Chapter 7

KNIGHT'S TOUR PROBLEMS

7.1 INTRODUCTION

The earliest serious attempt to find a knight's tour on the chessboard was made by Leonhard Euler in 1759 (Euler 1766). A knight must traverse all of the squares on an m x n chessboard but visit every square once and only once and return to the originated square where the knight moves in an L-shape route. Many great mathematicians including De Moivre, Vandermonde (Vandermonde 1774), Warnsdorff (Warnsdorff 1823), Pratt (Pratt 1825), Roget, Legendre (Legendre 1830), and De Lavernede (Lavernede 1839) attempted to solve the general problem. Their methods are based on either divide-and-conquer or backtracking dedicated for solving the 8 x 8 chessboard knight's tour problem. No general methods have been given to the problem in the last two centuries (Ball and Coxeter 1974). W. W. R. Ball and H. S. M. Coxeter reintroduced the problem in their book in 1892 and 1974. A parallel algorithm for finding a knight's tour on the m x n chessboard is presented in this paper. The problem is to find a constrained Hamiltonian circuit which belongs to NP-complete problems. However we are not sure whether the knight's tour problem is NP-complete or not. The artificial neural network computing makes it possible to provide the elegant algorithm in parallel which takes advantage of the simplied biological neural computation. Several examples are shown to demonstrate the capability of our parallel algorithm. This Chapter is based on a paper published in IEEE Trans. on System, Man, and Cybernetics (Lee and Takefuji 1992).

7.2 NEURAL REPRESENTATION AND MOTION EQUATIONS

In this Chapter a two-dimensional triangular neural network representation is introduced. The system requires p(p-1)/2 processing elements or hysteresis McCulloch-Pitts neurons where p is the number of squares on an m x n chessboard. Each state of the neurons represents a path. In order to establish a knight's path on a square on the chessboard, two neurons must be fired because one is for coming into the square and the other is for going out from the square. In our algorithm the following motion equation for the *ij*th neuron is used to solve the problem:

if $d_{ij} = 1$ then

$$\frac{dU_{ij}}{dt} = -\left(\sum_{k=1}^{p} V_{ik}d_{ik} - 2\right) - \left(\sum_{k=1}^{p} V_{kj}d_{kj} - 2\right)$$

if $d_{ij} = 0$ then

$$\frac{dU_{ij}}{dt} = 0 \qquad (7.1)$$

where the upper triangular elements in the two-dimensional array are only used: V_{ij} for i<j and V_{ij} for i>j is given by V_{ji}. The state of V_{ij} actually represents a knight's tour between the *i*th and the *j*th square. In other words, the upper triangular neural array represents the nondirected adjacency matrix in order to find a Hamiltonian circuit. In Eq. (7.1) d_{ij} is 1 if the move from the *i*th square to the *j*th square is legal or valid, and 0 otherwise. The first term and the second term are to determine a move between the *i*th and the *j*th square if it is valid. The first term forces the *ij*th neuron to have two valid moves from the *i*th square and the second term from the *j*th square respectively. The state of V_{ij} is updated by the hysteresis McCulloch-Pitts function: $V_{ij}=f(U_{ij})=1$ if $U_{ij}>3$, 0 if $U_{ij}<0$, and no changes otherwise.

Fig. 7-1 shows where the knight moves in an L-shape route. In a 3 x 4 chessboard knight's tour problem, 12 squares are numbered from left to right and from top to bottom as shown in Fig. 7-2a. From #2 square there are three valid moves: $V_{2,8}$, $V_{2,9}$ and $V_{2,11}$ where two moves are only needed to find the Hamiltonian circuit.

Fig. 7-2b shows the p(p-1)/2=66 neurons for this problem where the black squares indicate their outputs are one's. Fig. 7-2b depicts the two closed loops as shown in Fig. 7-2c which is not the Hamiltonian circuit. Unfortunately there is no solution for the 3 x 4 knight's tour problem (17). In 1943 Fred. Schuh stated that the number of squares has to be even which is necessary but not sufficient (17).

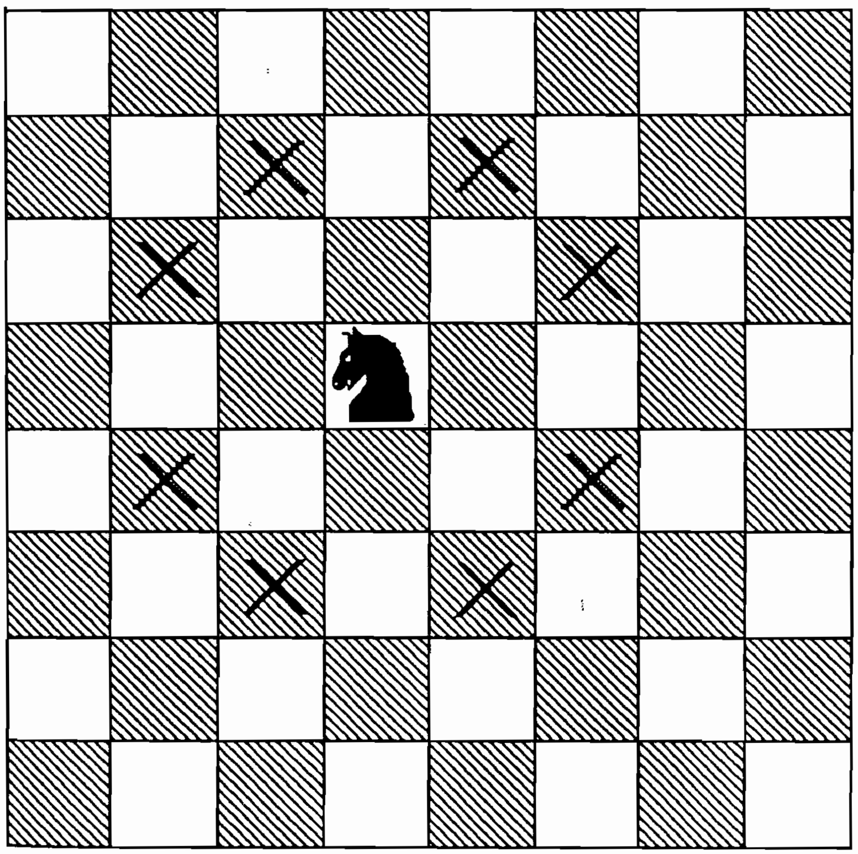

Fig. 7-1 (a) The knight moves in an L-shape route

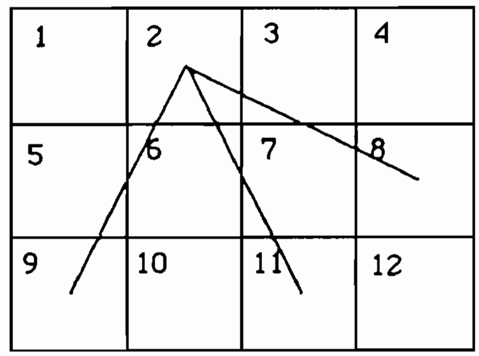

Fig. 7-2 (a) a 3 x 4 chessboard

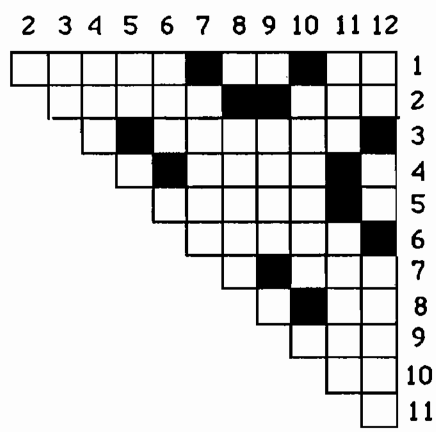

Fig. 7-2 (b) Neural representation

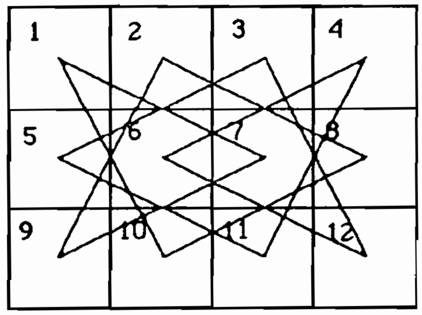

Fig. 7-2 (c) unsatisfactory solution

We have developed the simulator on a Macintosh SE/30 based on Eq. (7.1) to verify our algorithm. Fig. 7-3 through 7-10 show solutions for 6 x 6, 8 x 8, 10 x 10, 12 x 12, 14 x 14, 16 x 16, 18 x 18, and 20 x 20 chessboard knight's tour problems respectively.

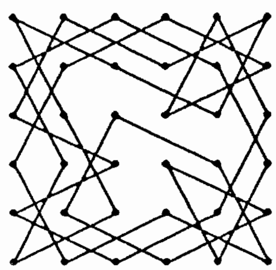

Fig. 7-3 6 x 6 chessboard

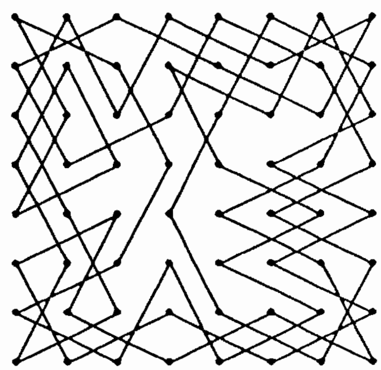

Fig. 7-4 8 x 8 chessboard

The average number of iteration steps takes less than 100 steps. We have observed that the larger problem the more often the state of the system converged to the unsatisfactory solution with several closed loops. Without the hysteresis property, the state of the system tends to oscillate and hardly converges to the solution.

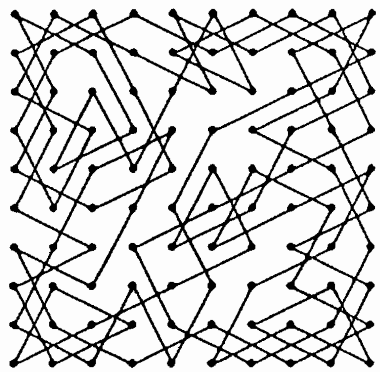

Fig. 7-5 10 x 10 chessboard

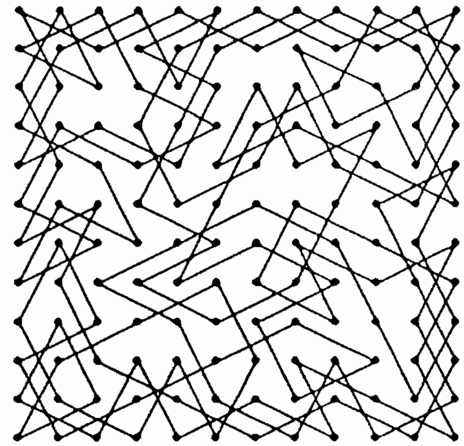

Fig. 7-6 12 x 12 chessboard

7.3 REFERENCES

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7.4 EXERCISES

1. Build a simulator to verify Eq. (7.1) and solve several knight's tour problems.
2. Discuss the minimum number of required neurons. Give the minimum number of neurons for the knight's tour problem on the 8 x 8 chessboard.

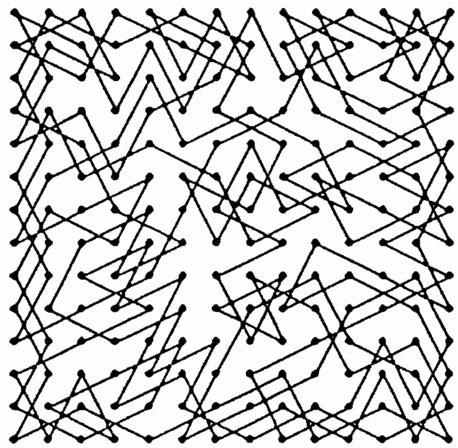

Fig. 7-7 14 x 14 chessboard

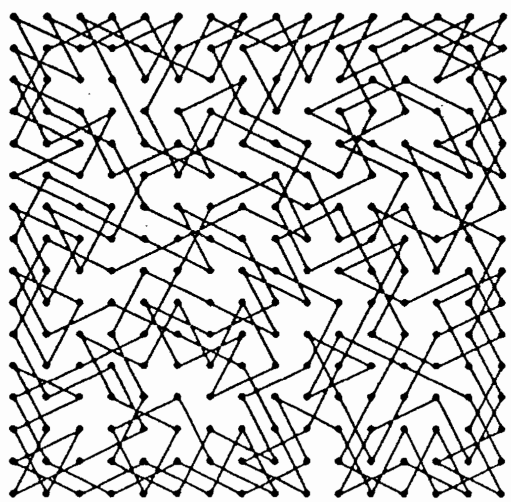

Fig. 7-8 16 x 16 chessboard

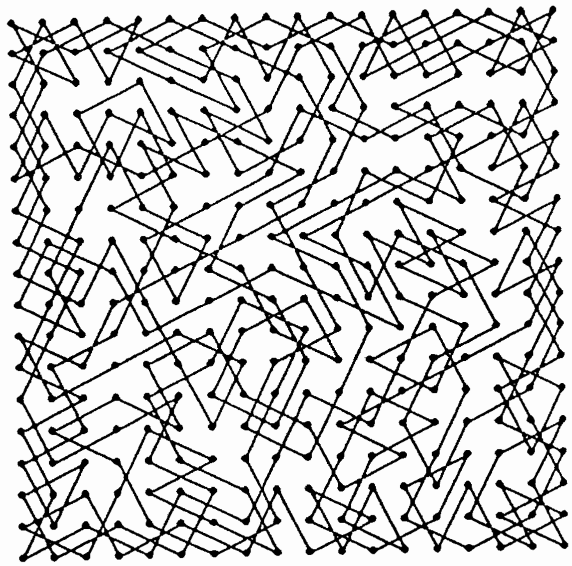

Fig. 7-9 18 x 18 chessboard

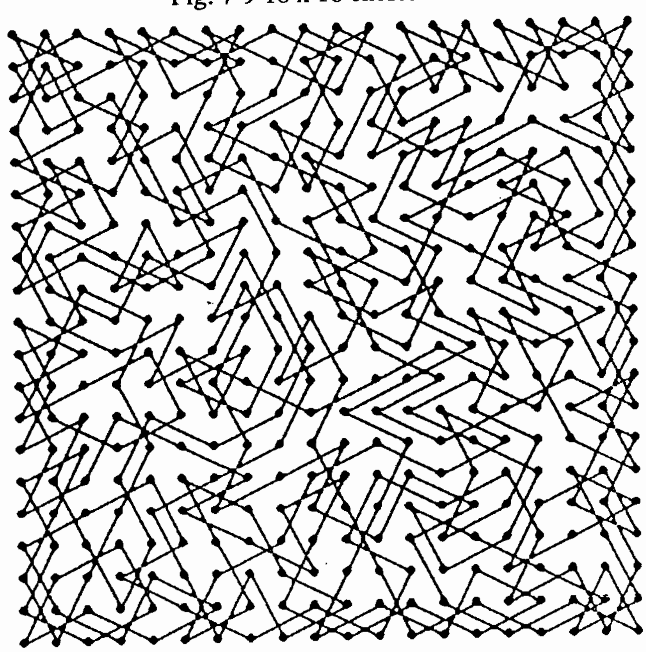

Fig. 7-10 20 x 20 chessboard