학과: 게임학부
학번: C077044
이름: 황태훈

85. (1) $(x_1\ x_2\ x_3) = k_1(1,2,1) + k_2(2,-1,0) + k_3(-3,-1,1)$

$$= (k_1, 2k_1, k_1) + (2k_2, -k_2, 0) + (-3k_3, -k_3, k_3)$$

$$= (k_1+2k_2-3k_3,\ 2k_1-k_2-k_3,\ k_1+k_3)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & x_1 \\ 2 & -1 & -1 & x_2 \\ 1 & 0 & 1 & x_3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & x_3 \\ 0 & 2 & -4 & x_1-x_3 \\ 0 & 1 & -3 & x_2-2x_3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & x_3 \\ 0 & 1 & -3 & x_2-2x_3 \\ 0 & 2 & -4 & x_1-x_3 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & x_3 \\ 0 & 1 & -3 & x_2-2x_3 \\ 0 & 0 & 1 & \frac{x_1-2x_2+3x_3}{2} \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{-x_1+2x_2-x_3}{2} \\ 0 & 1 & 0 & \frac{3x_1-4x_2+5x_3}{2} \\ 0 & 0 & 1 & \frac{x_1-2x_2+3x_3}{2} \end{array}\right)$$

$$\therefore (x_1, x_2, x_3) = \frac{-x_1+2x_2-x_3}{2}V_1 + \frac{3x_1-4x_2+5x_3}{2}V_2 + \frac{x_1-2x_2+3x_3}{2}V_3$$

$$T(x_1, x_2, x_3) = \frac{-x_1+2x_2-x_3}{2}T(V_1) + \frac{3x_1-4x_2+5x_3}{2}T(V_2) + \frac{x_1-2x_2+3x_3}{2}T(V_3)$$

$$= \frac{-x_1+2x_2-x_3}{2}(0,-1) + \frac{3x_1-4x_2+5x_3}{2}(1,1) + \frac{x_1-2x_2+3x_3}{2}(1,0)$$

$$= \left(0, \frac{x_1-2x_2+x_3}{2}\right) + \left(\frac{3x_1-4x_2+5x_3}{2}, \frac{3x_1-4x_2+5x_3}{2}\right) + \left(\frac{x_1-2x_2+3x_3}{2}, 0\right)$$

$$= \left(\frac{4x_1-6x_2+8x_3}{2}, \frac{4x_1-6x_2+6x_3}{2}\right) = (2x_1-3x_2+4x_3,\ 2x_1-3x_2+3x_3)$$

$$\therefore T(10,5,17) = (33, 26)$$

85. (2) $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & -3 & 4 \\ 2 & -3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ 2 \end{pmatrix}x_1 + \begin{pmatrix} -3 \\ -3 \end{pmatrix}x_2 + \begin{pmatrix} 4 \\ 3 \end{pmatrix}x_3$

$\mathrm{Span}\left\{\begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ -3 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \end{pmatrix}\right\}$

$\therefore$ T의 치역 $= \begin{pmatrix} 2 \\ 2 \end{pmatrix}x_1 + \begin{pmatrix} -3 \\ -3 \end{pmatrix}x_2 + \begin{pmatrix} 4 \\ 3 \end{pmatrix}x_3$

T의 치역의 기저 $= \mathrm{Span}\left\{\begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ -3 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \end{pmatrix}\right\}$

(3) $\left(\begin{array}{ccc|c} 2 & -3 & 4 & 0 \\ 2 & -3 & 3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -\frac{3}{2} & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$ $x_3 = 0$, $x_2 = t$, $x_1 = \frac{3}{2}t$

$\therefore$ T의 핵 $= 3x_1 + 2x_2$, T의 핵의 기저 $= \mathrm{Span}\{(3, 2, 0)\}$

86. $L(x,y) = (4x-2y, -6x+3y) = \begin{pmatrix} 4 & -2 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(1) $\ker(L) = \{(x,y) \mid L(x,y) = (0,0)\}$

$\left(\begin{array}{cc|c} 4 & -2 & 0 \\ -6 & 3 & 0 \end{array}\right) \Rightarrow$ $AX=0$ 이 될려면 A는 가역이여야 되지만 A가 가역이 아니므로 단사가 아니다.

$\left(\begin{array}{cc|c} 4 & -2 & 0 \\ -6 & 3 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right)$ $x = \frac{1}{2}t$, $y = t$

$\therefore \ker(L) = \{(t, 2t) \in \mathbb{R}^2 \mid t \in \mathbb{R}\}$

$\mathrm{Span}(\ker(L)) = \left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$, L이 단사가 아니다.

(2) $\mathrm{Im}(L) = \left\{\begin{pmatrix} 4 & -2 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \middle| (x,y) \in \mathbb{R}^2\right\}$.

L이 단사가 아니면 전사가 아니므로 L은 전사가 아니다.

$\therefore \mathrm{Im}(L) = \left\{\begin{pmatrix} 4 \\ -6 \end{pmatrix}x + \begin{pmatrix} -2 \\ 3 \end{pmatrix}y \middle| (x,y) \in \mathbb{R}^2\right\}$

$\mathrm{Span}(\mathrm{Im}(L)) = \left\{\begin{pmatrix} 4 \\ -6 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \end{pmatrix}\right\}$, L이 전사가 아니다.

87. $T(x,y,z) = (x-y+z, -2x+y-3z) = \begin{pmatrix} 1 & -1 & 1 \\ -2 & 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

(1) $\ker(L) = \{(x,y,z) \mid T(x,y,z) = (0,0)\}$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -2 & 1 & -3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$x = -2t, y = -t, z = t \Rightarrow$

$\therefore \ker(L) = \{(-2,-1,1)t \mid t \in R\}$, $\mathrm{span}(\ker(L)) = \left\{\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}\right\}$

(0,0,0) 뿐만 아니라 (-2,-1,1)이 있으므로 단사가 아닙니다.

(2) $\mathrm{Im}(L) = \left\{ \begin{pmatrix} 1 & -1 & 1 \\ -2 & 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \,\middle|\, (x,y,z) \in R^3 \right\}$

↳ 위에 단사가 아니므로 전사도 아닙니다.

$\therefore \mathrm{Im}(L) = \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} x + \begin{pmatrix} -1 \\ 1 \end{pmatrix} y + \begin{pmatrix} 1 \\ -3 \end{pmatrix} z \,\middle|\, (x,y,z) \in R^3 \right\}$

$\mathrm{Span}(\mathrm{Im}(L)) = \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \end{pmatrix} \right\}$, 전사가 아닙니다.

88. $T(x,y) = (2x-3y, 3x+y, -6x+9y) = \begin{pmatrix} 2 & -3 \\ 3 & 1 \\ -6 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(1) $\ker(T) = \{(x,y) \mid T(x,y) = (0,0)\}$

$$\left(\begin{array}{cc|c} 2 & -3 & 0 \\ 3 & 1 & 0 \\ -6 & 9 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & \frac{11}{2} & 0 \\ 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$x=0$, $y=0$ 이므로 단사이다.

$\therefore \ker(T) = \{ \emptyset \}$, $\mathrm{span}\{\ker(T)\} = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$, 단사

(2) $\mathrm{Im}(T) = \left\{ \begin{pmatrix} 2 & -3 \\ 3 & 1 \\ -6 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \middle| (x,y) \in \mathbb{R}^2 \right\}$

위에 단사라고 증명되었으므로 전사이다.

$\therefore \mathrm{Im}(T) = \left\{ \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix} x + \begin{pmatrix} -3 \\ 1 \\ 9 \end{pmatrix} y \middle| (x,y) \in \mathbb{R}^2 \right\}$

$\mathrm{span}(\mathrm{Im}(T)) = \left\{ \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 9 \end{pmatrix} \right\}$, 전사이다.

89. $T(x,y) = (-6x+3y,\ 2x-y,\ 4x-2y)$

$$= \begin{pmatrix} -6 & 3 \\ 2 & -1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(1) $\ker(T) = \{(x,y) \mid T(x,y) = (0,0)\}$

$$\left(\begin{array}{cc|c} -6 & 3 & 0 \\ 2 & -1 & 0 \\ 4 & -2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} x = \frac{1}{2}t \\ y = t \end{array}$$

$\therefore \ker(T) = \{(1,2)t \mid t \in R\}$, $\mathrm{Span}(\ker(T)) = \left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$

단사라면 (0,0)만 가능하지만 (1,2)도 존재하기 때문에
단사가 아니다.

(2) $Im(T) = \left\{ \begin{pmatrix} -6 & 3 \\ 2 & -1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; (x,y) \in R^2 \right\}$

위에 문제에 증명을 통해 단사가 아니므로 전사가 아닙니다.

$\therefore Im(T) = \left\{ \begin{pmatrix} -6 \\ 2 \\ 4 \end{pmatrix} x + \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} y \;\middle|\; (x,y) \in R^2 \right\}$

$\mathrm{Span}(Im(T)) = \left\{ \begin{pmatrix} -6 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} \right\}$, 전사가 아닙니다.

9. $T(x,y,z) = (x+3z,\ 3x+y+4z,\ -2x+2y+4z) = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \\ -2 & 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

(1) $\ker(T) = \{(x,y,z) \mid T(x,y,z) = (0,0,0)\}$

$$\begin{vmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \\ -2 & 2 & 4 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2+4-2(4-6) = 10$$

$\det \neq 0$ 이므로 가역행렬이므로 단사이다.

$\therefore \ker(T) = \{0\}$, $\mathrm{span}(\ker(T)) = \{(0,0)\}$, 단사이다.

(2) $\mathrm{Im}(T) = \left\{ \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \\ -2 & 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \,\middle|\, (x,y,z) \in \mathbb{R}^3 \right\}$

위문제를 통해서 단사이므로 전사이다.

$\therefore \mathrm{Im}(T) = \left\{ \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}x + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}y + \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}z \,\middle|\, (x,y,z) \in \mathbb{R}^3 \right\}$

$\mathrm{span}(\mathrm{Im}(T)) = \left\{ \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} \right\}$, 전사이다.

91. $T(x,y) = \begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(1) $T(E_1) = \begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ $T(E_2) = \begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ $[T]_E = \begin{pmatrix} 2 & -4 \\ -3 & 1 \end{pmatrix}$

$P_{B\to E} \Rightarrow \left(\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 1 \end{array}\right) = \begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix} = P_{B\to E}$

정답! $\boxed{[T]_E = \begin{pmatrix} 2 & -4 \\ -3 & 1 \end{pmatrix},\ P_{B\to E} = \begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix}}$

(2) $P_{B\to E}\begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix}$ $P_{E\to B} = -\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -3 \\ -1 & -2 \end{pmatrix}$

$[T]_B = P_{E\to B}[T]_E P_{B\to E} = \begin{pmatrix} -1 & -3 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 2 & -4 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 7 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -20 \\ 6 & -10 \end{pmatrix}$

정답! $\boxed{[T]_B = \begin{pmatrix} 5 & -20 \\ 6 & -10 \end{pmatrix}}$

(3) $\begin{pmatrix} 5 & -20 \\ 6 & -10 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 45 \\ 32 \end{pmatrix}$

92.(1) $T(x,y) = (2x-y,\ 4x+3y) = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$$\left.\begin{aligned} T(e_1) &= \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2e_1 + 4e_2 \\ T(e_2) &= \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} = -e_1 + 3e_2 \end{aligned}\right\} [T]_E = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}$$

$$P_{B\to E} \Rightarrow \begin{array}{c} \left(\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right) \\ \quad E \qquad\quad B \end{array} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} = P_{B\to E}$$

(2) $P_{B\to E} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \quad P_{E\to B} = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$

$$[T]_B = P_{E\to B}\,[T]_E\,P_{B\to E} = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 3 & 1 \\ 0 & 10 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 5 & 4 \\ -10 & 10 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} & \frac{4}{3} \\ -\frac{10}{3} & \frac{10}{3} \end{pmatrix}$$

(3) $\begin{pmatrix} \frac{5}{3} & \frac{4}{3} \\ -\frac{10}{3} & \frac{10}{3} \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$

93. (1) $T(1-2x) = 1-2(x+1) = 1-2x-2 = -2x-1 \Rightarrow -P_1+4P_2$

$T(-x) = 0-(x+1) = -x-1 \Rightarrow -P_1+3P_2$

$[T]_B = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}$

$$P_{B'\to B} \Rightarrow \begin{array}{c} \left(\begin{array}{cc|cc} 3 & 0 & 1 & -2 \\ 1 & -1 & 0 & -1 \end{array}\right) \\ \quad B \qquad\quad B' \end{array} \rightarrow \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{3} & -\frac{1}{3} \\ -1 & 1 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \end{array}\right)$$

정답!

$$\boxed{[T]_B = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix},\ P_{B'\to B} = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}}$$

(2) $P_{B\to B'} = \frac{1}{9}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$

$$[T]_{B'} = \frac{1}{27}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$

$$= \frac{1}{27}\begin{pmatrix} -3 & 11 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} = \frac{1}{27}\begin{pmatrix} 8 & 25 \\ -1 & -2 \end{pmatrix}$$

정답!

$$\boxed{[T]_{B'} = \frac{1}{27}\begin{pmatrix} 8 & 25 \\ -1 & -2 \end{pmatrix}}$$