

McGill U2S3 Lectures

Hy Vu

Honours Analysis I

Lecture Notes

Prof: Dr. Hundemer



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McGill Honour Analysis

Lectures (MATH 254)

Hy Vu

Professor: Dr. Axel Hundemer

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Foreword

Mathematics & Statistics (Sci) : Properties of \mathbb{R} . Cauchy and monotone sequences, Bolzano–Weierstrass theorem. Limits, limsup, liminf of functions. Pointwise, uniform continuity: Intermediate Value theorem. Inverse and monotone functions. Differentiation: Mean Value theorem, L'Hospital's rule, Taylor's Theorem.

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Prerequisites: MATH 141

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1 Preliminaries

Notations Used Frequently:

- (Natural Numbers) $\mathbb{N} = \{1, 2, 3, \dots\}$.
If 0 is included, we write \mathbb{N}_0 .
- (Integers) $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$ ¹
- (Rational Numbers) $\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N} \right\}$.
- (Real Numbers) $\mathbb{R} = \{x : -\infty < x < \infty\}$.
- (Complex Numbers) $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, i = \sqrt{-1}\}$.
- (Closed Interval) $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$.²
- (Open Interval) $]a, b[:= \{x \in \mathbb{R} : a < x < b\} = (a, b)$.

Remark 1.1. The symbol ":" means "such that" which can also be written as ":" or "s.t."

1.1 Sample Proofs

Definition 1.1. A **direct proof** is a proof where we start with a true statement and derive from this via a finite sequence of implications (\Rightarrow) the desired result.

Example 1.1.1. Show that $x \mapsto x^2$ is strictly increasing on $[0, \infty[$. In other word, show that $f : [0, \infty[\rightarrow \mathbb{R}$, $f(x) = x^2$ is strictly increasing.³

Before beginning the proof, let's establish some background. **Under what condition** is $x \mapsto x^2$ strictly increasing on $[0, \infty[$? Well...the condition is as follows $\forall 0 \leq x < y : x^2 < y^2$

¹ Z stands for "Zahl" which is number in German.

² ":" means "equal by definition"

³ " \mapsto " means "maps to". The interval $[0, \infty[$ can also be written \mathbb{R}_0^+ (which is read "all positive real number including 0").

Proof. Let $0 \leq x < y$, then $y^2 - x^2 = \overbrace{(y-x)}^{x < y} \cdot \overbrace{(y+x)}^{x,y \geq 0}$. Because of the condition shown, both results will be > 0 which means their multiplication is also > 0 .
 $\Rightarrow y^2 > x^2 \Rightarrow x^2 < y^2$ i.e. f is strictly increasing. \square

Definition 1.2. A **proof by contradiction** is a proof where we assume the opposite of the desired result. This leads to a contradiction via a finite sequence of implication.

Example 1.1.2. Prove that $\sqrt{2}$ is irrational.

Historical Background: This was proven by Euclid around 300BC and it is one of the most famous arithmetic proof. This can be considered the first proof of mathematics ever.

Proof. Assume that $\sqrt{2}$ is rational then $\exists a, b \in \mathbb{N} : \gcd(a, b) = 1$ and $\sqrt{2} = \frac{a}{b}$.

$$\begin{aligned}\Rightarrow 2 &= \frac{a^2}{b^2} \\ \Rightarrow 2b^2 &= a^2\end{aligned}$$

The left hand side is even then right hand side is also even which means a^2 is even $\Rightarrow a$ is even which means $\exists c \in \mathbb{N} : a = 2c$

$$\begin{aligned}\Rightarrow 2b^2 &= (2c)^2 = 4c^2 \\ \Rightarrow b^2 &= 2c^2\end{aligned}$$

So now the right hand side is even which means $b^2 \Rightarrow b$ is even. So both a and b are even since $\gcd(a, b) = 1$.⁴ Thus, the assumption is wrong which means $\sqrt{2}$ is irrational. \square

Definition 1.3. Let $P(n)$ be a statement on natural numbers. A **proof by induction** is a proof where we can prove that:

- $P(1)$ i.e. P holds for $n = 1$.
- $\forall n \in \mathbb{N} : P(n) \Rightarrow P(n+1)$ i.e. if P holds for n then it also holds for $n + 1$.

Then $P(n)$ holds $\forall n \in \mathbb{N}$.

Example 1.1.3. Prove that $\sum_{i=1}^n i = 1 + 2 + \dots + n = \frac{n(n+1)}{2}$.

⁴" $\not\models$ " is a symbol used when there's a contradiction in the proof. Another symbol used for this can be \sharp .

Proof. We will prove by induction:

- $(n = 1 \text{ :}) 1 = \frac{1+2}{2} = 1$
- $(n \implies n + 1 \text{ :})$ Assume the formula holds for some $n \in \mathbb{N}$ (*inductive hypothesis*). We have to show the formula holds for $n + 1$:

$$\begin{aligned} 1 + 2 + \dots + n + (n + 1) &= \frac{n(n + 1)}{2} + (n + 1) \\ &= \frac{n(n + 1)}{2} + \frac{2(n + 1)}{2} \\ &= \frac{(n + 1)(n + 2)}{2} = \frac{(n + 1)[(n + 1) + 1]}{2} \end{aligned}$$

Thus $\sum_{i=1}^n i = 1 + 2 + \dots + n = \frac{n(n+1)}{2} \forall n \in \mathbb{N}$. □

Remark 1.2. When finishing a proof, we must place either \square , \blacksquare or just Q.E.D (stands for *quod erat demonstrandum*) at the end.

Theorem 1.1. (Bernoulli's Inequalities). $\forall x \geq -1, \forall n \in \mathbb{N} : (1 + x)^n \geq 1 + nx$

Proof. Induction on n .

Base case ($n = 1$): $(1 + x)^1 = 1 + x = 1 + 1 \cdot x$ i.e. $(1 + x)^1 = 1 + 1 \cdot x$ especially $(1 + x)^1 \geq 1 + 1 \cdot x$.

$n \rightarrow n + 1$: Assume the inequalities holds for some $n \in \mathbb{N}$ i.e.

$$\begin{aligned} (1 + x)^n \geq 1 + nx \implies (1 + x)^{n+1} &= \underbrace{(1 + x)^n}_{\geq 1 + nx} \cdot \underbrace{(1 + x)}_{\geq 0} \\ &\geq (1 + nx)(1 + x) \\ &= 1 + x + nx + nx^2 \\ &= 1 + (n + 1)x + nx^2 \\ &\geq 1 + (n + 1)x \end{aligned}$$

$$\implies (1 + x)^{n+1} \geq 1 + (n + 1)x$$

□

Remark 1.3. For a specific case of $x = 1$, then $\forall n \in \mathbb{N} : 2^n \geq n + 1 \iff \forall n \in \mathbb{N} : n < 2^n$

Theorem 1.2. (*The Well-Ordering Principle of \mathbb{N}*). Every non-empty subset of \mathbb{N} has a least element (or minimal element or minimal) i.e. $\exists s \in S, \forall t \in S : s \leq t$.

Remark 1.4. Most other sets of numbers do not have this property e.g. $\mathbb{Z}, S = \{\dots, -3, -2, -1, 0\}$.

Example 1.1.4. Continuing from remark 1.3, we can have $\mathbb{Q}, S = \{\frac{1}{n} : n \in \mathbb{Q}\}$. In this case, numbers gets arbitrarily close to 0 i.e. it does not have a minimal elements (0 is not part of this set).

Claim: S does not have a least element

Proof. (By Contradiction) Assume $s \in S$ is a least element. Then $s > 0$. Let $n \in \mathbb{N}$ s.t. $n > \frac{1}{s}$. Then, $\frac{1}{n} < s \implies s$ is not a least element \nexists . Thus S does not have a least element. \square

Proof. (To Theorem 1.2) We'll first consider 2 cases:

1. S is a finite subset of \mathbb{N} : We will prove the result by induction on $n := |S|$.⁵

- Base case ($n = 1$): then $S = \{a_1\}$ for some $a_1 \in \mathbb{N}$. Then a_1 is the least element of S .
- $n \rightarrow n+1$: Ind. Hyp: Every subset of \mathbb{N} of cardinality n has a least element for some $n \in \mathbb{N}$. Let $S = \{a_1, a_2, \dots, a_n, a_{n+1}\} \subseteq \mathbb{N}$. Let $S' := \{a_1, a_2, \dots, a_n\}$, by inductive hypothesis, S' has a least element s . If $s < a_{n+1}$ then, $\forall 1 < k \leq n+1 : s \leq a_k$ so s is the least element. If $a_{n+1} < s$ then, $\forall 1 \leq k \leq n : a_{n+1} < s \leq a_k \implies \forall 1 \leq k \leq n+1, a_{n+1} \leq a_k \implies a_{n+1}$ is the least element of S .

So in either case, S has a least element. We've now proven for all finite subsets of \mathbb{N} .

2. S is infinite: let $t_0 \in S$ be arbitrary and let $S' := S \cap \{1, 2, 3, \dots, t_0\} \implies S'$ is finite and non-empty (since t_0 is in the intersection). Because S' is finite, by first case, S' has a least element s . We will now prove that s is also the least element of S . Let $t \in S$ be arbitrary

- (a) If $t \in S' \implies s \leq t$ since s is the least element of S' .
- (b) If $t \notin S'$, Then $t > t_0$ and $\underbrace{s \leq t_0}_{t_0 \in S'} < t$ and s is the least element of S'
 $S' \implies s < t \implies s \leq t$.

⁵ $|...|$ is called the cardinality or number of element of a set.

In either case, $s \leq t \implies s$ is the least element of S .

Therefore, no matter if S is finite or infinite, it will have a least element. \square

1.2 Brief Introduction to Logic

Definition 1.4. A **statement** is any expression that is either true or false

Example 1.2.1. "7 is a prime" is a true statement. "27 is a prime" is a false statement.

Definition 1.5. Compound statement: Let p and q be statements then $P \wedge Q$, called " P and Q ", is true iff both P and Q are true. Similarly $P \vee Q$, called " P or Q ", iff at least 1 P, Q is true. See the **truth table** below

P	Q	$P \wedge Q$	P	Q	$P \vee Q$
T	T	T	T	T	T
T	F	F	T	F	T
F	T	F	F	T	T
F	F	F	F	F	F

Definition 1.6. $\neg P$, called "not P ", is true iff P is false. See truth table below

P	$\neg P$
T	F
F	T

Theorem 1.3. (De Morgan's Law). Let P and Q be statements then

$$(a) \quad \neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q).^6$$

$$(b) \quad \neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q).$$

Proof. We will only prove (a) using truth table:

P	Q	$P \vee Q$	$\neg(P \wedge Q)$	P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
T	T	T	F	T	T	F	F	F
T	F	F	T	T	F	F	T	T
F	T	F	T	F	T	T	F	T
F	F	F	T	F	F	T	T	T

As we can see, truth table for $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ is equivalence. \square

⁶ \equiv means logically equivalent i.e. both sides have the same truth table.

Definition 1.7. **Implications** is denoted as $P \Rightarrow Q$ which is read as "if P then Q ". This is given as the truth table follows:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

The last 2 lines of the truth table is called *Ex Falso Quod Libet* i.e. "from falsehood, anything follows".

Example 1.2.2. $x = y \Rightarrow x^2 = y^2$ is true. Now let's plug in specific numbers

$$(a) \quad x = 1, y = -1: \underbrace{1 = -1}_{\text{false}} \Rightarrow \underbrace{1^2 = (-1)^2}_{\text{true}} \text{ is true.}$$

$$(b) \quad x = 1, y = 2: \underbrace{1 = 2}_{\text{false}} \Rightarrow \underbrace{1^2 = 2^2}_{\text{true}} \text{ is true.}$$

Exercise: Prove the following, $P \Rightarrow Q \equiv \neg P \vee Q$ and $\neg(P \Rightarrow Q) \equiv \neg((\neg P) \vee Q)$

Proof. We will prove the second one first directly from definition

$$\begin{aligned} \neg(P \Rightarrow Q) &\equiv \neg((\neg P) \vee Q) \\ &\equiv (\neg \neg P) \wedge Q (\neg Q) \equiv P \wedge (\neg Q) \end{aligned}$$

For the first one, we'll prove using truth table:

P	Q	$P \Rightarrow Q$	P	Q	$\neg P$	$\neg P \vee Q$
T	T	T	T	T	F	T
T	F	F	T	F	F	F
F	T	T	F	T	T	T
F	F	T	F	F	T	T

As we can see, the truth table for $P \Rightarrow Q$ and $\neg P \vee Q$ is equivalence hence they're equivalence. \square

Remark 1.5. Another important result is the invert between the statement "for all" and "there exists": $\neg(\forall x : P(x)) \equiv \exists x : \neg P(x)$ and similarly $\neg(\exists x : P(x)) \equiv \forall x : \neg P(x)$.

1.3 Brief Introduction to Set Theory

Definition 1.8. A **set** is a collection of distinct objects called **elements**. Usually a set is represented as an upper case letter (e.g. A, B, etc.) while its elements are lower case letters.

Example 1.3.1. $A = \{a_1, a_2, a_3\}$ is a set.

Remark 1.6. A set is specified by either: listing its elements (if possible) or by stating a property that its elements have to satisfy.

Definition 1.9. Let A and B be sets then, the **union** of A is B is defined and written as:

$$A \cup B = \{x : x \in A \text{ or } x \in B\} \quad (1.1)$$

and the **intersection** of A and B is defined as:

$$A \cap B = \{x : x \in A \text{ and } x \in B\} \quad (1.2)$$

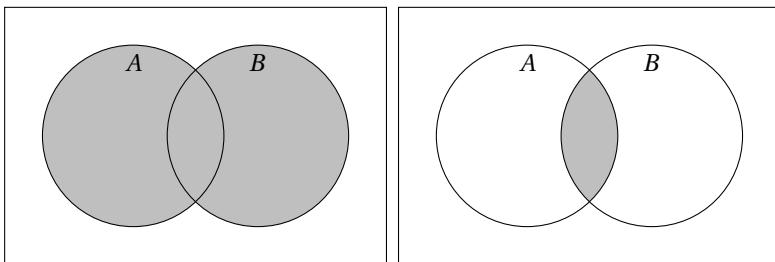


Figure 1.1: Left figure represent set union while the right represent set intersection.

Example 1.3.2. Let $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$ and $C = \{2, 4, 6\}$, then $A \cap B = \{1, 3\}$. However, when $B \cap C$, we will see that there's no shared element which means this will create a **empty set** symbolized as \emptyset .

Example 1.3.3. Using the same sets: A, B and C then we can see that $A \cup B = \{1, 2, 3, 5\}$.

Definition 1.10. Let A and B be sets. Then, A is contained or is a **subset** of B if every elements in A is in B . This relation is denoted as

$$A \subset B \quad (1.3)$$

Example 1.3.4. Let $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$, $C = \{2, 4, 6\}$ and $D = \{1, 2, 3, 4, 5, 6\}$, then $A \subset D$, $B \subset D$ and $C \subset D$.

Definition 1.11. If a set contain all elements under consideration in a particular context, we call it a **universal set (S)**.⁷

Definition 1.12. If a set consists of no elements, we call it an **empty set (Ø)**.

Definition 1.13. Let $A \subset S$. Then, the **complement** of A , written as A^c or A' is the set of elements in S but not in A . That is,

$$A' = \{a \in S : a \notin A\} \quad (1.4)$$

Definition 1.14. For unions and intersection of finitely many sets. Then, let $n \in \mathbb{N}$, The **union of finitely many sets** is given as

$$A_1 \cup A_2 \cup \dots \cup A_n := \{x : x \in A_1 \vee x \in A_2 \vee \dots \vee x \in A_n\} \quad (1.5)$$

or simply

$$A_1 \cup A_2 \cup \dots \cup A_n := \{x : \exists 1 \leq i \leq n : x \in A_i\} = \bigcup_{i=1}^n A_i \quad (1.6)$$

We can also define similarly for the **intersection of finitely many sets** as

$$A_1 \cap A_2 \cap \dots \cap A_n := \{x : x \in A_1 \wedge x \in A_2 \wedge \dots \wedge x \in A_n\} \quad (1.7)$$

or simply

$$A_1 \cap A_2 \cap \dots \cap A_n := \{x : \forall 1 \leq i \leq n : x \in A_i\} = \bigcap_{i=1}^n A_i \quad (1.8)$$

Definition 1.15. For unions and intersection of infinitely many sets. Then, let A_1, A_2, \dots be sets, the **infinite union** of these is defined as

$$A_1 \cup A_2 \cup \dots = \bigcup_{i=\mathbb{N}} A_i := \{x : \exists i \in \mathbb{N} : x \in A_i\} \quad (1.9)$$

And we define the **infinite intersection** in the same manner

$$A_1 \cap A_2 \cap \dots = \bigcap_{i=\mathbb{N}} A_i := \{x : \forall i \in \mathbb{N} : x \in A_i\} \quad (1.10)$$

Remark 1.7. We can definite unions or intersections for any (non-empty) index set

⁷Some other notation of universal set can be: U, ξ or \mathcal{U}

Example 1.3.5. $\bigcap_{x \in \mathbb{R}^+} [0, x] = \{0\}$

Proof. We must show that RHS \subseteq LHS and vice versa. $\forall x > 0 : 0 \in [0, x] \implies 0 \in \bigcap_{x > 0} [0, x]$. We now need to show that LHS \subseteq RHS. Let $x_0 \in \bigcap_{x > 0} [0, x]$ $\implies x_0 \geq 0$. Assume $x_0 > 0$, let $x := \frac{x_0}{2}$ then $0 \leq x < x_0 \implies x_0 \notin [0, x] \implies x \notin \bigcap_{x > 0} [0, x]$. This means $x_0 = 0$.

Since $\bigcap_{x \in \mathbb{R}^+} [0, x] \subseteq \{0\}$ and $\{0\} = \bigcap_{x \in \mathbb{R}^+} [0, x]$ Thus $\bigcap_{x \in \mathbb{R}^+} [0, x] = \{0\}$ \square

Definition 1.16. Let A and B be sets then the **complement of A and B** is given as

$$A \setminus B := \{x : x \in A \wedge x \notin B\} \quad (1.11)$$

Theorem 1.4. (De Morgan's Laws), Let A, B and C be sets then

1. $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
2. $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Proof. Proof will be given in tutorials. \square

Definition 1.17. Generalization of the complement for infinitely many set is give as

$$A \setminus \bigcup_{i \in S} B_i = \bigcap_{i \in S} (A \setminus B_i) \quad (1.12)$$

and

$$A \setminus \bigcap_{i \in S} B_i = \bigcup_{i \in S} (A \setminus B_i) \quad (1.13)$$

1.3.1 Function

Definition 1.18. Let D and E be sets, a **function** $f : D \rightarrow E$ is a rule that assigns to each $x \in D$ and uniquely determined $y \in E$. If $y \in E$ is assigned to x via f , we write

$$y = f(x) \quad (1.14)$$

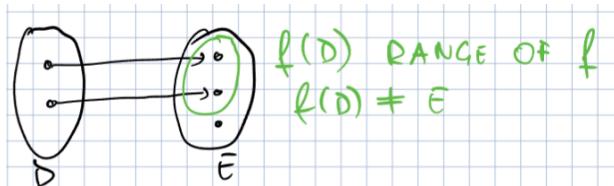
The set D is called the **domain** and E is called the **codomain**.⁸

Remark 1.8. $f(D)$ can be a true subset of E i.e. $f(D) \subseteq E$ and $f(D) \neq E$. We can notate it as $f(D) \subsetneq E$ or $f(D) \subset E$.⁹

Example 1.3.6. $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x^2$. $f(\mathbb{R}) = \mathbb{R}_0^+ = [0, \infty \subset \mathbb{R}]$

⁸There are other way to call a codomain such as range, image, targets.

⁹We will not use \subset since some misinterpret it that \subset and \subseteq are interchangeable.

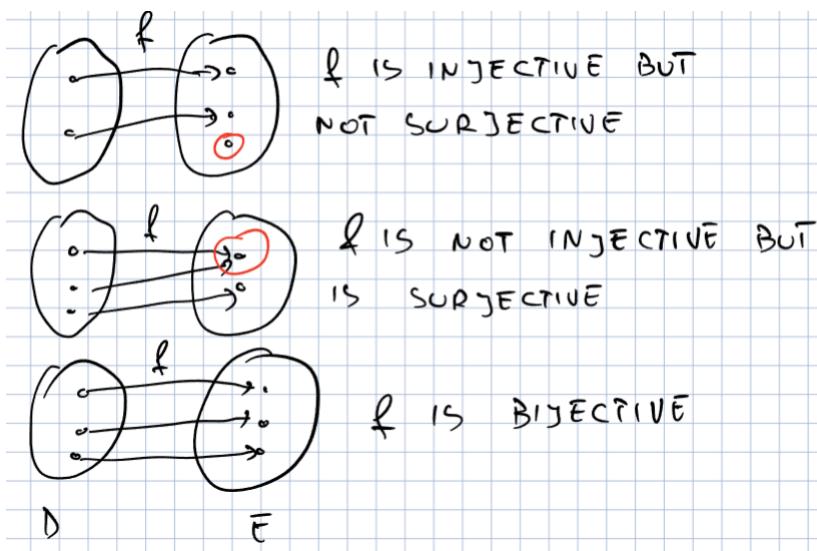
**Figure 1.2:** Range (image) vs codomain

Definition 1.19. A function $f : D \rightarrow E$ where $f(D) = E$ is called **surjective**.

Definition 1.20. Let $f : D \rightarrow E$ be a function such that $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2), \forall x_1, x_2 \in D$ is called **injective (one-to-one)**.

Definition 1.21. A function that is both injective and surjective is called **bijective**.

Example 1.3.7. The followings are injective, surjective and bijective functions:



Theorem 1.5. A function is invertible (i.e. has an inverse function) iff it is bijective

The Absolute Value Function

Definition 1.22. An **absolute value function** $f: \mathbb{R} \rightarrow \mathbb{R}_0^+, x \mapsto |x|$ is defined as

$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} \quad (1.15)$$

Theorem 1.6. (Properties of function): $\forall x, y \in \mathbb{R}$:

1. $|x| = \sqrt{x^2}$
2. $|xy| = |x| \cdot |y|$
3. $-|x| \leq x \leq |x|$
4. $|x+y| \leq |x| + |y|$ (**Triangle inequalities**)

Proof. We'll prove only the 2. $|xy| = \sqrt{(xy)^2} = \sqrt{x^2 \cdot y^2} = \sqrt{x^2} \cdot \sqrt{y^2} = |x| \cdot |y|$. \square

1.4 Completeness

We've essentially done with all of all preliminaries, so now we'll be moving toward closer concept of analysis.

Definition 1.23. A non-empty set $S \subseteq \mathbb{R}$ is said to be **bounded from above** if $\exists u \in \mathbb{R}: \forall s \in S, s \leq u$. Any such u is called the **upper bound of S** .

Example 1.4.1. $S = [0, 1]$ then any number $u \geq 1$ is an upper bound. Contrarily, $S := [0, \infty[$ is not bounded from above.

Proof. Assume an upper bound u exists. Then, $u \geq 0 \implies u+1 \geq 0 \implies u+1 \in S$ but $u < u+1$, \nexists to the definition of upper bound. Thus S does not have an upper bound. \square

Remark 1.9. Upper bound is not uniquely defined as there can be an upper bound larger than another upper bound.

Definition 1.24. Similarly, $S \subseteq \mathbb{R}$ is called **bounded from below** if $\exists v \in \mathbb{R}: \forall s \in S, s \geq v$. Any such v is called a **lower bound of S** .

Definition 1.25. Let $S \neq \emptyset$ and $S \subseteq \mathbb{R}$. We say that

- (a) u is the **maximum/greatest element** of S , if u is an upper bound of S and $u \in S$. We denote it as $u = \max S$.
- (b) v is the **minimum/least element** of S , if v is a lower bound of S and $v \in S$. We denote it as $v = \min S$.

Example 1.4.2. $S = [a, b]$. we will show that $a = \min S$, but that does S does not have a maximum.

Lemma 1.1. If $x < y$ then $x < \frac{x+y}{2} < y$ where $\frac{x+y}{2}$ is called the **arithmetic mean** of x and y .

Proof. $x = \frac{1}{2}x + \frac{1}{2}x < \frac{1}{2}x + \frac{1}{2}y = \frac{x+y}{2} < \frac{1}{2}y + \frac{1}{2}y = y \implies x < \frac{x+y}{2} < y$. \square

Remark 1.10. This shows that between each 2 non equal real numbers. There always lies another real number. Especially there are no "immediate neighbours" on the number line.

In fact, between any 2 real numbers, there even lie infinitely many real numbers.

Proof. To be proven in exercise. The general idea is that you keep halving the distance between 2 numbers etc. \square

Returning back to example 1.4.2

Proof. ($a = \min S$ for $[a, b]$). Let $x \in [a, b]$. Then especially $a \leq x$ i.e. a is a lower bound of S ; also, $a \in S \implies a \in \min S$. \square

Proof. (S does not have a max). Assume that S has a max and let m be this max. Then $m \in [a, b]$ i.e. $a \leq m < b$. Now consider $m' = \frac{m+b}{2}$. By lemma 1.1, we have that $a \leq m < m' < b \implies m' \in [a, b]$ and $m < m' \implies m'$ is the max of $[a, b]$. Thus S does not have a max. \square

Definition 1.26. Consider 2 situations:

- (a) Let $S \neq \emptyset$ and $S \subseteq \mathbb{R}$ be bounded from above. We say that $s \in R$ is the **supremum/least upper bound** of S if
 - (i) s is an upper bound of S , and
 - (ii) $s \leq u$ for all upper bounds u of S .
- (b) Let $S \neq \emptyset$ and $S \subseteq \mathbb{R}$ be bounded from below. We say that $t \in R$ is the **infimum/greatest lower bound** of S if

- (i) t is a lower bound of S , and
- (ii) $t \geq u$ for all lower bounds u of S .

We denote supremum and infimum of S as: $\sup S$ and $\inf S$ respectively.

Example 1.4.3. $S = [a, b]$. We will show that $\inf S = a$ and $\sup S = b$.

Proof. First, we will show for infimum:

- (i) Let $x \in [a, b]$. Then, $a \leq x \implies a$ is a lower bound of $[a, b]$.
- (ii) Now let u be an arbitrary lower bound of $[a, b]$. Then, $\forall x \in [a, b] : v \leq x$. Especially $v \leq a \implies a \in \inf S$

Now we will show for supremum:

- (i) Let $x \in [a, b]$. Then, $x < b \implies b$ is an upper bound of $[a, b]$.
- (ii) Let u be an arbitrary upper bound of $[a, b]$. Assume that $u < b$. Then, $a \leq u < b$ thus by lemma 1.1., $a \leq u < \underbrace{\frac{u+b}{2}}_{\in [a,b]} < b \implies u$ is not an upper bound of $[a, b]$ $\not\implies u \geq b \implies b = \sup S$.

□

Theorem 1.7. Let $S \neq \emptyset$ and $S \subseteq \mathbb{R}$. Then,

- (a) If S has a maximum s , then s is also $\sup S$ (but the converse does not hold).
- (b) If S has a minimum t , then t is also $\inf S$ (but the converse does not hold).

Proof. (a) Let $s = \max S \implies s$ is an upper bound of $S \implies$ definition 1.26.a(i) holds. Let u be any upper bound of S . Then $\forall x \in S : x \leq u$, Especially, since $s \in S$, it follows that $s \leq u \implies$ definition 1.26.a(ii) holds $\implies s = \sup S$.

(b) will follow similar structure of proof. □

Example 1.4.4. $[a, b]; a = \min[a, b] \implies a = \inf[a, b]$.

Question: If $S \neq \emptyset$ and $S \subseteq \mathbb{R}$ is bounded from above, does it necessarily have a supremum?

Answer: Well,,, we cannot answer this question formally since we do not have a proper definition of \mathbb{R} available. In factm in modern mathematics, \mathbb{R} is defined to be the completion of \mathbb{Q} i.e. as the smallest set of numbers containing \mathbb{Q} that is complete, meaning that any non-empty and bounded from above subset has a supremum in this set.

Intuitively, \mathbb{R} is ddefined by "filling the gaps" in the numbers left by \mathbb{Q} . This is done through **Dedekind cuts**. For us, we will state the completeness as an axiom instead:

Axiom of Completeness. Let $\emptyset \neq S \subseteq \mathbb{R}$ and S is bounded from above. Then, S has a supremum in \mathbb{R} .

1.4.1 Consequences of Completeness

Theorem 1.8. (*The Archimedean Property of \mathbb{R}*). *Let $x \in \mathbb{R}$ be arbitrary. Then $\exists n \in \mathbb{N} : n > x$.*

Proof. Assume $\exists x \in \mathbb{R} : \forall n \in \mathbb{N} : n \leq x$. Now consider $S := \{x \in \mathbb{R} : \forall n \in \mathbb{N} : n \leq x\}$. NOte that $S \neq \emptyset$ by our assumption. Furthermore, S is bounded from below (e.g. 0 is the lower bound of S since $\forall n \in \mathbb{N} : n > x$ if $x \leq 0$, i.e. No $x \leq 0$ is in S).

It follows from completeness that S has an infimum in \mathbb{R} ; let $t := \inf S$. t is thus the greatest lower bound of $S \implies t + 1$ is not a lower bound for S . Thus $\exists x \in S : x < t + 1 \implies x - 1 < t$; since t is a lower bound of S , $x - 1$ cannot be in S i.e. $x - 1 \notin S \implies \exists n \in \mathbb{N} : n > x - 1 \implies \underbrace{n+1}_{\in \mathbb{N}} > \underbrace{x}_{\in S}$ (by construction of S).

Thus our assumption was wrong \therefore we've shown that $\forall x \in \mathbb{R}, \exists n \in \mathbb{N} : n > x$. \square

Corollary 1.1. *Let $0 < x$. Then, $\exists n \in \mathbb{N} : \frac{1}{n} < x$.*

Proof. Let $0 < x$. By the Archimedean property, $\exists n \in \mathbb{N} : n > \frac{1}{x} \iff \frac{1}{n} < x$. \square

Example 1.4.5. Define the set $S := \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$. Find $\sup S$ and $\inf S$.

Solution: Note that S is bounded from above by 1 and bounded from below by 0. Thus S has both a sup and an inf by completeness. NOte also that $1 \in S$

and 1 is upper bound of $S \implies 1 = \max S \implies 1 = \sup S$. We will now show that $0 = \inf S$.

Proof. Let $t := \inf S$. Since 0 is lower bound of $S \implies t \geq 0$. Assume that $t > 0$. By corrolary ... $\exists n \in \mathbb{N} : \underbrace{\frac{1}{n}}_{\in S} < t \implies t$ is not a lower bound of $S \not\implies t = 0 \implies \inf S = 0$. \square

1.4.2 Density of \mathbb{Q} in \mathbb{R}

Definition 1.27. A subset $S \subseteq \mathbb{R}$ is called **dense** in \mathbb{R} if $\forall a, b \in \mathbb{R}, a < b, \exists x \in S : a < x < b$ i.e. no matter how close a and b , you can always find x if S is dense in \mathbb{R} .

Theorem 1.9. \mathbb{Q} is dense in \mathbb{R}

Proof. Let a and $b \in \mathbb{R} : a < b$. Then, we'll be considering the following cases

1. $0 \leq a < b$: Let $n > \frac{1}{b-a}$ (by Archemidean property) $\iff \frac{1}{n} < b-a$. Consider the set $\left\{ \frac{1}{n}, \frac{2}{n}, \dots \right\}$

Claim: $\exists k_0 \in \mathbb{N} : \frac{k_0}{n} > a \iff k_0 > na$ (Arch prop.) Thus, $S := \left\{ j \in \mathbb{N} : \frac{j}{n} > a \right\}$ is non-empty $\implies k_0 \in S$. By well-ordering principle of \mathbb{N} , S has a least element k . Then, $a < \frac{k}{n}$. Also, $\frac{k-1}{n} \leq a$ **why?** Well...if $k > 1$, then $k-1 \in \mathbb{N}$ and by construction of k being the least of element of S , we must have that $\frac{k-1}{n} \leq a$. If $k = 1$, then $\frac{k-1}{n} = 0 \leq a$. In all cases, we thus have $\frac{k-1}{n} = \frac{k}{n} - \frac{1}{n} \leq a \implies \frac{k}{n} \leq a + \underbrace{\frac{1}{n}}_{< b-a} < a + (b-a) \implies \frac{k}{n} < b$.

Thus $a < \frac{k}{n} < b$, we define $x := \frac{k}{n}$ is a rational number between a and b

2. $a < b \leq 0$: Then, we have $0 \leq -b < -a$. By the first case, $\exists x \in \mathbb{Q} : -b < x < -a \implies a < \underbrace{-x}_{\in \mathbb{Q}} < b$. Therefore, we've found a rational number between a and b .

3. $a < 0 < b$: Since $0 \in \mathbb{Q}$, then we can define $x := 0$.

Thus we've found $x \in \mathbb{Q}$ in all cases such that $a < x < b \implies \mathbb{Q}$ is dense in \mathbb{R} . \square

Theorem 1.10. (The Nested Interval Property). *Let $a_1, a_2, \dots, b_1, b_2, \dots \in \mathbb{R}, \forall n \in \mathbb{N} : a_n \leq b_n, [a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \dots$. Then*

$$\bigcap_{n \in \mathbb{N}} [a_n, b_n] \neq \emptyset$$

Proof. Let $\forall n \in \mathbb{N} : I_n := [a_n, b_n]$, we have $I_1 \supseteq I_2 \supseteq \dots \implies a_1 \leq a_2 \leq \dots$ and $b_1 \geq b_2 \geq \dots$. Also, $\forall n \in \mathbb{N} : a_n \leq b_n$. We will have the following lemma:

Lemma 1.2. *We also have that $\forall n, k \in \mathbb{N} : a_n \leq b_k$*

Proof. We will consider the following cases:

1. $n \leq k$: Then, $a_n \leq a_k \leq b_k \implies a_n \leq b_k$.
2. $n > k$: Then, $a_n \leq b_n \leq b_k \implies a_n \leq b_k$

Thus in all cases, we have $a_n \leq b_k$ \square

Returning to our proof, let $A := \{a_1, a_2, a_3, \dots\}$. By lemma 1.2, $\forall n \in \mathbb{N} : a_n \leq b_1 \implies b_1$ is an upper bound of $A \implies A \neq \emptyset$ and is bounded from above. By the completeness of \mathbb{R} , A has a supremum $a := \sup A$. Similarly, the set $B := \{b_1, b_2, b_3, \dots\}$ is non-empty and bounded from below (e.g. by a_1). By completeness, B has an infimum $b := \inf B$. Then, consider the following lemmas:

Lemma 1.3. $\forall k \in \mathbb{N} : a \leq b_k$

Proof. By the Lemma 1.2, $\forall n, k \in \mathbb{N} : a_n \leq b_k \implies \forall k \in \mathbb{N} : b_k$ is upper bound of A . Since a is the least upper bound of A , we have $\forall k \in \mathbb{N}, a \leq b_k$. \square

Lemma 1.4. $a \leq b$

Proof. By lemma 1.3, $\forall k \in \mathbb{N} : a \leq b_k \implies a$ is the lower bound of B . Since b is the greatest lower bound of B , we then have $a \leq b$. \square

Returning to the proof again, let $n \in \mathbb{N}$. Then, $a_n \leq a \leq b \leq b_n$ (since a is the upper and b is the lower bound). In other words, $\forall n \in \mathbb{N}, [a, b] \subseteq [a_n, b_n] \implies [a, b] \subseteq \bigcap_{n \in \mathbb{N}} [a_n, b_n] \implies \bigcap_{n \in \mathbb{N}} [a_n, b_n] \neq \emptyset$. \square

Remark 1.11. *It can be shown that the intersection $\bigcap_{n \in \mathbb{N}} [a_n, b_n] = [a, b]$*

If $a = b$, the intersection consists of a single point. If $a < b$ then the intersection has infinitely many points.

What if the nested sequence consists of intervals that are unbounded?
Well...the intersection can be empty

Example 1.4.6. $\bigcap_{n \in \mathbb{N}} [0, \frac{1}{n}] = \emptyset$. **Hint:** Consider 2 cases: $x > 0$ and $x = 0$. ¹⁰

End of Lecture

1.5 Cardinality

Goal: "Measure" the size of a set.

Definition 1.28. If S is a finite site, then the **cardinality** of S is the number of element of S . We notate as

$$|S| = n \quad (1.16)$$

where n is the number of element.

Example 1.5.1. $|\emptyset| = 0$ and $|\{1, 3, 5\}| = 3$.

Georg Cantor's idea was to use functions to compare the sizes of sets.

- **Finite case:** Supposed that $|A| = n$ and $|B| = k$ where $A = \{a_1, a_2, \dots, a_k, a_{k+1}, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_k\}$. If $n > k \exists$ surjective map from $f : A \rightarrow B$, $f(a_j) = b_j$ if $1 \leq j \leq k$, and $f(a_j) = b_k$ if $k+1 \leq j \leq n$. However, \nexists injective map from A to B **why?** Well...we'll prove it through the following

Proof. Assume that \exists injective $f : A \rightarrow B, \forall b_j = f(a_j)$. Then, $f(A) = \{b_1, \dots, b_n\}$ where $|f(A)| = n$ since the b_1, \dots, b_n are *pairwise distinct*. Contrarily, $\{b_1, \dots, b_n\} \subseteq B$ and $|B| = k \implies \{b_1, \dots, b_n\} \leq k < n \notin$. Therefore no such injective map can exist. \square

Similarly, if $|A| < |B|, \exists$ an injective map from A to B but there is no surjective map from A to B . Thus, if $|A| \neq |B|$, there cannot exist a bijective map from A to B . Otherwise, **if $|A| = |B|$, there exist a bijective map from A to B .**

We can now extend this idea to infinite sets

¹⁰ $[a, \infty]$ is closed as it's the same as $\{x : a \leq x\}$

Definition 1.29. Let A and B be sets. We say that A and B have the same cardinality i.e. $|A| = |B|$ if $f : A \rightarrow B$ and f is bijective.

Furthermore, we say that the cardinality of A is strictly less than the cardinality of B i.e. $|A| < |B|$, $\exists f : A \rightarrow B$, f is surjective.

Example 1.5.2. Consider the following sets \mathbb{N} and \mathbb{N}_0 . Then, $\mathbb{N} \subsetneq \mathbb{N}_0$. We define a map $f : \mathbb{N} \rightarrow \mathbb{N}_0$, $f(n) := \underbrace{n - 1}_{\in \mathbb{N}_0}$. We state that $|\mathbb{N}| = |\mathbb{N}_0|$

Proof. We need to check if it's injective. Let $f(n) = f(k) \implies n - 1 = k - 1 \implies n = k$. Thus, f is injective. We now check for surjectivity. Let $n \in \mathbb{N}_0$ be arbitrary. Then, $n + 1 \in \mathbb{N}$ and $f(n + 1) = n \implies f$ is surjective. Because f is injective and surjective, it's bijective $\implies |\mathbb{N}| = |\mathbb{N}_0|$ \square

Remark 1.12. This cannot happen in finite sets.

Example 1.5.3. Consider \mathbb{Z} . We'll construct a bijective map from \mathbb{N} to \mathbb{Z} even though \mathbb{Z} is seemingly "twice as big" as \mathbb{N} .

Consider this complete list of integers:

$$0, 1, -1, 2, -2, 3, -3, \dots$$

and let $a_1 = 0, a_2 = 1, a_3 = -1, \dots$. Define $f : \mathbb{N} \rightarrow \mathbb{Z}$ by: $f(n) = a_n, \forall n \in \mathbb{N}$. f is injective since all the numbers in the list are distinct; it is surjective since it's complete $\implies f$ is bijective $\implies |\mathbb{N}| = |\mathbb{Z}|$.

We can even explicitly construct f as

$$f(n) := \begin{cases} \frac{n}{2} & \text{if } n \text{ even} \\ -\frac{n-1}{2} & \text{if } n \text{ odd} \end{cases}$$

Definition 1.30. A set A is called **countably infinite** if $|A| = |\mathbb{N}|$ i.e. $\exists f : \mathbb{N} \rightarrow A$ is bijective.

Definition 1.31. A set B is called **countable** if B is either finite or countably infinite.

Example 1.5.4. \mathbb{N}_0 and \mathbb{Z} are countably infinite and countable. On the other hand, $\{1, 3, 5\}$ is countable but not countably infinite.

Theorem 1.11. Let A be a subset of \mathbb{N} . Then, A is countable.

Proof. Assignment 3 \square

Theorem 1.12. Let A be a set s.t. \exists surjective map from \mathbb{N} to A . Then A is countable.

Proof. If A is finite then no need to prove. Let A be infinite and let $a \in A$ be arbitrary. Since f is surjective, $\mathbb{N} \supseteq f^{-1}(\{a\}) \neq \emptyset$. Since $f^{-1}(\{a\}) \neq \emptyset$, it has a least element $n_a \subseteq \mathbb{N}$. Let $S = \{n_a : a \in A\} \subset \mathbb{N}$.

Consider the restriction $f|_S$ of f to S i.e. $f|_S : S \rightarrow A, n_a \mapsto a$. We will prove that $f|_S$ is bijective.¹¹

- Injective: Let $\underbrace{f(n_a)}_{=a} = \underbrace{f(n_b)}_{=b} \implies a = b \implies n_a = n_b \implies f$ is injective.
- Surjective: Let $a \in A$ be arbitrary. Then, $n_a \in S$ and by definition, $f(n_a) = a \implies f$ is surjective.

Since $f : S \rightarrow A$ is injective and surjective, it's also bijective $\implies |S| = |A|$ but $S \subseteq \mathbb{N} \implies S$ is countable by theorem 1.11 $\implies A$ is countable. \square

Theorem 1.13. \mathbb{Q} is countably infinite.

Proof. First, we'll prove that the set of all positive rational number \mathbb{Q}^+ is countably infinite. Consider the matrix-like arrangement

1	2	3	4	...
1	1	1	1	
1	2	3	4	
2	2	2	2	
1	2	3	4	
3	3	3	3	
1	2	3	4	
4	4	4	4	
...

Every positive rational number appears infinitely often in this scheme. Counting along the diagonal yield the following enumeration with repetition of \mathbb{Q}^+ :

$$\underbrace{\frac{1}{1}}_{1. \text{ diag.}}, \underbrace{\frac{2}{1}, \frac{1}{2}}_{2. \text{ diag.}}, \underbrace{\frac{3}{1}, \frac{2}{2}, \frac{1}{3}}_{3. \text{ diag.}}, \dots$$

Let $a_1 = 1$. diag., $a_2 = 2$. diag.,.... Let $f : \mathbb{N} \rightarrow \mathbb{Q}, n \mapsto a_n$. Since the scheme contains every positive rational number (even infinitely often) f is surjective (not injective). By theorem 1.12, \mathbb{Q}^+ is countable but $\mathbb{N} \subseteq \mathbb{Q}^+ \implies \mathbb{Q}^+$ is

¹¹ $f|_S$ is called the restriction of f to s

infinite $\Rightarrow \mathbb{Q}^+$ is countably infinite.

Now, we need to show that \mathbb{Q} is countably infinite. To do so, we use the same trick we used for \mathbb{Z} from last lecture. Let a_1, a_2, a_3, \dots be an enumeration of \mathbb{Q}^+ . Then, $0, a_1, -a_1, a_2, -a_2, \dots$ is an enumeration of \mathbb{Q} . Thus \mathbb{Q} is countably infinite \square

All of these results are not as shocking as the Cantor's result that \mathbb{R} is uncountable.

Theorem 1.14. *Every interval $[a, b] : a < b$ is uncountable.*

Proof. Assume that $[a, b]$ is countably infinite. Then, x_1, x_2, x_3, \dots be an enumeration of $[a, b]$. Construct a nested sequence of closed and bounded intervals as follows:

Divide $[a, b]$ into 3 closed subintervals of equal width. Then, at least one of these subintervals does not contain x_1 , we'll call it I_1 . ¹² Now divide I_1 into 3 subintervals of equal width; at least one of these subintervals will not contain x_2 ; we'll call it I_2 . Divide I_2 into 3 subintervals \dots etc. We obtain a nested sequence of $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ of closed and bounded intervals with property $\forall n \in \mathbb{N} : x_n \notin I_n$.

By nested interval property of \mathbb{R} , $\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$. Let $x \in \bigcap_{n \in \mathbb{N}} I_n$. Then, $x \in [a, b]$. Thus $\exists n_0 \in \mathbb{N} : x = x_{n_0}$. But $x = x_{n_0} \notin I_{n_0} \Rightarrow x = x_{n_0} \notin \bigcap_{n \in \mathbb{N}} I_n$. We thus have both $x \in \bigcap_{n \in \mathbb{N}} I_n$ and $x \notin \bigcap_{n \in \mathbb{N}} I_n$. Thus $[a, b]$ cannot be countably infinite $\Rightarrow [a, b]$ is uncountable. \square

Because of theorem 1.14, we know that $\mathbb{R} \geq [a, b] \Rightarrow \mathbb{R}$ is also uncountable.

¹²We need 3 subintervals since x_1 can be contained in at most 2 of these subintervals and thus cannot be contained in the third.

2 Sequences

Definition 2.1. (Calculus). A **sequence** is an ordered list of numbers, often denoted as

$$(a_1, a_2, a_3, \dots) \quad (2.1)$$

or sometimes

$$a_1, a_2, a_3, \dots \quad (2.2)$$

As seen in the previous section, an ordered list can be identified with functions with domain \mathbb{N} . Thus we can make the definition of a sequence more rigorous as

Definition 2.2. A **sequence** in S is a function from \mathbb{N} to S , conventionally denote as: if $f : \mathbb{N} \rightarrow S$, we write $\forall n \in \mathbb{N} : a_n := f(n)$ and denote the sequence by

$$\left(\underbrace{a_1}_{f(1)}, \underbrace{a_2}_{f(2)}, \underbrace{a_3}_{f(3)}, \dots \right) =: (a_n)$$

or even $(a_n)_{n \in \mathbb{N}}$ or $(a_n)_{n=1}^\infty$

In this course, S is usually \mathbb{R} or a subset of \mathbb{R} . Sometimes it's convenient to not start a sequence at $n = 1$ but rather at $n = 0$ or any arbitrary integer n_0 ; we write $(a_n)_{n \in \mathbb{N}_0}$ or $(a_n)_{n=0}^\infty$; for $n_0 : (a_n)_{n=n_0}^\infty$

Example 2.0.1. Consider the following sequences

- $(1, 4, 9, 16, \dots) = (n^2) = (n^2)_{n \in \mathbb{N}} = (n^2)_{n=1}^\infty$
- $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots) = \left(\frac{1}{n}\right) = \left(\frac{1}{n}\right)_{n \in \mathbb{N}} = \left(\frac{1}{n}\right)_{n=1}^\infty$

2.1 Limits of Sequences

Intuitively from calculus, the definition of a limit is given as

Definition 2.3. (Calculus). Let (a_n) be a sequence of real numbers and let $L \in \mathbb{R}$. We say that (a_n) converges to L , or that L is the limit of (a_n) , if the a_n gets arbitrarily close to L as n grows without bounds.

This is a good definition...for the elementary level of calculus. We need to construct a more rigorous and precise definition of said limit for modern

mathematics.

Recall that $|x - y|$ measures the distance between 2 points $x, y \in \mathbb{R}$. Then we will get the modern definition of a sequence as

Definition 2.4. Let (a_n) be a sequence of real numbers and let $L \in \mathbb{R}$. we say (a_n) **converges** to L if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N : |a_n - L| < \varepsilon \quad (2.3)$$

We call $\exists N \in \mathbb{N}$ the **threshold index** and $|a_n - L|$ the **distance between** a_n and L .

Definition 2.5. Furthermore, L is called the **limit** of (a_n) , denotes as

$$L = \lim(a_n) = \left[\lim_{n \rightarrow \infty} (a_n) = \lim_{n \rightarrow \infty} a_n \right] \quad (2.4)$$

If the sequence (a_n) has a limit, we say that (a_n) **converges**; otherwise, we say that (a_n) **diverges**

Example 2.1.1. 1) Show that $\lim\left(\frac{1}{n}\right) = 0$.

Proof. Let $\varepsilon > 0$ be arbitrary. Then,

$$|a_n - L| = \left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \varepsilon \implies n > \frac{1}{\varepsilon}$$

Now let $N > \frac{1}{\varepsilon}$. Then we have $\forall n \geq N$ that

$$\left| \frac{1}{n} - 0 \right| < \varepsilon \implies \lim\left(\frac{1}{n}\right) = 0$$

□

2) Show that $\lim\left(\frac{n^2}{n^2+1}\right) = 1$.

Proof. Let $\varepsilon > 0$. Then

$$\left| \frac{n^2}{n^2+1} \right| = \left| \frac{n^2 - (n^2 + 1)}{n^2 + 1} \right| = \left| \frac{-1}{n^2 + 1} \right| = \underbrace{\frac{1}{n^2 + 1}}_{>n^2} < \frac{1}{n^2} < \varepsilon \iff n^2 > \frac{1}{\varepsilon}$$

Thus $n > \frac{1}{\sqrt{\varepsilon}}$. Now, let $N > \frac{1}{\sqrt{\varepsilon}}$. Then, we have that $\forall n \geq N$

$$\left| \frac{n^2}{n^2+1} - 1 \right| < \varepsilon \implies \lim\left(\frac{n^2}{n^2+1}\right) = 1$$

□

Definition 2.6. Let $a \in \mathbb{R}$ and $\varepsilon > 0$, the **ε -neighbourhood** about a is defined as

$$V_\varepsilon(a) := \{x \in \mathbb{R} : |x - a| < \varepsilon\} =]a - \varepsilon, a + \varepsilon[\quad (2.5)$$

Using the definition of ε -neighbourhood, we can rewrite the definition of the limit of a sequence as follows

Definition 2.7. (Renew definition)

$$\lim(a_n) = L \iff \forall \varepsilon > 0, \exists N, \forall n \in \mathbb{N} : a_n \in V_\varepsilon(L) \quad (2.6)$$

Example 2.1.2. Show that $(-1)^n$ diverges.

Proof. Let $L \in \mathbb{R}$ be arbitrary, we need to show that $(-1)^n$ does not converge to L . Assume $\lim((-1)^n) = L$. Let $\varepsilon = 1$. Then, $\exists N \forall n \geq N : |(-1)^n - L| < \varepsilon = 1$. Let $n \geq N$, we consider 2 cases

- If n is even, then

$$|(-1)^n - L| = |1 - L| = |L - 1| < 1 \iff -1 < L - 1 < 1 \iff 0 < L < 2$$

- If n is odd, then

$$|(-1)^n - L| = |-1 - L| = |L + 1| < 1 \iff -1 < L + 1 < 1 \iff -2 < L < 0$$

Adding these 2 inequalities, we obtain: $0 < L < 0$. This $(-1)^n$ diverges. \square

We will see a shorter proof of this eventually.

Theorem 2.1. Let $0 < a < 1$. Then, $\lim(a^n) = 0$

Proof. (First). Let $\varepsilon > 0$. Then, $|a^n - 0| = a^n < \varepsilon$

$$\iff \ln(a^n) < \ln \varepsilon$$

$$\iff \underbrace{n \ln(a)}_{<0} < \ln \varepsilon$$

$$\iff n > \frac{\ln \varepsilon}{\ln a}$$

Let $N > \frac{\ln \varepsilon}{\ln a}$. Then $\forall n \geq N : |a^n - 0| < \varepsilon \implies \lim(a^n) = 0$ \square

Remark 2.1. You are NOT allowed to use logarithm or a^x , x irrational, in your proof in this course.¹

¹This is because they're not properly defined.

Because of the above remark, we need to perform a different proof that does not use log.

Proof. (Second). $0 < a < 1 \implies \frac{1}{a} > 1$. Let $b := \frac{1}{a} - 1 > 0$. Then, $\frac{1}{a} = b + 1 \iff a = \frac{1}{1+b} \iff a^n = \frac{1}{(1+b)^n}$. Recall Bernoulli's inequalities, $\frac{1}{(1+b)^n} < \frac{1}{1+nb} \iff a^n < \frac{1}{1+nb} < \frac{1}{nb} < \varepsilon$. Then,

$$\begin{aligned} &\iff nb > \frac{1}{\varepsilon} \\ &\iff n > \frac{1}{b\varepsilon} \end{aligned}$$

Let $N > \frac{1}{b\varepsilon}$. Then, $\forall n \geq N : |a^n - 0| < \varepsilon \implies \lim(a^n) = 0$ \square

Exercise: Prove that $\lim(a^n) = 0, \forall |a| < 1$

2.1.1 Limits Laws

Theorem 2.2. Let (a_n) be a convergent sequence, then a_n is bounded. i.e. $\exists M > 0 : \forall n \in \mathbb{N}, |a_n| \leq M$

Remark 2.2. The converse of this theorem is false

Example 2.1.3. $((-1)^n)$ is bounded by 1 since $\forall n \in \mathbb{N} : |(-1)^n| = 1$. However, $((-1)^n)$ diverges.

Proof. (Theorem 2.2). Let $\lim(a_n) = L$ and $\varepsilon := 1$. Then, $\exists N, \forall n \geq N : |a_n - L| < 1$. Then,

$$|a_n| = |(a_n - L) + L| \leq \underbrace{|a_n - L|}_{< \varepsilon} + |L| < 1 + |L|$$

Thus $\forall n \geq N : |a_n| < 1 + |L|$. Let $M := \max\{|a_1|, |a_2|, \dots, |a_{N-1}|, 1 + |L|\}$. Then, $\forall n \in \mathbb{N}, |a_n| \leq M \implies (a_n)$ is bounded \square

Exercise: Show that (n) diverges.

Theorem 2.3. (Algebraic Limit Laws). Let (a_n) and (b_n) be convergent sequences and $c \in \mathbb{R}$. Then,

- (a) $(a_n + b_n)$ converges and $\lim(a_n + b_n) = \lim(a_n) + \lim(b_n)$
- (b) $(a_n - b_n)$ converges and $\lim(a_n - b_n) = \lim(a_n) - \lim(b_n)$
- (c) $(c \cdot a_n)$ converges and $\lim(c \cdot a_n) = c \cdot \lim(a_n)$

(d) $(a_n \cdot b_n)$ converges and $\lim(a_n \cdot b_n) = \lim(a_n) \cdot \lim(b_n)$

(b) If $\forall n \in \mathbb{N} : b_n \neq 0$ and $\lim(b_n) \neq 0$; then, $(\frac{a_n}{b_n})$ converges and $\lim(\frac{a_n}{b_n}) = \frac{\lim(a_n)}{\lim(b_n)}$

Proof. We will begin with the sum.

- (a) Let $\varepsilon > 0$. (a_n) and (b_n) converges and let $\lim(a_n) = a$ and $\lim(b_n) = b$. Thus $\exists N_1, \forall n \geq N : |a_n - a| < \frac{\varepsilon}{2}$ and, $\exists N_2, \forall n \geq N : |b_n - b| < \frac{\varepsilon}{2}$. Let $N := \max\{N_1, N_2\}$. Then $\forall n \geq N : |a_n - a| < \frac{\varepsilon}{2} \wedge |b_n - b| < \frac{\varepsilon}{2}$.

Now,

$$\begin{aligned} |(a_n + b_n) - (a + b)| &= |(a_n - a) + (b_n - b)| \\ &\leq \underbrace{|(a_n - a)|}_{< \frac{\varepsilon}{2}} + \underbrace{|(b_n - b)|}_{< \frac{\varepsilon}{2}} < \varepsilon \end{aligned}$$

i.e. $\forall n \geq N : |(a_n + b_n) - (a + b)| < \varepsilon \implies (a_n + b_n)$ converges and $\lim(a_n + b_n) = a + b = \lim(a_n) + \lim(b_n)$

- (d) Let $\varepsilon > 0$. Since, (a_n) and (b_n) converges then let $\lim(a_n) = a$ and $\lim(b_n) = b$. Then,

$$\begin{aligned} |a_n \cdot b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &= |b_n(a_n - a) + a(b_n - b)| \\ &\leq |b_n(a_n - a)| + |a(b_n - b)| \\ &= \underbrace{|b_n| |(a_n - a)|}_{\text{call it (1)}} + \underbrace{|a| |(b_n - b)|}_{\text{call it (2)}} \end{aligned}$$

We will then treat 1 and 2 separately.

- for 1: $|b_n| |a_n - a|$. $|b_n|$ is bounded $\implies \exists M > 0, \forall n \in \mathbb{N} : |b_n| \leq M$. Thus, $\exists N_1, \forall n \geq N_1 : |a_n - a| < \frac{\varepsilon}{2M}$. Then, $\forall n \geq N_1 :$

$$|a_n - a| |b_n| < \frac{\varepsilon}{2M} \cdot M = \frac{\varepsilon}{2}$$

i.e. $\forall n \geq N_1 : |a_n - a| |b_n| < \frac{\varepsilon}{2}$.

- for 2: $|a_n| |b_n - b|$. Let $M' > 0 : |a| \leq M'$. Thus, $\exists N_2, \forall n \geq N_2 : |b_n - b| < \frac{\varepsilon}{2M'}$

$$\implies |a| |b_n - b| < M' \cdot \frac{\varepsilon}{2M'} = \frac{\varepsilon}{2}$$

i.e. $\forall n \geq N_1 : |b_n - b| |a| < \frac{\varepsilon}{2}$.

Let $N := \max\{N_1, N_2\}$. Then $\forall n \geq N : (1) < \frac{\varepsilon}{2} \wedge (2) < \frac{\varepsilon}{2} \implies \forall n \geq N :$

$$|a_n b_n - ab| < (1) - (2) < \varepsilon$$

Thus $(a_n b_n)$ converges and $\lim(a_n b_n) = ab$

- (c) Let $c \in \mathbb{R}$ and $\forall n \in \mathbb{N}, b_n = c$. Then $(a_n b_n)$ converges by the above proof and $\lim(a_n b_n) = \lim(a_n) \cdot \lim(b_n) = \lim(a_n) \cdot \lim(c) = c \lim(a_n) \implies \lim(ca_n) = c \lim(a_n)$.
- (b) Consider the convergent sequences (a_n) and (b_n) . Then, $a_n - b_n = a_n + (-1)b_n$. We know that $(-1)b_n$ would converge by (c) and by (a), $(a_n + (-1)b_n) = (a_n - b_n)$ converges also.
And,

$$\begin{aligned}\lim(a_n - b_n) &= \lim(a_n + (-1)b_n) \\ &= \lim(a_n) + (-1)\lim(b_n) \\ &= \lim(a_n) - \lim(b_n)\end{aligned}$$

For (e) will be proof at tutorials. □

Definition 2.8. A sequence (a_n) of real numbers is called a **null sequence** if $\lim(a_n) = 0$

Example 2.1.4. Show that $\forall k \in \mathbb{N}, \left(\frac{1}{n^k}\right)$ is a null sequence.

Proof. We will use induction on k .

1. $(k = 1)$: $\lim\left(\frac{1}{n}\right) = 0$ is correct.

2. $(k \implies k+1)$: Assume that $\lim\left(\frac{1}{n^k}\right) = 0$. Then, $\frac{1}{n^{k+1}} = \frac{1}{n} \cdot \frac{1}{n^k}$. Since both $\frac{1}{n}$ and $\frac{1}{n^k}$ converges to 1, by theorem 2.3(d), it must be the case that $\frac{1}{n} \cdot \frac{1}{n^k}$ converges to 0 and is thus a null sequence.

Therefore, $\frac{1}{n^k}$ is a null sequence. □

Theorem 2.4. Let (a_n) be a null sequence and let $c \in \mathbb{R}$. Then, $(c \cdot a_n)$ is a null sequence.

Proof. $\lim(a_n) = 0 \implies \lim(c \cdot a_n) = c \cdot \lim(a_n) = c \cdot 0 = 0 \implies (c \cdot a_n)$ is a null sequence. □

Theorem 2.5. (*A convergence criterion for sequence*).

Let (a_n) be a sequence, let $L \in \mathbb{R}$ and let (b_n) be a non-negative null sequence. If $\exists k \in \mathbb{N} : \forall n \geq k : |a_n - L| \leq b_n$. It follows that (a_n) converges to L .

Proof. Let $\varepsilon > 0$. Since (b_n) is a null sequence, $\exists \tilde{N}, \forall n \geq \tilde{N} : |b_n - 0| = |b_n| = b_n < \varepsilon$. Let $N := \max\{\tilde{N}, k\}$. Then, $\forall n \geq N : |a_n - L| \leq b_n < \varepsilon \implies \forall n \geq N : |a_n - L| < \varepsilon \implies (a_n)$ converges to L . \square

2.2 Limits and Orders

Theorem 2.6. Let (a_n) be a convergent sequence. If $\exists K \in \mathbb{N}, \forall n \geq K : a_n \geq 0$. Then, $\lim(a_n) \geq 0$.

Proof. Let $a := \lim(a_n)$; assume that $a < 0$. Let $\varepsilon := -a > 0$. Then, $\exists \tilde{N} \forall n \geq \tilde{N} : |a_n - a| < \varepsilon = -a$. Recall that $\forall x \in \mathbb{R} : x \leq |x|$. Thus,

$$a_n - a \leq |a_n - a| < -a \implies a_n - a < -a \implies a_n < 0$$

Thus, $\forall n \geq \tilde{N} : a_n < 0$. Let $N := \max\{K, \tilde{N}\}$. Then, $\forall n \geq N : a_n \geq 0 \wedge a_n < 0$. Thus $a \geq 0$. \square

Theorem 2.7. Let $(a_n), (b_n)$ be convergent sequences. If $\exists K \in \mathbb{N} \forall n \geq K : a_n \leq b_n$. Then $\lim(a_n) \leq \lim(b_n)$.

Proof. Consider the sequence (c_n) where $c_n := b_n - a_n$. Then, $\forall n \geq K : c_n = b_n - a_n \geq 0$. Furthermore, (c_n) converges and $\lim(c_n) = \lim(b_n) - \lim(a_n)$. Then, by the previous theorem, $\lim(c_n) = \lim(b_n) - \lim(a_n) \geq 0 \implies \lim(a_n) \leq \lim(b_n)$. \square

Remark 2.3. If $a_n < b_n \forall n \geq K$, we cannot, in general, conclude that $\lim(a_n) < \lim(b_n)$ but we can conclude that $\lim(a_n) \leq \lim(b_n)$.

Example 2.2.1. $(a_n) := \left(-\frac{1}{n}\right), (b_n) := \left(\frac{1}{n}\right)$. Then, $\forall n \in \mathbb{N} : a_n < b_n$ but $\lim(a_n) = 0 = \lim(b_n)!$

Theorem 2.8. Let (b_n) be a convergent sequence, let $a, c \in \mathbb{R}$. If $\exists K \in \mathbb{N}, \forall n \geq K : a \leq b_n \leq c$. Then, $a \leq \lim(b_n) \leq c$.

Proof. Exercise. \square

Interpretation: If the terms of the convergent sequence (b_n) eventually lie within the interval $[a, c]$ i.e. $\exists K \in \mathbb{N} \forall n \geq K : b_n \in [a, c]$, then $\lim(b_n) \in [a, c]$.

Caution: This does NOT hold for non-closed intervals.

Example 2.2.2. $(b_n) = \left(\frac{1}{n}\right)$, interval $]0, 1]$. Then, $\forall n \in \mathbb{N} : b_n \in]0, 1]$, but $0 = \lim(b_n) \notin]0, 1]$.

Theorem 2.9. (Squeeze Theorem). Let $(a_n), (b_n), (c_n)$ be sequences such that:

- $\exists K \in \mathbb{N}, \forall n \geq K : a_n \leq b_n \leq c_n$
- (a_n) and (c_n) converge and have the same limit. Then (b_n) converges and $\lim(b_n) = \lim(a_n) = \lim(c_n)$.

Proof. Let $L := \lim(a_n) = \lim(c_n)$. Then,

$$\begin{aligned} \exists N_1, \forall n \geq N_1 : |a_n - L| < \varepsilon \\ \exists N_2, \forall n \geq N_2 : |c_n - L| < \varepsilon \end{aligned}$$

Let $N := \max\{N_1, N_2, K\}$. Then, $\forall n \geq N$:

$$a_n \leq b_n \leq c_n \wedge \underbrace{|a_n - L| < \varepsilon}_{L - \varepsilon < a_n < L + \varepsilon} \wedge \underbrace{|c_n - L| < \varepsilon}_{L - \varepsilon < c_n < L + \varepsilon}$$

So,

$$\begin{aligned} &\implies L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon \\ &\implies L - \varepsilon < b_n < L + \varepsilon \\ &\implies -\varepsilon < b_n - L < \varepsilon \\ &\implies |b_n - L| < \varepsilon \end{aligned}$$

i.e. $\forall n \geq N : |b_n - L| < \varepsilon$. Thus, (b_n) converges and $\lim(b_n) = L$. \square

Example 2.2.3. Show that $\left(\frac{\sin n}{n}\right)$ converges to 0. Recall that $-1 \leq \sin n \leq 1$ for all $n \in \mathbb{N}$,

$$\underbrace{-\frac{1}{n}}_{\rightarrow 0} \leq \frac{\sin n}{n} \leq \underbrace{\frac{1}{n}}_{\rightarrow 0}$$

By squeeze theorem, $\left(\frac{\sin n}{n}\right)$ converges and its limits is 0.

2.3 Monotone Sequence

Definition 2.9. Let (a_n) be a sequence. If

- $\forall n \in \mathbb{N} : a_n \leq a_{n+1}$, we say that (a_n) is **increasing**.

- $\forall n \in \mathbb{N} : a_{n+1} \leq a_n$, we say that (a_n) is **decreasing**.

Noticeably, a sequence that's either decreasing or increasing is called **monotone sequence**.

Theorem 2.10. (*Monotone Convergent Sequence*).

- Let (a_n) be increasing and bounded from above. Then (a_n) converges and $\lim(a_n) = \sup\{a_n : n \in \mathbb{N}\}$.
- Let (a_n) be decreasing and bounded from below. Then, (a_n) converges and $\lim(a_n) = \inf\{a_n : n \in \mathbb{N}\}$.

Proof. Let's prove the first part. Since (a_n) is bounded from above, the set $\{a_n : n \in \mathbb{N}\}$ has a supremum a . We will show that (a_n) converges to a .

Let $\varepsilon > 0$. Then, $a - \varepsilon$ is not an upper bound of $S := \{a_n : n \in \mathbb{N}\}$ since a is the least upper bound of S i.e. $\exists N \in \mathbb{N} : a - \varepsilon < a_N$.

Since (a_n) is increasing, we have that

$$a - \varepsilon < a_n \leq a_{N+1} \leq a_{N+2} \leq \cdots \leq a$$

Then,

$$\begin{aligned} &\implies \forall n \geq N : a - \varepsilon < a_n \leq a < a + \varepsilon \\ &\implies \forall n \geq N : a - \varepsilon < a_n < a + \varepsilon \\ &\implies \forall n \geq N : a_n \in V_\varepsilon(a) \end{aligned}$$

$\implies (a_n)$ converges to a . Second part will be left as an exercise. □

2.3.1 Euler's Number

In this lecture, we will prove that the sequence $\left(\left(1 + \frac{1}{n}\right)^n\right)$ converges of which its limit is denoted as e called **Euler's number (constant)**.

Proving this is not a trivial task, we will use the following strategy:

Let $a_n := \left(1 + \frac{1}{n}\right)^n$ and $b_n := \left(1 + \frac{1}{n}\right)^{n+1}$ for all $n \in \mathbb{N}$. We will show that

- (a_n) is strictly increasing
- (b_n) is strictly decrease
- $\forall n, k \in \mathbb{N} : a_n \leq b_k$

- (a_n) is bounded from above and (b_n) is bounded from below
- (a_n) and (b_n) both converge and have the same limit

We will define $e := \lim(a_n) = \lim(b_n)$

1. Step: We will show that the sequence (a_n) is strictly increasing.

We will achieve this by proving that $\forall n \in \mathbb{N}$ it holds that $\frac{a_{n+1}}{a_n} > 1$ (which is equivalent to $\forall n \in \mathbb{N}: a_n < a_{n+1}$). Note that

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{\left(1 + \frac{1}{n}\right)^n} = \frac{\left(\frac{n+2}{n+1}\right)^{n+1}}{\left(\frac{n+1}{n}\right)^n} = \frac{(n+2)^{n+1} \cdot n^n}{(n+1)^{n+1} \cdot (n+1)^n} = \frac{n+2}{n+1} \cdot \frac{(n+2)^n \cdot n^n}{(n+1)^{2n}} \\ &= \frac{n+2}{n+1} \cdot \left[\frac{(n+2) \cdot n}{(n+1)^2} \right]^n = \frac{n+2}{n+1} \cdot \left[\frac{n^2 + 2n}{n^2 + 2n + 1} \right]^n = \frac{n+2}{n+1} \cdot \left[1 - \frac{1}{n^2 + 2n + 1} \right]^n \\ &= \frac{n+2}{n+1} \cdot \left[1 + \underbrace{\left(-\frac{1}{(n+1)^2} \right)}_{\geq -1} \right]^n \geq \frac{n+2}{n+1} \cdot \left[1 - n \frac{1}{(n+1)^2} \right] \\ &= \frac{n+2}{n+1} \cdot \frac{n^2 + 2n + 1 - n}{(n+1)^2} = \frac{(n+2) \cdot (n^2 + n + 1)}{(n+1)^3} \\ &= \frac{n^3 + n^2 + n + 2n^2 + 2n + 2}{n^3 + 3n^2 + 3n + 1} = \frac{n^3 + 3n^2 + 3n + 2}{n^3 + 3n^2 + 3n + 1} > 1\end{aligned}$$

This proves that (a_n) is strictly increasing.

2. Step: We will show that the sequence (b_n) is strictly decreasing.

We will get this by proving that $\forall b \in \mathbb{N}$ it holds that $\frac{b_n}{b_{n+1}} > 1$. Note that

$$\begin{aligned}\frac{b_n}{b_{n+1}} &= \frac{\left(1 + \frac{1}{n}\right)^{n+1}}{\left(1 + \frac{1}{n+1}\right)^{n+2}} = \frac{\left(\frac{n+1}{n}\right)^{n+1}}{\left(\frac{n+2}{n+1}\right)^{n+2}} = \frac{(n+1)^{n+1} \cdot (n+1)^{n+2}}{n^{n+1} \cdot (n+2)^{n+2}} \\ &= \frac{n+1}{n+2} \cdot \frac{(n+1)^{2(n+1)}}{n^{n+1} \cdot (n+2)^{n+1}} = \frac{n+1}{n+2} \cdot \left[\frac{(n+1)^2}{n \cdot (n+2)} \right]^{n+1} \\ &= \frac{n+1}{n+2} \cdot \left[\frac{n^2 + 2n + 1}{n^2 + 2n} \right]^{n+1} = \frac{n+1}{n+2} \cdot \left[1 + \frac{1}{n^2 + 2n} \right]^{n+1} \\ &\geq \frac{n+1}{n+2} \cdot \left[1 + (n+1) \frac{1}{n^2 + 2n} \right]^{n+1} = \frac{n+1}{n+2} \cdot \frac{n^2 + 2n + n + 1}{n^2 + 2n} \\ &= \frac{n+1}{n+2} \cdot \frac{n^2 + 3n + 1}{n^2 + 2n} = \frac{n^3 + 3n^2 + n + n^2 + 3n + 1}{n^3 + 2n^2 + 2n^2 + 4n} = \frac{n^3 + 4n^2 + 4n + 1}{n^3 + 4n^2 + 4n} > 1\end{aligned}$$

This proves that (b_n) is strictly decreasing.

3. Step: We will show that $\forall n, k \in \mathbb{N} : a_n \leq b_k$.

It is clear that $\forall n \in \mathbb{N} : a_n = \left(1 + \frac{1}{n}\right)^n \leq \left(1 + \frac{1}{n}\right)^{n+1} = b_n$. Now let $n, k \in \mathbb{N}$ be arbitrary. We will distinguish 2 cases:

1. $n \leq k$: Since (a_n) is increasing, it holds that $a_n \leq a_k \leq b_k$, Especially, it follows that $a_n \leq b_k$.
2. $n > k$: Since (b_k) is decreasing, it holds that $a_n \leq b_n \leq b_k$, Especially, it follows that $a_n \leq b_k$.

Thus, it follows in all cases that $a_n \leq b_k \forall n, k \in \mathbb{N}$

4. Step: We will show that (a_n) is bounded from above and that (b_n) is bounded from below.

It follows from the previous step that $a_n \leq b_1$ and $a_1 \leq b_n \forall n \in \mathbb{N}$. b_1 is thus an upper bound for (a_n) and a_1 is a lower bound for (b_n) . This proves that (a_n) is bounded from above and that (b_n) is bounded from below.

5. Step: We will show that (a_n) and (b_n) both converge and have the same limit.

It was shown in the previous steps that (a_n) is increasing and bounded from above; we've also shown that (b_n) is decreasing and bounded from below. It now follows from the monotone convergence theorem that both (a_n) and (b_n) converge. Furthermore,

$$b_n = \left(1 + \frac{1}{n}\right)^{n+1} = \left(1 + \frac{1}{n}\right)^n \cdot \left(1 + \frac{1}{n}\right) a_n \cdot \left(1 + \frac{1}{n}\right)$$

Thus,

$$\lim(b_n) = \lim(a_n) \cdot \lim \underbrace{\left(1 + \frac{1}{n}\right)}_{=1} = \lim(a_n)$$

We can thus now define

$e := \lim \left(\left(1 + \frac{1}{n}\right)^n \right) = \lim \left(\left(1 + \frac{1}{n}\right)^{n+1} \right)$

(2.7)

Finally, we will give a numerical estimates for e . It was shown in 3. step that $a_n \leq b_k \forall n, k \in \mathbb{N}$. Thus b_k is an upper bound for $S := \{a_n : n \in \mathbb{N}\} \forall k \in \mathbb{N}$. Note that since (a_n) is bounded from above, S has a supremum i.e. a

least upper bound. Thus, $\sup S \leq b_k \forall k \in \mathbb{N}$. Furthermore, it follows from the monotone convergence theorem that $e = \lim(a_n) = \sup S$. Combining these results, we obtain that $e \leq b_k \forall k \in \mathbb{N}$.

In a very similar manner, it follows that $a_n \leq e \forall n \in \mathbb{N}$. This finally proves that

$$\forall n, k \in \mathbb{N} : a_n \leq e \leq b_k$$

A quick calculation yields that e.g. $b_5 = (1 + \frac{1}{5})^6 = 2.98\dots < 3$ and $a_6 = (1 + \frac{1}{6})^6 = 2.52\dots > \frac{5}{2}$. This shows immediately that

$$\boxed{\frac{5}{2} < e < 3} \quad (2.8)$$

This estimate is sufficiently good for the purpose of this course. Choosing larger values for n and k , one can show that $e = 2.718281828459\dots$

2.4 Subsequences

Definition 2.10. Let $n_1, n_2, n_3, \dots \in \mathbb{N} : n_1 < n_2 < n_3 < \dots$ and let $(a_n) = (a_1, a_2, a_3, \dots)$ be a sequence. Then, $(a_{n_k}) = (a_{n_1}, a_{n_2}, a_{n_3}, \dots)$ is called a **subsequence** of (a_n) .

Example 2.4.1. Let (a_n) be a sequence. Then,

1. $(a_{2k}) = (a_2, a_4, a_6, \dots)$ is a subsequence of (a_n) called the *subsequence of even indices* of which we often write (a_{2n}) instead of (a_{2k}) .
2. $(a_{2k-1}) = (a_1, a_3, a_5, \dots)$ is a subsequence of (a_n) called the *subsequence of odd indices*.
3. $(a_{k+1}) = (a_2, a_3, a_4, \dots)$ is a subsequence of (a_n) called the **1-tail** of (a_n) ²

Similarly, $(a_{k+1}) = (a_{k+1}, a_{k+2}, a_{k+3}, \dots)$ is called the **k-tail** of (a_n) .

Lemma 2.1. Let $n_1, n_2, n_3, \dots \in \mathbb{N} : n_1 < n_2 < \dots$. Then, $\forall k \in \mathbb{N} : n_k \geq k$

Proof. We will prove this via induction:

- $k = 1$: $n_1 \geq 1$ is true.
- $k \rightarrow k+1$: Assume that $n_k \geq k$ for some $k \in \mathbb{N}$. Then, $n_{k+1} > n_k \geq k \implies n_{k+1} > k \implies n_{k+1} \geq k+1$.

²Once again, we also usually write (a_{k+1}) as (a_{n+1}) .

This is all we had to show. \square

Theorem 2.11. Let (x_n) be a convergent sequence and let (x_{n_k}) be an arbitrary subsequence of (x_n) . Then, (x_{n_k}) converges and $\lim(x_{n_k}) = \lim(x_n)$.

Proof. Let $x := \lim(x_n)$. Let $\varepsilon > 0$. Since (x_n) converges to x , $\exists n \in \mathbb{N} \forall n \geq N : |x_n - x| < \varepsilon$. Then $\forall k \geq N$, by the above lemma and construction, we have

$$|x_{n_k} - x| < \varepsilon$$

$$\implies (x_{n_k}) \text{ converges and } \lim(x_{n_k}) = x. \quad \square$$

Corollary 2.1. Let (x_n) be a sequence; let (x_{n_k}) and (x_{n_j}) be convergent sequences s.t. $\lim(x_{n_k}) \neq \lim(x_{n_j})$. Then, (x_n) diverge.

Proof. If (x_n) would converge. Then, $\lim(x_{n_k}) = \lim(x_n) = \lim(x_{n_j})$, which is not the case. Thus (x_n) diverges. \square

Example 2.4.2. Show that $\lim a^n = 0$, $0 < a < 1$. From last class, we used subsequence with even indices to prove this. This class we will use 1-tail.

Proof. The 1-tail of (a^n) (from last class we know that a^n converges) will simply be $a^{n+1} \implies (a^{n+1})$ converges and $\lim(a^n) = \lim(a^{n+1}) =: x$. Then,

$$\lim(a \cdot a^n) = a \cdot \lim(a^n) = a \cdot x$$

$$\implies ax = x \implies x - ax = 0 \implies x \underbrace{(1-a)}_{\neq 0} = 0 \implies x = 0 \quad \square$$

Exercise: Show that $\lim(\sqrt[n]{a}) = 1$

Hint: Use subsequence of even indices.

Example 2.4.3. Consider the sequence (x_n) , recursively defined by $x_1 := 2$, $x_{n+1} := 2 - \frac{1}{x_n}$. Show $\forall n \in \mathbb{N} : 1 < x_n \leq 2$. (especially, $\forall n \in \mathbb{N} : x_n \neq 0$, which means the sequence is well-defined). Then show that (x_n) is decreasing and bounded from below; finally, determine $\lim(x_n)$.

Proof. We will first use induction to prove $1 < x_n \leq 2$:

- $n = 1$: $x_1 = 2$

- $n \rightarrow n+1$: Assume that $1 < x_n \leq 2$. Then,

$$\begin{aligned} &\implies \frac{1}{2} \leq \frac{1}{x_n} < 1 \\ &\implies -1 < -\frac{1}{x_n} \leq -\frac{1}{2} \\ &\implies 1 < 2 - \frac{1}{x_n} = x_{n+1} \leq \frac{3}{2} < 2. \end{aligned}$$

$$\implies \forall n \in \mathbb{N}: 1 < x_n \leq 2.$$

Now, we need to prove that (x_n) is decreasing. Consider $x_n - x_{n+1} = x_n - 2 + \frac{1}{x_n} = \frac{x_n^2 - 2x_n + 1}{x_n} = \frac{(x_n - 1)^2}{x_n} > 0$.

$\implies \forall n \in \mathbb{N}: x_{n+1} < x_n$. Thus (x_n) is strictly decreasing and is bounded below by 1 (proven above) \implies convergent by monotone convergence theorem.

Let $x := \lim(x_n)$; $x_{n+1} = 2 - \frac{1}{x}$

$$\begin{aligned} &\implies \lim(x_{n+1}) = \lim\left(2 - \underbrace{\frac{1}{x_n}}_{\neq 0}\right) \\ &\implies \underbrace{x}_{\text{1-tail}} = 2 - \frac{1}{x} \\ &\implies x^2 = 2x - 1 \implies x^2 - 2x + 1 = 0 \\ &\implies (x - 1)^2 = 0 \implies x = 1 \end{aligned}$$

$$\implies \lim(x_n) = 1$$

□

Theorem 2.12. (Bolzano-Weierstrass). Every bounded sequence of real numbers has a convergent subsequence.

Proof. Let (x_n) be bounded; let $M > 0 : \forall n \in \mathbb{N} : |x_n| \leq M$ i.e. $\forall n \in \mathbb{N} : x_n \in [-M, M]$. Divide this interval into 2 closed sub-interval of equal width. At least 1 of these subintervals contains infinitely many terms of (x_n) . Pick one such subinterval and call I_1 . Divide I_1 into 2 other subintervals of equal width. At least 1 of these subintervals, called I_2 , contains infinitely many terms of (x_n) etc.

We thus obtained a nested-sequence $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ of closed and bounded intervals. So by the nested interval property, $\bigcap_{k \in \mathbb{N}} I_k \neq \emptyset$. Let

$x \in \bigcap_{k \in \mathbb{N}} I_k$. We will now construct a subsequence (x_{n_k}) of (x_n) that converge to x . Let $x_{n_1} \in I_1$ (which contains infinitely many terms of (x_n)). Since I_2 also contains infinitely many terms of (x_n) , $\exists n_2 \in \mathbb{N}, n_2 > n_1 : x_{n_2} \in I_2$. Similarly, $\exists n_3 \in \mathbb{N}, n_3 > n_2 : x_{n_3} \in I_3$, etc.

So we've obtained that $n_1 < n_2 < n_3 < \dots$ s.t. $\forall k \in \mathbb{N} : x_{n_k} \in I_k$. Note that $|I_k| = \frac{2M}{2^k} \forall k \in \mathbb{N}$ (Prove this by induction).

$$\left| \underbrace{x_{n_k}}_{\in I_k} - \underbrace{x}_{\in I_k} \right| \leq |I_k| = \frac{2M}{2^k} = 2M \underbrace{\left(\frac{1}{2} \right)^k}_{\text{Null Seq.}} \underbrace{\left(\frac{1}{2} \right)^k}_{\text{Null Seq.}}$$

By the null sequence convergence criterium, (x_{n_k}) converges to x . \square

2.5 Cauchy Sequences

Definition 2.11. A sequence (x_n) of real numbers is called a **Cauchy sequence** if

$$\forall \varepsilon > 0, \exists N \forall n, m \geq N : |x_n - x_m| < \varepsilon$$

Example 2.5.1. Show that $x_n := \frac{1}{n}$ is a Cauchy sequence

Proof. Let $\varepsilon > 0$. $|x_n - x_m| = \left| \frac{1}{n} - \frac{1}{m} \right| = \left| \frac{1}{n} + \left(-\frac{1}{m} \right) \right| \leq \left| \frac{1}{n} \right| + \left| -\frac{1}{m} \right| = \frac{1}{n} + \frac{1}{m}$. Then, $\frac{1}{n} < \frac{\varepsilon}{2} \iff n > \frac{2}{\varepsilon}$. Now, let $N > \frac{2}{\varepsilon}, n, m \geq N$. Then, $\frac{1}{n} < \frac{\varepsilon}{2} \wedge \frac{1}{m} < \frac{\varepsilon}{2} \implies \forall n, m \geq N : \frac{1}{n} + \frac{1}{m} < \varepsilon \implies \left| \frac{1}{n} - \frac{1}{m} \right| < \varepsilon \implies (x_n)$ is Cauchy. \square

Theorem 2.13. Every convergent sequence of real numbers is a Cauchy sequence.

Proof. Let (x_n) be convergent and define $x := \lim(x_n)$. Let $\varepsilon > 0$. Then, $\exists N \forall n \geq N : |x_n - x| < \frac{\varepsilon}{2}$. Let $n, m \geq N$. Then $|x_n - x| < \frac{\varepsilon}{2} \wedge |x_m - x| < \frac{\varepsilon}{2}$. Thus, $|x_n - x_m| = |(x_n - x) + (x - x_m)| \leq |x_n - x| + |x_m - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$. Thus, $\forall n, m \geq N : (x_n - x_m) < \varepsilon \implies (x_n)$ is Cauchy. \square

Theorem 2.14. Every Cauchy Sequence is bounded

Proof. Let (x_n) be a Cauchy sequence, let $\varepsilon := 1$. Then, $\exists N \forall n, m \geq N : |x_n - x_m| < 1$. Let $m := N$. Then, $\forall n \geq N : |x_m - x_N| < 1$. Now $|x_n| = |x_n - x_N + x_N| \leq \underbrace{|x_n - x_N|}_{< 1} + |x_N| < 1 + |x_N|$ i.e. $\forall n \geq N : |x_n| < 1 + |x_N|$. Let

$M := \max\{|x_1|, |x_2|, \dots, |x_{N-1}|, 1 + |x_N|\}$. Then $\forall n \in \mathbb{N} : |x_n| \leq M \implies (x_n)$ is bounded. \square

Theorem 2.15. *Every Cauchy sequence converges*

Proof. Let (x_n) be a Cauchy sequence. Our first task is to find a candidate for the limit of (x_n) . Since (x_n) is a Cauchy sequence, it is bounded. Thus, by Bolzano-Weierstrass, (x_n) has a convergent subsequence (x_{n_k}) . Let $x := \lim(x_{n_k})$. We will show that x is not only the limit of (x_{n_k}) but is even the limit of (x_n) .

Let $\varepsilon > 0$. We have the following:

- Since (x_{n_k}) converges to x : $\exists k \forall K \geq K : |x_{n_k} - x| < \frac{\varepsilon}{2}$ (*)
- Since (x_n) is Cauchy: $\exists \tilde{N} \forall n, m \geq \tilde{N} : |x_n - x_m| < \frac{\varepsilon}{2}$ (**)

Let $N := \max\{K, \tilde{N}\}$. Let $n \geq N$. Then,

$$|x_n - x| = |x_n - x_{n_N} + x_{n_N} - x| \leq \underbrace{|x_n - x_{n_N}|}_{(1)} + \underbrace{|x_{n_N} - x|}_{(2)}$$

(2) : $|x_{n_N} - x| < \frac{\varepsilon}{2}$ by (*) and $N \geq K$.

(1) : $|x_n - x_{n_N}| < \frac{\varepsilon}{2}$ by (**), $n \geq N \geq \tilde{N}$ and $n_N \geq N \geq \tilde{N}$.

$\implies \forall n \geq N : |x_n - x| \leq (1) + (2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \implies (x_n)$ converges to x . \square

Now, we can combine this with the other theorem to get

Theorem 2.16. *A sequence converges \iff it is a Cauchy sequence.*

Remark 2.4. *The definition of Cauchy sequence is symmetrical in n and m . Thus, we can assume without loss of generality that $n \geq m$ v.v.*

2.6 Contractive Sequence

Definition 2.12. A sequence (x_n) is called **contractive** if $\exists 0 < c < 1$ s.t. $\forall n \in \mathbb{N} : |x_{n+2} - x_{n+1}| \leq c \cdot |x_{n+1} - x_n|$

Theorem 2.17. *Every contractive sequence converges.*

Before we do this, a quick reminder of sums of finite geometric series.

Definition 2.13. Let $a \in \mathbb{R}$ and $a \neq 1$. Consider the following series

$$1 + a + a^2 + \cdots + a^n \quad (2.9)$$

This is called a **finite geometric series**

What is the sum of the geometric series? Well...Define the follows:

$$I : s = 1 + a + a^2 + a^3 + \cdots + a^{n-1} + a^n$$

$$II : as = a + a^2 + a^3 + \cdots + a^{n-1} + a^n + a^{n+1}$$

$$\Rightarrow I - II : s - as = 1 - a^{n+1} \Rightarrow s(1 - a) = 1 - a^{n+1} \Rightarrow s = \frac{1 - a^{n+1}}{1 - a} \quad (a \neq 1).$$

Now let's back to the proof.

Proof. Let (x_n) be contractive. Since we don't have a candidate for $\lim(x_n)$, we will show that (x_n) is a Cauchy sequence. Then,

$$\begin{aligned} |x_{n+2} - x_{n+1}| &\leq c \cdot |x_{n+1} - x_n| \leq c [c \cdot |x_n - x_{n+1}|] = c^2 |x_n - x_{n+1}| \\ &\leq c^3 |x_{n-1} - x_{n-2}| \leq \cdots \leq c^n |x_2 - x_1| \end{aligned} \quad (*)$$

Without loss of generality, let $m \geq n$ and $\varepsilon > 0$. Then,

$$\begin{aligned} |x_m - x_n| &= |x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2} - \cdots + x_{n+1} - x_n| \\ &\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\ &\leq c^{m-2} |x_2 - x_1| + c^{m-3} |x_2 - x_1| + \cdots + c^{n-1} |x_2 - x_1| \quad \text{by (*)} \\ &= |x_2 - x_1| \cdot (c^{n-1} + c^n + \cdots + c^{m-2}) \\ &= |x_2 - x_1| \cdot c^{n-1} (1 + c + c^2 + \cdots + c^{(m-2)-(n-1)}) \\ &= c^{n-1} |x_2 - x_1| \cdot \underbrace{(1 + c + c^2 + \cdots + c^{(m-n-1)})}_{\text{finite geometric series}} \\ &= c^{n-1} |x_2 - x_1| \cdot \underbrace{\frac{1 - c^{m-n}}{1 - c}}_{>0} \quad (c \neq 1) \\ &< c^{n-1} |x_2 - x_1| \cdot \frac{1}{1 - c} \\ &= \underbrace{\frac{1}{c(c-1)} \cdot |x_2 - x_1|}_{\text{constant}} \cdot \underbrace{c^n}_{\text{null seq.}} \quad 0 < c < 1 \\ &= \underbrace{\frac{1}{c(c-1)} \cdot |x_2 - x_1|}_{\text{null seq.}} \cdot c^n \end{aligned}$$

Thus $\exists N \forall n \geq N : \frac{1}{c(c-1)} |x_2 - x_1| \cdot c^n < \varepsilon$. Thus $\forall m \geq n \geq N : |x_m - x_n| < \varepsilon$. Thus (x_n) is a Cauchy sequence $\Rightarrow (x_n)$ converges. \square

Example 2.6.1. Let $x_1 = 2, x_{n+1} = 2 + \frac{1}{x_n}, \forall n \in \mathbb{N}$ (i.e. $x_2 = 2 + 1/2 = 2.5, x_3 = 2 + \frac{1}{2.5} = 2.4$, etc.). Prove that this sequence converges and find its limit.

Proof. Note that (x_n) is not monotone thus our method of using monotone sequence to prove convergence will not work. Instead, we will show that it is contractive.

Claim: $\forall n \in \mathbb{N}, n \geq 2$.

We will prove this using induction:

- $n = 1$: $x_1 = 2 \geq 2$.
- $n \implies n+1$ Suppose that $x_n \geq 2$. Then,

$$x_{n+1} = 2 + \frac{1}{x_n} > 2 \geq 2$$

Notice that this also shows that x_n is well-defined as $x_n \neq 0 \implies x_{n+1} = 2 + \frac{1}{x_n}$ exists for all $n \in \mathbb{N}$.

Now, we will show that it's contractive:

$$\begin{aligned} |x_{n+2} - x_{n+1}| &= \left| \left(2 + \frac{1}{x_{n+1}}\right) - \left(2 + \frac{1}{x_n}\right) \right| = \left| \frac{1}{x_{n+1}} - \frac{1}{x_n} \right| \\ &= \left| \frac{x_n - x_{n+1}}{x_n x_{n+1}} \right| = \frac{|x_{n+1} - x_n|}{\underbrace{|x_n|}_{>0} \cdot \underbrace{|x_{n+1}|}_{>0}} \\ &= \underbrace{\frac{1}{x_n \cdot x_{n+1}}}_{\geq 2} |x_{n+1} - x_n| \leq \frac{1}{2 \cdot 2} |x_{n+1} - x_n| = \frac{1}{4} |x_{n+1} - x_n| \end{aligned}$$

We can define $c := \frac{1}{4} \implies \forall n \in \mathbb{N}, \exists 0 < c < 1 : |x_{n+2} - x_{n+1}| \leq c \cdot |x_{n+1} - x_n| \implies (x_n)$ is contractive \implies it converges by the above theorem.

Let $x := \lim(x_n)$. Then, $\forall n \in \mathbb{N} : x_{n+1}$

$$\begin{aligned} &\implies \underbrace{\lim(x_{n+1})}_{=x} = \lim\left(2 + \frac{1}{x_n}\right) = 2 + \frac{1}{\lim(x_n)} = 2 + \frac{1}{x} \\ &\implies x = 2 + \frac{1}{x} \implies x^2 - 2x - 1 = 0 \\ &\implies x = \frac{2 \pm \sqrt{4 - 4(-1)}}{2} = 1 \pm \sqrt{2} \end{aligned}$$

Now, notice that we've shown before that $(x_n) \geq 2 \implies \lim(x_n) \neq 1 - \sqrt{2} < 0 \implies \lim(x_n) = 1 + \sqrt{2}$. Thus (x_n) converges to $1 + \sqrt{2}$. \square

2.7 Sequences Diverging to Infinity

Definition 2.14. Let (x_n) be a sequence of real numbers.

- We say that (x_n) diverges to $+\infty$ denoted as

$$\lim(x_n) = +\infty \quad (2.10)$$

If $\forall M \in \mathbb{R}, \exists N \in \mathbb{N} \forall n \geq N : x_n \geq M$

- We say that (x_n) diverges to $-\infty$ denoted as

$$\lim(x_n) = -\infty \quad (2.11)$$

If $\forall M \in \mathbb{R}, \exists N \in \mathbb{N} \forall n \geq N : x_n < M$

Example 2.7.1. Show that $\lim(n) = +\infty$

Proof. Let $M \in \mathbb{R}$. By the Archimedean property, $\exists N \in \mathbb{N} : N > M$. Then, $\forall n \geq N : n \geq N > M$ i.e. $\forall n \geq N : n > M$. Thus (n) diverges to $+\infty$. \square

Show that $\lim(-n) = -\infty$

Proof. Let $M \in \mathbb{R}$. By the Archimedean property, $\exists N \in \mathbb{N} : N > -M$. Then, $\forall n \geq N : n \geq N > -M$ i.e. $\forall n \geq N : n > -M \iff -n < M$. Thus $(-n)$ diverges to $-\infty$. \square

Let $a > 1$. Show that $\lim(a^n) = +\infty$.

Proof. If $a > 1 \implies 0 < \frac{1}{a} < 1 \implies \lim\left(\left(\frac{1}{a}\right)^n\right) = 0$. Let $M \in \mathbb{R}$. If $M \leq 0$. Then, $\forall n \in \mathbb{N} : a^n > M$. Now, let $M > 0$ and $\varepsilon := \frac{1}{M} > 0$. Then, $\exists N \forall n \geq N : \left|\left(\frac{1}{a}\right)^n - 0\right| = \frac{1}{a^n} < \varepsilon = \frac{1}{M} \implies \forall n \geq N : \frac{1}{a^n} < \frac{1}{M} \iff \forall n \geq N : a^n > M$. Thus, $\lim(a^n) = +\infty$. \square

Theorem 2.18. If $\lim(x_n) = +\infty$ or $-\infty$. Then, (x_n) diverges

Theorem 2.19. Let $(a_n), (b_n)$ be sequences s.t. $\lim(b_n) = +\infty$ and $\exists K \forall n \geq K : a_n \geq b_n$. Then, $\lim(a_n) = +\infty$

Proof. Let $M \in \mathbb{R}$. Since $\lim(b_n) = +\infty, \exists \tilde{N} \forall n \geq \tilde{N} : b_n \geq M$. Let $N = \max\{K, \tilde{N}\}$. Then, $\forall n \geq N : a_n \geq b_n > M \implies a_n > M \implies \lim(a_n) = +\infty$. \square

Example 2.7.2. Let $a > 1$. Then, $\lim(a^n) = +\infty$

Proof. Consider the sequence $\left(\frac{1}{a^n}\right) = \left(\left(\frac{1}{a^n}\right)\right)^k$ where $0 < \frac{1}{a} < 1$. Thus $\lim\left(\frac{1}{a^n}\right) = 0$.

Let $M \in \mathbb{R}$. If $M \leq 0$, then $\forall n \in M, a^n > M$. Assume that $M > 0$. Then, $\lim\left(\frac{1}{a^n}\right) = 0$ thus $\exists N \forall n \geq N : 0 < \frac{1}{a^n} < \varepsilon := \frac{1}{M} > 0 \implies \forall n \geq N : a^n > M \implies \lim(a^n) = +\infty$. \square

Show that $\lim((1 + \frac{1}{n})^{n^2}) = +\infty$

Proof. Consider the sequence:

$$\left(1 + \frac{1}{n}\right)^{n^2} = \left[\left(1 + \frac{1}{n}\right)^n\right]^n$$

The sequence within is increasing and is equal to 2 for $n = 1$ thus $\geq 2 \forall n \in \mathbb{N}$. i.e. $\forall n \in \mathbb{N}$:

$$\left(1 + \frac{1}{n}\right)^{n^2} \geq 2^n$$

Where $\lim(2^n) = +\infty$ and thus by the last theorem, we get that $\lim((1 + \frac{1}{n})^{n^2}) = +\infty$ \square

3 Point-Set Topology

We will start by generalize the notion of open and close interval.

3.1 Open- and Closeness

Definition 3.1. A subset $\mathcal{U} \subseteq \mathbb{R}$ is called **open** if $\forall x \in \mathcal{U}, \exists \varepsilon > 0 : V_\varepsilon(x) \subseteq \mathcal{U}$

Theorem 3.1. Every open interval is open

Proof. Let I be an open interval. Then, consider 4 cases:

1. $I = \mathbb{R}$: This is trivial.

2. $I =]a, +\infty[$, $a \in \mathbb{R}$: Let $x \in I$ and $\varepsilon := x - a > 0$. Then,

$$V_\varepsilon(x) =]x - \varepsilon, x + \varepsilon[=]x - (x - a), x + (x - a)[=]a, 2x - a[\subseteq]a, +\infty[$$

$$\implies V_\varepsilon(x) \subseteq I \implies I \text{ is open}$$

3. $I =]-\infty, b[$, $b \in \mathbb{R}$: Let $x \in I$ and $\varepsilon := b - x > 0$. Then,

$$V_\varepsilon(x) =]x - (\varepsilon), x + (\varepsilon)[=]2x - b, x[\subseteq]-\infty, b[$$

$$\implies V_\varepsilon(x) \subseteq I \implies I \text{ is open.}$$

4. $I =]a, b[$: Let $\varepsilon := \min\{x - a, b - x\} > 0$. Then,

$$\begin{aligned} V_\varepsilon(x) &=]x - \varepsilon, x + \varepsilon[\\ &\subseteq]x - \varepsilon, b[& \varepsilon < b - x \\ &\subseteq]a, b[& \varepsilon < x - a \end{aligned}$$

$$\implies V_\varepsilon(x) \subseteq I =]a, b[\implies I \text{ is open.}$$

□

Example 3.1.1. Consider the following open sets:

- (1) \emptyset is open by definition of openness.
- (2) \mathbb{R} is open by the above trivial case.
- (3) $\mathbb{R} \setminus \{0\}$ is open

Proof. Let $x \in \mathbb{R} \setminus \{0\}$. Then, $x > 0$ or $x < 0$.

If $x > 0$, then $x \in]0, +\infty[$ which is an open interval and thus open.

$$\implies \exists \varepsilon > 0 : V_\varepsilon(x) \subseteq]0, +\infty[\subseteq \mathbb{R} \setminus \{0\}.$$

If $x < 0$ then $x \in]-\infty, 0[$ which is open $\implies \exists \varepsilon > 0 : V_\varepsilon(x) \subseteq]-\infty, 0[\subseteq \mathbb{R} \setminus \{0\}$

$$\implies \mathbb{R} \setminus \{0\} \text{ is open. } \square$$

Theorem 3.2. *Arbitrary unions of open sets are open i.e. Let J be an arbitrary index set and \mathcal{U}_j be open $\forall j \in J$. Then,*

$$\mathcal{U} = \bigcup_{j \in J} \mathcal{U}_j$$

is open.

Proof. Let $x \in \mathcal{U}$. Then, $\exists j \in J : x \in \mathcal{U}_j$. Since \mathcal{U}_j is open, $\exists \varepsilon > 0 : V_\varepsilon(x) \subseteq \mathcal{U}_j \subseteq \bigcup_{j \in J} \mathcal{U}_j = \mathcal{U} \implies V_\varepsilon(x) \subseteq \mathcal{U} \implies \mathcal{U}$ is open. \square

Theorem 3.3. *Finite intersection of open sets are open i.e. If $\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_n : n \in \mathbb{N}$ are open. Then,*

$$\mathcal{U} := \bigcap_{j=1}^n \mathcal{U}_j$$

is open.

Proof. Let $x \in \mathcal{U}$. Then, $\forall 1 \leq j \leq n : x \in \mathcal{U}_j$; each \mathcal{U}_j is open. Therefore $\exists \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n : V_{\varepsilon_1}(x) \subseteq \mathcal{U}_1, \dots, V_{\varepsilon_n}(x) \subseteq \mathcal{U}_n$. Let $\varepsilon := \min\{\varepsilon_1, \dots, \varepsilon_n\}$. Then, $\varepsilon > 0$, and $V_\varepsilon(x) \subseteq V_{\varepsilon_1}(x) \subseteq \mathcal{U}_1, \dots, V_\varepsilon(x) \subseteq V_{\varepsilon_n}(x) \subseteq \mathcal{U}_n$
 $\implies \forall 1 \leq j \leq n : V_\varepsilon(x) \subseteq \mathcal{U}_j \implies V_\varepsilon(x) \subseteq \bigcap_{j=1}^n \mathcal{U}_j = \mathcal{U}$
 $\implies V_\varepsilon(x) \subseteq \mathcal{U} \implies \mathcal{U}$ is open \square

Remark 3.1. *This result does not, in general, extend to infinite intersection.*

Example 3.1.2. $\mathcal{U}_n :=]-\frac{1}{n}, \frac{1}{n}[, n \in \mathbb{N}$. Then, all \mathcal{U}_n are open but $\bigcap_{n \in \mathbb{N}} \mathcal{U}_n = \{0\}$ which is not open.

Definition 3.2. A subset $A \subseteq \mathbb{R}$ is called **closed** if $A^c := \mathbb{R} \setminus A$ is open.

Theorem 3.4. *Every closed interval is closed.*

Proof. Consider the following cases:

1. $A = \mathbb{R}$: $A^c = \emptyset$ which is open $\implies \mathbb{R}$ is closed.

2. $A = [a, +\infty[$: $A^c =]-\infty, a[$ which is open $\implies A$ is closed.

3. $A =]-\infty, a]$: $A^c =]a, +\infty[$ which is open $\implies A$ is closed.
4. $A = [a, b]$: $A^c = \underbrace{]-\infty, a]}_{\text{open}} \cup \underbrace{]b, +\infty[}_{\text{open}}$ $\implies A^c$ is open $\implies A$ is closed.

Thus, in all cases, A is closed. \square

Example 3.1.3. $\emptyset, A = \emptyset \implies A^c = \mathbb{R}$ is open $\implies \emptyset$ is closed.

Theorem 3.5. *The only subsets that can be both open and closed are \emptyset and \mathbb{R} .*

Proof. We need not to prove as this theorem is unimportant in this class. \square

Theorem 3.6. *Consider the following*

1. *Finite unions of closed sets are closed.*
2. *Arbitrary intersections of closed sets are closed.*

Proof. We will first prove for 1. Let A_1, \dots, A_n be closed and $A := \bigcup_{i=1}^n A_i$. Then $A^c = \mathbb{R} \setminus A = \mathbb{R} \setminus \bigcup_{i=1}^n A_i = \bigcap_{i=1}^n (\mathbb{R} \setminus A_i)$ is open since \mathbb{R} is open $\implies A$ is closed.

For 2, let I be an index set $\forall i \in I : A_i$ is closed, $A = \bigcap_{i \in I} A_i$. Then, $A^c = \mathbb{R} \setminus A = \mathbb{R} \setminus \bigcap_{i \in I} A_i = \bigcup_{i \in I} (\mathbb{R} \setminus A_i)$ is open since \mathbb{R} is open $\implies A$ is closed. \square

Remark 3.2. *The structure of closed sets is more complicated than that of open sets, e.g. there are closed subsets of \mathbb{R} which are not countable union of closed interval.*

Example 3.1.4. Cantor's set.

Definition 3.3. Let $A \subseteq \mathbb{R}$. We say that a sequence (x_n) is **in A** if $\forall n \in \mathbb{N}, x_n \in A$.

Theorem 3.7. *Let $A \subseteq \mathbb{R}$ be closed and let (x_n) be a convergent sequence in A . Let $x := \lim(x_n)$. Then $x \in A$.*

Proof. Let (x_n) be a sequence in A . Assume that $x \notin A \implies x \in A^c$ where A^c is open. Thus $\exists \varepsilon > 0 : V_\varepsilon(x) \subseteq A^c$. Since $x = \lim(x_n), \exists N \forall n \geq N : x_n \in V_\varepsilon(x) \subseteq A^c \implies \forall n \geq N : x_n \notin A$. Thus $x \in A$. \square

Remark 3.3. *The condition that A is closed is essential: Consider $A =]0, 1]$ which is not closed (nor is it closed). Let $x_n := \frac{1}{n}$. Then, (x_n) is a convergent sequence in A but $0 = \lim(x_n) \notin A$.*

3.2 Boundary and Compactness

Definition 3.4. Let $A \subseteq \mathbb{R}$. A point $x \in R$ is called a **boundary point** of A if $\forall \varepsilon > 0 : V_\varepsilon(x) \cap A \neq \emptyset \wedge V_\varepsilon(x) \cap A^c \neq \emptyset$.

The set of all boundary points of A is called the **boundary** of A and is denoted as ∂A .

Example 3.2.1. Let $I = [a, \infty[$. Then, $\partial I = \{a\}$.

Proof. Let $x > a$. We know that $]a, \infty[$ is open. So, $x \in]a, \infty[\implies \exists \varepsilon > 0 : V_\varepsilon(x) \subseteq]a, \infty[\subseteq I \implies V_\varepsilon(x) \cap I^c = \emptyset \implies x \notin \partial I$.

Now, let $x < a$. Then, consider $] - \infty, a[$ which is open. So $x \in] - \infty, a[\implies \exists \varepsilon > 0 : V_\varepsilon(x) \subseteq] - \infty, a[\subseteq I \implies V_\varepsilon(x) \cap I^c = \emptyset \implies x \notin \partial I$.

Finally, $\forall \varepsilon > 0 : V_\varepsilon(a) \cap I \neq \emptyset$, since $a + \frac{\varepsilon}{2} \in V_\varepsilon(a) \cap I \neq \emptyset$ and similarly $a - \frac{\varepsilon}{2} \in V_\varepsilon(a) \cap I^c \implies V_\varepsilon(x) \cap I^c \neq \emptyset \implies V_\varepsilon(x) \cap I \neq \emptyset \wedge V_\varepsilon(x) \cap I^c \neq \emptyset \implies a \in \partial I \implies \partial I = \{a\}$. (since any x less or larger than a is not a boundary point). \square

Example 3.2.2. $\partial[a, b] = \partial[a, b[= \partial]a, b] = \partial]a, b[= \{a, b\}$

Theorem 3.8. Let $A \subseteq \mathbb{R}$. Then,

1. A is open iff A does not contain any of its boundary points i.e. $A \cap \partial A = \emptyset \iff \partial A \subseteq A^c$.
2. A is closed iff A contains all of its boundary points i.e. $\partial A \subseteq A$.

Caution: Most subsets of \mathbb{R} are neither open nor closed!

Example 3.2.3. $I :=]0, 1]$ is neither open nor closed: $\partial(]0, 1]) = \{0, 1\}$, where $0 \notin I$ but $1 \in I \implies I$ is neither open nor closed.

3.3 Interior and Closure

Definition 3.5. Let $S \subseteq \mathbb{R}$.

- The **interior** of S , denoted as \mathring{S} or $\text{int}(S)$ is defined as $\mathring{S} := S \setminus \partial S$.
- The **Closure** of S , denoted as \overline{S} is defined as $\overline{S} := S \cup (\partial S)^c$.

Since $A \setminus B = A \cap B^c$, we can also express the interior as $\mathring{S} = S \cap (\partial S)^c$, which makes the definitions of interior and closure more symmetrical.

The reason for naming these concepts will become clear after the next theorem.

Theorem 3.9. Let $S \subseteq \mathbb{R}$. Then,

$$(a) \quad \text{int } S = \bigcup_{\substack{U \subseteq S \\ U \text{ open}}} U$$

$$(b) \quad \text{cl } S = \bigcap_{\substack{A \supseteq S \\ A \text{ closed}}} A$$

Remark 3.4. Since arbitrary unions of open sets are open, it follows directly from this theorem that $\text{int } S$ is open, and that it is, in fact, the largest open subset of S . Similarly, since arbitrary intersection of closed sets are closed, it follows that $\text{cl } S$ is the smallest closed superset of S .

Proof. (a) (\implies) Let $x \in S \setminus \partial S$. Since $x \notin \partial S$, we have:

$$\begin{aligned} & \neg \forall \varepsilon > 0 : V_\varepsilon(x) \cap S \neq \emptyset \wedge V_\varepsilon(x) \cap S^C \neq \emptyset \\ & \equiv \exists \varepsilon > 0 : V_\varepsilon(x) \cap S = \emptyset \vee V_\varepsilon(x) \cap S^C = \emptyset \end{aligned}$$

But since $x \in S$, it holds that $x \in V_\varepsilon(x) \cap S$ i.e. $V_\varepsilon(x) \cap S \neq \emptyset$. Hence, it must hold that $V_\varepsilon(x) \cap S^C = \emptyset$, which means that $x \in V_\varepsilon(x) \subseteq S$. Since $V_\varepsilon(x)$ is open, this implies that $x \in \bigcup_{\substack{U \subseteq S \\ U \text{ open}}} U$.

(\Leftarrow) Let $x \in \bigcup_{\substack{U \subseteq S \\ U \text{ open}}} U$. Then $x \in S$, And: $\exists U \subseteq S$ open s.t. $x \in U$. Let $\varepsilon > 0$ s.t.

$V_\varepsilon(x) \subseteq U$. Since $U \subseteq S$, this implies that $V_\varepsilon(x) \cap S^C = \emptyset$, which means that $x \notin \partial S$. We thus have both $x \in S$ and $x \notin \partial S$, which shows that $x \in S \setminus \partial S$.

(b) We will prove this using part (a) and taking complements. Note that

$$(\text{int } S)^C = (S \cap (\partial S)^C)^C = S^C \cup \partial S = S^C \cup (\partial S^C) = \text{cl } S$$

Thus,

$$\text{cl } S = (\text{int } S)^C = \left[\bigcup_{\substack{U \subseteq S \\ U \text{ open}}} U \right]^C = \bigcap_{\substack{U \subseteq S \\ U \text{ open}}} (U^C)$$

Hence

$$\text{cl } S = \bigcap_{\substack{U \subseteq S^C \\ U \text{ open}}} (U^C)$$

(where $U \subseteq S^C$, U open $\iff U^C \supseteq S$, U^C closed)

$$= \bigcap_{\substack{U^C \supseteq S \\ U^C \text{ closed}}} (U^C) = \bigcap_{\substack{A \supseteq S \\ A \text{ closed}}} A$$

□

3.4 Sequences and Topology

In this section, we will discuss some important connections between the topology of a set and the properties of sequences within this set.

Definition 3.6. Let $S \subseteq \mathbb{R}$, and let (x_n) be a sequence of real numbers. We say that (x_n) is **in** S , if $\forall n \in \mathbb{N} : x_n \in S$.

Theorem 3.10. Let $A \subseteq \mathbb{R}$ be closed and let (x_n) be a convergent sequence in A . Then, $\lim(x_n) \in A$ as well.¹

Proof. Assume that $x := \lim(x_n) \notin A$. Then, $x \in A^C$, which is open. Hence, $\exists \varepsilon > 0 : V_\varepsilon(x) \subseteq A^C$. Furthermore, since $x = \lim(x_n)$, we have that $\exists N \forall n \geq N : x_n \in V_\varepsilon(x)$, which means that $\forall n \geq N : x_n \notin A$. Since (x_n) is a sequence in A . Thus our assumption was wrong and hence $\lim(x_n) \in A$. □

Definition 3.7. A set $A \subseteq \mathbb{R}$ is **compact** if A is both closed and bounded.

Definition 3.8. A set $A \subseteq \mathbb{R}$ is called **sequentially compact** if every sequence (x_n) in A has a convergent subsequence (x_{n_k}) whose limit lies in A .

Theorem 3.11. A set $A \subset \mathbb{R}$ is compact iff it is sequentially compact.

Proof. (\implies) Let A be compact. Then, A is bounded. Let (x_n) be arbitrary sequence in A . By Bolzano-Weierstrass theorem, (x_n) has convergent subsequence (x_{n_k}) . Define $x := \lim(x_{n_k})$. Since A is compact and thus closed and (x_{n_k}) a sequence in A , it follows from a theorem proven in class that $x \in A \implies A$ is sequentially compact.

(\impliedby) Let A be sequentially compact. We will show that A is both closed and bounded

¹This theorem states that closed subsets of \mathbb{R} are closed under taking limits of sequences, which is the property they got their name from.

A is bounded: Assume that A is unbounded. Then, $\forall n \in \mathbb{N}, \exists x_n \in A : |x_n| \geq n$. Consider the sequence (x_n) . Let (x_{n_k}) be any subsequence of (x_n) . Then, $\forall k \in \mathbb{N} : |x_{n_k}| \geq n_k \geq k$. This implies that (x_{n_k}) is unbounded and thus diverges. We've shown that (x_n) doesn't have any convergent subsequence \nrightarrow since A is sequentially compact. Thus A is bounded.

A is closed: Assume that A is not closed. Then, A^C is not open which means that

$$\exists x \in A^C \forall \varepsilon > 0 : V_\varepsilon(x) \not\subseteq A^C \implies \exists x \in A^C \forall \varepsilon > 0 : V_\varepsilon(x) \cap A \neq \emptyset \quad (3.1)$$

Select one such x and let $\varepsilon := \frac{1}{n}$. It follows from (1) that $V_{\frac{1}{n}}(x) \cap A \neq \emptyset$ and consider the sequence (x_n) . Then, $\forall n \in \mathbb{N} : |x_n - x| < \frac{1}{n}$, where $(\frac{1}{n})$ is a null sequence. This implies that (x_n) converges to $x \in A^C$. Every subsequence (x_{n_k}) of (x_n) thus also converges to $x \notin A \nrightarrow$ since A is sequentially compact. Thus, A is closed.

Since A is closed and bounded, it's compact. □

3.5 Accumulation Points of Sequences

Definition 3.9. Let (x_n) be a sequence of real numbers. A point $x \in \mathbb{R}$ is called an **accumulation point (limit point)** of (x_n) , if $\exists (x_{n_k})$ of (x_n) that converges to x .

Theorem 3.12. (Bolzano-Weierstrass). *Every bounded sequence has at least one accumulation point.*

Proof. By the original version of Bolzano-Weierstrass, every bounded sequence has at least one convergent subsequence, and thus at least one accumulation point. □

Theorem 3.13. *If a sequence (x_n) of real numbers converges to x , then x is the only accumulation point of (x_n) .*

Proof. Since (x_n) is a subsequence of itself, x is indeed an accumulation point. Furthermore, since every subsequence (x_{n_k}) of (x_n) also converges to x , no other point can be an accumulation point. Consequently, x is the one and only accumulation point of (x_n) . □

Theorem 3.14. *Let (x_n) be a sequence. Then, $x \in \mathbb{R}$ is an accumulation point of (x_n) iff $\forall \varepsilon > 0 : V_\varepsilon(x)$ contains infinitely many terms of (x_n) .*

Proof. We need to prove it both ways:

(\Rightarrow) Assume that x is an accumulation point of (x_n) . Then, \exists subsequence (x_{n_k}) of (x_n) such that $\lim(x_{n_k}) = x \implies \forall \varepsilon > 0, \exists K \forall k \geq K : x_{n_k} \in V_\varepsilon(x) \implies V_\varepsilon(x)$ contains infinitely many terms of (x_n)

(\Leftarrow) Let $x \in \mathbb{R} : \forall \varepsilon > 0 : V_\varepsilon(x)$ contains infinitely many terms of (x_n) . Let $\varepsilon := 1$ and $n_1 \in \mathbb{N} : x_{n_1} \in V_1(x)$. Let $\varepsilon := \frac{1}{2}$, since there are infinitely many terms of $(x_n) \in V_{\frac{1}{2}}(x), \exists n_2 > n_1 : x_{n_2} \in V_{\frac{1}{2}}(x), \dots$, Let $\varepsilon := \frac{1}{k} : \exists$ infinitely many x_n is $V_{\frac{1}{k}}(x) \implies \exists n_k > n_{k-1} : x_{n_k} \in V_{\frac{1}{k}}(x), \dots$, Since $n_1 < n_2 < \dots$ and (x_{n_k}) is a subsequence of (x_n) and $\forall k \in \mathbb{N} : x_{n_k} \in V_{\frac{1}{k}}(x) \iff |x_{n_k} - x| < \frac{1}{k}$. By the null sequence criterion, (x_{n_k}) converges with $\lim(x_{n_k}) = x \implies x$ is an accumulation point of (x_n) . \square

Corollary 3.1. Let (x_n) be a sequence. $x \in \mathbb{R}$ is not an accumulation point of (x_n) iff $\exists \varepsilon > 0 : V_\varepsilon(x)$ contains at most finitely many terms of (x_n) .

Proof. Immediate deduction from last theorem (contrapositive). \square

Theorem 3.15. Let (x_n) be a sequence and A be the set of accumulation points of (x_n) . Then, A is always closed.

Proof. We will show that A^c is open.

Let $x \notin A \iff x \in A^c$. By the corollary 3.1. $\exists \varepsilon > 0 : V_\varepsilon(x)$ contains finitely many term of (x_n) . Let $y \in V_\varepsilon(x)$ be arbitrary. $V_\varepsilon(x)$ is open; thus, $\exists \tilde{\varepsilon} > 0 : V_{\tilde{\varepsilon}}(y) \subseteq V_\varepsilon(x)$. Since $V_\varepsilon(x)$ at most finitely many terms of (x_n) , then so does $V_{\tilde{\varepsilon}}(y)$. By corollary 3.1. y is not an accumulation point of $(x_n) \implies y \in A^c \implies V_\varepsilon(x) \subseteq A^c \implies A^c$ is open $\implies A$ is closed. \square

Example 3.5.1. Consider the following examples:

1. If (x_n) converges to x , we know that the set of all accumulation $A = \{x\}$, which is closed.
2. $(x_n) = ((-1)^n) \implies A = \{-1, 1\}$ is closed.
3. (x_n) enumeration of \mathbb{Q} ; $A = \mathbb{R}$ is closed.
4. $(x_n) := (1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$. Consider $S := \{x_n : n \in \mathbb{N}\} = \{\frac{1}{n} : n \in \mathbb{N}\}$. The followings are subsequences of (x_n) :

$$(1, 1, 1, \dots), \left(\frac{1}{2}, \frac{1}{2}, \dots\right), \dots$$

i.e. $\forall k \in \mathbb{N} : (x_n)$ contains the constant subsequence $(\frac{1}{k}, \frac{1}{k}, \dots)$ with limits $\frac{1}{k} \implies \forall k \in \mathbb{N} : \frac{1}{k} \in A \implies S \subseteq A$. Note that S is not closed! So A cannot be equal to S . Then, $\overline{S} = S \cup \{0\} \implies \overline{S} \subseteq A$. Thus 0 must be an accumulation point. A concrete example of a subsequence of (x_n) with limit 0 is $(1, \frac{1}{2}, \frac{1}{3}, \dots)$.

In fact, $A = \overline{S}$. Prove this as an exercise.

Theorem 3.16. Let (x_n) be a sequence and A be the set of accumulation points of (x_n)

1. If (x_n) is bounded from above then A is bounded from above.
2. If (x_n) is bounded from below then A is bounded from below.
3. If (x_n) is bounded then A is bounded.

Proof. We will only prove part 1.

Let $m > 0 : \forall n \in \mathbb{N}, x_n \leq M$. Let $x \in A$, then \exists a subsequence (x_{n_k}) of (x_n) with $\lim(x_{n_k}) = x$. We have the followings: $\forall k \in \mathbb{N} : x_{n_k} \leq M$. By a theorem from "limits and order" section, this implies that $x := \lim(x_{n_k}) \leq M \implies \forall x \in A : x \leq M \implies A$ is bounded from above.

Part 2 and 3. will be left as exercise. \square

Corollary 3.2. Let (x_n) be a bounded sequence; A , the set of accumulation points of (x_n) . Then, A is compact.

Proof. By Theorem 3.10. A is both closed and bounded and is thus compact. \square

Let (x_n) be bounded from above and A be the set of accumulation points of (x_n) . Then, A is bounded from above $\implies A$ has a supremum $x^* := \sup A$. By last class, $x^* \in \partial A$. But A is closed $\implies x^* \in A$ i.e. x^* is itself an accumulation point. Especially, $x^* = \max A$.

Similarly, if (x_n) is bounded from below. Then, A is bounded from below $\implies A$ has an infimum which is actually a minimum ($x_* := \min A$). We can put what we've just did in a more formal definitions

3.5.1 Limsup and Liminf

Definition 3.10. Let (x_n) be a sequence and A be the set of accumulation point:

1. If (x_n) is bounded from above, we call the greatest accumulation points, the *limes superior* or **limit superior**; which is denoted as

$$x^* = \limsup(x_n) \quad (3.2)$$

2. If (x_n) is bounded from below, we call the least accumulation points, the *limes inferior* or **limit inferior**; which is denoted as

$$x_* = \liminf(x_n) \quad (3.3)$$

Example 3.5.2. Consider the following examples:

1. If (x_n) converges. Then, $\lim(x_n) = \limsup(x_n) = \liminf(x_n)$.
2. $(x_n) = ((-1)^n)$. Then, $\limsup(x_n) = 1$ and $\liminf(x_n) = -1$.
3. $(x_n) := (1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$. We know that $A = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. Then, $\limsup(x_n) = 1$ and $\liminf(x_n) = 0$.

\limsup and \liminf play an important role in analysis e.g. consider the power series

$$\sum_{n=1}^{\infty} a_n(x - x_0)^n \quad (3.4)$$

Its radius of convergence R can be calculated as:

$$R = \frac{1}{\limsup(\sqrt[n]{|a_n|})} \quad (3.5)$$

With the understanding that $R = 0$ if $(\sqrt[n]{|a_n|})$ is not bounded from above.

Theorem 3.17. Let (x_n) be bounded. Then, $\forall \varepsilon > 0, \exists N \forall n \geq N : x_n \in]x_* - \varepsilon, x^* + \varepsilon[$.

Remark 3.5. If (x_n) is convergent, $x = x^* = x_*$ and the theorem reduces to the definition of convergent i.e. $\forall \varepsilon > 0, \exists N \forall n \geq N : x_n \in]x - \varepsilon, x + \varepsilon[$

Proof. (Theorem 3.11) Let $\varepsilon > 0$. Assume the conclusion of the theorem does not hold. Then, \exists infinitely many terms of (x_n) lies which are not in $]x_* - \varepsilon, x^* + \varepsilon[\implies \exists$ infinitely many x_n with $x_n \geq x^* + \varepsilon$ or \exists infinitely many x_n with $x_n \leq x_* - \varepsilon$. Consider the 2 cases:

1. Assume \exists infinitely many x_n with $x_n \geq x^* + \varepsilon$ (call this (1)): Collect these terms into a subsequence (x_{n_k}) . Then, (x_{n_k}) is bounded. By Bolzano-Weierstrass theorem, (x_{n_k}) has a sub-subsequence $(x_{n_{k_j}})$; let $x := \lim(x_{n_{k_j}})$. By (1), $\forall j \in \mathbb{N} : (x_{n_{k_j}}) \geq x^* + \varepsilon$ (call this (3)). But we also have $x \in A \implies x \leq x^*$ (call this (2)). We can see that (2) and (3) contradict each other
Thus $\exists N \forall n \geq N : x_n \in]x_* - \varepsilon, x^* + \varepsilon[$

2. Assume \exists infinitely many x_n with $x_n \leq x_* - \varepsilon$: Exercise.

This completes the theorem. □

4 Limits of Functions

Goal: Given a function $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$, we want to define the notion of a limit of f as x approaches c i.e. $\lim_{x \rightarrow c} f(x)$ or $\lim_{x \rightarrow c} f$ and we do it in 2 ways.

Definition 4.1. ($\varepsilon - \delta$) by Weierstrass. Let $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$. We say that L is the **limit** of f as x approaches c , if

$$\begin{aligned} \forall \varepsilon > 0, \exists \delta > 0 \forall x \in D \setminus \{c\} : |x - c| < \delta \implies |f(x) - L| < \varepsilon \\ \equiv \forall \varepsilon > 0, \exists \delta > 0 \forall x \in D : 0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon \end{aligned}$$

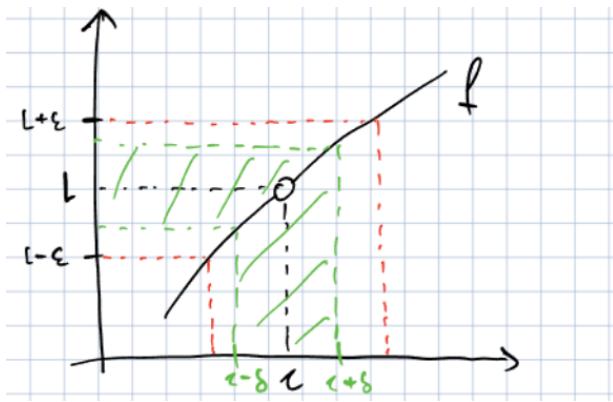


Figure 4.1: Brief illustration of definition 4.1.

Definition 4.2. Let $c \in \mathbb{R}, \varepsilon > 0$. Then, $V_\varepsilon^*(c) := V_\varepsilon(c) \setminus \{c\}$ is called the **punctured ε -neighbourhood** about c .

Using the definition 4.2, we can rephrase definition 4.1 of a limit as the followings:

$$\begin{aligned} \forall \varepsilon > 0, \exists \delta > 0 \forall x \in D : 0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon \\ \equiv \forall \varepsilon > 0, \exists \delta > 0 \forall x \in D : x \in V_\delta^*(c) \implies f(x) \in V_\varepsilon(L) \\ \equiv \forall \varepsilon > 0, \exists \delta > 0 \forall x \in V_\delta^*(c) \cap D \implies f(x) \in V_\varepsilon(L) \\ \boxed{\equiv \forall \varepsilon > 0, \exists \delta > 0 : f(V_\delta^*(c) \cap D) \subseteq V_\varepsilon(L)} \end{aligned}$$

Example 4.0.1. Consider the following examples:

1. $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto 2x$. Let $c \in \mathbb{R}$. Show that $\lim_{x \rightarrow c} f = 2c$.

Proof. Let $\varepsilon > 0$ and $\delta > 0$ be arbitrary for now. Let $0 < |x - c| < \delta$. Then,

$$|f(x) - L| = |2x - 2c| = 2|x - c| < 2\delta < \varepsilon \iff \delta < \frac{\varepsilon}{2}$$

Now, let $\delta < \frac{\varepsilon}{2}$. Then, $\forall x \in \mathbb{R} : 0 < |x - c| < \delta$, it holds that $|f(x) - 2c| < \varepsilon$. Thus, $\lim_{x \rightarrow c} f = 2c$. \square

2. $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x^2$. Let $c \in \mathbb{R}$. Show that, $\lim_{x \rightarrow c} f = c^2$.

Proof. Let $\varepsilon > 0$ and $\delta > 0$ be arbitrary for now. Suppose that $0 < |x - c| < \delta$. Then,

$$\begin{aligned} |f(x) - L| &= |x^2 - c^2| = |(x - c)(x + c)| \\ &= |x - c| \cdot |x + c| \\ &< \delta \cdot |x - c + c + c| \\ &\leq \delta \cdot (|x - c| + |2c|) \\ &< \delta(\delta + 2|c|) \end{aligned}$$

let $\delta < 1$. Then,

$$< \delta(1 + 2|c|) < \varepsilon \iff \delta < \underbrace{\frac{\varepsilon}{1 + 2|c|}}_{> 0}$$

Now, let $\delta < \min \left\{ 1, \frac{\varepsilon}{1+2|c|} \right\}$. Then, $\forall x \in \mathbb{R} : 0 < |x - c| < \delta$, it holds that $|f(x) - c^2| < \varepsilon$. Thus, $\lim_{x \rightarrow c} f = c^2$. \square

3. $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}, x \mapsto \frac{1}{x}$. Let $c \in \mathbb{R} \setminus \{0\}$. Show that $\lim_{x \rightarrow c} f = \frac{1}{c}$.

Proof. Let $\varepsilon > 0$ and $\delta > 0$ be arbitrary for now. Suppose that $0 < |x - c| < \delta$ and $x \neq 0$. Then,

$$|f(x) - L| = \left| \frac{1}{x} - \frac{1}{c} \right| = \left| \frac{x - c}{xc} \right| = \frac{|x - c|}{|x| \cdot |c|} < \frac{\delta}{|x| \cdot |c|} \quad (1)$$

Since $\frac{1}{|x|}$ can be arbitrarily large if x is chosen close to 0, we need to "bound x away from 0". To do so, let $\delta < \frac{|c|}{2}$. Then, $|x| = |x - c + c| =$

$|c - (c - x)| \geq |c| - |c - x| = |c| - \underbrace{|x - c|}_{< \delta} > |c| - \delta > |c| - \frac{|c|}{2} = \frac{|c|}{2}$ i.e. $|x| > \frac{|c|}{2}$

if $\delta < \frac{|c|}{2}$. Thus,

$$(1) = \frac{\delta}{|x| \cdot |c|} < \frac{\delta}{\frac{|c|}{2} \cdot |c|} = \frac{2\delta}{|c|^2} = \frac{2\delta}{c^2} < \varepsilon \iff \delta < \varepsilon \cdot \frac{c^2}{2}$$

Now, let $\delta < \min \left\{ \frac{|c|}{2}, \varepsilon \cdot \frac{c^2}{2} \right\}$. Then, $\forall x \in \mathbb{R} : 0 < |x - c| < \delta$, it holds that by above $|f(x) - \frac{1}{x}| < \varepsilon$ whenever $x \in \mathbb{R} \setminus \{0\} \cap V_\delta^*(c)$. Thus, $\lim_{x \rightarrow c} f = \frac{1}{c}$. \square

4. Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$. Show that $\lim_{x \rightarrow 0} f = 0 \neq 1 = f(0)$.

Proof. Let $\varepsilon > 0$ and $\delta > 0$ be arbitrary for now. Suppose that $0 < |x - c| < \delta \iff 0 < |x| < \delta$. Then,

$$|f(x) - L| = |0 - 0| = 0 < \varepsilon \quad (x \neq 0 \implies f(x) = 0)$$

Thus, for any $\delta > 0$, it follows that $|f(x) - L| < \varepsilon$ whenever $x \in V_\delta^*(0) \implies \lim_{x \rightarrow 0} f = 0$. \square

Definition 4.3.

Limit of a Function By Sequences. Let $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$. We say that L is the limit of f as x approaches c , denoted as

$$\lim_{x \rightarrow c} f = L \quad (4.1)$$

if for all sequences (x_n) in $D \setminus \{c\}$ with $\lim(x_n) = c$. It holds that $\lim(f(x_n)) = L$



Figure 4.2: Illustration of definition 4.3

Example 4.0.2. Consider the following examples:

1. $\lim_{x \rightarrow c} x^2 = c^2$. Let (x_n) be any sequence in $\mathbb{R} \setminus \{c\}$: $\lim(x_n) = c$. Then, $\lim(x_n^2) = [\lim(x_n)]^2 = c^2 \implies \lim_{x \rightarrow c} x^2 = c^2$.

2. $\lim_{x \rightarrow c} \frac{1}{x} = \frac{1}{c}$; $c \neq 0$. Let (x_n) be any sequence in $\mathbb{R} \setminus \{0, c\}$: $\lim(x_n) = c$. Then, $\lim\left(\frac{1}{x_n}\right) = \frac{1}{\lim(x_n)} = \frac{1}{c}$. Thus, $\lim_{x \rightarrow c} \frac{1}{x} = \frac{1}{c}$.

End of Lecture —

Theorem 4.1. *The $\varepsilon - \delta$ and sequential definition of the limit of a function are equivalent.*

Proof. We need to prove this in 2 directions:

(\Rightarrow) Let $f : D \subseteq \mathbb{R} \rightarrow BR, c \in \mathbb{R}$ such that $\lim_{x \rightarrow c} f = L$ by $\varepsilon - \delta$ definition i.e. $\forall \varepsilon > 0, \exists \delta > 0 \forall x \in V_\delta^*(c) \cap D : |f(x) - L| < \varepsilon$ call this (*). Let (x_n) be a sequence in $D \setminus \{c\}$: $\lim(x_n) = c$. Let $\varepsilon > 0$ and $\delta > 0$ as in (*), then $\exists N \forall n \geq N : x_n \in V_\delta(c)$, and $\forall n \in \mathbb{N} : x_n \in D \setminus \{c\}$. Then,

$$\begin{aligned} &\Rightarrow \forall n \geq N : x_n \in V_\delta^*(c) \cap D \\ &\Rightarrow \forall n \geq N : |f(x_n) - L| < \varepsilon \\ &\Rightarrow \lim(f(x_n)) = L \end{aligned}$$

(\Leftarrow) Let $f : D \subseteq \mathbb{R} \rightarrow BR, c, L \in \mathbb{R}$ such that $\lim_{x \rightarrow c} f(x) = L$ by sequential definition. Assume that the $\varepsilon - \delta$ definition of the limit is not satisfied i.e.

$$\begin{aligned} &\neg \forall \varepsilon > 0, \exists \delta > 0 \forall x \in V_\delta^*(c) \cap D : |f(x) - L| < \varepsilon \\ &\equiv \exists \varepsilon > 0, \forall \delta > 0 \exists x \in V_\delta^*(c) \cap D : |f(x) - L| \geq \varepsilon \end{aligned} \quad (**)$$

Let ε as in (**) and $\delta = 1$. Then, $\exists x_1 \in V_1^*(c) \cap D : |f(x_1) - L| \geq \varepsilon$. Now, let $\delta = \frac{1}{2}$. Then, $\exists x_2 \in V_{\frac{1}{2}}^*(c) \cap D : |f(x_2) - L| \geq \varepsilon$, etc... Now, let $\delta = \frac{1}{n}$. Then, $\exists x_n \in V_{\frac{1}{n}}^*(c) \cap D : |f(x_n) - L| \geq \varepsilon$. Consider the sequence (x_n) . Then, $\forall n \in \mathbb{N} : x_n \in V_{\frac{1}{n}}^*(c) \cap D \Rightarrow \forall n \in \mathbb{N} : |x_n - c| < \frac{1}{n}$. By the Null-sequence criterion, $\lim(x_n) = c$ and $\forall n \in \mathbb{N}, x_n \in D \setminus \{c\}$. We can see that all condition satisfies then, by sequential definition, $\lim(f(x_n)) = L \Rightarrow \exists N \forall n \geq N : |f(x_n) - L| < \varepsilon$. But by construction previously, $\forall n \in \mathbb{N} : |f(x_n) - L| \geq \varepsilon$ \nexists . Thus, our assumption was wrong. Therefore, $\lim_{x \rightarrow c} f = L$ by $\varepsilon - \delta$ definition. \square

Theorem 4.2. (2-Sequence Criterion)¹. *Let $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}, c \in \mathbb{R}$. If $\exists (x_n), (u_n)$ in $D \setminus \{c\}$: $\lim(x_n) = c = \lim(u_n)$ and such that $\lim(f(x_n))$ and $\lim(f(u_n))$ exist but are different. Then, $\lim_{x \rightarrow c} f$ does not exists (DNE).*

¹for the non-existence of the limit of a function

Proof. Immediate consequence of the sequential definition. \square

Example 4.0.3. Dirichlet function: Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) := \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

Show that, $\lim_{x \rightarrow c} f$ DNE for any $c \in \mathbb{R}$.

Proof. Let $c \in \mathbb{R}$, as seen on the midterm, $\exists (x_n) \in \mathbb{Q} \setminus \{c\} : \lim(x_n) = c$ (Due to density of \mathbb{Q} in \mathbb{R}). Similarly, $\exists (u_n) \in (\mathbb{R} \setminus \mathbb{Q}) \setminus \{c\} : \lim(u_n) = c$. Then, $f(x_n) = 1, \forall n \in \mathbb{N}$. Similarly, $f(u_n) = 0, \forall n \in \mathbb{N} \implies \lim(f(x_n)) = 1$ and $\lim(f(u_n)) = 0$ which are not the same. Then, by the 2-sequence criterion, $\lim_{x \rightarrow c} f$ DNE. \square

Theorem 4.3. (1-Sequence Criterion)². Let $f : D \rightarrow \mathbb{R}, c \in \mathbb{R}$. Let (x_n) be a sequence in $D \setminus \{c\} : \lim(x_n) = 0$. If $\lim(f(x_n))$ DNE, then $\lim_{x \rightarrow c} f$ DNE.

Proof. Immediate from sequential definition. \square

Example 4.0.4. Let $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}, x \mapsto \frac{1}{x}$. Show that $\lim_{x \rightarrow 0} f$ DNE.

Proof. Consider $(x_n) = \left(\frac{1}{n}\right)$. Then, $\forall n \in \mathbb{N} : x_n \in \mathbb{R} \setminus \{0\}$ and $\lim(x_n) = 0$.

But, $\lim(f(x_n)) = \lim\left(\frac{1}{\frac{1}{n}}\right) = \lim(n) = +\infty$ which DNE. Then, by 1-sequence criterion, $\lim_{x \rightarrow 0} f$ DNE. \square

4.1 Uniqueness of the Limit of a Function

Example 4.1.1. Let $D := \{0\} \cup [1, 2]$ and $f : D \rightarrow \mathbb{R}, f \equiv 0$ on D .³ Let $L \in \mathbb{R}$ be arbitrary. Show that $\lim_{x \rightarrow 0} f = L$.

Proof. (First). We will prove this using sequential definition. For any sequence $(x_n) \in D \setminus \{0\} = [1, 2]$ with $\lim(x_n) = 0$, we need to show that $\lim(f(x_n)) = L$. Is this true? Well...yes, since there's no such sequence! So we get that $\lim(f(x_n)) = L$. \square

Proof. (Second). We will prove this using $\varepsilon - \delta$ definition. We need to show that $\forall \varepsilon > 0, \exists \delta \forall x \in V_\delta^*(c) \cap D \implies |f(x) - L| < \varepsilon$. Let $\varepsilon > 0$ be arbitrary and $\delta := 1$. Then, $V_\delta^*(0) \cap D = V_1^*(0) \cap D = \emptyset$. Thus, $\forall x \in \underbrace{V_\delta^*(0) \cap D}_{=\emptyset} : |f(x) - L| < \varepsilon$.

Thus, $\lim_{x \rightarrow c} f = L$ \square

²for the non-existence of the limit of a function

³ $f \equiv 0$ means f is constantly 0.

Conjecture. Maybe this happens because 0 is an isolated point in D .

Definition 4.4. A point $x \in D \subseteq \mathbb{R}$ is called an **isolated point** of D if $\exists \delta > 0 : V_\delta^*(x) \cap D = \emptyset$.

Definition 4.5. A point $x \in \mathbb{R}$ is called a **limit point** or a **cluster point** if $\exists (x_n)$ in $D \setminus \{x\}$ with $\lim(x_n) = x$

Example 4.1.2. Let $D := \{0\} \cup [1, 2]$, then 0 is an isolated point since $V_1^*(0) \cap D = \emptyset$. Any point in $[1, 2]$ is a cluster point (exercise to prove).

Theorem 4.4. Let $D \subseteq \mathbb{R}$. Then,

- (a) $c \in D$ is isolated iff $\exists \delta > 0 : V_\delta^*(c) \cap D = \emptyset$.
- (b) $c \in \mathbb{R}$ is cluster point iff $\forall \delta > 0 : V_\delta^*(c) \cap D \neq \emptyset$.

Proof. We will only prove b. Let c be a cluster point.

(\Rightarrow) Let (x_n) be a sequence in $D \setminus \{c\}$ such that $\lim(x_n) = c$. Let $\delta > 0$ be arbitrary, then $\exists N \forall n \geq N : x_n \in V_\delta(c)$ but $\forall n \in \mathbb{N} : x_n \in D$ and $x_n \neq c \implies \forall n \geq N : x_n \in V_\delta^*(c) \cap D \implies V_\delta^*(c) \cap D \neq \emptyset$.

(\Leftarrow) Assume that $\forall \delta > 0 : V_\delta^*(c) \cap D \neq \emptyset$. Let $n \in \mathbb{N}, \delta := \frac{1}{n}$; then $V_{\frac{1}{n}}^*(c) \cap D \neq \emptyset$. Let $x_n \in V_{\frac{1}{n}}^*(c) \cap D$; consider the sequence (x_n) . Then, $\forall n \in \mathbb{N} : x_n \in D \setminus \{c\}$ and $\forall n \in \mathbb{N} : x_n \in V_{\frac{1}{n}}(c) \iff |x_n - c| < \frac{1}{n}$ (a null sequence) $\implies \lim(x_n) = c$. \square

Theorem 4.5. Let $D \subseteq \mathbb{R}$, $f : D \rightarrow \mathbb{R}$, $c \in \mathbb{R}$ such that $\lim_{x \rightarrow c} f$ exists.

- (a) If c is a cluster point of D . Then, $\lim_{x \rightarrow c} f$ is uniquely determined.
- (b) If c is isolated point of D . Then, $\lim_{x \rightarrow c} f$ is arbitrary.

Proof. (a) c is a cluster point and let (x_n) be a sequence in $D \setminus \{c\}$ s.t. $\lim(x_n) = c$. Let L_1, L_2 be limits of f at c . Then, $L_1 = \lim_{x \rightarrow c} f = \lim(f(x_n))$. Similarly, $L_2 = \lim_{x \rightarrow c} f = \lim(f(x_n))$. We've proven that limit of a sequence is uniquely determined $\implies L_1 = L_2 \implies \lim_{x \rightarrow c} f$ is uniquely determined.

(b) is left as an exercise. \square

We'll assume from now on that c is a cluster point.

4.1.1 Limit Laws

Theorem 4.6. (Algebraic Limit Laws). Let $f, g : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and c a cluster point of D . Suppose that $\lim_{x \rightarrow c} f$ and $\lim_{x \rightarrow c} g$ exist. Then,

$$(a) \lim_{x \rightarrow c} (f + g) = \lim_{x \rightarrow c} f + \lim_{x \rightarrow c} g \text{ i.e. } \lim_{x \rightarrow c} (f + g) \text{ exists and has value } \lim_{x \rightarrow c} f + \lim_{x \rightarrow c} g.$$

$$(b) \lim_{x \rightarrow c} (f - g) = \lim_{x \rightarrow c} f - \lim_{x \rightarrow c} g.$$

$$(c) \lim_{x \rightarrow c} (C \cdot f) = C \cdot \lim_{x \rightarrow c} f, \text{ for all } C \in \mathbb{R}.$$

$$(d) \lim_{x \rightarrow c} (f \cdot g) = \lim_{x \rightarrow c} f \cdot \lim_{x \rightarrow c} g.$$

$$(e) \forall x \in D : g(x) \neq 0 \text{ and } \lim_{x \rightarrow c} g \neq 0. \text{ Then, } \lim_{x \rightarrow c} \left(\frac{f}{g} \right) = \frac{\lim_{x \rightarrow c} f}{\lim_{x \rightarrow c} g}.$$

Proof. (a) Let (x_n) be an arbitrary sequence in $D \setminus \{c\}$: $\lim(x_n) = c$. Then, $\lim_{x \rightarrow c} f + \lim_{x \rightarrow c} g = \lim(f(x_n)) + \lim(g(x_n)) = \lim[f(x_n) + g(x_n)] = \lim((f + g)(x_n)) = \lim_{x \rightarrow c} (f + g)$.

(b), (c) and (d) will be left as exercises.

(e) Let $\forall x \in D : g(x) \neq 0$ and $\lim_{x \rightarrow c} g \neq 0$. Then,

$$\frac{\lim_{x \rightarrow c} f}{\lim_{x \rightarrow c} g} = \frac{\lim(f(x_n))}{\underbrace{\lim(g(x_n))}_{>0}}.$$

and $\forall n \in \mathbb{N} : g(x_n) \neq 0$ since $g \neq 0$.

$$\begin{aligned} &= \lim \left(\frac{f(x_n)}{g(x_n)} \right) \\ &= \lim \left(\frac{f}{g} \right) (x_n) = \lim_{x \rightarrow c} \frac{f}{g} \end{aligned}$$

□

Theorem 4.7. (The Squeeze Theorem). Let $f, g, h : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$, c is a cluster point of D . Suppose that:

- $\forall x \in D, f(x) \leq g(x) \leq h(x)$.

- $\lim_{x \rightarrow c} f = \lim_{x \rightarrow c} h = L$.

Then, $\lim_{x \rightarrow c} g = L$

Proof. $\lim_{x \rightarrow c} f = \lim(f(x_n)) = L$ and $\lim_{x \rightarrow c} h = \lim(h(x_n)) = L$. $\forall n \in \mathbb{N}$: $f(x_n) \leq g(x_n) \leq h(x_n)$. By the squeeze theorem for sequences, we conclude that $\lim(g(x_n))$ exists and equals to L . But $L = \lim(g(x_n)) = \lim_{x \rightarrow c} g$ i.e. $\lim_{x \rightarrow c} g = L$. \square

Example 4.1.3. Let $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$, $f(x) = x \cdot \sin\left(\frac{1}{x}\right)$. Show that $\lim_{x \rightarrow 0} f = 0$

Proof. First of all, 0 is a cluster point of $D = \mathbb{R} \setminus \{0\}$ since $\lim\left(\frac{1}{n}\right) = 0$. Note that $\forall x \in \mathbb{R} \setminus \{0\} : -1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \implies -|x| = -x \leq x \sin\left(\frac{1}{x}\right) \leq x = |x|$ of $x > 0$. And if $x < 0$, $-x \geq x \sin\left(\frac{1}{x}\right) \geq x \iff x \leq x \sin\left(\frac{1}{x}\right) \leq -x \iff -|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|$. Thus, $\forall x \in \mathbb{R} \setminus \{0\} : -|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|$.

Lemma 4.1. $\lim_{x \rightarrow 0} |x| = 0$

Proof. Let $\varepsilon > 0$ and $\delta := \varepsilon$. Let $0 < |x - 0| < \delta \iff 0 < |x| < \delta = \varepsilon$. Then, $||x| - 0| = |x| < \delta = \varepsilon \implies \lim_{x \rightarrow 0} |x| = 0$ \square

Now, we know that $-|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|$ but $|x| \rightarrow 0$. Then, By squeeze theorem,

$$\lim_{x \rightarrow 0} \left(x \cdot \sin\left(\frac{1}{x}\right) \right) = 0$$

\square

5 Continuity

We will define continuity in several ways (in this lecture, 3 different definitions).

Definition 5.1. (**Limit Definition of Continuity**). Let $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$, $c \in D$. We say that f is **continuous at c** if

$$\lim_{x \rightarrow c} f = f(c) \quad (5.1)$$

i.e. the limit of f at c exists and equal the function value. In any other cases, we say that f is **discontinuous at c** .

Remark 5.1. There are 2 ways that functions can be discontinuous at c :

1. $\lim_{x \rightarrow c} f$ exists but differs from $f(c)$.
2. $\lim_{x \rightarrow c} f$ DNE.

Definition 5.2. If $f : D \rightarrow \mathbb{R}$ is continuous at all $c \in D$, we simply say f is **continuous** (on D).

Example 5.0.1. x^2 is continuous on \mathbb{R} . Similarly, $\frac{1}{x}$ is continuous on $\mathbb{R} \setminus \{0\}$ since $\lim_{x \rightarrow c} \frac{1}{x} = \frac{1}{c}$. And, \sqrt{x} is continuous on \mathbb{R}_0^+ by assignment 9.

Example 5.0.2. Consider the following examples:

1. $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2$, $c \in \mathbb{R}$ arbitrary. We've seen that $\lim_{x \rightarrow c} x^2 = c^2$ i.e. $\lim_{x \rightarrow c} f = f(c) \implies f$ is continuous at any $c \in \mathbb{R}$.
2. $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) := \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$. Then, $\lim_{x \rightarrow 0} f = 0$ since $f \equiv 0$ on $\mathbb{R} \setminus \{0\}$. But $f(0) = 1 \neq 0 = \lim_{x \rightarrow 0} f \implies f$ is discontinuous at 0.
3. $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) := \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$. Claim: $\lim_{x \rightarrow 0} f$ DNE.

Proof. We'll use the 2-sequence criterion. Define $(x_n) := (\frac{1}{n})$; $\forall n \in \mathbb{N} : x_n \neq 0$, then $\lim(x_n) = 0$ and $\lim(\underbrace{f(x_n)}_{=1}) = 1$. Similarly, consider,

$(u_n) := (-\frac{1}{n})$; $\forall n \in \mathbb{N} : u_n \neq 0$, then $\lim(u_n) = 0$ and $\lim(\underbrace{f(u_n)}_{=0}) = 0 \neq 1$

$1 \implies \lim_{x \rightarrow 0} f$ DNE $\implies f$ is discontinuous at 0. □

Definition 5.3. ($\epsilon - \delta$). Let $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$, $c \in \mathbb{R}$. We say that f is **continuous at c** if $\forall \epsilon > 0 \exists \delta > 0 \forall x \in D : |x - c| < \delta \implies |f(x) - f(c)| < \epsilon$.

Definition 5.4. (Sequential). Let $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$, $c \in \mathbb{R}$. We say that f is **continuous at c** if $\forall (x_n)$ in D with $\lim(x_n) = c$, it holds that $\lim(f(x_n)) = f(c)$.

Theorem 5.1. All 3 definitions are equivalent

Proof. (\Leftarrow) Let $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$, $c \in \mathbb{R}$ be continuous by $\epsilon - \delta$ definition i.e. $\forall \epsilon > 0 \exists \delta > 0 \forall x \in D : |x - c| < \delta \implies |f(x) - f(c)| < \epsilon$. Thus, we have especially that $\forall \epsilon > 0 \exists \delta > 0 \forall x \in D \setminus \{c\} : |x - c| < \delta \implies |f(x) - f(c)| < \epsilon \implies \lim_{x \rightarrow c} f(x) = f(c) \implies f$ is constant at c

(\Rightarrow) $\lim_{x \rightarrow c} f$ exists and equals $f(c)$. Assume f is continuous by the limit definition then, $\forall \epsilon > 0, \exists \delta > 0 \forall x \in V_\delta^*(c) \cap D : |f(x) - f(c)| < \epsilon$. If $x = c$ then $x \in V_\epsilon(c) \cap D$ true and $|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon$ true. Then with them combined together you get $\forall \epsilon > 0, \exists \delta > 0 \forall x \in V_\delta^*(c) \cap D : |f(x) - f(c)| < \epsilon$.

Limit definition is equivalent to sequential definition as exercise. \square

Exercise. Show that $f : \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto |x|$ is continuous at 0.

Proof. We'll use the $\delta - \epsilon$ definition. Let $\epsilon > 0$ and $\delta := \epsilon$. Then, $|x - 0| = |x| < \delta = \epsilon$. Then, $|f(x) - f(0)| = ||x| - 0| = |x| < \delta = \epsilon \implies f$ is continuous at 0. \square

Theorem 5.2. (Algebraic Continuous Theorem). Let $f, g : D \rightarrow \mathbb{R}$, $c \in D$ s.t. f, g is continuous at c . Then,

1. $f + g$ is continuous at c .
2. $f - g$ is continuous at c .
3. $C \cdot f$ is continuous at c , $\forall C \in \mathbb{R}$.
4. $f g$ is continuous at c .
5. $\frac{f}{g}$ is continuous at c if $\forall x \in D : g(x) \neq 0$.

Proof. 1) f, g is continuous at $c \implies \lim_{x \rightarrow c} f = f(c)$ and similarly for $g(c) \implies \lim_{x \rightarrow c} (f + g) = \lim_{x \rightarrow c} f + \lim_{x \rightarrow c} g = f(c) + g(c) = (f + g)(c)$ i.e. $\lim_{x \rightarrow c} (f + g) = (f + g)(c)$. Therefore $f + g$ is continuous at c .

2-5. are left as exercises. \square

Applications

Theorem 5.3. All polynomials are continuous (on \mathbb{R}).

Proof. Let $P : \mathbb{R} \rightarrow \mathbb{R}$, $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. Thus, the function $x \mapsto x$ is continuous (continue as exercise). By theorem 5.1(4), x^n is continuous $\forall n \in \mathbb{N}$ and by theorem 5.1(3) x_nx^n is continuous $\forall n \in \mathbb{N}$. Lastly, by theorem 5.1(1), $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is continuous at any $c \in \mathbb{R}$. \square

Theorem 5.4. All rational functions $\frac{P(x)}{Q(x)}$, where P, Q are continuous polynomials, are continuous everywhere except at roots of Q .

Proof. This follows immediately from theorem 5.1(5). \square

Theorem 5.5. Compositions of continuous functions are continuous i.e. Let $f : A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, $c \in A$, and $g : B \subseteq \mathbb{R} \rightarrow \mathbb{R}$, $d \in B$ s.t. $f(A) \subseteq B$ and $f(c) = d$; and f is continuous at c while g is continuous at d . Then, $g \circ f : A \rightarrow \mathbb{R}$ is continuous at c .

Proof. (First). We will use sequential definition. Let (x_n) be a sequence in A such that $\lim(x_n) = c$. f is continuous at $c \implies \lim(f(x_n)) = f(c) = d$ i.e. $(f(x_n))$ is a sequence in B s.t. $\lim(f(x_n)) = d$. g is continuous at d . Then, $\lim g(f(x_n)) = g(d) \implies \lim((g \circ f)(x_n)) = (g \circ f)(c)$. i.e. For any sequence (x_n) in A with $\lim(x_n) = c$, we have $\lim((g \circ f)(x_n)) = (g \circ f)(c) \implies g \circ f$ is continuous at c . \square

Proof. (Second). We will use the $\varepsilon - \delta$ definition. Let $\varepsilon > 0$ and $y = g(u)$ be continuous at d . Then, $\exists \gamma > 0 : \forall u \in V_\gamma(d) \cap B : |g(u) - g(d)| < \varepsilon$ (*). Now let $u = f(x)$ be continuous at c . Then, $\exists \delta : \forall x \in V_\delta(c) \cap A : |f(x) - f(c)| < \gamma$ i.e. $|u - d| < \gamma \implies u \in V_\gamma(d) \cap B$. By (*), $|g(u) - g(d)| < \varepsilon \implies |g(f(x)) - g(f(c))| < \varepsilon \implies |(g \circ f)(x) - (g \circ f)(c)| < \varepsilon$. Summary: $\forall x \in V_\delta(c) \cap A : |(g \circ f)(x) - (g \circ f)(c)| < \varepsilon \implies g \circ f$ is continuous at c . \square

Example 5.0.3. By assignment 9, $\sqrt{} : \mathbb{R}_0^+ \rightarrow \mathbb{R}, x \mapsto \sqrt{x}$ is continuous. Thus $\sqrt{x^2 + 1}$ is continuous on \mathbb{R} . as the composition of the polynomials $x^2 + 1$ with $\sqrt{}$.

5.1 Continuity and Topology

Let $f : D \rightarrow \mathbb{R}$ be continuous,

1. If D is open, is $f(D)$ open?

2. If D is closed, is $f(D)$ closed?
3. If D is bounded, is $f(D)$ bounded?

Answer: NO, In general, for all 3 questions.

Example 5.1.1. Consider the following examples:

1. $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2$. Then, $] -1, 1[$ is open but $f(] -1, 1[) = [0, 1[$ not open, or $f(\mathbb{R}) = \mathbb{R}_0^+$.
2. $f : \mathbb{R}^+ \rightarrow \mathbb{R}$, $f(x) = \frac{1}{x}$. Then, $[1, \infty[$ is closed but $f([1, \infty[) =]0, 1]$ which is not closed.
3. From 2, $f(]0, 1]) = [1, \infty[$ which is unbounded since $1/x$ is its own inverse.

Nevertheless, there's an interesting exception.

Theorem 5.6. (Preservation of Compactness). Let $f : A \rightarrow \mathbb{R}$, A is compact and f is continuous. Then, $f(A)$ is compact.

Proof. Using sequential compactness. A is compact \implies it's sequentially compact i.e. every sequence (x_n) in A has a convergent subsequence (x_{n_k}) s.t. $\lim(x_{n_k}) \in A$. Let (y_n) in $f(A)$ be arbitrary. $f : A \rightarrow f(A)$ is surjective, thus every $y \in f(A)$ has at least 1 pre-image in A . Let $\forall n \in \mathbb{N} : x_n$ be a pre-image of y_n under f . Consider (x_n) and A is sequentially compact. Thus, $\exists (x_{n_k})$ of (x_n) and it is convergent where $x := \lim(x_{n_k})$ s.t. $x \in A$. Let $y := f(x)$. $\lim(x_{n_k}) = x \in A$. f is continuous on A , especially continuous at x . Then, by sequentially definition of continuity, $\lim(f(x_{n_k})) = f(x) = \underbrace{\lim(y_{n_k})}_{=y_n} = y$

$y \implies \lim(y_{n_k}) = y \in f(A)$. Thus, for any sequence (y_n) in $f(A)$, we can find a convergent subsequence (y_{n_k}) whose limits $y \in f(A)$. Therefore, $f(A)$ is sequentially compact and thus compact. \square

Definition 5.5. Let $f : D \rightarrow \mathbb{R}$. Then,

- (a) $x_0 \in D$ is called **global/absolute maximum** of f if $\forall x \in D : f(x) \leq f(x_0)$.
- (b) $x_1 \in D$ is called **global/absolute minimum** of f if $\forall x \in D : f(x) \geq f(x_1)$.

Theorem 5.7. Let $f : D \rightarrow \mathbb{R}$, D is compact and non-empty, where f is continuous. Then, f has both an absolute minimum and maximum on D .

Remark 5.2. All conditions in theorem 5.6 are essential i.e. f is continuous, D closed and bounded.

Example 5.1.2. Suppose the followings:

1. Let $D =]0, 1[$ and $y = x$. Thus, f is continuous but D is only bounded but not closed. In such case, f has neither an absolute min nor max.
2. Let $D = [0, 1]$ and $y = \begin{cases} x & x \in]0, 1[\\ \frac{1}{2} & x \in \{0, 1\} \end{cases}$. Thus, D is compact but D is discontinuous. We can see that it also has neither an absolute min nor max.
3. Let $D = [1, \infty[$ and $y = \frac{1}{x}$. Thus, f is continuous but D is only closed but not bounded. For this function, we have a maximum (at $x = 1$) but not a minimum, it only has an infimum of 0.

Proof. (Theorem 5.6) Let D be compact and f continuous, we conclude that $f(D)$ is compact. D is non-empty $\Rightarrow f(D)$ is non-empty. Especially, $f(D)$ is non-empty and bounded \Rightarrow has both infimum and supremum. As seen in the chapter on topology, inf and sup are boundary points of $f(D)$. Furthermore, since $f(D)$ is compact, it's closed \Rightarrow contains all of its boundary points, especially, its inf and sup i.e. $\inf f(D) \in f(D)$ and $\sup f(D) \in f(D)$. Thus, the $\inf f(D)$ is the $\min f(D)$, and $\sup f(D)$ is the $\max f(D)$. i.e. $\forall x \in D : \underbrace{\min f(D)}_{\in f(D)} \leq f(x) \leq \underbrace{\max f(D)}_{\in f(D)}$.

Thus, $\exists x_0, x_1 \in D : f(x_0) = \min f(D) \leq f(x) \leq \max f(D) = f(x_1) \Rightarrow x_0$ is an absolute min and x_1 is an absolute max of f on D . \square

5.2 The Intermediate Value Theorem

Theorem 5.8. (Localization of Roots). Let $f : [a, b] \rightarrow \mathbb{R}$, f is continuous such that f has opposite signs at a and b i.e. either $f(a) > 0$ and $f(b) < 0$ or, $f(a) < 0$ and $f(b) > 0$. Then, $\exists c \in]a, b[: f(c) = 0$.

Proof. Let $I_0 = [a, b]$, $a_0 = a$, $b_0 = b$, divides I_0 into 2 sub-intervals of equal width. If $f(\frac{a_0+b_0}{2}) = 0$, we've found the root of f . If not, there are 2 possibilities:

- $f(\frac{a+b}{2}) > 0$, let $a_1 := \frac{a_0+b_0}{2}$ and $b_1 = b_0$ and $I_1 = [a_1, b_1]$.
- $f(\frac{a+b}{2}) < 0$, let $a_1 = a_0$ and $b_1 := \frac{a_0+b_0}{2}$, $I_1 := [a_1, b_1]$.

Then divide I_1 into 2 sub-intervals of equal width. If $f(\frac{a_1+b_1}{2}) = 0$, we've found a root. If not, there are 2 cases as above etc.

In the end, either this algorithm terminates after finitely many steps with a root of f , or this algorithm doesn't terminate, then we obtain a nested sequence: $I_0 \supseteq I_1 \supseteq I_2 \supseteq \dots$ of closed and bounded intervals. By the nested interval property of \mathbb{R} . $\cap_{n \in \mathbb{N}} I_n \neq \emptyset$. Let $c \in \cap_{n \in \mathbb{N}} I_n$. Then, $\forall n \in \mathbb{N}_0, c \in [a_n, b_n] \implies |a_n - c| \leq b_n - a_n$ and $|b_n - c| \leq b_n - a_n = |I_n| = \underbrace{\frac{1}{2^n} \cdot |I_0|}_{\text{Null seq.}}$

Thus, by the null sequence criterion, $\lim(a_n) = c = \lim(b_n)$. Now, $\forall n \in \mathbb{N}, f(a_n) > 0$ and $f(b_n) < 0$ and f is continuous on D , especially at $c \implies \lim(\underbrace{f(a_n)}_{>0}) = f(c) = \lim(\underbrace{f(b_n)}_{<0}) \implies$ By limits and orders section, we'd get that $\lim(f(a_n)) \geq 0$ and $\lim(f(b_n)) \leq 0 \implies 0 \leq f(c) \leq 0 \implies f(c) = 0$. \square

Theorem 5.9. (Intermediate Value Theorem [IVT]). Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. Let $y_0 \in \mathbb{R} : y_0 \in]f(a), f(b)[$ i.e. $f(a) < y_0 < f(b)$ if $f(a) < f(b)$ or $f(b) < y_0 < f(a)$ if $f(a) < f(b)$. Then, y_0 is also a value under f i.e. $\exists x_0 \in]a, b[: f(x_0) = y_0$.

Remark 5.3. Theorem 5.7 is the special case where $y_0 = 0$ of the IVT.

Proof. (IVT). Let $g : [a, b] \rightarrow \mathbb{R}, g(x) \mapsto f(x) - y_0$. Then, g is continuous on $[a, b]$; $g(a) = f(a) - y_0$ and $g(b) = f(b) - y_0$. Then, $g(a) > 0$ and $g(b) < 0$ or $g(a) < 0$ and $g(b) > 0$. By localization of roots theorem, $\exists x_0 \in]a, b[: g(x_0) = 0 \implies 0 = g(x_0) = f(x_0) - y_0 \implies f(x_0) = y_0$. \square

This result will be used to prove the following:

Theorem 5.10. (Preservation of Intervals). Let $I \subseteq \mathbb{R}$ be a non-empty interval and $f : I \rightarrow \mathbb{R}$ be continuous. Then, $f(I)$ is an interval.

The proof of theorem 5.10, in turn, is based on the following result:

Theorem 5.11. (Characterization of Intervals). A non-empty subset $I \subseteq \mathbb{R}$ is an interval if and only if $\forall x, y \in I : x < y$, it holds that $[x, y] \subseteq I$.

Proof. (Theorem 5.10.) Assignment 10. \square

5.3 Uniform Continuity

Definition 5.6. A function $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be **uniformly continuous** (on D) if $\forall \varepsilon \exists \delta \forall x, u \in D : |x - u| < \delta \implies |f(x) - f(u)| < \varepsilon$.

Example 5.3.1. Let $a > 0$. Show that x^2 is uniformly continuous on $[-a, a]$.

Proof. Let $x, u \in [-a, a]$ such that $|x - u| < \delta$. Then,

$$\begin{aligned} |x^2 - u^2| &= |(x - u)(x + u)| = |x - u| \cdot |x + u| \\ &= \delta \cdot |x + u| \\ &\leq \delta(|x| + |u|) \\ &\leq \delta(a + a) \\ &\leq 2a\delta < \varepsilon \implies \delta < \frac{\varepsilon}{2a} \end{aligned}$$

Let $\delta < \frac{\varepsilon}{2a}$. Then, whenever $|x - u| < \delta \implies |f(x) - f(u)| < \varepsilon \implies x^2$ is continuous on $[-a, a]$ \square

5.3.1 Sequential Criterion For the Absence of Uniform Continuity

Theorem 5.12. Let $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is not uniformly continuous iff $\exists \varepsilon > 0$ and sequences $(x_n), (u_n)$ in D s.t. $\lim(x_n - u_n) = 0$ and $\forall n \in \mathbb{N} : |f(x_n) - f(u_n)| \geq \varepsilon$.

Proof. (\implies) Let f not be uniformly continuous. Then, $\neg(\forall \varepsilon \exists \delta \forall x, u \in D : |x - u| < \delta \implies |f(x) - f(u)| < \varepsilon) \equiv \exists \varepsilon \forall \delta \exists x, u \in D : |x - u| < \delta \implies |f(x) - f(u)| \geq \varepsilon$ (*). Select 1 such ε ; let $n \in \mathbb{N}$ be arbitrary and $\delta := \frac{1}{n}$. Then, $x_n, u_n \in D : |x_n - u_n| < \delta = \frac{1}{n}$ and $|f(x_n) - f(u_n)| \geq \varepsilon$. Consider (x_n) and (u_n) . Then, $\forall n \in \mathbb{N} : |x_n - u_n| < \frac{1}{n} \implies \lim(x_n - u_n) = 0$ and $\forall n \in \mathbb{N} : |f(x_n) - f(u_n)| \geq \varepsilon$. Which is what we need to show.

(\impliedby) Let $\varepsilon > 0, (x_n), (u_n) : \lim(x_n - u_n) = 0$ and $\forall n \in \mathbb{N} : |f(x_n) - f(u_n)| \geq \varepsilon$. Assume that f is uniformly continuous. Then, $\exists \delta > 0 \forall x, u \in D : |x - u| < \delta \implies |f(x) - f(u)| < \varepsilon$. $\lim(x_n - u_n) = 0 \implies \exists N \in \mathbb{N} : \forall n \geq N : |(x_n - u_n) - 0| = |x_n - u_n| < \delta$. Thus, $|f(x_n) - f(u_n)| < \varepsilon$ \sharp This f os not uniformly continuous. \square

Example 5.3.2. Show that $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}, x \mapsto \frac{1}{x}$ is not uniformly continuous on $[0, 1]$.

Proof. Let $x_n := \frac{1}{2n}$, $u_n = \frac{1}{n} \forall n \in \mathbb{N}$. Then, $|x_n - u_n| < |\frac{1}{2n} - \frac{1}{n}| = \frac{1}{2n} \iff \lim(x_n - u_n) = 0$ and $|f(x_n) - f(u_n)| = \left| \frac{1}{1/2n} - \frac{1}{1/n} \right| = |2n - n| = n \geq 1$. Let $\varepsilon := 1$. Then, $\forall n \in \mathbb{N} : |f(x) - f(u)| \geq 1 \implies f$ is not uniformly continuous on $]0, 1]$ and thus also not uniformly continuous on $\mathbb{R} \setminus \{0\}$. \square

Remark 5.4. By definition, every uniformly continuous function on $D \subseteq \mathbb{R}$ is continuous on D . The above example shows that the converse does not hold, in general. We say that uniform continuity is stronger than "ordinary" continuity.

Example 5.3.3. Show that $x \mapsto x^2$ is not uniformly continuous on \mathbb{R} .

Proof. $\forall n \in \mathbb{N}$, let $x_n := n + \frac{1}{n}$, $u_n := n$. Then, $\lim(x_n - u_n) = \lim(1/n) = 0$. Then, $|f(x_n) - f(u_n)| = |x_n^2 - u_n^2| = |(n + \frac{1}{n})^2 - n^2| = 2 + \frac{1}{n^2} > 2$. Let $\varepsilon := 2$. Then, $\forall n \in \mathbb{N} : |x_n^2 - u_n^2| > \varepsilon \iff x^2$ is not uniformly continuous on $[0, \infty[$ and thus also not uniformly continuous on \mathbb{R} . \square

Theorem 5.13. Let $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be uniformly continuous. Then, f maps Cauchy sequence to Cauchy sequence i.e. if (x_n) is a Cauchy sequence in D . Then, $(f(x_n))$ is a Cauchy sequence (in $f(D)$).

Proof. Let $\varepsilon > 0$. Since f is uniformly continuous $\exists \delta > 0 \forall x, u \in D : |x - u| < \delta \implies |f(x) - f(u)| < \varepsilon$. Let (x_n) be Cauchy, thus, $\exists N \forall n, m \geq N : |x_n - x_m| < \delta$. Thus, $\forall n, m \geq N : |f(x_n) - f(x_m)| < \varepsilon$ which is the definition of $f(x_n)$ being Cauchy. \square

Example 5.3.4. Show that $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}, x \mapsto \frac{1}{x}$ is not uniformly continuous.

Proof. Let $\forall n \in \mathbb{N} : x_n := \frac{1}{n}$. Then, (x_n) is a Cauchy sequence in $\mathbb{R} \setminus \{0\}$. But $\forall n \in \mathbb{N}, f(x_n) = n$. Thus $f(x_n)$ diverges and is not Cauchy $\implies \frac{1}{x}$ is not uniformly continuous. \square

Remark 5.5. This example shows that "ordinary" continuity functions do not, in general, map Cauchy to Cauchy sequence.

Theorem 5.14. (Preservation of Boundedness). Let $f : D \rightarrow \mathbb{R}$ be uniformly continuous and D be bounded. Then, $f(D)$ is bounded.

Proof. Assume that $f(D)$ is unbounded i.e. $\forall n \in \mathbb{N} \exists x_n \in D : |f(x_n)| \geq n$. Consider (x_n) in D where D is bounded. Then, by Bolzano-Weierstrass, (x_n) has a convergent subsequence (x_{n_k}) . Then, by last theorem, $f(x_{n_k})$ converges and is thus bounded. But, we said that $|f(x_{n_k})| \geq k \geq k \forall k \in \mathbb{N} \Rightarrow f(x_{n_k})$ is unbounded \nRightarrow Therefore, $f(D)$ is bounded. \square

Theorem 5.15. Let $f, g : D \rightarrow \mathbb{R}$ be uniformly continuous. Then,

1. $f + g$ is uniformly continuous.
2. $f - g$ is uniformly continuous.
3. $\forall k \in \mathbb{R}, k \cdot f$ is uniformly continuous.

Remark 5.6. Surprisingly, in general, the product of 2 uniformly continuous function is not uniformly continuous!

Example 5.3.5. Let $f, g : D \rightarrow \mathbb{R}$ both be equal to the identity function i.e. $f(x) = x$ and $g(x) = x$ for all $x \in \mathbb{R}$. Then, f and g are uniformly continuous. However x^2 is not (proven by the above).

Theorem 5.16. Let $A \subseteq \mathbb{R}$ be compact and let $f : A \rightarrow \mathbb{R}$ be continuous. Then, f is uniformly continuous.

Proof. Assume that f is not uniformly continuous. Then, $\exists \varepsilon > 0 : (x_n), (u_n)$ in A s.t. $\lim(x_n - u_n) = 0$. But $\forall n \in \mathbb{N} : |f(x_n) - f(u_n)| \geq \varepsilon$ (*). A is compact $\Rightarrow A$ is sequentially compact. Thus, (x_n) has a convergence subsequence (x_{n_k}) whose limit x lies in A . Consider the corresponding subsequence (u_{n_k}) . Then,

$$\begin{aligned}\lim(u_{n_k}) &= \lim(x_{n_k} - (x_{n_k} - u_{n_k})) \\ &= \underbrace{\lim(x_{n_k})}_{=x} - \underbrace{\lim(x_{n_k} - u_{n_k})}_{=0} \\ &= x\end{aligned}$$

i.e. both (x_{n_k}) and (u_{n_k}) converges to $x \in A$. f is continuous on A and thus especially at $x \Rightarrow \lim f(x_{n_k}) = f(x)$ and $\lim f(u_{n_k}) = f(u) \Rightarrow \lim(f(x_{n_k}) - f(u_{n_k})) = f(x) - f(u) = \Rightarrow \exists K \forall k \geq K : |f(x_{n_k}) - f(u_{n_k})| < \varepsilon$ \nRightarrow to (*). Thus, f is uniformly continuous (on A). \square

Remark 5.7. On compact domain, the properties of continuity and uniform continuity are identical.

Theorem 5.17. (*In Analysis II*). Every continuous function on $[a, b]$, $a < b$ is Riemann integrable.

Proof. Relies heavily on theorem 5.16. \square

Corollary 5.1. Let $f, g : A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, A is compact and f, g is uniformly continuous. Then, $f \cdot g$ is uniformly continuous.

Proof. Since f, g are uniformly continuous, they are continuous. Thus, $f \cdot g$ is continuous (on A). Since A is compact, it follows from the above theorem that $f \cdot g$ is uniformly continuous (on A). \square

Remark 5.8. The definition of uniform continuity of a function $f : D \rightarrow \mathbb{R}$ i.e. $\forall \varepsilon \exists \delta \forall x, u \in D : |x - u| < \delta \implies |f(x) - f(u)| < \varepsilon$ is symmetrical in x and u . We can thus assume without loss of generality that $x \geq u$ or v.v.

Lemma 5.1. $\forall x, u \in \mathbb{R}, x \geq u \geq 0$. Then, $\sqrt{x} - \sqrt{u} \leq \sqrt{x-u}$.

Proof. Recall from our first lecture that $\forall a, b \geq 0 : \sqrt{a+b} \geq \sqrt{a} + \sqrt{b}$. Let $a := u$ and $b := x - u \geq 0$. Then, $\sqrt{a+b} = \sqrt{x} \leq \sqrt{a} + \sqrt{b} = \sqrt{u} + \sqrt{x-u} \implies \sqrt{x} - \sqrt{u} \leq \sqrt{x-u}$. \square

Example 5.3.6. Show that $x \mapsto \sqrt{u}$ is uniformly continuous on $[0, \infty[$.

Proof. Let $\varepsilon > 0$ and $\delta > 0$ be arbitrary for now. Let $x, u \in \mathbb{R} : x \geq u \geq 0$. Then,

$$|\sqrt{x} - \sqrt{u}| = \sqrt{x} - \sqrt{u} \leq \sqrt{x-u} = \sqrt{|x-u|} < \sqrt{\delta} < \varepsilon \iff \delta < \varepsilon^2 \quad (5.2)$$

Let $\delta < \varepsilon^2$. Then, $|\sqrt{x} - \sqrt{u}| < \varepsilon$ whenever $|x - u| < \delta \implies \sqrt{x}$ is uniformly continuous on $[0, \infty[$. \square

Remark 5.9. It can be shown that $x \mapsto x^a$ is uniformly continuous on its domain (\mathbb{R}_0^+ or \mathbb{R}) if $0 < a \leq 1$ and not uniformly continuous if $a > 1$.

5.4 Lipschitz Continuity

Definition 5.7. $f : D \rightarrow \mathbb{R}$ is said to be **Lipschitz (continuous)** if $\exists k > 0 \forall x, u \in D : |f(x) - f(u)| < k \cdot |x - u|$. Additionally, k is called a **Lipschitz constant** of f

Theorem 5.18. Every Lipschitz function is uniformly continuous i.e. Let $f : D \rightarrow \mathbb{R}$ be Lipschitz. Then, f is uniformly continuous.

Proof. Let $\varepsilon > 0$, $\delta := \frac{\varepsilon}{k}$, and $|x - u| < \delta$. Then, $|f(x) - f(u)| \leq k \cdot |x - u| < k \cdot \delta = k \cdot \frac{\varepsilon}{k} = \varepsilon$. Thus, $|f(x) - f(u)| < \varepsilon \implies f$ is uniformly continuous. \square

Remark 5.10. *The converse does not hold!*

Example 5.4.1. Show that \sqrt{x} is not Lipschitz continuous on $[0, 1]$.

Proof. Assume that f is Lipschitz continuous on $[0, 1]$. Let k be a Lipschitz constant for f . Let $x := \frac{1}{4k^2}$ and $u := 0$. Then, $k \cdot |x - u| = k \cdot \frac{1}{4k^2} = \frac{1}{4k}$. On the other hand, $|f(x) - f(u)| = |\sqrt{x} - \sqrt{u}| = \left|\frac{1}{2\sqrt{x}} - 0\right| = \frac{1}{2\sqrt{x}} > \frac{1}{4k} \implies |f(x) - f(u)| \leq k \cdot |x - u|$. Thus f is not Lipschitz continuous on $[0, 1]$. \square

Thus, we can see that Lipschitz continuity is stronger than uniform continuity which is stronger than ordinary continuity. In this case, "stronger" implies the conditions are more restrictive but important.

6 Differentiation

Definition 6.1. Let $I \subseteq \mathbb{R}$ be an interval, let $f : I \rightarrow \mathbb{R}$ and $c \in I$. We say that f is **differentiable** at c if $\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$ exists. In this case, we denote

$$f'(c) := \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \quad (6.1)$$

the **derivative** of f at c .

Remark 6.1. If f is differentiable at all $c \in I$. Then, we simply say that f is **differentiable**.

Note: Geometrically speaking, $f'(x)$ represents the tangent line to the graph of f at c .

Example 6.0.1. Consider the following examples:

1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto x^2$. Then, let c be arbitrary, we get that

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = \lim_{x \rightarrow c} \frac{x^2 - c^2}{x - c} = \lim_{x \rightarrow c} \frac{(x - c)(x + c)}{x - c} = \lim_{x \rightarrow c} x + c = 2c$$

Thus $\forall c \in \mathbb{R}$, $f'(c) = 2c$ or simply $f'(x) = 2x$ on \mathbb{R} .

2. $f :]0, \infty[\rightarrow \mathbb{R}$, $x \mapsto \frac{1}{x}$. Let $c \in]0, \infty[$. Then,

$$\begin{aligned} \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} &= \lim_{x \rightarrow c} \frac{\frac{1}{x} - \frac{1}{c}}{x - c} = \lim_{x \rightarrow c} \frac{\frac{c - x}{xc}}{x - c} \\ &= \lim_{x \rightarrow c} \frac{-(x - c)}{xc} \frac{1}{x - c} \\ &= \lim_{x \rightarrow c} \frac{-1}{xc} = -\frac{1}{c^2} \end{aligned}$$

$$\implies f'(x) = -\frac{1}{x^2} \text{ on }]0, \infty[.$$

Exercise. Find the derivative of \sqrt{x} , $\frac{1}{\sqrt{x}}$ and $\frac{1}{x^2}$.

Theorem 6.1. (Caratheodory). Consider the followings:

1. Let $f : I \rightarrow \mathbb{R}$. Then, f is differentiable at c iff $\exists \varphi : I \rightarrow \mathbb{R}$ which is continuous at c such that $\forall x \in I : f(x) = f(c) + \varphi(x) \cdot (x - c)$.

2. In case φ does exists, it holds that $\varphi(c) = f'(c)$.

Proof. (\Leftarrow) Let $\varphi : I \rightarrow \mathbb{R}$ be continuous at c : $\forall x \in I, f(x) = f(c) + \varphi(x)(x - c)$. Then, $\varphi(x) = \frac{f(x) - f(c)}{x - c} \forall x \in I \setminus \{c\}$. Since f is continuous at c , $\varphi(x) = \lim_{x \rightarrow c} \varphi(x) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$. Thus, f is differentiable at c and $f'(c) = \varphi(c)$.
 (\Rightarrow) Let f be differentiable at c . Define $\varphi : I \rightarrow \mathbb{R}$ as follows:

$$\varphi(x) := \begin{cases} \frac{f(x) - f(c)}{x - c} & x \neq c \\ f'(c) & x = c \end{cases}$$

We then have that if $x \neq c$, $f(x) = f(c) + \varphi(x)(x - c)$ and thus it's proven. If $x = c$, $f(x) = f(c) + \underbrace{\varphi(x)(x - c)}_{=0} = f(c)$. The functional equation thus holds for all $x \in I$ and

$$\lim_{x \rightarrow c} \varphi(x) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = f'(c) = f(c)$$

Thus, φ is continuous at c . Lastly, $\varphi(x) = f'(x)$ always holds by the above implications. \square

Example 6.0.2. $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x^2$. Use Caratheodory theorem to show that f is differentiable.

Proof. Let $c \in \mathbb{R}$ be arbitrary. If $x \neq c$, then,

$$f(x) = \frac{x^2 - c^2}{x - c} = \frac{(x - c)(x + c)}{x - c} = x + c$$

where $x \mapsto x + c$ is continuous on \mathbb{R} . Thus, $\varphi(x) = x + c$ is the caratheodory function for $f \Rightarrow f$ is differentiable at c and $f'(c) = \varphi(c) = c + c = 2c$. \square

Theorem 6.2. Let $f : I \rightarrow \mathbb{R}$ and $c \in I$. If f is differentiable at c then f is continuous at c .

Proof. Let f be differentiable at c . Then, $\exists \varphi : I \rightarrow \mathbb{R}$ that is continuous at c . Furthermore, $\forall x \in I, f(x) = f(c) + \underbrace{\varphi(x)}_{\substack{\text{cont. at } c}} \underbrace{(x - c)}_{\substack{\text{cont. at } c}}$
 $\qquad\qquad\qquad \underbrace{\qquad\qquad\qquad}_{\text{cont. at } c}$

Thus, f is continuous at c . \square

Remark 6.2. The converse will NOT hold.

Example 6.0.3. $f : \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto |x|$ is not differentiable at 0 but is continuous at 0 since

$$\lim_{x \rightarrow 0} \frac{|x| - |0|}{x - 0} = \lim_{x \rightarrow 0} \frac{|x|}{x} \quad \text{where} \quad \frac{|x|}{x} = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

Let $(x_n) := (1/n)$ and $(u_n) := (-1/n)$. Then, $\lim(x_n) = \lim(u_n) = 0$. However, $\lim(g(x_n)) = \lim(1) = 1$ while $\lim(g(u_n)) = \lim(-1) = -1$. Thus, by the 2-sequence criteria for non-existence of the limit of function, $\lim_{x \rightarrow 0} \frac{|x|}{x}$ DNE. Thus, $|x|$ is not differentiable at 0 however, we've seen that $|x|$ is continuous at 0.

Remark 6.3. This can be expressed as "differentiability is a strictly stronger condition than continuity".

Theorem 6.3. Let $f, g : I \rightarrow \mathbb{R}$, $c \in I$, f, g is differentiable at c . Then,

1. $f + g$ is differentiable at c and $(f + g)'(c) = f'(c) + g'(c)$.
2. $\forall k \in \mathbb{R}$, kf is differentiable at c and $(kf)'(x) = kf'(x)$.
3. $f - g$ is differentiable at c and $(f - g)'(c) = f'(c) - g'(c)$.
4. fg is differentiable at c and $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.
5. If $g \neq 0$ on I , f/g is differentiable and $\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c) - g'(c)f(c)}{g^2(c)}$.

Proof. This proof directly follows from the limit definition of derivative. One can even use Caratheodory theorem. \square

Theorem 6.4. Let $f : I \rightarrow \mathbb{R}$, $g : J \rightarrow \mathbb{R}$ where $c \in I$, $d := f(c)$. f is differentiable at c and g is differentiable at d . Then, $g \circ f : I \rightarrow \mathbb{R}$ is differentiable at $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.

Proof. Left as an exercise. Hint: Use Caratheodory theorem. \square

6.1 Mean Value Theorem

Definition 6.2. Let $f : D \rightarrow \mathbb{R}$, $c \in D$ is said to be a **local maximum** of f if $\exists \delta > 0 : f$ has an absolute maximum at c on $V_\delta(c) \cap D$. Similarly c is a **local minimum** if $\exists \delta > 0 : c$ is an absolute minimum of f on $V_\delta(c) \cap D$.

Remark 6.4. The collective term for both minimum and maximum is "extremum".

Lemma 6.1. Let $f : D \rightarrow \mathbb{R}$, $c \in D$ such that $\lim_{x \rightarrow c} f > 0$. Then, $\exists \delta > 0 : \forall x \in V_\delta^*(c) \cap D : f(x) > 0$. A similar result holds if $\lim_{x \rightarrow c} f < 0$.

Proof. Let $L := \lim_{x \rightarrow c} f > 0$ and $\varepsilon := \frac{L}{2} > 0$. Then, $\exists \delta > 0 : \forall x \in V_\delta^*(c) \cap D : |f(x) - L| < \varepsilon = \frac{L}{2} \iff -\frac{L}{2} < f(x) - L < \frac{L}{2} \iff 0 < f(x) - \frac{L}{2} < L$. Thus, $\forall x \in V_\delta^*(c) \cap D : f(x) > \frac{L}{2} > 0$. \square

Theorem 6.5. (Fermat's). Let $f : I \rightarrow \mathbb{R}$, $c \in I$ and $c \notin \partial I$, f is differentiable at c , and f has a local extremum at c . Then, the $f'(c) = 0$.

Remark 6.5. $c \notin \partial I$ is necessary e.g. let $f : [a, b] \rightarrow \mathbb{R}$, $f(x) \mapsto x$ then $f'(a) \neq 0$ and $f'(b) \neq 0$.

Proof. (Fermat's). Assume without loss of generality that f has a local max at c (otherwise consider $-f$). Now, assume that $f'(c) > 0$ i.e. $\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} > 0$. Then, by the above lemma, $\exists \delta_1 > 0 : \forall x \in V_{\delta_1}^*(c) \cap D : \frac{f(x) - f(c)}{x - c} > 0$; furthermore, $c \notin \partial I \implies \exists \delta_2 > 0 : V_{\delta_2}(c) \subseteq I$. Let $\delta := \min\{\delta_1, \delta_2\}$. Then, $\forall x \in V_\delta^*(c) \cap I = V_\delta^*(c) : \frac{f(x) - f(c)}{x - c} > 0$. Especially, $\forall x \in]c, c + \delta[: \frac{f(x) - f(c)}{x - c} > 0$. So $\forall x \in]c, c + \delta[\subseteq V_\delta(c) : f(x) > f(c)$ since c is a local maximum. Consequently, $f'(x) \leq 0$.

Now assume that $f'(c) < 0$. As in the first case, $\exists \delta > 0 : \forall x \in V_\delta^*(c) \cap I : \frac{f(x) - f(c)}{x - c} < 0$. Especially, $\forall x \in]c - \delta, c[: \frac{f(x) - f(c)}{x - c} < 0 \implies \forall x \in]c - \delta, c[\subseteq V_\delta(c) : f(x) > f(c)$ since c is local maximum.

$$\therefore f'(x) = 0 \quad \square$$

Note: Looking at the above remark, we cannot exclude that $f'(c) > 0$ in this case (1st assumption of the proof) indeed $f'(c) > 0$.

Theorem 6.6. (Rolle's). Let $f : [a, b] \rightarrow \mathbb{R}$, f is continuous on $[a, b]$, is differentiable on $]a, b[$ and $f(a) = f(b) = 0$. Then, $\exists c \in]a, b[: f'(c) = 0$.

Proof. If $f \equiv 0$ on $[a, b]$ then any c works. Assume that $f \not\equiv 0$ on $[a, b]$. Let $f(x_0) \neq 0$. Assume, without loss of generality, that $f(x_0) > 0$ (otherwise consider $-f$). f is continuous on the compact set $[a, b] \implies f$ has an absolute maximum on $[a, b]$, supposedly at c . Then, $f(c) \geq f(x_0) > 0 \implies c \neq a, c \neq b \implies c \notin \partial[a, b]$ i.e. $c \in]a, b[$. Especially, f is differentiable at c and c is also a local maximum. Thus, by Fermat's theorem, $f'(c) = 0$. \square

Theorem 6.7. (Mean Value Theorem [MVT]). Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$, differentiable $]a, b[$. Then, $\exists c \in]a, b[: f'(c) = \frac{f(b) - f(a)}{b - a}$.

Proof. Let $L(x)$ be the secant line through the points $(a, f(a))$ and $(b, f(b))$ i.e. $L(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$. First, check $L(a) = f(a)$ and $L(b) = f(b)$. Now, consider the function $g : [a, b] \rightarrow \mathbb{R}$, $g = f - L$. Then, g is continuous on $[a, b]$ and also g is differentiable on (a, b) . Furthermore, $g(a) = f(a) - L(a) = 0$ and $g(b) = f(b) - L(b) = 0$. Then, by Rolle's theorem, $\exists c \in]a, b[$: $g'(c) = 0$, where $g'(x) = f'(x) - L'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$. Then, $g'(c) = 0 = f'(c) - \frac{f(b)-f(a)}{b-a} \implies f'(c) = \frac{f(b)-f(a)}{b-a}$. \square

Note: The MVT can be used to deduce properties of f from properties of f' .

Example 6.1.1. Let $f : I \rightarrow \mathbb{R}$ be differentiable s.t. $\forall x \in I : f'(x) \geq 0$. Then, f is increasing on I .

Proof. Let $a, b \in I : a < b$. We need to show that $f(a) \leq f(b)$. Apply the MVT to f on $[a, b]$ (we can do this since f is differentiable and thus continuous on $[a, b]$). By MVT, $\exists c \in]a, b[$: $f'(c) = \frac{f(b)-f(a)}{b-a} \implies \frac{f(b)-f(a)}{b-a} \geq 0$. Since $b-a > 0 \implies f(b)-f(a) \geq 0 \iff f(a) \leq f(b)$. Thus, f is increasing on I . \square

Note to Author: Good luck on your final exam on December 19th, 2024!!!



Properties of R. Cauchy and monotone sequences, Bolzano- Weierstrass theorem. Limits, limsup, liminf of functions. Pointwise, uniform continuity: Intermediate Value theorem. Inverse and monotone functions. Differentiation: Mean Value theorem, L'Hospital's rule, Taylor's Theorem.

