Ordinary Least Squares (OLS) Estimation

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Introduction

In this document, we derive the Ordinary Least Squares (OLS) estimators for both Simple Linear Regression and Multiple Linear Regression. The OLS method minimizes the sum of squared residuals to find the best-fit parameters.

1. Simple Linear Regression

A Simple Linear Regression model can be expressed as:

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \quad i = 1, 2, \ldots, n$$

where:

- y_i: Dependent variable (response).
- x_i: Independent variable (predictor).
- β_0, β_1: Parameters to estimate.
- ϵ_i: Error term.

The goal is to minimize the sum of squared residuals:

$$S = \sum_{i=1}^{n} (y_i - (\beta_0 + \beta_1 x_i))^2$$

Taking partial derivatives with respect to β_0 and β_1 and setting them to zero gives:

$$\frac{\partial S}{\partial \beta_0} = -2 \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) = 0$$

$$\frac{\partial S}{\partial \beta_1} = -2 \sum_{i=1}^{n} x_i (y_i - \beta_0 - \beta_1 x_i) = 0$$

Solving these equations, the estimators are:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}, \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

2. Multiple Linear Regression

The Multiple Linear Regression model is expressed as:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

where:

- $\mathbf{y}$: $n \times 1$ vector of responses.
- $\mathbf{X}$: $n \times p$ matrix of predictors (with a column of 1s for the intercept).
- $\boldsymbol{\beta}$: $p \times 1$ vector of coefficients.
- $\boldsymbol{\epsilon}$: $n \times 1$ vector of errors.

The OLS objective is to minimize:

$$S = (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})^\top (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})$$

Taking the derivative with respect to $\boldsymbol{\beta}$ and setting it to zero:

$$\frac{\partial S}{\partial \boldsymbol{\beta}} = -2\mathbf{X}^\top (\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) = 0$$

Solving for $\boldsymbol{\beta}$:

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{y}$$

Conclusion

The OLS method provides a straightforward approach to estimating regression coefficients by minimizing the residual sum of squares. The derived estimators for both Simple and Multiple Linear Regression are widely used in statistical modeling.