第六章 内积空间

之前的讨论中推广了多维空间中的线性结构(加法和标乘)。但是忽略了不同向量之间的关系(角度)，向量本身的不变量(长度)。

1 内积与范数

1.1 内积

定义 1.1. *R^n上的范数（norm）*

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

定义 1.2. *R^n上的点积(dot product)*

$$\begin{gathered} x, y \in R^n, x \cdot y \colon (V \times V \to F).\, x = (x_1, \ldots, x_n); y = (y_1, \ldots, y_n) \\ x \cdot y = x_1 y_1 + \cdots + x_n y_n \end{gathered}$$

定理 1.3. *R^n上点积的一些性质*

$$\begin{array}{cc} 1 & \forall x \in R^n, x \cdot x = \|x\|^2 \\ 2 & \forall x \in R^n \to x \cdot x \geqslant 0 \\ 3 & x \cdot x = 0 \Leftrightarrow x = \mathbf{0} \\ 4 & \forall y \in R^n.\, f \colon R^n \to F, f(x) = x \cdot y \in \mathcal{L}(R^n, F) \\ 5 & \forall x, y \in R^n.\, x \cdot y = y \cdot x \end{array}$$

定义 1.4. *C^n上的范数*

$$z \in C^n, \|z\| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$$

Remark: $\|z\|^2 = z_1 \bar{z}_1 + \cdots + z_n \bar{z}_n$

定义 1.5. *内积(inner product)*

$$\begin{gathered} V\text{上的内积是一个函数}\langle\rangle.\langle\rangle \colon V^2 \to F \\ \langle\rangle(u, v) = u_1 \bar{v_1} + \cdots + u_n \bar{v_n} \\ \text{中缀形式}\langle u, v\rangle := \langle\rangle(u, v) \end{gathered}$$

定理 1.6. *内积公理*

$$\begin{array}{ccc} 1 & \text{正性} & \forall v \in V \to \langle v, v\rangle \geqslant 0 \\ 2 & \text{定性} & \langle v, v\rangle = 0 \Leftrightarrow v = \mathbf{0} \\ 3 & \text{第一个位置加性} & \langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle \\ 4 & \text{第一个位置齐性} & \forall \lambda \in F.\, \langle \lambda u, v\rangle = \lambda \langle u, v\rangle \\ 5 & \text{共轭对称性} & \langle u, v\rangle = \overline{\langle v, u\rangle} \end{array}$$

例 1.7. 一些内积

$$\begin{array}{cc} 1 & F^n\text{上的欧几里得内积}: \langle (w_1, \ldots, w_n), (z_1, \ldots, z_n)\rangle = w_1 \bar{z_1} + \cdots + w_n \bar{z_n} \\ 2 & c_1, \ldots, c_n > 0.\, F^n\text{上有内积}: \langle \boldsymbol{w}, \boldsymbol{z}\rangle = c_1 w_1 \bar{z_1} + \cdots + c_n w_n \bar{z_n} \\ 3 & [-1, 1]\text{上的所有实连续函数空间有内积}: \langle f, g\rangle = \int_{-1}^{1} f(x) g(x) \mathrm{d}x \\ 4 & \mathcal{P}(R)\text{上有内积}: \langle p, q\rangle = \int_0^{\infty} p(x) q(x) e^{-x} \mathrm{d}x \end{array}$$

定义 1.8. *内积空间(inner product space)*

$$\text{内积空间} := \text{带有内积的向量空间}V$$

定理 1.9. *内积的基本性质*

$$\begin{array}{cc} 1 & \forall u \in V.\, f\colon V \to F, f(v) = \langle v, u\rangle \in \mathcal{L}(V, F) \\ 2 & \forall u \in V \to \langle 0, u\rangle = 0 \\ 3 & \forall u \in V \to \langle u, 0\rangle = 0 \\ 4 & \forall u, v, w \in V \to \langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle \\ 5 & \forall \lambda \in F, \forall u, v \in V \to \langle u, \lambda v\rangle = \bar{\lambda}\langle u, v\rangle \end{array}$$

证明.

$$\begin{array}{cc} 1 & f(x+y) = \langle x + y, u\rangle = \langle x, u\rangle + \langle y, u\rangle = f(x) + f(y) \\ & f(\lambda x) = \langle \lambda x, u\rangle = \lambda\langle x, u\rangle = \lambda f(x) \\ 2 & f \in \mathcal{L}(V, F) \to f(0) = \langle 0, u\rangle = 0 \in F \\ 3 & \langle u, 0\rangle = \overline{\langle 0, u\rangle} = \bar{0} = 0 \\ 4 & \langle u, v + w\rangle = \overline{\langle v + w, u\rangle} = \overline{\langle v, u\rangle + \langle w, u\rangle} = \overline{\langle v, u\rangle} + \overline{\langle w, u\rangle} = \langle u, v\rangle + \langle u, w\rangle \\ 5 & \langle u, \lambda v\rangle = \overline{\langle \lambda v, u\rangle} = \overline{\lambda\langle v, u\rangle} = \bar{\lambda}\overline{\langle v, u\rangle} = \bar{\lambda}\langle u, v\rangle \end{array}$$

□

定义 1.10. *范数(norm).* $\|v\|$

$$\forall v \in V, v\text{的范数}: \|v\| = \sqrt{\langle v, v\rangle}$$

例 1.11. 一些范数

1. $\boldsymbol{z} \in F^n$. $\|z\| = \|(z_1, \ldots, z_n)\| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$
2. $[-1, 1]$上的实连续函数: $\|f\| = \sqrt{\int_{-1}^{1} (f(x))^2 \mathrm{d}x}$

定理 1.12. *范数的基本性质*

$$\begin{array}{cc} & v \in V \\ 1 & \|v\| = 0 \Leftrightarrow v = 0 \\ 2 & \forall \lambda < F, \|\lambda v\| = |\lambda|\, \|v\| \end{array}$$

证明.

$$\begin{array}{cl} 1 & \|v\| = 0 \Leftrightarrow \sqrt{\langle v, v\rangle} = 0 \Leftrightarrow \langle v, v\rangle = 0 \Leftrightarrow v = 0 \\ 2 & \|\lambda v\|^2 = \langle \lambda v, \lambda v\rangle \\ & \quad = \lambda\langle v, \lambda v\rangle \\ & \quad = \lambda\bar{\lambda}\langle v, v\rangle \\ & \quad = |\lambda|^2 \|v\|^2 \\ & \Leftrightarrow \|\lambda v\| = |\lambda| \|v\| \end{array}$$

□

定义 1.13. *正交(orthogonal)*

$$u, v \in V. u, v\text{称为相互正交的} := \langle u, v\rangle = 0$$

Remark: 在欧氏空间中的内积$\langle u, v\rangle = \|u\|\, \|v\| \cos\theta$,因此在欧氏空间中正交表示两向量垂直

定理 1.14. **0**的正交性

$$\begin{array}{cc} 1 & \forall v \in V, \mathbf{0} \perp v \\ \to & \mathbf{0} \perp \mathbf{0} \end{array}$$

证明.

$$\begin{gathered}\forall v\in V,\langle \mathbf{0},v\rangle=0\to\mathbf{0}\perp v\\ v\in V\wedge\langle v,v\rangle=0\to v=\mathbf{0}\to\mathbf{0}\perp\mathbf{0}\end{gathered}$$

□

定理 1.15. *勾股定理*

$$u,v\in V,u\perp v\to\|u+v\|^2=\|u\|^2+\|v\|^2$$

证明.

$$\begin{gathered}\|u+v\|^2=\langle u+v,u+v\rangle\\ =\langle u,u+v\rangle+\langle v,u+v\rangle\\ =\langle u,u\rangle+\langle u,v\rangle+\langle v,u\rangle+\langle v,v\rangle\\ =\|u\|^2+\|v\|^2+\langle u,v\rangle+\langle v,u\rangle\\ =\|u\|^2+\|v\|^2\end{gathered}$$

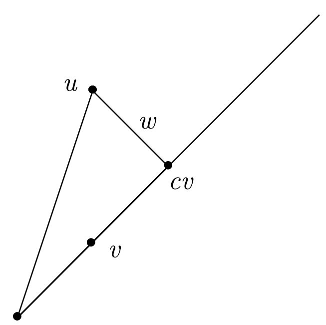
□

remark: 上述证明表明，勾股定理 $\Leftrightarrow\langle u,v\rangle+\langle v,u\rangle=0\Leftrightarrow 2\mathrm{Re}\langle u,v\rangle=0$. *因此逆命题在实内积空间成立*

定理 1.16. *向量的正交分解*

$$\begin{gathered}u,v\in V\wedge v\neq 0.c=\frac{\langle u,v\rangle}{\|v\|^2},w=u-\frac{\langle u,v\rangle}{\|v\|^2}v\\ \to\langle w,v\rangle=0\wedge u=cv+w\end{gathered}$$

证明.



$$\begin{gathered}u=cv+(u-cv)\\ 0=\langle u-cv,v\rangle=\langle u,v\rangle-\langle cv,v\rangle\\ =\langle u,v\rangle-c\|v\|^2\\ \to c=\tfrac{\langle u,v\rangle}{\|v\|^2}\\ \to u=\tfrac{\langle u,v\rangle}{\|v\|^2}v+\left(u-\tfrac{\langle u,v\rangle}{\|v\|^2}v\right)\end{gathered}$$

□

定理 1.17. *柯西-施瓦茨不等式*

$$u,v\in V.\ |\langle u,v\rangle|\leqslant\|u\|\,\|v\|.\text{等号成立}\Leftrightarrow u=\lambda v$$

证明.

$$\begin{aligned}
v=0\to\langle u,v\rangle=0\wedge\|v\|=0\to\text{等式成立}\\
v\neq 0\to u=\tfrac{\langle u,v\rangle}{\|v\|^2}v+w,w\perp v\\
\|u\|^2=\left\|\tfrac{\langle u,v\rangle}{\|v\|^2}v\right\|^2+\|w\|^2\\
=\left\langle\tfrac{\langle u,v\rangle}{\|v\|^2}v,\tfrac{\langle u,v\rangle}{\|v\|^2}v\right\rangle+\|w\|^2\\
=\left(\tfrac{\langle u,v\rangle}{\|v\|^2}\right)^2\|v\|^2+\|w\|^2\\
=\tfrac{|\langle u,v\rangle|^2}{\|v\|^2}+\|w\|^2\\
\geqslant\tfrac{|\langle u,v\rangle|^2}{\|v\|^2}\\
\to\|u\|^2\|v\|^2\geqslant|\langle u,v\rangle|^2\\
\to\|u\|\,\|v\|\geqslant|\langle u,v\rangle|
\end{aligned}$$

Remark: 若等式成立，需要$w=0$即u向量到v的正交分解没有其它分量。即u是v的标量倍 □

例 1.18. 柯西-施瓦茨不等式的例子

$$\begin{array}{ll}
1 & x_1,\ldots,x_n;y_1,\ldots,y_n\in R\\
 & |x_1y_1+\cdots+x_ny_n|^2\leqslant(x_1^2+\cdots+x_n^2)(y_1^2+\cdots+y_n^2)\\
 & |x_1y_1+\cdots+x_ny_n|\leqslant\sqrt{x_1^2+\cdots+x_n^2}\cdot\sqrt{y_1^2+\cdots+y_n^2}\\
2 & f,g\text{是}[-1,1]\text{上的实连续函数}\\
 & \left|\int_{-1}^1 f(x)g(x)\mathrm{d}x\right|\leqslant\sqrt{\int_{-1}^1(f(x))^2\mathrm{d}x}\cdot\sqrt{\int_{-1}^1(g(x))^2\mathrm{d}x}\\
 & \left|\int_{-1}^1 f(x)g(x)\mathrm{d}x\right|^2\leqslant\int_{-1}^1(f(x))^2\mathrm{d}x\cdot\int_{-1}^1(g(x))^2\mathrm{d}x
\end{array}$$

定理 1.19.

$$u,v\in V.\ \|u+v\|\leqslant\|u\|+\|v\|.\text{等号成立}\Leftrightarrow u,v\text{是之一是另一个的非负标量倍}$$

证明.

$$\begin{aligned}
\|u+v\|^2=\langle u+v,u+v\rangle\\
=\langle u,u\rangle+\langle v,v\rangle+\langle u,v\rangle+\langle v,u\rangle\\
=\langle u,u\rangle+\langle v,v\rangle+2\mathrm{Re}\langle u,v\rangle\\
\leqslant\langle u,u\rangle+\langle v,v\rangle+2|\langle u,v\rangle|\\
\leqslant\|u\|^2+\|v\|^2+2\|u\|\,\|v\|\\
=(\|u\|+\|v\|)^2\\
\to\|u+v\|\leqslant\|u\|+\|v\|
\end{aligned}$$

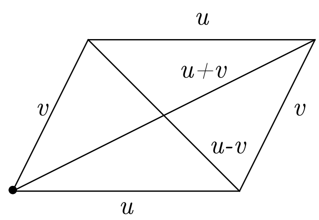
Remark: 上面证明表示等号成立$\Leftrightarrow\langle u,v\rangle=\|u\|\,\|v\|$ □

定理 1.20. *平行四边形恒等式*

$$u,v\in V\to\|u+v\|^2+\|u-v\|^2=2(\|u\|^2+\|v\|^2)$$

证明.

$$\begin{aligned}
\|u+v\|^2+\|u-v\|^2=\langle u+v,u+v\rangle+\langle u-v,u-v\rangle\\
=\|u\|^2+\|v\|^2+\langle u,v\rangle+\langle v,u\rangle+\|u\|^2+\|v\|^2-\langle u,v\rangle-\langle v,u\rangle\\
=2(\|u\|^2+\|v\|^2)
\end{aligned}$$



□

习题6.A

2 规范正交基

定义 2.1. *规范正交的(orthonormal)*

$$\text{向量组的规范正交}\quad \text{向量组}\boldsymbol{v}.\forall i\in 1\ldots n\to\|v_i\|=1\text{且}\forall i,j,i\neq j\to\langle v_i,v_j\rangle=0$$
$$\langle e_i,e_j\rangle=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$$

例 2.2. 一些规范正交组

1	F^n的标准基是规范正交组
2	$\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right),\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)$是$F^3$中的规范正交组
3	$\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right),\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right),\left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}}\right)$是$F^3$中的规范正交组

定理 2.3. *规范正交组的范数*

$$e_1,\ldots,e_n\text{是}V\text{中的规范正交组}\to\forall a_1,\ldots a_n\in F\to\|a_1e_1+\cdots+a_ne_n\|^2=|a_1|^2+\cdots+|a_n|^2$$

证明.

$$\begin{aligned}\|a_1e_1+a_2e_2+\cdots+a_ne_n\|^2&=\|a_1e_1\|^2+\|a_2e_2+\cdots+a_ne_n\|^2\\&=|a_1|^2\|e_1\|+\cdots+\|a_2e_2+\cdots+a_ne_n\|^2\\&=|a_1|^2+\|a_2e_2+\cdots+a_ne_n\|^2\\&\to=|a_1|^2+\cdots+|a_n|^2\end{aligned}$$

□

定理 2.4. *规范正交组是线性无关的*

证明.

$$
\begin{aligned}
&e_1,\ldots,e_n\text{是}V\text{中的规范正交组}, a_1,\ldots,a_n\in F\\
&a_1e_1+\cdots+a_ne_n=\mathbf{0}\\
&\to\|\mathbf{0}\|^2=\|a_1e_1+\cdots+a_ne_n\|^2\\
&\to|a_1|^2+\cdots+|a_n|^2=0\\
&\to a_1=\cdots=a_n=0\\
&\to e_1,\ldots,e_n\text{线性无关}
\end{aligned}
$$

□

定义 2.5. *规范正交基(orthonormal basis)*

$$V\text{的规范正交基是}V\text{中的规范正交向量组成的基}$$

定理 2.6.

$$\operatorname{length}(\boldsymbol{e})=\dim V\Leftrightarrow\boldsymbol{e}\text{是}V\text{的规范正交基}$$

定理 2.7. *向量是规范正交基的线性组合，并且范数的计算具有良好性质*

$$
\begin{gathered}
e_1,\ldots,e_n\text{是}V\text{的规范正交基},\forall v\in V\\
v=\langle v,e_1\rangle e_1+\cdots+\langle v,e_n\rangle e_n\\
\|v\|^2=|\langle v,e_1\rangle|^2+\cdots+|\langle v,e_n\rangle|^2
\end{gathered}
$$

证明.

$$
\begin{aligned}
&\boldsymbol{e}\text{是}V\text{的基}\to v=a_1e_1+\cdots+a_ne_n\\
&\to\langle v,e_i\rangle=\langle a_1e_1+\cdots+a_ne_n,e_i\rangle\\
&=\langle a_1e_1,e_i\rangle+\cdots+\langle a_ne_n,e_i\rangle\\
&=a_i\\
&\to v=\langle v,e_1\rangle e_1+\cdots+\langle v,e_n\rangle e_n
\end{aligned}
$$

□

带入勾股定理即可得到$\|v\|^2=|\langle v,e_1\rangle|^2+\cdots+|\langle v,e_n\rangle|^2$

定理 2.8. *格拉姆-施密特过程*

$$
\begin{gathered}
v_1,\ldots,v_n\text{是}V\text{中的线性无关的向量组}.\, e_1=\frac{v_1}{\|v_1\|}\\
e_i=\frac{v_i-\langle v_i,e_1\rangle e_1-\cdots-\langle v_i,e_{i-1}\rangle e_{i-1}}{\|v_i-\langle v_i,e_1\rangle e_1-\cdots-\langle v_i,e_{i-1}\rangle e_{i-1}\|},i\in 2\ldots n\\
\to e_1,\ldots,e_n\text{是}V\text{中的规范正交组}\wedge\forall i\in 1\ldots n\to\operatorname{span}(e_1,\ldots,e_i)=\operatorname{span}(v_1,\ldots,v_i)
\end{gathered}
$$

证明.

$$
\begin{gathered}
\text{使用归纳法}\\
i=1\to e_1=\tfrac{v_1}{\|v_1\|}\to\operatorname{span}(v_1)=\operatorname{span}(e_1)\\
\text{假设}1<i<m.\operatorname{span}(v_1,\ldots,v_{i-1})=\operatorname{span}(e_1,\ldots,e_{i-1})\\
v_1,\ldots,v_n\text{线性无关}\to v_i\notin\operatorname{span}(v_1,\ldots,v_{i-1})\\
\to v_i\notin\operatorname{span}(e_1,\ldots,e_{i-1})\\
\to v_i-\langle v_i,e_1\rangle e_1-\cdots-\langle v_i,e_{i-1}\rangle e_{i-1}\neq 0\\
\to\|e_i\|=0\\
\forall j\in 1\ldots i\\
\langle e_i,e_j\rangle=\left\langle\tfrac{v_i-\langle v_i,e_1\rangle e_1-\cdots-\langle v_i,e_{i-1}\rangle e_{i-1}}{\|v_i-\langle v_i-e_1\rangle e_1-\cdots-\langle v_i,e_{i-1}\rangle e_{i-1}\|},e_j\right\rangle\\
=\tfrac{\langle v_i,e_j\rangle-\langle v_i,e_j\rangle}{\|v_i-\langle v_i-e_1\rangle e_1-\cdots-\langle v_i,e_{i-1}\rangle e_{i-1}\|}\\
=0\\
\to e_1,\ldots,e_i\text{是规范正交组}\\
v_i\in\operatorname{span}(e_1,\ldots,e_i)\\
\to\operatorname{span}(v_1,\ldots,v_i)\subset\operatorname{span}(e_1,\ldots e_i)\\
\operatorname{length}v_1,\ldots,v_i=\operatorname{length}e_1,\ldots,e_i\\
\to\operatorname{span}(v_1,\ldots,v_i)=\operatorname{span}(e_1,\ldots,e_i)
\end{gathered}
$$

□

例 2.9. $\mathcal{P}_2(R)$在内积$\langle p,q\rangle=\int_{-1}^{1}p(x)q(x)\mathrm{d}x$

$$1,x,x^2\text{是}\mathcal{P}_2(R)\text{的一组基}$$
$$\|1\|^2=\int_{-1}^{1}1^2\mathrm{d}x=2$$
$$\rightarrow e_1=\frac{v_1}{\|v_1\|}=\frac{1}{\sqrt{2}}$$
$$e_2=\frac{v_2-\langle v_2,e_1\rangle e_1}{\|v_2-\langle v_2,e_1\rangle e_1\|}$$
$$\langle v_2,e_1\rangle=\int_{-1}^{1}x\cdot\frac{1}{\sqrt{2}}\mathrm{d}x=0$$
$$\text{分子}=x-0\cdot\frac{1}{\sqrt{2}}=x$$
$$\text{分母}=\|x\|=\sqrt{\langle x,x\rangle}=\sqrt{\int_{-1}^{1}x^2\mathrm{d}x}=\sqrt{\frac{2}{3}}$$
$$\rightarrow e_2=x\cdot\sqrt{\frac{3}{2}}$$
$$e_3=\frac{v_3-\langle v_3,e_1\rangle e_1-\langle v_3,e_2\rangle e_2}{\|v_3-\langle v_3,e_1\rangle e_1-\langle v_3,e_2\rangle e_2\|}$$
$$v_3=x^2,e_1=\frac{1}{\sqrt{2}}\rightarrow\langle v_3,e_1\rangle=\int_{-1}^{1}x^2\frac{1}{\sqrt{2}}\mathrm{d}x=\frac{1}{\sqrt{2}}\cdot\frac{2}{3}$$
$$e_2=x\sqrt{\frac{3}{2}}\rightarrow\langle v_3,e_2\rangle=\int_{-1}^{1}x^2x\sqrt{\frac{3}{2}}\mathrm{d}x=\sqrt{\frac{3}{2}}\int_{-1}^{1}x^3\mathrm{d}x=0$$
$$\text{分子}=x^2-\frac{2}{3\sqrt{2}}\cdot\frac{1}{\sqrt{2}}-0\cdot x\sqrt{\frac{3}{2}}=x^2-\frac{1}{3}$$
$$\text{分母}^2=\left\|x^2-\frac{1}{3}\right\|^2=\int_{-1}^{1}(x^2-\frac{1}{3})^2\mathrm{d}x$$
$$=\int_{-1}^{1}x^4-\frac{2}{3}x^2+\frac{1}{9}\mathrm{d}x$$
$$=(\frac{1}{5}x^5-\frac{2}{3}\cdot\frac{1}{3}x^3+\frac{1}{9}x)|_{-1}^{1}$$
$$=(\frac{1}{5}-\frac{2}{9}+\frac{1}{9})-(-\frac{1}{5}+\frac{2}{9}-\frac{1}{9})$$
$$=(\frac{1}{5}-\frac{1}{9})-(\frac{1}{9}-\frac{1}{5})$$
$$=\frac{1}{5}-\frac{1}{9}-\frac{1}{9}+\frac{1}{5}$$
$$=\frac{2}{5}-\frac{2}{9}=\frac{18-10}{45}=\frac{8}{45}$$
$$\rightarrow e_3=\sqrt{\frac{45}{8}}\cdot(x^2-\frac{1}{3})$$
$$e_1=\frac{1}{\sqrt{2}};e_2=\sqrt{\frac{3}{2}}x;e_3=\sqrt{\frac{45}{8}}(x^2-\frac{1}{3})$$

定理 2.10. *每个有限维内积空间都有规范正交基*

证明.

$$V\text{是有限维的}\rightarrow V\text{必有基}\boldsymbol{v}.$$
$$\text{对}\boldsymbol{v}\text{执行格拉姆}-\text{施密特过程得到}\boldsymbol{e}$$
$$\operatorname{length}\boldsymbol{e}=\dim V\rightarrow\boldsymbol{e}\text{是基}$$
$$\rightarrow\boldsymbol{e}\text{是规范正交基}$$

□

定理 2.11. *有限维向量空间. 规范正交组必可扩充为规范正交基*

证明.

$$\boldsymbol{e}\text{是}V\text{中的规范正交组}\rightarrow\boldsymbol{e}\text{是线性无关的}$$
$$\rightarrow\boldsymbol{e},\boldsymbol{v}\text{是}V\text{的基}$$
$$\text{对}(\boldsymbol{e},\boldsymbol{v})\text{执行格拉姆}-\text{施密特过程得到}(\boldsymbol{e},\boldsymbol{f})$$
$$\rightarrow(\boldsymbol{e},\boldsymbol{f})\text{是规范正交基}$$

□

定理 2.12. *有上三角矩阵的线性映射必有规范正交基使得线性映射也是上三角矩阵*

$T\in\mathcal{L}(V)$.T关于V的某个基具有上三角矩阵，T关于V的某个规范正交基也有上三角矩阵

证明.

$$
\begin{aligned}
&T\text{关于}V\text{的基}v_1,\ldots,v_n\text{具有上三角矩阵，}\forall i=1\ldots n\rightarrow\operatorname{span}(v_1,\ldots,v_i)\text{在}T\text{下不变}\\
&v_1,\ldots,v_n\text{执行格拉姆}-\text{施密特过程}\rightarrow\boldsymbol{e}\text{，}\boldsymbol{e}\text{是}V\text{的规范正交基}\\
&\forall i\in 1\ldots n\rightarrow\operatorname{span}(e_1,\ldots,e_i)=\operatorname{span}(v_1,\ldots,v_i)\\
&\rightarrow\operatorname{span}(e_1,\ldots,e_i)\text{都是}T\text{的不变子空间}\\
&\rightarrow T\text{关于}\boldsymbol{e}\text{有上三角矩阵}
\end{aligned}
$$

□

定理 2.13. *Schur.舒尔定理*

$$V\text{是有限维复向量空间，算子}T\in\mathcal{L}(V).T\text{关于}V\text{的某个规范正交基必有上三角矩阵}$$

证明.

$$
\begin{aligned}
&V\text{是有限维复向量空间}\rightarrow V\text{必有基使得}T\text{的矩阵为上三角矩阵}\\
&\rightarrow V\text{必有规范正交基使得}T\text{的矩阵为上三角矩阵}
\end{aligned}
$$

□

2.1 内积空间上的线性泛函

定义 2.14. *线性泛函(linear functional)*

$$V\text{上的线性泛函是}V\rightarrow F\text{的线性映射}.\text{线性泛函是}\mathcal{L}(V,F)\text{中的元素}$$

例 2.15. $\mathcal{P}_2(R)$上的线性泛函

$$
\begin{gathered}
\varphi(p):\mathcal{P}_2(R)\rightarrow R\\
\varphi(p)=\int_{-1}^{1}p(t)\cos(\pi t)\mathrm{d}t\\
\varphi(p+q)=\int_{-1}^{1}(p+q)(t)\cos(\pi t)\mathrm{d}t\\
=\int_{-1}^{1}p(t)\cos(\pi t)+q(t)\cos(\pi t)\,\mathrm{d}t\\
=\int_{-1}^{1}p(t)\cos(\pi t)\mathrm{d}t+\int_{-1}^{1}q(t)\cos(\pi t)\mathrm{d}t\\
=\varphi(p)+\varphi(q)\\
\forall\lambda\in R,\varphi(\lambda p)=\int_{-1}^{1}(\lambda p)(t)\cos(\pi t)\mathrm{d}t\\
=\int_{-1}^{1}\lambda\,(p(t)\cos(\pi t))\mathrm{d}t\\
=\lambda\int_{-1}^{1}p(t)\cos(\pi t)\mathrm{d}t\\
=\lambda\varphi(p)\\
\rightarrow\varphi\in\mathcal{L}(\mathcal{P}_2(R)\rightarrow R)
\end{gathered}
$$

定理 2.16. *Reisz. 里斯表示定理。线性泛函必可表示成线性算子的内积*

$$V\text{是有限维的，}\varphi\text{是}V\text{上的线性泛函}\rightarrow\exists u\in V,\forall v\in V,\varphi(v)=\langle v,u\rangle$$

证明.

$$
\begin{gathered}
\text{存在性}\\
\text{设}\boldsymbol{e}\text{是}V\text{的规范正交基}\\
\varphi(v)=\varphi(\langle v,e_1\rangle e_1+\cdots+\langle v,e_n\rangle e_n)\\
=\langle v,e_1\rangle\varphi(e_1)+\cdots+\langle v,e_n\rangle\varphi(e_n)\\
=\langle v,\overline{\varphi(e_1)}e_1+\cdots+\overline{\varphi(e_n)}e_n\rangle\\
u=\overline{\varphi(e_1)}e_1+\cdots+\overline{\varphi(e_n)}e_n\\
\rightarrow\varphi(v)=\langle v,u\rangle\\
\langle v,\overline{\varphi(e_1)}e_1+\overline{\varphi(e_2)}e_1\rangle\\
=\overline{\langle\overline{\varphi(e_1)}e_1+\overline{\varphi(e_2)}e_1,v\rangle}\\
=\overline{\langle\overline{\varphi(e_1)}e_1,v\rangle}+\overline{\langle\overline{\varphi(e_2)}e_2,v\rangle}\qquad\text{这里主要问题是}\varphi(e_1)\in F,\text{导致可以提出去}\\
=\overline{\overline{\varphi(e_1)}\langle e_1,v\rangle}+\overline{\overline{\varphi(e_2)}\langle e_2,v\rangle}\qquad\text{泛函的作用}\\
=\varphi(e_1)\langle v,e_1\rangle+\varphi(e_2)\langle v,e_2\rangle
\end{gathered}
$$

□

$$\text{唯一性}$$
$$u_1, u_2 \in V \to \forall v \in V, \varphi(v) = \langle v, u_1 \rangle = \langle v, u_2 \rangle$$
$$0 = \langle v, u_1 \rangle - \langle v, u_2 \rangle = \overline{\langle u_1, v \rangle} - \overline{\langle u_2, v \rangle} = \overline{\langle u_1 - u_2, v \rangle} = \langle v, u_1 - u_2 \rangle$$
$$\text{let: } v = u_1 - u_2 \to 0 = \|u_1 - u_2\|^2 \to u_1 - u_2 = 0 \to u_1 = u_2$$

例 2.17. 计算$u \in \mathcal{P}_2(R)$的关于内积$\int_{-1}^{1} p(t)\cos(\pi t)\mathrm{d}t$的线性表示

$$\int_{-1}^{1} p(t)\cos(\pi t)\mathrm{d}t = \int_{-1}^{1} p(t)u(t)\mathrm{d}t$$
$$\mathcal{P}_2(R)\text{的一组规范正交基}$$
$$e_1 = \frac{1}{\sqrt{2}}; e_2 = \sqrt{\frac{3}{2}}x; e_3 = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$$

$$u(x) = \overline{\varphi(e_1)}e_1 + \overline{\varphi(e_2)}e_2 + \overline{\varphi(e_3)}e_3$$
$$\begin{aligned}\varphi(e_1)e_1 &= \left(\int_{-1}^{1} \sqrt{\frac{1}{2}}\cos(\pi t)\mathrm{d}t\right)\sqrt{\frac{1}{2}}\\ &= \frac{1}{2}\int_{-1}^{1}\cos(\pi t)\mathrm{d}t\\ &= 0\end{aligned}$$
$$\begin{aligned}\varphi(e_2)e_2 &= \left(\int_{-1}^{1} \sqrt{\frac{3}{2}}t\cos(\pi t)\mathrm{d}t\right)\sqrt{\frac{3}{2}}x\\ &= \frac{3}{2}x\left(\int_{-1}^{1} t\cos(\pi t)\mathrm{d}t\right)\end{aligned}$$

$$\begin{aligned}\varphi(e_3)e_3 &= \left(\int_{-1}^{1} \sqrt{\frac{45}{8}}\left(t^2 - \frac{1}{3}\right)\cos(\pi t)\mathrm{d}t\right)\sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)\\ &= \frac{45}{8}\left(x^2 - \frac{1}{3}\right)\left(\int_{-1}^{1} t^2\cos(\pi t)\mathrm{d}t\right)\\ &= \frac{45}{8}\cdot\frac{-4}{\pi^2}\left(x^2 - \frac{1}{3}\right)\\ &= -\frac{45}{2\pi^2}\left(x^2 - \frac{1}{3}\right)\end{aligned}$$

$$\to u(x) = \overline{\varphi_3}e_3 = -\frac{45}{2}\left(x^2 - \frac{1}{3}\right)$$

$$\forall p \in \mathcal{P}_2(R) \to \int_{-1}^{1} p(t)\cos(\pi t)\mathrm{d}t = -\frac{45}{2\pi^2}\int_{-1}^{1} p(t)\left(t^2 - \frac{1}{3}\right)\mathrm{d}t$$

Remark: 从计算过程上看 线性泛函可被表示成向量空间元素的内积。元素$u = \overline{\varphi(e_1)}e_1 + \cdots + \overline{\varphi(e_n)}e_n$依赖于规范正交基的选择.然而Reisz表示定理 表明向量u是唯一的，跟规范正交基的选择无关。

6.B

1. $\theta \in R$.

   a. Proof: $(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)$和$(\cos\theta, \sin\theta), (\sin\theta, -\cos\theta)$都是$R^2$的规范正交基

$$\langle v_1, v_2 \rangle = \cos\theta \cdot -\sin\theta + \sin\theta \cdot \cos\theta = 0$$
$$\|v_1\| = \sqrt{(\cos\theta)^2 + (\sin\theta)^2} = \sqrt{1} = 1$$
$$\|v_2\| = \sqrt{(-\sin\theta)^2 + (\cos\theta)^2} = \sqrt{1} = 1$$
$$\to v_1, v_2\text{是规范的，}v_1, v_2\text{相互正交}$$
$$\dim R^2 = \mathrm{length}\,(v_1, v_2) = 2$$
$$\to v_1 = (\cos\theta, \sin\theta); v_2 = (-\sin\theta, \cos\theta)\text{是}R^2\text{的基}$$

$$\langle v_1, v_2 \rangle = \cos\theta \cdot \sin\theta + \sin\theta \cdot -\cos\theta = 0$$
$$\|v_1\| = \sqrt{(\cos\theta)^2 + (\sin\theta)^2} = \sqrt{1} = 1$$
$$\|v_2\| = \sqrt{(\sin\theta)^2 + (-\cos\theta)^2} = \sqrt{1} = 1$$

b. Proof: R^2的任意规范正交基必是a中的形式之一

$$
\begin{gathered}
\forall v_1, v_2 \text{是} R^2 \text{的规范正交基} \\
\|v_1\|^2 = \langle v_1, v_1 \rangle = 1 \\
\|v_2\|^2 = \langle v_2, v_2 \rangle = 1 \\
\langle v_1, v_2 \rangle = 0 \\
v_1 = (x, y) \to \langle v_1, v_1 \rangle = x^2 + y^2 = 1 \\
v_2 = (p, q) \to \langle v_2, v_2 \rangle = p^2 + q^2 = 1 \\
\text{由于} x^2 + y^2 = 1 \text{对应} R^2 \text{上的单位圆} \\
\to x \text{和} \cos\theta, y \text{和} \sin\theta \text{是} f: [0, 2\pi) \to (x, y) \text{上的一一映射} \\
\to v_1 = (\cos\theta, \sin\theta) \text{是一个范数为1的任意向量} \\
\langle v_1, v_2 \rangle = 0 \to xp + yq = 0 \\
\to p\cos\theta + q\sin\theta = 0 \\
\to p = \sin\theta, q = -\cos\theta \vee p = -\sin\theta, q = \cos\theta \\
\text{对于任意} \theta \text{只有此两个解} \\
\to v_1, v_2 \text{必然是a中的形式之一}
\end{gathered}
$$

3 正交补与极小化问题

3.1 正交补

定义 3.1. *正交补(orthogonal complement), $U^\perp$*

$$
\begin{gathered}
U \text{是} V \text{的子集}, U \text{的正交补} U^\perp \text{是} V \text{中与} U \text{的每个向量都正交的向量组成的集合} \\
U^\perp = \{v \in V : \forall u \in U, \langle u, v \rangle = 0\}
\end{gathered}
$$

定理 3.2. *正交补的基本性质*

$$
\begin{array}{cc}
1 & U \text{是} V \text{的子集} \to U^\perp \text{是} V \text{的子空间} \\
2 & \{0\}^\perp = V \\
3 & V^\perp = \{0\} \\
4 & U \text{是} V \text{的子集} \to U \cap U^\perp \subset \{0\} \\
5 & U \text{和} W \text{都是} V \text{的子集} \wedge U \subset W \to W^\perp \subset U^\perp
\end{array}
$$

证明.

$$
\begin{array}{lc}
1 & U \text{是} V \text{的子集} \to \forall u \in U, \langle 0, u \rangle = 0 \to 0 \in U^\perp \\
 & \forall u, w \in U^\perp. u \in U \to \langle v + w, u \rangle = \langle v, u \rangle + \langle w, u \rangle = 0 + 0 = 0 \\
 & \to v + w \in U^\perp \\
 & \forall v \in U^\perp \to \langle \lambda v, u \rangle = \lambda \langle v, u \rangle = \lambda 0 = 0 \\
 & \to \lambda v \in U^\perp \\
 & \to U^\perp \text{是子空间}
\end{array}
$$

$\square$

$$
\begin{array}{lc}
2 & v \in V. \langle v, 0 \rangle = 0 \to v \in \{0\}^\perp \to \{0\}^\perp = V \\
3 & v \in V \wedge v \in V^\perp \to \langle v, v \rangle = 0 \to v = 0 \to V^\perp = \{0\} \\
4 & \forall v \in U \cap U^\perp \to \langle v, v \rangle = 0 \to v = 0 \to U \cap U^\perp \subset \{0\} \\
5 & v \in W^\perp, \forall u \in W \to \langle u, v \rangle = 0 \\
 & \to \forall u \in U \to \langle u, v \rangle = 0 \\
 & \to v \in U^\perp \\
 & \to W^\perp \subset U^\perp
\end{array}
$$

定理 3.3. *子空间与其正交补的和是直和且和为*V

$$U\text{是}V\text{的有限维子空间} \to V = U \oplus U^{\perp}$$

证明.

$$\begin{gathered}
\forall v \in V, e_1, \ldots, e_n\text{是}U\text{的规范正交基} \\
v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n + v - \langle v, e_1\rangle e_1 - \cdots - \langle v, e_n\rangle e_n \\
u = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n \\
w = v - \langle v, e_1\rangle e_1 - \cdots - \langle v, e_n\rangle e_n \\
\forall i \in 1\ldots n \to \langle w, e_i\rangle = \langle v, e_i\rangle - \langle v, e_i\rangle = 0 \\
\to \langle w, \operatorname{span}(\boldsymbol{e})\rangle = 0 \\
\to w \in U^{\perp} \\
\to v = u + w, u \in U \wedge w \in U^{\perp} \\
\to V = U + U^{\perp} \\
U \cap U^{\perp} = \{0\} \to V = U \oplus U^{\perp}
\end{gathered}$$

□

定理 3.4. *正交补的维数*

$$V\text{是有限维的且}U\text{是}V\text{的子空间}. \dim U^{\perp} = \dim V - \dim U$$

证明.

$V = U \oplus U^{\perp} \to \dim V = \dim U + \dim U^{\perp}$ □

定理 3.5. *有限维子空间的正交补的正交补是原子空间*

$$U\text{是}V\text{的有限维子空间} \to U = (U^{\perp})^{\perp}$$

证明.

1
$$\begin{gathered}
U \subset (U^{\perp})^{\perp} \\
\forall u \in U, \forall v \in U^{\perp}, \langle u, v\rangle = 0 \\
u\text{正交于}\forall v \in U^{\perp} \to u \in (U^{\perp})^{\perp}
\end{gathered}$$

2
$$\begin{gathered}
(U^{\perp})^{\perp} \subset U \\
\forall v \in (U^{\perp})^{\perp}, v = u + w, u \in U, w \in U^{\perp} \\
v \in (U^{\perp})^{\perp}, u \in U \subset (U^{\perp})^{\perp} \\
\to v - u \in (U^{\perp})^{\perp} \\
\to v - u \in (U^{\perp})^{\perp} \cap U^{\perp} \\
\to v - u = 0 \\
\to v \in U \\
\to (U^{\perp})^{\perp} \subset U
\end{gathered}$$

$$\to U = (U^{\perp})^{\perp}$$

□

定义 3.6. *正交投影(orthogonal projection),* P_U

$$\begin{gathered}
U\text{是}V\text{的有限维子空间}. V\text{到}U\text{上的正交投影是算子}P_U \in \mathcal{L}(V) \\
\forall v \in V, v = u + w, u \in U \wedge w \in U^{\perp}; P_U(v) = u
\end{gathered}$$

例 3.7. 正交投影

$$x \in V, x \neq 0 \wedge U = \operatorname{span}(x);\ \ P_U(v) = \frac{\langle v, x\rangle}{\|x\|^2}x$$

证明.

$$\begin{aligned}
v \in V . v &= \tfrac{\langle v, x\rangle}{\|x\|^2}x + \left(v - \tfrac{\langle v, x\rangle}{\|x\|^2}x\right) \\
\tfrac{\langle v, x\rangle}{\|x\|^2}x &\in \operatorname{span}(x) \subset U \\
\left\langle v - \tfrac{\langle v, x\rangle}{\|x\|^2}x, \tfrac{\langle v, x\rangle}{\|x\|^2}x\right\rangle &= \left\langle v, \tfrac{\langle v, x\rangle}{\|x\|^2}\right\rangle - \left\langle \tfrac{\langle v, x\rangle}{\|x\|^2}x, \tfrac{\langle v, x\rangle}{\|x\|^2}x\right\rangle \\
&= \left\langle v, \tfrac{\langle v, x\rangle}{\|x\|^2}x\right\rangle - \left|\tfrac{\langle v, x\rangle}{\|x\|^2}\right|^2\|x\|^2 \\
&= \overline{\tfrac{\langle v, x\rangle}{\|x\|^2}}\langle v, x\rangle - \tfrac{|\langle v, x\rangle|^2}{\|x\|^2} \\
&= \|x\|^{-2}(\overline{\langle v, x\rangle}\langle v, x\rangle - |\langle v, x\rangle|^2) \\
&= \|x\|^{-2}(|\langle v, x\rangle|^2 - |\langle v, x\rangle|^2) \\
&= \|x\|^{-2}0 = 0 \\
&\to v - \tfrac{\langle v, x\rangle}{\|x\|^2}x \text{和} \tfrac{\langle v, x\rangle}{\|x\|^2}x \text{是正交的} \\
&\to \left(v - \tfrac{\langle v, x\rangle}{\|x\|^2}x\right) \in U^{\perp} \\
&\to P_U(v) = \tfrac{\langle v, x\rangle}{\|x\|^2}x
\end{aligned}$$

□

定理 3.8. *正交投影P_U的性质*

$$\begin{array}{lc}
 & U\text{是}V\text{的有限维子空间}, v \in V \\
1 & P_U \in \mathcal{L}(V) \\
2 & \forall u \in U \to P_U(u) = u \\
3 & \forall w \in U^{\perp} \to P_U(w) = 0 \\
4 & \operatorname{range} P_U = U \\
5 & \operatorname{null} P_U = U^{\perp} \\
6 & v - P_U(v) \in U^{\perp} \\
7 & P_U \circ P_U = P_U^2 = P_U \\
8 & \|P_U v\| \leqslant \|v\| \\
9 & U\text{的规范正交基}\boldsymbol{e} \to P_U(v) = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n
\end{array}$$

证明.

$$\begin{aligned}
1 \qquad & P_U \in \mathcal{L}(V) \\
& \forall v_1, v_2 \in V \to P_U(v_1 + v_2) \\
&= P_U(u_1 + w_1 + u_2 + w_2) \\
&= P_U((u_1 + u_2) + (w_1 + w_2)) \\
&= u_1 + u_2 \\
&= P_U(u_1 + w_1) + P_U(u_2 + w_2) \\
& \forall v \in V, \forall \lambda \in F \to P_U(\lambda v) \\
&= P_U(\lambda u + \lambda w) \\
&= \lambda u \\
&= \lambda P_U(u + w) \\
&\to P_U \in \mathcal{L}(V)
\end{aligned}$$

$$\begin{aligned}
2 \qquad & \forall u \in U \to P_U(u) = u \\
& \forall u \in U, u = u + w \to w = 0 \\
&\to P_U(u) = u
\end{aligned}$$

3 $\forall w \in U^{\perp}, P_U(w) = 0$
$\forall w \in U^{\perp} \to w = 0 + w$
$\to P_U(w) = 0$

4 $\text{range}\, P_U = U$
$\forall u \in U, P_U(u) = u$
$\to U \subset \text{range}\, P_U$
$\forall v \in V, P_U(v) \in U$
$\to \text{range}\, P_U \subset U$
$\to \text{range}\, P_U = U$

5 $\text{null}\, P_U = U^{\perp}$
$\forall w \in U^{\perp} \to P_U(w) = 0$
$\to U^{\perp} \subset \text{null}\, P_U$
$\forall v \in \text{null}\, P_U, v = u + w$
$P_U(v) = 0 \to u = 0$
$\to v = w \in U^{\perp}$
$\to \text{null}\, P_U \subset U^{\perp}$

6 $v - P_U(v) \in U^{\perp}$
$\forall v \in V, v = u + w$
$u + w - P(u + w) = w \in U^{\perp}$

7 $P_U \circ P_U = P_U^2 = P_U$
$\forall v \in V, v = u + w$
$P_U^2(v) = P_U(P_U(v))$
$= P_U(P_U(u + w))$
$= P_U(u) = u$
$= P_U(v)$

8 $\|P_U v\| \leqslant \|v\|$
$\forall v \in V, v = u + w$
$\|P_U(v)\| = \|u\| \leqslant \|u + w\|$

9 U的规范正交基$\boldsymbol{e} \to P_U(v) = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n$
$v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n + v - \langle v, e_1\rangle e_1 - \cdots - \langle v, e_n\rangle e_n$ □
$\to P_U(v) = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n$

3.2 极小化问题

给定V的子空间U和点$v \in V$.求$u \in U$使得$\|v - u\|$最小

定理 3.9. *到子空间的最小距离是到子空间上的投影*

$$U\text{是}V\text{的有限维子空间}, v \in V \wedge u \in U \to \|v - P_U v\| \leqslant \|v - u\|$$

等号成立当且仅当$u = P_U(v)$

证明.

$\|v - P_U v\|^2 \leqslant \|v - P_U v\|^2 + \|P_U v - u\|^2$
$= \|(v - P_U v) + (P_U v - u)\|^2$ $\quad \langle v - P_U v, u\rangle = 0$，勾股定理 □
$= \|v - u\|^2$

等号成立 $\to \|P_U(v) - u\|^2 = 0 \to P_U(v) = u$

例 3.10. 求$p\in\mathcal{P}_5(R)\to[-\pi,\pi]$上逼近$\sin x$

$$\int_{-\pi}^{\pi}|\sin x-p(x)|^2\mathrm{d}x\text{最小}$$

$C_R[-\pi,\pi]$是$[-\pi,\pi]$上的实连续函数构成的实内积空间

$$\begin{gathered}
\text{内积}: \langle f,g\rangle=\int_{-\pi}^{\pi}f(x)g(x)\mathrm{d}x\\
\mathcal{P}_5(R)\text{的基}: 1,x,x^2,x^3,x^4,x^5\\
\text{格拉姆}-\text{施密特过程}\\
\|e_1\|^2=\int_{-\pi}^{\pi}1^2\mathrm{d}x=2\pi\\
e_1=\frac{1}{\sqrt{2\pi}}\\
\\
e_2=\frac{v_2-\langle v,e_1\rangle e_1}{\|v_2-\langle v-e_1\rangle e_1\|}\\
\langle v_2,e_1\rangle=\int_{-\pi}^{\pi}\frac{1}{\sqrt{2\pi}}x\mathrm{d}x=0\\
v_2-\langle v_2,e_1\rangle e_1=x\\
\|v_2\|^2=\int_{-\pi}^{\pi}x^2\mathrm{d}x=\frac{2}{3}\pi^2\\
e_2=\pi\sqrt{\frac{3}{2}}x\\
\\
e_3=\frac{v_3-\langle v_3,e_1\rangle e_1-\langle v_3,e_2\rangle e_2}{\|v_3-\langle v_3,e_1\rangle e_1-\langle v_3,e_2\rangle e_2\|}\\
\langle v_3,e_1\rangle=\int_{-\pi}^{\pi}x^2\frac{1}{\sqrt{2\pi}}=\frac{\sqrt{2}}{3}\pi^{\frac{5}{2}}\\
\langle v_3,e_2\rangle=\int_{-\pi}^{\pi}x^2\pi\sqrt{\frac{3}{2}}x\mathrm{d}x=0\\
\text{分子}=x^2-\frac{\sqrt{2}}{3}\pi^{\frac{5}{2}}\\
\text{分母}^2=\int_{-\pi}^{\pi}\Big(x^2-\frac{\sqrt{2}}{3}\pi^{\frac{5}{2}}\Big)\mathrm{d}x=\frac{2}{3}\Big(\pi^3-\sqrt{2}\pi^{\frac{7}{2}}\Big)\\
e_3=\\
\\
e_4=\frac{v_4-\langle v_4,e_1\rangle e_1-\langle v_4,e_2\rangle e_2-\langle v_4,e_3\rangle e_3}{\|v_4-\langle v_4,e_1\rangle e_1-\langle v_4,e_2\rangle e_2-\langle v_4,e_3\rangle e_3\|}\\
\langle v_4,e_1\rangle=\int_{-\pi}^{\pi}x^3\frac{1}{\sqrt{2}}\mathrm{d}x=0\\
\langle v_4,e_2\rangle=\int_{-\pi}^{\pi}x^3\sqrt{\frac{3}{2}}x\mathrm{d}x=
\end{gathered}$$

???算了，等学学mma再算把。inner都错了

6.C