第四章 多项式

1 复共轭与绝对值

定义 1. *复数Complex number; 实部real part,Re z; 虚部imaginary part, Im z*

$$\begin{array}{lll} & & z=a+bi, a, b\in R \\ \text{实部} & \operatorname{Re} z & a \\ \text{虚部} & \operatorname{Im} z & b \end{array}$$

$\forall z\in C. z=\operatorname{Re} z+i\cdot\operatorname{Im} z$

定义 2. *复共轭Complex conjugate, $\bar{z}$; 绝对值 Absolute value, $|z|$*

$$\begin{array}{ccc} & z\in C. & \\ \bar{z} & \operatorname{Re} z-i\cdot\operatorname{Im} z & a-bi \\ |z| & \sqrt{(\operatorname{Re} z)^2+(\operatorname{Im} z)^2} & \sqrt{a^2+b^2} \end{array}$$

定理 3. *复数的性质*

$$\begin{array}{lcc} & & w, z\in C. \\ 1 & \text{共轭和} & z+\bar{z}=2\operatorname{Re} z \\ 2 & \text{共轭差} & z-\bar{z}=2i\operatorname{Im} z \\ 3 & \text{共轭积} & z\bar{z}=|z|^2 \\ 4 & \text{共轭加、乘} & \overline{z+w}=\bar{z}+\bar{w}, \overline{zw}=\bar{z}\cdot\bar{w} \\ 5 & \text{共轭复合} & \bar{\bar{z}}=z \\ 6 & \text{绝对值,部分小于整体} & |\operatorname{Re} z|\leqslant|z|, |\operatorname{Im} z|\leqslant|z| \\ 7 & \text{共轭的绝对值} & |\bar{z}|=|z| \\ 8 & \text{绝对值积} & |zw|=|z|\cdot|w| \\ 9 & \text{三角不等式} & |w+z|\leqslant|w|+|z| \end{array}$$

证明.

$$\begin{aligned} |w+z|^2=&(w+z)(\bar{w}+\bar{z}) \\ =&w\bar{w}+z\bar{z}+w\bar{z}+z\bar{w} \\ =&|w|^2+|z|^2+2\operatorname{Re}(w\bar{z}) \\ \leqslant&|w|^2+|z|^2+2|w\bar{z}| \\ =&|w|^2+|z|^2+2|w|\cdot|z| \\ =&(|w|+|z|)^2 \\ \to&|w+z|\leqslant|w|+|z| \end{aligned}$$

□

2 多项式系数的唯一性

定理 4. *零多项式的所有系数为0*

$$p\in\mathcal{P}(R), p=\mathbf{0}\to a_0=a_1=\cdots=a_n=0$$

证明.

$$\begin{gathered} \text{设并非所有系数为0, let:} a_m\neq 0 \\ z=\frac{|a_0|+|a_1|+\cdots+|a_{m-1}|}{|a_m|}+1 \\ z\geqslant 1\to\forall i<m-1, z^i\leqslant z^{m-1} \\ \to|a_0+a_1z+\cdots+a_{m-1}z^{m-1}|\leqslant(|a_0|+\cdots+|a_{m-1}|)z^{m-1}<|a_mz^m| \\ \to a_0+a_1z+\cdots+a_{m-1}z^{m-1}\neq-a_mz^m \\ \to a_0+a_1z+\cdots+a_{m-1}z^{m-1}+a_mz^m\neq 0 \\ \text{矛盾}\to a_m=0 \end{gathered}$$

注意 5. 这表明多项式系数必唯一，否则两个多项式相减 $=0$ 但 $a_m \neq 0$ 矛盾 $\square$

定义 6. *规定0次多项式的次数为 $-\infty$，这使得一些运算是平凡的。eg $\deg(pq)=\deg(p)+\deg(q)$*

3 多项式的带余除法

定义 7. *多项式的带余除法*

$$p, s \in \mathcal{P}(F), s \neq 0. \exists \text{唯一} q, r \in \mathcal{P}(F) \rightarrow p = sq + r \wedge \deg r < \deg s$$

证明.

$$n = \deg p, m = \deg s.$$

$\deg p < \deg s$

$$n < m, q = 0 \rightarrow r = p.$$

$\deg p \geqslant \deg s$

$$\begin{aligned}
&m \geqslant n. \\
&\text{let}: T: \mathcal{P}_{n-m}(F) \times \mathcal{P}_{m-1}(F) \rightarrow \mathcal{P}_n(F) \\
&T(q, r) = sq + r \\
T(q_1+q_2, r_1+r_2) &= s(q_1+q_2) + (r_1+r_2) \\
&= sq_1 + r_1 + sq_2 + r_2 \\
&= T(q_1, r_1) + T(q_2, r_2) \\
\\
T(\lambda q, \lambda r) &= s(\lambda q) + \lambda r \\
&= \lambda sq + \lambda r \\
&= \lambda(sq + r) \\
&= \lambda T(q, r)
\end{aligned}$$

$$\begin{gathered}
\rightarrow T(q, r) \in \mathcal{L}(\mathcal{P}_{n-m}(F) \times \mathcal{P}_{m-1}(F), \mathcal{P}_n(F)) \\
(q, r) \in \operatorname{null} T \rightarrow sq + r = 0 \rightarrow q = 0 \wedge r = 0 \\
\rightarrow \dim \operatorname{null} T = 0 \\
\rightarrow T \text{是单射} \rightarrow q, r \text{是唯一的} \\
\\
\dim(\mathcal{P}_{n-m}(F) \times \mathcal{P}_{m-1}(F)) = (n - m + 1) + (m - 1 + 1) = n + 1 \\
\rightarrow \dim \operatorname{range} T = n + 1 = \dim \mathcal{P}_n(F) \\
\rightarrow \exists q \in \mathcal{P}_{n-m}(F), r \in \mathcal{P}_{m-1}(F) \rightarrow p = T(q, r) = sq + r
\end{gathered}$$

$\square$

4 多项式的零点

定义 8. 多项式的零点 zero of polynomial

$\lambda \in F$ 是多项式 $p \in \mathcal{P}(F)$ 的零点(根), $p(\lambda) = 0$

定义 9. *因式 factor*

$$s \in \mathcal{P}(F) \text{是多项式} p \in \mathcal{P}(F) \text{的因式}: \exists q \in \mathcal{P}(F) \rightarrow p = sq$$

定理 10. *多项式每个零点对应一个一次因式*

$$p \in \mathcal{P}(F), \lambda \in F. p(\lambda) = 0 \Leftrightarrow \exists q \in \mathcal{P}(F), \forall z \in F, p(z) = (z - \lambda) q(z)$$

证明.

$$
\begin{aligned}
p(z) &= (z-\lambda)q(z) \\
p(\lambda) &= (\lambda-\lambda)q(z) = 0
\end{aligned}
$$

$$
\begin{aligned}
p(\lambda) = 0 \to \exists q \in \mathcal{P}(F) &\to p = (z-\lambda)q(z) \\
p(z) &= (z-\lambda)q(z) + r \\
\deg r &< \deg(z-\lambda) = 1 \\
\\
p(\lambda) &= 0 \to r = 0 \\
&\to p(z) = (z-\lambda)q(z)
\end{aligned}
$$

□

定理 11. *多项式零点的个数不超过它的次数*

$$p \in \mathcal{P}(F)\text{是}m\text{次多项式}, m \geqslant 0. p\text{在}F\text{中最多有}m\text{个互不相同的零点}$$

证明.

$$
\begin{aligned}
m = 0, p(z) = a_0 \neq 0, p&\text{没有零点} \\
m = 1, p(z) = a_0 + a_1 z, a_1 \neq 0 \to p&\text{有一个零点} - \tfrac{a_0}{a_1} \\
\forall m > 1, m\text{使用归纳法.设}m-1\text{次多项式最多有}m-1&\text{个不同的零点} \\
\text{若}p\text{在}F\text{中没有零点} \to &\text{结论成立} \\
\text{若}p\text{有一个零点} \to &\exists p(z) = q(z)(z-\lambda) \\
\deg q(z) = m-1 \to &q(z)\text{至多有}m-1\text{个零点} \\
\to &p\text{至多有}m-1+1\{\lambda\}\text{个零点}
\end{aligned}
$$

□

5 C上多项式的分解

定理 12. *代数学基本定理*

$$\forall p \in \mathcal{P}(C), p \neq 0 \to p\text{有零点}$$

Remark: 这里需要强调是复多项式必有零点，而不是其它F

证明. 使用复变中的刘维尔定理

$$
\begin{aligned}
p \in \mathcal{P}(C)&, p \neq 0. \\
\text{Assume: } p\text{没有零点}, &\tfrac{1}{p}\text{是}C\text{上的解析函数} \\
|z| \to \infty, &|p(z)| = \infty \\
\to &\lim_{|z|\to\infty} \tfrac{1}{p} = 0 \\
\to \tfrac{1}{p}\text{是}C&\text{上的有界解析函数.} \\
\to &\tfrac{1}{p}\text{是常数} && \text{刘维尔定理} \\
\to &p\text{是常数; 矛盾} \\
\to &p\text{有零点}
\end{aligned}
$$

□

注意 13. 三次求根公式(但这个不重要)

$$
\begin{aligned}
u &= \frac{9abc - 2b^3 - 27a^2 d}{54a^3} \\
v &= u^2 + \left(\frac{3ac - b^2}{9a^2}\right)^3 \\
v \geqslant 0 \to &-\frac{b}{3a} + \sqrt[3]{u + \sqrt{v}} + \sqrt[3]{u - \sqrt{v}}\text{是}p\text{的零点}
\end{aligned}
$$

定理 14. *C上的多项式必可分解为$deg\ p$个一次多项式之积*

$$
\begin{gathered}
p \in \mathcal{P}(C), p \neq 0. \\
\rightarrow \quad \exists c, \lambda_i \in C \rightarrow p(z) = c(z-\lambda_1)\cdots(z-\lambda_m)
\end{gathered}
$$

证明.

$$
\begin{gathered}
p \in \mathcal{P}(C), p \neq 0, m = \deg p \\
p\text{必有一个根}\lambda \qquad (12) \\
\rightarrow p(z) = q(z)(z-\lambda) \\
\rightarrow \text{若}\deg q > 0 \rightarrow q\text{必有根}\lambda_1 \\
\text{重复} \rightarrow p(z) = (z-\lambda_0)(z-\lambda_1)\ldots q_n \\
\deg q_n = 0 \rightarrow q_n = c \\
\rightarrow p(z) = c(z-\lambda_0)(z-\lambda_1)\ldots(z-\lambda_m)
\end{gathered}
$$

□

唯一性: $\quad$ let: $p(z) = c_1(z-\lambda_0)(z-\lambda_1)\ldots(z-\lambda_m) = c_2(z-\tau_0)(z-\tau_1)\ldots(z-\tau_m)$

$$
\begin{gathered}
c_1 \neq c_2 \rightarrow a_0 \neq b_0\text{矛盾} \\
\rightarrow c_1 = c_2
\end{gathered}
$$

这里对个因式先排序，乘法具有交换律所以可以执行这种操作

$$
\begin{gathered}
\text{Assume: } \lambda_i \neq \tau_i \\
p(\lambda_i) = c(z-\lambda_0)\ldots(\lambda_i-\lambda_i)\ldots(\lambda_i-\lambda_m) = 0 \neq c(z-\tau_0)(z-\tau_1)\ldots(z-\tau_m) \\
\text{矛盾} \rightarrow \lambda_i = \tau_i
\end{gathered}
$$

6 R上多项式的分解

R上的零点性质不如C上的那么完美，比如$1+x^2$没有零点

定理 15. *实系数多项式的非实零点成对出现*

$$
p \in \mathcal{P}(R). \lambda \in C\text{是}p\text{的零点} \rightarrow \bar{\lambda}\text{是}p\text{的零点}
$$

证明.

$$
\begin{gathered}
\lambda \in R \rightarrow \bar{\lambda} = \lambda\text{是}p\text{的零点} \\
p(z) = a_0 + a_1 z + \cdots + a_n z^n \\
p(\lambda) = 0 = a_0 + a_1\lambda + \cdots + a_n\lambda^n \\
\bar{0} = a_0 + a_1\bar{\lambda} + \cdots + a_n\overline{\lambda^n} \\
\rightarrow p(\bar{\lambda}) = 0
\end{gathered}
$$

□

定理 16. *实二次多项式的分解*

$$
b^2 \geqslant 4ac \Leftrightarrow \exists \lambda_1, \lambda_2 \in R \rightarrow x^2 + bx + c = (x-\lambda_1)(x-\lambda_2)
$$

证明.

$$
\begin{gathered}
x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right) \\
b^2 < 4ac \rightarrow c - \frac{b^2}{4} > 0 \\
\rightarrow \text{没有实解} \\
b^2 \geqslant 4ac \rightarrow \text{有实解}
\end{gathered}
$$

□

定理 17. *实多项式可被唯一分解为一次项和二次项*

$$
\begin{gathered}
p \in \mathcal{P}(R), p \neq 0. \\
\rightarrow \quad p = c(x-\lambda_1)(x-\lambda_2)\ldots(x-\lambda_m)(x^2 + b_1 x + c_1)\ldots(x^2 + b_M x + c_M) \\
c, \lambda_1, \ldots \lambda_m, b_1, \ldots, b_m, c_1, \ldots, c_M \in R. b_i^2 < 4c_i
\end{gathered}
$$

证明.

存在性

$$将p视为C中的元素,则p有唯一分解$$
$$p(x)=c(x-\lambda_0)(x-\lambda_1)\ldots(x-\lambda_m)$$
$$设c\notin R\rightarrow p(x)\notin\mathcal{P}(R)矛盾$$
$$\rightarrow c\in R.$$

$$\lambda是零点\rightarrow\bar{\lambda}是零点$$
$$\rightarrow\lambda_i\notin R\rightarrow\lambda\neq\bar{\lambda}$$
$$\rightarrow p(x)=c(x-\lambda_1)(x-\lambda_2)\ldots(x-\mu_1)(x-\overline{\mu_1})\ldots(x-\mu_n)(x-\overline{\mu_n})$$
$$(x-\lambda)(x-\bar{\lambda})=x^2-x\lambda-x\bar{\lambda}+\lambda\bar{\lambda}$$
$$=x^2-2x\operatorname{Re}(\lambda)+|\lambda|^2$$
$$2\operatorname{Re}(\lambda)\in R, |\lambda|^2\in R$$
$$\rightarrow(x-\mu_1)(x-\overline{\mu_1})=(x^2-2\operatorname{Re}(\mu_1)x+|\mu_1|^2)$$
$$p(x)=(x-\lambda)(x-\bar{\lambda})q(x)$$
$$q(x)=\frac{p(x)}{x^2-2(\operatorname{Re}\lambda)x+|\lambda|^2}$$
$$\rightarrow q(x)\in\mathcal{P}(R)$$
$$\rightarrow 0=\operatorname{Im}q(x)=\operatorname{Im}(a_0)+\operatorname{Im}(a_1)x+\cdots+\operatorname{Im}(a_{n-2})x^{n-2}$$
$$\rightarrow\operatorname{Im}a_0=\cdots=\operatorname{Im}(a_{n-2})=0$$

???没整明白书上的证明，但这不重要，写出来一个存在性证明简单

□

唯一性

$$b_i<4c_i,\rightarrow x^2+b_ix+c_i有唯一的(x-\lambda_i)(x-\bar{\lambda_i})$$
$$b_i\neq b_j\vee c_i\neq c_j\rightarrow\lambda_i\neq\lambda_j$$
$$\rightarrow若p在R上有两个不同的分解\rightarrow p在C上有两个不同的分解$$
$$矛盾\rightarrow在R上的分解是唯一的$$

习题

1.

2. Proof or Counter: $m\in N^+, E=\{0\}\cup\{p\in\mathcal{P}(F)\colon\deg p=m\}$是$\mathcal{P}(F)$的子空间

$$0\in E$$
$$\forall p_1,p_2\in E$$
$$p_1=x^2+x, p_2=-x^2. p_1+p_2=x\notin E$$
$$\forall p\in E,\lambda\in F,\deg(\lambda p)=m\rightarrow\lambda p\in E$$
$$\rightarrow E不是子空间$$

3. Proof or Counter: $E=\{0\}\cup\{p\in\mathcal{P}(F)\colon\deg p是偶数\}, E$是$\mathcal{P}(F)$的子空间

$$0\in E.$$
$$p_1=x^2+x+1, p_2=-x^2+x, p_1+p_2=1\notin E$$
$$\rightarrow E不是子空间$$

4. Proof:$m,n\in N^+, m\leqslant n.\lambda_1,\ldots,\lambda_m\in F$. Proof: $\exists p\in\mathcal{P}(F),\deg p=n\wedge 0=p(\lambda_1)=\cdots=p(\lambda_m)\wedge$ p没有其它零点

$$p=(x-\lambda_1)(x-\lambda_2)\ldots(x-\lambda_m)(x-\lambda_m)^{n-m-1}$$
$$\deg p=m+n+1-m-1=n$$

5. Proof: $m\in N.z_1,\ldots,z_{m+1}\in F$, 且$z_i$互不相同.$w_1,\ldots,w_{m+1}\in F$. Proof: $\exists p\in\mathcal{P}_m(F)\rightarrow p(z_i)=w_i$

$$p(z_i)-w_i=0$$
$$\operatorname{span}(1,z,\ldots,z^m)=\mathcal{P}_m(F)$$
$$???$$

得构建一个线性映射，并证明是$1-1$的

6. Proof: $p\in\mathcal{P}(C).\deg p=m.$ Proof: p有m个不同的零点$\Leftrightarrow p$和p'没有公共零点

$$p\text{有}m\text{个不同的零点}\rightarrow p\text{和}p'\text{没有公共零点}$$
$$p=(z-p_1)(z-p_2)\ldots(z-p_m)$$
$$p'=(z-p_2)\ldots(z-p_m)+(z-p_1)\ldots(z-p_m)+\cdots+(z-p_1)\ldots(z-p_m)$$
$$\text{Assume: } p'(p_i)=p(p_i)=0$$
$$\rightarrow p'(t)=(p_i-p_1)\ldots(p_i-p_{i-1})(p_i-p_{i+1})\ldots(p_i-p_m)$$
$$\rightarrow\exists j\neq i\rightarrow p_i=p_j\text{矛盾}$$
$$\rightarrow p'(p_i)\neq p(p_i)\text{即没有公共零点}$$

$$p\text{和}p'\text{没有公共零点}\rightarrow p\text{有}m\text{个不同的零点}$$
$$p'=\sum_1^m\left(\prod_1^{i-1}x_i\cdot\prod_{i+1}^m x_i\right)$$
$$\forall p_i,p(p_i)=0.\,p'(p_i)\neq 0$$
$$\rightarrow\forall i\in 1\ldots m\rightarrow(p_i-p_1)\ldots(p_i-p_{i-1})(p_i-p_{i+1})\ldots(p_i-p_m)\neq 0$$
$$\rightarrow\forall j\neq i\rightarrow p_j\neq p_i$$
$$\rightarrow\forall i,\forall j\neq i\rightarrow p_j\neq p_i$$
$$\rightarrow p\text{的零点}p_i\text{互不相同}$$

7. Proof: 奇数次多项式必有实零点

$$p(x)=(x-\lambda_1)(x-\lambda_2)\ldots(x-\lambda_n),\lambda_i\in C,n=\deg p$$
$$\text{Assume: } p(x)=(x^2+b_1x+c_1)(x^2+b_2x+c_2)\ldots(x^2+b_mx+c_m)$$
$$\deg p=2m\neq 2m+1$$
$$\text{矛盾}\rightarrow p(x)\text{必有一次项}$$

8. $T:\mathcal{P}(R)\rightarrow R^R.Tp=\begin{cases}\frac{p-p(3)}{x-3} & x\neq 3\\ p'(3) & x=3\end{cases}.$ Proof: $\forall p\in\mathcal{P}(R),Tp\in\mathcal{P}(R)\wedge T\in\mathcal{L}(\mathcal{P}(R),R^R)$

$$\forall p\in\mathcal{P}(R).\,p=a_0+a_1x+a_2x^2+\cdots+a_nx^n$$
$$\lim\frac{p-p(3)}{x-3}(3)=p'(3)$$

$$p-p(3)=p(x)-p(3)=a_0+a_1x+\cdots+a_nx^n-(a_0+a_13+\cdots+a_n3^n)$$
$$=0+a_1(x-3)+\cdots+a_n(x^n-3^n)$$
$$(a^n-b^n)=(a-b)(a^{n-1}b+a^{n-2}b^2+\cdots+ab^{n-1})$$
$$\rightarrow(p-p(3))=(x-3)q(x)$$

$$\rightarrow\frac{p-p(3)}{x-3}\in\mathcal{P}(R)$$

$$T(x^a+x^b)=\frac{x^a+x^b-3^a-3^b}{x-3}=\frac{x^a-3^a}{x-3}+\frac{x^b-3^b}{x-3}$$
$$=T(x^a)+T(x^b)$$
$$T(ax^n+bx^m)=\frac{ax^n+bx^m-a3^n+b3^m}{x-3}$$
$$=\frac{ax^n-a3^n}{x-3}+\frac{bx^m-b3^m}{x-3}$$
$$=T(ax^n)+T(bx^m)$$

$$\rightarrow T(p_1+p_2)=T((a_0+b_0)+(a_1+b_1)x^2+\cdots+(a_n+b_n)x^n)$$
$$=T(a_0)+T(b_0)+T(a_1x^2)+T(b_1x^2)+\cdots+T(a_nx^m)+T(b_nx^n)$$
$$=T(a_0)+T(a_1x)+\cdots+T(a_nx^m)+T(b_0)+T(b_1x)+\cdots+T(b_nx^n)$$
$$=T(a_0+a_1x+\cdots+a_nx^n)+T(b_0+b_1x+\cdots+b_nx^n)$$
$$=T(p)+T(q)$$

$$T(ap)=\frac{ap-ap(3)}{x-3}=a\frac{p-p(3)}{x-3}=aT(p)$$
$$\rightarrow T\in\mathcal{L}(\mathcal{P}(R),R^R)$$

9. Proof: $p\in\mathcal{P}(C).q:C\rightarrow C,q(z)=p(z)\overline{p(\bar{z})}.$ Proof; $q(z)$是实多项式

$$
\begin{aligned}
q(z) &= p(z)\overline{p(\bar{z})} \\
p(z) &= (z-\lambda_1)\ldots(z-\lambda_n) \\
p(\bar{z}) &= (\bar{z}-\lambda_1)\ldots(\bar{z}-\lambda_n) \\
\overline{p(\bar{z})} &= \overline{(\bar{z}-\lambda_1)\ldots(\bar{z}-\lambda_n)} \\
&= \overline{(\bar{z}-\lambda_1)}\ldots\overline{(\bar{z}-\lambda_n)} \\
&= (z-\bar{\lambda}_1)\ldots(z-\overline{\lambda_n}) \\
q(z) &= (z-\lambda_1)(z-\bar{\lambda}_1)\ldots(z-\lambda_n)(z-\overline{\lambda_n}) \\
&= (z^2-2\mathrm{Re}\,\lambda_1 z+|\lambda_1|^2)\ldots(z^2-2\mathrm{Re}\,\lambda_n+|\lambda_n|^2) \\
&\rightarrow q(z)\text{是实多项式}
\end{aligned}
$$

10. Proof: $m\in N.\, p\in\mathcal{P}_m(C).\exists x_0,\ldots,x_m\in R,\wedge$互不相同 $\rightarrow p(x_i)\in R.$ Proof: p的系数均为实数

    ???貌似和第5题用同一个trick

11. $p\in\mathcal{P}(F), p\neq 0. U=\{pq: q\in\mathcal{P}(F)\}$

    a. Proof: $\dim\mathcal{P}(F)/U=\deg p$

    $$
    \begin{gathered}
    0\in\mathcal{P}(F)\rightarrow 0p=0\in U \\
    \forall x,y\in U.\, x+y=p_1q+p_2q=(p_1+p_2)q\rightarrow x+y\in U \\
    x\in U.\, x=pq.\, \lambda x=(\lambda p)\,q\rightarrow\lambda x\in U \\
    \rightarrow U\text{是子空间}
    \end{gathered}
    $$

    $$
    \begin{gathered}
    \mathcal{P}(F)/U=\{x+U: x\in\mathcal{P}(F)\}.\, u\in U, u=pq. \\
    ???
    \end{gathered}
    $$

    b. Calculate: $\mathcal{P}(F)/U$的一个基

    ???