Homework 4

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Question 1. Any quantity which is left unchanged by the Lorentz Transformation is called a *Lorentz Invariant*. Show that Δs, called the "interval" is a Lorentz invariant, where

$$\Delta s^2 = (\Delta x^2 + \Delta y^2 + \Delta z^2) - (c\Delta t)^2$$

Here Δt is the time interval between two events and $(\Delta x^2 + \Delta y^2 + \Delta z^2)^{1/2}$ is the distance between them in the same inertial system. Assume the usual Lorentz Transformation between the "lab" frame and a frame moving in the positive x-direction with velocity v.

Proof. If the interval is a Lorentz invariant is true, then we must prove that

$$\Delta s^2 = (\Delta x'^2 + \Delta y'^2 + \Delta z'^2) - (c\Delta t')^2, \tag{1}$$

where the space-time coordinates (x', y', z', t') can be expressed under the following Lorentz Transformation:

$$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma\left(t - \frac{v}{c^2}x\right). \end{cases}$$

Given $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $\beta = \frac{v}{c}$, expanding the right-hand side of (1) gives us

$$\begin{aligned}
(\Delta x'^2 + \Delta y^2 + \Delta z^2) - (c\Delta t')^2 &= \Delta y^2 + \Delta z^2 + \gamma^2\left[(\Delta x - v\Delta t)^2 - c^2\left(\Delta t - \frac{v}{c^2}\Delta x\right)^2\right] \\
&= \Delta y^2 + \Delta z^2 + \gamma^2\Big(\Delta x^2 \cancel{-2v\Delta x\Delta t} + v^2\Delta t^2 \\
&\quad - c^2\Delta t^2 + \cancel{2v\Delta x\Delta t} - \frac{v^2}{c^2}\Delta x^2\Big) \\
&= \Delta y^2 + \Delta z^2 + \gamma^2\Delta x^2(1-\beta^2) + \gamma^2 c^2\Delta t^2(\beta^2 - 1) \\
&= \Delta y^2 + \Delta z^2 + \frac{\Delta x^2(1-\beta^2)}{1-\beta^2} + \frac{c^2\Delta t^2(\beta^2-1)}{1-\beta^2} \\
&= \boxed{\Delta x^2 + \Delta y^2 + \Delta z^2 - (c\Delta t)^2 = \Delta s^2.}
\end{aligned}$$

Hence, the interval is a Lorentz invariant.

$\square$

Question 2. (similar to KK 13.5) A particle of mass m whose total energy is twice its rest energy collides with an identical particle at rest. If they stick together, what is the mass and velocity of the resulting composite particle?

Solution. In this problem, we will use the laboratory frame. Additionally, linear momentum and total energy is conserved as such.

$$\begin{cases} \gamma m v = \gamma' m' v' \\ \gamma m c^2 + m c^2 = \gamma' m' c^2. \end{cases}$$

However, we know $\gamma = 2$, and c^2 can be factored out in the 2nd-equation, thus getting the system of equations

$$2mv = \gamma' m' v' \tag{1}$$
$$3m = \gamma' m'. \tag{2}$$

Let us go back to $\gamma = 2$. Based on this, we can say that

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = 2 \Rightarrow \beta = \sqrt{1 - \frac{1}{\gamma^2}} = \frac{\sqrt{3}}{2}.$$

Hence, this makes $v = \frac{\sqrt{3}}{2}c$. This can be substituted to the equation when dividing the first equation over the second as such:

$$\frac{2mv}{3m} = \frac{\gamma' m' v'}{\gamma m'} \quad \Rightarrow \quad \boxed{v' = \frac{2}{3}v = \frac{2}{3}\left(\frac{\sqrt{3}}{2}c\right) = \frac{\sqrt{3}}{3}c.}$$

Now, let us find the mass m'. Since we found v', it is implied that

$$\gamma' = \frac{1}{\sqrt{1-\beta'^2}} = \frac{1}{\sqrt{1 - \left(\frac{\sqrt{3}}{3}\right)^2}} = \sqrt{\frac{3}{2}}.$$

Finally, we can use this to get m' by rearranging (2):

$$\boxed{m' = \frac{3m}{\gamma'} = \sqrt{\frac{2}{3}}(3m) = \sqrt{6}m.}$$

$\square$

Question 3. (Eisberg and Resnick (ER) 2.1)

(a) The energy required to remove an electron from sodium is 2.3 eV. Does sodium show a photoelectric effect for light with a wavelength of 5890 angstroms?

(b) What is the cutoff wavelength for photoelectric emission from sodium?

Solution. For part (a), for sodium to show a photoelectric effect, the energy of the photon with wavelength 5890×10^{-10}m must be greater than 2.3 eV. The energy of the photon E_γ is

$$\begin{aligned}
E_\gamma = \frac{hc}{\lambda} &= \frac{(6.63 \times 10^{-34}\ \text{J} \cdot \text{s})(3 \times 10^{8}\ \text{m/s})}{5.89 \times 10^{-7}\ \text{m}} \cdot \frac{1\ \text{eV}}{1.6 \times 10^{-19}\ \text{J}} \\
&\approx \frac{(6 \times 10^{-34}\ \text{J} \cdot \text{s})(3 \times 10^{8}\ \text{m/s})}{6 \times 10^{-7}\ \text{m}} \cdot \frac{1\ \text{eV}}{1.6 \times 10^{-19}\ \text{J}} \approx 1.9\ \text{eV} < 2.3\ \text{eV}.
\end{aligned}$$

$\boxed{\text{Therefore, there is no photoelectric effect present.}}$

For part (b), the cutoff wavelength λ can be obtained by rearranging the equation $\frac{hc}{\lambda} = \phi$ as

$$\begin{aligned}
\lambda = \frac{hc}{\phi} &= \frac{(6.63 \times 10^{-34}\ \text{J} \cdot \text{s})(3 \times 10^{8}\ \text{m/s})}{2.3\ \text{eV}} \cdot \frac{1\ \text{eV}}{1.6 \times 10^{-19}\ \text{J}} \\
&\approx \frac{(6.4 \times 10^{-34}\ \text{J} \cdot \text{s})(3 \times 10^{8}\ \text{m/s})}{2.3\ \text{eV}} \cdot \frac{1\ \text{eV}}{1.6 \times 10^{-19}\ \text{J}} \approx 5 \times 10^{-7}\ \text{m} = \boxed{5000\ \text{angstroms}}
\end{aligned}$$

□

Question 4. (ER 2.22) What is the maximum possible kinetic energy of a recoiling Compton electron in terms of the incident photon energy hf and the electron's rest energy mc^2?

Solution. The change in wavelength for Compton scattering is given by:

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta)$$

To find the scattered photon energy E', we substitute $\lambda = c/f$ and $\lambda' = c/f'$:

$$\frac{c}{f'} - \frac{c}{f} = \frac{h}{mc}(1 - \cos\theta)$$

Dividing both sides by c and multiplying by h to convert frequencies to energies ($E = hf$) gives us

$$\frac{1}{E'} - \frac{1}{E} = \frac{1}{mc^2}(1 - \cos\theta).$$

Hence, we expand

$$\frac{1}{E'} = \frac{1}{E} + \frac{1 - \cos\theta}{mc^2} = \frac{mc^2 + E(1 - \cos\theta)}{E \cdot mc^2},$$
$$E' = \frac{E \cdot mc^2}{mc^2 + E(1 - \cos\theta)} = \frac{E}{1 + \frac{E}{mc^2}(1 - \cos\theta)}.$$

Thus, by the conservation of energy, the kinetic energy of an electron is

$$K_{e^+} = \Delta E = E - E' = hf - \frac{hf}{1 + \frac{hf}{mc^2}(1 - \cos\theta)}.$$

For maximum kinetic energy K_{max}, the photon must transfer the most energy possible, which occurs when $\cos(180°) = -1$ so that

$$K_{max} = hf - \frac{hf}{1 + \frac{2hf}{mc^2}} = \boxed{\frac{2(hf)^2}{mc^2 + 2hf}}.$$

□

Question 5. Solar radiation falls on Earth's surface at a rate of 1400 W/m^2.

(a) Assuming that the radiation has an average wavelength of 550 nm, how many photons per square meter per second fall on the surface of Earth?

(b) How does the total force of sunlight compare with the sun's gravitational force on earth?

Solution. For part (a), recall that watts has units W = J / s. Additionally, since light comes in discrete photons, the energy of one photon is

$$E_\gamma = \frac{hc}{\lambda} = \frac{(6.63 \times 10^{-34} \text{ J} \cdot \text{s })(3 \times 10^8 \text{ m/s})}{5.5 \times 10^{-7} \text{ m}} \approx 3.6 \times 10^{-19} \text{ J}.$$

Hence, by, dimensional analysis,

$$1400 \frac{\text{J}}{\text{m}^2 \cdot \text{s}} \cdot \frac{1 \text{ photon}}{3.6 \times 10^{-19} \text{ J}} \approx \boxed{4 \times 10^{21} \frac{\text{photons}}{\text{m}^2 \cdot \text{s}}.}$$

For part (b) we must first find the radiation pressure. Since $P = F \cdot v$, we can thus find the pressure

$$p_{rad} = \frac{P/A}{c} = \frac{F \cdot \cancel{c}}{A \cdot \cancel{c}} = \frac{1400 \text{ W/m}^2}{3.00 \times 10^8 \text{ m/s}} \approx 4.67 \times 10^{-6} \text{ N/m}^2.$$

The total force F_{sun} is therefore, assuming $R_E \approx 6.37 \times 10^6$ m,

$$F_{rad} = p_{rad} \cdot \pi R_E^2 \approx (4.67 \times 10^{-6} \text{ N/m}^2) \cdot \pi (6.37 \times 10^6 \text{ m})^2 \approx 5.95 \times 10^8 \text{ N}.$$

Next, we calculate the Sun's gravitational force on Earth using Newton's law of universal gravitation

$$F_g = \frac{GM_S M_E}{R^2},$$

where $G \approx 6.67 \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$, $M_S \approx 1.99 \times 10^{30}$ kg, $M_E \approx 5.97 \times 10^{24}$ kg, and the distance $R \approx 1.50 \times 10^{11}$ m.

$$F_g = \frac{(6.67 \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2})(1.99 \times 10^{30} \text{ kg})(5.97 \times 10^{24} \text{ kg})}{(1.50 \times 10^{11} \text{ m})^2} \approx 3.54 \times 10^{22} \text{ N}.$$

Comparing the two:

$$\frac{F_g}{F_{rad}} \approx \frac{3.54 \times 10^{22} \text{ N}}{5.95 \times 10^8 \text{ N}} \approx 6 \times 10^{13}.$$

$\boxed{\text{The gravitational force is approximately 60 trillion times stronger than the force of sunlight.}}$

□