1 矩阵

1.1 基本性质

矩阵 同构(加法和数乘) 线性映射　因此矩阵空间是线性空间

运算法则

加法

$$\begin{array}{cc} \text{交换} & A+B=B+A \\ \text{结合} & (A+B)+C=A+(B+C) \\ \text{单位元} & A+O=A \\ \text{逆元} & A+(-A)=O \end{array}$$

乘法

$$\begin{array}{cc} \text{结合} & (AB)C=A(BC) \\ \text{单位元} & AE=EA=A \\ \text{不可交换} & AB\neq BA\text{无法判定} \\ \text{有非}O\text{的零因子} & AB=0\nrightarrow A\neq 0\wedge B\neq 0 \\ \text{没有消去律} & AB=O\wedge B\neq O\nrightarrow A=O \end{array}$$

数乘

$$\begin{array}{cc} \text{定义} & \lambda A=\left(\ \lambda a_{i,j}\ \right)=(\lambda E)A \\ \text{结合} & k(lA)=(kl)A \\ \text{矩阵加法和数乘的分配率} & k(A+B)=kA+kB \\ & (k+l)A=kA+kB \\ \text{仅有0元是零因子} & \lambda A=O\rightarrow(\lambda=0\vee A=O) \end{array}$$

转置(对偶映射或伴随映射)

$$\begin{array}{cc} & (A+B)'=A'+B' \\ & (\lambda A)'=\lambda A' \\ & (AB)'=B'A' \\ & (A')'=A \\ \text{对称} & A=A' \\ \text{反对称} & A=-A' \end{array}$$

分配律

$$\begin{array}{c} (A+B)C=AC+BC \\ A(B+C)=AB+AC \\ A^2-\lambda=(A+\lambda)(A-\lambda) \end{array}$$

其它运算律(tricks)这里需要用抽象代数

$$A^2-\lambda=(A+\lambda)(A-\lambda)$$

$$A^n-E=(A-E)\left(\sum_{i=0}^{n-1}A^i\right)=(A-E)(E+A+\cdots+A^{n-1})$$

$$A^n+E=(A+E)\left(\sum_{i=0}^{n-1}(-1)^iA^i\right)=(A+E)(E-A+A^2-\cdots+(-1)^{n-1}A^{n-1})$$

tricks矩阵方程

用逆矩阵变换到目标结果

$$\begin{array}{c} p(A),p,q\in\mathcal{P}(F^{n,n});\text{求}q(A).\text{或}q(A)\text{的性质} \\ (A+xE)(A+yE)=A^2+(x+y)A+xyE \\ \rightarrow A^2+mA+nE=B \\ m=x+y;n=B-xy;\text{ 通常}B\text{有良好的性质} \end{array}$$

tricks只有一个非零元的矩阵的作用

左乘	行变换	$a_{i,j}=1$ 把第j行换到第i行	列 行 换到 行 行
右乘	列变换	$a_{i,j}=1$ 把第i列换到第j列	行 列 换到 列 列

$$\text{左乘} \qquad \times\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & 1_{i,j} & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{m,m} \times \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}_{m\times n}$$

$$= \qquad \begin{array}{c|ccc} \text{行号}\backslash\text{列号} & 1 & & n \\ \hline 1 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ i & a_{j,1} & \cdots & a_{j,n} \\ \vdots & & & \\ 0 & 0 & & 0 \end{array}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{特殊的反向重排行} \qquad \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}_{m\times m} \times A = \begin{pmatrix} a_{m,1} & \cdots & a_{m,n} \\ a_{m-1,1} & \cdots & a_{m-1,n} \\ \vdots & & \vdots \\ a_{1,1} & \cdots & a_{1,n} \end{pmatrix}$$

$$\text{右乘} \qquad \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}_{m\times n} \times \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & 1_{i,j} & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{n\times n}$$

$$= \qquad \begin{array}{c|ccccc} \text{行号}\backslash\text{列号} & 1 & \cdots & j & \cdots & n \\ \hline 1 & 0 & \cdots & a_{1,i} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{m,i} & \cdots & 0 \end{array}$$

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{1,2} & 0 & 0 \\ a_{2,2} & 0 & 0 \\ a_{3,2} & 0 & 0 \end{pmatrix}$$

$$\text{特殊的反向重排列} \quad A \times \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}_{n\times n} = \begin{pmatrix} a_{1,n} & a_{1,n-1} & \cdots & a_{1,2} & a_{1,1} \\ a_{2,n} & a_{2,n-1} & \cdots & a_{2,2} & a_{2,1} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m,n} & a_{m,n-1} & \cdots & a_{m,2} & a_{m,1} \end{pmatrix}$$

1.2 秩的性质

1. 秩是最大线性无关向量的个数，一切不为0的自己的最高阶数

2. 行秩=列秩=矩阵的秩

3. 对运算的性质

$$\begin{array}{ll}
1 & \operatorname{rank} A = \operatorname{rank} A' \\
2 & \lambda \neq 0 \to \operatorname{rank} \lambda A = \operatorname{rank} A \\
3 & \operatorname{rank} A^* \leqslant \operatorname{rank} A \\
 & \operatorname{rank} A^* = \begin{cases} n & \operatorname{rank} A = n \\ 1 & \operatorname{rank} A = n-1 \\ 0 & \operatorname{rank} A < n-1 \end{cases} \\
4 & \operatorname{rank} A \pm B \leqslant \operatorname{rank} A + \operatorname{rank} B \\
5 & \operatorname{rank} AB \leqslant \min(\operatorname{rank} A, \operatorname{rank} B) \\
6 & \text{初等变换不改变矩阵的秩} \\
7 & \operatorname{rank} A_{m\times n} = r \to \exists \text{可逆} P_{m\times m}, \text{可逆} Q_{n\times n}, PAQ = \begin{pmatrix} E_r & O \\ O & O_{n-r} \end{pmatrix}
\end{array}$$

一些奇怪的等式不等式

$$\begin{array}{ll}
\text{Sylverster} & A_{s\times n}, B_{n\times m} \to \operatorname{rank} A + \operatorname{rank} B - n \leqslant \operatorname{rank} AB \\
\text{Frobenius} & \operatorname{rank} ABC \geqslant \operatorname{rank} AB + \operatorname{rank} BC - \operatorname{rank} B \\
 & \operatorname{rank} A_{n\times n} = n \to \operatorname{rank} AB = \operatorname{rank} B \\
\to & \operatorname{rank} A_{n\times n} = n \to \operatorname{rank} BA = \operatorname{rank} B \\
 & \\
 & \operatorname{rank} A_{n\times n} = n \wedge \operatorname{rank} AB = O \to B = O \\
\to & \operatorname{rank} A_{n\times n} = n \wedge \operatorname{rank} BA = O \to B = O
\end{array}$$

1.3 伴随矩阵和逆矩阵的性质

伴随矩阵

1. $A^* = \begin{pmatrix} A_{1,1} & & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{pmatrix}, A_{i,j} = (-1)^{i+j}\det(\sim)$

2. $AA^* = A^*A = |A|E$

3. $|A^*| = |A|^{n-1}$

4. $\operatorname{rank} A^* = \begin{cases} n & \operatorname{rank} A = n \\ 1 & \operatorname{rank} A = n-1 \\ 0 & \operatorname{rank} A < n-1 \end{cases}$

5. $(AB)^* = B^*A^*$

逆矩阵

1. $A^{-1} := AB = BA = E \to A^{-1} := B$

2. A可逆$. A^{-1} = \frac{1}{|A|}A^*$

3. $(A^{-1})^{-1} = A$

4. $(A^{-1})' = (A')^{-1}$

5. $(AB)^{-1} = B^{-1}A^{-1}$

6. $(kA)^{-1} = \frac{1}{k}A^{-1}. kA = \begin{pmatrix} ka_{1,1} & \cdots & ka_{1,n} \\ \vdots & & \vdots \\ ka_{n,1} & \cdots & ka_{n,n} \end{pmatrix}. |kA| = k^n|A| \Leftrightarrow \left|\frac{1}{k}A^{-1}\right| = \left(\frac{1}{k}\right)^n |A^{-1}| = (k^n|A|)^{-1}$

7. 广义逆$:= \exists G_{n\times m}, A_{m\times n} G_{n\times m} A_{m\times n} = \begin{pmatrix} E_r & O \\ O & O \end{pmatrix}_{m\times n}$

求逆方法

1. 定义法
2. $A^{-1}=\frac{1}{|A|}A^*$
3. 初等变换法$\begin{pmatrix} a_{1,1} & \cdots & a_{1,n} & 1 & & \\ \vdots & \ddots & \vdots & & \ddots & \\ a_{n,1} & \cdots & a_{n,n} & & & 1 \end{pmatrix} \to \begin{pmatrix} 1 & & & b_{1,1} & \cdots & b_{1,n} \\ & \ddots & & \vdots & \ddots & \vdots \\ & & 1 & b_{n,1} & \cdots & b_{n,n} \end{pmatrix}$
4. 乘开，解方程组

一些可逆矩阵

$$\begin{array}{ll} 1 & \operatorname{rank} A = A. \\ 2 & \operatorname{null} A = \{0\} \\ 3 & \text{严格对角占优.} \sum_{j,i\neq j} a_{i,j} < a_{i,i} \text{ Or } \sum_{i,j\neq i} a_{i,j} < a_{i,i} \\ 4 & \sum m_i = n. m_i\text{是第}i\text{个特征值的重数} \qquad ? \\ 5 & \end{array}$$

Tricks 利用矩阵的行列式是元素间的多项式，可以对行列式使用多项式的分析性质

Eg：$(AB)^* = B^*A^*$

Pr
$$\begin{gathered} \text{Assume: } |AB| \neq 0 \to |A| \neq 0 \wedge |B| \neq 0 \\ (AB)^* = |AB|(AB)^{-1} = |A|\,|B|B^{-1}A^{-1} = |B|B^{-1}|A|A^{-1} = B^*A^* \\ \text{Another: } |AB| = 0 \\ A(\lambda) = A - \lambda E; B(\lambda) = B - \lambda E. \\ \to \exists\text{无穷}\lambda \to |A(\lambda)| \neq 0 \wedge |B(\lambda)| \neq 0 \\ \to (A(\lambda)B(\lambda))^* = B(\lambda)^*A(\lambda)^* \\ (A(\lambda)B(\lambda))^* \in \mathcal{P}(R); B(\lambda)^*, A(\lambda)^* \in \mathcal{P}(R) \\ \to (A(\lambda)B(\lambda))^*\text{和}B(\lambda)^*A(\lambda)^*\text{在}\lambda \in R\text{上连续} \\ \to \lim_{\lambda\to 0}(A(0)B(0))^* = \lim_{\lambda\to 0} B(0)^*A(0)^* \\ \to (AB)^* = B^*A^* \end{gathered}$$

1.4 一些特殊的矩阵和矩阵间关系

$$\mathcal{P}_n(R)\text{上的微分算子}D$$

$$\mathcal{M}(D,(1,x,\ldots,x^n))(p) = \begin{pmatrix} 0 & & & & \\ 1 & \ddots & & & \\ & 2 & \ddots & & \\ & & & \ddots & \\ & & & n-1 & 0 \end{pmatrix} \times p$$

$$D(1,x,x^2)' = \begin{pmatrix} 0 & & \\ 1 & 0 & \\ & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2x \end{pmatrix}$$

$$D(p) = p \times \begin{pmatrix} 0 & 1 & & & \\ & \ddots & 2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & n-1 \\ & & & & 0 \end{pmatrix}$$

正交阵	A的每个行向量或列向量都是V上的规范正交基, $AA' = A'A = E$
合同	A合同$B := \exists T, A = T'BT$
相似	$A \sim B := A = Q^{-1}BQ$
等价	$\operatorname{rank} A = \operatorname{rank} B$

对称阵和反对称阵

$$\begin{array}{ll} \text{对称} & A = A' \\ \text{反对称} & A = -A' \end{array}$$

对称阵的关系

	实	复
合同	$\operatorname{rank} A = \operatorname{rank} B$	正惯性指数和负惯性指数分别相等
相似	具有相同的特征值	

2 分块矩阵

2.1 分块矩阵的变换

1. 矩阵分块根据问题进行分块，不唯一

2. 常见分法

$$\begin{array}{lll} 1 & \text{行向量} & A = (\alpha_1, \ldots, \alpha_n)'. \alpha_i := A\text{的第}i\text{行} \\ 2 & \text{列向量} & A = (\alpha_1, \ldots, \alpha_n). \alpha_i := A\text{的第}i\text{列} \\ 3 & \text{分两个} & A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \end{pmatrix} \\ 4 & \text{分四个} & A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \end{array}$$

3. 广义初等变换

交换分块的两行或两列	秩不变
用可逆阵乘分块阵的某一行或列	秩不变
用某矩阵乘某一行(列)加到另一行(列)	行列式不变

4. 广义初等阵

$$\begin{array}{lcc} 1 & \begin{pmatrix} O & E_m \\ E_n & O \end{pmatrix} & \text{交换} \\ 2 & \begin{pmatrix} D & O \\ O & E \end{pmatrix}, \begin{pmatrix} E & O \\ O & G \end{pmatrix} D, G\text{均可逆} & \text{可逆阵乘} \\ 3 & \begin{pmatrix} E & O \\ M & E \end{pmatrix}, \begin{pmatrix} E & H \\ O & E \end{pmatrix}. & \text{矩阵乘再到其它行(列)} \end{array}$$

5. 分块阵求逆

$$\begin{array}{lcc} 1 & \text{定义} & \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_m \end{pmatrix}^{-1} = \begin{pmatrix} A_1^{-1} & & \\ & \ddots & \\ & & A_m^{-1} \end{pmatrix} \\ 2 & \text{广义初等变换} & \begin{pmatrix} A_{1,1} & A_{1,2} & E & O \\ A_{2,1} & A_{2,2} & O & E \end{pmatrix} \to \begin{pmatrix} E & O & B_{1,1} & B_{1,2} \\ O & E & B_{2,1} & B_{2,2} \end{pmatrix} \\ 3 & \text{解方程组} & \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix} = \begin{pmatrix} E & O \\ O & E \end{pmatrix} \\ & \to & \begin{cases} A_{1,1}B_{1,1} + A_{1,2}B_{2,1} = E \\ A_{1,1}B_{1,2} + A_{1,2}B_{2,2} = O \\ A_{2,1}B_{1,1} + A_{2,2}B_{2,1} = O \\ A_{2,1}B_{1,2} + A_{2,2}B_{2,2} = E \end{cases} \end{array}$$

6. 分块矩阵的秩

$$
\begin{array}{ll}
1 & \operatorname{rank}\begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_m \end{pmatrix} = \operatorname{rank}(A_1) + \cdots + \operatorname{rank}(A_m) \\
2 & \operatorname{rank}\begin{pmatrix} A_{1,1} & \cdots & A_{1,s} \\ \vdots & \ddots & \vdots \\ A_{r,1} & \cdots & A_{r,s} \end{pmatrix} \geqslant \operatorname{rank}(A_{i,j}) \\
3 & \operatorname{rank}\begin{pmatrix} A & O \\ C & B \end{pmatrix} \geqslant \operatorname{rank}(A) + \operatorname{rank}(B)
\end{array}
$$

7. 分块矩阵的行列式

$$
\begin{array}{ll}
1 & \det\begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_m \end{pmatrix} = \det(A_1) \times \cdots \times \det(A_m) \\
2 & \det\begin{pmatrix} O & B_{n\times n} \\ A_{m\times m} & O \end{pmatrix} = (-1)^{m\times n}\,|A|\,|B|
\end{array}
$$

变对角阵

$$
\begin{aligned}
& \begin{pmatrix} E_s & -A \\ O & E_n \end{pmatrix}\begin{pmatrix} E_s & A \\ B & E_n \end{pmatrix}\begin{pmatrix} E_s & O \\ -B & E_n \end{pmatrix} \\
= & \begin{pmatrix} E_s - AB & E_sA - AE_n \\ B & E_n \times E_n \end{pmatrix}(\sim) \\
= & \begin{pmatrix} E_s - AB & O \\ B & E_n \end{pmatrix}\begin{pmatrix} E_s & O \\ -B & E_n \end{pmatrix} \\
= & \begin{pmatrix} E_s - AB & O \\ BE_s - E_nB & E_n \end{pmatrix} = \begin{pmatrix} E_s - AB & O \\ O & E_n \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
& \begin{pmatrix} E_s & O \\ -B & E_n \end{pmatrix}\begin{pmatrix} E_s & A \\ B & E_n \end{pmatrix}\begin{pmatrix} E_s & -A \\ O & E_n \end{pmatrix} \\
= & \begin{pmatrix} E_s & O \\ O & E_n - BA \end{pmatrix}
\end{aligned}
$$

3 方阵的特征多项式

3.1 一些特殊的公式

1 Sylvester $A_{m,n}, B_{n,m} \rightarrow \lambda^n f_{AB}(\lambda) = \lambda^m f_{BA}(\lambda)$. f_X是X的特征多项式
2