Lecture 11 – Multiple Linear Regression and Feature Engineering

DSC 40A, Fall 2022 @ UC San Diego
Dr. Truong Son Hy, with help from **many others**

Announcements

- Look at the readings linked on the course website!
- Groupwork Relsease Day: Thursday afternoon
  Groupwork Submission Day: Monday midnight
  Homework Release Day: Friday after lecture
  Homework Submission Day: Friday before lecture
- See `dsc40a.com/calendar` for the Office Hours schedule.

Midterm study strategy

- Review the solutions to previous homeworks and groupworks.
- Re-watch lecture, post on Campuswire, come to office hours.
- Look at the past exams at `https://dsc40a.com/resources`.
- Study in groups.
- **Remember:** it's just an exam.

Agenda

- Many demo.
- Recap of linear regression and linear algebra.
- Using multiple features.

Many demo

Regression and linear algebra

- Last time, we used linear algebra to fit a prediction rule of the form

$$H(x) = w_0 + w_1 x$$

- To do so, we first defined a **design matrix** X, **parameter vector** $\vec{w}$, and **observation vector** $\vec{y}$ as follows:

$$X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \dots & \dots \\ 1 & x_n \end{bmatrix}, \qquad \vec{w} = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}, \qquad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix}$$

- We also re-wrote our prediction rule as a matrix-vector multiplication, defining the **hypothesis vector** $\vec{h}$ as

$$\vec{h} = X\vec{w}$$

Minimizing mean squared error

- With our new linear algebra formulation of regression, our mean squared error now looks like:

$$R_{sq}(\vec{w}) = ||\vec{y} - X\vec{w}||^2$$

- To find $\vec{w}^*$, the optimal parameter vector, we took the gradient of $R_{sq}(\vec{w})$ with respect to $\vec{w}$, set it equal to 0, and solved.
- The result is the **normal equations**:

$$X^T X \vec{w}^* = X^T y$$

- When $X^T X$ is invertible, an equivalent form is

$$\vec{w}^* = (X^T X)^{-1} X^T y$$

Using multiple features

Using multiple features

- How do we predict salary given **multiple** features?
- We believe salary is a function of experience *and* GPA.
- In other words, we believe there is a function H so that:

$$\text{salary} \approx H(\text{years of experience}, \text{GPA})$$

- Recall: H is a **prediction rule**.
- **Our goal**: find a good prediction rule, H.

Example prediction rules

$$H_1(\text{experience}, \text{GPA}) = \$2,000 \times (\text{experience}) + \$40{,}000 \times \frac{\text{GPA}}{4.0}$$

$$H_2(\text{experience}, \text{GPA}) = \$60{,}000 \times 1.05^{(\text{experience}+\text{GPA})}$$

$$H_3(\text{experience}, \text{GPA}) = \cos(\text{experience}) + \sin(\text{GPA})$$

Linear prediction rules

- We'll restrict ourselves to **linear** prediction rules:

$$H(\text{experience}, \text{GPA}) = w_0 + w_1(\text{experience}) + w_2(\text{GPA})$$

- This is called **multiple linear regression.**

- Note that H is **linear in the parameters** w_0, w_1, w_2.
  - H is a linear combination of features (1, experience, GPA) with ws as the coefficients (w_0, w_1, and w_2).

- As a result, we can solve the **normal equations** to find w_0^*, w_1^*, and w_2^*!

- Linear regression with multiple features is called **multiple linear regression**.

Geometric interpretation

Question: The prediction rule

$$H(\text{experience}) = w_0 + w_1(\text{experience})$$

looks like a line in 2D.

1. How many dimensions do we need to graph

$$H(\text{experience}, \text{GPA}) = w_0 + w_1(\text{experience}) + w_2(\text{GPA})$$

2. What is the shape of the prediction rule?

Example dataset

- For each of n people, collect each feature, plus salary:

Person #	Experience	GPA	Salary
1	3	3.7	85,000
2	6	3.3	95,000
3	10	3.1	105,000

- We represent each person with a **feature vector**:

$$\vec{x}_1 = \begin{bmatrix} 3 \\ 3.7 \end{bmatrix}, \qquad \vec{x}_2 = \begin{bmatrix} 6 \\ 3.3 \end{bmatrix}, \qquad \vec{x}_3 = \begin{bmatrix} 10 \\ 3.1 \end{bmatrix}$$

The hypothesis vector

- When our prediction rule is

$$H(\text{experience}, \text{GPA}) = w_0 + w_1(\text{experience}) + w_2(\text{GPA}),$$

the hypothesis vector $\vec{h} \in \mathbb{R}^n$ can be written

$$\vec{h} = \begin{bmatrix} H(\text{experience}_1, \text{GPA}_1) \\ H(\text{experience}_2, \text{GPA}_2) \\ \dots \\ H(\text{experience}_n, \text{GPA}_n) \end{bmatrix} = \begin{bmatrix} 1 & \text{experience}_1 & \text{GPA}_1 \\ 1 & \text{experience}_2 & \text{GPA}_2 \\ \dots & \dots & \dots \\ 1 & \text{experience}_n & \text{GPA}_n \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix}$$

How do we find $\vec{w}^*$?

- To find the best parameter vector, $\vec{w}^*$, we can use the design matrix and observation vector

$$X = \begin{bmatrix} 1 & \text{experience}_1 & \text{GPA}_1 \\ 1 & \text{experience}_2 & \text{GPA}_2 \\ \dots & \dots & \dots \\ 1 & \text{experience}_n & \text{GPA}_n \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix}$$

and solve the **normal equations**

$$X^T X \vec{w}^* = X^T \vec{y}$$

- Notice that the rows of the design matrix are the (transposed) feature vectors, with an additional 1 in front.

Notation for multiple linear regression

- We will need to keep track of multiple[1] features for every individual in our data set.
- As before, subscripts distinguish between individuals in our data set. We have n individuals (or **training examples**).
- Superscripts distinguish between features.[2] We have d features.
  - experience = $x^{(1)}$
  - GPA = $x^{(2)}$

[1] In practice, we might use hundreds or even thousands of features.
[2] Think of them as new variable names, such as new letters.

Augmented feature vectors

- The **augmented feature vector** $\text{Aug}(\vec{x})$ is the vector obtained by adding a 1 to the front of feature vector $\vec{x}$:

$$\vec{x} = \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(d)} \end{bmatrix} \qquad \text{Aug}(\vec{x}) = \begin{bmatrix} 1 \\ x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(d)} \end{bmatrix} \qquad \vec{w} = \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \\ w_d \end{bmatrix}$$

- Then, our prediction rule is

$$\begin{aligned} H(\vec{x}) &= w_0 + w_1 x^{(1)} + w_2 x^{(2)} + \ldots + w_d x^{(d)} \\ &= \vec{w} \cdot \text{Aug}(\vec{x}) \end{aligned}$$

The general problem

- We have n data points (or **training examples**): $(\vec{x}_1, y_1), \dots, (\vec{x}_n, y_n)$ where each $\vec{x}_i$ is a feature vector of d features:

$$\vec{x}_i = \begin{bmatrix} x_i^{(1)} \\ x_i^{(2)} \\ \dots \\ x_i^{(d)} \end{bmatrix}$$

- We want to find a good linear prediction rule:

$$\begin{aligned} H(\vec{x}) &= w_0 + w_1 x^{(1)} + w_2 x^{(2)} + \dots + w_d x^{(d)} \\ &= \vec{w} \cdot \text{Aug}(\vec{x}) \end{aligned}$$

The general solution

- Use design matrix

$$X = \begin{bmatrix} 1 & x_1^{(1)} & x_1^{(2)} & \dots & x_1^{(d)} \\ 1 & x_2^{(1)} & x_2^{(2)} & \dots & x_2^{(d)} \\ \dots & \dots & \dots & & \dots \\ 1 & x_n^{(1)} & x_n^{(2)} & \dots & x_n^{(d)} \end{bmatrix} = \begin{bmatrix} \mathrm{Aug}(\vec{x}_1)^T \\ \mathrm{Aug}(\vec{x}_2)^T \\ \dots \\ \mathrm{Aug}(\vec{x}_n)^T \end{bmatrix}$$

and observation vector to solve the **normal equations**

$$X^T X \vec{w}^* = X^T \vec{y}$$

to find the optimal parameter vector.

Interpreting the parameters

- With d features, $\vec{w}$ has $d + 1$ entries.
- w_0 is the **bias**, also known as the **intercept**.
- $w_1, \dots, w_d$ each give the **weight**, i.e. **coefficient**, of a feature.

$$H(\vec{x}) = w_0 + w_1 x^{(1)} + \dots + w_d x^{(d)}$$

- The sign of w_i tells us about the relationship between ith feature and the output of our prediction rule.