Chapter 17

BY 多元函数微分学

1 Def & Theo

1.1 可微性与微分

1. 二元函数在某一点可微：

$$
\begin{gathered}
z=f(x,y)\text{在}P_0=(x_0,y_0)\text{的领域}U_{P_0}\text{有定义}\\
\forall P\in U_{P_0},\Delta f=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)\\
=A\Delta x+B\Delta y+o\left(\sqrt{x^2+y^2}\right)\\
\text{记为：}\mathrm{d}z|_{P_0}=\mathrm{d}f(x_0,y_0)=A\Delta x+B\Delta y\\
\Delta z=A\Delta x+B\Delta y+\alpha\Delta x+\beta\Delta y;(\Delta x,\Delta y)\to(0,0)\Rightarrow\alpha=\beta=0
\end{gathered}
$$

2. 区域上的可微：

$$
\begin{gathered}
\text{二元函数在区域}D\text{上每个点都可微，称为在}D\text{上可微}\\
\mathrm{d}f=\frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x+\frac{\mathrm{d}f}{\mathrm{d}y}\mathrm{d}y
\end{gathered}
$$

3. 在某一点的偏导数：

$$
\begin{gathered}
z=f(x,y),f(x,y_0)\text{在}x_0\text{的某个领域上存在}\\
\lim_{x\to x_0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{x-x_0}\text{存在，称为}f\text{在}(x_0,y_0)\text{关于}x\text{的偏导数}\\
\text{记为：}\left.\frac{\partial f}{\partial x}\right|(x_0,y_0);f_x(x_0,y_0);\\
\text{同样可定义出关于}y\text{的偏导数}
\end{gathered}
$$

4. 区域上的偏导函数：

$$
\begin{gathered}
\text{函数}f\text{在区域}D\text{上每一点都存在对}x\text{或}y\text{的偏导数，在每一点的偏导数构成}D\text{上的函数}\\
\text{记为：}f_x(x,y);\frac{\partial f}{\partial x}
\end{gathered}
$$

5. 可微的必要条件:

$$\begin{gathered}
若f在点P=(x_0,y_0)可微\Rightarrow关于此点任意方向的方向导数都存在\\
特殊的，任意变量的偏导数也存在\\
\mathrm{d}f=A\mathrm{d}x+B\mathrm{d}y;A=\frac{\partial f}{\partial x};B=\frac{\partial f}{\partial y};
\end{gathered}$$

6. 可微的充分条件:

$$\begin{aligned}
&f关于任意自变量的偏导数在U_P上存在且连续\Rightarrow f在P可微\\
\mathrm{Pr}\quad&\Delta z=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)\\
&=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0+\Delta y)\\
&+f(x_0,y_0+\Delta y)-f(x_0,y_0)\\
&\Rightarrow\Delta z=f_x(x_0+\theta_1\Delta x,y_0+\Delta y)\Delta x+f_y(x_0,y_0+\theta_2\Delta y)\Delta y\\
&f_x,f_y连续\Rightarrow\Delta z=f_x(x_0,y_0)\Delta x+f_y(x_0,y_0)\Delta y
\end{aligned}$$

7. 中值定理:

$$\begin{gathered}
若f在U_{(x_0,y_0)}内存在偏导数\\
\Rightarrow\forall(x,y)\in U_{(x_0,y_0)},f(x,y)-f(x_0,y_0)=f_x(\xi,y)(x-x_0)+f_y(x_0,\eta)(y-y_0)\\
\xi=x_0+\theta_1(x-x_0);\eta=\theta_2(y-y_0);\theta_i\in(0,1)
\end{gathered}$$

1.2 可微性的几何意义

1. 在$P=(x_0,y_0)$可微$\Leftrightarrow$在$(x_0,y_0,f(x_0,y_0))$有不平行于z轴的切平面

2. 在一点的切平面方程(几乎隐含了可微):

$$z-z_0=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

3. 在一点的法线方程:

$$\frac{x-x_0}{f_x(x_0,y_0)}=\frac{y-y_0}{f_y(x_0,y_0)}=\frac{z-z_0}{-1}$$

1.3 复合函数求导法:

1. 多元函数的复合:

$$\begin{gathered}
x=\varphi(s,t);y=\phi(s,t);在D上有定义;\\
f(x,y)在\{(x,y)|x=\varphi;y=\phi,(s,t)\in D\}\subset D_1上有定义\\
f(\varphi,\phi)称为D上的复合函数
\end{gathered}$$

2. 若内外函数都可微，则复合函数也可微:(链式法则)

$$\begin{gathered}
x=\varphi(s,t);y=\phi(s,t);在D上都可微\\
z=f(x,y)在(\varphi,\phi)[x,y]可微\\
则复合函数f(\varphi,\phi)在(s,t)可微
\end{gathered}$$

$$\begin{gathered}
对s偏导数:\ \frac{\partial f}{\partial s}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}\\
对t偏导数:\ \frac{\partial f}{\partial t}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}
\end{gathered}$$

3. 偏导数存在性只需要内函数具有相应的偏导数即可，不需要内函数的可微性；

$$x=\varphi(s,t);y=\phi(s,t);\text{在}D\text{上存在关于}x\text{，}y\text{的偏导数}$$
$$f(x,y)\text{在}(\varphi(s,t),\phi(s,t))\text{可微}$$
$$\text{则}f(\varphi,\phi)\text{在}(s,t)\text{点关于}x\text{，}y\text{的偏导数存在}$$

Re：这里外函数的可微性是必要的，反例：

$$f(x,y)=\begin{cases}\frac{x^2y}{x^2+y^2} & x^2+y^2\neq 0\\ 0 & x^2+y^2=0\end{cases}$$
$$f\text{在}(0,0)\text{不可微(但连续),但两个偏导数均存在且等于}0$$
$$\text{令}x=t,y=t\text{是内函数则得到复合函数}\frac{t^3}{2t^2}=\frac{t}{2};\frac{\partial}{\partial t}f(t)=\frac{1}{2}\neq 0$$

1.4 复合函数的全微分

1. 全微分：

$$\mathrm{d}f=\frac{\partial f}{\partial x}\mathrm{d}x+\frac{\partial f}{\partial y}\mathrm{d}y$$
$$x=\varphi(s,t);y=\phi(s,t)$$
$$\mathrm{d}x=\frac{\partial\varphi}{\partial s}\mathrm{d}s+\frac{\partial\varphi}{\partial t}\mathrm{d}t;\mathrm{d}y=\frac{\partial\phi}{\partial s}\mathrm{d}s+\frac{\partial\phi}{\partial t}\mathrm{d}t$$
$$\mathrm{d}f=\frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial s}\mathrm{d}s+\frac{\partial x}{\partial t}\mathrm{d}t\right)+\frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial s}+\frac{\partial y}{\partial t}\mathrm{d}t\right)$$
$$=\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}\right)\mathrm{d}s+\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\right)\mathrm{d}t$$

2. 一阶微分不变性：

$$\text{由于}\mathrm{d}z=\frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial s}\mathrm{d}s+\frac{\partial x}{\partial t}\mathrm{d}t\right)+\frac{\partial z}{\partial y}\left(\frac{\partial x}{\partial s}\mathrm{d}s+\frac{\partial y}{\partial t}\mathrm{d}t\right)$$
$$\text{从而：}\mathrm{d}z=\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y$$

1.5 方向导数与梯度

1. 方向导数是指从任意方向逼近多元函数的某一个点时的差分极限

$$n\text{元函数}f\text{在点}P\text{的领域上有定义(貌似不需要这么强吧)}$$
$$l\text{为一个}n\text{维向量;}$$
$$\lim_{\|l\|\to 0^+}\frac{f(P_0+l)-f(P_0)}{\|l\|}\text{称为}f\text{沿着方}l\text{的方向导数}$$
$$\text{记作：}\left.\frac{\partial f}{\partial l}\right|_{P_0},f_l(P_0)\text{或}f_l(x_0,y_0,z_0)$$

Re:$l=(1,0,0)\Rightarrow\frac{\partial f}{\partial l}=\frac{\partial f}{\partial x};l=(-1,0,0)\Rightarrow\frac{\partial f}{\partial l}=-\frac{\partial f}{\partial x}$

2. 若f在一点可微，则在此点沿着任意方向的方向导数都存在

$$f_l(P_0)=\sum f_{x_i}(P_0)\cos\alpha_i;\cos\alpha_i=\frac{x_i}{\|l\|}$$
$$\text{对于三元函数有：}$$
$$l=(x,y,z);\cos\alpha=\frac{x}{\|l\|}=\frac{x}{\sqrt{x^2+y^2+z^2}}$$
$$\cos\beta=\frac{y}{\|l\|};\cos\gamma=\frac{z}{\|l\|}$$
$$f_l=f_x(P_0)\cos\alpha+f_y(P_0)\cos\beta+f_z(P_0)\cos\gamma$$

Re：函数在某一点的任意方向都有方向导数，不能表明在该点可微

Re：函数在一点的任意方向导数都存在，不能表明函数在该点连续

$$f(x,y)=\begin{cases}1 & 0<y<x^2\\ 0 & \text{else}\end{cases}$$
$$f\text{在}(0,0)\text{不连续；也不可微，但在该点的方向导数都存在且为}0$$
$$(y<x^2\text{在}0\text{处的界点不是函数的定义域；这是一个开区域上的阶梯})$$

3. 梯度(grad)：

$$\text{若}f\text{在点}P\text{存在任意自变量的偏导数}$$
$$\text{称向量}(f_x(P),f_y(P),f_z(P))\text{称为}f\text{在}P\text{点的梯度}$$
$$\mathrm{grad}\ f=(f_x,f_y,f_z)(P)$$
$$|\mathrm{grad}\ f|=\|\mathrm{grad}\ f\|=\sqrt{(f_x(P))^2+(f_y(P))^2+(f_z(P))^2}$$

4. 方向导数与梯度的关系：

$$l\text{方向的单位向量为}e=\frac{l}{\|l\|}$$
$$f_l(P)=\mathrm{grad}\ f(P_0)\cdot e=|\mathrm{grad}\ f(P)|\cos\theta;\theta\text{是梯度向量与}l\text{的夹角}$$
$$\cos\theta=\frac{x_1x_2+y_1y_2+z_1z_2}{\|l\|\cdot\|\mathrm{grad}\ f\|};$$

1.6 高阶偏导数

1. 高阶偏导数的定义：

$$\text{二元函数的一阶偏导数有两个}$$
$$f_x;f_y$$
$$\text{二阶偏导数有}$$
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}=f_{xx}(x,y)$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x\partial y}=f_{xy}(x,y)$$
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y\partial x}=f_{yx}(x,y)$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}=f_{yy}(x,y)$$

Re：这里$f_{xyz}=((f_x)_y)_z$和函数复合的顺序一致;$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$与函数复合顺序相反

Re：对不同变量的求的导数称为混合偏导数

2. 混合偏导数在某点连续，则混合偏导数相等

$$f_{xy}(x,y),f_{yx}(x,y)\text{在}(x_0,y_0)\text{连续}\Rightarrow f_{xy}(x_0,y_0)=f_{yx}(x_0,y_0)$$

Re：对于更多元的函数也成立

$$\begin{gathered}f_{xyz},f_{xzy},f_{yxz},f_{yzx},f_{zxy},f_{zyz}\text{都在点}P\text{连续}\\ \Rightarrow f_{xyz},f_{xzy},f_{yxz},f_{yzx},f_{zxy},f_{zyz}\text{在}P\text{的值相等}\end{gathered}$$

3. 复合函数的高阶偏导数：

$$\begin{gathered}z=f(x,y);x=\varphi(s,t);y=\phi(s,t)\\ \text{若}f,\varphi,\phi\text{都存在二阶连续偏导数.}\\ \frac{\partial f}{\partial s}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial s};\frac{\partial f}{\partial t}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\\ \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial s}\right)=\frac{\partial^2 f}{\partial s^2}=\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}\right)\\ =\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{\partial x}{\partial s}+\frac{\partial f}{\partial x}\cdot\frac{\partial}{\partial s}\left(\frac{\partial x}{\partial s}\right)+\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial y}\right)\cdot\frac{\partial y}{\partial s}+\frac{\partial f}{\partial y}\cdot\frac{\partial}{\partial s}\left(\frac{\partial y}{\partial s}\right)\\ =\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial s}+\frac{\partial^2 f}{\partial x\partial y}\frac{\partial y}{\partial s}\right)\cdot\frac{\partial x}{\partial s}+\frac{\partial f}{\partial x}\cdot\frac{\partial^2 x}{\partial s^2}+\left(\frac{\partial^2 f}{\partial y\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial s}\right)\cdot\frac{\partial y}{\partial s}+\frac{\partial f}{\partial y}\cdot\frac{\partial^2 y}{\partial s^2}\\ =\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial x}{\partial s}\right)^2+2\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial s}\cdot\frac{\partial y}{\partial s}+\frac{\partial^2 f}{\partial y^2}\left(\frac{\partial y}{\partial s}\right)^2+\frac{\partial f}{\partial x}\cdot\frac{\partial^2 x}{\partial s^2}+\frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s^2}\end{gathered}$$

$$\begin{gathered}\text{同样的}\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right)=\sim\\ \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial s}\right)=\sim\\ \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial t}\right)=\sim\end{gathered}$$

1.7 中值定理和泰勒公式

1. 凸区域：区域D内任意两点之间的连线都在区域D内，称D为凸区域

$$P(x_1+\lambda(x_2-x_1),y_1+\lambda(y_2-y_1))\in D;\lambda\in[0,1]$$

2. 中值定理：

$$\begin{gathered}\text{二元函数}f\text{在凸开域}D\subset R^2\text{上连续，在}D\text{的所有内点都可微}\\ \Rightarrow\forall P(a,b),Q(a+h,b+k)\in D,\exists\theta\in(0,1)\\ \to f(a+h,b+k)-f(a,b)\\ =f_x(a+\theta h,b+\theta k)h+f_y(a+\theta h,b+\theta k)k\end{gathered}$$

Pr
$$\begin{gathered}\Phi(t)=f(a+th,b+tk);\\ \Phi(1)-\Phi(0)=\Phi'(\theta)\cdot 1\end{gathered}$$

Re：若D是闭凸域，则对D上的任意两点$P_1,P_2,\forall\lambda\in(0,1)$,都有$P(x_1+\lambda(x_2-x_1),y_1+\lambda(y_2-y_1))\in\operatorname{int}D$,则对$D$上连续，$\operatorname{int}D$内可微的函数$f$，只要$P,Q\in D,\exists\theta\in(0,1)$成立中值定理

Re：若f在区域D上存在偏导数，且$f_x = f_y \equiv 0$则在区域D上为常量函数(区域必能被凸剖分)

3. 泰勒定理：

$$\begin{array}{cc} & \text{函数} f \text{在} P_0(x_0,y_0) \text{在某领域} U_{P_0} \text{上有直到} n+1 \text{阶的连续偏导数} \\ \Rightarrow & \text{对} U_{P_0} \text{内任意一点} (x_0+h, y_0+k) \text{存在相应的} \theta \in (0,1) \\ \rightarrow & f(x_0+h, y_0+k) \\ = & f(x_0,y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0,y_0) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0) \\ & + \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0,y_0) + \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h, y_0+\theta k) \\ & \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^m f(x_0,y_0) = \sum_{i=0}^{m} C_m^i \frac{\partial^m}{\partial x^i \partial y^{m-i}} f(x_0,y_0) h^i k^{m-i} \end{array}$$

Re：公式中的余项$R_n = o(\rho^n), \rho = \sqrt{h^2+k^2} = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h, y_0+\theta k)$

1.8 极值问题

1. $f: U_{P_0} \rightarrow R, \forall P \in U_{P_0}, f(P) \leqslant f(P_0)$,称$P_0$为$f$的极大值点，相应的$f(P) \geqslant f(P_0)$

   Re：这里的极值点只限于定义域的内点

2. 极值的必要条件：f在点P_0存在偏导数，且在P_0取得极值$\Rightarrow f_x(P) = f_y(P) = 0$

3. 稳定点：$f_x(P) = f_y(P) = 0$的所有点

4. 黑塞矩阵：

$$\begin{gathered} f \text{在} P \text{具有二阶连续偏导数} \\ H_f(P_0) = \begin{pmatrix} f_{xx}(P_0) & f_{xy}(P_0) \\ f_{yx}(P_0) & f_{yy}(P_0) \end{pmatrix} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}_{P_0} \end{gathered}$$

5. 极值充分条件：

$$\begin{array}{cc} & \text{设二元函数} f \text{在} P_0 \text{的某领域} U_{P_0} \text{上具有二阶连续偏导数，} P_0 \text{是} f \text{的稳定点} \\ \Rightarrow & H_f(P_0)\text{是}\begin{cases} \text{正定矩阵，} f \text{在} P_0 \text{取得极小值} \\ \text{负定矩阵，} f \text{在} P_0 \text{取得极大值} \\ \text{不定矩阵，} f \text{在} P_0 \text{不取得极值} \end{cases} \end{array}$$

$$\begin{array}{cc} \text{Pr} & f(x,y) - f(x_0,y_0) = \frac{1}{2}(\Delta x, \Delta y) H_f(P_0) (\Delta x, \Delta y)^T + o(\Delta x^2 + \Delta y^2). \\ & H_f(P_0)\text{正定} \Rightarrow \text{二次型} Q(\Delta x, \Delta y) = (\Delta x, \Delta y) H_f(P_0) (\Delta x, \Delta y)^T > 0 \\ & \frac{Q(\Delta x, \Delta y)}{(\Delta x^2 + \Delta y^2)} = (u,v) H_f(P_0) (u,v)^T = \Phi(u,v) \\ & u = \frac{\Delta x}{\sqrt{\Delta x^2 + \Delta y^2}}; v = \frac{\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}} \\ & \Phi\text{是}(u,v)\text{的连续函数}. u^2+v^2 = 1 \text{因此在单位圆上必有最小值} 2q \geqslant 0. (u,v) \neq (0,0) \rightarrow q > 0 \\ & Q\,(\Delta x, \Delta y) \geqslant 2q(\Delta x^2 + \Delta y^2) \\ & \rightarrow f(x,y) - f(x_0,y_0) \geqslant q(\Delta x^2 + \Delta y^2) + o(\Delta x^2 + \Delta y^2) \geqslant 0 \\ & \rightarrow f\text{在}(x_0,y_0)\text{取得最小值} \end{array}$$

同理H_f在P_0是负定矩阵时，则f取得最大值

对于二元函数的特殊情况：

$$\begin{array}{cl}
f_{xx}(P_0)>0,(f_{xx}f_{yy}-f_{xy}^2)(P_0)>0 & f\text{在}P_0\text{取得极小值}\\
f_{xx}(P_0)<0,(f_{xx}f_{yy}-f_{xy}^2)(P_0)>0 & f\text{在}P_0\text{取得极大值}\\
(f_{xx}f_{yy}-f_{xy}^2)(P_0)<0 & f\text{在}P_0\text{不取得极值}\\
(f_{xx}f_{yy}-f_{xy}^2)(P_0)=0 & f\text{在}P_0\text{处不能判断}
\end{array}$$

Re：这只是正定矩阵的理论而已

$$\begin{array}{cc}
\text{矩阵的顺序主子式全大于0} & \text{矩阵正定}\\
\text{矩阵的顺序主子式在正负交替}\wedge|a_{11}|<0 & \text{矩阵负定}\\
\text{顺序主子式中最后一个为0，前面都为正定或负定} & \text{矩阵不定}\\
\text{顺序主子式前面为0} & \text{矩阵半正定}
\end{array}$$

2 Trick

1. 最小二乘法理论：

$$\begin{gathered}
\text{一系列观测点}(x_i,y_i),\text{确定直线使得}\\
f(a,b)=\sum_{i=1}^{n}(ax_i+b-y_i)^2\text{取得最小值} \quad y\text{上取最小;整体最小}\\
f_a=2\sum_{i=1}^{n}x_i(ax_i+b-y_i)=0\\
f_b=2\sum_{i=1}^{n}(ax_i+b-y_i)=0\\
\rightarrow\begin{cases} a\sum x_i^2+b\sum x_i=\sum x_iy_i\\ a\sum x_i+bn=\sum y_i\end{cases}\\
\rightarrow\bar{a}=\frac{n\sum x_iy_i-(\sum x_i)(\sum y_i)}{n\sum x_i^2-(\sum x_i)^2}\\
\bar{b}=\frac{(\sum x_i^2)(\sum y_i)-(\sum x_iy_i)(\sum x_i)}{n\sum x_i^2-(\sum x_i)^2}\\
\text{验证此点确实是极小值点}\\
A=f_{aa}=2\sum x_i^2\\
B=f_{ab}=2\sum x_i\\
C=f_{bb}=2n\\
D=AC-B^2=4n\sum x_i^2-4\big(\sum x_i\big)^2>0\\
\rightarrow H_f\text{是正定阵}\rightarrow f\text{在此点取得极小值}
\end{gathered}$$