Homework 1

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PHYS 15
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Question K&K 3.15. The gravitational force on a body located at distance R from the center of a uniform spherical mass is due solely to the mass lying at distance $r \leq R$, measured from the center of the sphere. This mass exerts a force as if it were a point mass at the origin.

Use the above result to show that if you drill a hole through the Earth and then fall in, you will execute simple harmonic motion about the Earth's center. Find the time it takes you to return to your point of departure and show that this is the time needed for a satellite to circle the Earth in a low orbit with $r \approx R_e$. In deriving this result, treat the Earth as a uniformly dense sphere, neglect friction, and neglect any effects due to the Earth's rotation.

Solution. Recall that the scalar form of Newton's Law of Universal Gravitation is

$$F(r) = m\ddot{r} = -\frac{GMm}{r^2} \tag{1}$$

As you get closer to Earth's core, the force exerted decreases because there is less mass and density is uniform. In fact, this relation is shown by

$$\frac{M(r)}{M_e} = \frac{r^3}{R_e^3}.$$

Rearranging this equation yields

$$M(r) = M_e \left(\frac{r}{R_e}\right)^3. \tag{2}$$

By substituting equation (2) into (1), and canceling out m, we get

$$\ddot{r} = -\frac{GM_e r}{R_e^3}\hat{r}, \quad \ddot{r} + \frac{GM_e r}{R_e^3}\hat{r} = 0 \tag{3}$$

This is a second-order differential equation, and therefore we can identify

$$\omega^2 = \frac{GM_e}{R_e^3} \Rightarrow \omega = \sqrt{\frac{GM_e}{R_e^3}}.$$

The time it takes to return from the point of departure is simply the period T:

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R_e^3}{GM_e}}.$$

For a satellite to circle the Earth in a low orbit with $r \approx R_e$, the satellite must go under uniform circular motion with equation

$$F_{net} = m\frac{v^2}{r} = mr\omega^2 = \frac{GM_e m}{r^2} \Rightarrow \omega^2 = \frac{GM_e}{r^3}.$$

At low orbit $r \approx R_e$,

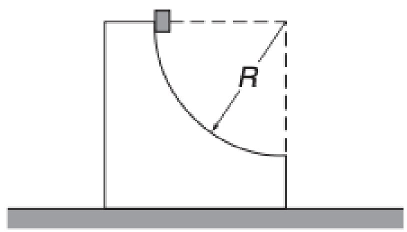
$$\omega^2 = \frac{GM_e}{R_e^3}.$$

The angular velocity ω is the same angular frequency from the simple harmonic motion equation mentioned earlier in the problem. Thus, the time needed for a satellite to complete 1 full orbit at $r \approx R_e$ is equal to the period T obtained earlier.

$\square$

Question K&K 5.4. A small cube of mass m slides down a circular path of radius R cut into a large block of mass M, as shown. M rests on a table, and both blocks move without friction. The blocks are initially at rest, and m starts from the top of the path.

Find the velocity v of the cube as it leaves the block.



Solution. It is important to note that both blocks (denoted as small cube m and large block M) move without friction. Thus, energy and momentum must be strictly conserved.

Using the reference of when both blocks are initially at rest and the cube leaving the block, we get an equation

$$U = K_M + K_m \Rightarrow mgR = \frac{1}{2}Mv_M^2 + \frac{1}{2}mv_m^2. \tag{4}$$

We also get an equation based on the conservation of momentum:

$$p_{\text{net},x} = 0 = Mv_M - mv_m \tag{5}$$

Rearranging equation (5) in terms of v_b gets us

$$v_M = \frac{m}{M}v_m,$$

and substituting this into equation 4 gives us

$$mgR = \frac{1}{2}M\left(\frac{m}{M}v_m\right)^2 + \frac{1}{2}mv_m^2.$$

Therefore, rearranging the equation, the velocity of the cube v_m is

$$v_m = \sqrt{\left(\frac{M}{m+M}\right)(2gR)}.$$

□

Question K&K 5.9. A block of mass M on a horizontal frictionless table is connected to a spring (spring constant k). The block is set in motion so that it oscillates about its equilibrium point with a certain amplitude A_0. The period of motion is $T_0 = 2\pi\sqrt{M/k}$.

(a) A lump of sticky putty of mass m is dropped onto the block. The putty sticks without bouncing. The putty hits M at the instant when the velocity of M is zero. Find

(1) The new period.
(2) The new amplitude.
(3) The change in the mechanical energy of the system.

(b) Repeat part (a), but this time assume that the sticky putty hits M at the instant when M has its maximum velocity.

Solution. (a)

(1) After the putty sticks, the oscillating mass is $M + m$, so

$$T_f = 2\pi\sqrt{\frac{M+m}{k}}.$$

(2) The putty lands when $v = 0$, i.e. at a turning point where $x = \pm A_0$. By conservation of momentum, immediately after the collision, the position is still $x = \pm A_0$ and the velocity is still zero, so the turning point remains the same:

$$A_f = A_0.$$

(3) Since the amplitude is the same and total mechanical energy is proportional to A^2,

$$\Delta E = 0.$$

(b)
(1) The period is still

$$T_f = 2\pi\sqrt{\frac{M+m}{k}}.$$

(2) However, the putty lands at equilibrium ($x = 0$) when the block has maximum speed $v_0 = \omega_0 A_0 = \sqrt{\frac{k}{M}}\,A_0$. Conserving momentum for the perfectly inelastic collision,

$$Mv_0 = (M+m)v_f \quad \Rightarrow \quad v_f = \frac{M}{M+m}v_0 = \frac{M}{M+m}\sqrt{\frac{k}{M}}A_0.$$

Immediately after impact the energy of the new oscillator is

$$\frac{1}{2}(M+m)v_f^2 = \frac{1}{2}kA_f^2,$$

so

$$A_f = \sqrt{\frac{M}{M+m}}\,A_0.$$

(3) The initial energy is $E_i = \frac{1}{2}kA_0^2$, and the final energy is

$$E_f = \frac{1}{2}kA_f^2 = \frac{1}{2}kA_0^2\frac{M}{M+m}.$$

Therefore

$$\Delta E = E_f - E_i = -\frac{1}{2}kA_0^2\frac{m}{M+m},$$

so the energy lost is $\frac{1}{2}kA_0^2\frac{m}{M+m}$. $\square$

Question 4. (Similar to K&K 5.12) The potential energy of two atoms in a diatomic molecule is given by

$$U(r) = \frac{-a}{r^6} + \frac{b}{r^{12}}$$

where r is the distance between the two atoms and a and b are positive constants. Assume the atoms are identical and have mass m.
(a) At what values of r is $U(r)$ a minimum?
(b) At what values of r is $U(r) = 0$?
(c) Plot the function $U(r)$
(d) Describe the motion of one atom with respect to the second atom in the two cases when $E < 0$ and when $E > 0$.
(e) Find the frequency of small oscillations about the minimum you found in part (a).

Solution. For all solutions, we must assume $r > 0$ because distance cannot be negative between two atoms.

(a) When $U(r)$ is at a minimum, $\frac{dU}{dr} = 0$ and $\frac{d^2U}{dr^2} > 0$. The derivative $\frac{dU}{dr}$ is as such:

$$\frac{dU}{dr} = \frac{6a}{r^7} - \frac{12b}{r^{13}} = 0$$

Rearranging this equation gives us

$$6ar^6 = 12b \Rightarrow r = \sqrt[6]{\frac{2b}{a}}.$$

At $r = \sqrt[6]{\frac{2b}{a}}$,

$$\frac{d^2U}{dr^2} = -\frac{42a}{r^8} + \frac{156b}{r^{14}}, \quad \frac{d^2U(\sqrt[6]{2b/a})}{dr^2} = -\frac{42a}{r^8} + \frac{156b}{r^8\left(\frac{2b}{a}\right)} = \frac{36a}{r^8} > 0.$$

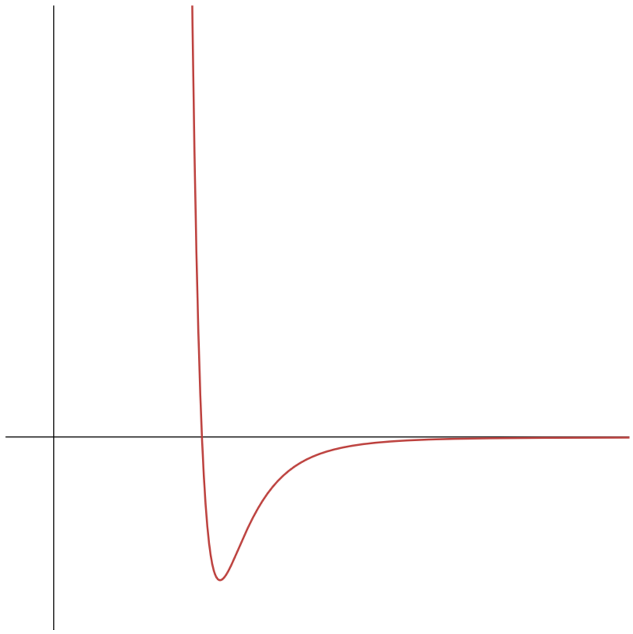
Hence, at $r = \sqrt[6]{\frac{2b}{a}}$, there is a local minimum.

(b) The equation is set up as

$$U(r) = -\frac{a}{r^6} + \frac{b}{r^{12}} = 0$$

$$b = ar^6, \quad r = \sqrt[6]{\frac{b}{a}}.$$

(c) We can plot $U(r)$ as shown below:



(d) Total mechanical energy E is defined as

$$E = K + U.$$

When $E < 0$, $K+U < 0$. As $r \to \infty$, U approaches zero as shown by the graph. However, this is not possible because K can never be negative. Thus, the two atoms oscillate while bounded, and can never escape the potential "well". However, when $E > 0$, the atoms are no longer bounded because $K+U > 0$. Thus, the atoms interact by first attracting together and then repelling to an infinite distance.

(e) The spring constant is

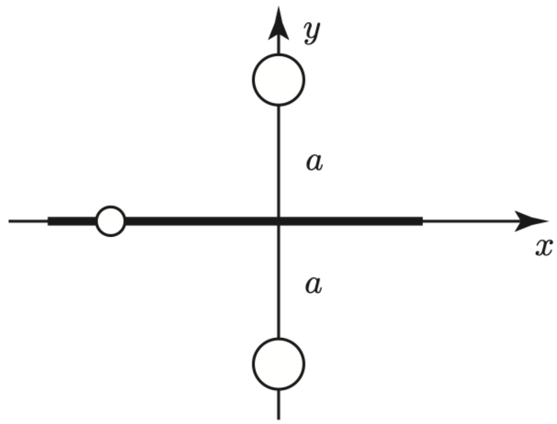
$$k = \frac{d^2U}{dr^2} = \frac{36a}{\left(\frac{2b}{a}\right)^{8/6}} = \frac{18a^{7/3}}{2^{1/3}b^{4/3}}.$$

The rest mass m_r is $m_r = \frac{m*m}{m+m} = \frac{m}{2}$. Therefore, the frequency is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{36a^{7/3}}{2^{1/3}b^{4/3}\,(m/2)}} = \frac{6a^{7/6}}{2^{1/6}(b^{2/3})(m)}.$$

$\square$

Question 5. (similar to K&K 6.1) A bead of mass m slides without friction on a smooth rod along the x-axis. The rod is equidistant between two spheres of mass M. The spheres are located at $x = 0, y = \pm a$ as shown, and attract the bead gravitationally.



(a) Find the potential energy of the bead as a function of its position, x.
(b) The bead is released at $x = 3a$ with velocity v_0 toward the origin. Find the speed as it passes through the origin and briefly describe qualitatively (in words) the motion of the bead.
(c) Find the frequency of small oscillations of the bead about the origin.

Solution. (a) The potential energy from one point mass is $U = -\frac{GMm}{r}$, and the distance $r = \sqrt{x^2 + a^2}$ Therefore, the potential energy is

$$U = -\frac{2GMm}{\sqrt{x^2 + a^2}}$$

(b) The mechanical energy of the system is conserved, thus

$$E = U + K = \frac{1}{2}mv_0^2 - \frac{2GMm}{\sqrt{(3a)^2 + a^2}} = \frac{1}{2}mv_f^2 - \frac{2GMm}{a}.$$

Rearranging the equation gives

$$v_f^2 = v_0^2 + \frac{4GM}{a}\left(1 - \frac{1}{\sqrt{10}}\right) \Rightarrow v_f = \sqrt{v_0^2 + \frac{4GM}{a}\left(1 - \frac{1}{\sqrt{10}}\right)}.$$

By similar logic to problem 4(d), If $E < 0$, the system is bounded and the bead oscillates about $x = 0$. If $E > 0$, then the bead passes $x = 0$ without enough speed to approach $-\infty$.

(c) Let us use power rules to find $\frac{d^2U}{dx^2}$:

$$\begin{aligned}\frac{dU}{dx} = -2GMm\frac{d}{dx}(x^2+a^2)^{-1/2} &= (-2GMm)\left[-\frac{1}{2}(x^2+a^2)^{-3/2}\right](2x) \\ &= 2GMmx(x^2+a^2)^{-3/2}\end{aligned}$$

$$\frac{d^2U}{dx^2} = 2GMm\frac{d}{dx}(x(x^2+a^2)^{-3/2}) = (2GMm)\left[-\frac{3}{2}x(x^2+a^2)^{-5/2}+(x^2+a^2)^{-3/2}\right]$$

At $x=0$,

$$\frac{d^2U}{dx^2} = (2GMm)\left[0+(0^2+a^2)^{-3/2}\right] = \frac{2GMm}{a^3}.$$

Thus, we are approximating the spring constant $k = \frac{2GMm}{a^3}$. Therefore, the frequency of oscillation is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2GM}{a^3}}.$$

□