Chapter 6

BY 微分中值定理

1 Def

费马

$$f'(x_0)\text{存在}.f\text{在}x_0\text{处取得极值点}\rightarrow f'(x_0)=0$$

$$\begin{aligned}\text{Pr}\quad &\forall x\in U_{x_0}^+(\delta),x>x_0\rightarrow\frac{f(x)-f(x_0)}{x-x_0}\leqslant 0\\ &\forall x\in U_{x_0}^-(\delta),x<x_0\rightarrow\frac{f(x)-f(x_0)}{x-x_0}\geqslant 0\\ \rightarrow\quad &f'(x)\text{存在}\rightarrow\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=0\end{aligned}$$

罗尔

$$\begin{array}{ll}1 & f\text{在闭区间}[a,b]\text{连续}\\ 2 & f\text{在开区间}(a,b)\text{可导}\quad\rightarrow\quad\exists\xi\in(a,b)\wedge f'(\xi)=0\\ 3 & f(a)=f(b)\end{array}$$

$$\text{Pr}\quad f\not\equiv 0;f(x_0)>0.M=\max\{f(x_0)\}\text{存在}.\text{由费马定理：}f'(x_0)=0$$

拉格朗日

$$\text{Def}\quad\begin{array}{l}f\text{在闭区间}[a,b]\text{上连续}\quad\rightarrow\quad\exists\xi\in(a,b)\wedge f'(\xi)=\dfrac{f(b)-f(a)}{b-a}\\ f\text{在开区间}(a,b)\text{上可导}\end{array}$$

$$\text{Pr}\quad F(x)=f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a).\text{使用罗尔定理}$$

$$\begin{array}{lc}\text{推论1} & f'(I)\equiv 0\rightarrow f\text{在}I\text{上为常数}\\ \text{Pr} & \forall x\in I,f(x)=f(x_0)+f'(\xi)(x-x_0)=f(x_0)\\ \text{推论2} & f,g\text{在}I\text{上可导},\forall x\in I,f'(x)\equiv g'(x)\rightarrow f-g\equiv c.\\ \text{Pr} & \text{对}f-g\text{使用推论1}\end{array}$$

$$\text{推论3（导数极限定理）}\quad\begin{array}{l}f\text{在}U_{x_0}(\delta)\text{上连续}\\ \text{在}U_{x_0}^0(\delta)\text{上可导}\\ \text{极限}\lim\limits_{x\to x_0}f'(x)\text{存在}\end{array}\quad\rightarrow\quad f\text{在}x_0\text{可导}\wedge f'(x_0)=\lim_{x\to x_0}f'(x)$$

$$\begin{array}{lc}\text{Pr} & \forall x\in U_{x_0}^+,f\text{在}[x_0,x]\text{上满足拉格朗日条件}.\\ \rightarrow & f'(\xi)=\dfrac{f(x)-f(x_0)}{x-x_0}\\ & \lim\limits_{x\to x_0}\dfrac{f(x)-f(x_0)}{x-x_0}=\lim\limits_{x\to x_0}f'(x_0+\theta(x-x_0))=f_+'(x_0)\\ & \text{同理}.f_-'(x_0)=\lim\limits_{x\to x_0}f'(x_0-\theta(x-x_0))\\ & \lim\limits_{x\to x_0}f'(x_0)\text{存在}\Leftrightarrow f_+'(x_0)=f_-'(x_0)\\ \rightarrow & \text{(可导充要条件)}\rightarrow f'(x_0)=\lim\limits_{x\to x_0}f'(x)\end{array}$$

区间上可导函数与单调性的关系

区间上的可导函数：单调增$\Leftrightarrow$导数$\geqslant 0$

开区间可导函数：严格增$\Leftrightarrow\forall x\in(a,b),f'(x)\geqslant 0\wedge\forall(p,q)\subset(a,b),f'((p,q)\not\equiv 0$

Remark: 这暗示了一维实数空间上开集的本质结构

达布定理(导数的介值性)

$$\left.\begin{array}{l} f\text{在}[a,b]\text{可导} \\ f'_+(a)\neq f'_-(b) \\ c\in(f'_+(a),f'_-(b)) \end{array}\right. \to \exists\xi\in(a,b)\wedge f'(\xi)=c$$

Pr $\exists\xi\in[a,b], F(\xi)$是最大值.

$$F(x)=f-cx.F\text{在}[a,b]\text{上可导}$$
$$F'_+(a)\cdot F'_-(b)<0$$
$\to$ $\exists x_1, F(x_1)>F(a);\exists x_2, F(x_2)>F(b)$

$\to$ a,b不是F的最值点$\to\xi$是最值点$\to F'(\xi)=0$

$\to$ $f'(\xi)=c$

推论 f在区间I上$f'(x)\neq 0\to f$在I上严格单调

柯西中值定理

$$f,g\quad \left.\begin{array}{l} \text{在}[a,b]\text{上连续} \\ \text{在}(a,b)\text{上可导} \\ f',g'\text{不同时为}0 \\ g(a)\neq g(b) \end{array}\right. \to \exists\xi\in(a,b),\frac{f'(\xi)}{g'(\xi)}=\frac{f(b)-f(a)}{g(b)-g(a)}$$

Pr $F(x)=f(x)-f(a)-\dfrac{f(b)-f(a)}{g(b)-g(a)}(g(x)-g(a))$.使用罗尔定理

洛必达法则

$$\frac{0}{0}\quad f,g\quad \left.\begin{array}{l} \lim\limits_{x\to x_0}f(x)=\lim\limits_{x\to x_0}g(x)=0 \\ \exists U^0_{x_0}(\delta),f,g\text{可导}\wedge g'(x)\neq 0 \\ \lim\limits_{x\to x_0}\dfrac{f'(x)}{g'(x)}=A.(A\in R\cup\pm\infty\}) \end{array}\right. \to \lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}=A$$

Pr 补充x_0处f和g的定义. $f(x_0)=0;g(x_0)=0$

在$[x_0,x];[x,x_0]$上使用柯西定理$\dfrac{f(x)}{g(x)}=\dfrac{f(x)-f(x_0)}{g(x)-g(x_0)}=\dfrac{f'(\xi)}{g'(\xi)}$.取极限得结论

扩展 $x\to x_0^+;x\to x_0^-;x_0\to+\infty;x_0\to-\infty$时，上述证法无影响

$$\text{在}U^0_{x_0^+}\text{上可导}\wedge g'(x)\neq 0$$

$\dfrac{\cdot}{\infty}$ $\quad f,g \quad \lim\limits_{x\to x_0} g(x)=\pm\infty \quad \to \lim\limits_{x\to x_0^+}\dfrac{f(x)}{g(x)}=A$

$$\lim_{x\to x_0^+}\frac{f'(x)}{g'(x)}=A.\,(A\in R\cup\{\pm\infty\})$$

Pr $\quad A\in R.\forall\varepsilon>0, x_1\in U^0_{x_0^+},\forall x\in(x_0,x_1)\to A-\dfrac{\varepsilon}{2}<\dfrac{f'(x)}{g'(x)}<A+\dfrac{\varepsilon}{2}$

$$f,g\text{在}[x_0,x_1]\text{上满足柯西条件}$$

$\to \quad \exists\xi\in(x,x_1)\subset(x_0,x_1)\to A-\dfrac{\varepsilon}{2}<\left(\dfrac{f(x)}{g(x)}-\dfrac{f(x_1)}{g(x)}\right)\left(\dfrac{g(x)}{g(x)-g(x_1)}\right)$

$$=\frac{f(x)-f(x_1)}{g(x)-g(x_1)}=\frac{f'(\xi)}{g'(\xi)}<A+\frac{\varepsilon}{2}$$

$$\lim_{x\to x_0}\frac{g(x)}{g(x)-g(x_1)}=1\to\exists U^+_{x_1}(\delta_1),\forall x\in\sim,g(x)>0$$

$\to \quad \left(1-\dfrac{g(x_1)}{g(x)}\right)\left(A-\dfrac{\varepsilon}{2}\right)+\dfrac{f(x_1)}{g(x)}<\dfrac{f(x)}{g(x)}<\left(1-\dfrac{g(x_1)}{g(x)}\right)\left(A+\dfrac{\varepsilon}{2}\right)+\dfrac{f(x_1)}{g(x)}$

$$\lim_{x\to x_0}g(x)=\infty\to\lim_{x\to x_0}\left(1-\frac{g(x_1)}{g(x)}\right)=1;\ \lim_{x\to x_0}\frac{f(x_1)}{g(x)}=0$$

$\to \quad A-\dfrac{\varepsilon}{2}<\dfrac{f(x)}{g(x)}<A+\dfrac{\varepsilon}{2}$

$\to \quad \lim\limits_{x\to x_0^+}\dfrac{f(x)}{g(x)}=A$

Rem 使用乘积法则绕过$g\to\infty$的情况，在区间上找个点变成比值消去

扩展 $\quad x\to x_0^-;x\to x_0;x\to\pm\infty;x\to\infty$都可以使用上述证法.

注意 $\quad$ 若$\lim\limits_{x\to x_0}\dfrac{f'(x)}{g'(x)}$不存在，不能推出$\dfrac{f(x)}{g(x)}$不存在

ex $\quad \lim\limits_{x\to+\infty}\dfrac{x+\sin x}{x}=1+\dfrac{\sin x}{x}=1\neq\dfrac{1+\cos x}{1}=\text{DNE}.$

泰勒公式.在某一点上用多项式逼近函数

f在x_0处有n阶导数 $\to \quad f(x)=T_n(x)+o((x-x_0)^n)$

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)+o((x-x_0)^n)$$

Pr $\quad R_n=f(x)-T_n.\,Q_n(x)=(x-x_0)^n.$

$$\lim_{x\to x_0}\frac{R_n(x)}{Q_n(x)}=0$$

$$R_n(x_0)=R_n'(x_0)=\cdots=R^{(n)}(x_0)=0;$$

$$Q_n(x_0)=Q_n'(x_0)=\cdots=Q_n^{(n-1)}(x_0)=0,Q_n^{(n)}(x_0)=n!$$

$\to \quad \lim\limits_{x\to x_0}\dfrac{R_n}{Q_n}=\lim\limits_{x\to x_0}\dfrac{R^{(n-1)}(x)}{Q^{(n-1)}(x)}$

$$=\frac{f^{(n-1)}(x)-f^{(n-1)}(x_0)-f^{(n)}(x_0)(x-x_0)}{n!\,(x-x_0)}$$

$$=\frac{1}{n!}\lim_{x-x_0}\left(\frac{f^{(n-1)}(x)-f^{(n-1)}(x_0)}{x-x_0}-f^{(n)}(x_0)\right)$$

$$=0;$$

皮亚诺余项: $o((x-x_0)^n)$

泰勒定理

Def $\begin{matrix} f\text{在}[a,b]\text{上存在}n\text{阶连续导函数} \\ f\text{在}(a,b)\text{上存在}(n+1)\text{阶导函数} \end{matrix} \Rightarrow \forall x, x_0 \in [a,b], \exists \xi \in (a,b) \to f(x) = T_n(x) + R_n(x)$

余项 $R_n =$

- 拉格朗日 $\dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$
- 柯西 $\dfrac{f^{(n+1)}(x_0+\theta(x-x_0))}{n!}(1-\theta)^n(x-x_0)^{n+1}$
- 施勒米希－洛希 $\dfrac{f^{(n+1)}(x_0+\theta(x-x_0))}{n!\,p}(1-\theta)^{n+1-p}(x-x_0)^{n+1}$

Pr

$$F(t) = f(x) - \left[f(t) + \frac{f'(t)}{1!}(x-t) + \cdots + \frac{f^{(n)}(t)}{n!}(x-t)^n\right]$$

$$G(t) = (x-t)^{n+1}$$

柯西中值定理：$\dfrac{F(x_0)}{G(x_0)} = \dfrac{F(x_0)-F(x)}{G(x_0)-G(x)} = \dfrac{F'(\xi)}{G'(\xi)} = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}$. (分子求导后前后相消)

一元可导函数极值

def 极值: $\exists U_{x_0}, \forall x \in U_{x_0}, f(x) \leqslant f(x_0)$

必要 f在U_{x_0}可导，x_0是极值点$\Rightarrow f'(x_0) = 0$

充分 $f\ \begin{matrix} \text{在}x_0\text{连续} \\ \text{在}U^0_{x_0}\text{可导} \end{matrix} \wedge \begin{matrix} x \in (x_0-\delta, x_0), f'(x) \leqslant 0; x \in (x_0, x_0+\delta), f'(x) \geqslant 0 & \Rightarrow & f\text{在}x_0\text{极小} \\ x \in (x_0-\delta, x_0), f'(x) \geqslant 0; x \in (x_0, x_0+\delta), f'(x) \leqslant 0 & \Rightarrow & f\text{在}x_0\text{极大} \end{matrix}$

充分 $f\ \begin{matrix} \text{在}U_{x_0}\text{可导} \\ \text{在}x_0\text{二阶可导} \\ f'(x_0) = 0 \end{matrix} \wedge \begin{matrix} f''(x_0) < 0 & \Rightarrow & f\text{在}x_0\text{极大} \\ f''(x_0) > 0 & \Rightarrow & f\text{在}x_0\text{极小} \end{matrix}$

充分 $f\ \begin{matrix} \text{在}U_{x_0}\text{有}n-1\text{阶导数} \\ \text{在}x_0\text{处}n\text{阶可导} \\ \forall k \in 1\ldots n-1, f^{(k)}(x_0) = 0 \end{matrix} \wedge \begin{matrix} n\text{是偶数} & \begin{matrix} f^{(n)}(x_0) < 0 \Rightarrow f(x_0)\text{极大} \\ f^{(n)}(x_0) > 0 \Rightarrow f(x_0)\text{极小} \end{matrix} \\ n\text{是奇数} & f\text{在}x_0\text{不取极值} \end{matrix}$

凸函数

Def $f: I \to R. \forall x_1, x_2 \in I, \forall \lambda \in (0,1); f(\lambda x_1 + (1-\lambda)x_2) \leqslant \lambda f(x_1) + (1-\lambda) f(x_2)$

充要 f是I上的凸函数$\Leftrightarrow \forall x_1 < x_2 < x_3 \in I, \dfrac{f(x_2)-f(x_1)}{x_2-x_1} \leqslant \dfrac{f(x_3)-f(x_2)}{x_3-x_2}$.(差分递增)

充要 f是I上的凸函数$\Leftrightarrow \forall x_1 < x_2 < x_3 \in I,$

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \leqslant \frac{f(x_3)-f(x_1)}{x_3-x_1} \leqslant \frac{f(x_3)-f(x_2)}{x_3-x_2}.\text{(差分递增)}$$

可导 f凸$\Leftrightarrow f'$在I上单调增$\Leftrightarrow \forall x_1 < x_2 \in I, f(x_2) \geqslant f(x_1) + f'(x_1)(x_2-x_1)$

f'' f凸$\Leftrightarrow \forall x \in I, f''(x) \geqslant 0$

Jensen f是$[a,b]$上的凸函数, $\forall x_i \in [a,b], \lambda_i > 0 \wedge \sum \lambda_i = 1$

$\Rightarrow$ $f\left(\sum \lambda_i x_i\right) \leqslant \sum \lambda_i f(x_i)$

单导数 开区间上的凸函数在任意点都存在左右导数

Pr $\forall 0 < h_1 < h_2, \dfrac{f(x_0+h_1)-f(x_0)}{h_1} \leqslant \dfrac{f(x_0+h_2)-f(x_0)}{h_2}$

而h_1固定时, h_2的函数在$h_2 \to h_1$时有单调有下界$\to$右导数存在

Def 拐点$:=f$在x_0有穿过曲线的切线, 在x_0两侧分别是严格凸和严格凹的

必要 x_0是拐点$\Rightarrow f''(x_0) = 0$

2 Fomula

	一些函数的麦克劳林展开式
$(1+x)^\alpha$	$1+\frac{\alpha}{1!}x+\frac{\alpha(\alpha-1)}{2!}x^2+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n+o(x^n)$
e^x	$1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+o(x^n)$
$\ln(1+x)$	$x-\frac{x^2}{2}+\frac{x^3}{3}+\cdots+(-1)^{n-1}\frac{x^n}{n}+o(x^n)$
$\sin x$	$x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots+(-1)^{m-1}\frac{x^{2m-1}}{(2m-1)!}+o(x^{2m})$
$\cos x$	$1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots+(-1)^m\frac{x^{2m}}{(2m)!}+o(x^{2m+1})$
$\tan x$	$x+\frac{x^3}{3}+\frac{2x^5}{15}+\cdots+o(x^{2m+1})$
$\arcsin x$	$x+\frac{1}{6}x^3+\frac{3}{40}x^5+o(x^6)$
$\arccos x$	$\frac{\pi}{2}-x-\frac{1}{6}x^3-\frac{3}{40}x^5+o(x^6)$
$\arctan x$	$x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots+(-1)^{m-1}\frac{x^{2m-1}}{2m-1}+o(x^{2m})$
$\sinh x$	$x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots+\frac{x^{2m-1}}{(2m-1)!}+o(x^{2m})$
$\cosh x$	$1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots+\frac{x^{2m}}{(2m)!}+o(x^{2m+1})$
$\tanh x$	$x-\frac{x^3}{3}+\frac{2x^5}{15}+o(x^6)$
$\operatorname{arcsinh} x$	$x+\frac{x^3}{3!}+\cdots+\frac{x^{2m-1}}{(2m-1)!}+o(x^{2m})$
$\operatorname{arccosh} x$	$i\left(\frac{\pi}{2}-x-\frac{x^3}{6}-\frac{3x^5}{40}+o(x^6)\right)$
$\operatorname{arctanh} x$	$x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots+\frac{x^{2m-1}}{(2m-1)!}+o(x^{2m})$

3 Trick

1. $\sqrt{\xi}f'(\xi)=\frac{f(x)-f(y)}{\sqrt{x}-\sqrt{y}}$.证$\sqrt{x}f'(x)$的一致连续性

2. 不定式化为能使用洛必达的形式

$0\cdot\infty$	$f\cdot g\to\frac{f}{\frac{1}{g}}\in\frac{0}{0}\vee\frac{g}{\frac{1}{f}}=\frac{\infty}{\infty}$
1^∞	$e^{g\cdot\ln f}=e^{0\cdot\infty}$.根据连续性化为$0\cdot\infty$的形式
0^0	$e^{g\cdot\ln f}=e^{0\cdot\infty}$.根据连续性化为$0\cdot\infty$的形式
∞^0	$\ln f\cdot g=0\cdot\infty$. 根据连续性化为$0\cdot\infty$的形式
$\infty-\infty$	通分等手段化为一般式

3. 不能使用洛必达的例子

$$f=\begin{cases}\frac{g(x)}{x} & x\neq 0\\ 0 & x=0\end{cases}.\ g(0)=g'(0);\ g''(0)=3.\text{求}f'(0)$$

$$f'(0)=\lim_{x\to 0}\frac{g(x)}{x^2}.\ g'(x)\text{在0处连续}(g''\text{存在}),$$

$$g'(x)=0,(x^2)'(0)=0;\text{可以使用洛必达}$$

$$\to\lim_{x\to 0}\frac{g(x)}{x^2}=\lim_{x\to 0}\frac{g'(x)}{2x}$$

这里只有g''在0处存在，但不在0的任何领域内存在因此不能用洛必达

$$\to\lim_{x\to 0}\frac{g'(x)}{2x}=\frac{1}{2}\lim_{x\to 0}\frac{g'(x)-g'(0)}{x-0}=\frac{1}{2}f''(0)=\frac{3}{2}$$

4. 使用柯西中值定理处理对称差

f在a处二阶连续可导

$$\lim_{h\to 0}\frac{f(a+h)+f(a-h)-2f(a)}{h^2}=f''(a)$$

Pr $\quad F(x)=f(a+x)+f(a-x)-2f(a);G(x)=x^2$

F,G在$U_a(\delta)$上可导,在$U_a(\delta)$内闭可导$\to$在$[a,a+h]$上可导

$G'(x)=2x$在$(0,h)$内不为0, $G(0)\neq G(h)$

$$\to\quad\frac{F(h)-F(0)}{G(h)-G(0)}=\frac{F'(\xi)}{G'(\xi)}$$

$$F(h)=\sim;F(0)=0;G(0)=0;G(h)=h^2$$

$$F'(x)=f'(a+h)-f'(a-h);G'(x)=2x$$

$$\to\quad\frac{f(a+h)+f(a-h)-2f(a)}{h^2}=\frac{f'(a+\theta h)-f'(a-\theta h)}{2\theta h}$$

在$[a,a+\theta h]$上使用拉格朗日

$$\frac{f'(a+\theta h)-f'(a)+f'(a)-f'(a-\theta h)}{2\theta h}=\frac{1}{2}\left(\frac{f''(a+\eta_1 h)(\theta h)+f''(a-\eta_2 h)(\theta h)}{\theta h}\right)$$

$$=\frac{1}{2}(f''(a+\eta_1 h)+f''(a-\eta_2 h))$$

取极限：$\lim_{x\to x_0}\frac{f''(a+\eta_1 h)+f''(a-\eta_2 h)}{2}=\frac{f''_+(a)+f''_-(a)}{2}$

由于f''在a点连续$\to f''_-(a)=f''_+(a)=f''(a)$

$$\to\quad\frac{f''_+(a)+f''_-(a)}{2}=\frac{f''(a)+f''(a)}{2}=f''(a)$$

5. 处理对称差

$$\frac{f(a+h)+f(a-h)-2f(h)}{h^2}=\frac{f''(a+\theta h)+f''(a-\theta h)}{2}$$

$$\text{根据柯西定理：}\frac{f(a+h)+f(a-h)-2f(h)}{h^2}=\frac{f'(a+\eta h)-f'(a-\eta h)}{2\eta}$$

$$F(x)=f'(a+hx)-f'(a-hx);G(x)=2x$$

$$\frac{F(\eta)-F(0)}{G(\eta)-G(0)}=\frac{F'(\xi)}{G'(\xi)}$$

$$\frac{f'(a+\eta h)-f'(a-\eta h)}{2\eta}=\frac{f''(a+\xi h)+f''(a-\xi h)}{2}$$

6.

$$e\text{是无理数}$$

$$e=\sum_{i=0}^{n}\frac{1}{i!}+\frac{e^\theta}{(n+1)!}.\theta\in(0,1)$$

$$\to n!e-\left(n!+n!+\frac{n!}{2!}+\cdots+\frac{n!}{n!}\right)=\frac{e^\theta}{n+1}$$

$$e=\frac{p}{q},n>q\text{时},n!e\in Z^+,\frac{n!}{i!}\in Z^+$$

$$\to\frac{e^\theta}{n+1}\in Z$$

$$\text{而}\frac{e^\theta}{n+1}<\frac{e}{n+1}<\frac{3}{n+1},\text{在}n\geqslant 2\text{时为非整数}$$

$$\to e\notin Q$$

7. 根据已知展开式，求在其它点的展开式(这里没考虑泰勒级数的收敛性情况...)

$$\ln(x)\text{在}2\text{处的展开式}$$

$$\ln(x)=\ln(2+(x-2))=\ln\left(2\left(1+\frac{x-2}{2}\right)\right)=\ln2+\ln\left(1+\frac{x-2}{2}\right)$$

$$\to\ln x=\ln2+\left(1+\frac{x-2}{2}\right)-\frac{(1+\frac{x-2}{2})^2}{2}+\frac{(1+\frac{x-2}{2})^3}{3}+\cdots+(-1)^{n-1}\frac{\sim^n}{n!}+o(\sim^n)$$

8. 紧集上一元函数求最值$\begin{cases}f'=0\\ \text{不可导点}\\ \text{区间端点}\end{cases}$

9. 牛顿切线法求数值解

$$\begin{array}{ll}
 & f\text{是}[a,b]\text{上的二阶可导函数};\forall x\in I,\,f'(x)\cdot f''(x)\neq 0,\,f(a)\cdot f(b)<0\\
\text{理论依据} & \text{根据连通性，}f\text{在}(a,b)\text{必有解}\\
\text{构造依据} & f'\text{不为}0\Rightarrow f\text{严格单调}\\
 & f''\text{不为}0,\text{则必为凸或凹函数}\Rightarrow\text{切线在函数曲线单侧}\Rightarrow x_n>x_0\\
 & \text{压缩映像原理}\\
\text{Step} & x_0=a,x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}\\
\text{误差估计} & f(x_n)=f(x_n)-f(\xi)=f'(\eta)(x_n-\xi).x_n<\eta<\xi\\
 & x_n-\xi=\dfrac{f(x_n)}{f'(\eta)}\\
 & m=\min\limits_{x\in[a,b]}\{|f'(x)|\}\\
 & |x_n-\xi|\leqslant\dfrac{|f(x_n)|}{m}
\end{array}$$