Chapter11

BY 反常积分

1 Def&Theo

1.1 Def: 无穷积分和瑕积分

1. 定积分的两类极限:

$$\begin{array}{lll} 1 & \text{无穷区间} f \text{在} [a,u] \text{内可积} & \lim\limits_{u\to+\infty}\int_a^u f(x)\mathrm{d}x = J := \int_a^{+\infty} f(x)\mathrm{d}x = J \\ 2 & \text{瑕点} a, f \text{在} [u,b] \text{内可积} & \lim\limits_{u\to a}\int_u^b f(x)\mathrm{d}x = J := \int_a^b f(x)\mathrm{d}x = J \end{array}$$

1.2 Property

1. 柯西准则:

$$\begin{array}{ll} \text{无穷} & \int_a^{+\infty} f \text{收敛} \Leftrightarrow \forall\varepsilon>0, \exists G\geqslant a, \forall u_1,u_2>G \to \left|\int_{u_1}^{u_2} f\right| < \varepsilon \\ \text{瑕} a & \int_a^b f \text{收敛} \Leftrightarrow \forall\varepsilon>0, \exists U_a^+(\delta), x_1,x_2\in U \to \left|\int_{u_1}^{u_2} f\right| < \varepsilon \end{array}$$

2. 线性性:

$$\begin{array}{ll} \text{无穷} & \int_a^{+\infty}(\lambda_1 f_1+\lambda_2 f_2) = \lambda_1\int_a^{+\infty} f_1 + \lambda_2\int_a^{+\infty} f_2 \\ \text{瑕} a & \int_a^b(\lambda_1 f_1+\lambda_2 f_2) = \lambda_1\int_a^b f_1 + \lambda_2\int_a^b f_2 \end{array}$$

3. 区间可加性:

$$\begin{array}{lll} \text{无穷} & \int_a^{+\infty} f = \int_a^b f + \int_b^{+\infty} f & \int_a^b f \text{为定积分} \\ \text{瑕} a & \int_a^b f = \int_a^c f + \int_c^b f & \int_c^b f \text{为定积分} \end{array}$$

4. 绝对收敛⇒收敛

$$\begin{array}{ll} \text{无穷} & \int_a^{+\infty} |f| \text{收敛} \Rightarrow \int_a^{+\infty} f \text{收敛} \\ \text{瑕} a & \int_a^b |f| \text{收敛} \Rightarrow \int_a^b f \text{收敛} \\ \text{Pr} & \text{使用积分不等式} \left|\int f\right| \leqslant \int |f| \end{array}$$

5. 非负反常积分和反常积分

$$\begin{array}{ll} \text{无穷} & f\geqslant 0 \Rightarrow F(x) = \int_a^x f \text{是单调增的} \Rightarrow \int_a^{+\infty} f \text{有上界必收敛} \\ \text{瑕} a & f\geqslant 0 \Rightarrow F(x) = \int_x^b f \text{是单调减的} \Rightarrow \int_x^b f \text{有下界必收敛} \end{array}$$

比较原则:

$$\begin{array}{ll} \text{无穷} & f,g \text{在} [a,u] \text{可积} \wedge \begin{cases} f\leqslant g \wedge \int_a^{+\infty} g \text{收敛} \Rightarrow \int_a^{+\infty} f \text{收敛} \\ f\geqslant g \wedge \int_a^{+\infty} g \text{发散} \Rightarrow \int_a^{+\infty} f \text{发散} \end{cases} \\ \text{瑕} a & f,g \text{在} [u,b] \text{可积} \wedge \begin{cases} f\leqslant g \wedge \int_a^b g \text{收敛} \Rightarrow \int_a^b f \text{收敛} \\ f\geqslant g \wedge \int_a^b g \text{发散} \Rightarrow \int_a^b f \text{发散} \end{cases} \end{array}$$

与另一个函数相比较:

$$\text{无穷}\quad f,g\text{在}[a,u]\text{上可积}\wedge f\geqslant 0\wedge g>0\wedge \lim_{x\to+\infty}\frac{f}{g}=c$$
$$\wedge\begin{cases} c\in R^+\Rightarrow & \Rightarrow\ \int f,\int g\text{同敛散}\\ c=0\wedge\int_a^{+\infty}g\text{收敛} & \Rightarrow\ \int_a^{+\infty}f\text{收敛}\\ c=+\infty\wedge\int_a^{+\infty}g\text{发散} & \Rightarrow\ \int_a^{+\infty}f\text{发散}\end{cases}$$

$$\text{瑕}a\quad f,g\text{在}[u,b]\text{上可积}\wedge f\geqslant 0\wedge g>0\wedge \lim_{x\to a^+}\frac{f}{g}=c$$
$$\wedge\begin{cases} c\in R^+ & \Rightarrow\ f,g\text{同敛散}\\ c=0\wedge\int g\text{收敛} & \Rightarrow\ \Rightarrow\int f\text{收敛}\\ c=+\infty\wedge\int g\text{发散} & \Rightarrow\ \int f\text{发散}\end{cases}$$

与$\frac{1}{x^p}$相比较:

$$\text{无穷}\quad\begin{cases} 0\leqslant f\leqslant\frac{1}{x^p}\wedge p>1 & \Rightarrow\ \int_a^{+\infty}f\text{收敛}\\ f\geqslant\frac{1}{x^p}\wedge p\leqslant 1 & \Rightarrow\ \int_a^{+\infty}f\text{发散}\end{cases}$$

$$\text{瑕}a\quad\begin{cases} 0\leqslant f\leqslant\frac{1}{(x-a)^p}\wedge p\in(0,1) & \Rightarrow\ \int f\text{收敛}\\ f\geqslant\frac{1}{(x-a)^p}\wedge p\geqslant 1 & \Rightarrow\ \int f\text{发散}\end{cases}$$

与$\frac{1}{x^p}$比较的极限形式:

$$\text{无穷}\quad \lim_{x\to+\infty}x^pf(x)=\lambda\wedge\begin{cases} p>1\wedge\lambda\in[0,+\infty)\Rightarrow\int f\text{收敛}\\ p\leqslant 1\wedge\lambda\in(0,+\infty)\Rightarrow\int f\text{发散}\end{cases}$$
$$\text{瑕}a\quad \lim_{x\to a^+}(x-a)^pf(x)=\lambda\wedge\begin{cases} p<1\wedge\lambda\in[0,+\infty)\Rightarrow\int f\text{收敛}\\ p\geqslant 1\wedge\lambda\in(0,+\infty)\Rightarrow\int f\text{发散}\end{cases}$$

6. 一般反常积分收敛的判别法:

狄利克雷

$$\text{无穷}\quad F(u)=\int_a^u f\text{在}[a,+\infty)\text{有界}\wedge g\text{在}[a,+\infty)\text{上单调}\wedge\lim_{x\to+\infty}g=0\Rightarrow\int_a^{+\infty}fg\text{收敛}$$
$$\text{瑕}a\quad F(u)=\int_u^b f\text{在}(a,b]\text{有界}\wedge g\text{在}(a,b]\text{上单调}\wedge\lim_{x\to a^+}g=0\Rightarrow\int_a^b fg\text{收敛}$$
$$\text{Pr}\quad \int_{u_1}^{u_2}fg=g(u_1)\int_{u_1}^{\xi}f+g(u_2)\int_{\xi}^{u_2}f$$
$$u_1,u_2\in U(\delta),\,g<\varepsilon.$$
$$\left|\int_{u_1}^{\xi}f\right|=\left|\int_a^{\xi}f-\int_a^{u_1}f\right|\leqslant 2M;\left|\int_{\xi}^{u_2}f\right|=\left|\int_a^{u_2}f-\int_a^{\xi}f\right|\leqslant 2M$$
$$\Rightarrow\left|\int_{u_1}^{u_2}f\right|<4M\varepsilon$$

阿贝尔:

$$\text{无穷}\quad\int_a^{+\infty}f\text{收敛}\wedge g\text{在}[a,+\infty)\text{单调有界}\Rightarrow\int_a^{+\infty}fg\text{收敛}$$
$$\text{瑕}a\quad\int_a^b f\text{收敛}\wedge g\text{在}(a,b]\text{单调有界}\Rightarrow\int_a^b fg\text{收敛}$$

2 Trick

1. 罗巴切夫斯基公式：

$$\forall x \in R, f(x) = f(\pi + x) = f(\pi - x)$$
$$\int_0^{+\infty} f(x) \cdot \frac{\sin}{x} \mathrm{d}x = \int_0^{2\pi} f(x) \mathrm{d}x$$

3 Formula

1. 欧拉-泊松积分

$$\int_0^{+\infty} e^{-x^2} \mathrm{d}x = \frac{\sqrt{\pi}}{2}$$

   直接计算使用Wallis公式；

   二维平面变换成$l^2 + r^2 = x^2$化为二重积分计算；

   使用Γ函数的余元公式