = (1-\lambda)(\lambda^2 - \lambda) \\ &= -\lambda(\lambda-1)^2 \end{aligned}$$

The eigenvalues are $\boxed{\lambda = 0, 1}$ (with $\lambda = 1$ having multiplicity 2).

For $\lambda = 0$:

$$A - 0I = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \implies \boxed{v_1 = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}.}$$

For $\lambda = 1$:

$$A - I = \begin{pmatrix} 1 & 0 & -1 \\ 4 & 0 & -4 \\ 2 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \implies \boxed{v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.}$$

Therefore, the basis for β for $A \in \mathbb{R}^3$ is

$$\boxed{\beta = \left\{ \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}.}$$

Now, let us find Q^{-1} using augmented matrix $[Q \mid I] \sim [I \mid Q^{-1}]$:

$$[Q \mid I] = \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \cdots \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 4 & 1 & -4 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right)$$

$$Q^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}$$

Therefore, we get that

$$\begin{aligned} D = Q^{-1}AQ &= \begin{pmatrix} -1 & 0 & 1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.} \end{aligned}$$

□

Question 9ab. Prove the following:

(a) Prove that a linear operator T on a finite-dimensional vector space is invertible if and only if zero is not an eigenvalue of T.

(b) Let T be an invertible linear operator. Prove that a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

Proof. For part(a), suppose T is invertible. If 0 were an eigenvalue of T, there would exist a non-zero vector $v \in V$ such that $T(v) = 0v = 0$. However, for an invertible operator, $T(v) = 0$ implies $v = T^{-1}(0) = 0$, which contradicts the requirement that an eigenvector must be non-zero. Thus, 0 is not an eigenvalue of T.

In the other direction, suppose 0 is not an eigenvalue of T. This implies that $T(v) = 0v = 0$ has only the trivial solution $v = 0$. Therefore, the null space $N(T) = \{0\}$, which means T is injective. Since V is finite-dimensional, an injective linear operator is also surjective and thus invertible.

For part (b), first assume that a scalar λ is an eigenvalue of T. Then this implies that there exists an eigenvector v such that $T(v) = \lambda v$. Since T is invertible, applying T^{-1} to both sides gives us $T^{-1}(T(v)) = v = T^{-1}(\lambda v) = \lambda T^{-1}(v)$. Rearranging this equation gives $T^{-1}(v) = \lambda^{-1}v$. Therefore, λ^{-1} is an eigenvalue of T^{-1}.

In the other direction, assume λ^{-1} is an eigenvalue of T^{-1}. Then, it implies that given eigenvector v, $T^{-1}(v) = \lambda^{-1}v$. Since, T is invertible, it is possible to apply T to both sides, implying $T(T^{-1}(v)) = v = T(\lambda^{-1}v) = \lambda^{-1}T(v)$. Rearramgomg the equation gives $T(v) = \lambda v$, and therefore λ is an eigenvalue of T. $\square$

Question 10. Prove that the eigenvalues of an upper triangular matrix M are the diagonal entries of M.

Proof. In a previous problem, we have proven that the determinant of an upper triangular matrix is A is the diagonal entries of A. It is obvious that $M - \lambda I$ is still an upper triangular matrix, therefore, assuming M is a $n \times n$ matrix,

$$\det(M - \lambda I) = \begin{pmatrix} m_{11} - \lambda & m_{12} & \cdots & m_{1n} \\ 0 & m_{22} - \lambda & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_{nn} - \lambda \end{pmatrix} = (m_{11} - \lambda)(m_{22} - \lambda) \ldots (m_{nn} - \lambda) = 0.$$

It is obvious that the solutions to this polynomial are the diagonal entries of M. Therefore,

$$\boxed{\lambda = m_{11}, m_{22}, \ldots, m_{nn}.}$$

□

Problem 12. A **scalar matrix** is a square matrix of the form λI for some scalar λ; that is, a scalar matrix is a diagonal matrix in which all the diagonal entries are equal.

(a) Prove that if a square matrix A is similar to a scalar matrix λI, then $A = \lambda I$.

(b) Show that a diagonalizable matrix having only one eigenvalue is a scalar matrix.

Solution. For part (a), if A is similar to λI, then it implies that there exists an invertible matrix Q such that $A = Q^{-1}(\lambda I)Q$. Since λ is a scalar, it can be commuted out of the equation such that $A = \lambda(PIP^{-1}) = \lambda(PP^{-1}) = \lambda I$. Thus, $A = \lambda I$.

For part (b), let A be a diagonalizable matrix with only one eigenvalue λ. Since A is diagonalizable, there exists an invertible matrix Q and a diagonal matrix D such that $A = QDQ^{-1}$. The diagonal entries of D are the eigenvalues of A. Since λ is the only eigenvalue, $D = \lambda I$. From part (a), we have $A = Q(\lambda I)Q^{-1} = \lambda I$. Thus, A is a scalar matrix. $\square$

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Question 8. Suppose that $A \in \mathrm{M}_{n\times n}(F)$ has two distinct eigenvalues, λ_1 and λ_2, and that $\dim(E_{\lambda_1}) = n-1$. Prove that A is diagonalizable.

Proof. We are given that λ_1 and λ_2 are distinct eigenvalues of A, and that $\dim(E_{\lambda_1}) = n-1$. It is true that $\dim(E_{\lambda_2}) \geq 1$. Additionally, the sum of the dimensions of eigenspaces corresponding to distinct eigenvalues must be less than or equal to n. Since distinct eigenvalues imply that E_{λ_1} and E_{λ_2} are linearly independent, we can then imply that $\dim(E_{\lambda_1})+\dim(E_{\lambda_2}) \leq n$. Substituting the given dimension for E_{λ_1} gives us

$$\begin{aligned}(n-1) + \dim(E_{\lambda_2}) &\leq n,\\ \dim(E_{\lambda_2}) &\leq 1.\end{aligned}$$

Since we already established $\dim(E_{\lambda_2}) \geq 1$, it must be that $\dim(E_{\lambda_2}) = 1$. Therefore, $\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) = (n-1)+1 = n$ and A is diagonalizable. $\square$

Question 10. Let T be a linear operator on a finite-dimensional vector space V with the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ and corresponding multiplicities $m_1, m_2, \ldots, m_k$. Suppose that β is a basis for V such that $[T]_\beta$ is an upper triangular matrix. Prove that the diagonal entries of $[T]_\beta$ are $\lambda_1, \lambda_2, \ldots, \lambda_k$ and that each λ_i occurs m_i times $(1 \leq i \leq k)$.

Proof. Let $A = [T]_\beta$ be the $n \times n$ upper triangular matrix that represents the linear operator T with respect to the basis β. In problem 10 of section 5.1, we already proved that the diagonal entries of an upper triangular matrix are the eigenvalues, so we will simply prove the latter regarding multiplicity.

The multiplicity m_i of an eigenvalue λ_i is defined as the number of times the factor $(\lambda_i - \lambda)$ appears in the factored characteristic polynomial for each $i \in \{1, 2, \ldots, k\}$, where k represents the total number of distinct eigenvalues. Because the factors of the characteristic polynomial $p(\lambda)$ are exactly $(a_{ii} - \lambda)$ for each diagonal entry a_{ii}, it follows that each eigenvalue λ_i must appear on the diagonal exactly m_i times. $\square$

Question 13. Let T be an invertible linear operator on a finite-dimensional vector space V.

(a) Recall that for any eigenvalue λ of T, λ^{-1} is an eigenvalue of T^{-1} (Exercise 8 of Section 5.1). Prove that the eigenspace of T corresponding to λ is the same as the eigenspace of T^{-1} corresponding to λ^{-1}.

(b) Prove that if T is diagonalizable, then T^{-1} is diagonalizable.

Proof. For part (a), let E_λ be the eigenspace of T for λ, and $E'_{\lambda^{-1}}$ be the eigenspace of T^{-1} for λ^{-1}. If $v \in E_\lambda$, then $T(v) = \lambda v$. Since T is invertible, $\lambda \neq 0$. Applying T^{-1} to both sides gives $v = T^{-1}(\lambda v) = \lambda T^{-1}(v)$, which implies $T^{-1}(v) = \lambda^{-1} v$. Thus $v \in E'_{\lambda^{-1}}$. Conversely, if $v \in E'_{\lambda^{-1}}$, then $T^{-1}(v) = \lambda^{-1} v$. Applying T to both sides gives $v = T(\lambda^{-1} v) = \lambda^{-1} T(v)$, which implies $T(v) = \lambda v$. Thus $v \in E_\lambda$. Therefore, $E_\lambda = E'_{\lambda^{-1}}$.

For part (b), if T is diagonalizable, there exists a basis β for V consisting of eigenvectors of T. By part (a), every eigenvector of T is also an eigenvector of T^{-1}. Thus, β is a basis for V consisting of eigenvectors of T^{-1}. By definition, T^{-1} is diagonalizable. $\square$

Question 14. Let $A \in \mathrm{M}_{n\times n}(F)$. Recall from Exercise 14 of Section 5.1 that A and A^t have the same characteristic polynomial and hence share the same eigenvalues with the same multiplicities. For any eigenvalue λ of A and A^t, let E_λ and E'_λ denote the corresponding eigenspaces for A and A^t, respectively.

(a) Show by way of example that for a given common eigenvalue, these two eigenspaces need not be the same.

(b) Prove that for any eigenvalue λ, $\dim(\mathrm{E}_\lambda) = \dim(\mathrm{E}'_\lambda)$.

(c) Prove that if A is diagonalizable, then A^t is also diagonalizable.

Proof. For part (a), let $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$. For $\lambda = 1$, the eigenspace of A is $E_1 = \mathrm{span}\{(1,0)^T\}$. For $A^t = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$, the eigenspace is $\mathrm{E}'_1 = \mathrm{span}\{(1,-1)^T\}$. Thus, $\mathrm{E}_1 \neq \mathrm{E}'_1$.

For part (b), note that $\dim(\mathrm{E}_\lambda) = n - \mathrm{rank}(A - \lambda I)$ and $\dim(\mathrm{E}'_\lambda) = n - \mathrm{rank}(A^t - \lambda I)$. Since $\mathrm{rank}(M) = \mathrm{rank}(M^t)$ for any matrix M, and $(A^t - \lambda I) = (A - \lambda I)^t$, it follows that $\mathrm{rank}(A - \lambda I) = \mathrm{rank}(A^t - \lambda I)$. Thus, $\dim(\mathrm{E}_\lambda) = \dim(\mathrm{E}'_\lambda)$.

For part (c), A is diagonalizable if the sum of the dimensions of its eigenspaces is n. Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues. Since A is diagonalizable, $\sum \dim(\mathrm{E}_{\lambda_i}) = n$. By part (b), $\dim(\mathrm{E}_{\lambda_i}) = \dim(\mathrm{E}'_{\lambda_i})$, so $\sum \dim(\mathrm{E}'_{\lambda_i}) = n$. Thus, A^t is diagonalizable. $\square$

3 5.4

Question 11. Let T be a linear operator on a vector space V, let v be a nonzero vector in V, and let W be the T-cyclic subspace of V generated by v. Prove that:

(a) W is T-invariant.

(b) Any T-invariant subspace of V containing v also contains W.

Proof. By definition, the T-cyclic subspace W generated by v is the span of the set $\{v, T(v), T^2(v), \dots\}$. Thus, any vector $w \in W$ can be expressed as a finite linear combination of the form:

$$w = \sum_{i=0}^{k} a_i T^i(v) = a_0 v + a_1 T(v) + \cdots + a_k T^k(v)$$

For part (a), to show W is T-invariant, we must show that for any $w \in W$, $T(w) \in W$. Applying T to the expression for w:

$$T(w) = T\left(\sum_{i=0}^{k} a_i T^i(v)\right) = \sum_{i=0}^{k} a_i T^{i+1}(v) = a_0 T(v) + a_1 T^2(v) + \cdots + a_k T^{k+1}(v)$$

Since each term $T^{i+1}(v)$ is an element of the set $\{v, T(v), T^2(v), \dots\}$, their linear combination $T(w)$ is in the span of that set. Therefore, $T(w) \in W$, confirming that W is T-invariant.

For part (b), let U be any T-invariant subspace of V that contains v. Since $v \in U$ and U is T-invariant, $T(v)$ must also be in U. By induction, if $T^k(v) \in U$, then $T(T^k(v)) = T^{k+1}(v) \in U$ because U is T-invariant. Thus, the entire set $\{v, T(v), T^2(v), \dots\}$ is contained within U. Since U is a subspace, it must be closed under linear combinations. Therefore, it contains the span of this set, which is W. Hence, $W \subseteq U$. $\square$

Question 13. Let T be a linear operator on a vector space V, let v be a nonzero vector in V, and let W be the T-cyclic subspace of V generated by v. For any $w \in V$, prove that $w \in W$ if and only if there exists a polynomial $g(t)$ such that $w = g(T)(v)$.

Proof. In the first direction, suppose $w \in W$. Since W is the span of the set $\{T^k(v) : k \geq 0\}$, w must be a finite linear combination of these vectors. Thus, there exist scalars $a_0, a_1, \ldots, a_n$ such that

$$w = a_0 v + a_1 T(v) + a_2 T^2(v) + \cdots + a_n T^n(v).$$

By the properties of linear operators and polynomial notation, we can factor out the vector v:

$$w = (a_0 I + a_1 T + a_2 T^2 + \cdots + a_n T^n)(v).$$

Let $g(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$. Then $w = g(T)(v)$, where $g(t)$ is a polynomial.

In the other direction, suppose there exists a polynomial $g(t) = b_0 + b_1 t + \cdots + b_m t^m$ such that $w = g(T)(v)$. Then:

$$w = (b_0 I + b_1 T + \cdots + b_m T^m)(v) = b_0 v + b_1 T(v) + \cdots + b_m T^m(v)$$

Since w is a finite linear combination of the vectors $\{v, T(v), T^2(v), \ldots\}$, it follows by the definition of a span that $w \in \text{span}(\{T^k(v) : k \geq 0\})$. Therefore, $w \in W$. $\square$