第六章 Riemann-Stieltjes 积分

Riemann积分的定义依赖于实数的序关系

1 积分的定义和存在性

定义 1.1. *区间的分法. Riemann积分*

$$\text{分法}\quad [a,b]\text{是给定区间},[a,b]\text{的分法指有限点集}\{x_1,\ldots,x_n\}$$
$$a=x_0\leqslant x_1\leqslant\cdots\leqslant x_{n-1}\leqslant x_n=b$$
$$\Delta x_i=x_i-x_{i-1},i\in 1\ldots n$$

Riemann积分

$$R\text{积分}\qquad f\text{是}[a,b]\text{上的有界实函数}.\forall[a,b]\text{的分法}P.$$
$$M_i=\sup f(x).\,(x_{i-1}\leqslant x\leqslant x_i)$$
$$m_i=\inf f(x).\,(x_{i-1}\leqslant x\leqslant x_i)$$

$$\text{达布上和}\qquad U(P,f)=\sum_{i=1}^{n}M_i\Delta x_i$$
$$\text{达布下和}\qquad L(P,f)=\sum_{i=1}^{n}m_i\Delta x_i$$

$$\text{上积分}\qquad \overline{\int_a^b} f\,\mathrm{d}x=\inf U(P,f).$$
$$\text{下积分}\qquad \underline{\int_a^b} f\,\mathrm{d}x=\sup L(P,f).$$

$$R\text{积分}\quad \overline{\int_a^b} f\mathrm{d}x=\underline{\int_a^b} f\mathrm{d}x\to\int_a^b f\mathrm{d}x\text{定义合理.称为}R\text{积分}$$
$$\mathfrak{R}\text{表示所有黎曼可积的函数构成的集合}$$

R积分的上下界问题

$$f\text{有界}\to m\leqslant f\leqslant M.\,(a\leqslant x\leqslant b)$$
$$\to m(b-a)\leqslant L(P,f)\leqslant U(P,f)\leqslant M(b-a)$$
$$\to\text{每个有界函数}f,\ f\text{的上积分和下积分都有意义.但它们不一定相等}$$

定义 1.2. *Stieltjes积分*

$$\text{函数}\alpha\text{是}[a,b]\text{上的单调增函数}.\forall[a,b]\text{的分法}P$$
$$\Delta\alpha_i=\alpha(x_i)-\alpha(x_{i-1}),\Delta\alpha_i\geqslant 0$$
$$\text{对于}[a,b]\text{上的有界实函数}f.$$
$$\text{达布上和}\qquad U(P,f,\alpha)=\sum_{i=1}^{n}M_i\Delta\alpha_i$$
$$\text{达布下和}\qquad L(P,f,\alpha)=\sum_{i=1}^{n}m_i\Delta\alpha_i$$
$$\text{上积分}\qquad \overline{\int_a^b} f\mathrm{d}\alpha=\inf U(P,f,\alpha)$$
$$\text{下积分}\qquad \underline{\int_a^b} f\mathrm{d}\alpha=\sup L(P,f,\alpha)$$

$$S\text{积分}\quad \overline{\int_a^b} f\mathrm{d}\alpha=\underline{\int_a^b} f\mathrm{d}\alpha\to\int_a^b f\mathrm{d}\alpha\text{定义合理.称为Stieltjes积分}$$

Remark: 一般情况下,α不一定是连续的.

由于一阶微分的不变性，$\mathrm{d}\alpha(x)$简记作$\mathrm{d}\alpha$.这样没有任何损失

定义 1.3. *分法的加细*

$$\text{分法}P\text{的加细}P^*.P\subset P^*.P_1,P_2\text{是分法},P_1\cup P_2\text{称为共同加细}$$

定理 1.4. P^*是P的加细

$$\begin{aligned} L(P,f,\alpha)&\leqslant L(P^*,f,\alpha) \quad \text{达布下和增长}\\ U(P,f,\alpha)&\geqslant U(P^*,f,\alpha) \quad \text{达布上和减小}\end{aligned}$$

证明.

$$\begin{gathered}
\text{设}P^*\text{比}P\text{增加了一个点}x^*.x_{i-1}<x^*<x_i\\
w_1=\inf f(x),(x_{i-1}\leqslant x\leqslant x^*)\\
w_2=\inf f(x),(x^*\leqslant x\leqslant x_i).\\
m_i=\inf f(x),(x_{i-1}\leqslant x\leqslant x_i).\\
\to w_1\geqslant m_i,w_2\geqslant m_i\\
\to L(P^*,f,\alpha)-L(P,f,\alpha)\\
=w_1[\alpha(x^*)-\alpha(x_{i-1})]+w_2[\alpha(x_i)-\alpha(x^*)]-m_i[\alpha(x_i)-\alpha(x_{i-1})]\\
=(w_1-m_i)[\alpha(x^*)-\alpha(x_{i-1})]+(w_2-m_i)[\alpha(x_i)-\alpha(x^*)]\\
\geqslant 0\\
\to L(P^*,f,\alpha)\geqslant L(P,f,\alpha)\\
\text{对于任意分法}P^*,\text{是}P\text{添加了}n\text{个点}x^*\text{由数学归纳法可得一般结论}
\end{gathered}$$

□

定理 1.5. $\underline{\int}f\mathrm{d}\alpha\leqslant\overline{\int}f\mathrm{d}\alpha$

证明.

$$\begin{gathered}
P^*\text{是}P_1,P_2\text{的共同加细}\\
L(P_1,f,\alpha)\leqslant L(P^*,f,\alpha)\leqslant U(P^*,f,\alpha)\leqslant U(P^2,f,\alpha)\\
\to L(P_1,f,\alpha)\leqslant U(P_2,f,\alpha)\\
\to \sup_{P\in\mathcal{P}}L(P,f,\alpha)=\underline{\int}f\mathrm{d}\alpha\leqslant U(P_2,f,\alpha)\\
\to\underline{\int}f\mathrm{d}\alpha\leqslant\overline{\int}f\mathrm{d}\alpha
\end{gathered}$$

□

定理 1.6. f在$[a,b]$上$f\in\mathfrak{R}(\alpha)\Leftrightarrow\forall\varepsilon>0,\exists P\to U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$

证明.

$$\begin{gathered}
\forall\varepsilon>0,\exists P\to U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon\to f\in\mathfrak{R}(\alpha)\\
\forall P,L(P,f,\alpha)\leqslant\underline{\int}f\mathrm{d}\alpha\leqslant\overline{\int}f\mathrm{d}\alpha\leqslant U(P,f,\alpha)\\
\to 0\leqslant\overline{\int}f\mathrm{d}\alpha-\underline{\int}f\mathrm{d}\alpha<\varepsilon\\
\to f\in\mathfrak{R}(\alpha).
\end{gathered}$$

$$\begin{gathered}
f\in\mathfrak{R}(\alpha)\to\forall\varepsilon>0,\exists P\to U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon\\
f\in\mathfrak{R}(\alpha),\forall\varepsilon>0,\exists P_1,P_2\\
\to U(P_2,f,\alpha)-\int f\mathrm{d}\alpha<\frac{\varepsilon}{2}\\
\int f\mathrm{d}\alpha-L(P_1,f,\alpha)<\frac{\varepsilon}{2}\\
P\text{是}P_1,P_2\text{的共同加细}\\
\to U(P,f,\alpha)\leqslant U(P_2,f,\alpha)<\int f\mathrm{d}\alpha+\frac{\varepsilon}{2}<L(P_1,f,\alpha)+\varepsilon\leqslant L(P,f,\alpha)+\varepsilon\\
\to U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon
\end{gathered}$$

□

定理 1.7.

$$U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$$

1. 对某个P和某个ε成立上式$\to$对此ε,P的任意加细此式也成立
2. 上式对P成立，s_i,t_i是$[x_{i-1},x_i]$内的任意点$\to\sum_1^n|f(s_i)-f(t_i)|\Delta\alpha_i<\varepsilon$
3. $f\in\mathfrak{R}(\alpha)$且2.成立$\to\left|\sum_1^n f(t_i)\Delta\alpha_i-\int_a^b f\mathrm{d}\alpha\right|<\varepsilon$

证明.

1. 1.4可得

2. $$s_i,t_i\in[x_{i-1},x_i]\to f(s_i),f(t_i)\in[m_i,M_i]$$
   $$\to\sum_1^n|f(s_i)-f(t_i)|\Delta\alpha_i\leqslant U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$$

3. $$L(P,f,\alpha)\leqslant\sum f(t_i)\Delta\alpha_i\leqslant U(P,f,\alpha)$$
   $$L(P,f,\alpha)\leqslant\int f\mathrm{d}\alpha\leqslant U(P,f,\alpha)$$
   $$\to\left|\sum f(t_i)\Delta\alpha_i-\int f\mathrm{d}\alpha\right|<\varepsilon$$

□

定理 1.8. 闭区间上的函数连续则R可积

$$f在[a,b]上连续\to f在[a,b]上\in\mathfrak{R}$$

证明.

$$\begin{aligned}
&f在[a,b]上连续\to f在[a,b]上一致连续\\
&\to\forall\eta>0,\exists\delta>0,x\in[a,b],t\in[a,b],|x-t|<\delta\to|f(x)-f(t)|<\eta\\
&P是\Delta x_i<\delta的分法\to M_i-m_i\leqslant\eta\\
&\to U(P,f,\alpha)-L(P,f,\alpha)=\sum(M_i-m_i)\Delta\alpha_i\\
&\leqslant\eta\sum\Delta\alpha_i\\
&=\eta[\alpha(b)-\alpha(a)]<\varepsilon\\
&\to f\in\mathfrak{R}(\alpha)
\end{aligned}$$

□

定理 1.9. f在$[a,b]$上单调,α在$[a,b]$上连续且单调$\to f\in\mathfrak{R}(\alpha)$

证明.

$$\begin{aligned}
&\forall\varepsilon>0,\forall n\in N^+,分法P\to\Delta\alpha_i=\frac{\alpha(b)-\alpha(a)}{n}.\ \alpha连续,所以这是能做到的\\
&f单调增\to M_i=f(x_i),m_i=f(x_{i-1})\\
&\exists n\in N^+\to U(P,f,\alpha)-L(P,f,\alpha)=\frac{\alpha(b)-\alpha(a)}{n}\sum_1^n[f(x_i)-f(x_{i-1})]\\
&=\frac{\alpha(b)-\alpha(a)}{n}[f(b)-f(a)]\\
&<\varepsilon\\
&\to f\in\mathfrak{R}(\alpha)
\end{aligned}$$

□

定理 1.10. f有界且只有有限个间断点.α在f的每个间断点处连续$\to f\in\mathfrak{R}(\alpha)$

证明.

$$\forall\varepsilon>0.M=\sup|f(x)|.E\text{是}f\text{的间断点集}.E\text{有限},\alpha\text{在}E\text{的每点连续}$$
$$\to\exists j\in N\to\exists[u_j,v_j]\subset[a,b]\text{覆盖}E.\wedge[u_i,v_i]\cap[u_j,v_j]=\varnothing$$
$$\text{let: }\sum_1^n(\alpha(v_i)-\alpha(u_i))<\varepsilon$$
$$\text{从}[a,b]\text{中去掉开区间}(u_j,v_j).\text{剩下的集}K\text{是闭的且有界的}\to K\text{是紧的}$$
$$\to\exists\delta>0\to s\in K,t\in K,d(s,t)<\delta\to d(f(s),f(t))<\varepsilon$$

□

$$\text{构造分法}: P=\{x_0,x_1,\ldots,x_n\}.\{u_i\}\subset P,\{v_i\}\subset P.\forall x\in(u_i,v_i),x\not\subset P$$
$$\forall x_{i-1}\neq u_i\to\Delta x_i<\delta$$
$$U(P,f,\alpha)-L(P,f,\alpha)\leqslant[\alpha(b)-\alpha(a)]\varepsilon+2M\varepsilon$$
$$\to f\in\Re(\alpha)$$

Remark: f和α有一个公共间断点,则f不一定是R可积的

定理 1.11. *闭区间内. 内函数可积, 外函数连续->复合可积*

$$f\text{在}[a,b]\text{上}\in\Re(\alpha),m\leqslant f\leqslant M.\phi\text{在}[m,M]\text{上连续}.\phi(f(x))\in\Re(\alpha)$$

证明.

$$\forall\varepsilon>0.\phi\text{在}[a,b]\text{一致连续}\to\exists\delta>0\wedge\delta<\varepsilon.|s-t|<\delta\to|\phi(s)-\phi(t)|<\varepsilon$$
$$f\in\Re(\alpha)\to\exists P=\{x_0,x_1,\ldots,x_n\}\to U(P,f,\alpha)-L(P,f,\alpha)<\delta^2$$
$$M_i^*,m_i^*\text{是}h\text{的子区间极值}.$$
$$M_i-m_i<\delta\Rightarrow i\in A.M_i-m_i\geqslant\delta\to i\in B$$
$$\forall i\in A.M_i^*-m_i^*<\varepsilon$$
$$\forall i\in B.M_i^*-m_i^*<2K.K=\sup|\phi(t)|,t\in[m,M]$$
$$\delta\sum_{i\in B}\Delta\alpha_i\leqslant\sum_{i\in B}(M_i-m_i)\Delta\alpha_i<\delta^2$$
$$\to\sum_{i\in B}\Delta\alpha_i<\delta$$
$$\to U(P,f,\alpha)-L(P,f,\alpha)$$
$$=\sum_{i\in A}(M_i^*-m_i^*)\Delta\alpha_i+\sum_{i\in B}(M_i^*-m_i^*)\Delta\alpha_i$$
$$\leqslant\varepsilon[\alpha(b)-\alpha(a)]+2K\delta$$
$$<\varepsilon[\alpha(b)-\alpha(a)+2K]$$
$$\to\phi(f(x))\in\Re(\alpha)$$

□

2 积分的性质

定理 2.1. *积分的性质*

$$[a,b]\text{上}f_1\in\Re(\alpha),f_2\in\Re(\alpha)$$

1加法封闭 $\quad f_1+f_2\in\Re(\alpha)$

2标乘封闭 $\quad \forall c\in R,cf\in\Re(\alpha)$

3线性性 $\quad \int f_1+f_2\,\mathrm{d}\alpha=\int f_1\mathrm{d}\alpha+\int f_2\mathrm{d}\alpha$

$$\int cf\,\mathrm{d}\alpha=c\int f\,\mathrm{d}\alpha$$

4保不等式 $\quad f_1\leqslant f_2\to\int f_1\leqslant\int f_2$

5区间可分 $\quad [a,b]\text{上}f\in\Re.c\in(a,b)\to f\text{在}[a,c],[c,b]\text{上都}R\text{可积}$

$$\int_a^bf=\int_a^cf+\int_c^bf$$

6 $\quad f\text{在}[a,b]\text{上}|f|\leqslant M\to\left|\int_a^bf\right|\leqslant M(b-a)$

7 $\quad f\in\Re(\alpha_1)\wedge f\in\Re(\alpha_2)\to\int_a^bf\mathrm{d}(\alpha_1+\alpha_2)=\int f\,\mathrm{d}\alpha_1+\int f\mathrm{d}\alpha_2$

8 $\quad f\in\Re(\alpha).c\in R^+,\int_a^bf\mathrm{d}(c\alpha)=\int cf\,\mathrm{d}\alpha=c\int f\,\mathrm{d}\alpha$

证明.

1
$$
\begin{gathered}
f_1, f_2 \in \mathfrak{R} \rightarrow \\
\exists P_1 \rightarrow U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha) < \varepsilon \\
\exists P_2 \rightarrow U(P_2, f_2, \alpha) - L(P_2, f_2, \alpha) < \varepsilon \\
\text{let: } P = P_1 \cup P_2 \\
M_i(f_1) + M_i(f_2) \geqslant M_i(f_1 + f_2) \\
m_i(f_1) + m_i(f_2) \leqslant m_i(f_1 + f_2) \\
\rightarrow U(P_1, f_1, \alpha) + U(P_2, f_2, \alpha) \geqslant U(P, f_1 + f_2, \alpha) \geqslant L(P, f_1 + f_2, \alpha) \geqslant L(P_1, f_1, \alpha) + L(P_2, f_2, \alpha) \\
\rightarrow U(P, f_1 + f_2, \alpha) - L(P, f_1 + f_2, \alpha) < 2\varepsilon \\
\rightarrow f_1 + f_2 \in \mathfrak{R}
\end{gathered}
$$

2
$$
\begin{gathered}
f \in \mathfrak{R} \rightarrow \exists P \rightarrow U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon \\
m_i(cf) = c m_i(f), M_i(cf) = c M_i(f) \\
\rightarrow cU(P, f, \alpha) \geqslant U(P, cf, \alpha) \geqslant L(P, cf, \alpha) \geqslant cL(P, f, \alpha) \\
\rightarrow U(P, cf, \alpha) - L(P, cf, \alpha) < c\varepsilon \\
\rightarrow cf \in \mathfrak{R}
\end{gathered}
$$

3
$$
\begin{gathered}
U(P, f_1, \alpha) + U(P, f_2, \alpha) \geqslant U(P, f_1 + f_2, \alpha) \geqslant L(P, f_1 + f_2, \alpha) \geqslant L(P, f_1, \alpha) + L(P, f_2, \alpha) \\
\rightarrow \lim U(P, f_1, \alpha) + U(P, f_2, \alpha) = \lim U(P, f_1 + f_2, \alpha) \\
\rightarrow \int f_1 + \int f_2 = \int (f_1 + f_2)
\end{gathered}
$$

$$
\begin{gathered}
cU(P, f, \alpha) \geqslant U(P, cf, \alpha) \geqslant L(P, cf, \alpha) \geqslant cL(P, f, \alpha) \\
\rightarrow \lim cU(P, f, \alpha) = \lim U(P, cf, \alpha) \\
\rightarrow \int cf = c \int f
\end{gathered}
$$

$\square$

4
$$
\begin{gathered}
f_1 \leqslant f_2 \\
\rightarrow M_i(f_1) \leqslant M_i(f_2), m_i(f_1) \leqslant m_i(f_2) \\
\rightarrow U(P, f_2, \alpha) \geqslant U(P, f_1, \alpha) \\
\rightarrow \int f_2 \geqslant \int f_1
\end{gathered}
$$

5
$$
\begin{gathered}
U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon \\
P' = P \cup c \\
U(P, f, \alpha) \geqslant U(P', f, \alpha) \geqslant L(P', f, \alpha) \geqslant L(P', f, \alpha) \\
\rightarrow \lim (U(P_1', f, \alpha) + U(P_2', f, \alpha)) = \lim U(P, f, \alpha) \\
\rightarrow \int_a^c f + \int_c^b f = \int_a^b f
\end{gathered}
$$

6
$$
\begin{gathered}
|f| \leqslant M \\
|m_i(f)| \leqslant M, |M_i(f)| \leqslant M \\
\rightarrow U(P, f) = \sum (b_i - a_i) M_i \\
\leqslant \sum (b_i - a_i) M \\
= (b - a) M
\end{gathered}
$$

7
$$
\begin{gathered}
U(P, f, \alpha) + U(P, f, \beta) = U(P, f, \alpha + \beta) \geqslant L(P, f, \alpha + \beta) \geqslant L(P, f, \alpha) + L(P, f, \beta) \\
\rightarrow \int f \mathrm{d}(\alpha + \beta) = \int f \mathrm{d}\alpha + \int f \mathrm{d}\beta
\end{gathered}
$$

8
$$
\begin{gathered}
c \in R^+ \rightarrow c\alpha \text{单调增} \\
c\alpha(x_1) - c\alpha(x_2) = c(\alpha(x_1) - \alpha(x_2)) \\
cU(P, f, \alpha) \geqslant U(P, f, c\alpha) \geqslant L(P, f, c\alpha) \geqslant cL(P, f, \alpha) \\
\rightarrow \int f \mathrm{d}(c\alpha) = c \int f \mathrm{d}\alpha = \int cf \mathrm{d}\alpha
\end{gathered}
$$

定理 2.2. *闭区间上可积函数. 逐点乘函数也可积,绝对值可积,满足绝对值不等式*

$$\begin{aligned} &\text{在}[a,b]\text{上},\ f\in\mathfrak{R},\ g\in\mathfrak{R} \\ 1.\quad & fg\in\mathfrak{R} \\ 2.\quad & |f|\in\mathfrak{R}(\alpha),\ \left|\int_a^b f\mathrm{d}\alpha\right|\leqslant\int_a^b|f|\mathrm{d}\alpha \end{aligned}$$

证明.

1. $\varphi(t)=t^2.\ \varphi\text{在}R\text{上连续}\rightarrow\varphi(f(x))=(f(x))^2\text{连续}$ 1.11
$$fg=\frac{(f+g)^2-(f-g)^2}{4}$$
$$\rightarrow fg\in\mathfrak{R}$$

2. $\varphi(t)=|t|\rightarrow\varphi\text{在}R\text{上连续}\rightarrow\varphi(f)\in\mathfrak{R}(\alpha)$
$$\int f\mathrm{d}\alpha\geqslant 0,\mathrm{let}\colon c=1;\int f\mathrm{d}\alpha<0,\mathrm{letc}=-1$$
$$\rightarrow c\int f\mathrm{d}\alpha\geqslant 0$$
$$cf\leqslant|f|$$
$$\left|\int f\mathrm{d}\alpha\right|=c\int f\mathrm{d}\alpha=\int cf\mathrm{d}\alpha\leqslant\int|f|\mathrm{d}\alpha$$

□

定义 2.3. *单位跃阶函数*

$$I\colon R\rightarrow R,\ I(x)=\begin{cases}0 & x\leqslant 0\\ 1 & x>0\end{cases}$$

定理 2.4. *跃阶函数作为S积分的增函数，在连续点处函数值是积分值*

$$a<s<b.\ f\text{在}[a,b]\text{有界},\ f\text{在}s\text{连续}\alpha(x)=I(x-s)\rightarrow\int_a^b f\mathrm{d}\alpha=f(s)$$

证明.

$$\text{分法}P=\{x_0,x_1,x_2,x_3\}.x_0=a,x_1=s<x_2<x_3=b.$$
$$M_0=0,M_1=0,M_3=1$$
$$m_0=0,m_1=0,m_3=1$$
$$\rightarrow\int_a^b f\mathrm{d}\alpha=\lim U(P,f,\alpha)=\lim L(P,f,\alpha)$$
$$U(P,f,\alpha)=M_2,L(P,f,\alpha)=m_2$$
$$f\text{在}s\text{连续}\rightarrow\lim_{x_2\rightarrow s}M_2=f(x_2)\cdot(I(x_2-s)-I(x_1-s))$$
$$=f(x_2)\cdot(1-0)$$
$$=f(s)$$
$$\rightarrow M_2(f)=f(s),m_2=f(s)$$

□

定理 2.5. *S积分与正项级数*

$$\begin{aligned} & n\in N^+,c_n\geqslant 0,\sum_\infty c_n\text{收敛}.\{s_n\}\in(a,b) && \text{收敛的正项级数，且和在开区间内} \\ & \alpha(x)=\sum_{n=1}^\infty c_nI(x-s_n) && \alpha(x)\text{是级数乘}s_n\text{的跃阶之和} \\ & f\text{在}[a,b]\text{上连续} && f\text{在闭区间连续} \\ \rightarrow\quad & \int_a^b f\mathrm{d}\alpha=\sum_{n=1}^\infty c_nf(s_n) && \text{积分等于级数乘函数在}s_n\text{点处的函数值} \end{aligned}$$

证明.

$$\alpha(x)=\sum_1^\infty c_n I(x-s_n)$$
$$I(t)\leqslant 1\rightarrow 0\leqslant c_n I(x-s_n)\leqslant c_n$$
$$\rightarrow 0\leqslant\sum c_n I(x-s_n)\leqslant\sum c_n$$
$$\rightarrow\sum c_n I(x-s_n)\text{收敛} \qquad \text{比较验敛法}$$
$$\alpha(a)=\sum c_n I(a-s_n)$$
$$=\sum c_n\cdot 0$$
$$=0$$
$$\alpha(b)=\sum c_n(b-s_n)$$
$$=\sum c_n$$
$$x_2>x_1\rightarrow\alpha(x_2)-\alpha(x_1)$$
$$=\sum c_n I(x_2-s_n)-\sum c_n I(x_1-s_n)$$
$$x_2>x_1\rightarrow I(x_2-s_n)\geqslant I(x_1-s_n)$$
$$\rightarrow\alpha(x_2)\geqslant\alpha(x_1)$$
$$\rightarrow\alpha\text{单调增}$$

□

let: $\alpha_1=\sum_1^N c_n I(x-s_n), \alpha_2=\sum_N^\infty c_n I(x-s_n)$.

$$\int_a^b f\mathrm{d}\alpha_1=\int_a^b f\mathrm{d}(\sum c_n I(x-s_n))$$
$$=\sum\left(\int_a^b f\mathrm{d}(c_n I(x-s_n))\right)$$
$$=\sum c_n(\int_a^b f\mathrm{d}(I(x-s_n)))$$
$$=\sum c_n f(s_n)$$
$$M=\sup_{x\in[a,b]}|f(x)|$$
$$\alpha_2(b)-\alpha_2(a)<\varepsilon\rightarrow\left|\int_a^b f\mathrm{d}\alpha_2\right|\leqslant M\varepsilon$$
$$\alpha=\alpha_1+\alpha_2$$
$$\rightarrow\left|\int_a^b f\mathrm{d}\alpha-\sum_1^N c_n f(s_n)\right|\leqslant M\varepsilon$$
$$\lim_{N\rightarrow\infty}\left|\int_a^b f\mathrm{d}\alpha-\sum_1^N c_n f(s_n)\right|\leqslant M\varepsilon=0$$
$$\rightarrow\int_a^b f\mathrm{d}\alpha=\sum_1^\infty c_n f(s_n)$$

定理 2.6. *S积分中的a是可微的->a'可积. f在闭区间上有界。 f可积<=>f a'可积*

$$\alpha\text{单调增}.\text{在}[a,b]\text{上}a'\in\mathfrak{R}.f\text{在}[a,b]\text{上有界}$$
$$f\in\mathfrak{R}\Leftrightarrow f\alpha'\in\mathfrak{R}$$
$$\int_a^b f\mathrm{d}\alpha=\int_a^b f(x)\alpha'(x)\mathrm{d}x$$

证明.

$$
\begin{aligned}
&f\alpha'\text{可积} \to \int f\mathrm{d}\alpha\text{存在}\\
&\alpha'\text{可积} \to \exists P \to U(P,\alpha') - L(P,\alpha') < \varepsilon\\
&\text{中值定理} \to \exists t_i \in [x_{i-1},x_i] \to \Delta\alpha_i = \alpha'(t_i)\Delta x_i\\
&\forall s_i \in [x_{i-1},x_i] \to \textstyle\sum_1^n |\alpha'(s_i) - \alpha'(t_i)|\Delta x_i < \varepsilon \qquad 1.7
\end{aligned}
$$

$$
\begin{aligned}
&M = \sup |f(x)|\\
&\textstyle\sum_1^n f(s_i)\Delta\alpha_i = \sum_1^n f(s_i)\alpha'(t_i)\Delta x_i\\
&|\textstyle\sum_1^n f(s_i)\Delta\alpha_i - \sum_1^n f(s_i)\alpha'(s_i)\Delta x_i|\\
&= |\textstyle\sum_1^n f(s_i)\alpha'(t_i)\Delta x_i - \sum_1^n f(s_i)\alpha'(s_i)\Delta x_i|\\
&= |\textstyle\sum_1^n f(s_i)(\alpha'(t_i) - \alpha'(s_i))\Delta x_i|\\
&\leqslant M\varepsilon\\
&\to \forall s_i \in [x_{i-1},x_i], \textstyle\sum_1^n f(s_i)\Delta\alpha_i \leqslant U(P,f\alpha') + M\varepsilon\\
&U(P,f,\alpha) \leqslant U(P,f\alpha') + M\varepsilon\\
&U(P,f\alpha') \leqslant U(P,f,\alpha) + M\varepsilon\\
&\to |U(P,f\alpha') - U(P,f,\alpha)| < M\varepsilon\\
&\to \left|\overline{\int_a^b} f\mathrm{d}\alpha - \overline{\int_a^b} f\alpha'\right| < M\varepsilon\\
&\varepsilon\text{的任意性} \to \forall f\text{在}[a,b]\text{上有界}\\
&\to \bar{\int}_a^b f\mathrm{d}\alpha = \bar{\int}_a^b f\alpha'\mathrm{d}x
\end{aligned}
$$

对于下积分也同理可证

$\square$

注意 2.7.

Stieltjes表现了比Riemann积分更一般的性质.若a是跃阶函数，那么积分就成了有限或无限的级数。若a可微，则积分变为Riemann积分. 这使得级数和积分的研究可以在一定程度上得到统一

定理 2.8. *换元法*

$$
\begin{gathered}
\varphi\text{是严格增的连续函数}, \varphi\colon [A,B] \to [a,b], \varphi\text{满}\\
\alpha\text{在}[a,b]\text{上单调增}, f\text{在}[a,b]\text{上}R\text{可积}\\
\text{在}[A,B]\text{上let}\colon \beta, g\\
\beta(y) = \alpha(\varphi(y)), g(y) = f(\varphi(y))
\end{gathered}
$$

$$
\to \qquad
\begin{gathered}
g \in \mathfrak{R}(\beta)\\
\int_a^b f\mathrm{d}\alpha = \int_A^B g\mathrm{d}\beta
\end{gathered}
$$

证明.

$$
\begin{aligned}
&\qquad [a,b]\text{的分法}P = \{x_0,\ldots,x_n\}, [A,B]\text{的分法}Q = [y_0,\ldots,y_n].\\
&\qquad x_i = \varphi(y_i)\\
&\qquad f(x_i) = f(\varphi(y_i)) = g(y_i)\\
&\to \qquad U(Q,g,\beta) = U(P,f,\alpha); L(Q,g,\beta) = L(P,f,\alpha)\\
&\qquad f \in \mathfrak{R}(\alpha), \exists P \to U(P,f,\alpha) \geqslant \textstyle\int f\mathrm{d}\alpha \geqslant L(P,f,\alpha)\\
&\to \qquad g \in \mathfrak{R}(\beta)\\
&\to \qquad \textstyle\int_a^b f\mathrm{d}\alpha = \int_A^B g\mathrm{d}\beta
\end{aligned}
$$

$$
\begin{aligned}
&\text{特殊的} \qquad \alpha(x) = x.\ \beta = \varphi, \varphi' \in \mathfrak{R}.\\
&\textstyle\int_a^b f(x)\mathrm{d}x = \int_A^B f(\varphi(y))\varphi'(y)\mathrm{d}y
\end{aligned}
$$

注意 2.9. *这表明Stieltjes积分中的a可微和换元法是等价的* $\square$

3 积分与微分

这表明某种程度上，实函数的积分和微分是互逆运算

定理 3.1. *f可积->变上限积分函数F连续（一致连续）.f在某点x连续->F在x可微*

$$
\begin{aligned}
& f\text{在}[a,b]\text{上}R\text{可积}, \forall a \leqslant x \leqslant b \\
& F(x) = \int_a^x f(t)\mathrm{d}t \\
1.\quad & F\text{在}[a,b]\text{上连续} \\
2.\quad & f\text{在}x_0\text{连续} \rightarrow F\text{在}x_0\text{可微} \wedge F'(x_0) = f(x_0)
\end{aligned}
$$

证明.

$$
\begin{aligned}
1.\quad & f \in \mathfrak{R}, f\text{有界}. \forall t \in [a,b], |f(t)| \leqslant M. a \leqslant x < y \leqslant b \\
& \rightarrow |F(y) - F(x)| = |\textstyle\int_x^y f(t)\mathrm{d}t| \leqslant M(y-x) \\
& |y - x| < \tfrac{\varepsilon}{M} \rightarrow |F(y) - f(x)| < \varepsilon
\end{aligned}
$$

$$
\begin{aligned}
2.\quad & f\text{在}x_0\text{连续} \rightarrow \forall \varepsilon > 0, \exists \delta > 0 \rightarrow |t - x_0| < \delta \wedge a \leqslant t \leqslant b \rightarrow |f(t) - f(x_0)| < \varepsilon \\
& s, x_0, t \in U_{x_0}(\delta) \wedge s \leqslant x_0 \leqslant t \\
& \left|\tfrac{F(t) - F(s)}{t - s} - f(x_0)\right| = \left|\tfrac{1}{t-s}\textstyle\int_s^t (f(u) - f(x_0))\mathrm{d}u\right| \leqslant \varepsilon \\
& \left|\tfrac{1}{t-s}(F(t) - F(s)) - f(x_0)\right| = \left|\tfrac{1}{t-s}\textstyle\int_s^t f(u)\mathrm{d}u - f(x_0)\right| \\
& |f(u) - f(x_0)| < \varepsilon, t - s < \delta \\
& \rightarrow \left|\tfrac{1}{t-s}\textstyle\int_s^t f(u)\mathrm{d}u\right| \leqslant (f(x_0) + \varepsilon) \cdot \tfrac{1}{\delta} \\
& \rightarrow \left|\tfrac{1}{t-s}\textstyle\int_s^t f(u)\mathrm{d}u - f(x_0)\right| \leqslant \varepsilon \\
& ???
\end{aligned}
$$

□

定理 3.2. *微积分基本定理*

$$
\begin{aligned}
& f\text{在闭区间}[a,b]R\text{可积}, \text{在}[a,b]\text{上有可微函数}F \wedge F' = f \\
\rightarrow \quad & \int_a^b f(x)\mathrm{d}x = F(b) - F(a)
\end{aligned}
$$

证明.

$$
\begin{aligned}
& \forall \varepsilon > 0. [a,b]\text{的分法}P = \{x_0, \ldots, x_n\} \rightarrow U(P,f) - L(P,f) < \varepsilon \\
& \text{由中值定理} \rightarrow \exists t_i \in [x_{i-1}, x_i] \rightarrow F(x_i) - F(x_{i-1}) = f(t_i)\Delta x_i \\
& \rightarrow \textstyle\sum_1^n f(t_i)\Delta x_i = F(b) - F(a) \\
& \rightarrow \left|F(b) - F(a) - \textstyle\int_a^b f(x)\mathrm{d}x\right| < \varepsilon \\
& \rightarrow F(b) - F(a) = \textstyle\int_a^b f(x)\mathrm{d}x
\end{aligned}
$$

□

定理 3.3. *分部积分法*

$$
\begin{aligned}
& F, G\text{是}[a,b]\text{上的可微函数}. F' = f \in \mathfrak{R}, G' = g \in \mathfrak{R} \\
\rightarrow \quad & \int_a^b F(x)g(x)\mathrm{d}x = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\mathrm{d}x
\end{aligned}
$$

证明.

$$
\begin{gathered}
\text{let: } H(x)=F(x)G(x). \\
f\text{可积}\to F\text{连续}\to F\text{可积} \\
F\text{可积，}g\text{可积}\to Fg\text{可积} \\
H'=(F(x)G(x))'=f(x)G(x)+F(x)g(x) \\
\to H'\text{可积}
\end{gathered}
$$

□

$$
\begin{gathered}
H'=f(x)G(x)+F(x)g(x) \\
H(b)-H(a)=\int_a^b H'(x)\mathrm{d}x \\
=\int_a^b (f(x)G(x)+F(x)g(x))\mathrm{d}x \\
=\int_a^b f(x)G(x)\mathrm{d}x+\int_a^b F(x)g(x)\mathrm{d}x \\
\to\int_a^b F(x)g(x)\mathrm{d}x=F(b)G(b)-F(a)G(a)-\int_a^b f(x)G(x)\mathrm{d}x
\end{gathered}
$$

4 向量值函数的积分

定义 4.1.

$$
\begin{gathered}
f_1,\ldots,f_k\text{是}[a,b]\text{上的实函数}.\boldsymbol{f}\colon[a,b]\to R^k,\ \boldsymbol{f}=(f_1,\ldots,f_k). \\
\alpha\text{在}[a,b]\text{上单调增}\to\boldsymbol{f}\in\mathfrak{R}(\alpha)\Leftrightarrow f_i\in\mathfrak{R}(\alpha) \\
\int_a^b\boldsymbol{f}\mathrm{d}\alpha=\left(\int_a^b f_1\mathrm{d}\alpha,\ldots,\int_a^b f_k\mathrm{d}\alpha\right)
\end{gathered}
$$

定理 4.2. *向量值函数也具有微积分学基本定理*

$$
\begin{gathered}
\boldsymbol{f},\boldsymbol{F}\colon[a,b]\to R^k.\ \boldsymbol{f}\text{可积}\wedge\boldsymbol{F}'=\boldsymbol{f} \\
\to\int_a^b\boldsymbol{f}(t)\mathrm{d}t=\boldsymbol{F}(b)-\boldsymbol{F}(a)
\end{gathered}
$$

证明. □

定理 4.3. *向量值函数的绝对值不等式等价于范数不等式*

$$
\begin{gathered}
\boldsymbol{f}\colon[a,b]\to R^k.\alpha\text{是}[a,b]\text{上的增函数},\ \boldsymbol{f}\text{可积}\to|\boldsymbol{f}|\text{可积} \\
\left|\int_a^b\boldsymbol{f}\mathrm{d}\alpha\right|\leqslant\int_a^b|\boldsymbol{f}|\mathrm{d}\alpha
\end{gathered}
$$

证明.

$$
\begin{gathered}
f_1,\ldots,f_n\text{是}f\text{的分量} \\
|\boldsymbol{f}|=(f_1^2+\cdots+f_n^2)^{1/2} \\
f_i\in\mathfrak{R}\to f_i^2\in\mathfrak{R}\to\sum f_i^2\in\mathfrak{R}\to\sqrt{\sum(f_i^2)}\in\mathfrak{R}
\end{gathered}
$$

$$
\begin{gathered}
|\boldsymbol{y}|^2=\sum y_i^2=\sum y_i\int f_i\mathrm{d}\alpha \\
=\int(\textstyle\sum y_if_i)\mathrm{d}\alpha \\
\sum y_if_i\leqslant|\boldsymbol{y}|\cdot|\boldsymbol{f}(t)| \qquad \text{Schwarz不等式} \\
\to|\boldsymbol{y}|^2\leqslant|\boldsymbol{y}|\cdot\int|\boldsymbol{f}|\mathrm{d}\alpha \\
\to|\boldsymbol{y}|\leqslant\int|\boldsymbol{f}|\mathrm{d}\alpha \\
y_i=\int_a^b f_i(x)\mathrm{d}\alpha\to\text{原式成立} \\
???
\end{gathered}
$$

□

5 可求长曲线

定义 5.1. *曲线、可求长曲线*

$$\begin{array}{cc} \gamma\text{曲线} & \gamma\colon [a,b]\to R^k \\ \text{弧} & \gamma\text{是}1-1\text{的} \\ \text{闭曲线} & \gamma(a)=\gamma(b) \end{array}$$

注意 **5.2.** *这里的曲线定义是映射而不是点集. γ的值域是几何上的曲线，但这里不同的曲线可以有相同的值域*

$$[a,b]\text{给定分法}P=\{x_0,\ldots,x_n\}\text{和曲线}\gamma$$

$$\Lambda(P,r)=\sum_1^n |\gamma(x_i)-\gamma(x_{i-1})|.\text{其中}||\text{为范数诱导的距离}$$

$$\text{这里可以理解为曲线上分点连接的折线长}$$

$$\begin{array}{cc} \text{曲线长} & \Lambda(\gamma)=\sup\limits_{P\in\mathcal{P}}\Lambda(P,\gamma) \\ \text{可求长} & \Lambda(\gamma)<\infty \\ R\text{积分表示曲线长} & \gamma'\text{连续}\to\gamma\text{的长可被}R\text{积分表示} \end{array}$$

定理 5.3. *曲线在闭区间上连续可微，则此曲线可求长*

$$\gamma'\text{在}[a,b]\text{上连续},\gamma\text{可求长}$$

$$\Lambda(\gamma)=\int_a^b|\gamma'(t)|\mathrm{d}t$$

证明.

$$a\leqslant x_{i-1}<x_i\leqslant b$$

$$\to|\gamma(x_i)-\gamma(x_{i-1})|=\left|\int_{x_{i-1}}^{x_i}\gamma'(t)\mathrm{d}t\right|\leqslant\int_{x_{i-1}}^{x_i}|\gamma'(t)|\mathrm{d}t$$

$$\forall P\in\mathcal{P}.\Lambda(P,\gamma)\leqslant\int_a^b|\gamma'(t)|\mathrm{d}t$$

$$\to\Lambda(\gamma)\leqslant\int_a^b|\gamma'(t)|\mathrm{d}t$$

$$\forall\varepsilon>0,\gamma'\text{在}[a,b]\text{上一致连续},\to\exists\delta>0,d(s,t)<\delta\to|\gamma'(s)-\gamma'(t)|<\varepsilon$$

$$\to P\text{是}[a,b]\text{的分法},\forall i,\Delta x_i<\delta,\forall x_{i-1}\leqslant t\leqslant x_i\to|\gamma'(t)|\leqslant|\gamma'(x_i)|+\varepsilon$$

$$\to\int_{x_{i-1}}^{x_i}|\gamma'(t)|\mathrm{d}t\leqslant|\gamma'(x_i)|\Delta x_i+\varepsilon\Delta x_i$$

$$=\left|\int_{x_{i-1}}^{x_i}(\gamma'(t)+\gamma'(x_i)-\gamma'(t))\mathrm{d}t\right|+\varepsilon\Delta x_i$$

$$\leqslant\left|\int_{x_{i-1}}^{x_i}\gamma'(t)\mathrm{d}t\right|+\left|\int_{x_{i-1}}^{x_i}(\gamma'(x_i)-\gamma'(t))\mathrm{d}t\right|+\varepsilon\Delta x_i$$

$$\leqslant|\gamma(x_i)-\gamma(x_{i-1})|+2\varepsilon\Delta x_i$$

$$\int_a^b|\gamma'(t)|\mathrm{d}t\leqslant\Lambda(P,\gamma)+2\varepsilon(b-a)$$

$$\leqslant\Lambda(\gamma)+2\varepsilon(b-a)$$

$$\to\int_a^b|\gamma'(t)|\mathrm{d}t\leqslant\Lambda(\gamma)$$

$$\to\int_a^b|\gamma'(t)|\mathrm{d}t=\Lambda(\gamma)$$

对R^2上弧的微分: $\mathrm{d}s=\sqrt{((\mathrm{d}x)^2+(\mathrm{d}y)^2)}=\sqrt{\mathrm{d}x^2+\mathrm{d}y^2}$

$\mathrm{d}^2x\colon\mathrm{d}(\mathrm{d}x).\mathrm{d}x^2\colon(\mathrm{d}x)^2$ □

习题

1. Proof: α在$[a,b]$上增. $a\leqslant x_0\leqslant b$, α在x_0连续. $f(x_0)=1$, $x\neq x_0\to f(x)=0$. Proof: $f\in\Re(\alpha)\wedge\int f\mathrm{d}\alpha=0$

$$
\begin{aligned}
U(P,f,\alpha) &= \sum M(f(x_i))\cdot\Delta\alpha_i\\
&= 1\Delta\alpha_{x_0}\\
&= \Delta\alpha(x-x_0)\\
\alpha\text{在}x_0\text{连续} &\to \alpha(x-x_0)=0\\
&\to U(P,f,\alpha)=0\\
L(P,f,\alpha) &= \sum m(f(x_i))\cdot\Delta\alpha_i\\
&= \sum 0\cdot\Delta\alpha_i\\
&= 0\\
\to U(P,f,\alpha) &= L(P,f,\alpha)=0\\
&\to\int f\mathrm{d}\alpha=0
\end{aligned}
$$

2.