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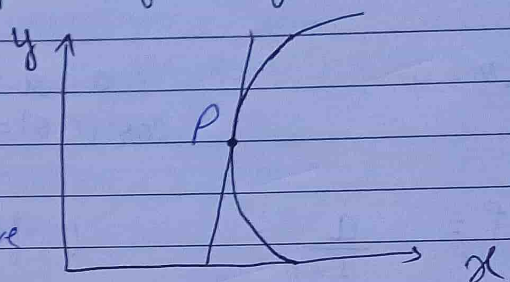
$$\rho = p + \frac{d^2p}{d\psi^2} \Rightarrow p - \frac{a^2}{4p^3} \Rightarrow \frac{4p^4 - a^2}{4p^3}$$

Concavity, Convexicity and point of inflexation:

A function of single variable is concave wrt x axis if every line segment joining two points on a graph on its curve doesn't lies above the graph.



i.e the arc of the curve containing the point P lies wholly inside the acute angle b/w the tanget at P and x-axis.

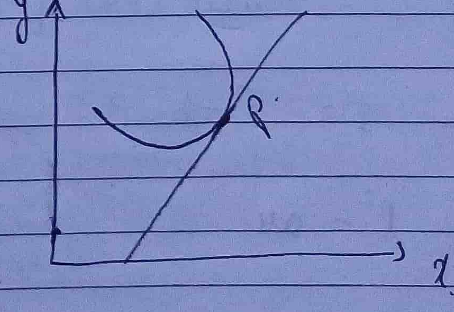
Criteria for concavitity:

1) Curve lie above the x axis, $y > 0$

$$y\frac{d^2y}{dx^2} < 0$$

Curve is lying below the x axis, $y < 0$

$$y\frac{d^2y}{dx^2} < 0$$



· Convexcity:

whole arc of the curve containing the point P lies only outside the tangent & P obtuse angle.

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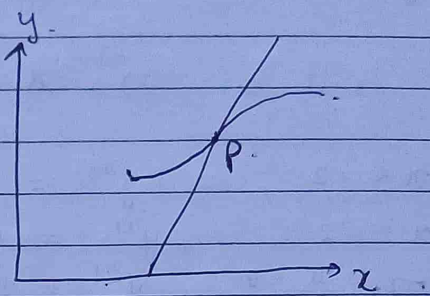
Criteria of Convexcity :
Curve lies above the x axis $y > 0$

$y \dfrac{d^2y}{dx^2} > 0$



Inflexion
Graph have regions which have concave and convex, thus there are often points at which the graph change from being concave to convex & vice versa these points are point of inflexion.

Criteria :
P = point of inflexion, convex on one side & concave on other side which is possible only when

$\dfrac{d^2y}{dx^2} = 0$, $\dfrac{d^3y}{dx^3} \neq 0$

Q) Show that curve $y^2 = 4x$ is concave wr.t x axis and convex wrt y axis at pt P (1, 2)

Wrt x-axis $2y \dfrac{dy}{dx} = 4$

$\dfrac{d^2y}{dx^2} = \dfrac{-2}{y^2} \times \dfrac{2}{y} \Rightarrow -\dfrac{1}{2} < 0$

$y \times \dfrac{d^2y}{dx^2} \Rightarrow -1 < 0$

Concave

wrt y-axis $2y = 4\dfrac{dx}{dy}$

$\dfrac{d^2x}{dy^2} = \dfrac{1}{2} \times \dfrac{y}{2} > 0$

$x \dfrac{d^2x}{dy^2} = \dfrac{1}{2} > 0$

Convex

Ans = (0, 2, -2)

① Find pt. of inflexion of curve $3x^5 - 40x^3 + 3x - 20 = 0$

$y = 3x^5 - 40x^3 + 3x - 20$

$y' = 15x^4 - 120x^2 + 3$, $y'' = 60x^3 - 240x = 0$

$\Rightarrow x = 0, \quad x = \pm 2$

$x < -2$, $y'' = -900$ (concave down)

$x = -2$, $y'' = 0$ (inflection point) ✓

$-2 < x < 0$, $y'' = 180$ (concave up)

$x = 0$, $y'' = 0$ (inflection) ✓

$0 < x < 2$, $y'' = -180$ (concave down)

$x = 2$, $y'' = 0$ (inflection) ✓

$x > 2$, $y'' = 900$ (concave up)

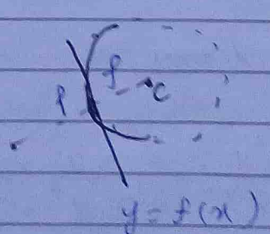
Point of inflection $\Rightarrow$ $(-2, 198)$, $(0, -20)$, $(2, -238)$

Centre and Circle of Curvature

Let $P(x,y)$ be a point on the curve $y = f(x)$, the normal drawn in the direction to the curve, cuts of the length $PC = \rho$ = radius of the curvature at P.

The point C is called centre of curvature at P

Radius drawn is circle of curvature at P



Let $(\bar{x}, \bar{y})$

$\cancel{\bar{y} = x - \frac{y'(1+y'^2)}{y''}}$

$\bar{x} = x - \frac{y'(1+y'^2)}{y''}$

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$$\bar{y} = y + \frac{(1+y'^2)}{y''}$$

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

Q Find the centre of curvature at point (x,y) of the parabola $y^2 = 4ax$

$C(\bar{x}, \bar{y})$

$$2y\frac{dy}{dx} = 4a$$

$$y' = \frac{2a}{y}$$

$$y'' = -\frac{2a}{y^2} \times \frac{2a}{y} \Rightarrow \frac{-4a^2}{y^3}$$

$$\bar{x} = x - \frac{y'(1+y'^2)}{y''}$$

$$\bar{x} = \frac{y^2}{4a} + \frac{\frac{2a}{y}\left(1+\frac{4a^2}{y^2}\right)}{\frac{4a^2}{y^3}}$$

$$\bar{x} \Rightarrow \frac{3y^2+8a^2}{4a} \Rightarrow \underline{\underline{3x+2a}}$$

$$\bar{y} = y + \frac{\left[1+\left(\frac{2a}{y}\right)^2\right]}{-\frac{4a^2}{y^3}} \Rightarrow$$

$$\bar{y} = y - \frac{(y^3 + 4a^2y^2)}{4a^2} \Rightarrow \frac{y^3}{2a}$$

$$\left(x - \frac{y^2+4a^2}{2a}, \frac{y^3}{2a}\right)$$

Chord of Curvature :

① Chord of the curvature at 'P' ||el to x axis

$$= \frac{2y'(1+y'^2)}{y''}$$

② ||el to y axis $= \frac{2(1+y'^2)}{y''}$

Q) Show that the chord of curvature at $(0,0)$ ||el to y axis on the curve $y = mx + \frac{x^2}{a}$ is $(1+m^2)a$.

$y' = m + \frac{2x}{a}$ $\quad y'' = \frac{2}{a}$.

||el to y axis $= \frac{2\left(1 + m^2 + \frac{4x^2}{a^2} + \frac{4mx}{a}\right)}{\frac{2}{a}}$

at $(0,0)$

Chord $= (1+m^2)a$

Curve Tracing

Multiple points = a point on a curve through which more than two branch of curve pass is called a multiple points.

Double points = A point on curve is called double point if only two branch of curve passes through it.

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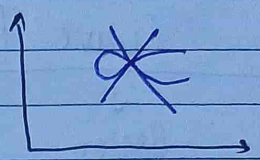
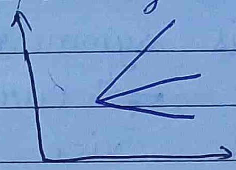
Types of Double points :

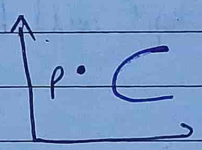
(1) Node : If the two branches through a double point on the curve are here and the tangent to them are not coincident.



(2) Cusp = If the two branches through a double point on a curve are here and the tangent to them are coincident.



(3) Conjugate = If there are no real point on the curve in the never hold of a point P . P is called as an isolated point



Condition for a double point

Let $f(x,y) = 0$ eq^n of a curve, $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$

If $\frac{\partial^2 f}{\partial x \partial y} \geq 0$, then point is double points. (Node, Cusp)

$\frac{\partial^2 f}{\partial x \partial y} < 0$, then point is conjugate.

Method of Tracing Curve :

When the eq^n is in Cartisian form:

(1) Symmetry about a x-axis = if eq^n remains the same while replacing $y = -y$, then y should have even powers.

(iii) Symmetry about y-axis = replace $x = -x$ and x should be of even powers.

Curve Tracing.

Point of intersection | Region | Tangent | Asymptotes | Origin | Symmetry

Curve is symmetric above line $y = x$. if the equation of the ~~eq^n~~ curve remains unchanged. by replacing $y = x$ vice versa for $y = -x$.

$$x^3 - y^3 = 3axy$$ [replace it by $y = -x$.]

eq^n will be same, Eq^n is symmetric

Procedure for curve Tracing

① Nature of origin =

$f(0,0) = 0$ is stasify i.e $0 = 0$ then the curve passes from origin.

② Tangent = If $f(x, y) = 0$ equating the lowest degree term to 0, gives the tangent at origin, ~~its double point~~.

- If this result is quadratic eq^n, the origin is double point.
- If the eq^n has 2 real and distinct factors that is tangent are real and diff.
- Origin is cusp if eq^n has 2 real and coincident factors i.e. tanget are real and same factors.

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Imaginary roots then origin is a conjugate points

Tangent at Intersection point.
Suppose $(a, 0)$ is a intersection point with x-axis shift the origin to $(a, 0)$ by putting

$y = Y, \quad x = X + a.$

and then calculate the lowest degree term in the equation to zero

Point of intersection with coordinate axis :
* Put $y = 0$ in the given eqⁿ and determine the interesection point ~~of~~ the curve with x-axis.
* If we put $x = 0$ in the eqⁿ, determine intersection point with y axis

Q 1) $y^2(2a - x) = x^3$
* Step 1) Symmetry)

Since the eqⁿ of the curve has only even power in 'y' so the curve is symmetric about x axis
* Origin at $(0, 0)$ eqⁿ will be $0 = 0$ so curve passes through origin - calucating lowest degree term to zero,

$2ay^2 = 0 \Rightarrow y = 0, 0$

Thus, the curve has 2 coincident tangent at origin $y = 0$, that's x axis, so origin is a Cusp.

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Asymptotes

||el to x axis, $(2a-x)=0 \Rightarrow x=2a$

||el to y axis, coff. of x^3 is constant, so curve has no ||el asym. to y axis

Curve has no oblique asymptotes.

Curve Intersect the coordinate at origin only.

Reason: Solving the eqn for y ~~axis~~

$y^2(2a-x)=x^3$ $\quad y^2=\dfrac{x^3}{(2a-x)}$

$y = \pm\sqrt{\dfrac{x^3}{2a-x}}$

Here, y is imaginary when, $\dfrac{x^3}{2a-x}$ is negative.

$(x<0$ or $x>2a)$

Region $(0 \le x < 2a)$

Plotting the points:

$y^2(2a-x)=x^3$

$(2ay^2 - y^2x = x^3 \Rightarrow 4ayy' - 2yy'x + y^2 = 3x^2)^*$

$y^2 = \dfrac{x^3}{2a-x} \Rightarrow 2y \times \dfrac{dy}{dx} = \dfrac{3(2a-x)x^2 - x^3(-1)}{(2a-x)^2}$

$\dfrac{dy}{dx} = \dfrac{1 \times 6ax^2 - 3x^3 + x^3}{2y(2a-x)^2} \Rightarrow \dfrac{2(3ax^2 - x^3)}{2y(2a-x)^2}$

$\dfrac{dy}{dx} = \dfrac{x^2(3a-x)}{y(2a-x)^2}$

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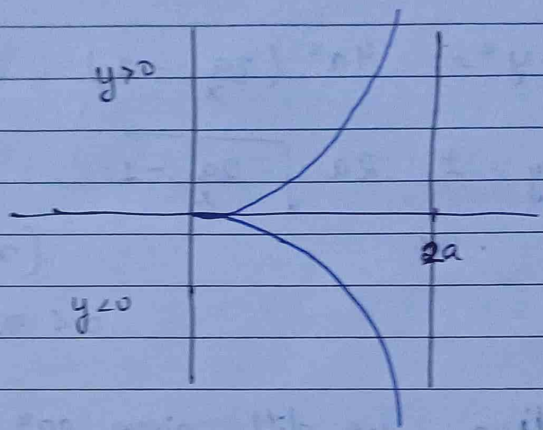
for $y>0$, $0\le x<2a$
$\frac{dy}{dx}>0$ Curve is increasing.

if $y<0$ $\frac{dy}{dx}<0$ Curve is decreasing



Q2) $xy^2 = 4a^2(2a-x)$

Step 1) Symmetry

y has even powers, so it is symmetrical to x-axis.

Step 2) Origin (0,0)

$0 \neq 8a^3$

Curve doesn't passes through origin.

Step 3) Asymptotes

||el to x-axis , y-axis

$xy^2 - 8a^3 + 4a^2x = 0$

$\boxed{x=0}$ ||el to y axis

$xy^2 = 4a^2(2a-x)$

$\cancel{y^2 = 4a^2\left(\frac{2a}{x}-1\right)}$

$x(y^2+4a^2) = 8a^3$

$y^2+4a^2 = 0$

y = imaginary so no oblique asy.

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One asymptotes $x=0$

for $y=0$ $\quad 8a^3 = 4a^2 x.$

$x = 2a$

Region $y^2 = 4a^2\left(\frac{2a}{x} - 1\right)$

$$y = \pm\, 2a\sqrt{\frac{2a}{x} - 1}$$

[$x=0$, $y=$ not defined.]

$0 < x \leq 2a.$

For plotting we diff given eqⁿ wrt x

$$2y\,y' = 4a^2\,\frac{x(-1) + 1(2a-x)}{x^2}$$

$$y' = \frac{2a^2}{y}\left[\frac{-x + 2a - x}{x^2}\right]$$

=