Algebra

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1 Numbers and Sets

1.1 Sets

Definition 1.1 (Set(Class))**.** .

Symbol 1.1.1. $\in, \subseteq, \subset, [A, B](\cap), \wedge(\cup)$

1.2 Mappings. Cardinality

Definition 1.2 (Function)**.** $\forall a \in \mathcal{M}, \exists$only one$\phi(a) \in \mathcal{N}$.

$$\begin{aligned} \text{image: } & \{\phi(a) : a \in \mathcal{M}\} \\ \text{preimage: } & \{a : \exists \phi(a) \in \text{image}(\phi)\} \\ \text{surjective(onto): } & \forall y \in \text{image}(\phi), \exists a \to \phi(a) = y \\ \text{injective(one-one): } & \phi(a) = \phi(b) \to a = b \\ \text{inverse: } & \phi^{-1}\phi = I_{\mathcal{M}}; \phi\phi^{-1} = I_{\mathcal{N}} \end{aligned}$$

Theorem 1.2.1 (Bernstein)**.**

$$|\mathcal{M}| = |\mathcal{N}| := \exists \text{bijective } \phi : \mathcal{M} \to \mathcal{N}$$

We say $\mathcal{M}$ and $\mathcal{N}$ are equipotent or have same cardinality.

1.3 The Number Sequence

Axiom 1.3.1 (Peano)**.**

$$\begin{aligned} & 1 \text{ is a natural number.} \\ & \forall \text{natural number } a \text{ has a definite successor } a^+ \in N \\ & \forall a \in N, a^+ \neq 1 \\ & \forall a, b \in N, a^+ = b^+ \to a = b \\ & X \subseteq N.1 \in X, \forall a \in X, a^+ \in X \to X = N \end{aligned}$$

The five axiom is called the principle of mathematical induction.

Sum of two numbers.

Definition 1.3 (Addition $+$)**.**

$$+(x, y) := \begin{cases} \forall x, x + 1 = x^+ \\ \forall x, y, x + y^+ = (x + y)^+ \end{cases}$$

Proposition 1.3.1 (Addition)**.**

$$\begin{aligned} \text{Associative :} & (a + b) + c = a + (b + c) \\ \text{Commutative :} & a + b = b + a \\ \text{Cancellative :} & a + b = a + c \to b = c \end{aligned}$$

Definition 1.4 (Product $\cdot$)**.**

$$\cdot(x,y) := \begin{cases} \forall x, x \cdot 1 = x \\ \forall x, y, x \cdot y^{+} = x \cdot y + x \end{cases}$$

Proposition 1.3.2 (Product)**.**

$$\begin{aligned} \text{Associative :} & (a \cdot b) \cdot c = a \cdot (b \cdot c) \\ \text{Commutative :} & a \cdot b = b \cdot a \\ \text{Cancellative :} & a \cdot b = a \cdot c \to b = c \end{aligned}$$

Proposition 1.3.3 (Addition and Product, Distributive)**.**

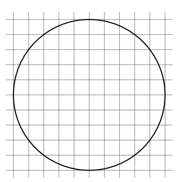
$$a \cdot (b + c) = a \cdot b + a \cdot c$$

Definition 1.5 (Order, Greater, Less)**.** $a > b := \exists u \in N, a = b + u$.

Proposition 1.3.4 (Order)**.**

$$\begin{aligned} & \forall a, b, \text{only one relation } a < b, a = b, a > b \text{ hold.} \\ & a < b \wedge b < c \to a < c \\ & a < b \to a + c < b + c \\ & a < b \to ac < bc \end{aligned}$$

Theorem 1.3.1. Every nonempty set of natural numbers contains a least number.



1.4 Finite and Countable (Denumerable) Sets

1.5 Partitions