Chapter3

BY 数列极限

1 Def

定义 1. *数列:* 函数$f: D \to R; D = N \vee D = N^+$

定义 2. *数列极限 lim:*

$$\forall \varepsilon > 0, \exists N \in N, n > N \to |a_n - a| < \varepsilon \colon \lim_{n \to \infty} a_n = a$$
$$\forall \varepsilon > 0, \operatorname{card}(\{a_n\} \cap (U_a(\varepsilon))^c) \in N^+$$

定义 3. *无穷小数列:* $\lim a_n = 0$

定理 4. *数列.* $\lim a_n = a \Leftrightarrow \lim (a_n - a) = 0$

推论 5. *收敛数列的性质*

1. *唯一性*
2. *有界性:* $\forall a_n, |a_n| < M$
3. *保号，保不等式*
4. *迫敛性:* $a_n \leqslant x_n \leqslant b_n \wedge a_n \to a \wedge b_n \to a \Rightarrow x_n \to a$

定义 6. *数列极限与基本运算*

$$a_n \to a; b_n \to b$$
$$\begin{array}{ll} 1 & \lim_{n \to \infty} (a_n \pm b_n) = \lim_{n \to \infty} a_n \pm \lim_{n \to \infty} b_n \\ 2 & \lim_{n \to \infty} a_n b_n = \lim_{n \to \infty} a_n \times \lim_{n \to \infty} b_n \\ 3 & b_n \neq 0 \wedge b \neq 0. \lim_{n \to \infty} \dfrac{a_n}{b_n} = \dfrac{\lim a_n}{\lim b_n} \end{array}$$

定理 7. *数列.* 收敛$\Leftrightarrow$任意子列收敛

定理 8. *数列 单调有界原理. 单调有界，必有极限*

定理 9. *任何数列都有单调子列.(选择公理。有上界必有单调减子列，无上界必有单调增子列)*

定理 10. *致密性定理：有界数列必有收敛子列*

定理 11. *Cauchy准则:* $a_n \to a \Leftrightarrow \forall \varepsilon > 0, \exists N \in N^+, \forall n, m > N \to |a_n - a_m| < \varepsilon$

定理 12. *Stolz定理*

$$
\begin{aligned}
& \text{单调增数列} y_n \to \infty \wedge \lim_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} = A (A \in R \cup \pm\infty) \\
\Rightarrow \quad & \lim_{n\to\infty} \frac{x_n}{y_n} = A \\
\text{Pr} \quad & n > N \to \left| \frac{x_n - x_{n-1}}{y_n - y_{n-1}} - A \right| < \frac{\varepsilon}{2} \\
\to \quad & \left| \frac{x_{N+1} - x_N + \cdots + x_{n+1} - x_n}{y_{N+1} - y_N + \cdots + y_n - y_{n-1}} \right| = \left| \frac{x_n - x_N}{y_n - y_N} - A \right| < \frac{\varepsilon}{2} \\
\to \quad & \frac{x_n}{y_n} - A = \frac{x_N - A y_N}{y_n} + \left(1 - \frac{y_N}{y_n}\right)\left(\frac{x_n - x_N}{y_n - y_N} - A\right) \\
& y_n \to \infty \to \frac{x_n}{y_n} - A = 0 + 1 \cdot A - A = 0 \\
& \to \left| \frac{x_n}{y_n} - A \right| < \varepsilon
\end{aligned}
$$

2 Tricks

1. $\sqrt[n]{n}$. $\sqrt[n]{n} - 1 = h_n$. $(h_n+1)^n > \frac{n(n-1)}{2} h_n^2 \Rightarrow 0 < h_n < \sqrt{\frac{2}{n-1}} \Rightarrow 1 \leqslant a_n = 1 + h_n \leqslant 1$

2.

$$
\begin{gathered}
\lim \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} \\
2 = \frac{1+3}{2} \geqslant \sqrt{1 \cdot 3} \\
4 = \frac{3+5}{2} \geqslant \sqrt{3 \cdot 5} \\
\cdots \\
2n = \frac{2n-1+2n+1}{2} \geqslant \sqrt{(2n-1)(2n+1)} \\
\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} \leqslant \frac{1 \cdot 3 \cdot 5 \cdots 2n-1}{\sqrt{1}\sqrt{3}\sqrt{3}\sqrt{5} \cdots \sqrt{2n-1}\sqrt{2n+1}} \\
= \frac{1}{\sqrt{2n+1}} \\
0 \leqslant a_n \leqslant \frac{1}{\sqrt{2n+1}} \Rightarrow a_n \to 0 \\
(2n-1)!! = 2^n \left(\frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \cdots \cdot \frac{1}{2} + n\right) = 2^n \Gamma\left(\frac{1}{2} + n\right) \\
(2n)!! = 2^n n! = 2^n \Gamma(1+n) \\
\Rightarrow \text{Ori} = \frac{\Gamma\left(\frac{1}{2} + n\right)}{\Gamma(1+n)} \\
\text{Stirling公式:} \frac{\sqrt{2\pi\left(n - \frac{1}{2}\right)} \cdot \left(\frac{n-1/2}{e}\right)^{n-1/2}}{\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n} = \frac{e^{1/2-n} \cdot (n-1/2)^n}{e^{-n} \cdot n^{n+\frac{1}{2}}} \\
\Rightarrow 0 = \frac{1}{\sqrt{n - \frac{1}{2}}} = \frac{\left(n - \frac{1}{2}\right)^n}{\left(n - \frac{1}{2}\right)^{n+\frac{1}{2}}} \leqslant e^{1/2} \frac{\left(n - \frac{1}{2}\right)^n}{n^{n+\frac{1}{2}}} \leqslant \sim \frac{n^n}{n^{n+\frac{1}{2}}} = \sim \frac{1}{\sqrt{n}} \to 0 \\
\Rightarrow \text{Oir} \to 0 \\
\text{一般的}\Gamma\text{函数} \prod \left(\frac{a+bi}{c+di}\right) = \frac{b^n \cdot \Gamma\left(1 + \frac{a}{b} + n\right)}{d^n \cdot \Gamma\left(1 + \frac{c}{d} + n\right)}
\end{gathered}
$$

3.

$$
\begin{array}{rc}
& \lim (1+a^{2^0})(1+a^{2^1})\cdots(1+2^{2^n}).\ |a|<1 \\
\Rightarrow & \frac{1}{1-a}(1-a)\cdot(1+a^{2^0})\cdots \\
= & \frac{1}{1-a}(1-a^2)(1+a^2)\cdots \\
= & \frac{1}{1-a}(1-a^{2^2})\cdots \\
= & \frac{1}{1-a}(1-a^{2^{n+1}}) \\
& \lim a_n = \lim \frac{1}{1-a}(1-a^{2^{n+1}}) = \frac{1}{1-a}
\end{array}
$$

4.

$$
a_{n+1} = \sqrt{p\cdot a_n} \to a_i < \sqrt{p} \to a_{i+1} < \sqrt{p} \to a_n < \sqrt{p}
$$
$$
\frac{a_{n+1}}{a_n} = \frac{\sqrt{p a_n}}{a_n} = \sqrt{p}\frac{1}{\sqrt{a_n}} > \frac{\sqrt{p}}{\sqrt{p}} = 1 \to a_n \nearrow
$$

5.

$$
a_{n+1} = \sqrt{c+a_n}
$$
$$
a_1 < \sqrt{c}+1; a_{n+1} = \sqrt{c+a_n} < \sqrt{c+\sqrt{c}+1}
$$