McGill U2S1 Lectures

Hy Vu

Honours Analysis I

Lecture Notes

Prof: Dr. Hundemer

McGill

McGill Honour Analysis Lectures (MATH 254)

Hy Vu

Professor: Dr. Axel Hundemer

August 2024

Foreword

Mathematics & Statistics (Sci) : Properties of $\mathbb{R}$. Cauchy and monotone sequences, Bolzano–Weierstrass theorem. Limits, limsup, liminf of functions. Pointwise, uniform continuity: Intermediate Value theorem. Inverse and monotone functions. Differentiation: Mean Value theorem, L'Hospital's rule, Taylor's Theorem.

Term: FALL 2024

Prerequisites: MATH 141

Contents

1 Preliminaries

Notations Used Frequently:

- (Natural Numbers) $\mathbb{N} = \{1, 2, 3, \dots\}$.
  If 0 is included, we write $\mathbb{N}_0$.
- (Integers) $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$.[1]
- (Rational Numbers) $\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N}\right\}$.
- (Real Numbers) $\mathbb{R} = \{x : -\infty < 1 < \infty\}$.
- (Complex Numbers) $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, i = \sqrt{-1}\}$.
- (Closed Interval) $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$.[2]
- (Open Interval) $]a, b[:= \{x \in \mathbb{R} : a < x < b\} = (a, b)$.

Remark 1.1. *The symbol ":" means "such that" which can also be written as "|" or "s.t."*

1.1 Sample Proofs

Definition 1.1. A **direct proof** is a proof where we start with a true statement and derive from this via a finite sequence of implications ($\implies$) the desired result.

Example 1.1.1. Show that $x \mapsto x^2$ is strictly increasing on $[0, \infty[$. In other word, show that $f : [0, \infty[\to \mathbb{R}$, $f(x) = x^2$ is strictly increasing. [3]
Before beginning the proof, let's establish some background. **Under what condtion** is $x \mapsto x^2$ strictly increasing on $[0, \infty[$? Well...the condition is as follows $\forall 0 \leq x < y : x^2 < y^2$

[1] Z stands for "Zahl" which is number in German.
[2] ":=" means "equal by definition"
[3] "$\mapsto$" means "maps to". The interval $[0, \infty[$ can also be written $\mathbb{R}_0^+$ (which is read "all positive real number including 0").

Proof. Let $0 \le x < y$, then $y^2 - x^2 = \overbrace{(y-x)}^{x<y} \cdot \overbrace{(y+x)}^{x,y\ge 0}$. Because of the condition shown, both results will be > 0 which means their multiplication is also > 0. $\implies y^2 > x^2 \implies x^2 < y^2$ i.e. f is strictly increasing. □

Definition 1.2. A **proof by contradiction** is a proof where we assume the opposite of the desired result. This leads to a contradiction via a finite sequence of implication.

Example 1.1.2. Prove that $\sqrt{2}$ is irrational.
Historical Background: This was proven by Euclid around 300BC and it is one of the most famous arithmetic proof. This can be considered the first proof of mathematics ever.

Proof. Assume that $\sqrt{2}$ is rational then $\exists a, b \in \mathbb{N} : \gcd(a, b) = 1$ and $\sqrt{2} = \frac{a}{b}$.

$$\implies 2 = \frac{a^2}{b^2}$$
$$\implies 2b^2 = a^2$$

The left hand side is even then right hand side is also even which means a^2 is even $\implies$ a is even which means $\exists c \in \mathbb{N} : a = 2c$

$$\implies 2b^2 = (2c)^2 = 4c^2$$
$$\implies b^2 = 2c^2$$

So now the right hand side is even which means $b^2 \implies b$ is even. So both a and b are even $\lightning$ since $\gcd(a, b) = 1$.[4] Thus, the assumption is wrong which means $\sqrt{2}$ is irrational. □

Definition 1.3. Let $P(n)$ be a statement on natural numbers. A **proof by induction** is a proof where we can prove that:

- P(1) i.e. P holds for $n = 1$.
- $\forall n \in \mathbb{N} : P(n) \implies P(n+1)$ i.e. if P holds for n then it also holds for $n+1$.

Then $P(n)$ holds $\forall n \in \mathbb{N}$.

Example 1.1.3. Prove that $\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

[4] "$\lightning$" is a symbol used when there's a contradiction in the proof. Another symbol used for this can be $\lightning$.

Proof. We will prove by induction:

- $(n = 1:)\ 1 = \frac{1\cdot 2}{2} = 1$
- $(n \implies n+1:)$ Assume the formula holds for some $n \in \mathbb{N}$ (*inductive hypothesis*). We have to show the formula holds for $n+1:$

$$\begin{aligned} 1+2+\cdots+n+(n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} = \frac{(n+1\,[(n+1)+1])}{2} \end{aligned}$$

Thus $\sum_{i=1}^{n} i = 1+2+\cdots+n = \frac{n(n+1)}{2}\ \forall n \in \mathbb{N}$. □

End of Lecture

Remark 1.2. *When finishing a proof, we must place either* □, ■ *or just Q.E.D (stands for quod erat demonstrandum) at the end.*

Lecture 2: September 4^{th}, 2024

Theorem 1.1. ***(Bernoulli's Inequalities).*** $\forall x \geq -1, \forall n \in \mathbb{N} : (1+x)^n \geq 1+nx$

Proof. Induction on n.
Base case $(n = 1)$: $(1+x)^1 = 1+x = 1+1\cdot x$ i.e. $(1+x)^1 = 1+1\cdot x$ especially $(1+x)^1 \geq 1+1\cdot x$.
$n \to n+1$: Assume the inequalities holds for some $n \in \mathbb{N}$ i.e.

$$\begin{aligned} (1+x)^n \geq 1+nx \implies (1+x)^{n+1} &= \underbrace{(1+x)^n}_{\geq 1+nx} \cdot \underbrace{(1+x)}_{\geq 0} \\ &\geq (1+nx)(1+x) \\ &= 1+x+nx+nx^2 \\ &= 1+(n+1)x+nx^2 \\ &\geq 1+(n+1)x \end{aligned}$$

$\implies (1+x)^{n+1} \geq 1+(n+1)x$ □

Remark 1.3. *For a specific case of* $x = 1$, *then* $\forall n \in \mathbb{N} : 2^n \geq n+1 \iff \forall n \in \mathbb{N} : n < 2^n$

Theorem 1.2. ***(The Well-Ordering Principle of N).*** *Every non-empty subset of $\mathbb{N}$ has a least element (or minimal element or minimal) i.e. $\exists s \in S, \forall t \in S$: $s \leq t$.*

Remark 1.4. *Most other sets of numbers do not have this property e.g. $\mathbb{Z}$, $S = \{\ldots, -3, -2, -1, 0\}$.*

Example 1.1.4. Continuing from remark 1.3, we can have $\mathbb{Q}$, $S = \{\frac{1}{n} : n \in \mathbb{Q}\}$. In this case, numbers gets arbitrarily close to 0 i.e. it does not have a minimal elements (0 is not part of this set).
Claim: S does not have a least element

Proof. (By Contradiction) Assume $s \in S$ is a least element. Then $s > 0$. Let $n \in \mathbb{N}$ s.t. $n > \frac{1}{s}$. Then, $\frac{1}{n} < s \implies s$ is not a least element $\lightning$. Thus S does not have a least element. □

Proof. (To Theorem 1.2) We'll first consider 2 cases:

1. S is a finite subset of $\mathbb{N}$: We will prove the result by induction on $n := |S|$. [5]

   - <u>Base case ($n = 1$):</u> then $S = \{a_1\}$ for some $a_1 \in \mathbb{N}$. Then a_1 is the least element of S.
   - <u>$n \to n+1$:</u> Ind. Hyp: Every subset of $\mathbb{N}$ of cardinality n has a least element for some $n \in \mathbb{N}$. Let $S = \{a_1, a_2, \ldots, a_n, a_{n+1}\} \subseteq \mathbb{N}$. Let $S' := \{a_1, a_2, \ldots, a_n\}$, by inductive hypothesis, S has a least element s. If $s < a_{n+1}$ then, $\forall 1 < k \leq n+1 : s \leq a_k$ so s is the least element. If $a_{n+1} < s$ then, $\forall 1 \leq k \leq n : a_{n+1} < s \leq a_k \implies \forall 1 \leq k \leq n+1, a_{n+1} \leq a_k \implies a_{n+1}$ is the least element of S.

   So in either case, S has a least element. We've now proven for all finite subsets of $\mathbb{N}$.

2. S is infinite: let $t_0 \in S$ be arbitrary and let $S' := S \cap \{1, 2, 3, \ldots, t_0\} \implies S'$ is finite and non-empty (since t_0 is in the intersection). Because S' is finite, by first case, S' has a least element s. We will now prove that s is also the least element of S. Let $t \in S$ be arbitrary

   (a) If $t \in S' \implies s \leq t$ since s is the least element of S'.
   (b) If $t \notin S'$, Then $t > t_0$ and $\underbrace{s \leq t_0}_{t_0 \in S'} < t$ and s is the least element of $S' \implies s < t \implies s \leq t$.

[5] |...| is called the cardinality or number of element of a set.

In either case, $s \leq t \implies s$ is the least element of S.

Therefore, no matter if S is finite or infinite, it will have a least element. □

1.2 Brief Introduction to Logic

Definition 1.4. A **statement** is any expression that is either true or false

Example 1.2.1. "7 is a prime" is a true statement. "27 is a prime" is a false statement.

Definition 1.5. **Compound statement**: Let p and q be statements then $P \wedge Q$, called "P and Q", is true iff both P and Q are true. Similarly $P \vee Q$, called "P or Q", iff at least 1 P, Q is true. See the **truth table** below

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Definition 1.6. $\neg P$, called "not P", is true iff P is false. See truth table below

P	$\neg P$
T	F
F	T

Theorem 1.3. ***(De Morgan's Law).*** *Let P and Q be statements then*

(a) $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$. [6]

(b) $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$.

Proof. We will only prove (a) using truth table:

P	Q	$P \vee Q$	$\neg(P \wedge Q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

As we can see, truth table for $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ is equivalence. □

[6] $\equiv$ means logically equivalent i.e. both sides have the same truth table.

Definition 1.7. **Implications** is denoted as $P \implies Q$ which is read as "if P then Q". This is given as the truth table follows:

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

The last 2 lines of the truth table is called *Ex Falso Quod Libet* i.e. "from falsehood, anything follows".

Example 1.2.2. $x = y \implies x^2 = y^2$ is true. Now let's plug in specific numbers

(a) $x = 1, y = -1$: $\underbrace{1 = -1}_{\text{false}} \implies \underbrace{1^2 = (-1)^2}_{\text{true}}$ is true.

(b) $x = 1, y = 2$: $\underbrace{1 = 2}_{\text{false}} \implies \underbrace{1^2 = 2^2}_{\text{true}}$ is true.

Exercise: Prove the following, $P \implies Q \equiv \neg P \vee Q$ and $\neg(P \implies Q) \equiv \neg((\neg P) \vee Q)$

Proof. We will prove the second one first directly from definition

$$\begin{aligned} \neg(P \implies Q) &\equiv \neg((\neg P) \vee Q) \\ &\equiv (\neg\neg P) \wedge Q(\neg Q) \equiv P \wedge (\neg Q) \end{aligned}$$

For the first one, we'll prove using truth table:

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$\neg P$	$\neg P \vee Q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

As we can see, the truth table for $P \implies Q$ and $\neg P \vee Q$ is equivalence hence they're equivalence. □

End of Lecture

Remark 1.5. *Another important result is the invert between the statement "for all" and "there exists":* $\neg(\forall x : P(x)) \equiv \exists x : \neg P(x)$ *and similarly* $\neg(\exists x : P(x)) \equiv \forall x : \neg P(x)$.

Lecture 3: September 9th, 2024

1.3 Brief Introduction to Set Theory

Definition 1.8. A **set** is a collection of distinct objects called **elements**. Usually a set is represented as an upper case letter (e.g. A, B, etc.) while it's elements are lower case letters.

Example 1.3.1. $A = \{a_1, a_2, a_3\}$ is a set.

Remark 1.6. *A set is specified by either: listing its elements (if possible) or by stating a property that its elements have to satisfy.*

Definition 1.9. Let A and B be sets then, the **union** of A is B is defined and written as:

$$A \cup B = \{x : x \in A \text{ or } x \in B\} \tag{1.1}$$

and the **intersection** of A and B is defined as:

$$A \cap B = \{x : x \in A \text{ and } x \in B\} \tag{1.2}$$

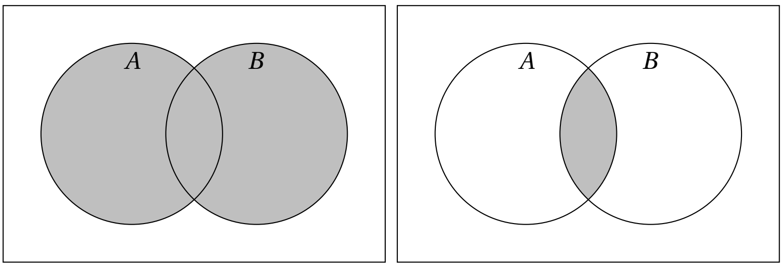


Figure 1.1: Left figure represent set union while the right represent set intersection.

Example 1.3.2. Let $A = \{1,2,3\}$, $B = \{1,3,5\}$ and $C = \{2,4,6\}$, then $A \cap B = \{1,3\}$. However, when $B \cap C$, we will see that there's no shared element which means this will create a **empty set** symbolized as Ø.

Example 1.3.3. Using the same sets: A, B and C then we can see that $A \cup B = \{1,2,3,5\}$.

Definition 1.10. Let A and B be sets. Then, A is contained or is a **subset** of B if every elements in A is in B. This relation is denoted as

$$A \subset B \tag{1.3}$$

Example 1.3.4. Let $A = \{1,2,3\}, B = \{1,3,5\}, C = \{2,4,6\}$ and $D = \{1,2,3,4,5,6\}$, then $A \subset D, B \subset D$ and $C \subset D$.

Definition 1.11. If a set contain all elements under consideration in a particular context, we call it a **universal set (S)**.[7]

Definition 1.12. If a set consists of no elements, we call it an **empty set** $(\emptyset)$.

Definition 1.13. Let $A \subset S$. Then, rthe **complement** of A, written as A^c or A' is the set of elements in S but not in A. That is,

$$A' = \{a \in S : a \notin A\} \tag{1.4}$$

Definition 1.14. For unions and intersection of finitely many sets. Then, let $n \in \mathbb{N}$, The **union of finitely many sets** is given as

$$A_1 \cup A_2 \cup \cdots \cup A_n := \{x : x \in A_1 \lor x \in A_2 \lor \cdots \lor x \in A_n\} \tag{1.5}$$

or simply

$$A_1 \cup A_2 \cup \cdots \cup A_n := \{x : \exists 1 \le i \le n : x \in A_i\} = \bigcup_{i=1}^{n} A_i \tag{1.6}$$

We can also define similarly for the **intersection of finitely many sets** as

$$A_1 \cap A_2 \cap \cdots \cap A_n := \{x : x \in A_1 \land x \in A_2 \land \cdots \land x \in A_n\} \tag{1.7}$$

or simply

$$A_1 \cap A_2 \cap \cdots \cap A_n := \{x : \forall 1 \le i \le n : x \in A_i\} = \bigcap_{i=1}^{n} A_i \tag{1.8}$$

Definition 1.15. For unions and intersection of infinitely many sets. Then, let $A_1, A_2, \dots$ be sets, the **infinite union** of these is defined as

$$A_1 \cup A_2 \cup \cdots = \bigcup_{i=\mathbb{N}} A_i := \{x : \exists i \in \mathbb{N} : x \in A_i\} \tag{1.9}$$

And we define the **infinite intersection** in the same manner

$$A_1 \cap A_2 \cap \cdots = \bigcap_{i=\mathbb{N}} A_i := \{x : \forall i \in \mathbb{N} : x \in A_i\} \tag{1.10}$$

Remark 1.7. *We can definite unions or intersections for any (non-empty) index set*

[7] Some other notation of universal set can be: U, ξ or $\mathscr{U}$

Example 1.3.5. $\bigcap_{x\in\mathbb{R}^+}[0,x] = \{0\}$

Proof. We must show that RHS⊆LHS and vice versa. $\forall x > 0 : 0 \in [0,x] \implies 0 \in \bigcap_{x>0}[0,x]$. We now need to show that LHS⊆RHS. Let $x_0 \in \bigcap_{x>0}[0,x] \implies x_0 \geq 0$. Assume $x_0 > 0$, let $x := \frac{x_0}{2}$ then $0 \leq x < x_0 \implies x_0 \notin [0,x] \implies x \notin \bigcap_{x>0}[0,x]$↯. This means $x_0 = 0$.
Since $\bigcap_{x\in\mathbb{R}^+}[0,x] \subseteq \{0\}$ and $\{0\} = \bigcap_{x\in\mathbb{R}^+}[0,x]$ Thus $\bigcap_{x\in\mathbb{R}^+}[0,x] = \{0\}$ □

Definition 1.16. Let A and B be sets then the **complement of A and B** is given as

$$A\backslash B := \{x : x \in A \wedge x \notin B\} \tag{1.11}$$

Theorem 1.4. ***(De Morgan's Laws)***, *Let A, B and C be sets then*

1. $A\backslash(B \cup C) = (A\backslash B) \cap (A\backslash C)$

2. $A\backslash(B \cap C) = (A\backslash B) \cup (A\backslash C)$

Proof. Proof will be given in tutorials. □

Definition 1.17. Generalization of the complement for infinitely many set is give as

$$A\backslash \bigcup_{i=S} B_i = \bigcap_{i\in S}(A\backslash B_i) \tag{1.12}$$

and

$$A\backslash \bigcap_{i=S} B_i = \bigcup_{i\in S}(A\backslash B_i) \tag{1.13}$$

1.3.1 Function

Definition 1.18. Let D and E be sets, a **function** $f : D \to E$ is a rule that assigns to each $x \in D$ and uniquely determined $y \in E$. If $y \in E$ is assigned to x via f, we write

$$y = f(x) \tag{1.14}$$

The set D is called the **domain** and E is called the **codomain**.[8]

Remark 1.8. *$f(D)$ can be a true subset of E i.e. $f(D) \subseteq E$ and $f(D) \neq E$. We can notate it as $f(D) \subsetneq E$ or $f(D) \subset E$.* [9]

Example 1.3.6. $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$. $f(\mathbb{R}) = \mathbb{R}_0^+ = [0,\infty[\subsetneq \mathbb{R}$

[8] There are other way to call a codomain such as range, image, targets.

[9] We will not use ⊂ since some misinterpret it that ⊂ and ⊆ are interchangeble.

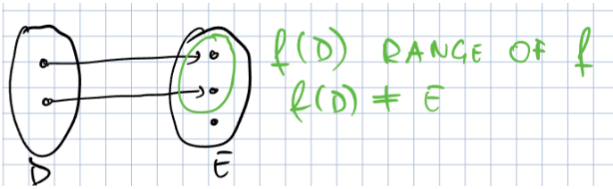


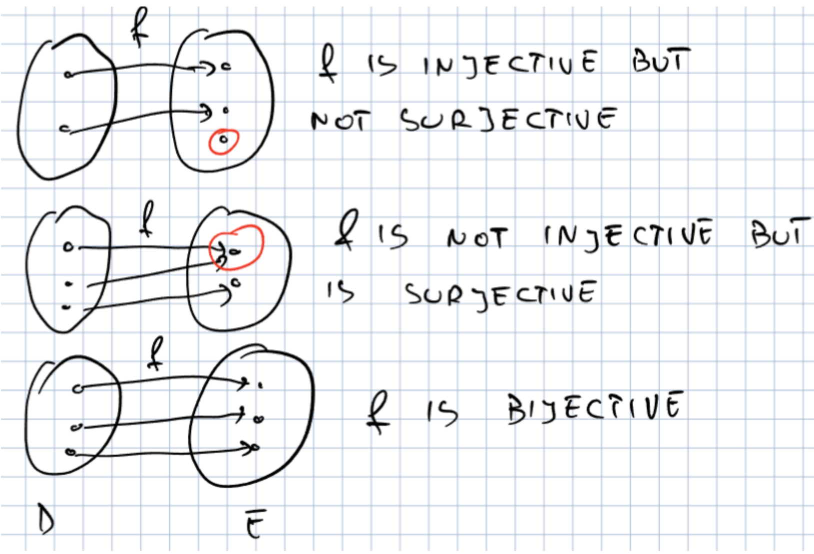
Figure 1.2: Range (image) vs codomain

Definition 1.19. A function $f : D \to E$ where $f(D) = E$ is called **surjective**.

Definition 1.20. Let $f : D \to E$ be a function such that $x_1 \neq x_2 \implies f(x_1) \neq f(x_2), \forall x_1, x_2 \in D$ is called **injective (one-to-one)**.

Definition 1.21. A function that is both injective and surjective is called **bijective**.

Example 1.3.7. The followings are injective, surjective and bijective functions:



Theorem 1.5. *A function is invertible (i.e. has an inverse function) iff it is bijective*

The Absolute Value Function

Definition 1.22. An **absolute value function** $f : \mathbb{R} \to \mathbb{R}_0^+, x \mapsto |x|$ is defined as

$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} \tag{1.15}$$

Theorem 1.6. ***(Properties of function):*** $\forall x, y \in \mathbb{R}$:

1. $|x| = \sqrt{x^2}$
2. $|xy| = |x| \cdot |y|$
3. $-|x| \leq x \leq |x|$
4. $|x + y| \leq |x| + |y|$ ***(Triangle inequalities)***

Proof. We'll prove only the 2. $|xy| = \sqrt{(xy)^2} = \sqrt{x^2 \cdot y^2} = \sqrt{x^2} \cdot \sqrt{y^2} = |x| \cdot |y|$. □

1.4 Completeness

We've essentially done with all of all preliminaries, so now we'll be moving toward closer concept of analysis.

Definition 1.23. A non-empty set $S \subseteq \mathbb{R}$ is said to be **bounded from above** if $\exists u \in \mathbb{R} : \forall s \in S, s \leq u$. Any such u is called the **upper bound of S**.

Example 1.4.1. $S = [0, 1]$ then any number $u \geq 1$ is an upper bound. Contrarily, $S := [0, \infty[$ is not bounded from above.

Proof. Assume an upper bound u exists. Then, $u \geq 0 \implies u + 1 \geq 0 \implies u + 1 \in S$ but $u < u + 1$, $\lightning$ to the definition of upper bound. Thus S does not have an upper bound. □

Remark 1.9. *Upper bound is not uniquely defined as there can be an upper bound larger than another upper bound.*

Definition 1.24. Similarly, $S \subseteq \mathbb{R}$ is called **bounded from below** if $\exists v \in \mathbb{R} : \forall s \in S, s \geq u$. Any such v is called a **lower bound of S**.

End of Lecture

Definition 1.25. Let $S \neq \emptyset$ and $S \subseteq \mathbb{R}$. We say that

Lecture 4: September 11^{th}, 2024.

(a) u is the **maximum/greatest element** of S, if u is an upper bound of S and $u \in \mathbb{S}$. We denote it as $u = \max S$.

(b) v is the **minimum/least element** of S, if v is a lower bound of S and $v \in \mathbb{S}$. We denote it as $u = \min S$.

Example 1.4.2. $S = [a, b[$. we will show that $a = \min S$, but that does S does not have a maximum.

Lemma 1.1. *If $x < y$ then $x < \frac{x+y}{2} < y$ where $\frac{x+y}{2}$ is caled the* ***arithmetic mean*** *of x and y.*

Proof. $x = \frac{1}{2}x + \frac{1}{2}x < \frac{1}{2}x + \frac{1}{2}y = \frac{x+y}{2} < \frac{1}{2}y + \frac{1}{2}y = y \implies x < \frac{x+y}{2} < y.$ □

Remark 1.10. *This shows that between each 2 non equal real numbers. There always lies another real number. Especially there aer no "immediate neighbours" on the number line.*

In fact, between any 2 real numbers, there even lie infinitely many real numbers.

Proof. To be proven in exercise. The general idea is that you keep halfing the distance between 2 numbers etc. □

Returning back to example 1.4.2

Proof. ($a = \min S$ for $[a, b[$). Let $x \in [a, b[$. Then especially $a \leq x$ i.e. a is a lower bound of S; also, $a \in S \implies a \min S$. □

Proof. (S does not have a max). Assume that S has a max and let m be this max. Then $m \in [a, b[$ i.e. $a \leq m < l$. Now consider $m' = \frac{m+b}{2}$. By lemma 1.1, we have that $a \leq m < m' < b \implies m' \in [a, b[$ and $m < m' \implies m$ is the the max of $[a, b[$ $\lightning$. Thus S does not have a max. □

Definition 1.26. Consider 2 situtations:

(a) Let $S \neq \emptyset$ and $S \subseteq \mathbb{R}$ be bounded from above. We say that $s \in R$ is the **supremum/least upper bound** of S if

  (i) s is an upper bound of S, and

  (ii) $s \leq u$ for all upper bounds u of S.

(b) Let $S \neq \emptyset$ and $S \subseteq \mathbb{R}$ be bounded from below. We say that $t \in R$ is the **infimum/greatest lower bound** of S if

(i) t is a lower bound of S, and

(ii) $t \geq u$ for all lower bounds u of S.

We denote supremum and infimum of S as: $\sup S$ and $\inf S$ respectively.

Example 1.4.3. $S = [a, b[$. We will show that $\inf S = a$ and $\sup S = b$.

Proof. First, we will show for infimum:

(i) Let $x \in [a, b[$. Then, $a \leq x \implies a$ is a lower bound of $[a, b[$.

(ii) Now let u be an arbitrary lower bound of $[a, b[$. Then, $\forall x \in [a, b[: v \leq x$. Especially $v \leq a \implies a \inf S$

Now we will show for supremum:

(i) Let $x \in [a, b[$. Then, $x < b \implies b$ is an upper bound of $[a, b, [$.

(ii) Let u be an arbitrary upper bound of $[a, b[$. Assume that $u < b$. Then, $a \leq u < b$ thus by lemma 1.1., $a \leq u < \underbrace{\frac{u+b}{2}}_{\in [a,b[} < b \implies u$ is not an upper bound of $[a, b[\lightning \implies u \geq b \implies b = \sup S$.

□

Theorem 1.7. *Let $S \neq \emptyset$ and $S \subseteq \mathbb{R}$. Then,*

(a) If S has a maximum s, then s is also $\sup S$ (but the converse does not hold).

(b) If S has a minimum t, then t is also $\inf S$ (but the converse does not hold).

Proof. (a) Let $s = \max S \implies s$ is an upper bound of $s \implies$ definition 1.26.a(i) holds. Let u be any upper bound of S. Then $\forall x \in S : x \leq u$, Especially, since $s \in S$, it follows that $s \leq u \implies$ definition 1.26.a(ii) holds $\implies s = \sup S$.
(b) will follow similar structure of proof. □

Example 1.4.4. $[a, b[; a = \min[a, b[\implies a = \inf[a, b[$.

Question: If $S \neq \emptyset$ and $S \subseteq \mathbb{R}$ is bounded from above, does it necessarily have a supremum?
Answer: Well,,, we cannot answer this question formally since we do not have a proper definition of $\mathbb{R}$ available. In factm in modern mathematics, $\mathbb{R}$ is defined to be the completion of $\mathbb{Q}$ i.e. as the smallest set of numbers containing $\mathbb{Q}$ that is <u>complete</u>, meaning that any non–empty and bounded from above subset has a supremum in this set.
Intuitively, $\mathbb{R}$ is ddefined by "filling the gaps" in the numbers left by $\mathbb{Q}$. This is done through **Dedekind cuts**. For us, we will state the completeness as an axiom instead:

Axiom of Completeness. Let $\emptyset \neq S \subseteq \mathbb{R}$ and S is bounded from above. Then, S has a supremum in $\mathbb{R}$.

1.4.1 Consequences of Completeness

Theorem 1.8. ***(The Archimedean Property of*** $\mathbb{R}$***).*** *Let* $x \in \mathbb{R}$ *be arbitrary. Then* $\exists n \in \mathbb{N} : n > x$.

Proof. Assume $\exists x \in \mathbb{R} : \forall n \in \mathbb{N} : n \leq x$. Now consider $S := \{x \in \mathbb{R} : \forall n \in \mathbb{N} : n \leq x\}$. NOte that $S \neq \emptyset$ by our assumption. Furthermore, S is bounded from below (e.g. 0 is the lower bound of S since $\forall n \in \mathbb{N} : n > x$ if $x \leq 0$, i.e. No $x \leq 0$ is in S).
It follows from completeness that S has an infimum in $\mathbb{R}$; let $t := \inf S$. t is thus the greatest lower bound of $S \implies t+1$ is not a lower bound for S. Thus $\exists x \in S : x < t+1 \implies x-1 < t$; since t is a lower bound of S, $x-1$ cannot be in S i.e. $x-1 \notin S \implies \exists n \in \mathbb{N} : n > x-1 \implies \underbrace{n+1}_{\in\mathbb{N}} > \underbrace{x}_{\in S}$ ↯(by construction of S).
Thus our assumption was wrong $\therefore$ we've shown that $\forall x \in \mathbb{R}, \exists n \in \mathbb{N} : n > x$. □

Corollary 1.1. *Let* $0 < x$. *Then,* $\exists n \in \mathbb{N} : \frac{1}{n} < x$.

Proof. Let $0 < x$. By the Archimedean property, $\exists n \in \mathbb{N} : n > \frac{1}{x} \iff \frac{1}{n} < x$. □

End of Lecture

Example 1.4.5. Define the set $S := \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Find $\sup S$ and $\inf S$.
Solution: Note that S is bounded from above by 1 and bounded from below by 0. Thus S has both a sup and an inf by completeness. NOte also that $1 \in S$

and 1 is upper bound of $S \implies 1 = \max S \implies 1 = \sup S$. We will now show that $0 = \inf S$.

Proof. Let $t := \inf S$. Since 0 is lower bound of $S \implies t \geq 0$. Assume that $t > 0$. By corrolary ... $\exists n \in \mathbb{N} : \underbrace{\frac{1}{n}}_{\in S} < t \implies t$ is not a lower bound of S

$\lightning \implies t = 0 \implies \inf S = 0$. ☐

1.4.2 Density of $\mathbb{Q}$ in $\mathbb{R}$

Definition 1.27. A subset $S \subseteq \mathbb{R}$ is called **dense** in $\mathbb{R}$ if $\forall a, b \in \mathbb{R}, a < b, \exists x \in S : a < x < b$ i.e. no matter how close a and b, you can always find x if S is dense in $\mathbb{R}$.

Theorem 1.9. *$\mathbb{Q}$ is dense in $\mathbb{R}$*

Proof. Let a and $b \in \mathbb{R} : a < b$. Then, we'll be considering the following cases

1. $\underline{0 \leq a < b}$: Let $n > \frac{1}{b-a}$ (by Archemidean property) $\iff \frac{1}{n} < b - a$. Consider the set $\{\frac{1}{n}, \frac{2}{n}, \cdots\}$
   Claim: $\exists k_0 \in \mathbb{N} : \frac{k_0}{n} > a \iff k_0 > na$ (Arch prop.) Thus, $S := \left\{ j \in \mathbb{N} : \frac{j}{n} > a \right\}$ is non–empty $\implies k_0 \in S$. By well-ordering principle of $\mathbb{N}$, S has a least element k. Then, $a < \frac{k}{n}$. Also, $\frac{k-1}{n} \leq a$ **why?** Well...if $k > 1$, then $k - 1 \in \mathbb{N}$ and by construction of k being the least of element of S, we must have that $\frac{k-1}{n} \leq a$. If $k = 1$, then $\frac{k-1}{n} = 0 \leq a$. In all cases, we thus have $\frac{k-1}{n} = \frac{k}{n} - \frac{1}{n} \leq a \implies \frac{k}{n} \leq a + \underbrace{\frac{1}{n}}_{<b-a} < a + (b-a) \implies \frac{k}{n} < b$.

   Thus $a < \frac{k}{n} < b$, we define $x := \frac{k}{n}$ is a rational number between a and b

2. $\underline{a < b \leq 0}$: Then, we have $0 \leq -b < -a$. By the first case, $\exists x \in \mathbb{Q} : -b < x < -a \implies a < \underbrace{-x}_{\in \mathbb{Q}} < b$. Therefore, we've found a rational number between a and b.

3. $\underline{a < 0 < b}$: Since $0 \in \mathbb{Q}$, then we can define $x := 0$.

Thus we've found $x \in \mathbb{Q}$ in all cases such that $a < x < b \implies \mathbb{Q}$ is dense in $\mathbb{R}$. □

Theorem 1.10. ***(The Nested Interval Property).*** *Let* $a_1, a_2, \ldots, b_1, b_2 \cdots \in \mathbb{R}, \forall n \in \mathbb{N} : a_n \le b_n$, $[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \cdots$. *Then*

$$\bigcap_{n \in \mathbb{N}} [a_n, b_n] \neq \emptyset$$

Proof. Let $\forall n \in \mathbb{N} : I_n := [a_n, b_n]$, we have $I_1 \supseteq I_2 \supseteq \cdots \implies a_1 \le a_2 \le \cdots$ and $b_1 \ge b_2 \ge \cdots$. Also, $\forall n \in \mathbb{N} : a_n \le b_n$. We will have the following lemma:

Lemma 1.2. *We also have that* $\forall n, k \in \mathbb{N} : a_n \le b_k$

Proof. We will consider the following cases:

1. $\underline{n \le k}$: Then, $a_n \le a_k \le b_k \implies a_n \le b_k$.
2. $\underline{n > k}$: Then, $a_n \le b_n \le b_k \implies a_n \le b_k$

Thus in all cases, we have $a_n \le b_k$ □

Returning to our proof, let $A := \{a_1, a_2, a_3, \ldots\}$. By lemma 1.2, $\forall n \in \mathbb{N} : a_n \le b_1 \implies b_1$ is an upper bound of $A \implies A \neq \emptyset$ and is bounded from above. By the completeness of $\mathbb{R}$, A has a supremum $a := \sup A$. Similarly, the set $B := \{b_1, b_2, b_3, \ldots\}$ is non–empty and bounded from below (e.g. by a_1). By completeness, B has an infimum $b := \inf B$. Then, consider the following lemmas:

Lemma 1.3. $\forall k \in \mathbb{N} : a \le b_k$

Proof. By the Lemma 1.2, $\forall n, k \in \mathbb{N} : a_n \le b_k \implies \forall k \in \mathbb{N} : b_k$ is upper bound of A. Since a is the least upper bound of A, we have $\forall k \in \mathbb{N}, a \le b_k$. □

Lemma 1.4. $a \le b$

Proof. By lemma 1.3, $\forall k \in \mathbb{N} : a \le b_k \implies a$ is the lower bound of B. Since b is the greatest lower bound of B, we then have $a \le b$. □

Returning to the proof again, let $n \in \mathbb{N}$. Then, $a_n \le a \le b \le b_n$ (since a is the upper and b is the lower bound). In other words, $\forall n \in \mathbb{N}, [a, b] \subseteq [a_n, b_n] \implies [a, b] \subseteq \bigcap_{n \in \mathbb{N}} [a_n, b_n] \implies \bigcap_{n \in \mathbb{N}} [a_n, b_n] \neq \emptyset$. □

Remark 1.11. *It can be shown that the intersection* $\bigcap_{n \in \mathbb{N}} [a_n, b_n] = [a, b]$

If $a = b$, the intersection consists of a single point. If $a < b$ then the intersection has infinitely many points.

What if the nested sequence consists of intervals that are unbounded? Well...the intersection can be empty

Example 1.4.6. $\bigcap_{n\in\mathbb{N}}]0, \frac{1}{n}[= \emptyset$. **Hint:** Consider 2 cases: $x > 0$ and $x = 0$. [10]

End of Lecture———

Lecture 6: September 18th, 2024.

1.5 Cardinality

Goal: "Measure" the size of a set.

Definition 1.28. If S is a finite site, then the **cardinality** of S is the number of element of S. We notate as

$$|S| = n \tag{1.16}$$

where n is the number of element.

Example 1.5.1. $|\emptyset| = 0$ and $|\{1, 3, 5\}| = 3$.

Georg Cantor's idea was to use functions to compare the sizes of sets.

- Finite case: Supposed that $|A| = n$ and $|B| = k$ where $A = \{a_1, a_2, \ldots, a_k, a_{k+1}, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_k\}$. If $n > k$ $\exists$ surjective map from $f : A \to B$, $f(a_j) = b_j$ if $1 \le j \le k$, and $f(a_j) = b_k$ if $k + 1 \le j \le n$. However, $\nexists$ injective map from A to B **why?** Well...we'll prove it through the following

  Proof. Assume that $\exists$ injective $f : A \to B, \forall b_j = f(a_j)$. Then, $f(A) = \{b_1, \ldots, b_n\}$ where $|f(A)| = n$ since the $b_1, \ldots, b_n$ are *pairwise distinct*. Contrarily, $\{b_1, \ldots, b_n\} \subseteq B$ and $|B| = k \implies \{b_1, \ldots, b_n\} \le k < n$ ↯. Therefore no such injective map can exist. □

  Similarly, if $|A| < |B|, \exists$ an injective map from A to B but there is no surjective map from A to B. Thus, if $|A| \ne |B|$, there cannot exist a bijective map from A to B. Otherwise, if $|A| = |B|$, there exist a bijective map from A to B.
  We can now extend this idea to infinite sets

[10] $[a, \infty[$ is closed as it's the same as $\{x : a \le x\}$

Definition 1.29. Let A and B be sets. We say that A and B have the same cardinality i.e. $|A| = |B|$ if $f : A \to B$ and f is bijective.

Furthermore, we say that the cardinality of A is strictly less than the cardinality of B i.e. $|A| < |B|$, $\nexists f : A \to B$, f is surjective.

Example 1.5.2. Consider the following sets $\mathbb{N}$ and $\mathbb{N}_0$. Then, $\mathbb{N} \subsetneq \mathbb{N}_0$. We define a map $f : \mathbb{N} \to \mathbb{N}_0, f(n) := \underbrace{n-1}_{\in \mathbb{N}_0}$. We state that $|\mathbb{N}| = |\mathbb{N}_0|$

Proof. We need to check if it's injective. Let $f(n) = f(k) \implies n-1 = k-1 \implies n = k$. Thus, f is injective. We now check for surjectivity. Let $n \in \mathbb{N}_0$ be arbitrary. Then, $n+1 \in \mathbb{N}$ and $f(n+1) = n \implies f$ is surjective. Because f is injective and surjective, it's bijective $\implies |\mathbb{N}| = |\mathbb{N}_0|$ □

Remark 1.12. *This cannot happen in finite sets.*

Example 1.5.3. Consider $\mathbb{Z}$. We'll construct a bijective map from $\mathbb{N}$ to $\mathbb{Z}$ even though $\mathbb{Z}$ is seemingly "twice as big" as $\mathbb{N}$.
Consider this complete list of integers:

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

and let $a_1 = 0, a_2 = 1, a_3 = -1, \ldots$. Define $f : \mathbb{N} \to \mathbb{Z}$ by: $f(n) = a_n, \forall n \in \mathbb{N}$. f is injective since all the numbers in the list are distinct; it is surjective since it's complete $\implies f$ is bijective $\implies |\mathbb{N}| = |\mathbb{Z}|$.
We can even explicitly construct f as

$$f(n) := \begin{cases} \frac{n}{2} & \text{if } n \text{ even} \\ -\frac{n-1}{2} & \text{if } n \text{ odd} \end{cases}$$

Definition 1.30. A set A is called **countably infinite** if $|A| = |\mathbb{N}|$ i.e. $\exists f : \mathbb{N} \to A$ is bijective.

Definition 1.31. A set B is called **countable** if B is either finite or countably infinite.

Example 1.5.4. $\mathbb{N}_0$ and $\mathbb{Z}$ are countably infinite and countable. On the other hand, $\{1, 3, 5\}$ is countable but not countably infinite.

Theorem 1.11. *Let A be a subset of $\mathbb{N}$. Then, A is countable.*

Proof. Assignment 3 □

Theorem 1.12. *Let A be a set s.t. $\exists$ surjective map from $\mathbb{N}$ to A. Then A is countable.*

Proof. If A is finite then no need to prove. Let A be infinite and let $a \in A$ be arbitrary. Since f is surjective, $\mathbb{N} \supseteq f^{-1}(\{a\}) \neq \emptyset$. Since $f^{-1}(\{a\}) \neq \emptyset$, it has a least element $n_a \subseteq \mathbb{N}$. Let $S = \{n_a : a \in A\} \subset \mathbb{N}$.
Consider the restriction $f\big|_S$ of f to S i.e. $f\big|_S : S \to A, n_a \mapsto a$. We will prove that $f\big|_S$ is bijective. [11]

- Injective: Let $\underbrace{f(n_a)}_{=a} = \underbrace{f(n_b)}_{=b} \implies a = b \implies n_a = n_b \implies f$ is injective.
- Surjective: Let $a \in A$ be arbitrary. Then, $n_a \in S$ and by definition, $f(n_a) = a \implies f$ is surjective.

Since $f : S \to A$ is injective and surjective, it's also bijective $\implies |S| = |A|$ but $S \subseteq \mathbb{N} \implies S$ is countable by theorem 1.11 $\implies A$ is countable. $\square$

Theorem 1.13. *$\mathbb{Q}$ is countably infinite.*

Proof. First, we'll prove that the set of all positive rational number $\mathbb{Q}^+$ is countably infinite. Consider the matrix–like arrangement

$$\begin{array}{ccccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \cdots \\
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \cdots \\
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \cdots \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Every positive rational number appears infinitely often in this scheme. Counting along the diagonal yield the following enumeration with repetition of $\mathbb{Q}^+$:

$$\underbrace{\frac{1}{1}}_{\text{1. diag.}}, \underbrace{\frac{2}{1}, \frac{1}{2}}_{\text{2. diag.}}, \underbrace{\frac{3}{1}, \frac{2}{2}, \frac{1}{3}}_{\text{3. diag.}}, \ldots$$

Let $a_1 = 1.$ diag.$, a_2 = 2.$ diag.$, \ldots$. Let $f : \mathbb{N} \to \mathbb{Q}, n \mapsto a_n$. Since the scheme contains every positive rational number (even infinitely often) f is surjective (not injective). By theorem 1.12, $\mathbb{Q}^+$ is countable but $\mathbb{N} \subseteq \mathbb{Q}^+ \implies \mathbb{Q}^+$ is

End of Lecture

[11] $f|_S$ is called the restriction of f to s

Lecture 7: September 23^{th}, 2024.

infinite $\implies \mathbb{Q}^+$ is countably infinite.

Now, we need to show that $\mathbb{Q}$ is countably infinite. To do so, we use the same trick we used for $\mathbb{Z}$ from last lecture. Let $a_1, a_2, a_3, \ldots$ be an enumeration of $\mathbb{Q}^+$. Then, $0, a_1, -a_1, a_2, -a_2, \ldots$ is an enumeration of $\mathbb{Q}$. Thus $\mathbb{Q}$ is countably infinite □

All of these results are not as shocking as the Cantor's result that $\mathbb{R}$ is uncountable.

Theorem 1.14. *Every interval* $[a,b] : a < b$ *is uncountable.*

Proof. Assume that $[a,b]$ is countably infinite. Then, $x_1, x_2, x_3, \ldots$ be an enumeration of $[a,b]$. Construct a nested sequence of closed and bounded intervals as follows:

Divide $[a,b]$ into 3 closed subintervals of equal width. Then, at least one of these subintervals does not contain x_1, we'll call it I_1. [12] Now divide I_1 into 3 subintervals of equal width; at least one of these subintervals will not contain x_2; we'll call it I_2. Divide I_2 into 3 subintervals $\cdots$ etc. We obtain a nested sequence of $I_1 \geq I_2 \geq I_3 \geq \cdots$ of closed and bounded intervals with property $\forall n \in \mathbb{N} : x_n \notin I_n$.

By nested interval property of $\mathbb{R}$,$\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$. Let $x \in \bigcap_{x \in \mathbb{N}} I_n$. Then, $x \in [a,b]$. Thus $\exists n_0 \in \mathbb{N} : x = x_0$. But $x = x_{n_0} \notin I_{n_0} \implies x = x_0 \notin \bigcap_{x \in \mathbb{N}} I_n$. We thus have both $x \in \bigcap_{x \in \mathbb{N}} I_n$ and $x \notin \bigcap_{x \in \mathbb{N}} I_n$ ↯ . Thus $[a,b]$ cannot be countably infinite $\implies [a,b]$ is uncountable. □

Because of theorem 1.14, we know that $\mathbb{R} \geq [a,b] \implies \mathbb{R}$ is also uncountable.

[12] We need 3 subintervals since x_1 can be contained in at most 2 of these subintervals and thus cannot be contained in the third.

2 Sequences

Definition 2.1. (Calculus). A **sequence** is an ordered list of numbers, often denoted as

$$(a_1, a_2, a_3, \dots) \tag{2.1}$$

or sometimes

$$a_1, a_2, a_3, \dots \tag{2.2}$$

As seen in the previous section, an ordered list can be identified with functions with domain $\mathbb{N}$. Thus we can make the definition of a sequence more rigorous as

Definition 2.2. A **sequence** in S is a function from $\mathbb{N}$ to S, conventionally denote as: if $f : \mathbb{N} \to S$, we write $\forall n \in \mathbb{N} : a_n := f(n)$ and denote the sequence by

$$(\underbrace{a_1}_{f(1)}, \underbrace{a_2}_{f(2)}, \underbrace{a_3}_{f(3)}, \dots) =: (a_n)$$

or even $(a_n)_{n\in\mathbb{N}}$ or $(a_n)_{n=1}^{\infty}$

In this course, S is usually $\mathbb{R}$ or a subset of $\mathbb{R}$. Sometimes it's convenient to not start a sequence at $n = 1$ but rather at $n = 0$ or any arbitrary integer n_0; we write $(a_n)_{n\in\mathbb{N}_0}$ or $(a_n)_{n=0}^{\infty}$; for $n_0 : (a_n)_{n=n_0}^{\infty}$

Example 2.0.1. Consider the following sequences

- $(1, 4, 9, 16, \dots) = (n^2) = (n^2)_{n\in\mathbb{N}} = (n^2)_{n=1}^{\infty}$
- $\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right) = \left(\frac{1}{n}\right) = \left(\frac{1}{n}\right)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n=1}^{\infty}$

2.1 Limits of Sequences

Intuitively from calculus, the definition of a limit is given as

Definition 2.3. (Calculus). Let (a_n) be a sequence of real numbers and let $L \in \mathbb{R}$. We say that (a_n) converges to L, or that L is the limit of (a_n), if the a_n gets arbitrarily close to L as n grows without bounds.

This is a good definition...for the elementary level of calculus. We need to construct a more rigorous and precise definition of said limit for modern mathematics.

Recall that $|x-y|$ measures the distance between 2 points $x, y \in \mathbb{R}$. Then we will get the modern definition of a sequence as

Definition 2.4. Let (a_n) be a sequence of real numbers and let $L \in \mathbb{R}$. we say (a_n) **converges** to L if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N : |a_n - L| < \varepsilon \tag{2.3}$$

We call $\exists N \in \mathbb{N}$ the **threshold index** and $|a_n - L|$ the **distance between** a_n and L.

Definition 2.5. Furthermore, L is called the **limit** of (a_n), denotes as

$$L = \lim(a_n) = \left[\lim_{n\to\infty}(a_n) = \lim_{n\to\infty} a_n\right] \tag{2.4}$$

If the sequence (a_n) has a limit, we say that (a_n) **converges**; otherwise, we say that (a_n) **diverges**

Example 2.1.1. 1) Show that $\lim\left(\frac{1}{n}\right) = 0$.

Proof. Let $\varepsilon > 0$ be arbitrary. Then,

$$|a_n - L| = \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon \implies n > \frac{1}{\varepsilon}$$

Now let $N > \frac{1}{\varepsilon}$. Then we have $\forall n \geq N$ that

$$\left|\frac{1}{n} - 0\right| < \varepsilon \implies \lim\left(\frac{1}{n}\right) = 0$$

$\square$

2) Show that $\lim\left(\frac{n^2}{n^2+1}\right) = 1$.

Proof. Let $\varepsilon > 0$. Then

$$\left|\frac{n^2}{n^2+1}\right| = \left|\frac{n^2-(n^2+1)}{n^2+1}\right| = \left|\frac{-1}{n^2+1}\right| = \frac{1}{\underbrace{n^2+1}_{>n^2}} < \frac{1}{n^2} < \varepsilon \iff n^2 > \frac{1}{\varepsilon}$$

Thus $n > \frac{1}{\sqrt{\varepsilon}}$. Now, let $N > \frac{1}{\sqrt{\varepsilon}}$. Then, we have that $\forall n \geq N$

$$\left|\frac{n^2}{n^2+1} - 1\right| < \varepsilon \implies \lim\left(\frac{n^2}{n^2+1}\right) = 1$$

$\square$

End of Lecture

Definition 2.6. Let $a \in \mathbb{R}$ and $\varepsilon > 0$, the **ε−neighbourhood** about a is defined as

$$V_\varepsilon(a) := \{x \in \mathbb{R} : |x-a| < \varepsilon\} =]a-\varepsilon, a+\varepsilon[\tag{2.5}$$

Using the definition of ε−neighbourhood, we can rewrite the definition of the limit of a sequence as follows

Definition 2.7. (Renew definition)

$$\lim(a_n) = L \iff \forall \varepsilon > 0, \exists N, \forall n \in \mathbb{N} : a_n \in V_\varepsilon(L) \tag{2.6}$$

Example 2.1.2. Show that $(-1)^n$ diverges.

Proof. Let $L \in \mathbb{R}$ be arbitrary, we need to show that $(-1)^n$ does not converge to L. Assume $\lim((-1)^n) = L$. Let $\varepsilon = 1$. Then, $\exists N \forall n \geq N : |(-1)^n - L| < \varepsilon = 1$. Let $n \geq N$, we consider 2 cases

- If n is even, then

$$|(-1)^n - L| = |1-L| = |L-1| < 1 \iff -1 < L-1 < 1 \iff 0 < L < 2$$

- If n is odd, then

$$|(-1)^n - L| = |-1-L| = |L+1| < 1 \iff -1 < L+1 < 1 \iff -2 < L < 0$$

Adding these 2 inequalities, we obtain: $0 < L < 0$ ↯ . This $(-1)^n$ diverges. □

We will see a shorter proof of this eventually.

Theorem 2.1. *Let* $0 < a < 1$. *Then,* $\lim(a^n) = 0$

Proof. (First). Let $\varepsilon > 0$. Then, $|a^n - 0| = a^n < \varepsilon$

$$\begin{aligned} &\iff \ln(a^n) < \ln\varepsilon \\ &\iff n\overbrace{\ln(a)}^{<0} < \ln\varepsilon \\ &\iff n > \frac{\ln\varepsilon}{\ln a} \end{aligned}$$

Let $N > \frac{\ln\varepsilon}{\ln a}$. Then $\forall n \geq N : |a^n - 0| < \varepsilon \implies \lim(a^n) = 0$ □

Remark 2.1. *You are NOT allowed to use logarithm or a^x, x irrational, in your proof in this course.*[1]

[1] This is because they're not properly defined.

Because of the above remark, we need to perform a different proof that does not use log.

Proof. (Second). $0 < a < 1 \implies \frac{1}{a} > 1$. Let $b := \frac{1}{a} - 1 > 0$. Then, $\frac{1}{a} = b + 1 \iff a = \frac{1}{1+b} \iff a^n = \frac{1}{(1+b)^n}$. Recall Bernoulli's inequalities, $\frac{1}{(1+b)^n} < \frac{1}{1+nb} \iff a^n < \frac{1}{1+nb} < \frac{1}{nb} < \varepsilon$. Then,

$$\begin{aligned} &\iff nb > \frac{1}{\varepsilon} \\ &\iff n > \frac{1}{b\varepsilon} \end{aligned}$$

Let $N > \frac{1}{b\varepsilon}$. Then, $\forall n \geq N : |a^n - 0| < \varepsilon \implies \lim(a^n) = 0$ □

Exercise: Prove that $\lim(a^n) = 0, \ \forall |a| < 1$

2.1.1 Limits Laws

Theorem 2.2. *Let (a_n) be a convergent sequence, then a_n is bounded. i.e.* $\exists M > 0 : \forall n \in \mathbb{N}, \ |a_n| \leq M$

Remark 2.2. *The converse of this theorem is false*

Example 2.1.3. $((-1)^n)$ is bounded by 1 since $\forall n \in \mathbb{N} : |(-1)^n| = 1$. However, $((-1)^n)$ diverges.

Proof. (Theorem 2.2). Let $\lim(a_n) = L$ and $\varepsilon := 1$. Then, $\exists N, \forall n \geq N : |a_n - L| < 1$. Then,

$$|a_n| = |(a_n - L) + L| \leq \underbrace{|a_n - L|}_{<\varepsilon} + |L| < 1 + |L|$$

Thus $\forall n \geq N : |a_n| < 1 + |L|$. Let $M := \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 - |L|\}$. Then, $\forall n \in \mathbb{N}, |a_n| \leq M \implies (a_n)$ is bounded □

Exercise: Show that (n) diverges.

Theorem 2.3. ***(Algebraic Limit Laws).*** *Let (a_n) and (b_n) be convergent sequences and $c \in \mathbb{R}$. Then,*

(a) $(a_n + b_n)$ *converges and* $\lim(a_n + b_n) = \lim(a_n) + \lim(b_n)$

(b) $(a_n - b_n)$ *converges and* $\lim(a_n - b_n) = \lim(a_n) - \lim(b_n)$

(c) $(c \cdot a_n)$ *converges and* $\lim(c \cdot a_n) = c \cdot \lim(a_n)$

(d) $(a_n \cdot b_n)$ *converges and* $\lim(a_n \cdot b_n) = \lim(a_n) \cdot \lim(b_n)$

(b) *If* $\forall n \in \mathbb{N} : b_n \neq 0$ *and* $\lim(b_n) \neq 0$*; then,* $(\frac{a_n}{b_n})$ *converges and* $\lim(\frac{a_n}{b_n}) = \frac{\lim(a_n)}{\lim(b_n)}$

Proof. We will begin with the sum.

(a) Let $\varepsilon > 0$. (a_n) and (b_n) converges and let $\lim(a_n) = a$ and $\lim(b_n) = b$. Thus $\exists N_1, \forall n \geq N : |a_n - a| < \frac{\varepsilon}{2}$ and, $\exists N_2, \forall n \geq N : |b_n - b| < \frac{\varepsilon}{2}$. Let $N := \max\{N_1, N_2\}$. Then $\forall n \geq N : |a_n - a| < \frac{\varepsilon}{2} \wedge |b_n - b| < \frac{\varepsilon}{2}$. Now,

$$\begin{aligned} |(a_n + b_n) - (a+b)| &= |(a_n - a) + (b_n - b)| \\ &\leq \underbrace{|(a_n - a)|}_{<\frac{\varepsilon}{2}} + \underbrace{|(b_n - b)|}_{<\frac{\varepsilon}{2}} < \varepsilon \end{aligned}$$

i.e. $\forall n \geq N : |(a_n + b_n) - (a+b)| < \varepsilon \implies (a_n + b_n)$ converges and $\lim(a_n + b_n) = a + b = \lim(a_n) + \lim(b_n)$

(d) Let $\varepsilon > 0$. Since, (a_n) and (b_n) converges then let $\lim(a_n) = a$ and $\lim(b_n) = b$. Then,

$$\begin{aligned} |a_n \cdot b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &= |b_n(a_n - a) + a(b_n - b)| \\ &\leq |b_n(a_n - a)| + |a(b_n - b)| \\ &= \underbrace{|b_n||(a_n - a)|}_{\text{call it (1)}} + \underbrace{|a||(b_n - b)|}_{\text{call it (2)}} \end{aligned}$$

We will then treat 1 and 2 separately.

- <u>for 1:</u> $|b_n||a_n - a|$. $|b_n|$ is bounded $\implies \exists M > 0, \forall n \in \mathbb{N} : |b_n| \leq M$. Thus, $\exists N_1, \forall n \geq N_1 : |a_n - a| < \frac{\varepsilon}{2 \cdot M}$. Then, $\forall n \geq N_1 :$

$$|a_n - a||b_n| < \frac{\varepsilon}{2M} \cdot M = \frac{\varepsilon}{2}$$

  i.e. $\forall n \geq N_1 : |a_n - a||b_n| < \frac{\varepsilon}{2}$.

- <u>for 2:</u> $|a_n||b_n - b|$. Let $M' > 0 : |a| \leq M'$. Thus, $\exists N_2, \forall n \geq N_2 : |b_n - b| < \frac{\varepsilon}{2M'}$

$$\implies |a||b_n - b| < M' \cdot \frac{\varepsilon}{2M'} = \frac{\varepsilon}{2}$$

  i.e. $\forall n \geq N_1 : |b_n - b||a| < \frac{\varepsilon}{2}$.

Let $N := \max\{N_1, N_2\}$. Then $\forall n \geq N : (1) < \frac{\varepsilon}{2} \wedge (2) < \frac{\varepsilon}{2} \implies \forall n \geq N :$

$$|a_n b_n - ab| < (1) - (2) < \varepsilon$$

Thus $(a_n b_n)$ converges and $\lim(a_n b_n) = ab$

(c) Let $c \in \mathbb{R}$ and $\forall n \in \mathbb{N}, b_n = c$. Then $(a_n b_n)$ converges by the above proof and $\lim(a_n b_n) = \lim(a_n) \cdot \lim(b_n) = \lim(a_n) \cdot \lim(c) = c \lim(a_n) \implies \lim(ca_n) = c \lim(a_n)$.

(b) Consider the convergent sequences (a_n) and (b_n). Then, $a_n - b_n = a_n + (-1)b_n$. We know that $(-1)b_n$ would converge by (c) and by (a), $(a_n + (-1)b_n) = (a_n - b_n)$ converges also.
And,

$$\begin{aligned} \lim(a_n - b_n) &= \lim(a_n + (-1)b_n) \\ &= \lim(a_n) + (-1)\lim(b_n) \\ &= \lim(a_n) - \lim(b_n) \end{aligned}$$

For (e) will be proof at tutorials. □

End of Lecture

Lecture 9: September 30^{th}, 2024.

Definition 2.8. A sequence (a_n) of real numbers is called a **null sequence** if $\lim(a_n) = 0$

Example 2.1.4. Show that $\forall k \in \mathbb{N}, \left(\frac{1}{n^k}\right)$ is a null sequence.

Proof. We will use induction on k.

1. $\underline{(k = 1):}$ $\lim\left(\frac{1}{n}\right) = 0$ is correct.

2. $\underline{(k \implies k+1):}$ Assume that $\lim\left(\frac{1}{n^k}\right) = 0$. Then, $\frac{1}{n^{k+1}} = \frac{1}{n} \cdot \frac{1}{n^k}$. Since both $\frac{1}{n}$ and $\frac{1}{n^k}$ converges to 1, by theorem 2.3(d), it must be the case that $\frac{1}{n} \cdot \frac{1}{n^k}$ converges to 0 and is thus a null sequence.

Therefore, $\frac{1}{n^k}$ is a null sequence. □

Theorem 2.4. *Let (a_n) be a null sequence and let $c \in \mathbb{R}$. Then, $(c \cdot a_n)$ is a null sequence.*

Proof. $\lim(a_n) = 0 \implies \lim(c \cdot a_n) = c \cdot \lim(a_n) = c \cdot 0 = 0 \implies (c \cdot a_n)$ is a null sequence. □

Theorem 2.5. ***(A convergence criterion for sequence).***
Let (a_n) be a sequence, let $L \in \mathbb{R}$ and let (b_n) be a non-negative null sequence. If $\exists k \in \mathbb{N} : \forall n \geq k : |a_n - L| \leq b_n$. It follows that (a_n) conveges to L.

Proof. Let $\varepsilon > 0$. Since (b_n) is a null sequence, $\exists \tilde{N}, \forall n \geq \tilde{N} : |b_n - 0| = |b_n| = b_n < \varepsilon$. Let $N := \max\{\tilde{N}, k\}$. Then, $\forall n \geq N : |a_n - L| \leq b_n < \varepsilon \implies \forall n \geq N : |a_n - L| < \varepsilon \implies (a_n)$ converges to L. □

2.2 Limits and Orders

Theorem 2.6. *Let (a_n) be a convergent sequence. If $\exists K \in \mathbb{N}, \forall n \geq K : a_n \geq 0$. Then,* $\lim(a_n) \geq 0$.

Proof. Let $a := \lim(a_n)$; assume that $a < 0$. Let $\varepsilon := -a > 0$. Then, $\exists \tilde{N} \forall n \geq \tilde{N} : |a_n - a| < \varepsilon = -a$. Recall that $\forall x \in \mathbb{R} : x \leq |x|$. Thus,

$$a_n - a \leq |a_n - a| < -a \implies a_n - a < -a \implies a_n < 0$$

Thus, $\forall n \geq \tilde{N} : a_n < 0$. Let $N := \max\{K, \tilde{N}\}$. Then, $\forall n \geq N : a_n \geq 0 \wedge a_n < 0$ ↯ . Thus $a \geq 0$. □

Theorem 2.7. *Let $(a_n), (b_n)$ be convergent sequences. If $\exists K \in \mathbb{N} \forall n \geq K : a_n \leq b_n$. Then* $\lim(a_n) \leq \lim(b_n)$.

Proof. Consider the sequence (c_n) where $c_n := b_n - a_n$. Then, $\forall n \geq K : c_n = b_n - a_n \geq 0$. Furthermore, (c_n) converges and $\lim(c_n) = \lim(b_n) - \lim(a_n)$. Then, by the previous theorem, $\lim(c_n) = \lim(b_n) - \lim(a_n) \geq 0 \implies \lim(a_n) \leq \lim(b_n)$. □

Remark 2.3. *If $a_n < b_n \forall n \geq K$, we cannot, in general, conclude that* $\lim(a_n) < \lim(b_n)$ *but we can conclude that* $\lim(a_n) \leq \lim(b_n)$.

Example 2.2.1. $(a_n) := \left(-\frac{1}{n}\right), (b_n) := \left(\frac{1}{n}\right)$. Then, $\forall n \in \mathbb{N} : a_n < b_n$ but $\lim(a_n) = 0 = \lim(b_n)$!

Theorem 2.8. *Let (b_n) be a convergent sequence, let $a, c \in \mathbb{R}$. If $\exists K \in \mathbb{N}, \forall n \geq K : a \leq b_n \leq c$. Then, $a \leq$* $\lim(b_n)$ *$\leq c$.*

Proof. Exercise. □

Interpretation: If the terms of the convergent sequence (b_n) eventually lie within the interval $[a, c]$ i.e. $\exists K \in \mathbb{N} \forall n \geq K : b_n \in [a, c]$, then $\lim(b_n) \in [a, c]$.

Caution: This does NOT hold for non-closed intervals.

Example 2.2.2. $(b_n) = \left(\frac{1}{n}\right)$, interval $]0,1]$. Then, $\forall n \in \mathbb{N} : b_n \in]0,1]$, but $0 = \lim(b_n) \notin]0,1]$.

Theorem 2.9. ***(Squeeze Theorem).*** *Let $(a_n), (b_n), (c_n)$ be sequences such that:*

- $\exists K \in \mathbb{N}, \forall n \geq K : a_n \leq b_n \leq c_n$
- *(a_n) and (c_n) converge and have the same limit. Then (b_n) converges and* $\lim(b_n) = \lim(a_n) = \lim(c_n)$.

Proof. Let $L := \lim(a_n) = \lim(c_n)$. Then,

$$\exists N_1, \forall n \geq N_1 : |a_n - L| < \varepsilon$$
$$\exists N_2, \forall n \geq N_2 : |c_n - L| < \varepsilon$$

Let $N := \max\{N_1, N_2, K\}$. Then, $\forall n \geq N$:

$$a_n \leq b_n \leq c_n \land \underbrace{|a_n - L| < \varepsilon}_{L-\varepsilon<a_n<L+\varepsilon} \land \underbrace{|c_n - L| < \varepsilon}_{L-\varepsilon<c_n<L+\varepsilon}$$

So,

$$\begin{aligned} &\implies L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon \\ &\implies L - \varepsilon < b_n < L + \varepsilon \\ &\implies -\varepsilon < b_n - L < \varepsilon \\ &\implies |b_n - L| < \varepsilon \end{aligned}$$

i.e. $\forall n \geq N : |b_n - L| < \varepsilon$. Thus, (b_n) converges and $\lim(b_n) = L$. □

Example 2.2.3. Show that $\left(\frac{\sin n}{n}\right)$ converges to 0. Recall that $-1 \leq \sin n \leq 1$ for all $n \in \mathbb{N}$,

$$\underbrace{-\frac{1}{n}}_{\to 0} \leq \frac{\sin n}{n} \leq \underbrace{\frac{1}{n}}_{\to 0}$$

By squeeze theorem, $\left(\frac{\sin n}{n}\right)$ converges and its limits is 0.

2.3 Monotone Sequence

Definition 2.9. Let (a_n) be a sequence. If

- $\forall n \in \mathbb{N} : a_n \leq a_{n+1}$, we say that (a_n) is **increasing**.

- $\forall n \in \mathbb{N} : a_{n+1} \leq a_n$, we say that (a_n) is **decreasing**.

Noticeably, a sequence that's either decreasing or increasing is called **monotone sequence**.

Theorem 2.10. ***(Monotone Convergent Sequence).***

- *Let* (a_n) *be increasing and bounded from above. Then* (a_n) *converges and* $\lim(a_n) = \sup\{a_n : n \in \mathbb{N}\}$.

- *Let* (a_n) *be decreasing and bounded from below. Then,* (a_n) *converges and* $\lim(a_n) = \inf\{a_n : n \in \mathbb{N}\}$.

Proof. Let's prove the first part. Since (a_n) is bounded from above, the set $\{a_n : n \in \mathbb{N}\}$ has a supremum a. We will show that (a_n) convergest to a.
Let $\varepsilon > 0$. Then, $a - \varepsilon$ is not an upper bound of $S := \{a_n : n \in \mathbb{N}\}$ since a is the least upper bound of S i.e. $\exists N \in \mathbb{N} : a - \varepsilon < a_N$.
Since (a_n) is increasing, we have that

$$a - \varepsilon < a_n \leq a_{N+1} \leq a_{N+2} \leq \cdots \leq a$$

Then,

$$\begin{aligned}
&\implies \forall n \geq N : a - \varepsilon < a_n \leq a < a + \varepsilon \\
&\implies \forall n \geq N : a - \varepsilon < a_n < a + \varepsilon \\
&\implies \forall n \geq N : a_n \in V_\varepsilon(a)
\end{aligned}$$

$\implies (a_n)$ converges to a. Second part will be left as an exercise. $\square$

End of Lecture

2.3.1 Euler's Number

In this lecture, we will prove that the sequence $\left(\left(1 + \frac{1}{n}\right)^n\right)$ converges of which its limit is denoted as e called **Euler's number (constant)**.
Proving this is not a trivial task, we will use the following strategy:
Let $a_n := \left(1 + \frac{1}{n}\right)^n$ and $b_n := \left(1 + \frac{1}{n}\right)^{n+1}$ for all $n \in \mathbb{N}$. We will show that

- (a_n) is strictly increasing

- (b_n) is strictly decrease

- $\forall n, k \in \mathbb{N} : a_n \leq b_k$

Lecture 10: October 2^{nd}, 2024.

- (a_n) is bounded from above and (b_n) is bounded from below
- (a_n) and (b_n) both converge and have the same limit

We will define $e := \lim(a_n) = \lim(b_n)$

1. Step: We will show that the sequence (a_n) is strictly increasing.
We will achieve this by proving that $\forall n \in \mathbb{N}$ it holds that $\frac{a_{n+1}}{a_n} > 1$ (which is equivalent to $\forall n \in \mathbb{N}: a_n < a_{n+1}$). Note that

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n}} = \frac{\left(\frac{n+2}{n+1}\right)^{n+1}}{\left(\frac{n+1}{n}\right)^{n}} = \frac{(n+2)^{n+1}\cdot n^n}{(n+1)^{n+1}\cdot(n+1)^n} = \frac{n+2}{n+1}\cdot\frac{(n+2)^n\cdot n^n}{(n+1)^{2n}} \\
&= \frac{n+2}{n+1}\cdot\left[\frac{(n+2)\cdot n}{(n+1)^2}\right]^n = \frac{n+2}{n+1}\cdot\left[\frac{n^2+2n}{n^2+2n+1}\right]^n = \frac{n+2}{n+1}\cdot\left[1-\frac{1}{n^2+2n+1}\right]^n \\
&= \frac{n+2}{n+1}\cdot\Bigg[1+\underbrace{\left(-\frac{1}{(n+1)^2}\right)}_{\geq -1}\Bigg]^n \geq \frac{n+2}{n+1}\cdot\left[1-n\frac{1}{(n+1)^2}\right] \\
&= \frac{n+2}{n+1}\cdot\frac{n^2+2n+1-n}{(n+1)^2} = \frac{(n+2)\cdot(n^2+n+1)}{(n+1)^3} \\
&= \frac{n^3+n^2+n+2n^2+2n+2}{n^3+3n^2+3n+1} = \frac{n^3+3n^2+3n+2}{n^3+3n^2+3n+1} > 1
\end{aligned}$$

This proves that (a_n) is strictly increasing.

2. Step: We will show that the sequence (b_n) is strictly decreasing.
We will get this by proving that $\forall b \in \mathbb{N}$ it holds that $\frac{b_n}{b_{n+1}} > 1$. Note that

$$\begin{aligned}
\frac{b_n}{b_{n+1}} &= \frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n+1}\right)^{n+2}} = \frac{\left(\frac{n+1}{n}\right)^{n+1}}{\left(\frac{n+2}{n+1}\right)^{n+2}} = \frac{(n+1)^{n+1}\cdot(n+1)^{n+2}}{n^{n+1}\cdot(n+2)^{n+2}} \\
&= \frac{n+1}{n+2}\cdot\frac{(n+1)^{2(n+1)}}{n^{n+1}\cdot(n+2)^{n+1}} = \frac{n+1}{n+2}\cdot\left[\frac{(n+1)^2}{n\cdot(n+2)}\right]^{n+1} \\
&= \frac{n+1}{n+2}\cdot\left[\frac{n^2+2n+1}{n^2+2n}\right]^{n+1} = \frac{n+1}{n+2}\cdot\left[1+\frac{1}{n^2+2n}\right]^{n+1} \\
&\geq \frac{n+1}{n+2}\cdot\left[1+(n+1)\frac{1}{n^2+2n}\right]^{n+1} = \frac{n+1}{n+2}\cdot\frac{n^2+2n+n+1}{n^2+2n} \\
&= \frac{n+1}{n+2}\cdot\frac{n^2+3n+1}{n^2+2n} = \frac{n^3+3n^2+n+n^2+3n+1}{n^3+2n^2+2n^2+4n} = \frac{n^3+4n^2+4n+1}{n^3+4n^2+4n} > 1
\end{aligned}$$

This proves that (b_n) is strictly decreasing.

3. Step: We will show that $\forall n,k \in \mathbb{N} : a_n \leq b_k$.
It is clear that $\forall n \in \mathbb{N} : a_n = \left(1+\frac{1}{n}\right)^n \leq \left(1+\frac{1}{n}\right)^{n+1} = b_n$. Now let $n,k \in \mathbb{N}$ be arbitrary. We will distinguish 2 cases:

1. $\underline{n \leq k}$: Since (a_n) is increasing, it holds that $a_n \leq a_k \leq b_k$, Especially, it follows that $a_n \leq b_k$.
2. $\underline{n > k}$: Since (b_k) is decreasing, it holds that $a_n \leq b_n \leq b_k$, Especially, it follows that $a_n \leq b_k$.

Thus, it follows in all cases that $a_n \leq b_k \; \forall n,k \in \mathbb{N}$

4. Step: We will show that (a_n) is bonded from above and that (b_n) is bounded from below.
It follows from the previous step that $a_n \leq b_1$ and $a_1 \leq b_n \; \forall n \in \mathbb{N}$. b_1 is thus an upper bound for (a_n) and a_1 is a lower bound for (b_n). This proves that (a_n) is bounded from above and that (b_n) is bounded from below.

5. Step: We will show that (a_n) and (b_n) both converge and have the same limit.
It was shown in the previous steps that (a_n) is increasing and bounded from above; we've also shown that (b_n) is decreasing and bounded from below. It now follows from the monotone convergence theorem that both (a_n) and (b_n) converge. Furthermore,

$$b_n = \left(1+\frac{1}{n}\right)^{n+1} = \left(1+\frac{1}{n}\right)^n \cdot \left(1+\frac{1}{n}\right) a_n \cdot \left(1+\frac{1}{n}\right)$$

Thus,

$$\lim(b_n) = \lim(a_n) \cdot \underbrace{\lim\left(1+\frac{1}{n}\right)}_{=1} = \lim(a_n)$$

We can thus now define

$$\boxed{e := \lim\left(\left(1+\frac{1}{n}\right)^n\right) = \lim\left(\left(1+\frac{1}{n}\right)^{n+1}\right)} \tag{2.7}$$

Finally, we will five a numerical estimates for e. It was shown in 3. step that $a_n \leq b_k \forall n,k \in \mathbb{N}$. Thus b_k is an upper bound for $S := \{a_n : n \in \mathbb{N}\} \; \forall k \in \mathbb{N}$. Note that since (a_n) is bounded from above, S has a supremum i.e. a

least upper bound. Thus, $\sup S \leq b_k \ \forall k \in \mathbb{N}$. Furthermore, it follows from the monotone convergence theorem that $e = \lim(a_n) = \sup S$. Combining these results, we obtain that $e \leq b_k \ \forall k \in \mathbb{N}$.
In a very similar manner, it follows that $a_n \leq e \ \forall n \in \mathbb{N}$. This finally proves that

$$\forall n, k \in \mathbb{N} : a_n \leq e \leq b_k$$

A quick calculation yields that e.g. $b_5 = \left(1 + \frac{1}{5}\right)^6 = 2.98... < 3$ and $a_6 = \left(1 + \frac{1}{6}\right)^6 = 2.52... > \frac{5}{2}$. This shows immediately that

$$\boxed{\frac{5}{2} < e < 3} \tag{2.8}$$

This estimate is sufficiently good for the purpose of this course. Choosing larger values for n and k, one can show that $e = 2.718281828459\ldots$

2.4 Subsequences

Definition 2.10. Let $n_1, n_2, n_3, \dots \in \mathbb{N} : n_1 < n_2 < n_3 < \cdots$ and let $(a_n) = (a_1, a_2, a_3, \dots)$ be a sequence. Then, $(a_{n_k}) = (a_{n_1}, a_{n_2}, a_{n_3}, \dots)$ is called a **subsequence** of (a_n).

Example 2.4.1. Let (a_n) be a sequence. Then,

1. $(a_{2k}) = (a_2, a_4, a_6, \dots)$ is a subsequence of (a_n) called the *subsequence of even indices* of which we often write (a_{2n}) instead of (a_{2k}).

2. $(a_{2k-1}) = (a_1, a_3, a_5, \dots)$ is a subsequence of (a_n) called the *subsequence of odd indices.*

3. $(a_{k+1}) = (a_2, a_3, a_4, \dots)$ is a subsequence of (a_n) called the **1–tail** of (a_n)[2]
   Similarly, $(a_{k+1}) = (a_{k+1}, a_{k+2}, a_{k+3}, \dots)$ is called the **k–tail** of (a_n).

Lemma 2.1. *Let $n_1, n_2, n_3, \dots \in \mathbb{N} : n_1 < n_2 < \cdots$. Then, $\forall k \in \mathbb{N} : n_k \geq k$*

Proof. We will prove this via induction:

- $\underline{k = 1}$: $n_1 \geq 1$ is true.

- $\underline{k \to k+1}$: Assume that $n_k \geq k$ for some $k \in \mathbb{N}$. Then, $n_{k+1} > n_k \geq k \implies n_{k+1} > k \implies n_{k+1} \geq k+1$.

[2]Once again, we also usually wrte (a_{k+1}) as (a_{n+1}).

This is all we had to show. $\square$

Theorem 2.11. *Let (x_n) be a convergent sequence and let (x_{n_k}) be an arbitrary subsequence of (x_n). Then, (x_{n_k}) converges and* $\lim(x_{n_k}) = \lim(x_n)$.

Proof. Let $x := \lim(x_n)$. Let $\varepsilon > 0$. Since (x_n) converges to $x, \exists n \; \forall n \geq N :$ $|x_n - x| < \varepsilon$. Then $\forall k \geq N$, by the above lemma and construction, we have

$$|x_{n_k} - x| < \varepsilon$$

$\implies (x_{n_k})$ converges and $\lim(x_{n_k}) = x$. $\square$

End of Lecture ——

Corollary 2.1. *Let (x_n) be a sequence; let (x_{n_k}) and (x_{n_j}) be convergent sequences s.t.* $\lim(x_{n_k}) \neq \lim(x_{n_j})$. *Then, (x_n) diverge.*

Proof. If (x_n) would converge. Then, $\lim(x_{n_k}) = \lim(x_n) = \lim(x_{n_j})$, which is not the case. Thus (x_n) diverges. $\square$

Lecture 11: October 7th, 2024.

Example 2.4.2. Show that $\lim a^n = 0$, $0 < a < 1$. From last class, we used subsequence with even indices to prove this. This class we will use 1-tail.

Proof. The 1-tail of (a^n) (from last class we know that a^n converges) will simply be $a^{n+1} \implies (a^{n+1})$ converges and $\lim(a^n) = \lim(a^{n+1}) =: x$. Then,

$$\lim(a \cdot a^n) = a \cdot \lim(a^n) = a \cdot x$$

$$\implies ax = x \implies x - ax = 0 \implies x\underbrace{(1-a)}_{\neq 0} = 0 \implies x = 0$$ $\square$

Exercise: Show that $\lim(\sqrt[n]{a}) = 1$
Hint: Use subsequence of even indices.

Example 2.4.3. Consider the sequence (x_n), recursively defined by $x_1 := 2, x_{n+1} := 2 - \frac{1}{x_n}$. Show $\forall n \in \mathbb{N} : 1 < x_n \leq 2$. (especially, $\forall n \in \mathbb{N} : x_n \neq 0$, which means the sequence is well-defined). Then show that (x_n) is decreasing and bounded from below; finally, determine $\lim(x_n)$.

Proof. We will first use induction to prove $1 < x_, \leq 2:$

- $\underline{n = 1}$: $x_1 = 2$

- $\underline{n \to n+1:}$ Assume that $1 < x_n \le 2$. Then,

$$\begin{aligned} \implies \frac{1}{2} \le \frac{1}{x_n} &< 1 \\ \implies -1 < -\frac{1}{x_n} &\le -\frac{1}{2} \\ \implies 1 < 2 - \frac{1}{x_n} &= x_{n+1} \le \frac{3}{2} < 2. \end{aligned}$$

$\implies \forall n \in \mathbb{N} : 1 < x_n \le 2$.
Now, we need to prove that (x_n) is decreasing. Consider $x_n - x_{n+1} = x_n - 2 + \frac{1}{x_n} = \frac{x_n^2 - 2x_n + 1}{x_n} = \frac{(x_n - 1)^2}{x_n} > 0$.
$\implies \forall n \in \mathbb{N} : x_{n+1} < x_n$. Thus (x_n) is strictly decreasing and is bounded below by 1 (proven above) $\implies$ convergent by monotone convergence theorem.
Let $x := \lim(x_n)$; $x_{n+1} = 2 - \frac{1}{x}$

$$\begin{aligned} \implies \lim(x_{n+1}) &= \lim\left(2 - \underbrace{\frac{1}{x_n}}_{\neq 0}\right) \\ \implies \underbrace{x}_{\text{1-tail}} &= 2 - \frac{1}{x} \\ \implies x^2 &= 2x - 1 \implies x^2 - 2x + 1 = 0 \\ \implies (x-1)^2 &= 0 \implies x = 1 \end{aligned}$$

$\implies \lim(x_n) = 1$ □

Theorem 2.12. ***(Bolzano-Weierstrass).*** *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. Let (x_n) be bounded; let $M > 0 : \forall n \in \mathbb{N} : |x_n| \le M$ i.e. $\forall n \in \mathbb{N} : x_n \in [-M, M]$. Divide this interval into 2 closed sub-interval of equal width. At least 1 of these subintervals contains infinitely many terms of (x_n). Pick one such subinterval and call I_1. Divide I_1 into 2 other subintervals of equal width. At least 1 of these subintervals, called I_2, contains infinitely many terms of (x_n) etc.

We thus obtained a nested-sequence $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ of closed and bounded intervals. So by the nested interval property, $\bigcap_{k \in \mathbb{N}} I_k \neq \emptyset$. Let

$x \in \bigcap_{k\in\mathbb{N}} I_k$. We will now construct a subsequence (x_{n_k}) of (x_n) that converge to x. Let $x_{n_1} \in I_1$ (which contains infinitely many terms of (x_n)). Since I_2 also contains infinitely many terms of (x_n), $\exists n_2 \in \mathbb{N}, n_2 > n_1 : x_{n_2} \in I_2$. Similarly, $\exists n_3 \in \mathbb{N}, n_3 > n_2 : x_{n_3} \in I_3$, etc.

So we've obtained that $n_1 < n_2 < n_3 < \cdots$ s.t. $\forall k \in \mathbb{N} : x_{n_k} \in I_k$. Note that $|I_k| = \frac{2M}{2^k} \forall k \in \mathbb{N}$ (Prove this by induction).

$$\Big|\underbrace{x_{n_k}}_{\in I_k} - \underbrace{x}_{\in I_k}\Big| \le |I_k| = \frac{2M}{2^k} = \underbrace{2M\underbrace{\left(\frac{1}{2}\right)^k}_{\text{Null Seq.}}}_{\text{Null Seq.}}$$

By the null sequence convergence criterium, (x_{n_k}) converges to x. □

2.5 Cauchy Sequences

Definition 2.11. A sequence (x_n) of real numbers is called a **Cauchy sequence** if

$$\forall \varepsilon > 0, \exists N \forall n, m \ge N : |x_n - x_m| < \varepsilon$$

Example 2.5.1. Show that $x_n := \frac{1}{n}$ is a Cauchy sequence

Proof. Let $\varepsilon > 0$. $|x_n - x_m| = \left|\frac{1}{n} - \frac{1}{m}\right| = \left|\frac{1}{n} + \left(-\frac{1}{m}\right)\right| \le \left|\frac{1}{n}\right| + \left|-\frac{1}{m}\right| = \frac{1}{n} + \frac{1}{m}$. Then, $\frac{1}{n} < \frac{\varepsilon}{2} \iff n > \frac{\varepsilon}{2}$. Now, let $N > \frac{2}{\varepsilon}, n, m \ge N$. Then, $\frac{1}{n} < \frac{\varepsilon}{2} \wedge \frac{1}{m} < \frac{\varepsilon}{2} \implies \forall n, m \ge N : \frac{1}{n} + \frac{1}{m} < \varepsilon \implies |\frac{1}{n} - \frac{1}{m}| < \varepsilon \implies (x_n)$ is Cauchy. □

Theorem 2.13. *Every convergent sequence of real numbers is a Cauchy sequence.*

Proof. Let (x_n) be convergent and define $x := \lim(x_n)$. Let $\varepsilon > 0$. Then, $\exists N \forall n \ge N : |x_n - x| < \frac{\varepsilon}{2}$. Let $n, m \ge N$. Then $|x_n - x| < \frac{\varepsilon}{2} \wedge |x_m - x| < \frac{\varepsilon}{2}$. Thus, $|x_n - x_m| = |(x_n - x) + (x - x_m)| \le |x_n - x| + |x_m - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$.
Thus, $\forall n, m \ge N : (x_n - x_m) < \varepsilon \implies (x_n)$ is Cauchy. □

End of Lecture

Theorem 2.14. *Every Cauchy Sequence is bounded*

Proof. Let (x_n) be a Cauchy sequence, let $\varepsilon := 1$. Then, $\exists N \forall n, m \ge N : |x_n - x_m| < 1$. Let $m := N$. Then, $\forall n \ge N : |x_m - x_N| < 1$. Now $|x_n| = |x_n - x_N + x_N| \le \underbrace{|x_n - x_N|}_{<1} + |x_n| < 1 + |x_N|$ i.e. $\forall n \ge N : |x_n| < 1 + |x_N|$. Let

$M := \max\{|x_1|, |x_2|, \dots, |x_{N-1}|, 1 + |x_N|\}$. Then $\forall n \in \mathbb{N} : |x_n| \le M \implies (x_n)$ is bounded. □

Lecture 12: October 9th, 2024.

Theorem 2.15. *Every Cauchy sequence converges*

Proof. Let (x_n) be a Cauchy sequence. Our first task is to finda candidate for the limit of (x_n). Since (x_n) is a Cauchy sequence, it is bounded. Thus, by Bolzano-Weierstrass, (x_n) has a convergent subsequence (x_{n_k}). Let $x := \lim(x_{n_k})$. We will show that x is not only the limit of (x_{n_k}) but is even the limit of (x_n).
Let $\varepsilon > 0$. We have the following:

- Since (x_{n_k}) converges to $x : \exists k \forall k \ge K : |x_{n_k} - x| < \frac{\varepsilon}{2}$ $\quad (*)$
- Since (x_n) is Cauchy : $\exists \tilde{N} \forall n, m \ge \tilde{N} : |x_n - x_m| < \frac{\varepsilon}{2}$ $\quad (**)$

Let $N := \max\{K, \tilde{N}\}$. Let $n \ge N$. Then,

$$|x_n - x| = |x_n - x_{n_N} + x_{n_N} - x| \le \underbrace{|x_n - x_{n_N}|}_{(1)} + \underbrace{|x_{n_N} - x|}_{(2)}$$

$(2) : |x_{n_N} - x| < \frac{\varepsilon}{2}$ by $(*)$ and $N \ge K$.
$(1) : |x_n - x_{n_N}| < \frac{\varepsilon}{2}$ by $(**)$, $n \ge N \ge \tilde{N}$ and $n_N \ge N \ge \tilde{N}$.
$\implies \forall n \ge N : |x_n - x| \le (1) + (2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \implies (x_n)$ converges to x. □

Now, we can combine this with the other theorem to get

Theorem 2.16. *A sequence converges $\iff$ it is a Cauchy sequence.*

Remark 2.4. *The definition of Cauchy sequence is symmetrical in n and m. Thus, we can assume without loss of generality that $n \ge m$ v.v.*

2.6 Contractive Sequence

Definition 2.12. A sequence (x_n) is called **contractive** if $\exists 0 < c < 1$ s.t. $\forall n \in \mathbb{N} : |x_{n+2} - x_{n+1}| \le c \cdot |x_{n+1} - x_n|$

Theorem 2.17. *Every contractive sequence converges.*

Before we do this, a quick reminder of sums of finite geometric series.

Definition 2.13. Let $a \in \mathbb{R}$ and $a \neq 1$. Consider the following series

$$1 + a + a^2 + \cdots + a^n \tag{2.9}$$

This is called a **finite geometric series**

What is the sum of the geometric series? Well...Define the follows:

$$I : s = 1 + a + a^2 + a^3 + \cdots + a^{n-1} + a^n$$
$$II : as = a + a^2 + a^3 + \cdots + a^{n-1} + a^n + a^{n+1}$$

$\implies I - II : s - as = 1 - a^{n+1} \implies s(1-a) = 1 - a^{n+1} \implies s = \frac{1-a^{n+1}}{1-a}$ $(a \neq 1)$.
Now let's back to the proof.

Proof. Let (x_n) be contractive. Since we don't have a candidate for $\lim(x_n)$, we will show that (x_n) is a Cauchy sequence. Then,

$$\begin{aligned} |x_{n+2} - x_{n+1}| &\leq c \cdot |x_{n+1} - x_n| \leq c\,[c \cdot |x_n - x_{n+1}|] = c^2 |x_n - x_{n+1}| \\ &\leq c^3 |x_{n-1} - x_{n-2}| \leq \cdots \leq c^n |x_2 - x_1| \qquad (*) \end{aligned}$$

Without loss of generality, let $m \geq n$ and $\varepsilon > 0$. Then,

$$\begin{aligned} |x_m - x_n| &= |x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2} - \cdots + x_{n+1} - x_n| \\ &\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\ &\leq c^{m-2}|x_2 - x_1| + c^{m-3}|x_2 - x_1| + \cdots + c^{n-1}|x_2 - x_1| && \text{by}(*) \\ &= |x_2 - x_1| \cdot (c^{n-1} + c^n + \cdots + c^{m-2}) \\ &= |x_2 - x_1| \cdot c^{n-1}(1 + c + c^2 + \cdots + c^{(m-2)-(n-1)}) \\ &= c^{n-1}|x_2 - x_1| \cdot (\underbrace{1 + c + c^2 + \cdots + c^{(m-n-1}}_{\text{finite geometric series}}) \\ &= c^{n-1}|x_2 - x_1| \cdot \frac{1 - \overbrace{c^{m-n}}^{>0}}{1-c} && (c \neq 1) \\ &< c^{n-1}|x_2 - x_1| \cdot \frac{1}{1-c} \\ &= \underbrace{\underbrace{\frac{1}{c(c-1)} \cdot |x_2 - x_1|}_{\text{constant}} \cdot \underbrace{c^n}_{\text{null seq.}}}_{\text{null seq}} && 0 < c < 1 \end{aligned}$$

Thus $\exists N \forall n \geq N : \frac{1}{c(1-c)}|x_2 - x_1| \cdot c^n < \varepsilon$. Thus $\forall m \geq n \geq N : |x_m - x_n| < \varepsilon$.
Thus (x_n) is a Cauchy sequence $\implies$ (x_n) converges. □

Example 2.6.1. Let $x_1 = 2, x_{n+1} = 2 + \frac{1}{x_n}, \forall n \in \mathbb{N}$ (i.e. $x_2 = 2 + 1/2 = 2.5, x_3 = 2 + \frac{1}{2.5} = 2.4$, etc.). Prove that this sequence converges and find its limit.

Proof. Note that (x_n) is not monotone thus our method of using monotone sequence to prove convergence will not work. Instead, we will show that it is contractive.
Claim: $\forall n \in \mathbb{N}, n \geq 2$.
We will prove this using induction:

- $\underline{n = 1:}$ $x_1 = 2 \geq 2$.
- $\underline{n \implies n+1}$ Suppose that $x_n \geq 2$. Then,

$$x_{n+1} = 2 + \frac{1}{x_n} > 2 \geq 2$$

Notice that this also shows that x_n is well-defined as $x_n \neq 0 \implies x_{n+1} = 2 + \frac{1}{x_n}$ exists for all $n \in \mathbb{N}$.
Now, we will show that it's contractive:

$$\begin{aligned} |x_{n+2} - x_{n+1}| &= \left|(2 + \frac{1}{x_{n+1}}) - (2 + \frac{1}{x_n})\right| = \left|\frac{1}{x_{n+1}} - \frac{1}{x_n}\right| \\ &= \left|\frac{x_n - x_{n+1}}{x_n x_{n+1}}\right| = \frac{|x_{n+1} - x_n|}{\underbrace{|x_n|}_{>0} \cdot \underbrace{|x_{n+1}|}_{>0}} \\ &= \frac{1}{\underbrace{x_n}_{\geq 2} \cdot \underbrace{x_{n+1}}_{\geq 2}} |x_{n+1} - x_n| \leq \frac{1}{2 \cdot 2} |x_{n+1} - x_n| = \frac{1}{4} |x_{n+1} - x_n| \end{aligned}$$

We can define $c := \frac{1}{4} \implies \forall n \in \mathbb{N}, \exists 0 < c < 1 : |x_{n+2} - x_{n+1}| \leq c \cdot |x_{n+1} - x_n| \implies (x_n)$ is contractive $\implies$ it converges by the above theorem.

Let $x := \lim(x_n)$. Then, $\forall n \in \mathbb{N} : x_{n+1}$

$$\implies \underbrace{\lim(x_{n+1})}_{=x} = \lim\left(2 + \frac{1}{x_n}\right) = 2 + \frac{1}{\lim(x_n)} = 2 + \frac{1}{x}$$

$$\implies x = 2 + \frac{1}{x} \implies x^2 - 2x - 1 = 0$$

$$\implies x = \frac{2 \pm \sqrt{4 - 4(-1)}}{2} = 1 \pm \sqrt{2}$$

Now, notice that we've shown before that $(x_n) \geq 2 \implies \lim(x_n) \neq 1 - \sqrt{2} < 0 \implies \lim(x_n) = 1 + \sqrt{2}$. Thus (x_n) converges to $1 + \sqrt{2}$. $\square$

End of Lecture

Lecture 13: October 21st, 2024.

2.7 Sequences Diverging to Infinity

Definition 2.14. Let (x_n) be a sequence of real numbers.

- We say that (x_n) diverges to $+\infty$ denoted as

$$\lim(x_n) = +\infty \tag{2.10}$$

  If $\forall M \in \mathbb{R}, \exists N \in \mathbb{N} \forall n \geq N : x_n \geq M$

- We say that (x_n) diverges to $-\infty$ denoted as

$$\lim(x_n) = -\infty \tag{2.11}$$

  If $\forall M \in \mathbb{R}, \exists N \in \mathbb{N} \forall n \geq N : x_n < M$

Example 2.7.1. Show that $\lim(n) = +\infty$

Proof. Let $M \in \mathbb{R}$. By the Archimedian property, $\exists N \in \mathbb{N} : N > M$. Then, $\forall n \geq N : n \geq N > M$ i.e. $\forall n \geq N : n > M$. Thus (n) diverges to $+\infty$. □

Show that $\lim(-n) = -\infty$

Proof. Let $M \in \mathbb{R}$. By the Archimedian property, $\exists N \in \mathbb{N} : N > -M$. Then, $\forall n \geq N : n \geq N > -M$ i.e. $\forall n \geq N : n > -M \iff -n < M$. Thus $(-n)$ diverges to $-\infty$. □

Let $a > 1$. Show that $\lim(a^n) = +\infty$.

Proof. If $a > 1 \implies 0 < \frac{1}{a} < 1 \implies \lim\left(\left(\frac{1}{a}\right)^n\right) = 0$. Let $M \in \mathbb{R}$. If $M \leq 0$. Then, $\forall n \in \mathbb{N} : a^n > M$. Now, let $M > 0$ and $\varepsilon := \frac{1}{M} > 0$. Then, $\exists N \forall n \geq N$: $\left|\left(\frac{1}{a}\right)^n - 0\right| = \frac{1}{a^n} < \varepsilon = \frac{1}{M} \implies \forall n \geq N : \frac{1}{a^n} < \frac{1}{M} \iff \forall n \geq N : a^n > M$. Thus, $\lim(a^n) = +\infty$ □

Theorem 2.18. *If* $\lim(x_n) = +\infty$ *or* $-\infty$. *Then,* (x_n) *diverges*

Theorem 2.19. *Let* $(a_n), (b_n)$ *be sequences s.t.* $\lim(b_n) = +\infty$ *and* $\exists K \forall n \geq K : a_n \geq b_n$. *Then,* $\lim(a_n) = +\infty$

Proof. Let $M \in \mathbb{R}$. Since $\lim(b_n) = +\infty, \exists \tilde{N} \forall n \geq \tilde{N} : b_n \geq M$. Let $N = \max\{K, \tilde{N}\}$. Then, $\forall n \geq N : a_n \geq b_n > M \implies a_n > M \implies \lim(a_n) = +\infty$ □

Example 2.7.2. Let $a > 1$. Then, $\lim(a^n) = +\infty$

Proof. Consider the sequence $\left(\frac{1}{a^n}\right) = \left(\left(\frac{1}{a^n}\right)\right)^k$ where $0 < \frac{1}{a} < 1$. Thus $\lim(\frac{1}{a^n}) = 0$.
Let $M \in \mathbb{R}$. If $M \leq 0$, then $\forall n \in M, a^n > M$. Assume that $M > 0$. Then, $\lim(\frac{1}{a^n}) = 0$ thus $\exists N \forall n \geq N : 0 < \frac{1}{a^n} < \varepsilon := \frac{1}{M} > 0 \implies \forall n \geq N : a^n > M \implies \lim(a^n) = +\infty$. □

Show that $\lim((1+\frac{1}{n})^{n^2}) = +\infty$

Proof. Consider the sequence:

$$\left(1+\frac{1}{n}\right)^{n^2} = \left[\left(1+\frac{1}{n}\right)^n\right]^n$$

The sequence within is increasing and is equal o 2 for $n = 1$ thus $\geq 2 \forall n \in \mathbb{N}$. i.e. $\forall n \in \mathbb{N}$:

$$\left(1+\frac{1}{n}\right)^{n^2} \geq 2^n$$

Where $\lim(2^n) = +\infty$ and thus by the last theorem, we get that $\lim((1+\frac{1}{n})^{n^2}) = +\infty$ □

3 Point-Set Topology

We will start by generalize the notion of open and close interval.

3.1 Open- and Closeness

Definition 3.1. A subset $\mathscr{U} \subseteq \mathbb{R}$ is called **open** if $\forall x \in \mathscr{U}, \exists \varepsilon > 0 : V_\varepsilon(x) \subseteq \mathscr{U}$

Theorem 3.1. *Every open interval is open*

Proof. Let I be an open interval. Then, consider 4 cases:

1. $\underline{I = \mathbb{R}:}$ This is trivial.
2. $\underline{I =]a, +\infty[, a \in \mathbb{R}}$: Let $x \in I$ and $\varepsilon := x - a > 0$. Then,

$$V_\varepsilon(x) =]x - \varepsilon, x + \varepsilon[=]x - (x - a), x + (x - a)[=]a, 2x - a[\subseteq]a, +\infty[$$

$$\implies V_\varepsilon(x) \subseteq I \implies I \text{ is open}$$

3. $\underline{I =]-\infty, b[, b \in \mathbb{R}}$: Let $x \in I$ and $\varepsilon := b - x$. Then,

$$V_\varepsilon(x) =]x - (b - x), x + (b - x)[=]2x - b, x[\subseteq]-\infty, b[$$

$$\implies V_\varepsilon(x) \subseteq I \implies I \text{ is open.}$$

4. $\underline{I =]a, b[}$: Let $\varepsilon := \min\{x - a, b - x\}$. Then,

$$\begin{aligned} V_\varepsilon(x) &=]x - \varepsilon, x + \varepsilon[\\ &\subseteq]x - \varepsilon, b[&& \varepsilon < b - x \\ &\subseteq]a, b[&& \varepsilon < x - a \end{aligned}$$

$$\implies V_\varepsilon(x) \subseteq I =]a, b[\implies I \text{ is open.}$$

□

Example 3.1.1. Consider the following open sets:

(1) $\emptyset$ is open by definition of openess.

(2) $\mathbb{R}$ is open by the above trivial case.

(3) $\mathbb{R} \backslash \{0\}$ is open

Proof. Let $x \in \mathbb{R}\backslash\{0\}$. Then, $x > 0$ or $x < 0$.
If $x > 0$, then $x \in]0, +\infty[$ which is an open interval and thus open.
$\implies \exists \varepsilon > 0 : V_\varepsilon x \subseteq]0, +\infty[\subseteq \mathbb{R}\backslash\{0\}$.
If $x < 0$ then $x \in]-\infty, 0[$ which is open $\implies \exists \varepsilon > 0 : V_\varepsilon(x) \subseteq]-\infty, 0[\subseteq \mathbb{R}\backslash\{0\}$
$\implies \mathbb{R}\backslash\{0\}$ is open. □

Theorem 3.2. *Arbitrary unions of open sets are open i.e. Let J be an arbitrary index set and $\mathcal{U}_j$ be open $\forall j \in J$. Then,*

$$\mathcal{U} = \bigcup_{j \in J} \mathcal{U}_j$$

is open.

Proof. Let $x \in \mathcal{U}$. Then, $\exists j \in J : x \in \mathcal{U}_j$. Since $\mathcal{U}_j$ is open, $\exists \varepsilon > 0 : V_\varepsilon x \subseteq \mathcal{U}_j \subseteq \bigcup_{j \in J} \mathcal{U}_j = \mathcal{U} \implies V_\varepsilon(x) \subseteq \mathcal{U} \implies \mathcal{U}$ is open. □

Theorem 3.3. *<u>Finite</u> intersection of open sets are open i.e. If $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n : n \in \mathbb{N}$ are open. Then,*

$$\mathcal{U} := \bigcap_{j=1}^{n} \mathcal{U}_j$$

is open.

Proof. Let $x \in \mathcal{U}$. Then, $\forall 1 \leq j \leq n : x \in \mathcal{U}_j$; each $\mathcal{U}_j$ is open. Therefore $\exists \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n : V_{\varepsilon_1}(x) \subseteq \mathcal{U}_1, \ldots, V_{\varepsilon_n}(x) \subseteq \mathcal{U}_n$. Let $\varepsilon := \min\{\varepsilon_1, \ldots, \varepsilon_n\}$. Then, $\varepsilon > 0$, and $V_\varepsilon(x) \subseteq V_{\varepsilon_1}(x) \subseteq \mathcal{U}_1 \ldots, V_\varepsilon(x) \subseteq V_{\varepsilon_n}(x) \subseteq \mathcal{U}_n$
$\implies \forall 1 \leq j \leq n : V_\varepsilon(x) \subseteq \mathcal{U}_j \implies V_\varepsilon(x) \subseteq \bigcap_{j=1}^{n} \mathcal{U}_j = \mathcal{U}$
$\implies V_\varepsilon(x) \subseteq \mathcal{U} \implies \mathcal{U}$ is open □

Remark 3.1. *This result does not, in general, extend to infinite intersection.*

Example 3.1.2. $\mathcal{U}_n :=]-\frac{1}{n}, \frac{1}{n}[, n \in \mathbb{N}$. Then, all $\mathcal{U}_n$ are open but $\bigcap_{n \in \mathbb{N}} \mathcal{U}_n = \{0\}$ which is not open.

Definition 3.2. A subset $A \subseteq \mathbb{R}$ is called **closed** if $A^c := \mathbb{R}\backslash A$ is open.

Theorem 3.4. *Every closed interval is closed.*

Proof. Consider the following cases:

1. <u>$A = \mathbb{R}$:</u> $A^c = \emptyset$ which is open $\implies \mathbb{R}$ is closed.

2. <u>$A = [a, +\infty[$:</u> $A^c =]-\infty, a[$ which is open $\implies A$ is closed.

3. $\underline{A =]-\infty, a]}$: $A^c =]a, +\infty[$ which is open $\implies$ A is closed.
4. $\underline{A = [a, b]}$: $A^c = \underbrace{]-\infty, a[}_{\text{open}} \cup \underbrace{]b, +\infty[}_{\text{open}} \implies A^c$ is open $\implies$ A is closed.

Thus, in all cases, A is closed. □

Example 3.1.3. $\emptyset, A = \emptyset \implies A^c = \mathbb{R}$ is open $\implies \emptyset$ is closed.

Theorem 3.5. *The only subsets that can be both open and closed are $\emptyset$ and $\mathbb{R}$.*

Proof. We need not to prove as this theorem is unimportant in this class. □

End of Lecture —

Lecture 14: October 23rd, 2024.

Theorem 3.6. *Consider the following*

1. *Finite unions of closed sets are closed.*
2. *Arbitrary intersections of closed sets are closed.*

Proof. We will first prove for 1. Let $A_1, \ldots, A_n$ be closed and $A := \bigcup_{i=1}^n A_i$. Then $A^c = \mathbb{R} \setminus A = \mathbb{R} \setminus \bigcup_{i=1}^n A_i = \bigcap_{i=1}^n (\mathbb{R} \setminus A_i)$ is open since $\mathbb{R}$ is open $\implies$ A is closed.
For 2, let I be an index set $\forall i \in I : A_i$ is closed, $A = \bigcap_{i \in I} A_i$. Then, $A^c = \mathbb{R} \setminus A = \mathbb{R} \setminus \bigcap_{i \in I} A_i = \bigcup_{i \in I} (\mathbb{R} \setminus A_i)$ is open since $\mathbb{R}$ is open $\implies$ A is closed. □

Remark 3.2. *The structure of closed sets is more complicated than that of open sets, e.g. there are closed subsets of $\mathbb{R}$ which are not countable union of closed interval.*

Example 3.1.4. Cantor's set.

Definition 3.3. Let $A \subseteq \mathbb{R}$. We say that a sequence (x_n) is **in A** if $\forall n \in \mathbb{N}, x_n \in A$.

Theorem 3.7. *Let $A \subseteq \mathbb{R}$ be closed and let (x_n) be a convergent sequence in A. Let $x := \lim(x_n)$. Then $x \in A$.*

Proof. Let (x_n) be a sequence in A. Assume that $x \notin A \implies x \in A^c$ where A^c is open. Thus $\exists \varepsilon > 0 : V_\varepsilon(x) \subseteq A^c$. Since $x = \lim(x_n), \exists N \forall n \geq N : x_n \in V_\varepsilon(x) \subseteq A^c \implies \forall n \geq N : x_n \notin A$ ↯. Thus $x \in A$. □

Remark 3.3. *The condition that A is closed is essential: Consider $A =]0, 1]$ which is not closed (nor is it closed). Let $x_n := \frac{1}{n}$. Then, (x_n) is a convergent sequence in A but $0 = \lim(x_n) \notin A$.*

3.2 Boundary and Compactness

Definition 3.4. Let $A \subseteq \mathbb{R}$. A point $x \in R$ is called a **boundary point** of A if $\forall \varepsilon > 0 : V_\varepsilon(x) \cap A \neq \emptyset \wedge V_\varepsilon(x) \cap A^c \neq \emptyset$.
The set of all boundary points of A is called the **boundary** of A and is denoted as ∂A.

Example 3.2.1. Let $I = [a, \infty[$. Then, $\partial I = \{a\}$.

Proof. Let $x > a$. We know that $]a, \infty[$ is open. So, $x \in]a, \infty[\implies \exists \varepsilon > 0 : V_\varepsilon(x) \subseteq]a, \infty[\subseteq I \implies V_\varepsilon(x) \cap I^c = \emptyset \implies x \notin \partial I$.
Now, let $x < a$. Then, consider $]-\infty, a[$ which is open. So $x \in]-\infty, a[\implies \exists \varepsilon > 0 : V_\varepsilon(x) \subseteq]-\infty, a[\subseteq I \implies V_\varepsilon(x) \cap I^c = \emptyset \implies x \notin \partial I$.
Finally, $\forall \varepsilon > 0 : V_\varepsilon(a) \cap I \neq \emptyset$, since $a + \frac{\varepsilon}{2} \in V_\varepsilon(a) \cap I \neq \emptyset$ and similarly $a - \frac{\varepsilon}{2} \in V_\varepsilon(a) \cap I^c \implies V_\varepsilon(x) \cap I^c \neq \emptyset \implies V_\varepsilon(x) \cap I \neq \emptyset \wedge V_\varepsilon(x) \cap I^c \neq \emptyset \implies a \in \partial I \implies \partial I = \{a\}$. (since any x less or larger than a is not a boundary point). $\square$

Example 3.2.2. $\partial[a, b] = \partial[a, b[= \partial]a, b] = \partial]a, b[= \{a, b\}$

Theorem 3.8. *Let $A \subseteq \mathbb{R}$. Then,*

1. *A is open iff A does not contain any of its boundary points i.e. $A \cap \partial A = \emptyset \iff \partial A \subseteq A^c$.*
2. *A is closed iff A contain all of its boundary points i.e. $\partial A \subseteq A$.*

Caution: Most subsets of $\mathbb{R}$ are neither open nor closed!

Example 3.2.3. $I :=]0, 1]$ is neither open nor closed: $\partial(]0, 1]) = \{0, 1\}$, where $0 \notin I$ but $1 \in I \implies I$ is neither open nor closed.

End of Lecture

3.3 Interior and Closure

Definition 3.5. Let $S \subseteq \mathbb{R}$.

- The **interior** of S, denoted as $\mathring{S}$ or $\operatorname{int}(S)$ is defined as $\mathring{S} := S \setminus \partial S$.
- The **Closure** of S, denoted as $\overline{S}$ is defined as $\overline{S} := S \cup (\partial S)^C$.

Since $A \setminus B = A \cap B^C$, we can also express the interior as $\mathring{S} = S \cap (\partial S)^C$, which meaks the definitions of interior and closure more symmetrical.
The reason for naming these concepts will become clear after the next theorem.

Lecture 15: October 28th

Theorem 3.9. *Let $S \subseteq \mathbb{R}$. Then,*

(a) $\mathring{S} = \bigcup_{\substack{U \subseteq S \\ U \text{ open}}} U$

(b) $\overline{S} = \bigcap_{\substack{A \supseteq S \\ A \text{ closed}}} A$

Remark 3.4. *Since arbitrary unions of open sets are open, it follows directly from this theorem that $\mathring{S}$ is open, and that it is, in fact, the largest open subset of S. Similarly, since arbitrary intersection of closed sets are closed, it follows that $\overline{S}$ is the smallest closed superset of S.*

Proof. (a) ($\Longrightarrow$) Let $x \in S \setminus \partial S$. Since $x \notin \partial S$, we have:

$$\neg \forall \varepsilon > 0 : V_\varepsilon(x) \cap S \neq \emptyset \wedge V_\varepsilon(x) \cap S^C \neq \emptyset$$
$$\equiv \exists > 0 : V_\varepsilon(x) \cap S = \emptyset \vee V_\varepsilon(x) \cap S^C = \emptyset$$

But since $x \in S$, it holds that $x \in V_\varepsilon(x) \cap S$ i.e. $V_\varepsilon(x) \cap S \neq \emptyset$. Hence, it must hold that $V_\varepsilon(x) \cap S^C = \emptyset$, which means that $x \in V_\varepsilon(x) \subseteq S$. Since $V_\varepsilon(x)$ is open, this implies that $x \in \bigcup_{\substack{U \subseteq S \\ U \text{ open}}} U$.

($\Longleftarrow$) Let $x \in \bigcup_{\substack{U \subseteq S \\ U \text{ open}}} U$. Then $x \in S$, And: $\exists U \subseteq S$ open s.t. $x \in U$. Let $\varepsilon > 0$ s.t. $V_\varepsilon(x) \subseteq U$. Since $U \subseteq S$, this implies that $V_\varepsilon(x) \cap S^C = \emptyset$, which means that $x \notin \partial S$. We thus have both $x \in S$ and $x \notin \partial S$, which shows that $x \in S \setminus \partial S$.

(b) We will prove this using part (a) and taking complements. Note that

$$(\mathring{S})^C = (S \cap (\partial S)^C)^C = S^C \cup \partial S = S^C \cup (\partial S^C) = \overline{S^C}$$

Thus,

$$\overline{S^C} = (\mathring{S})^C = \left[\bigcup_{\substack{U \subseteq S \\ U \text{ open}}} U \right]^C = \bigcap_{\substack{U \subseteq S \\ U \text{ open}}} (U^C)$$

Hence

$$\overline{S} = \bigcap_{\substack{U \subseteq S^C \\ U \text{ open}}} (U^C)$$

(where $U \subseteq S^C$, U open $\iff U^C \supseteq S, U^C$ closed)

$$= \bigcap_{\substack{U^C \supseteq S \\ U^C \text{ closed}}} (U^C) = \bigcap_{\substack{A \supseteq S \\ A \text{ closed}}} A$$

$\square$

3.4 Sequences and Toplogy

In this section, we will discuss some important connections between the topology of a set and the properties of sequences within this set.

Definition 3.6. Let $S \subseteq \mathbb{R}$, and let (x_n) be a sequence of real numbers. We say that (x_n) is **in** S, if $\forall n \in \mathbb{N} : x_n \in S$.

Theorem 3.10. *Let $A \subseteq \mathbb{R}$ be closed and let (x_n) be a convergent sequence in A. Then,* $\lim(x_n) \in A$ *as well.* [1]

Proof. Assume that $x := \lim(x_n) \notin A$. Then, $x \in A^C$, which is open. Hence, $\exists \varepsilon > 0 : V_\varepsilon(x) \subseteq A^C$. Furthermore, since $x = \lim(x_n)$, we have that $\exists N \forall n \geq N : x_n \in V_\varepsilon(x)$, which means that $\forall n \geq N : x_n \notin A$. $\lightning$ since (x_n) is a sequence in A. Thus our assumption was wrong and hence $\lim(x_n) \in A$. $\square$

Definition 3.7. A set $A \subseteq \mathbb{R}$ is **compact** if A is both closed and bounded.

Definition 3.8. A set $A \subseteq \mathbb{R}$ is called **sequentially compact** if every sequence (x_n) in A has a convergent subsequence (x_{n_k}) whose limit lies in A.

Theorem 3.11. *A set $A \subset \mathbb{R}$ is compact iff it is sequentially compact.*

Proof. ($\Longrightarrow$) Let A be compact. Then, A is bounded. Let (x_n) be arbitrary sequence in A. By Bolzano-Weierstrass theorem, (x_n) has convergent subsequence (x_{n_k}). Define $x := \lim(x_{n_k})$. Since A is compact and thus closed and (x_{n_k}) a sequence in A, it follows from a theorem proven in class that $x \in A \implies A$ is sequentially compact.

($\Longleftarrow$) Let A be sequentially compact. We will show that A is both closed and bounded

[1] This theorem states that closed subsets of $\mathbb{R}$ are closed under taking limits of sequences, which is the property they got their name from.

A is bounded: Assume that A is unbounded. Then, $\forall n \in \mathbb{N}, \exists x_n \in A : |x_n| \geq n$. Consider the sequence (x_n). Let (x_{n_k}) be any subsequence of (x_n). Then, $\forall k \in \mathbb{N} : |x_{n_k}| \geq n_k \geq k$. This implies that (x_{n_k}) is unbounded and thus diverges. We've shown that (x_n) doesn't have any convergent subsequence $\lightning$ since A is sequentially compact. Thus A is bounded.

A is closed: Assume that A is not closed. Then, A^C is not open which means that

$$\exists x \in A^C \forall \varepsilon > 0 : V_\varepsilon(x) \not\subseteq A^C \implies \exists x \in A^C \forall \varepsilon > 0 : V_\varepsilon(x) \cap A \neq \emptyset \quad (3.1)$$

Select one such x and let $\varepsilon := \frac{1}{n}$. It follows from (1) that $V_{\frac{1}{n}}(x) \cap A$ and consider the sequence (x_n). Then, $\forall n \in \mathbb{N} : |x_n - x| < \frac{1}{n}$, where $\left(\frac{1}{n}\right)$ is a null sequence. This implies that (x_n) converges to $x \in A^C$. Every subsequence (x_{n_k}) of (x_n) thus also converges to $x \notin A$ $\lightning$ since A is sequentially compact. Thus, A is closed.

Since A is closed and bounded, it's compact. □

3.5 Accumulation Points of Sequences

Definition 3.9. Let (x_n) be a sequence of real numbers. A point $x \in \mathbb{R}$ is called an **accumulation point (limit point)** of (x_n), if $\exists (x_{n_k})$ of (x_n) that converges to x.

Theorem 3.12. ***(Bolzano-Weierstrass).*** *Every bounded sequence has at least one accumulation point.*

Proof. By the original version of Bolzano-Weierstrass, every bounded sequence has at least one convergent subsequence, and thus at least one accumulation point. □

Theorem 3.13. *If a sequence (x_n) of real numbers converges to x, then x is the only accumulation point of (x_n).*

Proof. Since (x_n) is a subsequence of itself, x is indeed an accumulation point. Furthermore, since every subsequence (x_{n_k}) of (x_n) also converges to x, no other point can be an accumulation point. Consequently, x is the one and only accumulation point of (x_n). □

Theorem 3.14. *Let (x_n) be a sequence. Then, $x \in \mathbb{R}$ is an accumulation point of (x_n) iff $\forall \varepsilon > 0 : V_\varepsilon(x)$ contains infinitely many terms of (x_n).*

Proof. We need to prove it both ways:
($\Longrightarrow$) Assume that x is an accumulation point of (x_n). Then, $\exists$ subsequence (x_{n_k}) of (x_n) such that $\lim(x_{n_k}) = x \implies \forall \varepsilon > 0, \exists K \forall k \geq K : x_{n_k} \in V_\varepsilon(x) \implies$ $V_\varepsilon(x)$ contains infinitely many terms of (x_n)

($\Longleftarrow$) Let $x \in \mathbb{R} : \forall \varepsilon > 0 : V_\varepsilon(x)$ contains infinitely many terms of (x_n). Let $\varepsilon := 1$ and $n_1 \in \mathbb{N} : x_{n_1} \in V_1(x)$. Let $\varepsilon := \frac{1}{2}$, since there are infinitely many terms of $(x_n) \in V_{\frac{1}{2}}(x), \exists n_2 > n_1 : x_{n_2} \in V_{\frac{1}{2}}(x), \ldots,$ Let $\varepsilon := \frac{1}{k} : \exists$ infinitely many x_n is $V_{\frac{1}{k}}(x) \implies \exists n_k > n_{k01} : x_{n_k} \in V_{\frac{1}{k}}(x), \ldots,$
Since $n_1 < n_2 < \ldots$ and (x_{n_k}) is a subsequence of (x_n) and $\forall k \in \mathbb{N} : x_{n_k} \in V_{\frac{1}{k}}(x) \iff |x_{n_k} - x| < \frac{1}{k}$. By the null sequence criterion, (x_{n_k}) converges with $\lim(x_{n_k}) = x \implies x$ is an accumulation point of (x_n). □

Corollary 3.1. *Let (x_n) be a sequence. $x \in \mathbb{R}$ is not an accumulation point of (x_n) iff $\exists \varepsilon > 0 : V_\varepsilon(x)$ contains at most finitely many terms of (x_n).*

Proof. Immediate deduction from last theorem (contrapositive). □

Theorem 3.15. *Let (x_n) be a sequence and A be the set of accumulation points of (x_n). Then, A is always closed.*

Proof. We will show that A^c is open.
Let $x \notin A \iff x \in A^c$. By the corollary 3.1. $\exists \varepsilon > 0 : V_\varepsilon(x)$ contains finitely many term of (x_n). Let $y \in V_\varepsilon(x)$ be arbitrary. $V_\varepsilon(x)$ is open; thus, $\exists \tilde{\varepsilon} > 0 :$ $V_{\tilde{\varepsilon}}(y) \subseteq V_\varepsilon(x)$. Since $V_\varepsilon(x)$ at most finitely many terms of (x_n), then so does $V_{\tilde{\varepsilon}}(y)$. By corollary 3.1. y is not an accumulation point of $(x_n) \implies y \in A^c \implies V_\varepsilon(x) \subseteq A^c \implies A^c$ is open $\implies A$ is closed. □

Example 3.5.1. Consider the following examples:

1. If (x_n) converges to x, we know that the set of all accumulation $A = \{x\}$, which is closed.

2. $(x_n) = ((-1)^n) \implies A = \{-1, 1\}$ is closed.

3. (x_n) enumeration of $\mathbb{Q}$; $A = \mathbb{R}$ is closed.

4. $(x_n) := \left(1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$. Consider $S := \{x_n : n \in \mathbb{N}\} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. The followings are subsequences of (x_n):

$$(1, 1, 1, \ldots), \left(\frac{1}{2}, \frac{1}{2}, \ldots\right), \ldots$$

i.e. $\forall k \in \mathbb{N} : (x_n)$ contains the constant subsequence $\left(\frac{1}{k}, \frac{1}{k}, \ldots\right)$ with limits $\frac{1}{k} \implies \forall k \in \mathbb{N} : \frac{1}{k} \in A \implies S \subseteq A$. Note that S is not closed! So A cannot be equal to S. Then, $\overline{S} = S \cup \{0\} \implies \overline{S} \subseteq A$. Thus 0 must be an accumulation point. A concrete example of a subsequence of (x_n) with limit 0 is $\left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$.

In fact, $A = \overline{S}$. Prove this as an exercise.

Theorem 3.16. *Let (x_n) be a sequence and A be the set of accumulation points of (x_n)*

1. *If (x_n) is bounded from above then A is bounded from above.*
2. *If (x_n) is bounded from below then A is bounded from below.*
3. *If (x_n) is bounded then A is bounded.*

Proof. We will only prove part 1.
Let $m > 0 : \forall n \in \mathbb{N}, x_n \leq M$. Let $x \in A$, then $\exists$ a subsequence (x_{n_k}) of (x_n) with $\lim(x_{n_k}) = x$. We have the followings: $\forall k \in \mathbb{N} : x_{n_k} \leq M$. By a theorem from "limits and order" section, this implies that $x := \lim(x_{n_k}) \leq M \implies \forall x \in A : x \leq M \implies A$ is bounded from above.
Part 2 and 3. will be left as exercise. □

Corollary 3.2. *Let (x_n) be a bounded sequence; A, the set of accumulation points of (x_n). Then, A is compact.*

Proof. By Theorem 3.10. A is both closed and bounded and is thus compact. □

Let (x_n) be bounded from above and A be the set of accumulation points of (x_n). Then, A is bounded from above $\implies$ A has a supremum $x^* := \sup A$. By last class, $x^* \in \partial A$. But A is closed $\implies x^* \in A$ i.e. x^* is itself an accumulation point. Especially, $x^* = \max A$.
Similarly, if (x_n) is bounded from below. Then, A is bounded from below $\implies$ A has an infimum which is actually a minimum ($x_* := \min A$). We can put what we've just did in a more formal definitions

3.5.1 Limsup and Liminf

Definition 3.10. Let (x_n) be a sequence and A be the set of accumulation point:

1. If (x_n) is bounded from above, we call the greatest accumulation points, the *limes superior* or **limit superior**; which is denoted as

$$x^* = \limsup(x_n) \tag{3.2}$$

2. If (x_n) is bounded from below, we call the least accumulation points, the *limes inferior* or **limit inferior**; which is denoted as

$$x_* = \liminf(x_n) \tag{3.3}$$

Example 3.5.2. Consider the following examples:

1. If (x_n) converges. Then, $\lim(x_n) = \limsup(x_n) = \liminf(x_n)$.

2. $(x_n) = ((-1)^n)$. Then, $\limsup(x_n) = 1$ and $\liminf(x_n) = -1$.

3. $(x_n) := \left(1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$. We know that $A = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Then, $\limsup(x_n) = 1$ and $\liminf(x_n) = 0$.

$\limsup$ and $\liminf$ play an important role in analysis e.g. consider the power series

$$\sum_{n=1}^{\infty} a_n (x - x_0)^n \tag{3.4}$$

Its radius of convergence R can be calculated as:

$$R = \frac{1}{\limsup(\sqrt[n]{|a_n|})} \tag{3.5}$$

With the understanding that $R = 0$ if $(\sqrt[n]{|a_n|})$ is not bounded from above.

Theorem 3.17. *Let (x_n) be bounded. Then, $\forall \varepsilon > 0, \exists N \forall n \geq N : x_n \in]x_* - \varepsilon, x^* + \varepsilon[$.*

Remark 3.5. *If (x_n) is convergent, $x = x^* = x_*$ and the theorem reduces to the definition of convergent i.e. $\forall \varepsilon > 0, \exists N \forall n \geq N : x_n \in]x - \varepsilon, x + \varepsilon[$*

Proof. (Theorem 3.11) Let $\varepsilon > 0$. Assume the conclusion of the theorem does not hold. Then, $\exists$ infinitely many terms of (x_n) lies which are not in $]x_* - \varepsilon, x^* + \varepsilon[\implies \exists$ infinitely many x_n with $x_n \geq x^* + \varepsilon$ or $\exists$ infinitely many x_n with $x_n \leq x_* - \varepsilon$. Consider the 2 cases:

1. Assume $\exists$ infinitely many x_n with $x_n \geq x^* + \varepsilon$ (call this (1)): Collect these terms into a subsequence (x_{n_k}). Then, (x_{n_k}) is bounded. By Bolzano-Weierstrass theorem, (x_{n_k}) has a sub-subsequence $(x_{n_{k_j}})$; let $x := \lim(x_{n_{k_j}})$. By (1), $\forall j \in \mathbb{N} : (x_{n_{k_j}}) \geq x^* + \varepsilon$ (call this (3)). But we also have $x \in A \implies x \leq x^*$ (call this (2)). We can see that (2) and (3) contradict each other
   Thus $\exists N \forall n \geq N : x_n \in]x_* - \varepsilon, x^* + \varepsilon[$

2. Assume $\exists$ infinitely many x_n with $x_n \leq x_* - \varepsilon$: Exercise.

This completes the theorem. $\square$

End of Lecture

Lecture 17: November 4^{th}, 2024.

4 Limits of Functions

Goal: Given a function $f : D \subseteq \mathbb{R} \to \mathbb{R}$, we want to define the notion of a limit of f as x approaches c i.e. $\lim_{x\to c} f(x)$ or $\lim_{x\to c} f$ and we do it in 2 ways.

Definition 4.1. ($\varepsilon - \delta$) by Weierstrass. Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$. We say that L is the **limit** of f as x approaches c, if

$$\begin{aligned}
&\forall \varepsilon > 0, \exists \delta > 0 \forall x \in D \setminus \{c\} : |x - c| < \delta \implies |f(x) - L| < \varepsilon \\
&\equiv \forall \varepsilon > 0, \exists \delta > 0 \forall x \in D : 0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon
\end{aligned}$$

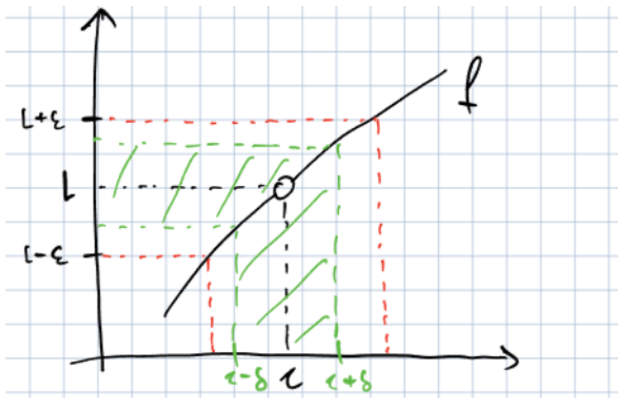


Figure 4.1: Brief illustration of definition 4.1.

Definition 4.2. Let $c \in \mathbb{R}, \varepsilon > 0$. Then, $V_\varepsilon^*(c) := V_\varepsilon(c) \setminus \{c\}$ is called the **punctured ε-neighbourhood** about c.

Using the definition 4.2, we can rephrase definition 4.1 of a limit as the followings:

$$\begin{aligned}
&\forall \varepsilon > 0, \exists \delta > 0 \forall x \in D : 0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon \\
&\equiv \forall \varepsilon > 0, \exists \delta > 0 \forall x \in D : x \in V_\delta^*(c) \implies f(x) \in V_\varepsilon(L) \\
&\equiv \forall \varepsilon > 0, \exists \delta > 0 \forall x \in V_\delta^*(c) \cap D \implies f(x) \in V_\varepsilon(L) \\
&\equiv \boxed{\forall \varepsilon > 0, \exists \delta > 0 : f(V_\delta^*(c) \cap D) \subseteq V_\varepsilon(L)}
\end{aligned}$$

Example 4.0.1. Consider the following examples:

1. $f : \mathbb{R} \to \mathbb{R}, x \mapsto 2x$. Let $c \in \mathbb{R}$. Show that $\lim\limits_{x \to c} f = 2c$.

   Proof. Let $\varepsilon > 0$ and $\delta > 0$ be arbitrary <u>for now</u>. Let $0 < |x - c| < \delta$. Then,

   $$|f(x) - L| = |2x - 2c| = 2|x - c| < 2\delta < \varepsilon \iff \delta < \frac{\varepsilon}{2}$$

   Now, let $\delta < \frac{\varepsilon}{2}$. Then, $\forall x \in \mathbb{R} : 0 < |x - c| < \delta$, it holds that $|f(x) - 2c| < \varepsilon$. Thus, $\lim\limits_{x \to c} f = 2c$. □

2. $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$. Let $c \in \mathbb{R}$. Show that, $\lim\limits_{x \to c} f = c^2$.

   Proof. Let $\varepsilon > 0$ and $\delta > 0$ be arbitrary for now. Suppose that $0 < |x - c| < \delta$. Then,

   $$\begin{aligned} |f(x) - L| = |x^2 - c^2| &= |(x - c)(x + c)| \\ &= |x - c| \cdot |x + c| \\ &< \delta \cdot |x - c + c + c| \\ &\leq \delta \cdot (|x - c| + |2c|) \\ &< \delta(\delta + 2|c|) \\ &\text{let } \delta < 1. \text{ Then,} \\ &< \delta(1 + 2|c|) < \varepsilon \iff \delta < \underbrace{\frac{\varepsilon}{1 + 2|c|}}_{>0} \end{aligned}$$

   Now, let $\delta < \min\left\{1, \frac{\varepsilon}{1+2|c|}\right\}$. Then, $\forall x \in \mathbb{R} : 0 < |x - c| < \delta$, it holds that $|f(x) - c^2| < \varepsilon$. Thus, $\lim\limits_{x \to c} f = c^2$. □

3. $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, x \mapsto \frac{1}{x}$. Let $c \in \mathbb{R} \setminus \{0\}$. Show that $\lim_{x \to c} f = \frac{1}{x}$.

   Proof. Let $\varepsilon > 0$ and $\delta > 0$ be arbitrary for now. Suppose that $0 < |x - c| < \delta$ and $x \neq 0$. Then,

   $$|f(x) - L| = \left|\frac{1}{x} - \frac{1}{c}\right| = \left|\frac{x - c}{xc}\right| = \frac{|x - c|}{|x| \cdot |c|} < \frac{\delta}{|x| \cdot |c|} \tag{1}$$

   Since $\frac{1}{|x|}$ can be arbitrarily large if x is chosen close to 0, we need to "bound x away from 0". To do so, let $\delta < \frac{|c|}{2}$. Then, $|x| = |x - c + c| =$

$$|c-(c-x)| \geq |c|-|c-x| = |c| - \underbrace{|x-c|}_{<\delta} > |c| - \delta > |c| - \frac{|c|}{2} = \frac{|c|}{2} \text{ i.e. } |x| > \frac{|c|}{2}$$

if $\delta < \frac{|c|}{2}$. Thus,

$$(1) = \frac{\delta}{|x|\cdot|c|} < \frac{\delta}{\frac{|c|}{2}\cdot|c|} = \frac{2\delta}{|c|^2} = \frac{2\delta}{c^2} < \varepsilon \iff \delta < \varepsilon \cdot \frac{c^2}{2}$$

Now, let $\delta < \min\left\{\frac{|c|}{2}, \varepsilon \cdot \frac{c^2}{2}\right\}$. Then, $\forall x \in \mathbb{R} : 0 < |x-c| < \delta$, it holds that by above $\left|f(x) - \frac{1}{x}\right| < \varepsilon$ whenever $x \in \mathbb{R} \setminus \{0\} \cap V_\delta^*(c)$. Thus, $\lim\limits_{x\to c} f = \frac{1}{c}$. □

4. Let $f : \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$. Show that $\lim\limits_{x\to 0} f = 0 \neq 1 = f(0)$.

   Proof. Let $\varepsilon > 0$ and $\delta > 0$ be arbitrary for now. Suppose that $0 < |x-c| < \delta \iff 0 < |x| < \delta$. Then,

$$|f(x) - L| = |0-0| = 0 < \varepsilon \qquad (x \neq 0 \implies f(x) = 0)$$

   Thus, for any $\delta > 0$, it follows that $|f(x) - L| < \varepsilon$ whenever $x \in V_\delta^*(0) \implies \lim\limits_{x\to 0} f = 0$. □

Definition 4.3.

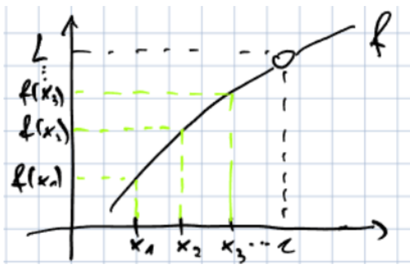


Figure 4.2: Illustration of definition 4,3

Limit of a Function By Sequences. Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$. We say that L is the limit of f as x approaches c, denoted as

$$\lim_{x\to c} f = L \tag{4.1}$$

if for all sequences (x_n) in $D \setminus \{c\}$ with $\lim(x_n) = c$. It holds that $\lim(f(x_n)) = L$

Example 4.0.2. Consider the following examples:

1. $\lim\limits_{x\to c} x^2 = c^2$. Let (x_n) be any sequence in $\mathbb{R} \setminus \{c\} : \lim(x_n) = c$. Then, $\lim(x_n^2) = [\lim(x_n)]^2 = c^2 \implies \lim\limits_{x\to c} x^2 = c^2$.

2. $\lim_{x\to c}\frac{1}{x}=\frac{1}{c}$; $c\neq 0$. Let (x_n) be any sequence in $\mathbb{R}\setminus\{0,c\}:\lim(x_n)=c$. Then, $\lim\left(\frac{1}{x_n}\right)=\frac{1}{\lim(x_n)}=\frac{1}{c}$. Thus, $\lim_{x\to c}\frac{1}{x}=\frac{1}{c}$.

End of Lecture ———

Lecture 18: November 6^{th}, 2024.

Theorem 4.1. *The $\varepsilon-\delta$ and sequential definition of the limit of a function are equivalent.*

Proof. We need to prove this in 2 directions:
($\Longrightarrow$) Let $f:D\subseteq\mathbb{R}\to B\mathbb{R}, c\in\mathbb{R}$ such that $\lim_{x\to c}f=L$ by $\varepsilon-\delta$ definition i.e. $\forall\varepsilon>0,\exists\delta>0\forall x\in V_\delta^*(c)\cap D:|f(x)-L|<\varepsilon$ call this (*). Let (x_n) be a sequence in $D\setminus\{c\}:\lim(x_n)=c$. Let $\varepsilon>0$ and $\delta>0$ as in (*), then $\exists N\forall n\geq N:x_n\in V_\delta(c)$, and $\forall n\in\mathbb{N}:x_n\in D\setminus\{c\}$. Then,

$$\begin{aligned}&\Longrightarrow \forall n\geq N:x_n\in V_\delta^*(c)\cap D\\&\Longrightarrow \forall n\geq N:|f(x_n)-L|<\varepsilon\\&\Longrightarrow \lim(f(x_n))=L\end{aligned}$$

($\Longleftarrow$) Let $f:D\subseteq\mathbb{R}\to B\mathbb{R}, c,L\in\mathbb{R}$ such that $\lim_{x\to c}f(x)=L$ by sequential definition. Assume that the $\varepsilon-\delta$ definition of the limit is <u>not satisfied</u> i.e.

$$\begin{aligned}&\neg\forall\varepsilon>0,\exists\delta>0\forall x\in V_\delta^*(c)\cap D:|f(x)-L|<\varepsilon\\&\equiv\exists\varepsilon>0,\forall\delta>0\exists x\in V_\delta^*(c)\cap D:|f(x)-L|\geq\varepsilon\end{aligned}\tag{**}$$

Let ε as in (**) and $\delta=1$. Then, $\exists x_1\in V_1^*(c)\cap D:|f(x_1)-L|\geq\varepsilon$. Now, let $\delta=\frac{1}{2}$. Then, $\exists x_2\in V_{\frac{1}{2}}^*(c)\cap D:|f(x_2)-L|\geq\varepsilon$, etc...Now, let $\delta=\frac{1}{n}$. Then, $\exists x_n\in V_{\frac{1}{n}}^*(c)\cap D:|f(x_n)-L|\geq\varepsilon$. Consider the sequence (x_n). Then, $\forall n\in\mathbb{N}:x_n\in V_{\frac{1}{n}}^*(c)\cap D\implies\forall n\in\mathbb{N}:|x_n-c|<\frac{1}{n}$. By the Null-sequence criterion, $\lim(x_n)=c$ and $\forall n\in\mathbb{N},x_n\in D\setminus\{c\}$. We can see that all condition satisfies then, by sequential definition, $\lim(f(x_n))=L\implies\exists N\forall n\geq N:|f(x_n)-L|<\varepsilon$. But by construction previously, $\forall n\in\mathbb{N}:|f(x_n)-L|\geq\varepsilon$ ↯ Thus, our assumption was wrong. Therefore, $\lim_{x\to c}f=L$ by $\varepsilon-\delta$ definition. □

Theorem 4.2. ***(2-Sequence Criterion)***[1]**.** *Let $f:D\subseteq\mathbb{R}\to\mathbb{R}, c\in\mathbb{R}$. If $\exists(x_n),(u_n)$ in $D\setminus\{c\}:\lim(x_n)=c=\lim(u_n)$ and such that $\lim(f(x_n))$ and $\lim(f(u_n))$ exist but are different. Then, $\lim_{x\to c}f$ does not exists (DNE).*

[1] for the non-exsistence of the limit of a function

Proof. Immediate consequence of the sequential definition. $\square$

Example 4.0.3. **Dirichlet function:** Let $f : \mathbb{R} \to \mathbb{R}, f(x) := \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$
Show that, $\lim_{x \to c} f$ DNE for any $c \in \mathbb{R}$.

Proof. Let $c \in \mathbb{R}$, as seen on the midterm, $\exists (x_n) \in \mathbb{Q} \setminus \{c\} : \lim(x_n) = c$ (Due to density of $\mathbb{Q}$ in $\mathbb{R}$). Similarly, $\exists (u_n) \in (\mathbb{R} \setminus \mathbb{Q}) \setminus \{c\} : \lim(u_n) = c$. Then, $f(x_n) = 1, \forall n \in \mathbb{N}$. Similarly, $f(u_n) = 0, \forall n \in \mathbb{N} \implies \lim(f(x_n)) = 1$ and $\lim(f(u_n)) = 0$ which are not the same. Then, by the 2-sequence criterion, $\lim_{x \to c} f$ DNE. $\square$

Theorem 4.3. ***(1-Sequence Criterion)***[2]**.** *Let $f : D \to \mathbb{R}, c \in \mathbb{R}$. Let (x_n) be a sequence in $D \setminus \{c\} : \lim(x_n) = 0$. If $\lim(f(x_n))$ DNE, then $\lim_{x \to c} f$ DNE.*

Proof. Immediate from sequential definition. $\square$

Example 4.0.4. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, x \mapsto \frac{1}{x}$. Show that $\lim_{x \to 0} f$ DNE.

Proof. Consider $(x_n) = \left(\frac{1}{n}\right)$. Then, $\forall n \in \mathbb{N} : x_n \in \mathbb{R} \setminus \{0\}$ and $\lim(x_n) = 0$. But, $\lim(f(x_n)) = \lim\left(\frac{1}{\frac{1}{n}}\right) = \lim(n) = +\infty$ which DNE. Then, by 1-sequence criterion, $\lim_{x \to 0} f$ DNE. $\square$

End of Lecture

4.1 Uniqueness of the Limit of a Function

Example 4.1.1. Let $D := \{0\} \cup [1,2]$ and $f : D \to \mathbb{R}, f \equiv 0$ on D.[3] Let $L \in \mathbb{R}$ be arbitrary. Show that $\lim_{x \to 0} f = L$.

Proof. (First). We will prove this using sequential definition. For any sequence $(x_n) \in D \setminus \{0\} = [1,2]$ with $\lim(x_n) = 0$, we need to show that $\lim(f(x_n)) = L$. Is this true? Well...yes, since there's no such sequence! So we get that $\lim(f(x_n)) = L$ $\square$

Proof. (Second). We will prove this using $\varepsilon - \delta$ definition. We need to show that $\forall \varepsilon > 0, \exists \delta \forall x \in V_\delta^*(c) \cap D \implies |f(x) - L| < \varepsilon$. Let $\varepsilon > 0$ be arbitrary and $\delta := 1$. Then, $V_\delta^*(0) \cap D = V_1^*(0) \cap D = \emptyset$. Thus, $\forall x \in \underbrace{V_\delta^*(0) \cap D}_{=\emptyset} : |f(x) - L| < \varepsilon$.

Thus, $\lim_{x \to c} f = L$ $\square$

[2] for the non-existence of the limit of a function
[3] $f \equiv 0$ means f is constantly 0.

Lecture 19: November 11$^{\text{th}}$, 2024.

Conjecture. Maybe this happens because 0 is an isolated point in D.

Definition 4.4. A point $x \in D \subseteq \mathbb{R}$ is called an **isolated point** of D if $\exists \delta > 0 : V^*_\delta(x) \cap D = \emptyset$.

Definition 4.5. A point $x \in \mathbb{R}$ is called a **limit point** or a **cluster point** if $\exists (x_n)$ in $D \setminus \{x\}$ with $\lim(x_n) = x$

Example 4.1.2. Let $D := \{0\} \cup [1,2]$, then 0 is an isolated point since $V^*_1(0) \cap D = \emptyset$. Any point in $[1,2]$ is a cluster point (exercise to prove).

Theorem 4.4. *Let $D \subseteq \mathbb{R}$. Then,*

(a) $c \in D$ is isolated iff $\exists \delta > 0 : V^*_\delta(c) \cap D = \emptyset$.

(b) $c \in \mathbb{R}$ is cluster point iff $\forall \delta > 0 : V^*_\delta(c) \cap D \neq \emptyset$.

Proof. We will only prove b. Let c be a cluster point.
($\Longrightarrow$) Let (x_n) be a sequence in $D \setminus \{c\} : \lim(x_n) = c$. Let $\delta > 0$ be arbitrary, then $\exists N \forall n \geq N : x_n \in V_\delta(c)$ but $\forall n \in \mathbb{N} : x_n \in D$ and $x_n \neq c \implies \forall n \geq N : x_n \in V^*_\delta(c) \cap D \implies V^*_\delta(c) \cap D \neq \emptyset$.
($\Longleftarrow$) Assume that $\forall \delta > 0 : V^*_\delta(c) \cap D \neq \emptyset$. Let $n \in \mathbb{N}, \delta := \frac{1}{n}$; then $V^*_{\frac{1}{n}}(c) \cap D \neq \emptyset$. Let $x_n \in V^*_{\frac{1}{n}}(c) \cap D$; consider the sequence (x_n). Then, $\forall n \in \mathbb{N} : x_n \in D \setminus \{c\}$ and $\forall n \in \mathbb{N} : x_n \in V_{\frac{1}{n}}(c) \iff |x_n - c| < \frac{1}{n}$ (a null sequence) $\implies \lim(x_n) = c$. □

Theorem 4.5. *Let $D \subseteq \mathbb{R}$. $f : D \to \mathbb{R}, c \in \mathbb{R}$ such that $\lim_{x \to c} f$ exists.*

(a) If c is a cluster point of D. Then, $\lim\limits_{x \to c} f$ is uniquely determined.

(b) If c is isolated point of D. Then, $\lim\limits_{x \to c} f$ is arbitrary.

Proof. (a) c is a cluster point and let (x_n) be a sequence in $D \setminus \{c\}$ s.t. $\lim(x_n) = c$. Let L_1, L_2 be limits of f at c. Then, $L_1 = \lim\limits_{x \to c} f = \lim(f(x_n))$. Similarly, $L_2 = \lim\limits_{x \to c} f = \lim(f(x_n))$. We've proven that limit of a sequence is uniquely determined $\implies L_1 = L_2 \implies \lim\limits_{x \to c} f$ is uniquely determined.
(b) is left as an exercise. □

We'll assume from now on that c is a cluster point.

4.1.1 Limit Laws

Theorem 4.6. ***(Algebraic Limit Laws).*** *Let $f, g : D \subseteq \mathbb{R} \to \mathbb{R}$ and c a cluster point of D. Suppose that $\lim_{x\to c} f$ and $\lim_{x\to c} g$ exist. Then,*

(a) $\lim\limits_{x\to c}(f+g) = \lim\limits_{x\to c} f + \lim\limits_{x\to c} g$ *i.e.* $\lim\limits_{x\to c}(f+g)$ *exists and has value* $\lim\limits_{x\to c} f + \lim\limits_{x\to c} g$.

(b) $\lim\limits_{x\to c}(f-g) = \lim\limits_{x\to c} f - \lim\limits_{x\to c} g$.

(c) $\lim\limits_{x\to c}(C\cdot f) = C\cdot \lim\limits_{x\to c} f$*, for all* $C \in \mathbb{R}$.

(d) $\lim\limits_{x\to c}(f\cdot g) = \lim\limits_{x\to c} f \cdot \lim\limits_{x\to c} g$.

(e) $\forall x \in D : g(x) \neq 0$ *and* $\lim_{x\to c} g \neq 0$*. Then,* $\lim\limits_{x\to c}\left(\dfrac{f}{g}\right) = \dfrac{\lim_{x\to c} f}{\lim_{x\to c} g}$.

Proof. (a) Let (x_n) be an arbitrary sequence in $D\setminus\{c\} : \lim(x_n) = c$. Then, $\lim_{x\to c} f + \lim_{x\to c} g = \lim(f(x_n)) + \lim(g(x_n)) = \lim[f(x_n)+g(x_n)] = \lim((f+g)(x_n)) = \lim_{x\to c}(f+g)$.
(b), (c) and (d) will be left as exercises.
(e) Let $\forall x \in D : g(x) \neq 0$ and $\lim_{x\to c} g \neq 0$. Then,

$$\frac{\lim_{x\to c} f}{\lim_{x\to c} g} = \underbrace{\frac{\lim(f(x_n))}{\lim(g(x_n))}}_{>0}.$$

and $\forall n \in \mathbb{N} : g(x_n) \neq 0$ since $g \neq 0$.

$$= \lim\left(\frac{f(x_n)}{g(x_n)}\right)$$

$$= \lim\left(\frac{f}{g}\right)(x_n) = \lim_{x\to c}\frac{f}{g}$$

□

Theorem 4.7. ***(The Squeeze Theorem).*** *Let $f, g, h : D \subseteq \mathbb{R} \to \mathbb{R}$, c is a cluster point of D. Suppose that:*

- $\forall x \in D, f(x) \le g(x) \le h(x)$.
- $\lim\limits_{x\to c} f = \lim\limits_{x\to c} h = L$.

Then, $\lim\limits_{x\to c} g = L$

Proof. $\lim_{x\to c} f = \lim(f(x_n)) = L$ and $\lim_{x\to c} h = \lim(h(x_n)) = L$. $\forall n \in \mathbb{N}$: $f(x_n) \le g(x_n) \le h(x_n)$. By the squeeze theorem for sequences, we conclude that $\lim(g(x_n))$ exists and equals to L. But $L = \lim(g(x_n)) = \lim\limits_{x\to c} g$ i.e. $\lim\limits_{x\to c} g = L$. □

Example 4.1.3. Let $f : \mathbb{R}\setminus\{0\} \to \mathbb{R}$, $f(x) = x\cdot\sin\left(\frac{1}{x}\right)$. Show that $\lim\limits_{x\to 0} f = 0$

Proof. First of all, 0 is a cluster point of $D = \mathbb{R}\setminus\{0\}$ since $\lim\left(\frac{1}{n}\right) = 0$. Note that $\forall x \in \mathbb{R}\setminus\{0\} : -1 \le \sin\left(\frac{1}{x}\right) \le 1 \implies -|x| = -x \le x\sin\left(\frac{1}{x}\right) \le x = |x|$ of $x > 0$. And if $x < 0$, $-x \ge x\sin\left(\frac{1}{x}\right) \ge x \iff x \le x\sin\left(\frac{1}{x}\right) \le -x \iff -|x| \le x\sin\left(\frac{1}{x}\right) \le |x|$. Thus, $\forall x \in \mathbb{R}\setminus\{0\} : -|x| \le x\sin\left(\frac{1}{x}\right) \le |x|$.

Lemma 4.1. $\lim\limits_{x\to 0} |x| = 0$

Proof. Let $\varepsilon > 0$ and $\delta := \varepsilon$. Let $0 < |x-0| < \delta \iff 0 < |x| < \delta = \varepsilon$. Then, $||x|-0| = |x| < \delta = \varepsilon \implies \lim\limits_{x\to 0} |x| = 0$ □

Now, we know that $-|x| \le x\sin\left(\frac{1}{x}\right) \le |x|$ but $|x| \to 0$. Then, By squeeze theorem,

$$\lim_{x\to 0}\left(x\cdot\sin\left(\frac{1}{x}\right)\right) = 0$$

□

5 Continuity

We will define continuity in several ways (in this lecture, 3 different definition).

Definition 5.1. (Limit Definition of Continuity). Let $f : D \subseteq \mathbb{R} \to \mathbb{R}, c \in D$. We say that f is **continuous at c** if

$$\lim_{x \to c} f = f(c) \tag{5.1}$$

i.e. the limit of f at c exists and equal the function value. In any other cases, we say that f is **discontinuous at c** .

Remark 5.1. *There are 2 ways that functions can be discontinuous at c:*

1. $\lim\limits_{x \to c} f$ *exists but differs from* $f(c)$.
2. $\lim\limits_{x \to c} f$ *DNE.*

Definition 5.2. If $f : D \to \mathbb{R}$ is continuous at all $c \in D$, we simply say f is **continuous** (on D).

Example 5.0.1. x^2 is continuous on $\mathbb{R}$. Similarly, $\frac{1}{x}$ is continuous on $\mathbb{R} \setminus \{0\}$ since $\lim_{x \to c} \frac{1}{x} = \frac{1}{c}$. And, $\sqrt{x}$ is continuous on $\mathbb{R}_0^+$ by assignment 9.

Example 5.0.2. Consider the following examples:

1. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2, c \in \mathbb{R}$ arbitrary. We've seen that $\lim_{x \to c} x^2 = c^2$ i.e. $\lim_{x \to c} f = f(c) \implies f$ is continuous at any $c \in \mathbb{R}$.
2. $f : \mathbb{R} \to \mathbb{R}, f(x) := \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$. Then, $\lim_{x \to 0} f = 0$ since $f \equiv 0$ on $\mathbb{R} \setminus \{0\}$. But $f(0) = 1 \neq 0 = \lim_{x \to 0} f \implies f$ is discontinuous at 0.
3. $f : \mathbb{R} \to \mathbb{R}, f(x) := \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$. Claim: $\lim_{x \to 0} f$ DNE.

   Proof. We'll use the 2-sequence criterion. Define $(x_n) := \left(\frac{1}{n}\right); \forall n \in \mathbb{N} : x_n \neq 0$, then $\lim(x_n) = 0$ and $\lim(\underbrace{f(x_n)}_{=1}) = 1$. Similarly, consider, $(u_n) := \left(-\frac{1}{n}\right); \forall n \in \mathbb{N} : u_n \neq 0$, then $\lim(u_n) = 0$ and $\lim(\underbrace{f(u_n)}_{=0}) = 0 \neq 1 \implies \lim_{x \to 0} f$ DNE $\implies f$ is discontinuous at 0. □

End of Lecture

Lecture 20: November 13^{th}, 2024.

Definition 5.3. **($\varepsilon - \delta$).** Let $f : D \subseteq \mathbb{R} \to \mathbb{R}, c \in \mathbb{R}$. We say that f is **continuous at** c if $\forall \varepsilon > 0 \exists \delta > 0 \forall x \in D : |x-c| < \delta \implies |f(x) - f(c)| < \varepsilon$.

Definition 5.4. **(Sequential).** Let $f : D \subseteq \mathbb{R} \to \mathbb{R}, c \in \mathbb{R}$. We say that f is **continuous at** c if $\forall (x_n)$ in D with $\lim(x_n) = c$, it holds that $\lim(f(x_n)) = f(c)$.

Theorem 5.1. *All 3 definitions are equivalent*

Proof. ($\impliedby$) Let $f : D \subseteq \mathbb{R} \to \mathbb{R}, c \in \mathbb{R}$ be continuous by $\varepsilon - \delta$ definition i.e. $\forall \varepsilon > 0 \exists \delta > 0 \forall x \in D : |x-c| < \delta \implies |f(x) - f(c)| < \varepsilon$. Thus, we have especially that $\forall \varepsilon > 0 \exists \delta > 0 \forall x \in D \setminus \{c\} : |x-c| < \delta \implies |f(x) - f(c)| < \varepsilon \implies \lim\limits_{x \to c} f(x) = f(c) \implies f$ is constant at c

($\implies$) $\lim_{x \to c} f$ exists and equals $f(c)$. Assume f is continuous by the limit definition then, $\forall \varepsilon > 0, \exists \delta > 0 \forall x \in V_\delta^*(c) \cap D : |f(x) - f(c)| < \varepsilon$. If $x = c$ then $x \in V_s(c) \cap D$ true and $|f(x) - f(c)| = |f(c) - f(c)| = 0 < \varepsilon true$. Then with them combined together you get $\forall \varepsilon > 0, \exists \delta > 0 \forall x \in V_\delta^*(c) \cap D : |f(x) - f(c)| < \varepsilon$.

Limit definition is equivalent to sequential definition as exercise. $\square$

Exercise. Show that $f : \mathbb{R} \to \mathbb{R}, x \mapsto |x|$ is continuous at 0.

Proof. We'll use the $\delta - \varepsilon$ definition. Let $\varepsilon > 0$ and $\delta := \varepsilon$. Then, $|x - 0| = |x| < \delta = \varepsilon$. Then, $|f(x) - f(0)| = ||x| - 0| = |x| < \delta = \varepsilon \implies f$ is continuous at 0. $\square$

Theorem 5.2. ***(Algebraic Continuous Theorem).*** *Let $f, g : D \to \mathbb{R}, c \in D$ s.t. f, g is continuous at c. Then,*

1. *$f + g$ is continuous at c.*
2. *$f - g$ is continuous at c.*
3. *$C \cdot f$ is continuous at $c, \forall C \in \mathbb{R}$.*
4. *fg is continuous at c.*
5. *$\frac{f}{g}$ is continuous at c if $\forall x \in D : g(x) \neq 0$.*

Proof. 1) f, g is continuous at $c \implies \lim_{x \to c} f = f(c)$ and similarly for $g(c) \implies \lim_{x \to c}(f+g) = \lim_{x \to c} f + \lim_{x \to c} g = f(c) + g(c) = (f+g)(c)$ i.e. $\lim_{x \to c}(f + g) = (f+g)(c)$. Therefore $f + g$ is continuous at c.
2-5) are left as exercises. $\square$

Applications

Theorem 5.3. *All polynomials are continuous (on $\mathbb{R}$).*

Proof. Let $P:\mathbb{R}\to\mathbb{R}, P(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$. Thus, the function $x\mapsto x$ is continuous (continue as exercise). By theorem 5.1(4), x^n is continuous $\forall n\in\mathbb{N}$ and by theorem 5.1(3) x_nx^n is continuous $\forall n\in\mathbb{N}$. Lastly, by theorem 5.1(1), $a_0+a_1x+a_2x^2+\cdots+a_nx^n$ is continuous at any $c\in\mathbb{R}$. □

Theorem 5.4. *All rational functions $\frac{P(x)}{Q(x)}$, where P, Q are continuous polynomials, are continuous everywhere except at roots of Q.*

Proof. This follows immediately from theorem 5.1(5). □

Theorem 5.5. *Compositions of continuous functions are continuous i.e. Let $f:A\subseteq\mathbb{R}\to\mathbb{R}, c\in A$, and $g:B\subseteq\mathbb{R}\to\mathbb{R}, d\in B$ s.t. $f(A)\subseteq B$ and $f(c)=d$; and f is continuous at c while g is continuous at d. Then, $g\circ f:A\to\mathbb{R}$ is continuous at c.*

Proof. *(First).* We will use sequential definition. Let (x_n) be a sequence in A such that $\lim(x_n)=c$. f is continuous at $c \implies \lim(f(x_n))=f(c)=d$ i.e. $(f(x_n))$ is a sequence in B s.t. $\lim(f(x_n))=d$. g is continuous at d. Then, $\lim g(f(x_n))=g(d) \implies \lim((g\circ f)(x_n))=(g\circ f)(c)$. i.e. For any sequence (x_n) in A with $\lim(x_n)=c$, we have $\lim((g\circ f)(x_n))=(g\circ f)(c) \implies g\circ f$ is continuous at c. □

Proof. *(Second).* We will use the $\varepsilon-\delta$ definition. Let $\varepsilon>0$ and $y=g(u)$ be continuous at d. Then, $\exists\gamma>0:\forall u\in V_\gamma(d)\cap B:|g(u)-g(d)|<\varepsilon$ (*). Now let $u=f(x)$ be continuous at c. Then, $\exists\delta:\forall x\in V_\delta(c)\cap A:|f(x)-f(c)|<\gamma$ i.e. $|u-d|<\gamma \implies u\in V_\gamma(d)\cap B$. By (*), $|g(u)-g(d)|<\varepsilon \implies |g(f(x))-g(f(c))|<\varepsilon \implies |(g\circ f)(x)-(g\circ f)(c)|<\varepsilon$. Summary: $\forall x\in V_\delta(c)\cap A: |(g\circ f)(x)-(g\circ f)(c)|<\varepsilon \implies g\circ f$ is continuous at c. □

Example 5.0.3. By assignment 9, $\sqrt{}:\mathbb{R}_0^+\to\mathbb{R}, x\mapsto\sqrt{x}$ is continuous. Thus $\sqrt{x^2+1}$ is continuous on $\mathbb{R}$. as the composition of the polynomials x^2+1 with $\sqrt{}$.

5.1 Continuity and Topology

Let $f:D\to\mathbb{R}$ be continuous,

1. If D is open, is $f(D)$ open?

2. If D is closed, is $f(D)$ closed?

3. If D is bounded, is $f(D)$ bounded?

Answer: NO, In general, for all 3 questions.

Example 5.1.1. Consider the following examples:

1. $f:\mathbb{R}\to\mathbb{R}, f(x)=x^2$. Then, $]-1,1[$ is open but $f(]-1,1[)=[0,1[$ not open, or $f(\mathbb{R})=\mathbb{R}_0^+$.

2. $f:\mathbb{R}^+\to\mathbb{R}, f(x)=\frac{1}{x}$. Then, $[1,\infty[$ is closed but $f([1,\infty[)=]0,1]$ which is not closed.

3. From 2, $f(]0,1])=[1,\infty[$ which is unbounded since $1/x$ is its own inverse.

Nevertheless, there's an interesting exception.

Theorem 5.6. ***(Preservation of Compactness).*** *Let $f:A\to\mathbb{R}$, A is compact and f is continuous. Then, $f(A)$ is compact.*

Proof. Using sequential compactness. A is compact $\implies$ it's sequentially compact i.e. every sequence (x_n) in A has a convergent subsequence (x_{n_k}) s.t. $\lim(x_{n_k})\in A$. Let (y_n) in $f(A)$ be arbitrary. $f:A\to f(A)$ is surjective, thus every $y\in f(A)$ has at least 1 pre-image in A. Let $\forall n\in\mathbb{N}: x_n$ be a pre-image of y_n under f. Consider (x_n) and A is sequentially compact. Thus, $\exists(x_{n_k})$ of (x_n) and it is convergent where $x:=\lim(x_{n_k})$ s.t. $x\in A$. Let $y:=f(x)$. $\lim(x_{n_k})=x\in A$. f is continuous on A, especially continuous at x. Then, by sequentially definition of continuity, $\lim(\underbrace{f(x_{n_k})}_{=y_{n_k}})=f(x)=y\implies\lim(y_{n_k})=y\in f(A)$. Thus, for any sequence (y_n) in $f(A)$, we can find a convergent subsequence (y_{n_k}) whose limits $y\in f(A)$. Therefore, $f(A)$ is sequentially compact and thus compact. □

End of Lecture

Definition 5.5. Let $f:D\to\mathbb{R}$. Then,

(a) $x_0\in D$ is called **global/absolute maximum** of f if $\forall x\in D: f(x)\le f(x_0)$.

(b) $x_1\in D$ is called **global/absolute minimum** of f if $\forall x\in D: f(x)\ge f(x_1)$.

Lecture 21: November 18^{th}, 2024.

Theorem 5.7. *Let $f : D \to \mathbb{R}, D$ is compact and non-empty, where f is continuous. Then, f has both an absolute minimum and maximum on D.*

Remark 5.2. *All conditions in theorem 5.6 are essential i.e. f is continuous, D closed and bounded.*

Example 5.1.2. Suppose the followings:

1. Let $D =]0,1[$ and $y = x$. Thus, f is continuous but D is only bounded but not closed. In such case, f has neither an absolute min nor max.

2. Let $D = [0,1]$ and $y = \begin{cases} x & x \in]0,1[\\ \frac{1}{2} & x \in \{0,1\} \end{cases}$. Thus, D is compact but D is discontinuous. We can see that it also has neither an absolute min nor max.

3. Let $D = [1,\infty[$ and $y = \frac{1}{x}$. Thus, f is continuous but D is only closed but not bounded. For this function, we have a maximum (at $x = 1$) but not a minimum, it only has an infimum of 0.

Proof. *(Theorem 5.6)* Let D be compact and f continuous, we conclude that $f(D)$ is compact. D is non-empty $\implies$ $f(D)$ is non-empty. Especially, $f(D)$ is non-empty and bounded $\implies$ has both infimum and supremum. As seen in the chapter on topology, inf and sup are boundary points of $f(D)$. Furthermore, since $f(D)$ is compact, it's closed $\implies$ contains all of its boundary points, especially, its inf and sup i.e. $\inf f(D) \in f(D)$ and $\sup f(D) \in f(D)$. Thus, the $\inf f(D)$ is the $\min f(D)$, and $\sup f(D)$ is the $\max f(D)$. i.e. $\forall x \in D : \underbrace{\min f(D)}_{\in f(D)} \le f(x) \le \underbrace{\max f(D)}_{\in f(D)}$.

Thus, $\exists x_0, x_1 \in D : f(x_0) = \min f(D) \le f(x) \le \max f(D) = f(x_1) \implies x_0$ is an absolute min and x_1 is an absolute max of f on D. □

5.2 The Intermediate Value Theorem

Theorem 5.8. ***(Localization of Roots).*** *Let $f : [a,b] \to \mathbb{R}, f$ is continuous such that f has opposite signs at a and b i.e. either $f(a) > 0$ and $f(b) < 0$ or, $f(a) < 0$ and $f(b) > 0$. Then, $\exists c \in]a,b[: f(c) = 0$.*

Proof. Let $I_0 = [a,b], a_0 = a, b_0 = b$, divides I_0 into 2 sub-intervals of equal width. If $f(\frac{a_0+b_0}{2}) = 0$, we've found the roof of f. If not, there are 2 possibilities:

- $f(\frac{a+b}{2}) > 0$, let $a_1 := \frac{a_0+b_0}{2}$ and $b_1 = b_0$ and $I_1 = [a_1, b_1]$.
- $f(\frac{a+b}{2}) < 0$, let $a_1 = a_0$ and $b_1 := \frac{a_0+b_0}{2}$, $I_1 := [a_1, b_1]$.

Then divide I_1 into 2 sub-intervals of equal width. If $f(\frac{a_1+b_1}{2}) = 0$, we've found a root. If not, there are 2 cases as above etc.
In the end, either this algorithm terminates after finitely many steps with a root of f, or this algorithm doesn't terminate, then we obtain a nested sequence: $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ of closed and bounded intervals. By the nested interval property of $\mathbb{R}$. $\cap_{n\in\mathbb{N}} I_n \neq \emptyset$. Let $c \in \cap_{n\in\mathbb{N}} I_n$. Then, $\forall n \in \mathbb{N}_0, c \in [a_n, b_n] \implies |a_n - c| \leq b_n - a_n$ and $|b_n - c| \leq b_n - a_n = |I_n| = \underbrace{\frac{1}{2^n}}_{\text{Null seq.}} \cdot |I_0|$.

Thus, by the null sequence criterion, $\lim(a_n) = c = \lim(b_n)$. Now, $\forall n \in \mathbb{N}, f(a_n) > 0$ and $f(b_n) < 0$ and f is continuous on D, especially at $c \implies \lim(\underbrace{f(a_n)}_{>0}) = f(c) = \lim(\underbrace{f(b_n)}_{<0}) \implies$ By limits and orders section, we'd get that $\lim(f(a_n)) \geq 0$ and $\lim(f(b_n)) \leq 0 \implies 0 \leq f(c) \leq 0 \implies f(c) = 0$. □

Theorem 5.9. ***(Intermediate Value Theorem [IVT]).*** *Let $f : [a,b] \to \mathbb{R}$ be continuous. Let $y_0 \in \mathbb{R} : y_0 \in]f(a), f(b)[$ i.e. $f(a) < y_0 < f(b)$ if $f(a) < f(b)$ or $f(b) < y_0 < f(a)$ if $f(a) < f(b)$. Then, y_0 is also a value under f i.e. $\exists x_0 \in]a,b[: f(x_0) = y_0$.*

Remark 5.3. *Theorem 5.7 is the special case where $y_0 = 0$ of the IVT.*

Proof. (IVT). Let $g : [a,b] \to \mathbb{R}, g(x) \mapsto f(x) - y_0$. Then, g is continuous on $[a,b]$; $g(a) = f(a) - y_0$ and $g(b) = f(b) - y_0$. Then, $g(a) > 0$ and $g(b) < 0$ or $g(a) < 0$ and $g(b) > 0$. By localization of roots theorem, $\exists x_0 \in]a,b[: g(x_0) = 0 \implies 0 = g(x_0) = f(x_0) - y_0 \implies f(x_0) = y_0$. □

This result will be used to prove the following:

Theorem 5.10. ***(Preservation of Intervals).*** *Let $I \subseteq \mathbb{R}$ be a non-empty interval and $f : I \to \mathbb{R}$ be continuous. Then, $f(I)$ is an interval.*

The proof of theorem 5.10, in turn, is based on the following result:

Theorem 5.11. ***(Characterization of Intervals).*** *A non-empty subset $I \subseteq \mathbb{R}$ is an interval if and only if $\forall x, y \in I : x < y$, it holds that $[x,y] \subseteq I$.*

Proof. (Theorem 5.10.) Assignment 10. □

End of Lecture

Lecture 22: November 20th, 2024.

5.3 Uniform Continuity

Definition 5.6. A function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is said to be **uniformly continuous** (on D) if $\forall \varepsilon \exists \delta \forall x, u \in D : |x - u| < \delta \implies |f(x) - f(u)| < \varepsilon$.

Example 5.3.1. Let $a > 0$. Show that x^2 is uniformly continuous on $[-a, a]$.

Proof. Let $x, u \in [-a, a]$ such that $|x - u| < \delta$. Then,

$$\begin{aligned} |x^2 - u^2| = |(x-u)(x+u)| = |x-u| \cdot |x+u| \\ &= \delta \cdot |x+u| \\ &\leq \delta(|x| + |u|) \\ &\leq \delta(a+a) \\ &\leq 2a\delta < \varepsilon \iff \delta < \frac{\varepsilon}{2a} \end{aligned}$$

Let $\delta < \frac{\varepsilon}{2a}$. Then, whenever $|x - u| < \delta \implies |f(x) - f(u)| < \varepsilon \implies x^2$ is continuous on $[-a, a]$ ☐

5.3.1 Sequential Criterion For the Absence of Uniform Continuity

Theorem 5.12. *Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is not uniformly continuous iff $\exists \varepsilon > 0$ and sequences $(x_n), (u_n)$ in D s.t. $\lim(x_n - u_n) = 0$ and $\forall n \in \mathbb{N} : |f(x_n) - f(u_n)| \geq \varepsilon$.*

Proof. ($\implies$) Let f not be uniformly continuous. Then, $\neg(\forall \varepsilon \exists \delta \forall x, u \in D : |x - u| < \delta \implies |f(x) - f(u)| < \varepsilon) \equiv \exists \varepsilon \forall \delta \exists x, u \in D : |x - u| < \delta \implies |f(x) - f(u)| \geq \varepsilon$ (*). Select 1 such ε; let $n \in \mathbb{N}$ be arbitrary and $\delta := \frac{1}{n}$. Then, $x_n, u_n \in D : |x_n - u_n| < \delta = \frac{1}{n}$ and $|f(x_n) - f(u_n)| \geq \varepsilon$. Consider (x_n) and (u_n). Then, $\forall n \in \mathbb{N} : |x_n - u_n| < \frac{1}{n} \implies \lim(x_n - u_n) = 0$ and $\forall n \in \mathbb{N} : |f(x_n) - f(u_n)| \geq \varepsilon$. Which is what we need to show.
($\impliedby$) Let $\varepsilon > 0, (x_n), (u_n) : \lim(x_n - u_n) = 0$ and $\forall n \in \mathbb{N} : |f(x_n) - f(u_n)| \geq \varepsilon$. Assume that f is uniformly continuous. Then, $\exists \delta > 0 \forall x, u \in D : |x - u| < \delta \implies |f(x) - f(u)| < \varepsilon$. $\lim(x_n - u_n) = 0 \implies \exists N \in \mathbb{N} : \forall n \geq N : |(x_n - u_n) - 0| = |x_n - u_n| < \delta$. Thus, $|f(x_n) - f(u_n)| < \varepsilon$ ↯ This f os not uniformly continuous. ☐

Example 5.3.2. Show that $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, x \mapsto \frac{1}{x}$ is not uniformly continuous on $]0, 1]$.

Proof. Let $x_n := \frac{1}{2n}$, $u_n = \frac{1}{n}\ \forall n \in \mathbb{N}$. Then, $|x_n - u_n| < |\frac{1}{2n} - \frac{1}{n}| = \frac{1}{2n} \Longleftarrow \lim(x_n - u_n) = 0$ and $|f(x_n) - f(u_n)| = \left|\frac{1}{1/2n} - \frac{1}{1/n}\right| = |2n - n| = n \geq 1$. Let $\varepsilon := 1$. Then, $\forall n \in \mathbb{N} : |f(x) - f(u)| \geq 1 \implies f$ is not uniformly continuous on $]0,1]$ and thus also not uniformly continuous on $\mathbb{R} \setminus \{0\}$. □

Remark 5.4. *By definition, every uniformly continuous function on $D \subseteq \mathbb{R}$ is continuous on D. The above example shows that the converse does not hold, in general. We say that uniform continuity is stronger than "ordinary" continuity.*

Example 5.3.3. Show that $x \mapsto x^2$ is not uniformly continuous on $\mathbb{R}$.

Proof. $\forall n \in \mathbb{N}$, let $x_n := n + \frac{1}{n}, u_n := n$. Then, $\lim(x_n - u_n) = \lim(1/n) = 0$. Then, $|f(x_n) - f(u_n)| = |x_n^2 - u_n^2| = \left|(n + \frac{1}{n})^2 - n^2\right| = 2 + \frac{1}{n^2} > 2$. Let $\varepsilon := 2$. Then, $\forall n \in \mathbb{N} : |x_n^2 - u_n^2| > \varepsilon \Longleftarrow x^2$ is not uniformly continuous on $[0,\infty[$ and thus also not uniformly continuous on $\mathbb{R}$. □

End of Lecture ——

Theorem 5.13. *Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be uniformly continuous. Then, f maps Cauchy sequence to Cauchy sequence i.e. if (x_n) is a Cauchy sequence in D. Then, $(f(x_n))$ is a Cauchy sequence (in $f(D)$).*

Proof. Let $\varepsilon > 0$. Since f is uniformly continuous $\exists \delta > 0 \forall x, u \in D : |x - u| < \delta \implies |f(x) - f(u)| < \varepsilon$. Let (x_n) be Cauchy, thus, $\exists N \forall n, m \geq N : |x_n - x_m| < \delta$. Thus, $\forall n, m \geq N : |f(x_n) - f(x_m)| < \varepsilon$ which is the definition of $f(x_n)$ being Cauchy. □

Example 5.3.4. Show that $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, x \mapsto \frac{1}{x}$ is not uniformly continuous.

Proof. Let $\forall n \in \mathbb{N} : x_n := \frac{1}{n}$. Then, (x_n) is a Cauchy sequence in $\mathbb{R} \setminus \{0\}$. But $\forall n \in \mathbb{N}, f(x_n) = n$. Thus $f(x_n)$ diverges and is not Cauchy $\implies \frac{1}{x}$ is not uniformly continuous. □

Remark 5.5. *This example shows that "ordinary" continuity functions do not, in general, map Cauchy to Cauchy sequence.*

Theorem 5.14. ***(Preservation of Boundedness).*** *Let $f : D \to \mathbb{R}$ be uniformly continuous and D be bounded. Then, $f(D)$ is bounded.*

Proof. Assume that $f(D)$ is unbounded i.e. $\forall n \in \mathbb{N} \exists x_n \in D : |f(x_n)| \geq n$. Consider (x_n) in D where D is bounded. Then, by Bolzano-Weierstrass, (x_n) has a convergent subsequence (x_{n_k}). Then, by last theorem, $f(x_{n_k})$ converges and is thus bounded. But, we said that $|f(x_{n_k})| \geq k \geq k \ \forall k \in \mathbb{N} \implies f(x_{n_k})$ is unbounded ↯ Therefore, $f(D)$ is bounded. □

Theorem 5.15. *Let $f, g : D \to \mathbb{R}$ be uniformly continuous. Then,*

1. *$f + g$ is uniformly continuous.*
2. *$f - g$ is uniformly continuous.*
3. *$\forall k \in \mathbb{R}, k \cdot f$ is uniformly continuous.*

Remark 5.6. *Surprisingly, in general, the product of 2 uniformly continuous function is not uniformly continuous!*

Example 5.3.5. Let $f, g : D \to \mathbb{R}$ both be equal to the identity function i.e. $f(x) = x$ and $g(x) = x$ for all $x \in \mathbb{R}$. Then, f and g are uniformly continuous. However x^2 is not (proven by the above).

Theorem 5.16. *Let $A \subseteq \mathbb{R}$ be compact and let $f : A \to \mathbb{R}$ be continuous. Then, f is uniformly continuous.*

Proof. Assume that f is not uniformly continuous. Then, $\exists > 0 : (x_n), (u_n)$ in A s.t. $\lim(x_n - u_n) = 0$. But $\forall n \in \mathbb{N} : |f(x_n) - f(u_n)| \geq \varepsilon \ (*)$. A is compact $\implies A$ is sequentially compact. Thus, (x_n) has a convergence subsequence (x_{n_k}) whose limit x lies in A. Consider the corresponding subsequence (u_{n_k}). Then,

$$\begin{aligned}
\lim(u_{n_k}) &= \lim(x_{n_k} - (x_{n_k} - u_{n_k})) \\
&= \underbrace{\lim(x_{n_k})}_{=x} - \underbrace{\lim(x_{n_k} - u_{n_k})}_{=0} \\
&= x
\end{aligned}$$

i.e. both (x_{n_k}) and (u_{n_k}) converges to $x \in A$. f is continuous on A and thus especially at $x \implies \lim f(x_{n_k}) = f(x)$ and $\lim f(u_{n_k}) = f(u) \implies \lim(f(x_{n_k}) - f(u_{n_k})) = f(x) - f(u) = \implies \exists K \forall k \geq K : |f(x_{n_k}) - f(u_{n_k})| < \varepsilon$ ↯ to $(*)$. Thus, f is uniformly continuous (on A). □

Remark 5.7. *On compact domain, the properties of continuity and uniform continuity are identical.*

Theorem 5.17. ***(In Analysis II).*** *Every continuous function on $[a,b], a < b$ is Riemann integral.*

Proof. Relies heavily on theorem 5.16. ☐

Corollary 5.1. *Let $f,g : A \subseteq \mathbb{R} \to \mathbb{R}$, A is compact and f,g is uniformly continuous. Then, $f \cdot g$ is uniformly compact.*

Proof. Since f,g are uniformly continuous, they are continuous. Thus, $f \cdot g$ is continuous (on A). Since A is compact, it follows from the above theorem that $f \cdot g$ is uniformly continuous (on A). ☐

Remark 5.8. *The definition of uniform continuity of a function $f : D \to \mathbb{R}$ i.e. $\forall \varepsilon \exists \delta \forall x,u \in D : |x-u| < \delta \implies |f(x) - f(u)| < \varepsilon$ is symmetrical in x and u. We can thus assume without loss of generality that $x \geq u$ or v.v.*

Lemma 5.1. $\forall x,u \in \mathbb{R}$, $x \geq u \geq 0$. *Then,* $\sqrt{x} - \sqrt{u} \leq \sqrt{x-u}$.

Proof. Recall from our first lecture that $\forall a,b \geq 0 : \sqrt{a+b} \geq \sqrt{a} + \sqrt{b}$. Let $a := u$ and $b := x-u \geq 0$. Then, $\sqrt{a+b} = \sqrt{x} \leq \sqrt{a} + \sqrt{b} = \sqrt{u} + \sqrt{x-u} \implies \sqrt{x} - \sqrt{u} \leq \sqrt{x-u}$. ☐

Example 5.3.6. Show that $x \mapsto \sqrt{u}$ is uniformly continuous on $[0,\infty[$.

Proof. Let $\varepsilon > 0$ and $\delta > 0$ be arbitrary for now. Let $x,u \in \mathbb{R} : x \geq u \geq 0$. Then,

$$|\sqrt{x} - \sqrt{u}| = \sqrt{x} - \sqrt{u} \leq \sqrt{x-u} = \sqrt{|x-u|} < \sqrt{\delta} < \varepsilon \iff \delta < \varepsilon^2 \tag{5.2}$$

Let $\delta < \varepsilon^2$. Then, $|\sqrt{x} - \sqrt{u}| < \varepsilon$ whenever $|x-u| < \delta \implies \sqrt{x}$ is uniformly continuous on $[0,\infty[$. ☐

Remark 5.9. *It can be shown that $x \mapsto x^a$ is uniformly continuous on its domain ($\mathbb{R}_0^+$ or $\mathbb{R}$) if $0 < a \leq 1$ and not uniformly continuous if $a > 1$.*

5.4 Lipschitz Continuity

Definition 5.7. $f : D \to \mathbb{R}$ is said to be **Lipschitz (continuous)** if $\exists k > 0 \forall x,u \in D : |f(x) - f(u)| < k \cdot |x-u|$. Additionally, k is called a **Lipschitz constant** of f

Theorem 5.18. *Every Lipschitz function is uniformly continuous i.e. Let $f : D \to \mathbb{R}$ be Lipschitz. Then, f is uniformly continuous.*

Proof. Let $\varepsilon > 0$, $\delta := \frac{\varepsilon}{k}$, and $|x-u| < \delta$. Then, $|f(x) - f(u)| \leq k \cdot |x-u| < k \cdot \delta = k \cdot \frac{\varepsilon}{k} = \varepsilon$. Thus, $|f(x) - f(u)| < \varepsilon \implies f$ is uniformly continuous. $\square$

Remark 5.10. *The converse does not hold!*

Example 5.4.1. Show that $\sqrt{x}$ is not Lipschitz continuous on $[0,1]$.

Proof. Assume that f is Lipschitz continuous on $[0,1]$. Let k be a Lipschitz constant for f. Let $x := \frac{1}{4k^2}$ and $u := 0$. Then, $k \cdot |x-u| = k \cdot \frac{1}{4k^2} = \frac{1}{4k}$. On the other hand, $|f(x) - f(u)| = |\sqrt{x} - \sqrt{u}| = \left|\frac{1}{2k} - 0\right| = \frac{1}{2k} > \frac{1}{4k} \implies |f(x) - f(u)| \leq k \cdot |x-u|$ ↯ Thus f is not Lipschitz continuous on $[0,1]$. $\square$

Thus, we can see that Lipschitz continuity is stronger than uniform continuity which is stronger than ordinary continuity. In this case, "stronger" implies the conditions are more restrictive but important.

6 Differentiation

Definition 6.1. Let $I \subseteq \mathbb{R}$ be an interval, let $f : I \to \mathbb{R}$ and $c \in \mathbb{R}$. We say that f is **differentiable** at c if $\lim\limits_{x \to x} \dfrac{f(x) - f(c)}{x - c}$ exists. In this case, we denote

$$f'(c) := \lim_{x \to x} \frac{f(x) - f(c)}{x - c} \tag{6.1}$$

the **derivative** of f at c.

Remark 6.1. *If f is differentiable at all $c \in I$. Then, we simply say that f is differentiable.*

Note: Geometrically speaking, $f'(x)$ represents the tangent line to the graph of f at c.

Example 6.0.1. Consider the following examples:

1. Let $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$. Then, let c be arbitrary, we get that

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{x^2 - c^2}{x - c} = \lim_{x \to c} \frac{(x - c)(x + c)}{x - c} = \lim_{x \to c} x + c = 2c$$

   Thus $\forall c \in \mathbb{R}, f'(c) = 2c$ or simply $f'(x) = 2x$ on $\mathbb{R}$.

2. $f :]0, \infty[\to \mathbb{R}, x \mapsto \frac{1}{x}$. Let $c \in]0, \infty[$. Then,

$$\begin{aligned} \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{\frac{1}{x} - \frac{1}{c}}{x - c} &= \lim_{x \to c} \frac{\frac{c - x}{xc}}{x - c} \\ &= \lim_{x \to c} \frac{-(x - c)}{xc} \frac{1}{x - c} \\ &= \lim_{x \to c} \frac{-1}{xc} = -\frac{1}{c^2} \end{aligned}$$

$\implies f'(x) = -\frac{1}{x^2}$ on $]0, \infty[$.

Exercise. Find the derivative of $\sqrt{x}$, $\frac{1}{\sqrt{x}}$ and $\frac{1}{x^2}$.

Theorem 6.1. ***(Caratheodory).*** *Consider the followings:*

1. *Let $f : I \to \mathbb{R}$. Then, f is differentiable at c iff $\exists \varphi : I \to R$ which is continuous at c such that $\forall x \in I : f(x) = f(c) + \varphi(x) \cdot (x - c)$.*

2. *In case φ does exists, it holds that $\varphi(c) = f'(c)$.*

Proof. ($\Longleftarrow$) Let $\varphi : I \to \mathbb{R}$ be continuous at $c : \forall x \in I, f(x) = f(c) + \varphi(x)(x - c)$. Then, $\varphi(x) = \frac{f(x)-f(c)}{x-c}$ $\forall x \in I \setminus \{c\}$. Since f is continuous at c, $\varphi(x) = \lim_{x\to c} \varphi = \lim_{x\to c} \dfrac{f(x) - f(c)}{x - c}$. Thus, f is differentiable at c and $f'(x) = \varphi(x)$.
($\Longrightarrow$) Let f be differentiable at c. Define $\varphi : I \to \mathbb{R}$ as follows:

$$\varphi(x) := \begin{cases} \frac{f(x)-f(c)}{x-c} & x \neq c \\ f'(x) & x = c \end{cases}$$

We then have that if $x \neq c$, $f(x) = f(c) + \varphi(x)(x - c)$ and thus it's proven. If $x = c$, $f(x) = f(c) + \underbrace{\varphi(x)(x - c)}_{=0} = f(c)$. The functional equation thus holds for all $x \in I$ and

$$\lim_{x\to c} \varphi(x) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c} = f'(c) = f(c)$$

Thus, φ is continuous at c. Lastly, $\varphi(x) = f'(x)$ always holds by the above implications. □

Example 6.0.2. $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$. Use Caratheodory theorem to show that f is differentiable.

Proof. Let $c \in \mathbb{R}$ be arbitrary. If $x \neq c$, then,

$$f(x) = \frac{x^2 - c^2}{x - c} = \frac{(x - c)(x + c)}{x - c} = x + c$$

where $x \mapsto x + c$ is continuous on $\mathbb{R}$. Thus, $\varphi(x) = x + c$ is the caratheodory function for $f \implies f$ is differentiable at c and $f'(c) = \varphi(c) = c + c = 2c$. □

Theorem 6.2. *Let $f : I \to \mathbb{R}$ and $c \in I$. If f is differentiable at c then f is continuous at c.*

Proof. Let f be differentiable at c. Then, $\exists \varphi : I \to \mathbb{R}$ that is continuous at c. Furthermore, $\forall x \in I, f(x) = \underbrace{f(c) + \underbrace{\varphi(x)}_{\text{cont. at } c} \underbrace{(x - c)}_{\text{cont. at } c}}_{\text{cont. at } c}$

Thus, f is continuous at c. □

Remark 6.2. *The converse will NOT hold.*

Example 6.0.3. $f : \mathbb{R} \to \mathbb{R},\ x \mapsto |x|$ is not differentiable at 0 but is continuous at 0 since

$$\lim_{x\to 0} \frac{|x| - |0|}{x - 0} = \lim_{x\to 0} \frac{|x|}{x} \qquad \text{where} \qquad \frac{|x|}{x} = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

Let $(x_n) := (1/n)$ and $(u_n) := (1/n)$. Then, $\lim(x_n) = \lim(u_n) = 0$. However, $\lim(g(x_n)) = \lim(1) = 1$ while $\lim(g(u_n)) = \lim(-1) = -1$. Thus, by the 2-sequence criteria for non-existence of the limit of function, $\lim_{x\to 0} \frac{|x|}{x}$ DNE. Thus, $|x|$ is not differentiable at 0 however, we've seen that $|x|$ is continuous at 0.

Remark 6.3. *This can be expressed as "differentiability is a strictly stronger condition than continuity".*

Theorem 6.3. *Let $f, g : I \to \mathbb{R}, c \in I$, f, g is differentiable at c. Then,*

1. *$f + g$ is differentiable at c and $(f+g)'(c) = f'(c) + g'(c)$.*
2. *$\forall k \in \mathbb{R}$, kf is differentiable at c and $(kf)'(x) = kf'(x)$.*
3. *$f - g$ is differentiable at c and $(f-g)'(c) = f'(c) - g'(c)$.*
4. *fg is differentiable at c and $(f\cdot g)'(c) = f'(c)g(c) + f(c)g'(c)$.*
5. *If $g \neq 0$ on I, f/g is differentiable and $\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c) - g'(c)f(c)}{g^2(c)}$.*

Proof. This proof directly follows from the limit definition of derivative. One can even use Caratheodory theorem. □

Theorem 6.4. *Let $f : I \to \mathbb{R}, g : J \to \mathbb{R}$ where $c \in I, d := f(c)$. f is differentiable at c and g is differentiable at d. Then, $g \circ f : I \to \mathbb{R}$ is differentiable at $(g\circ f)'(c) = g'(f(c)) \cdot f'(c)$.*

Proof. Left as an exercise. Hint: Use Caratheodory theorem. □

6.1 Mean Value Theorem

Definition 6.2. Let $f : D \to \mathbb{R}$, $c \in D$ is said to be a **local maximum** of f if $\exists \delta > 0 : f$ has an absolute maximum at c on $V_\delta(c) \cap D$. Similarly c is a **local minimum** if $\exists \delta > 0$: c is an absolute minimum of f on $V_\delta(c) \cap D$.

Remark 6.4. *The collective term for both minimum and maximum is "extremum".*

Lemma 6.1. *Let $f : D \to \mathbb{R}$, $c \in D$ such that* $\lim_{x\to c} f > 0$. *Then,* $\exists \delta > 0 : \forall x \in V^*_\delta(c) \cap D :\ f(x) > 0$. *A similar result holds if* $\lim_{x\to c} f < 0$.

Proof. Let $L := \lim_{x\to c} f > 0$ and $\varepsilon := \frac{L}{2} > 0$. Then, $\exists \delta > 0 : \forall x \in V^*_\delta(c) \cap D$: $|f(x) - L| < \varepsilon = \frac{L}{2} \iff -\frac{L}{2} < f(x) - L < \frac{L}{2} \iff 0 < f(x) - \frac{L}{2} < L$. Thus, $\forall x \in V^*_\delta(c) \cap D : f(x) > \frac{L}{2} > 0$. □

Theorem 6.5. ***(Fermat's).*** *Let $f : I \to \mathbb{R}, c \in I$ and $c \notin \partial I$, f is differentiable at c, and f has a local extremum at c. Then, the $f'(c) = 0$.*

Remark 6.5. *$c \notin \partial I$ is necessary e.g. let $f : [a, b] \to \mathbb{R}, f(x) \mapsto x$ then $f'(a) \neq 0$ and $f'(b) \neq 0$.*

Proof. (Fermat's). Assume without loss of generality that f has a local max at c (otherwise consider $-f$). Now, assume that $f'(c) > 0$ i.e. $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} > 0$. Then, by the above lemma, $\exists \delta_1 > 0 : \forall x \in V^*_{\delta_1}(c) \cap D : \frac{f(x)-f(c)}{x-c} > 0$; furthermore, $c \notin \partial I \implies \exists \delta_2 > 0 : V_{\delta_2}(c) \subseteq I$. Let $\delta := \min\{\delta_1, \delta_2\}$. Then, $\forall x \in V^*_\delta(c) \cap I = V^*_\delta(c) : \frac{f(x)-f(c)}{x-c} > 0$. Especially, $\forall x \in]c, c+\delta[: \frac{f(x)-f(c)}{x-c} > 0$. So $\forall x \in]c, c+\delta[\subseteq V_\delta(c) : f(x) > f(c)$ ↯ since c is a local maximum. Consequently, $f'(x) \leq 0$.
Now assume that $f'(c) < 0$. As in the first case, $\exists \delta > 0 : \forall x \in V^*_\delta(c) \cap I$: $\frac{f(x)-f(c)}{x-c} < 0$. Especially, $\forall x \in]c-\delta, c[: \frac{f(x)-f(c)}{x-c} < 0 \implies \forall x \in]c-\delta, c[\subseteq V_\delta(c) : f(x) > f(c)$ ↯ since c is local maximum.
$\therefore f'(x) = 0$ □

Note: Looking at the above remark, we cannot exclude that $f'(c) > 0$ in this case (1st assumption of the proof) indeed $f'(c) > 0$.

Theorem 6.6. ***(Rolle's).*** *Let $f : [a, b] \to \mathbb{R}$, f is continuous on $[a, b]$, is differentiable on $]a, b[$ and $f(a) = f(b) = 0$. Then, $\exists c \in]a, b[: f'(c) = 0$.*

Proof. If $f \equiv 0$ on $[a, b]$ then any c works. Assume that $f \not\equiv 0$ on $[a, b]$. Let $f(x_0) \neq 0$. Assume, without loss of generality, that $f(x_0) > 0$ (otherwise consider $-f$). f is continuous on the compact set $[a, b] \implies f$ has an absolute maximum on $[a, b]$, supposedly at c. Then, $f(c) \geq f(x_0) > 0 \implies c \neq a, c \neq b \implies c \notin \partial[a, b]$ i.e. $c \in]a, b[$. Especially, f is differentiable at c and c is also a local maximum. Thus, by Fermat's theorem, $f'(c) = 0$. □

Theorem 6.7. ***(Mean Value Theorem [MVT]).*** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, differentiable $]a, b[$. Then, $\exists c \in]a, b[: f'(c) = \frac{f(b)-f(a)}{b-a}$.*

Proof. Let $L(x)$ be the secant line through the points $(a, f(a))$ and $(b, f(b))$ i.e. $L(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$. First, check $L(a) = f(a)$ and $L(b) = f(b)$. Now, consider the function $g : [a, b] \to \mathbb{R}$, $g = f - L$. Then, g is continuous on $[a, b]$ and also g is differentiable on (a, b). Furthermore, $g(a) = f(a) - L(a) = 0$ and $g(b) = f(b) - L(b) = 0$. Then, by Rolle's theorem, $\exists c \in]a, b[$: $g'(c) = 0$, where $g'(x) = f'(x) - L'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$. Then, $g'(c) = 0 = f'(c) - \frac{f(b)-f(a)}{b-a} \implies f'(c) = \frac{f(b)-f(a)}{b-a}$. □

Note: The MVT can be used to deduce properties of f from properties of f'.

Example 6.1.1. Let $f : I \to \mathbb{R}$ be differentiable s.t. $\forall x \in I : f'(x) \geq 0$. Then, f is increasing on I.

Proof. Let $a, b \in I : a < b$. We need to show that $f(a) \leq f(b)$. Apply the MVT to f on $[a, b]$ (we can do this since f is differentiable and thus continuous on $[a, b]$). By MVT, $\exists c \in]a, b[$: $f'(c) = \frac{f(b)-f(a)}{b-a} \implies \frac{f(b)-f(a)}{b-a} \geq 0$. Since $b - a > 0 \implies f(b) - f(a) \geq 0 \iff f(a) \leq f(b)$. Thus, f is increasing on I. □

Note to Author: *Good luck on your final exam on December* 19^{th}*, 2024!!!*

Properties of R. Cauchy and monotone sequences, Bolzano- Weierstrass theorem. Limits, limsup, liminf of functions. Pointwise, uniform continuity: Intermediate Value theorem. Inverse and monotone functions. Differentiation: Mean Value theorem, L'Hospital's rule, Taylor's Theorem.

Pigeon’s Bindery