Chapter 1 2

BY 实数与实数列极限.

1 Def

1. 实数: 无限循环小数和无限不循环小数的并集;
2. 实数的性质：最小上界性的有序域
3. 三角不等式: $|a|-|b| \leqslant |a \pm b| \leqslant |a|+|b|$.(构成度量)
4. 上界: 非空实数集$A.\exists M \in R, \forall x \in A \wedge x < M$.称$M$为上界.
5. 上确界：M是A的上界.若$\forall m < M, \exists x \in A \wedge x > m$称$M$为上确界
6. 确界原理:非空实数集. 有上界必有上确界；有下界必有下确界(Pr: 二分法可以小于ε)
7. 函数：定义域、值域、对应法则。$f: D \rightarrow R$
8. 函数的运算和复合：逐点运算和复合
9. 反函数：$1-1$的区间上才有反函数
10. 函数性质：有界、单调、奇偶、周期
11. 数列：$f(N) \rightarrow R$
12. 数列的收敛，极限运算：$\exists a \in R. \forall \varepsilon > 0, \exists N \in N^+, n > N \rightarrow |a_n - a| < \varepsilon$.记$\lim_{n \rightarrow \infty} a_n = a$
13. 夹逼定理：$a_n \rightarrow a \wedge b_n \rightarrow a \wedge a_n \leqslant c_n \leqslant b_n \Rightarrow c_n \rightarrow a$.
14. 运算法则

$$\begin{gathered} a_n, b_n\text{全为收敛数列} \\ \lim a_n \pm b_n = \lim a_n \pm b_n \\ \lim a_n \times b_n = \lim a_n \times \lim b_n \\ b_n \neq 0 \wedge b \neq 0 \rightarrow \lim \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n} \end{gathered}$$

15. 数列与子列：数列. 收敛$\Leftrightarrow$任何子列收敛
16. 任何数列都存在单调子列.(有最大项则存在递减数列，无最大项存在递增数列)
17. 致密性定理：任何有界数列必有收敛子列
18. Cauchy：数列. 收敛$\Leftrightarrow \forall \varepsilon > 0, \exists N \in N^+, m, n > N \rightarrow |a_n - a_m| < \varepsilon$

2 Tricks

1. $|q| < 1, \lim q^n = 0$. $$\begin{gathered} h = \frac{1}{|q|} - 1, h > 0. \\ |q^n| = \frac{1}{(1+h)^n} \leqslant \frac{1}{1+nh} < \frac{1}{nh} < \varepsilon \end{gathered}$$

$$a>0,\lim\sqrt[n]{a}=1$$

2. $h=a^{1/n}-1\to a=(h+1)^n\geqslant 1+nh=1+n(a^{1/n}-1)$
$$\frac{a-1}{n}\geqslant a^{1/n}-1\to\sqrt[n]{a}-1\leqslant\frac{a-1}{n}<\varepsilon$$

3. $\sum_m^n q^i=\frac{q^{n+1}-q^m}{1-q}$

4. $\sum_m^n (a+bi)q^i=qS-S=(a+nb)q^{n+1}-(a+mb)q^i+b\,\frac{q^{n+1}-q^{m+1}}{1-q}$

5. $\prod_0^{n-1}(a_0+i)=\Gamma(1+a_0+n)$

6. $\prod_1^n(a+bi)=b^n\prod_1^n\frac{\frac{a}{b}+i}{i}=b^n\frac{\Gamma\left(1+\frac{a}{b}+n\right)}{\Gamma\left(1+\frac{a}{b}\right)}$

7. $\prod_1^n\frac{b\times i+a}{b\times i+c}=\frac{b^n}{b^n}\cdot\frac{\Gamma\left(1+\frac{a}{b}+n\right)}{\Gamma\left(1+\frac{a}{b}\right)\times\Gamma(1+n)}\cdot\frac{\Gamma\left(1+\frac{a}{b}\right)\times\Gamma(1+n)}{\Gamma\left(1+\frac{c}{b}+n\right)}=\frac{\Gamma\left(1+\frac{a}{b}+n\right)}{\Gamma\left(1+\frac{c}{b}+n\right)}$

8. $\prod_1^n\frac{b\times i-a}{b\times i}=\frac{b^n}{b^n}\cdot\frac{\Gamma\left(1+\frac{a}{b}+n\right)}{\Gamma\left(1+\frac{a}{b}\right)\times\Gamma(1+n)}\sim\frac{\Gamma\left(1+\frac{a}{b}+n\right)}{\Gamma(1+n)}$

9. Stirling公式: $\Gamma(x+1)\sim\left(\frac{x}{e}\right)^x\sqrt{2\pi x}$

10.
$$\begin{gathered}
\lim\left(1+\frac{1}{n}\right)^n<3\\
a_n=\left(1+\frac{1}{n}\right)^n.\\
a_n=1+C_n^1\frac{1}{n}+C_n^2\frac{1}{n^2}+\cdots+C_n^n\frac{1}{n^n}\\
=1+1+\frac{n(n-1)}{2!}\frac{1}{n^2}+\cdots+\frac{n(n-1)\cdots 1}{n!}\frac{1}{n^n}\\
=2+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)\\
<2+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\cdots+\frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right)\\
=a_{n+1}\\
\to a_n\nearrow\\
a_n<2+\frac{1}{2!}+\cdots+\frac{1}{n!}<2+\frac{1}{1\times 2}+\frac{1}{(n-1)n}\\
=2+1-\frac{1}{n}<3
\end{gathered}$$

11. $\left(1+\frac{1}{n}\right)^{n+1}<e,\left(1+\frac{1}{n}\right)^{n+1}\searrow$

12. 单调增函数的反函数也是单调增