<proof of Sample Correlation limiting distribution>

$(X_i, Y_i) \overset{iid}{\sim} (\mu, \sigma^2)$

for the sake of simplicity we assume that

$E(X_i)=0,\ E(Y_i)=0.$

we let

$$\begin{pmatrix} m_x \\ m_y \\ m_{xx} \\ m_{yy} \\ m_{xy} \end{pmatrix} = \frac{1}{n}\begin{pmatrix} \sum X_i \\ \sum Y_i \\ \sum X_i^2 \\ \sum Y_i^2 \\ \sum X_i Y_i \end{pmatrix}.$$

$$s_x^2 = m_{xx} - m_x^2,\quad s_y^2 = m_{yy} - m_y^2,\quad s_{xy} = m_{xy} - m_x m_y$$

then $\quad r = s_{xy} / s_x s_y.$

By CLT.

$$\sqrt{n}\left\{\begin{pmatrix} m_x \\ m_y \\ m_{xx} \\ m_{yy} \\ m_{xy} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \sigma_x^2 \\ \sigma_y^2 \\ \sigma_{xy} \end{pmatrix}\right\} \xrightarrow{d} N_5\left\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \underset{\Sigma}{\underset{\|}{\begin{pmatrix} Cov(X_1, X_1) & \cdots & Cov(X_1, X_1Y_1) \\ Cov(Y_1, X_1) & & \vdots \\ \vdots & & \\ Cov(X_1Y_1, X_1) & & Cov(X_1Y_1, X_1Y_1) \end{pmatrix}}}\right\}$$

We want to asymptotic distribution of r.

So define function $g : \mathbb{R}^5 \to \mathbb{R}^3$. which yields (s_x^2, s_y^2, s_{xy})

Then

$$\sqrt{n}\left(g\begin{pmatrix} m_x \\ m_y \\ \vdots \\ m_{xy} \end{pmatrix} - g\begin{pmatrix} 0 \\ 0 \\ \sigma_x^2 \\ \sigma_y^2 \\ \sigma_{xy} \end{pmatrix}\right) \xrightarrow{d} N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \nabla g^T \Sigma \nabla g\right)$$

which satisfy

$$\nabla g\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} -2a & 0 & -b \\ 0 & -2b & -a \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\boxed{\begin{array}{l} g(a,b,c,d,e) \\ = (c-a^2,\ d-b^2,\ e-ab) \end{array}}$$

So

$$\sqrt{n}\left(g\begin{pmatrix} m_X \\ m_Y \\ m_{XX} \\ m_{YY} \\ m_{XY} \end{pmatrix} - g\begin{pmatrix} 0 \\ 0 \\ \sigma_X^2 \\ \sigma_Y^2 \\ \sigma_{XY} \end{pmatrix}\right) \to N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \Sigma^*\right).$$

where

$$\Sigma^* = \begin{pmatrix} \mathrm{Cov}(X_1^2, X_1^2) & \mathrm{Cov}(X_1^2, Y_1^2) & \mathrm{Cov}(X_1^2, X_1Y_1) \\ \mathrm{Cov}(Y_1^2, X_1^2) & \mathrm{Cov}(Y_1^2, Y_1^2) & \mathrm{Cov}(Y_1^2, X_1Y_1) \\ \mathrm{Cov}(X_1Y_1, X_1^2) & \mathrm{Cov}(X_1Y_1, Y_1^2) & \mathrm{Cov}(X_1Y_1, X_1Y_1) \end{pmatrix}$$

precisely

$$\sqrt{n}\left(\begin{pmatrix} S_X^2 \\ S_Y^2 \\ S_{XY} \end{pmatrix} - \begin{pmatrix} \sigma_X^2 \\ \sigma_Y^2 \\ \sigma_{XY} \end{pmatrix}\right) \longrightarrow N_3(0, \Sigma^*).$$

As an aside, $\sqrt{n}(S_X^2 - \sigma_X^2) \xrightarrow{d} N(0, \underbrace{Var(X_1^2)}_{E((X_i-\mu)^4) - \sigma^4})$

Next, we use Delta method one more time.

which $h: \mathbb{R}^3 \to \mathbb{R}$, $h(a,b,c) = \frac{c}{\sqrt{a}\sqrt{b}}$

$$\nabla h(\cdot) = \begin{pmatrix} \frac{c}{\sqrt{b}} \cdot -\frac{1}{2}a^{-\frac{3}{2}} \\ \frac{c}{\sqrt{a}} \cdot -\frac{1}{2}b^{-\frac{3}{2}} \\ \frac{1}{\sqrt{ab}} \end{pmatrix} \Rightarrow \nabla h\begin{pmatrix} S_X^2 \\ S_Y^2 \\ S_{XY} \end{pmatrix} = \begin{pmatrix} \frac{S_{XY}}{S_Y} \cdot \frac{-1}{2} \cdot \frac{1}{S_X^3} \\ \frac{S_{XY}}{S_X} \cdot -\frac{1}{2} \cdot \frac{1}{S_Y^3} \\ \frac{1}{S_X S_Y} \end{pmatrix}$$

So that

$$\sqrt{n}(r - \rho) \xrightarrow{d} N(0, \nabla h^T \Sigma^* \nabla h)$$

When $E(X)=E(Y)=0$, $Var(X)=Var(Y)=1$, $\sqrt{n}(r-\rho) \xrightarrow{d} N(0, (1-\rho^2)^2)$