Optimization: introduction

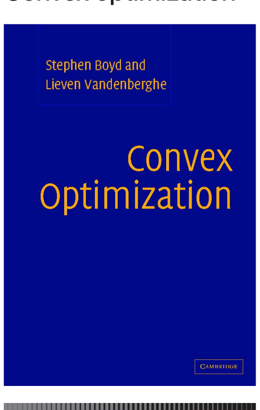
- Optimization or mathematical programming considers the problem

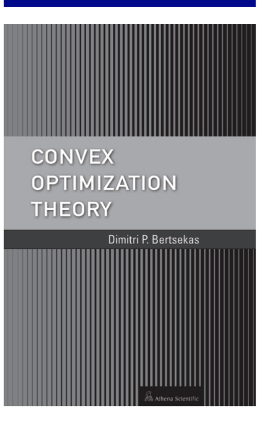
$$\begin{array}{ll} \text{minimize} & f(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in C \end{array}$$

  - Vector $\mathbf{x}$ in a vector space V is the *optimization variable*.
    - In most cases $V = \mathbb{R}^d$.
  - Function $f : V \to \mathbb{R}$ is the *objective function*.
  - $C \subset V$ is called the *constraint set*.
    - If $C = V$, the problem is *unconstrained*.
    - Otherwise it is constrained.
  - $\mathbf{x}^\star$ is called a (global) *solution* if $f(\mathbf{x}^\star) \le f(\mathbf{x})$ for all $\mathbf{x} \in C$.
    - Solution may not be unique (recall the least squares problem).
    - Solution may not even exist (e.g., $f(x) = 1/x$, $C = \{x \in \mathbb{R} : x \ge 0\}$).
- Possible confusion:
  - Statisticians mostly talk about **maximization**: $\max\ L(\theta)$.
  - Optimizers talk about **minimization**: $\min\ f(\mathbf{x})$.
- **Why** is optimization important in statistics?
  - Maximum likelihood estimation (MLE).
  - Maximum a posteriori (MAP) estimation in Bayesian framework.
  - Machine learning: minimize a loss + certain regularization.
  - ...
- Global optimization
  - Worst-case complexity grows exponentially with the d and n
  - Even small problems, with a few tens of variables, can take a very long time (e.g., hours or days) to solve.
- Local optimization
  - *Local* solution: $\mathbf{x}^\dagger$ such that $f(\mathbf{x}^\dagger) \le f(\mathbf{x})$ for all $\mathbf{x} \in C \cap \mathbf{N}(\mathbf{x}^\dagger)$, where $\mathbf{N}(\mathbf{x}^\dagger)$ is a certain neighborhood of $\mathbf{x}^\dagger$.
  - Can be found relatively easy, using only local information on f, e.g., gradients.
  - Local optimization methods can be fast, can handle large-scale problems, and are widely applicable.
- Our major **goal** (or learning objectives) is to
  - have a working knowledge of some commonly used optimization methods:
    - convex programming with emphasis in statistical applications
    - Newton-type algorithms
    - first-order methods
    - expectation-maximization (EM) algorithm
    - majorization-minimization (MM) algorithm
  - implement some of them in homework
  - get to know some optimization tools in Julia

Convex Optimization 101

Convex optimization





- Problem

$$\begin{array}{ll} \text{minimize} & f(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in C \end{array}$$

  - f is a *convex function*

$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \le \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in \text{dom} f, \forall \alpha \in [0,1].$$

  - C is a *convex set*

$$\mathbf{x}, \mathbf{y} \in C \implies \alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in C, \quad \forall \alpha \in [0,1].$$

- More familiar formulation: take $V = \mathbb{R}^d$,

$$\begin{array}{ll} \text{minimize} & f_0(\mathbf{x}) \\ \text{subject to} & f_i(\mathbf{x}) \le b_i, \quad i = 1, 2, \ldots, m. \end{array}$$

  where $f_i : \mathbb{R}^d \to \mathbb{R}$ are convex functions.

  - Equality constraint: $f_i(\mathbf{x}) = b_i \iff f_i(\mathbf{x}) \le b_i$ and $-f_i(\mathbf{x}) \le -b_i$
    - Hence only *linear* equality constraints $\mathbf{A}\mathbf{x} = \mathbf{b}$ are allowed (why?)
- Why do we care about convex optimization?

> **Fact**. Any local solution of a convex optimization problem is a global solution.

- Role of convex optimization
  - Initialization for local optimization
  - Convex heuristics for nonconvex optimization
  - Bounds for global optimization

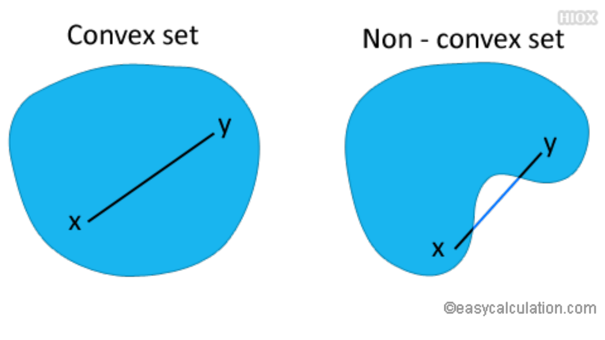
Convex sets

The underlying space is a vector space V unless otherwise stated.

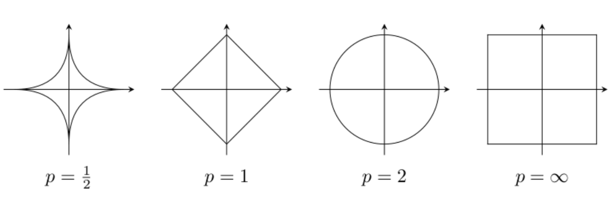
- Line segments: for given $\mathbf{x}$, $\mathbf{y}$,

$$\{\alpha\mathbf{x} + (1-\alpha)\mathbf{y} : 0 \le \alpha \le 1\}.$$

- A set C is convex if for every pair of points $\mathbf{x}$ and $\mathbf{y}$ lying in C the entire line segment connecting them also lies in C.



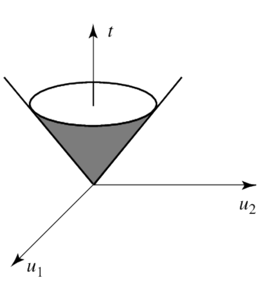
- Examples
  1. Singleton: $\{\mathbf{a}\}$.
  2. Euclidean space $\mathbb{R}^d$.
  3. Norm ball: $B_r(\mathbf{c}) = \{\mathbf{x} : \|\mathbf{x} - \mathbf{c}\| \le r\}$ for any proper norm $\|\cdot\|$.



     ℓ_p "norm" $\|\mathbf{x}\|_p = (\sum_{i=1}^d |x_i|^p)^{1/p}$, $0 < p < 1$, in $\mathbb{R}^d$ is *not* a proper norm.
  4. Hyperplane: $\{\mathbf{x} : \langle \mathbf{a}, \mathbf{x} \rangle = c\}$.
  5. Halfspace: $\{\mathbf{x} : \langle \mathbf{a}, \mathbf{x} \rangle \le c\}$ or $\{\mathbf{x} : \langle \mathbf{a}, \mathbf{x} \rangle < c\}$.
  6. Polyhedron: $\{\mathbf{x} : \langle \mathbf{a}_j, \mathbf{x} \rangle \le b_j,\ j = 1, \ldots, m\} = \{\mathbf{x} : \mathbf{A}\mathbf{x} \le \mathbf{b}\}$.
  7. Positive semidefinite cone $\mathbb{S}_+^d = \{\mathbf{X} \in \mathbb{R}^{d \times d} : \mathbf{X} = \mathbf{X}^T,\ \mathbf{X} \succeq \mathbf{0}\}$ and set of positive definite matrices $\mathbb{S}_{++}^d = \{\mathbf{X} \in \mathbb{R}^{d \times d} : \mathbf{X} = \mathbf{X}^T,\ \mathbf{X} \succ \mathbf{0}\}$. (What is V?)
  8. Translation: $C + \mathbf{a} = \{\mathbf{x} + \mathbf{a} : \mathbf{x} \in C\}$ if C is convex.
  9. Minkowski sum: $C + D = \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in C, \mathbf{y} \in D\}$ if C and D are convex.
  10. Cartesian product: $C \times D = \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in C, \mathbf{y} \in D\}$ if C and D are convex.
  11. Image $\mathcal{A}C$ of a convex set C under a linear map $\mathcal{A}$.
  12. Inverse image $\mathcal{A}^{-1}C$ of a convex set C under a linear map $\mathcal{A}$.

Convex cones

- A set C is a cone if for each $\mathbf{x}$ in C, the set $\{\alpha\mathbf{x} : \alpha > 0\} \subset C$.
  - Some authors (e.g., Boyd & Vandenberghe) use instead $\{\alpha\mathbf{x} : \alpha \ge 0\}$ to include the origin.
  - Is $\mathbb{S}_{++}^d$ a cone?
  - A cone is *unbounded*.
- A cone C is a convex cone if it is also convex. Equivalently, a set C is a convex cone if for any $\mathbf{x}, \mathbf{y} \in C$, $\{\alpha\mathbf{x} + \beta\mathbf{y} : \alpha, \beta > 0\} \subset C$.
- Examples of convex cone
  1. Any subspace
  2. Any hyperplane passing the origin.
  3. Any halfspace whose closure passes the origin.
  4. $\mathbb{S}_+^d$, set of positive semidefinite matrices.
  5. Second-order cone (Lorentz cone): $\{(\mathbf{x}, t) : \|\mathbf{x}\|_2 \le t\}$.



  6. Norm cone: $\{(\mathbf{x}, t) : \|\mathbf{x}\| \le t\}$, where $\|\cdot\|$ is any norm.
- Example of a nonconvex cone?

Affine sets

- A set C is a affine if for each $\mathbf{x}, \mathbf{y}$ in C, the whole line $\{\alpha\mathbf{x} + (1-\alpha)\mathbf{y} : \alpha \text{ is a scalar}\} \subset C$.
- Convex.
- $C = \mathbf{c} + S$ for some $\mathbf{c} \in V$ and subspace S.
  - We define $\dim C = \dim S$.
- Example
  - Any singleton.
  - Any subspace.
  - Set of solutions of a linear system of equations: $\{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}\}$.
    - In fact *any* affine set can be represented by the solution set of a linear system.

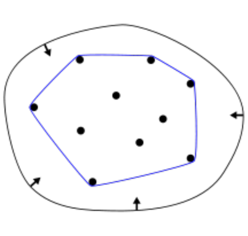
The intersection of an arbitrary collection of convex, affine, or conical sets is convex, affine, conical, respectively.

Generators

- Convex combination: $\{\sum_{i=1}^m \alpha_i \mathbf{x}_i : \alpha_i \ge 0,\ \sum_{i=1}^m \alpha_i = 1\}$
- Convex sets are closed under convex combination: any convex combination of points from a convex set C belongs to C
- Conic combination, affine conbination are defined similarly; similar closure properties also hold.
- **Convex hull**: the convex hull of a nonempty set C is the set of all convex combinations of points in C:

$$\text{conv}C = \{\sum_{i=1}^m \alpha_i \mathbf{x}_i : \mathbf{x}_i \in C,\ \alpha_i \ge 0, i = 1, \ldots, m \text{ for some } m,\ \sum_{i=1}^m \alpha_i = 1\}$$

  which is the *smallest convex set containing* C.



- **Conic hull** $\text{cone}C$ drops the sum condition, and **affine hull** $\text{aff}C$ drops the nonnegativity conditions on α_is.

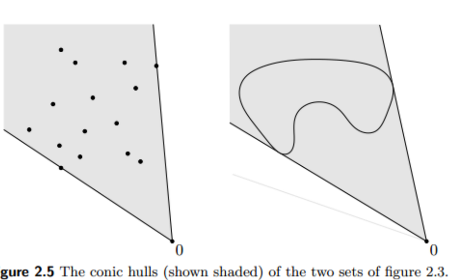


Figure 2.5 The conic hulls (shown shaded) of the two sets of figure 2.3.

- **Affine dimension**: $\dim(C) \triangleq \dim(\text{aff}C)$.
- Example: simplex

$$S = \text{conv}\{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k\}$$

  when $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k$ are *affinely independent*, i.e., $\mathbf{v}_1 - \mathbf{v}_0, \ldots, \mathbf{v}_k - \mathbf{v}_0$ are linearly independent.
  - $\dim(S) = k$
  - Unit simplex in $\mathbb{R}^d$: $\text{conv}\{\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_d\} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \ge \mathbf{0},\ \mathbf{1}^T\mathbf{x} \le 1\}$.
  - Probability simplex in $\mathbb{R}^d$: $\Delta_{d-1} = \text{conv}\{\mathbf{e}_1, \ldots, \mathbf{e}_d\} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \ge \mathbf{0},\ \mathbf{1}^T\mathbf{x} = 1\}$.

Relative interior

Most constraint sets in optimization does not have an interior, e.g, probability simplex. It is useful to define the interior relative to the affine hull:

$$\text{relint}C = \{\mathbf{x} \in C : (\exists r > 0) B(\mathbf{x}, r) \cap \text{aff}C \subset C\}$$

- What is the relative interior of the probability simplex in $\mathbb{R}^3$?

Convex functions

- Recall that a real-valued function f is convex if

$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \le \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in \text{dom} f, \forall \alpha \in [0,1],$$

  - If the inequality is strict for all $\alpha \in (0,1)$, then f is *strictly convex*.
  - f is concave if $-f$ is convex.
- Extended-value functions: it is often convenient to extend the domain of f to the whole V and allow to have the value ∞. Then $f : V \to \mathbb{R} \cup \{\infty\}$ and

$$\text{dom} f = \{\mathbf{x} : f(\mathbf{x}) < \infty\}$$

  is the **essential domain** of f.
- This extension allows us to consider the *indicator function* of a set:

$$\iota_C(\mathbf{x}) = \begin{cases} 0 & \mathbf{x} \in C \\ \infty & \mathbf{x} \notin C \end{cases}$$

  so that a constrained optimization problem is converted to an unconstrained problem:

$$\min_{\mathbf{x} \in C} f(\mathbf{x}) = \min_{\mathbf{x}} f(\mathbf{x}) + \iota_C(\mathbf{x})$$

- Properness: a function f is proper if $\text{dom} f \ne \emptyset$ and $f(\mathbf{x}) > -\infty$ for all $\mathbf{x}$.
- Examples
  1. Any affine function (also concave).
  2. Any norm
  3. Indicator function of a nonempty convex set.
  4. Exponential: $f(x) = e^{ax}$.
  5. Powers: $f(x) = x^\alpha$ on $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$. Convex if $\alpha \ge 1$, concave if $\alpha \in [0,1]$.
  6. Powers of absolute values: $f(x) = |x|^p$ on $\mathbb{R}$, if $p \ge 1$.
  7. Logarithm: $f(x) = \log x$ is *concave* in $\text{dom} f = \mathbb{R}_{++}$.
  8. Quadratic-over-linear function $f(x, y) = x^2/y$ with $\text{dom} f = \mathbb{R} \times \mathbb{R}_{++} = \{(x, y) : x \in \mathbb{R}, y > 0\}$.
  9. Maximum: $f(\mathbf{x}) = \max\{x_1, \ldots, x_d\}$.
  10. Log-sum-exp: $f(\mathbf{x}) = \log(e^{x_1} + \cdots + e^{x_d})$, a smoothed version of $\max\{x_1, \ldots, x_d\}$.
  11. Geometric mean: $f(\mathbf{x}) = \prod_{i=1}^d x_i^{1/d}$ is *concave*, with $\text{dom} f = \mathbb{R}_{++}^d$.
  12. Log-determinent: $f(\mathbf{X}) = \log\det\mathbf{X}$ is *concave* in $\text{dom} f = \mathbb{S}_{++}^d$.

Jensen's inequality

- Funciton f is convex **if and only if**

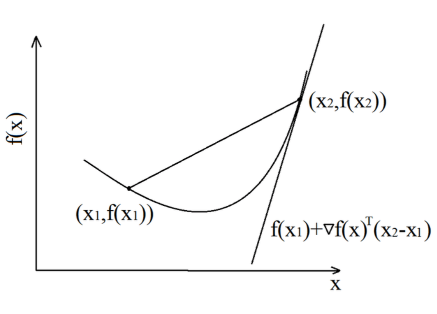
$$f(\sum_{i=1}^m \mathbf{x}_i) \le \sum_{i=1}^m \alpha_i f(\mathbf{x}_i), \quad \forall \mathbf{x}_1, \ldots, \mathbf{x}_m,\ \forall \alpha_i \ge 0,\ \sum_{i=1}^m \alpha_i = 1.$$

First-order condition (supporting hyperplane inequality)

- If f is differentiable (i.e., its gradient ∇f exists at each point in $\text{dom} f$, which is open), then f is convex **if and only if** $\text{dom} f$ is convex and

$$f(\mathbf{y}) \ge f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle$$

  for all $\mathbf{x}, \mathbf{y} \in \text{dom} f$.
  - f is strictly convex if and only if strict inequality holds for all $\mathbf{y} \ne \mathbf{x}$.



Second-order condition

- If f is twice differentiable (i.e., its Hessian $\nabla^2 f$ exists at each point in $\text{dom} f$, which is open), then f is convex **if and only if** $\text{dom} f$ is convex and its Hessian is positive semidefinite:, i.e,

$$\nabla^2 f(\mathbf{x}) \succeq \mathbf{0}$$

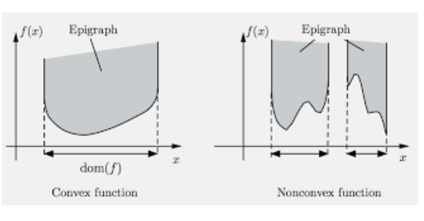
  for all $\mathbf{x} \in \text{dom} f$.
  - If $\nabla^2 f(\mathbf{x}) \succ \mathbf{0}$, then f is strictly convex.

Epigraph

- The epigraph of a function f is the set

$$\text{epi} f = \{(\mathbf{x}, t) : \mathbf{x} \in \text{dom} f,\ f(\mathbf{x}) \le t\}.$$

- A function f is convex **if and only if** $\text{epi} f$ is convex.



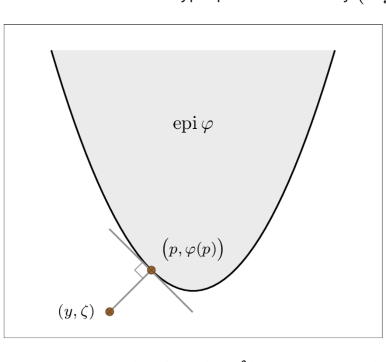
- If $(\mathbf{y}, t) \in \text{epi} f$, then from the supporting hyperplance inequality,

$$t \ge f(\mathbf{y}) \ge f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle$$

  or

$$\langle (\nabla f(\mathbf{x}), -1), (\mathbf{y}, t) - (\mathbf{x}, f(\mathbf{x})) \rangle \le 0.$$

  This means that the hyperplane defined by $(\nabla f(\mathbf{x}), -1)$ supports $\text{epi} f$ at the boundary point $(\mathbf{x}, f(\mathbf{x}))$:



- An extended-value function f is called *closed* if $\text{epi} f$ is closed.

Separating hyperplane theorem

Let A and B be two disjoint nonempty convex subsets of $\mathbb{R}^n$. Then there exist a nonzero vector $\mathbf{v}$ and a real number c such that

$$\mathbf{a}^T\mathbf{v} \le c \le \mathbf{b}^T\mathbf{v}, \quad \forall \mathbf{a} \in A,\ \forall \mathbf{b} \in B.$$

That is, the hyperplane $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{v} = c\}$ normal to $\mathbf{v}$ separates A and B.

If in addition both A and B are closed and one of them is bounded, then the separation is strict, i.e., there exist a nonzero vector $\mathbf{v}$ and real numbers c_1 and c_2 such that

$$\mathbf{a}^T\mathbf{v} < c_1 < c_2 < \mathbf{b}^T\mathbf{v}, \quad \forall \mathbf{a} \in A,\ \forall \mathbf{b} \in B.$$

Thus there exists a hyperplane that strictly separates (y, ζ) and $\text{epi}\varphi$ in the above plot if φ is closed. (why?)

Sublevel sets

- α-sublevel set of an extended-value function f is

$$S_\alpha = \{\mathbf{x} \in \text{dom} f : f(\mathbf{x}) \le \alpha\}.$$

- **If** f is convex, then S_α is convex for all α.
  - Converse is not true: $f(x) = -e^x$.
- Further if f is continuous, then all sublevel sets are closed.

Operations that preserve convexity

1. (Nonnegative weighted sums) If f and g are convex and α and β are nonnegative constants, then $\alpha f + \beta g$ is convex, with $\text{dom} f \cap \text{dom} g$.
   - Extension: If $f(\mathbf{x}, \mathbf{y})$ is convex in $\mathbf{x}$ for each fixed $\mathbf{y} \in \mathcal{A}$, and $w(\mathbf{y}) \ge 0$ for all $y \in \mathcal{A}$, then the integral $g(\mathbf{x}) = \int_{\mathcal{A}} w(\mathbf{y}) f(\mathbf{x}, \mathbf{y}) d\mathbf{y}$ is convex provided the integral exists.
2. (Composition with affine mapping) If f is convex, then composition $f(\mathbf{A}\mathbf{x} + \mathbf{b})$ of f with an affine map $\mathbf{x} \mapsto \mathbf{A}\mathbf{x} + \mathbf{b}$ is convex, with $\text{dom} g = \{\mathbf{x} : \mathbf{A}\mathbf{x} + \mathbf{b} \in \text{dom} f\}$.
3. (Pointwise maximum and supremum) If f_i is convex for each fixed $i = 1, \ldots, m$, then $g(\mathbf{x}) = \max_{i=1,\ldots,m} f_i(\mathbf{x})$ is convex, with $\text{dom} g = \cap_{i=1}^m \text{dom} f_i$.
   - Extension: If $f(\mathbf{x}, \mathbf{y})$ is convex in x, then $g(\mathbf{x}) = \sup_{\mathbf{y} \in \mathcal{A}} f(\mathbf{x}, \mathbf{y})$ is convex, with $\text{dom} g = \{\mathbf{x} : (\forall \mathbf{y} \in \mathcal{A})(\mathbf{x}, \mathbf{y}) \in \text{dom} f, \sup_{\mathbf{y} \in \mathcal{A}} f(\mathbf{x}, \mathbf{y}) < \infty\}$.
4. (Composition) For $h : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ and $g : V \to \mathbb{R} \cup \{\infty\}$, if h is convex and *nondecreasing*, and g is convex, then $f = h \circ g$ is convex, with $\text{dom} f = \{\mathbf{x} \in \text{dom} g : g(\mathbf{x}) \in \text{dom} h\}$.
5. (Paritial minimization) If $f(\mathbf{x}, \mathbf{y})$ is **jointly** convex in $(\mathbf{x}, \mathbf{y})$, then $g(\mathbf{x}) = \inf_{\mathbf{y} \in C} f(\mathbf{x}, \mathbf{y})$ is convex provided it is proper and C is nonempty convex, with $\text{dom} f = \{\mathbf{x} : (\exists \mathbf{y} \in C)(\mathbf{x}, \mathbf{y}) \in \text{dom} f\}$ (projection).
6. (Perspective) If $f(\mathbf{x})$ is convex and *finite*, then its *perspective* $g(\mathbf{x}, t) = t f(t^{-1}\mathbf{x})$ is convex, with $\text{dom} g = \{(\mathbf{x}, t) : t^{-1}\mathbf{x} \in \text{dom} f,\ t > 0\}$.

Examples

- A sum of convex functions is convex.
- $f(\mathbf{x}) = x_{(1)} + x_{(2)} + \cdots + x_{(k)}$, the some of k largest components of $\mathbf{x} \in \mathbb{R}^d$, is convex, because

$$f(\mathbf{x}) = \max\{x_{i_1} + \cdots + x_{i_k} : 1 \le i_1 < i_2 < \cdots < i_k \le n\}.$$

- Maximum eigenvalue of a symmetric matrix $\lambda_{\max} : \mathbb{S}^d \to \mathbb{R}$ is convex, because

$$\lambda_{\max}(\mathbf{X}) = \max_{\mathbf{v}^T\mathbf{v}=1} \mathbf{v}^T\mathbf{X}\mathbf{v}.$$

  (Rayleigh quotient). The maximand is linear (hence convex) in $\mathbf{X}$ for each $\mathbf{v}$.
  - Minimum eigenvalue of a symmetric matrix is concave.
- Sum of k largest eigenvalues of a symmetric matrix is convex, because

$$\sum_{i=1}^k \lambda_i(\mathbf{X}) = \max_{\mathbf{V}^T\mathbf{V} = \mathbf{I}_k, \mathbf{V} \in \mathbb{R}^{d \times k}} \text{tr}(\mathbf{V}^T\mathbf{X}\mathbf{V})$$

  (Ky Fan, 1949).
- Support function of a set: $\sigma_C(\mathbf{x}) = \sup_{\mathbf{y} \in C} \langle \mathbf{x}, \mathbf{y} \rangle$ is convex (with an obvious domain), because $\langle \mathbf{x}, \mathbf{y} \rangle$ is convex (linear) in $\mathbf{x}$ for each $\mathbf{y} \in C$.
  - Ex) dual norm: $\|\mathbf{x}\|_* = \sup_{\|\mathbf{y}\| \le 1} \langle \mathbf{x}, \mathbf{y} \rangle = \sigma_C(\mathbf{x})$, where $C = \{\mathbf{y} : \|\mathbf{y}\| \le 1\}$ is a norm ball.
  - Hence ℓ_p norm is a support function of the unit ℓ_q norm ball, $1/p + 1/q = 1$.
- Matrix perspective function $f(\mathbf{x}, \mathbf{Y}) = \mathbf{x}^T\mathbf{Y}^{-1}\mathbf{x}$ is convex with $\text{dom} f = \mathbb{R}^d \times \mathbb{S}_{++}^d$ because

$$\mathbf{x}^T\mathbf{Y}^{-1}\mathbf{x} = \sup_{\mathbf{z} \in \mathbb{R}^d} \{2\mathbf{x}^T\mathbf{z} - \mathbf{z}^T\mathbf{Y}\mathbf{z}\}.$$

- Kullback-Leibler divergence of two d-dimensional probability vectors

$$D(\mathbf{x}||\mathbf{y}) = \sum_{i=1}^d x_i \log(x_i/y_i)$$

  (taking $0 \log 0 = 0$ and $\log 0 = \infty$ by continuity) is convex on $\Delta_{d-1} \times \Delta_{d-1}$, because
  1. $g(x, t) = t\log(t/x)$ is convex since it is the perspective of $f(x) = -\log x$;
  2. $\tilde{D}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^d (x_i \log(x_i/y_i) - x_i + y_i)$ is convex on $\mathbb{R}_+^d \times \mathbb{R}_+^d$;
  3. $D(\mathbf{x}||\mathbf{y}) = \tilde{D}(\mathbf{x}, \mathbf{y}) + \iota_{\Delta_{d-1}}(\mathbf{x}) + \iota_{\Delta_{d-1}}(\mathbf{y})$.