

# 第三章 数列与级数

## 1 收敛序列

定义 1.1. 度量空间中的收敛序列。极限

度量空间中的序列  $\{x_n\}$ :  $\exists x \in X \rightarrow (\forall \varepsilon > 0, \exists N \in \mathbb{N}^+, n > N \rightarrow d(x_n, x) < \varepsilon)$ .

$\{x_n\}$  称为收敛序列

序列收敛

若  $\{x_n\}$  不收敛称序列发散

序列发散

收敛序列  $\{x_n\}$  与  $x$  定义极限:  $\lim_{n \rightarrow \infty} x_n = x$

极限

若  $\{x_n\}$  有界, 称序列有界

有界

Remark: 收敛序列的定义同时依赖于序列  $\{x_n\}$  和空间  $X$ .

Eg:  $\{1/n\}$  在  $\mathbb{R}$  中收敛于 0, 但在  $\mathbb{R}^+$  中不收敛

定理 1.2. 收敛序列的性质

1.  $\{x_n\}$  收敛于  $x \in X \Leftrightarrow \forall U_x, \text{card}(x_n - U_x) < \omega$ .  $x_n$  至多有限项在  $U_x$  外 致密
2.  $\forall x, y \in X, \lim_{n \rightarrow \infty} x_n = x, \lim_{n \rightarrow \infty} x_n = y \rightarrow x = y$  唯一性
3.  $\{x_n\}$  收敛  $\rightarrow \{x_n\}$  有界
4.  $E \subset X, p$  是  $E$  的极限点  $\rightarrow \exists \{p_n\} \in E, \lim_{n \rightarrow \infty} p_n = p$

证明.

1.  $\lim_{n \rightarrow \infty} x_n = x \rightarrow \forall U_x, x_n$  至多有限项在  $U_x$  外  
 $\lim_{n \rightarrow \infty} x_n = x: \forall \varepsilon > 0, \exists N \in \mathbb{N}^+, n > N \rightarrow d(x_n, x) < \varepsilon$   
 $\rightarrow n > N, x_n \in U_x(\varepsilon)$   
 由  $\varepsilon$  的任意性,  $U_x(\varepsilon)$  是任意邻域. 在  $U_x(\varepsilon)$  外的只有  $N$  项

$\forall U_x, x_n$  至多有限项在  $U_x$  外  $\rightarrow \lim_{n \rightarrow \infty} x_n = x$   
 至多有限项设最大的为  $x_N, n > N \rightarrow x_n \in U_x$   
 $\rightarrow d(x_n, x) < r$   
 由于  $r$  的任意性  $\rightarrow \lim_{n \rightarrow \infty} x_n = x$

2. Assume:  $x \neq y: x - y = r$   
 $U_x(\frac{r}{2}) \cap U_y(\frac{r}{2}) = \emptyset$   
 $\exists N \in \mathbb{N}^+, n > N \rightarrow d(x_n, x) < \frac{r}{2}, d(x_n, y) < \frac{r}{2}$   
 这与  $x_n - y \geq \frac{r}{2}$  矛盾  
 $\rightarrow x = y$

3.  $\{x_n\}$  收敛  $\rightarrow \exists x \in X, \forall \varepsilon > 0, \exists N \in \mathbb{N}^+, n > N \rightarrow d(x_n, x) < \varepsilon$   
 $x_1, \dots, x_N$  有界  
 $n > N, d(x_n, x) < \varepsilon \rightarrow x_n \in U_x(\varepsilon) \rightarrow x_n$  有界  
 $\rightarrow x_n$  有界

$\{x_n\}$  收敛,  $\exists N, n > N \rightarrow d(x_n, x) < 1$   
 $r = \max \{1, d(x_1, x), d(x_2, x), \dots\}$ , 此集合有界且必有  $r \in \mathbb{R}$   
 $\forall x_n, d(x_n, x) \leq r$   
 $\rightarrow \{x_n\}$  有界

4.  $E \subset X, p$  是  $E$  的极限点  $\rightarrow \exists \{p_n\} \in E, \lim_{n \rightarrow \infty} p_n = p$   
 用选择公理。  
 $p$  是  $E$  的极限点  $\rightarrow \forall U_p^0 \cap E \neq \emptyset$   
 构造序列:  $\varepsilon = \frac{1}{n}$ , 每次取  $U_p^0$  中的点  $p_n$   
 $\forall \varepsilon > 0, \exists N, n > N \rightarrow \frac{1}{n} < \varepsilon \rightarrow d(p_n, p) < \frac{1}{n} < \varepsilon$   
 $\rightarrow \lim_{n \rightarrow \infty} p_n = p$

□

**定理 1.3.** 序列收敛性与代数运算的关系

- $\{x_n\}, \{y_n\}$  都是复数序列,  $\lim_{n \rightarrow \infty} x_n = x, \lim_{n \rightarrow \infty} y_n = y$
1.  $\lim (x_n + y_n) = x + y$
  2.  $\forall \lambda \in F, \lim \lambda x_n = \lambda x; \lim (\lambda + x_n) = \lambda + x$
  3.  $\lim x_n \cdot y_n = x \cdot y$
  4.  $s_n \neq 0 \wedge s \neq 0 \rightarrow \lim \frac{1}{s_n} = \frac{1}{s}$

**证明.**

1.  $\lim (x_n + y_n) = x + y$   
 $\forall \varepsilon > 0, \exists N_x, n > N_x \rightarrow d(x_n, x) < \varepsilon; n > N_y \rightarrow d(y_n, y) < \varepsilon$   
 $\rightarrow n > \max(N_x, N_y) \rightarrow d(x_n, x) + d(y_n, y) < 2\varepsilon$   
 $\rightarrow \{x_n + y_n\}$  收敛于  $x + y$

2.  $\lim \lambda x_n = \lambda x$   
 $\forall \varepsilon > 0, n > N \rightarrow d(x_n, x) < \varepsilon$   
 $|\lambda x_n - \lambda x| = |\lambda| \cdot |x_n - x| < |\lambda| \varepsilon$   
 $|\lambda + x_n - \lambda - x| = |x_n - x| < \varepsilon$

这里对  $C$  的所有度量有点困难

3.  $\lim x_n y_n = x y$   
 $x_n y_n - x y = (x_n - x)(y_n - y) + x(y_n - y) + y(x_n - x)$   
 $\forall \varepsilon > 0, \exists N_x, N_y \rightarrow n > N_x \rightarrow d(x_n, x) < \varepsilon, d(y_n, y) < \varepsilon$   
 $n > \max(N_x, N_y) \rightarrow |(x_n - x)(y_n - y)| < \varepsilon^2$   
 $\rightarrow \lim (x_n - x)(y_n - y) = 0$   
 $\rightarrow \lim x_n y_n = \lim (x_n - x)(y_n - y) + x(y_n - y) + y(x_n - x) + x y$   
 $= 0 + x y = x y$

□

4.  $x_n \neq 0 \wedge x \neq 0 \rightarrow \lim \frac{1}{x_n} = \frac{1}{x}$   
 $\forall \varepsilon > 0, \exists N, n > N \rightarrow d(x_n, x) < \varepsilon$   
 $\left| \frac{1}{x_n} - \frac{1}{x} \right| = \left| \frac{x_n - x}{x_n x} \right| = \frac{|x_n - x|}{|x_n x|} < \frac{1}{|x_n x|} \varepsilon$

$x_n x$  是有界量

**定理 1.4.**  $R^k$  上的序列收敛条件, 性质

1.  $x_n \in R^k, x_n$  收敛与  $x \Leftrightarrow \forall i \in 1 \dots k, \lim_n x_{n,i} \rightarrow x_i$
2.  $\{x_n\}, \{y_n\}$  是  $R^k$  中的收敛序列,  $\{\lambda_n\}$  是  $R$  中的收敛序列  
 $\lim (x_n + y_n) = x + y$   
 $\lim x_n \cdot y_n = x \cdot y$   
 $\lim \lambda_n x_n = \lambda x$

**Remark:** 这里的  $R^k$  向量的度量为范数诱导的度量。

**证明.**

1.  $x_n \rightarrow x: |x_{n,i} - x_i| \leq |x_n - x| = (\sum_i (x_{n,i} - x_i)^2)^{1/2} < \varepsilon$   
 $\forall i \in 1 \dots k, |x_{n,i} - x_i| < \varepsilon \rightarrow |x_n - x| = (\sum_i (x_{n,i} - x_i)^2)^{1/2}$   
 $< (k\varepsilon^2)^{1/2} = \sqrt{k} \varepsilon$   
 $\rightarrow d(x_n, x) < \sqrt{k} \varepsilon \rightarrow x_n$  收敛

□

2.  $\lim (x_n + y_n) \Leftrightarrow \lim (x_{i,n} + y_{i,n}) = x + y$   
 $\dots$

## 2 子序列

**定义 2.1.** 子序列, 部分极限

序列  $\{x_n\}$ , 取正整数序列  $\{n_i\}, n_i < n_{i+1}$ . 称  $\{x_{n_i}\}$  为  $\{x_n\}$  的子序列 子序列  
若  $\{x_{n_i}\}$  收敛, 称为  $\{x_n\}$  的部分极限 部分极限

**定理 2.2.** 序列收敛与  $x \Leftrightarrow$  任意子序列收敛与  $x$

$$\begin{aligned} \lim x_n = x &\rightarrow \lim x_{n_i} = x \\ \lim x_n = x, \forall \varepsilon > 0, \exists N, n > N \rightarrow d(x_n, x) < \varepsilon \\ \forall \{x_{n_i}\}, \exists n_i > N, d(x_{n_i}, x) < \varepsilon &\quad n x > r \\ &\rightarrow \lim x_{n_i} = x \end{aligned}$$

$$\begin{aligned} \lim x_{n_i} = x &\rightarrow \lim x_n = x \\ \text{由 } n_i \text{ 的任意性令 } n_i &= i \\ &\rightarrow \lim x_i = x \end{aligned}$$

**定理 2.3.** 紧度量空间中的任意序列存在收敛于紧度量空间内部的子序列.

**证明.**

$\{p_n\}$  的值域有限  $\rightarrow \exists \{p_{n_i}\} = p$ . 否则与无限性矛盾  
 $\{p_n\}$  的值域  $E$  无限  $\rightarrow E$  在  $X$  中有极限点  $x$  紧集子集的极限点在紧集中. 第二章  
使用选择公理可以构造序列  $d(x_n, x) < \frac{1}{n}$  □  
 $\rightarrow \{x_n\} \subset \{p_n\}$   
 $\rightarrow \{p_{n_i}\} \rightarrow x$

**推论 2.4.** (致密性定理)  $R^k$  中的有界序列必有收敛子列. (参考第二章 Weierstrass 定理)

**定理 2.5.** 度量空间  $X$  里, 序列  $\{x_n\}$  的所有部分极限  $\{x_p\}$  是  $X$  中的闭集

**证明.**

$E = \{p: p \text{ 是 } \{x_n\} \text{ 的部分极限}\}$   
 $\Leftrightarrow q \text{ 是 } E \text{ 的极限点} \rightarrow q \in E$   
 $q \text{ 是 } E \text{ 的极限点: } \forall U_q^0(r) \cap E \neq \emptyset$   
选择公理:  $\exists \{q_n\} \in E, \lim q_n = q$   
 $\{q_n\} \subset \{x_n\}$   
极限点定义:  $n > N_d \rightarrow d(q_n, q) < \varepsilon$   
 $E$  的定义, 部分极限:  $\forall q_n \in E, \exists x_{n_i} \subset x_n, i > N_p \rightarrow d(x_{n_i}, q_n) < \varepsilon$   
 $\rightarrow n > \max(N_p, N_d) \rightarrow d(x_{n_i}, q) \leq d(x_{n_i}, q_n) + d(q_n, q) < 2\varepsilon$   
 $\rightarrow x_{n_i} \rightarrow q$  收敛定义  
 $\rightarrow q \in E$

证明是两步走: 1. 极限点则必然距离小于任意正数; 2. 序列中每个值必然是  $x_n$  的部分序列;

进而任意正数都有一个部分序列和极限点  $p$  的距离小于这个正数 □

### 3 Cauchy 序列

**定义 3.1.** Cauchy 序列

度量空间  $X$  中的序列  $\{x_n\}: \forall \varepsilon > 0, \exists N \in \mathbb{N}^+, \forall n \geq N, m \geq N \rightarrow d(x_n, x_m) < \varepsilon$  称为 Cauchy 序列

**定义 3.2.** 度量空间  $X$  的子集  $E$  的直径。  $\text{diam } E = \sup \{d(x, y): x, y \in E\}$

**定理 3.3.**  $X$  中的序列是 Cauchy 序列  $\Leftrightarrow \lim_{N \rightarrow \infty} \text{diam } \{x_N, x_{N+1}, \dots\} = 0$

**定理 3.4. 集合直径的性质**

- $$E \subset X, \text{diam } E = \text{diam } \bar{E}$$

$$E \subset \bar{E} \rightarrow \text{diam } E \leq \text{diam } \bar{E}$$

$$\bar{E} = E \cup E'$$

若  $\text{diam } \bar{E} > \text{diam } E$ , 那么至少有一个点在  $E'$  里

$$x \in E' \wedge y \in E:$$

$$d(x, y) \leq d(x, x_n) + d(x_n, y) \leq \varepsilon + \text{diam } E = \text{diam } E$$

$$x \in E' \wedge y \in E'$$

$$d(x, y) \leq d(x_n, x) + d(y_n, y) + d(x_n, y_n) = 2\varepsilon + \text{diam } E = \text{diam } E$$

$$\rightarrow \text{diam } E = \text{diam } \bar{E}$$
- $\{K_n\}$  是  $X$  中紧集的序列  $\wedge K_{n+1} \subset K_n \wedge \lim \text{diam } K_n = 0 \rightarrow \text{card } \bigcap_1^\infty K_n = 1$   
 let:  $K = \bigcap_1^\infty K_n$ . 若  $\exists K_n = \emptyset$  那么结果不是重要的  
 令  $K_n \neq \emptyset \rightarrow K \neq \emptyset$   
 $\exists x \neq y \in K \rightarrow \text{diam } K \geq d(x, y) > 0$ . 矛盾  
 $\rightarrow \text{card } K = 1$

**定理 3.5. Cauchy 序列的性质**

- 度量空间中: 收敛序列是 Cauchy 序列

$$\lim x_n = x \rightarrow \forall \varepsilon > 0, \exists N, n > N \rightarrow d(x_n, x) < \varepsilon$$

$$i, j > N \rightarrow d(x_i, x_j) \leq d(x_i, x) + d(x, x_j) < 2\varepsilon$$

$$\rightarrow x_n \text{ 是柯西序列}$$
- 紧度量空间  $X$  中的 Cauchy 序列  $\{x_n\}$  收敛于  $X$  的内部

$\{x_n\}$  是紧度量空间中的 Cauchy 序列:  $n > N \rightarrow d(x_i, x_j) < \varepsilon$

$$E_N = \{x_N, x_{N+1}, \dots\}$$

$$\lim_N (\text{diam } \bar{E}_N) = 0$$

$\bar{E}_N$  闭  $\wedge \bar{E}_N$  在紧空间中  $\rightarrow \bar{E}_N$  也是紧集

$$E_{N+1} \subset E_N \rightarrow \bar{E}_{N+1} \subset \bar{E}_N$$

$$\rightarrow \text{card } (\bigcap_1^\infty \bar{E}_N) = 1 \quad 3.4-2$$

$$\forall \varepsilon > 0, \exists N_0 \in N^+. n > N_0 \rightarrow \text{diam } \bar{E}_N < \varepsilon$$

let  $x \in \bigcap_1^\infty \bar{E}_N, x \in \bar{E}_N, \forall y \in \bar{E}_N, d(x, y) < \varepsilon$

$$\rightarrow d(x_n, x) < \varepsilon$$

$$\rightarrow \lim_{n \rightarrow \infty} x_n = x, x \in \bar{E}_N$$
- $R^k$  中的所有 Cauchy 序列收敛

$\{x_n\} \subset R^k$  是 Cauchy 序列:  $i, j > N \rightarrow d(x_i, x_j) < \varepsilon$

let:  $\varepsilon = 1 \rightarrow d(x_i, x_j) < 1$

$$\text{range } \{x_n\} \subset \text{range } \{x_1, \dots, x_N\} \cup U_{x_{N+1}}(1)$$

$$\rightarrow \{x_n\} \text{ 有界}$$

$$\rightarrow \exists F \subset R^k \wedge F \text{ 有界} \rightarrow \{x_n\} \subset F, F \text{ 紧}$$

$$\rightarrow \{x_n\} \text{ 在 } R^k \text{ 有收敛点}$$

**定义 3.6. 空间的完备性. 每个 Cauchy 序列都在  $X$  中收敛**

Remark: 所有紧度量空间是完备的; 欧氏空间是完备的; 完备度量空间的闭子集是完备的;

$Q$  不是完备的;

$R^k$  中的有界序列不一定收敛;  $R$  中的单调有界序列收敛;

**定义 3.7. 单调序列.**

实数序列  $\{x_n\}$

单调增  $x_n \leq x_{n+1} \quad n \in N^+$

单调减  $x_n \geq x_{n+1}$

定理 3.8. 单调序列. 收敛  $\Leftrightarrow$  有界

证明.

$$\begin{aligned} \text{单调增序列 } \{x_n\}. x_n \leq x_{n+1} &\rightarrow \exists x \in X, \lim x_n = x \\ \text{range } \{x_n\} \text{ 有界}. x = \sup \text{range } \{x_n\} \\ &\forall n \in N^+, x_n \leq x \\ \forall \varepsilon > 0, \exists N \in N^+, x - \varepsilon < x_n &\leq x \\ x_n \leq x_{n+1} \rightarrow n \geq N, x - \varepsilon < x_n &\leq x \\ &\rightarrow \lim x_n = x \end{aligned}$$

□

$$\lim x_n = x \rightarrow \text{range } \{x_n\} \text{ 有界.} \quad 1.2$$

## 4 上极限和下极限

定义 4.1. 特殊的发散. 收敛于  $-\infty, +\infty$

$$\begin{aligned} +\infty \quad \forall M \in R, \exists N \in N^+, n > N \rightarrow x_n \geq M \quad \lim x_n = +\infty \\ -\infty \quad \forall M \in R, \exists N \in N^+, n > N \rightarrow x_n \leq M \quad \lim x_n = -\infty \end{aligned}$$

定义 4.2. 上极限, 下极限

$$\begin{aligned} \{x_n\} \in R, E \text{ 是所有子序列 } \{x_{n_i}\} \text{ 的极限 } x \text{ 的集合. } E \text{ 还包括 } -\infty, +\infty \\ \text{上极限: } x^* = \sup E. \lim_{n \rightarrow \infty} \sup x_n = x^* \\ \text{下极限: } x_* = \inf E. \lim_{n \rightarrow \infty} \inf x_n = x_* \end{aligned}$$

定理 4.3. 上下极限的性质

1.  $x^*, x_* \in E$
2.  $y > x^*, \exists N \in N^+, n > N \rightarrow x_n < y$   
 $x^*$  是唯一满足上述两个性质的数

证明.

$$\begin{aligned} x^* = +\infty: E \text{ 无上界} &\rightarrow \exists x_{n_i}, \lim_i x_{n_i} = +\infty \\ &+\infty \in E \end{aligned}$$

$$\begin{aligned} x^* \in R: \\ E \text{ 闭} \\ E \text{ 上有界, } \sup E \in \bar{E} \\ \rightarrow x^* \in E = \bar{E} \end{aligned} \quad \begin{array}{l} 2.5 \\ \text{第二章定理2.28} \end{array}$$

$$\begin{aligned} x^* = -\infty: E = \{-\infty\} &\rightarrow E \text{ 没有部分极限} \\ \rightarrow \forall M \in R, x_n \text{ 中只有有限个 } &> M \\ \rightarrow \lim x_n &\rightarrow -\infty \\ \rightarrow x^* &\in E \end{aligned}$$

$$\begin{aligned} \text{设有无限多 } x_n > y > x^* \\ \exists x_{n_i} \subset x_n \rightarrow \lim x_{n_i} > y &\rightarrow \exists t \in R \cup \{-\infty, +\infty\} \rightarrow t > x^* \\ t \in E \text{ 与 } x^* = \sup E &\text{ 矛盾} \end{aligned}$$

唯一性

$$\begin{aligned} p \neq q \wedge p, q \text{ 满足 } 1.; 2. \\ p < q \rightarrow q = \sup E, \forall p < q \rightarrow p \neq \sup E \quad \text{依赖广义实数系上的 } \sup \text{ 唯一性} \\ \text{矛盾} \\ \rightarrow p \text{ 唯一} \end{aligned}$$

□

定理 4.4.

$$x_n, y_n. n > N, x_n \leq y_n. \rightarrow \liminf x_n \leq \liminf y_n \wedge \limsup x_n \leq \limsup y_n$$

例 4.5. 一些序列的上下极限

1.  $\text{range } \{x_n\} = Q.$  那么  $E = R \cup \{-\infty, +\infty\}.$   $\limsup x_n = +\infty, \liminf x_n = -\infty$
2.  $x_n = \frac{(-1)^n}{1+1/n}.$   $\limsup x_n = 1. \liminf x_n = -1$
3.  $\text{range } \{x_n\} \in R. \lim x_n = x \Leftrightarrow \limsup x_n = \liminf x_n = x$

## 5 一些特殊序列

定理 5.1. 一些特殊序列的极限

1.  $p > 0. \lim_n \frac{1}{n^p} = 0$  幂函数的倒数极限为0  
 $\forall \varepsilon > 0, n > \left(\frac{1}{\varepsilon}\right)^{1/p} \rightarrow x_n < \left(\left(\frac{1}{\varepsilon}\right)^{1/p}\right)^{-1} = \varepsilon$
2.  $p > 0. \lim_n \sqrt[p]{p} = 1$   
 $p = 1: \lim_n 1 = 1$   
 $p > 1: x_n = \sqrt[p]{p} - 1, x_n > 0$   
 $1 + nx_n \leq (1 + x_n)^n = p$   
 $\rightarrow 0 < x_n \leq \frac{p-1}{n}$   
 $\rightarrow x_n \rightarrow 0$  夹逼准则. 需要额外证一下, 没啥难度
3.  $\lim_n \sqrt[n]{n} = 1$   
 $x_n = \sqrt[n]{n} - 1 \rightarrow x_n > 0$   
 $n = (1 + x_n)^n = \sum_{i=0}^n C_n^i 1^{n-i} x_n^i$   
 $\geq C_n^2 x_n^2 = \frac{n(n-1)}{1 \cdot 2} x_n^2$   
 $1 \geq \frac{n-1}{2} x_n^2$   
 $\rightarrow 0 < x_n \leq \sqrt{\frac{2}{n-1}}$   
 $\rightarrow x_n \rightarrow 0$
4.  $p > 0, \alpha \in R. \lim_n \frac{n^\alpha}{(1+p)^n} = 0$  任意幂函数相对任意增指数函数为0  
 $\exists k \in N^+, k > \alpha. n > 2k$   
 $\rightarrow (1+p)^n > C_n^k p^k = \frac{n(n-1)\dots(n-k+1)}{k!} p^k > \frac{n^k p^k}{2^k k!}$   
 $(1+p)^n > \frac{n^k p^k}{2^k k!}$   
 $\rightarrow \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha-k}$   
 $\alpha < k \rightarrow \lim_n n^{\alpha-k} = 0$   
 $???\left(\frac{2}{p}\right)^k$  可以超过  $n^k$  的。循环论证
5.  $|x| < 1. \lim x^n = 0$   
 $|x| < 1. t = \frac{1}{x}. |t| > 1$   
 $t_n = t^n - 1$   
 $\lim t_n = 0 \rightarrow |t^n| = \infty$   
 $\lim x^n = 0$   
 $???$

## 6 级数

这里指复级数和有限维向量空间中的级数

定义 6.1. 无穷级数, 级数

$$\begin{array}{ll} \text{序列 } \{a_n\} & \\ \text{序列的部分和} & s_n = \sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n \\ \text{无穷级数} & \sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots \\ \text{级数的收敛} & \text{部分和序列 } s_n \text{ 收敛. 记作 } \sum_{i=1}^n a_n = s \\ \text{级数的发散} & \text{部分和序列 } s_n \text{ 不收敛} \end{array}$$

Remark: 级数的和是由加法和极限两个运算定义的。

定理 6.2. 级数的Cauchy准则

$$\sum a_n \text{ 收敛} \Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}^+, m \geq n \geq N \rightarrow |\sum_{i=n}^m a_n| < \varepsilon$$

Remark: 这里Cauchy序列一定收敛表明级数的空间已经被限定在有限维空间中

推论 6.3.  $\{a_n\}$  是Cauchy序列:  $m = n \rightarrow |a_n| < \varepsilon \rightarrow \lim a_n = 0$

Remark: 逆命题不一定成立.  $s_n = \sum \frac{1}{n}$

定理 6.4.  $a_n \geq 0 \wedge s_n$  收敛  $\Leftrightarrow s_n$  有界

定理 6.5. 比较验证法

1.  $\exists N \in \mathbb{N}^+, \forall n > N, |a_n| \leq b_n \wedge \sum b_n$  收敛  $\rightarrow \sum a_n$  收敛
2.  $\exists N \in \mathbb{N}^+, \forall n > N, a_n \geq b_n \geq 0 \wedge \sum b_n$  发散  $\rightarrow \sum a_n$  发散  $a_n$  是正项级数

证明.

$$\begin{array}{l} 1. \quad \sum b_n \text{ 收敛} \rightarrow \sum b_n \text{ 是Cauchy序列} \\ \rightarrow |a_n| < b_n \rightarrow |\sum_{i=n}^m a_n| \leq \sum_{i=n}^m |a_n| \leq \sum_{i=n}^m b_n < \varepsilon \\ \rightarrow |\sum_{i=n}^m a_n| < \varepsilon \\ \rightarrow a_n \text{ 是Cauchy序列} \rightarrow a_n \text{ 收敛} \end{array}$$

□

$$\begin{array}{ll} 2. & \text{若 } a_n \text{ 收敛} \rightarrow b_n \text{ 收敛} \\ & b_n \text{ 不收敛} \rightarrow a_n \text{ 不收敛} \end{array} \quad \begin{array}{l} 1 \\ \text{逆否} \end{array}$$

Remark: 应用比较验证法需要记忆常用的收敛或发散的项级数

## 7 非负项级数

定理 7.1. 几何级数的收敛性

$$\begin{array}{ll} \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} & 0^0 = 0, x^0 = 1 \\ 0 \leq x < 1 & \text{收敛} \\ x \geq 1 & \text{发散} \end{array}$$

证明.

$$\begin{array}{ll} x \neq 1 & \sum_{n=0}^k x^n = \frac{1-x^{k+1}}{1-x} \\ & \lim_{k \rightarrow \infty} \frac{1-x^{k+1}}{1-x} \\ 0 \leq x < 1 & \lim \frac{1-\lim x^{k+1}}{1-x} = \frac{1-0}{1-x} = \frac{1}{1-x} \quad \text{收敛} \\ x > 1 & \lim \frac{1-\lim x^{k+1}}{1-x} = \frac{1-\infty}{1-x} = \frac{-\infty}{-x} = \infty \quad \text{发散} \\ x = 1 & \sum x^n = 1 + 1 + \cdots = \infty \quad \text{发散} \end{array}$$

□

定理 7.2. Cauchy. 正项级数  $\sum a_n, a_{n+1} \leq a_n, \sum_1^{\infty} a_n$  收敛  $\Leftrightarrow \sum_0^{\infty} 2^n a_{2^n}$  收敛

证明.

$$\begin{aligned} \sum 2^n a_{2^n} \text{收敛} &\rightarrow \sum a_n \text{收敛} \\ s_n = \sum_{i=1}^n a_i; t_n &= \sum_{i=0}^n 2^i a_{2^i} \\ n < 2^k: s_n &\leq a_1 + (a_2 + a_3) + \cdots + (a_{2^i} + \cdots + a_{2^{i+1}-1}) \\ &\leq a_1 + 2a_2 + \cdots + 2^k a_{2^k} = t_k. \\ t_k \text{收敛} &\rightarrow \lim_k t_k = t \rightarrow t_k < t \rightarrow s_n \text{有界} \rightarrow s_n \text{收敛} \end{aligned}$$

□

$$\begin{aligned} \sum a_n \text{收敛} &\rightarrow \sum 2^n a_{2^n} \text{收敛} \\ n > 2^k: t_k &= a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k} \\ &\leq a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots + \\ &= s_{2^{k+1}} \leq 2s_n \\ &\rightarrow t_k \text{有界} \rightarrow t_k \text{收敛} \end{aligned}$$

定理 7.3.  $p$ 级数的敛散性.

$$\begin{aligned} p \text{级数: } \sum_n \frac{1}{n^p} \\ p > 1 &\text{ 收敛} \\ p \leq 1 &\text{ 发散} \\ p \leq 0 &\text{ 发散} \end{aligned}$$

证明.

$$\begin{aligned} p = 0: \rightarrow \lim_{n \rightarrow \infty} n^p = 1 \rightarrow \lim_{n \rightarrow \infty} \frac{1}{n^p} = \frac{1}{1} = 1 \quad a_n, a \text{都存在不为} 0 \rightarrow a_n^{-1} \rightarrow a^{-1} \\ \rightarrow \sum \frac{1}{n^p} \text{发散} \quad 6.3 \\ p < 0: -p > 0 \rightarrow \frac{1}{n^p} = n^{-p}. \\ \lim_{n \rightarrow \infty} n^{-p} = \infty \rightarrow \sum n^{-p} \text{发散} \quad 6.3 \end{aligned}$$

$$\begin{aligned} \sum n^{-p} \text{与} \sum_{k=0}^{\infty} 2^k \frac{1}{2^{kp}} = \sum_0^{\infty} 2^{k(1-p)} \text{同敛散} \quad 7.2 \\ \rightarrow \text{若收敛} \rightarrow \lim_k 2^{k(1-p)} = 0 \\ 2^{k(1-p)} = (2^{1-p})^k \rightarrow \text{后者是几何级数} \\ 0 \leq 2^{1-p} < 1 \text{时收敛} \\ \rightarrow 1-p < 0 \rightarrow p > 1 \\ \text{其它都发散} \\ p > 1: \text{收敛} \\ p < 1: \text{发散} \end{aligned}$$

□

定理 7.4. 级数  $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$  的敛散性

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{1}{n(\log n)^p} \\ p > 1 &\text{ 收敛} \\ p \leq 1 &\text{ 发散} \end{aligned}$$

Remark: 这里使用了  $\forall x > 1, \lim_{n \rightarrow \infty} \frac{\log n}{x^n} = 0$ . 对数函数增长速度低于幂函数.

证明.

$$\begin{aligned} n \log n \text{单调增} &\rightarrow \frac{1}{n \log n} \text{单调减} \\ \sum_{n=2}^{\infty} \frac{1}{n(\log n)^p} \text{等价于} \sum_{k=1}^{\infty} 2^k \frac{1}{2^{k(\log 2)^p}} \text{的敛散性} \\ \sum_{k=1}^{\infty} 2^k \frac{1}{2^{k(\log 2)^p}} &= \sum_{k=1}^{\infty} \frac{1}{(k \log 2)^p} = \sum_{k=1}^{\infty} \frac{1}{(\log 2)^p} \cdot \frac{1}{k^p} \\ &= \frac{1}{(\log 2)^p} \sum_{k=1}^{\infty} \frac{1}{k^p} \\ \frac{1}{(\log 2)^p} \in R &\rightarrow \text{级数等价于} p \text{级数} \\ \rightarrow p > 1: \text{收敛} \\ p \leq 1: \text{发散} \end{aligned}$$

□



这种级数构造法可以不断构造新的收敛速度更慢的级数.

$$\sum_{n=3}^{\infty} \frac{1}{n (\log n) (\log(\log(n)))^1} \quad \text{发散}$$

$$\sum_{n=3}^{\infty} \frac{1}{n (\log n) (\log(\log(n)))^2} \quad \text{收敛}$$

但是收敛级数和发散级数之间没有明确的界限。参考习题11, 12

Reference: Theory and Application of Infinite Series. Knopp. Chapter IX.

## 8 数e

定义 8.1.  $e$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

证明. 上述级数收敛

$$\begin{aligned} n > 4 &\rightarrow n! = 1 \times 2 \times 3 \times 4 \times \cdots \\ &> 1 \times 2 \times 2 \times 2^2 \times 2 \cdots = 2^n \\ 0! &= 2^0 = 1 \\ 1! &= 1 < 2^1 = 2 \\ 2! &= 2 < 2^2 = 4 \\ 3! &= 6 < 2^3 = 8 \end{aligned}$$

$$\begin{aligned} \sum_{n=1}^k x^n &= s \cdot x \cdot s = \sum_{n=2}^{k+1} x^n \\ x s - s &= \sum_{n=2}^{k+1} x^n - \sum_{n=1}^k x^n = x^{k+1} - x^1 \\ s &= \frac{x^{k+1} - x^1}{x - 1} \end{aligned}$$

□

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{n!} &= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots \\ &\leq 1 + \frac{1}{1} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \\ &= 1 + 1 + \frac{2^{-n-1} - 2^{-1}}{2^{-1} - 1}. \{= 2(2^{-1} - 2^{-n-1}) = 1 - 2^{-n}\} \end{aligned}$$

$$\begin{aligned} &\leq 1 + 1 + 1 = 3 \\ \frac{1}{n!} &> 0 \rightarrow \sum_{n=0}^{\infty} \frac{1}{n!} \text{收敛} \end{aligned}$$

有界  
单调

推论 8.2.  $\sum_{i=0}^{\infty} \frac{1}{n!}$  的收敛速度

$$\begin{aligned} e - s_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots \\ &< \frac{1}{(n+1)!} \left( 1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots \right) \\ &= \frac{1}{(n+1)!} \cdot \lim_{k \rightarrow \infty} \frac{(n+1)^{-k-1} - (n+1)^{-1}}{(n+1)^{-1} - 1} \\ &= \frac{1}{n!n} \end{aligned}$$

定理 8.3.  $e$  是无理数

证明.

$$\begin{aligned} \text{反证: } e &= p/q \\ \rightarrow e - s_n &< \frac{1}{n!n} \\ 0 &< q!(e - s_n) < \frac{1}{q} \\ q!(e - s_n) &= q!e - q!s_n \in N^+ \\ \text{但没有正整数在区间} &\left(0, \frac{1}{q}\right) \text{内} \\ \rightarrow e &\neq \frac{p}{q} \end{aligned}$$

Remark:  $e$ 不是代数数.

Reference:

*Irrational Numbers, Carus Mathematical Monograph No. 11. Niven, I.M.*

*Topics in Algebra. Herstein, I. N.*

□

定理 8.4.  $\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = e$

证明.

$$\begin{aligned} s_n &= \sum_{k=0}^n \frac{1}{k!}, \quad t_n = (1 + \frac{1}{n})^n \\ 1 + \frac{1}{n} &= \frac{n+1}{n} \\ t_n &= \sum_{i=0}^{n+1} C_n^i 1^i \frac{1}{n^{n+1-i}} \\ &= \sum_{i=0}^{n+1} \frac{(n+1)!}{i!(n+1-i)!} \cdot \frac{1}{n^{n+1-i}} = \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \\ &= 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + \frac{1}{3!}(1 - \frac{1}{n})(1 - \frac{2}{n}) + \cdots + \frac{1}{n!}(1 - \frac{1}{n})(1 - \frac{2}{n}) \cdots (1 - \frac{n-1}{n}) \\ &\rightarrow t_n \leq s_n \\ \lim_{n \rightarrow \infty} \sup t_n &\leq e \end{aligned} \quad 4.4$$

□

$$\begin{aligned} n \geq m \rightarrow t_{n,m} &\geq 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + \cdots + \frac{1}{m!}(1 - \frac{1}{n})(1 - \frac{2}{n}) \cdots (1 - \frac{m-1}{n}) \\ \lim_{n \rightarrow \infty} \inf t_n &\geq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{m!} \\ &\rightarrow s_m \leq \lim_{n \rightarrow \infty} \inf t_{n,m} \\ \rightarrow e &\leq \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \inf t_{n,m} \\ &\rightarrow \lim t_n = \lim s_n = e \end{aligned}$$

## 9 根值验敛法与比率验敛法

定理 9.1. 根值验敛法

级数  $\sum a_n$ .  $\alpha = \lim_{n \rightarrow \infty} \sup \sqrt[n]{|a_n|}$   
 $\alpha < 1$  时,  $\sum a_n$  收敛  
 $\alpha > 1$  时,  $\sum a_n$  发散  
 $\alpha = 1$  时,  $\sum a_n$  无法判断

证明.

$$\begin{aligned} \alpha < 1 \rightarrow \forall 0 < \beta < 1, \exists N \in \mathbb{N}^+, n \geq N \rightarrow \sqrt[n]{|a_n|} < \beta \\ &\rightarrow |a_n| < \beta^n \\ 0 \leq \beta < 1 \rightarrow \sum \beta^n &\text{收敛} \\ &\rightarrow \sum a_n \text{收敛} \end{aligned} \quad 6.5 \text{比较验敛法}$$

$$\begin{aligned} \alpha > 1 \rightarrow \lim_{n \rightarrow \infty} \sup \sqrt[n]{|a_n|} &> 1 \\ \rightarrow \exists n_i \rightarrow |a_{n_i}| &> 1 \\ \rightarrow \sum_{n=1}^{\infty} a_n &\text{发散} \end{aligned}$$

□

6.3

$$\begin{aligned} \alpha = 1: \sum \frac{1}{n} &\text{发散}, \sum \frac{1}{n^2} \text{收敛} \\ \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{n}} = 1; \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{n^2}} &= 1 \end{aligned}$$

定理 9.2. 比率验敛法

级数  $\sum a_n$

$\lim_{n \rightarrow \infty} \sup \left| \frac{a_{n+1}}{a_n} \right| < 1: \sum a_n$  收敛  
 $\exists N \in \mathbb{N}^+, n > N \rightarrow \left| \frac{a_{n+1}}{a_n} \right| \geq 1. \sum a_n$  发散.  $(\liminf \left| \frac{a_{n+1}}{a_n} \right| \geq 1)$   
 不满足上述条件的一切情况都无法判断

证明.

$$\begin{aligned} \alpha < 1 &\rightarrow \exists \alpha < \beta < 1, n > N \rightarrow \left| \frac{a_{n+1}}{a_n} \right| < \beta \\ &\rightarrow |a_{n+1}| < \beta |a_n| \\ &\rightarrow |a_n| < \beta^{-N} |a_N| \cdot \beta^n \\ &\rightarrow \sum \beta^n \text{收敛} \rightarrow |a_n| \text{收敛} \end{aligned} \quad 6.5$$

$$\begin{aligned} \alpha > 1 &\rightarrow \exists \alpha > \beta > 1, n > N \rightarrow \left| \frac{a_{n+1}}{a_n} \right| > \beta \\ &\rightarrow |a_{n+1}| > |a_n|. \exists a_n > 0 \rightarrow a_{n+1} > 0 \\ &\rightarrow \lim a_n > 0 \\ &\rightarrow \sum a_n \text{不收敛} \end{aligned} \quad 6.3$$

□

$$\begin{aligned} \alpha = 1 &\rightarrow \sum \frac{1}{n} \text{发散}, \sum \frac{1}{n^2} \text{收敛} \\ \limsup \frac{1/(n+1)}{1/n} &= \limsup \frac{n}{n+1} = 1 \\ \limsup \frac{1/(n+1)^2}{1/n^2} &= \limsup \frac{n^2}{(1+n^2)} = 1 \end{aligned}$$

**定理 9.3.** 正项级数. 比率收敛法能判断出级数收敛, 根值收敛法一定能判断出收敛性

$$\begin{aligned} &\forall \{c_n: c_n > 0\} \\ 1. \quad &\liminf \frac{c_{n+1}}{c_n} \leq \liminf \sqrt[n]{c_n} \\ 2. \quad &\limsup \sqrt[n]{c_n} \leq \limsup \frac{c_{n+1}}{c_n} \end{aligned}$$

证明.

$$\begin{aligned} 2. \quad &\alpha = \lim_{n \rightarrow \infty} \sup \frac{c_{n+1}}{c_n}. \alpha \geq 0 \\ &\alpha = +\infty \rightarrow \limsup \sqrt[n]{c_n} \leq +\infty \\ &\alpha \neq +\infty. \forall \alpha < \beta, n \geq N \rightarrow \frac{c_{n+1}}{c_n} < \beta \\ &\quad c_{N+k+1} \leq \beta c_{N+k} \\ &\quad \rightarrow c_{N+p} \leq \beta^p c_N \\ &\quad c_n \leq c_N \beta^{-N} \cdot \beta^n \\ &\quad \rightarrow \sqrt[n]{c_n} \leq \sqrt[n]{c_N} \beta^{-N/n} \cdot \beta \\ &\quad \rightarrow \lim_{n \rightarrow \infty} \sup \sqrt[n]{c_n} \leq \beta \\ &\beta \text{的任意性} \rightarrow \lim_{n \rightarrow \infty} \sup \sqrt[n]{c_n} \leq \alpha \end{aligned}$$

□

1. 与2.类似

*Remark:* 此定理表明比率收敛法能判断收敛的级数, 根值收敛法一定也能判断出其收敛

但这两个收敛法都不能判断级数发散。总是从  $n \rightarrow \infty, a_n \neq 0$  来判断发散性。

**例 9.4.** 比率收敛法失效, 根式收敛法有效的例子

$$1. \quad \frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots = \sum \left( \frac{1}{2^n} + \frac{1}{3^n} \right)$$

$$\begin{aligned}\liminf \frac{a_{n+1}}{a_n} &= \lim \left(\frac{2}{3}\right)^n = 0 && \text{没用} \\ \liminf \sqrt[n]{a_n} &= \lim \sqrt[2n]{\frac{1}{3^n}} = \frac{1}{\sqrt{3}} && \text{没用} \\ \limsup \frac{a_{n+1}}{a_n} &= \lim \left(\frac{3}{2}\right)^n = +\infty && \text{无法判断} \\ \limsup \sqrt[n]{a_n} &= \sqrt[2n]{\frac{1}{2^n}} = \frac{1}{\sqrt{2}} && \text{收敛}\end{aligned}$$

$$2. \frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + \frac{1}{128} + \frac{1}{64} + \cdots = \sum_0^\infty \left( \frac{1}{2^{2n+1}} + \frac{1}{2^{2n}} \right)$$

$$\begin{aligned}\liminf \frac{a_{n+1}}{a_n} &= \frac{1}{8} \\ \limsup \frac{a_{n+1}}{a_n} &= 2 \\ \limsup \sqrt[n]{a_n} &: \\ &\limsup \sqrt[n]{\frac{1}{2^{n+1}}} = 2^{-\frac{n+1}{n}} = 2^{-1} \\ &\limsup \sqrt[n]{\frac{1}{2^n}} = 2^{-\frac{n}{n}} = 2^{-1} \\ &\rightarrow \limsup \sqrt[n]{a_n} = \frac{1}{2}\end{aligned}$$

## 10 幂级数

定义 10.1. 幂级数

$$\{z_n: z \in C\}: \sum_{n=0}^{\infty} c_n z^n$$

称为幂级数。 $c_n$ 称为系数

定理 10.2. 幂级数的收敛性

$$\begin{array}{ll}\text{幂级数} \sum c_n z^n. \alpha = \lim_{n \rightarrow \infty} \sup \sqrt[n]{|c_n|}, R = \frac{1}{\alpha} & \alpha = 0 \rightarrow R = \infty, \alpha = +\infty, R = 0 \\ |z| < R & \text{收敛} \\ |z| > R & \text{发散} \\ |z| = R & \text{具体判断}\end{array}$$

Remark:  $R$ 叫级数的收敛半径

证明.

$$\begin{aligned}a_n &= c_n z^n. \text{根值验敛法} \\ \lim_{n \rightarrow \infty} \sup \sqrt[n]{|c_n z^n|} &= |z| \lim_{n \rightarrow \infty} \sup \sqrt[n]{|c_n|} = \frac{|z|}{R}\end{aligned}$$

□

例 10.3. 一些幂级数的收敛半径

1.  $\sum n^n z^n. R = 0$
2.  $\sum \frac{z^n}{n!}. R = +\infty$
3.  $\sum z^n. R = 1. |z| = 1$ 级数发散.  $\lim |z^n| \neq 0$
4.  $\sum \frac{z^n}{n}. R = 1. z = 1$ 时级数发散. 但在其它的 $|z| = 1$ 上收敛

5.  $\sum \frac{z^n}{n^2}, R=1, |z|=1$  的所有点收敛. 比较验敛法  $\sum \frac{z^n}{n^2} = \sum \frac{1}{n^2}$

## 11 分部求合法

处理积的级数. 数列  $\{a_n b_n\}$  的级数

定理 11.1. 分部求和公式

$$\begin{aligned} & \text{序列 } \{a_n\}, \{b_n\}. A_n = \sum_i^n a_i. A_{-1} = 0 \\ 0 \leq p \leq q \rightarrow \sum_{n=p}^q a_n b_n &= \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \\ & \sum_{n=p}^q a_n b_n = \sum_{n=p}^q (A_n - A_{n-1}) b_n \\ &= \sum_{n=p}^q A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1} \\ &= \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \end{aligned}$$

定理 11.2.  $\sum a_n b_n$  收敛的一个判断准则

- 1  $\sum a_n$  的部分和  $A_n$  有界
- 2  $b_0 \geq b_1 \geq b_2 \geq \dots, b_n$  单调减  $\rightarrow \sum a_n b_n$  收敛
- 3  $\lim b_n = 0$

证明. 虽然分部求和公式无法应用在无穷求和上, 但是可以用在 Cauchy 序列里. doge

$$\begin{aligned} & A_n \text{ 有界 } \rightarrow \exists M \in \mathbb{R}^+. |A_n| < M. \\ & \lim b_n = 0. \forall \varepsilon > 0. \exists N, n > N \rightarrow b_n < \varepsilon \\ & \quad \forall q \geq p \geq N \rightarrow \\ & |\sum_{n=p}^q a_n b_n| = |\sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p| \\ & \leq M |\sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q + b_p| \quad b_n - b_{n+1} \geq 0 \rightarrow \text{不改变方向} \\ & = 2M b_p \leq 2M \varepsilon \\ & \rightarrow \sum a_n b_n \text{ 是 Cauchy 的} \\ & \rightarrow \sum a_n b_n \text{ 收敛} \end{aligned} \quad \square$$

定理 11.3. 交错级数的一个判别法. Leibnitz 判别法

$$\begin{aligned} 1 & |c_1| \geq |c_2| \geq |c_3| \geq \dots \\ 2 & c_{2m-1} \geq 0, c_{2m} \leq 0 \rightarrow \sum c_n \text{ 收敛} \\ 3 & \lim c_n = 0 \end{aligned}$$

证明.

$$\begin{aligned} & a_n = (-1)^{n+1}, b_n = |c_n| \\ & |a_n| \text{ 有界 } \wedge b_n \text{ 单调减 } \wedge \lim b_n = 0 \rightarrow \sum a_n b_n \text{ 收敛} \quad 11.2 \end{aligned} \quad \square$$

定理 11.4. 幂级数收敛圆上的一个验敛法

$$\begin{aligned} 1 & \sum c_n z^n \text{ 的收敛半径 } = 1 \\ 2 & c_0 \geq c_1 \geq c_2 \geq \dots \rightarrow \sum c_n z^n \text{ 在收敛圆上除了 } z=1 \text{ 之外都收敛} \\ 3 & \lim c_n = 0 \end{aligned}$$

证明.

$$\begin{aligned}
a_n &= z^n, b_n = c_n, |z| = 1 \wedge z \neq 1 \\
\rightarrow |A_n| &= \left| \sum_{m=0}^n z^m \right| = \left| \frac{1-z^{n+1}}{1-z} \right| \leq \frac{2}{|1-z|} \\
&\rightarrow |A_n| \text{ 有界}
\end{aligned}$$

□

## 12 绝对收敛

定义 12.1. 绝对收敛.  $\sum |a_n|$  收敛

定理 12.2. 绝对收敛的级数一定收敛

证明.

$$\begin{aligned}
\sum |a_n| \text{ 收敛} &\rightarrow \sum_p^q |a_n| < \varepsilon \\
|\sum_p^q a_n| &\leq \sum_p^q |a_n| < \varepsilon \\
&\rightarrow \sum a_n \text{ 是柯西的} \\
&\rightarrow \sum a_n \text{ 收敛}
\end{aligned}$$

Remark: 正项级数的收敛就是绝对收敛

Remark: 幂级数在收敛圆内绝对收敛

□

注意 12.3. 比较验敛法、根值验敛法、比率验敛法都是绝对收敛的验敛法；不能处理条件收敛的级数。分部求和法有时可以处理条件收敛的级数。

## 13 级数的加法和乘法

定义 13.1. 级数的加法

$$\begin{aligned}
\sum a_n = A, \sum b_n = B &\rightarrow \sum a_n + b_n = A + B \\
\forall c \in F. \sum c a_n &= c A
\end{aligned}$$

证明.

$$\begin{aligned}
A_n &= \sum_{i=1}^n a_i; B_n = \sum_{i=1}^n b_i \\
A_n + B_n &= \sum_{i=1}^n (a_i + b_i) \\
\lim A_n = A \wedge \lim B_n = B &\rightarrow \lim A_n + B_n = A + B \quad 1.3
\end{aligned}$$

□

$$\lim A_n = A, \lim c A_n = c A$$

定义 13.2. 级数的乘法 Cauchy积

$$\begin{aligned}
&\sum a_n; \sum b_n \quad c_n = \sum_{k=0}^n a_k b_{n-k} \\
\text{definition } \sum c_n &= \sum a_n \sum b_n
\end{aligned}$$

来源

$$\begin{aligned}
&(a_1 + a_2 + \cdots)(b_1 + b_2 + \cdots) \\
&= a_1 b_1 + (a_1 b_2 + a_2 b_1) + (a_1 b_3 + a_2 b_2 + a_3 b_1) + \cdots
\end{aligned}$$

例 13.3. 收敛级数的积可以发散

$$\begin{aligned}
 A_n &= \sum_{i=0}^n a_i; B_n = \sum_{i=0}^n b_i; C_n = \sum_{i=0}^n c_i \\
 \lim A_n &= A; \lim B_n = B \\
 A_n B_n &= \sum_{i=0}^n \frac{(-1)^i}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \cdots \\
 A_n, B_n &\text{都收敛} \\
 A_n B_n &= 1 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}\sqrt{2}} + \frac{1}{\sqrt{3}}\right) - \left(\frac{1}{\sqrt{4}} + \frac{1}{\sqrt{3}\sqrt{2}} + \frac{1}{\sqrt{2}\sqrt{3}} + \frac{1}{\sqrt{4}}\right) + \cdots \\
 c_n &= (-1)^n \sum_{i=0}^n \frac{1}{\sqrt{(n-k+1)(k+1)}} \\
 (n-k+1)(k+1) &= \left(\frac{n}{2}+1\right)^2 - \left(\frac{n}{2}-1\right)^2 \leq \left(\frac{n+2}{2}\right)^2 \\
 \frac{1}{\sqrt{(n-k+1)(k+1)}} &\geq \frac{2}{n+2} \\
 |c_n| &\geq \sum_{i=0}^n \frac{2}{n+2} = \frac{2(n+1)}{n+2} \\
 \lim |c_n| &= 2 \geq 0 \\
 &\rightarrow \sum c_n \text{发散}
 \end{aligned}$$

11.3

定理 13.4. Mertens. 两个收敛级数中至少有一个绝对收敛则乘积收敛且收敛到和的乘积

$$\begin{aligned}
 1 \quad & \sum a_n \text{绝对收敛.} \sum a_n = A \\
 2 \quad & \sum b_n = B \quad \rightarrow \sum_{n=0}^{\infty} c_n = AB \\
 3 \quad & c_n = \sum_{k=0}^n a_k b_{n-k}
 \end{aligned}$$

证明.

$$\begin{aligned}
 \text{let: } A_n &= \sum_{i=0}^n a_i; B_n = \sum_{i=0}^n b_i; C_n = \sum_{i=0}^n c_i; \beta_n = B_n - B \\
 C_n &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) \\
 &= a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0 \\
 &= a_0 (B + \beta_n) + a_1 (B + \beta_{n-1}) + \cdots + a_n (B + \beta_0) \\
 &= A_n B + a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0 \\
 \text{let: } r_n &= a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0 \\
 \lim C_n &= AB \Leftrightarrow \lim r_n = 0 \\
 \alpha &= \sum |a_n| \\
 \forall \varepsilon > 0, n > N &\rightarrow |\beta_n| < \varepsilon \\
 |r_n| &\leq |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + |\beta_{N+1} a_{n-N-1} + \cdots + \beta_n a_0| \\
 &\leq |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + \varepsilon \alpha \\
 \lim a_n = 0: n > M &\rightarrow |a_n - a_{n-N}| < \varepsilon \\
 \lim n \rightarrow \infty |\beta_0 a_n + \cdots + \beta_N a_{n-N}| &\leq \beta_N N \varepsilon \\
 &\rightarrow \lim |r_n| \leq (\beta_N N + \alpha) \varepsilon \\
 &\rightarrow \lim \sup |r_n| = 0 \\
 &\rightarrow \lim r_n = 0
 \end{aligned}$$

□

定理 13.5. Abel.  $\sum a_n, \sum b_n, \sum c_n$  分别收敛与  $A, B, C$ .  $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 \rightarrow \sum c_n = AB$

证明. 需要依赖函数连续性的证明. 第八章

□

## 14 级数的重排

定义 14.1. 级数的重排

$k_n = Z^+$ , 且  $k_n$  是  $1-1$  的。

$$a'_n = a_{k_n}$$

$\sum a_{k_n}$  是  $\sum a_n$  的重排

Remark: 级数重排的部分和序列可能是完全不同的数组成的。

例 14.2. 收敛级数的重排不收敛于原来的收敛值的例子

$$\begin{aligned} \text{级数 } \sum \frac{(-1)^{n+1}}{n} &= \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots \\ \text{重排 } \sum \left( \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} \right) &= \frac{1}{1} + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \dots \\ \text{let: } s &= \sum \frac{(-1)^{n+1}}{n} \rightarrow s < 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6} \\ s'_n &= \sum \left( \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} \right), n \geq 1 \rightarrow \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} > 0 \\ &\rightarrow s'_n \text{ 是单调增的} \\ s'_3 &= \frac{1}{1} + \frac{1}{3} - \frac{1}{2} = \frac{5}{6}, s' > s'_3 \rightarrow s' \neq s \\ &\quad (s'_n \text{ 是收敛的}) \end{aligned}$$

定理 14.3. Riemann. 任意广义实数  $a, b$ , 条件收敛的实级数一定有重排使得上极限收敛到  $b$ , 下极限收敛到  $a$ 。

$$\begin{aligned} \text{实级数 } \sum a_n \text{ 条件收敛} \\ -\infty \leq \alpha \leq \beta \leq +\infty \quad \rightarrow \quad \exists \text{ 重排 } \sum a_{n_k} \\ \rightarrow \liminf s'_n = \alpha, \limsup s'_n = \beta \end{aligned}$$

证明.

$$\begin{aligned} \text{let: } p_n &= \frac{|a_n| + a_n}{2}, q_n = \frac{|a_n| - a_n}{2} \\ \rightarrow p_n - q_n &= a_n; p_n + q_n = |a_n| \end{aligned}$$

$$\begin{aligned} p_n, q_n \text{ 都发散: } \quad & \sum p_n, \sum q_n \text{ 收敛} \rightarrow \sum p_n + q_n = \sum |a_n| \text{ 收敛. 矛盾} \\ \text{Assume: } p_n \text{ 发散, } q_n \text{ 收敛} & \rightarrow \sum a_n = \sum (p_n - q_n) = \sum p_n - \sum q_n \\ & \rightarrow \sum a_n \text{ 发散. 矛盾} \\ & \rightarrow p_n, q_n \text{ 都发散} \end{aligned}$$

选取正项和负项:

$$\begin{aligned} a_n \geq 0: P_i &= a_n; \\ a_n < 0: Q_i &= -a_n; \\ P_n \text{ 与 } p_n \text{ 在 } a_n < 0 \text{ 的点上不同, 在 } a_n \geq 0 \text{ 的点上相同} & \rightarrow P_n \text{ 发散} \\ & \rightarrow Q_n \text{ 发散} \end{aligned}$$

□

$$\begin{aligned} \{m_n\}, \{k_n\} \rightarrow \text{级数 } P_1 + \dots + P_{m_1} - Q_1 - \dots - Q_{k_1} \\ + P_{m_1+1} + \dots + P_{m_2} - Q_{k_1+1} - \dots - Q_{k_2} + \dots \end{aligned}$$

$$\begin{aligned} \{m_n\}, \{k_n\} \text{ 存在: } \quad & \exists a_n, b_n \in R \wedge a_n \rightarrow a, b_n \rightarrow b, a_n < b_n, b_1 > 0 \\ & \exists m \rightarrow P_1 + \dots + P_m > b_1, m_1 = \min \{m\} \quad P_n \text{ 发散} \\ & \exists k \rightarrow P_1 + \dots + P_m - Q_1 - \dots - Q_k < a_1, k_1 = \min \{k\} \quad Q_n \text{ 发散} \\ & \text{在有限维空间中, 正项序列发散} \rightarrow \text{不是Cauchy序列} \\ & \rightarrow \text{可以序列取到 } \{m_n\}, \{k_n\} \\ \text{let: } \{x_n\} &= P_1 + \dots + P_{m_1} - Q_1 - \dots - Q_{k_1} + \dots + \\ & |x_n - b_n| \leq P_{m_n}; |x_n - a_n| \leq Q_{k_n} \\ & \lim_{n \rightarrow \infty} P_n = \lim_{n \rightarrow \infty} Q_n = 0 \\ & \rightarrow \liminf x_n = a, \limsup x_n = b \end{aligned}$$



定理 14.4. 绝对收敛的级数的任意重排收敛于原来值

$$\sum |a_n| \text{收敛} \rightarrow \forall \text{重排} \sum a_m \text{收敛}$$

证明.

$$\begin{aligned} \sum |a_n| \text{收敛} &\rightarrow \sum_{n=k}^m |a_n| \text{收敛} && \text{Cauchy} \\ \text{重排} \sum_{n=k}^m a'_n &\leq \sum_{n=k}^m |a'_n| \\ \text{部分和中包含的最大原项} &a'_n = a_n \text{是确定的} \\ \text{let: } \max \{a'_n = n\} &= N. \text{重排的前} n \text{项和中对应原数列中最大的} n \\ \sum_N^m |a'_n| &\leq \sum_k^m |a_n| < \varepsilon \\ &\rightarrow \sum a'_n \text{是柯西的} \\ &\rightarrow \sum a'_n \text{收敛} \end{aligned}$$

□

这里没写清楚。重排的部分和必有原级数中的项。  
选取  $N = \max \{\text{Cauchy选取的} N, \text{重排的最大项} N\}$ .  
则其余的Cauchy和必然小于前面的这些项之和。

$$\begin{aligned} \exists n \in N^+ \rightarrow |s_n - s'_n| \text{中的前} N \text{项相同的都被消掉} &\rightarrow |s_n - s'_n| \leq |s_m - s_n| = \varepsilon \\ &\rightarrow \lim s_n = \lim s'_n \end{aligned}$$

## 习题

1. Proof:  $\{a_n\}$ 收敛  $\rightarrow \{|a_n|\}$ 收敛;  $\{|a_n|\}$ 收敛  $\rightarrow \{a_n\}$ 不一定收敛

$$\begin{aligned} a_n \text{收敛: } \forall \varepsilon > 0, \exists N \in N^+, n > N &\rightarrow d(a_n, a) < \varepsilon \\ |a_n| \geq 0 \rightarrow a \geq 0; |a_n| < 0 &\rightarrow a \leq 0 \\ ||a_n| - |a|| \leq |a_n - a| < \varepsilon \\ &\rightarrow |a_n| \text{收敛} \\ a_n = (-1)^n. a_n \text{不收敛. 但} \lim |a_n| &= 1 \end{aligned}$$

2. Compute:  $\lim_{n \rightarrow \infty} (\sqrt{n^2 + n} - n)$

$$\begin{aligned} &\lim_{n \rightarrow \infty} \sqrt{n^2 + n} - n \\ &= \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2 + n} + n} \\ &\leftarrow \frac{\sqrt{n^2 + n} + n}{n} = \frac{\sqrt{n^2 + n}}{n} + 1 \\ &\sqrt{n^2} = n \leq \sqrt{n(n+1)} \leq n + \frac{1}{2} \\ &\lim \frac{\sqrt{n^2 + n}}{n} \leq \lim \frac{n + \frac{1}{2}}{n} = 1 \\ &1 = \lim \frac{n}{n} \leq \lim \frac{\sqrt{n(n+1)}}{n} \\ &\rightarrow 1 \leq \lim \frac{\sqrt{n(n+1)}}{n} \leq 1 \rightarrow \lim \frac{\sqrt{n(n+1)}}{n} = 1 \\ &\rightarrow \lim \frac{\sqrt{n^2 + n} + n}{n} = 1 + 1 = 2 \\ &\rightarrow \frac{n}{\sqrt{n^2 + n} + n} = \frac{1}{2} \end{aligned}$$

不能用连续性。构造夹逼准则，序列的夹逼准则可以绕过函数连续性

3. Proof:  $s_1 = \sqrt{2}$ ,  $s_{n+1} = \sqrt{2 + \sqrt{s_n}}$ . Proof:  $\{s_n\}$ 收敛  $\wedge s_n < 2$

$$\begin{aligned}
 s_1 &= \sqrt{2} < 2, s_{n+1} = \sqrt{2 + \sqrt{s_n}} < \sqrt{2 + 2} = 2 \\
 &\rightarrow \{s_n\} \text{有上界} 2. \\
 \frac{s_{n+1}}{s_n} &= \frac{\sqrt{2 + \sqrt{s_n}}}{s_n} = \sqrt{\frac{2 + \sqrt{s_n}}{s_n^2}} \geq \sqrt{\frac{2 + \sqrt{s_n}}{s_n}} = \sqrt{\frac{2}{s_n} + \frac{1}{\sqrt{s_n}}} \\
 &\geq \sqrt{1} = 1 \\
 &\rightarrow s_{n+1} > s_n \\
 &\rightarrow s_n \text{是单调数列} \\
 &\rightarrow s_n \text{收敛}
 \end{aligned}$$

4. Compute: 上下极限  $s_1 = 0$ ;  $s_{2m} = \frac{s_{2m-1}}{2}$ ;  $s_{2m+1} = \frac{1}{2} + s_{2m}$

$$\begin{aligned}
 s_1 &= 0; s_2 = \frac{0}{2} = 0; s_3 = \frac{1}{2} + 0 = \frac{1}{2}; s_4 = \frac{1}{4}; s_5 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}; s_6 = \frac{3}{8}; s_7 = \frac{7}{8} \dots \\
 s_{2m+1} &= \frac{1}{2} + s_{2m} \rightarrow s_{2m+1} > s_{2m} \\
 s_{2m} &= \frac{1}{2} s_{2m-1} \rightarrow s_{2m} \leq s_{2m-1} \\
 s_{2(m+1)+1} &= \frac{1}{2} + s_{2(m+1)} = \frac{1}{2} + \frac{1}{2} s_{2m+1} \\
 s_{2(m+1)} &= \frac{1}{2} s_{2m+1} = \frac{1}{2} \left( \frac{1}{2} + s_{2m} \right) \\
 s_1 &< 1 \\
 \text{设 } s_{2m-1} &< 1, s_{2m+1} = \frac{1}{2} + \frac{1}{2} s_{2m-1} < \frac{1}{2} + \frac{1}{2} = 1 \rightarrow s_{2m+1} < 1 \text{有界} \\
 s_{2m+1} &= \frac{1}{2} + \frac{1}{2} s_{2m-1} \geq \frac{1}{2} s_{2m-1} + \frac{1}{2} s_{2m-1} = s_{2m-1} \rightarrow s_{2m+1} \text{单调增} \\
 &\rightarrow s_{2m+1} \text{极限存在} \\
 s_{2m+1} &= \frac{1}{2} + s_{2m} > \frac{1}{2} \rightarrow s_{2m+1} \geq s_{2m}
 \end{aligned}$$

$$\begin{aligned}
 s_0 &< \frac{1}{2} \\
 \text{设 } s_{2m} &< \frac{1}{2}, s_{2(m+1)} = \frac{1}{2} \left( \frac{1}{2} + s_{2m} \right) \leq \frac{1}{2} \left( \frac{1}{2} + \frac{1}{2} \right) = \frac{1}{2} \rightarrow s_{2m} < \frac{1}{2} \text{有界} \\
 s_{2(m+1)} &= \frac{1}{2} \left( \frac{1}{2} + s_{2m} \right) \geq \frac{1}{2} (s_{2m} + s_{2m}) = s_{2m} \rightarrow s_{2m} \text{单调增} \\
 s_{2m} &\text{极限存在}
 \end{aligned}$$

$$\begin{aligned}
 \limsup s_n &= \lim s_{2m+1} = \frac{1}{2} + \frac{1}{2} (s_{2m-1}) = \frac{1}{2} + \frac{1}{2} \left( \frac{1}{2} + s_{2m-3} \right) \\
 &= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^4} + \dots + \\
 &= 1 \\
 \liminf s_n &= \lim s_{2m} = \frac{1}{2} \left( \frac{1}{2} + s_{2(m-1)} \right) = \frac{1}{2} \left( \frac{1}{2} + \frac{1}{2} \left( \frac{1}{2} + s_{2(m-2)} \right) \right) = \dots \\
 &= \frac{1}{2^2} + \frac{1}{2^3} + \dots \\
 &= \frac{1}{2}
 \end{aligned}$$

5. Proof:  $\forall \{a_n\}, \{b_n\}$ 收敛  $\wedge a+b$ 不是  $+\infty + -\infty$  这种类型。Proof:  $\limsup (a_n + b_n) \leq \limsup a_n +$

$$\limsup b_n$$

$$\limsup (a_n + b_n) = \sup \{a_{n_k} + b_{n_k} : n_k \text{ 是一个子序列}\}$$

$$\limsup a_n = \sup \{a_{n_i}\}, \limsup b_n = \sup \{b_{n_j}\}$$

若  $\limsup (a_n + b_n) = +\infty \rightarrow \forall M \in R^+, \exists N \in N^+, n > N \rightarrow a_n + b_n > M$

$a_n > M - b_n$ . 由于  $M$  的任意性,  $a_n > M$ . 同理  $b_n > M$

$\rightarrow \limsup a_n = +\infty, \limsup b_n = +\infty \rightarrow$  显然成立

$\limsup a_n = +\infty, \limsup b_n = b \in R \rightarrow$  显然成立

$\limsup a_n = a \in R, \limsup b_n = b \in R$

$\limsup a_n = a \rightarrow \forall a_{n_i}, a_{n_u} \geq a_{n_i}. \limsup b_n = b \rightarrow \forall b_{n_i}, b_{n_v} \geq b_{n_i}$

$\limsup a_n + b_n = a_{n_i} + b_{n_i} \leq a_{n_u} + b_{n_v} = \limsup a_n + \limsup b_n$

6. 验证.

a.  $a_n = \sqrt{n+1} - \sqrt{n}$

$$\sum (\sqrt{n+1} - \sqrt{n}) = \sqrt{n+1} - 1 = \infty$$

b.  $a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n}$

$$\frac{\sqrt{n+1} - \sqrt{n}}{n} = \frac{1}{n(\sqrt{n+1} + \sqrt{n})} \leq \frac{1}{n(2\sqrt{n})} = \frac{1}{2} \frac{1}{n^{3/2}}. p\text{-级数} \rightarrow \text{收敛}$$

c.  $a_n = (\sqrt[n]{n} - 1)^n$

$$\limsup \sqrt[n]{|(\sqrt[n]{n} - 1)^n|} = \lim \sqrt[n]{n} - 1 = 0 \rightarrow \text{根值验证法} \rightarrow \text{收敛}$$

d.  $a_n = \frac{1}{1+z^n}, z \in C$

$$\frac{a_{n+1}}{a_n} = \frac{1}{1+|z^{n+1}|} \cdot \frac{1+|z^n|}{1} = \frac{1+|z^n|}{1+|z^{n+1}|} < 1$$

$$\lim \sqrt[n]{\frac{1}{1+|z^n|}} \leq \sqrt[n]{\frac{1}{|z^n|}} = \frac{1}{|z|}$$

$\rightarrow |z| < 1$  时收敛,  $|z| > 1$  时发散

$|z| = 1 \wedge z \neq 1 \rightarrow z^n$  不是确定的数  $\rightarrow$  发散

$z = 1 \rightarrow \frac{1}{2} > 0 \rightarrow$  发散

7. Proof:  $a_n \geq 0, \sum a_n$  收敛. Proof:  $\sum \frac{\sqrt{a_n}}{n}$  收敛

$\forall \varepsilon > 0, \sum a_n$  收敛  $\rightarrow s_n$  单调有界

$$a_n \geq 0 \rightarrow \frac{\sqrt{a_n}}{n} \geq 0$$

$\rightarrow \sum \frac{\sqrt{a_n}}{n}$  单调

$\sqrt{x}$  是凸函数, Jensen:  $t\sqrt{x} + (1-t)\sqrt{y} \leq \sqrt{tx + (1-t)y}$

$$tf(a) + (1-t)f(b) \leq f(ta + (1-t)b)$$

$$\frac{1}{m}\sqrt{a} + \frac{1}{2}\sqrt{b} \leq \sqrt{\frac{a+b}{2}}$$

$$\frac{\sqrt{a_n}}{n} + \frac{\sqrt{a_{n+1}}}{n+1} + \dots + \frac{\sqrt{a_{n+m}}}{n+m} \leq \frac{\sqrt{a_n} + \sqrt{a_{n+1}} + \dots + \sqrt{a_{n+m}}}{n}$$

$$\leq \frac{1}{n} \sqrt{\frac{a_n + a_{n+1} + \dots + a_{n+m}}{m}} = \frac{1}{n} \sqrt{\frac{\varepsilon}{m}} < \frac{1}{nm} \sqrt{\varepsilon}$$

$\rightarrow \frac{\sqrt{a_n}}{n}$  收敛

8. Proof:  $a_n$ 收敛,  $b_n$ 单调有界. Proof:  $\sum a_n b_n$ 收敛

$$\begin{aligned} & b_n \text{ 单调有界} \rightarrow b_n \text{ 收敛} \\ & |b_n| < M \\ & |\sum_{n=i}^j a_n b_n| < M |\sum_{n=i}^j a_n| \\ & \quad = M\varepsilon \\ & \rightarrow \sum a_n b_n \text{ 收敛} \end{aligned}$$

9. Compute: 收敛半径

a.  $\sum n^3 z^n$

$$\begin{aligned} \limsup \sqrt[n]{|n^3|} &= \lim \sqrt[n]{n^3} = (\sqrt[n]{n})^3 = 1^3 = 1 \\ &\rightarrow R = \frac{1}{1} = 1 \end{aligned}$$

b.  $\sum \frac{2^n}{n!} z^n$

$$\begin{aligned} \limsup \sqrt[n]{\left|\frac{2^n}{n!}\right|} &= \lim \sqrt[n]{\frac{2^n}{n!}} = \frac{2}{\sqrt[n]{n!}} \\ \text{Stirling: } n! &\sim n^{n+1/2} \rightarrow R = +\infty. \end{aligned}$$

c.  $\sum \frac{2^n}{n^2} z^n$

$$\limsup \sqrt[n]{\left|\frac{2^n}{n^2}\right|} = \lim \sqrt[n]{\frac{2^n}{n^2}} = \lim \frac{2}{\sqrt[n]{n^2}} = 2 \rightarrow R = \frac{1}{2}$$

d.  $\sum \frac{n^3}{3^n} z^n$

$$\limsup \sqrt[n]{\left|\frac{n^3}{3^n}\right|} = \lim \sqrt[n]{\frac{n^3}{3^n}} = \frac{1}{3} \rightarrow R = 3$$

10. Proof: 幂级数  $\sum a_n z^n$ ,  $a_n \in \mathbb{Z}$ . 无限个  $a_n \neq 0$ . Proof: 收敛半径  $R \leq 1$

$$\begin{aligned} R > 1 &\rightarrow \limsup \sqrt[n]{|a_n|} < 1 \\ a_n &\neq 0 \rightarrow |a_n| \neq 0 \\ \forall x \in \mathbb{R}^+, \lim \sqrt[n]{x} &= 1 \rightarrow \exists n > N, \exists a_n \rightarrow \lim \sqrt[n]{a_n} = 1 \\ &\rightarrow \limsup \sqrt[n]{a_n} \geq 1 \\ &\rightarrow R \leq 1 \end{aligned}$$

11.  $a_n > 0$ ,  $s_n = \sum_{i=1}^n a_i$ .  $\sum a_i$  发散

a. Proof:  $\sum \frac{a_n}{1+a_n}$  发散

$$\begin{aligned} & \forall n \in \mathbb{N}^+, \exists i > j \rightarrow \sum_{n=j}^i a_n > \varepsilon \\ \left(\frac{x}{1+x}\right)' &= (x(1+x)^{-1})' = -x(1+x)^{-2} + (1+x)^{-1} < 0 \\ &\rightarrow \frac{1}{1+x} - \frac{x}{(1+x)^2} = \frac{1}{(1+x)^2} \geq 0 \\ &\rightarrow \frac{x}{1+x} \text{ 是凸函数.} \\ \text{Jensen: } t f(a) + (1-t) f(b) &\leq f(ta + (1-t)b) \\ \sum_{n=j}^i \frac{a_i}{1+a_i} &\leq \frac{a_j + a_{j+1} + \dots + a_{j+m}}{1+a_j + a_{j+1} + \dots + a_{j+m}} \\ &= \frac{\varepsilon}{1+\varepsilon} \\ &??? \end{aligned}$$

b. Proof:  $\frac{a_{N+1}}{s_{N+1}} + \dots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$ ;  $\sum \frac{a_n}{s_n}$  发散

c. Proof:  $\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$ ;  $\sum \frac{a_n}{s_n^2}$  收敛

d. Proof or Disproof:  $\sum \frac{a_n}{1+na_n}$  收敛;  $\sum \frac{a_n}{1+n^2a_n}$  收敛

12.  $a_n > 0 \wedge \sum a_n$  收敛.  $r_n = \sum_{m=n}^{\infty} a_m$

a. Proof:  $m < n \rightarrow \frac{a_m}{r_m} + \dots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$ ;  $\sum \frac{a_n}{r_n}$  发散

b. Proof:  $\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$ ;  $\sum \frac{a_n}{\sqrt{r_n}}$  收敛

13. Proof: 两个绝对收敛的级数的Cauchy积也绝对收敛

$$\begin{aligned} & \sum |a_n| \text{ 收敛}, \sum |b_n| \text{ 收敛} \\ & \rightarrow \sum |a_n| \sum |b_n| \text{ 收敛. } 13.4 \\ & |a_n| > 0, |b_n| \geq 0 \\ & \rightarrow |a_n| \cdot |b_{k-n}| \geq 0 \\ & \rightarrow |c_n| \geq 0 \\ & \rightarrow \text{绝对收敛} \end{aligned}$$

14.  $s_n$  为复数序列,  $\sigma_n = \frac{s_0 + s_1 + \dots + s_n}{n+1}$

a. Proof:  $\lim s_n = s$ . Proof:  $\lim \sigma_n = s$

$$\begin{aligned} & \lim s_n = s, i, j > N \rightarrow d(s_i, s_j) < \varepsilon \\ & d(\sigma_i, \sigma_j) = \frac{s_0 + \dots + s_i}{i+1} - \frac{s_0 + \dots + s_j}{j+1} \\ & \text{let: } i > j. d(\sigma_i, \sigma_j) \leq \frac{s_0 + \dots + s_i - (s_0 + \dots + s_j)}{j+1} \\ & \leq \frac{1}{j+1}(s_{i+1} + \dots + s_j) = \frac{1}{j+1}(j-i+1)\varepsilon \\ & = \frac{j-i+1}{j+1} < \varepsilon \\ & \rightarrow \lim \sigma_n \text{ 收敛} \end{aligned}$$

$$\begin{aligned} & \lim \frac{1}{n+1}(s_0 + \dots + s_n) - s \\ & = \lim \frac{1}{n+1}(s_0 + \dots + s_N) + \frac{1}{n+1}((n-N)(s-\varepsilon)) - s \\ & = 0 + \lim \frac{n-N}{n+1}(s-\varepsilon) - s \\ & = s - \varepsilon - s = -\varepsilon \\ & \rightarrow d\left(\lim \frac{1}{n+1}(\sum s_n), s\right) < \varepsilon \\ & \rightarrow \lim \frac{1}{n+1} \sum s_n = s \end{aligned}$$

b. Example:  $s_n, \lim \sigma_n = 0 \wedge s_n$  不收敛

$$s_n = (-1)^n. \lim \sigma_n = \frac{(-1)^n}{n+1} \text{ 收敛}$$

c. Proof or Counterexample:  $\exists s_n. \forall n \in \mathbb{N}^+. s_n > 0. \lim \sigma_n = 0 \wedge \limsup s_n = \infty$

$$\begin{aligned} & \text{不可能.} \\ & \limsup s_n = \infty \rightarrow \forall M \in \mathbb{R}^+. \exists n > N \rightarrow s_n > M \\ & \lim \sigma_n = 0. \lim \frac{1}{n+1} \sum s_n = 0 \\ & \rightarrow \sum_{m=i+1}^n \frac{1}{i+1} \sum s_i < \varepsilon \\ & \rightarrow i \in n, m \rightarrow s_i > n+1 \\ & \rightarrow \sum_{m=i+1}^n \frac{1}{i+1} \sum s_i \geq \frac{1}{n+1} \sum_{m=i+1}^n \sum s_i \geq \frac{1}{n+1}(n+1) = 1 \\ & \rightarrow \sigma_n \text{ 不是Cauchy的} \rightarrow \sigma_n \text{ 不收敛} \end{aligned}$$

d. Proof:  $n \geq 1, a_n = s_n - s_{n-1}$ . Proof:  $s_n - \sigma_n = \frac{1}{n+1} \sum_{i=1}^n i a_i$ ;  $\lim (n a_n) = 0 \wedge \sigma_n$  收敛  $\rightarrow s_n$  收敛.

$$\begin{aligned}
s_n - \sigma_n &= s_n - \frac{1}{n+1} \sum s_n = \frac{n}{1+n} s_n - \frac{1}{1+n} \sum_0^{n-1} s_n \\
a_n &= s_n - s_{n-1} \rightarrow s_n = a_n + s_{n-1} \\
s_n &= \sum a_n + s_0 \\
&\rightarrow \frac{n}{1+n} (\sum_1^n a_n + s_0) - \frac{1}{1+n} \sum_{i=0}^{n-1} (\sum_{j=1}^{i-1} (a_j + s_0)) \\
&= \frac{n}{1+n} (n s_0 + \sum_1^n a_n) - \frac{1}{1+n} \frac{(n-1)n}{2} s_0 - \frac{1}{1+n} \sum_{i=1}^{n-1} (n-i) a_i \\
&= \left( \frac{n^2}{1+n} - \frac{n^2-n}{(1+n)2} \right) s_0 + \frac{n}{1+n} \sum_1^n a_i - \frac{1}{1+n} \sum_{i=1}^{n-1} (n-i) a_i \\
&= \frac{n(n+1)}{2(1+n)} s_0 + \frac{1}{1+n} (\sum_1^n n a_i - \sum_1^{n-1} (n-i) a_i) \\
&= \frac{n}{2} s_0 + \frac{1}{1+n} \sum_1^n i a_i \\
&\quad ??? \\
\lim n a_n &= 0 \rightarrow \lim n (s_n - s_{n-1}) = 0 \\
\lim s_n &= \lim (s_0 + \sum a_i) \\
&= s_0 + \lim \sum a_i \\
&\leftarrow \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum i a_i \text{ 收敛} \\
\sum_n^m s_n &= \sum_{i=n}^m (s_0 + \sum_{j=1}^i a_j) \\
&= (m-n+1) s_0 + \sum_{i=1}^n \sum_{j=1}^i a_j + \sum_{i=n}^m \sum_{j=n}^i a_j \\
&= (m-n+1) s_0 + \sum_{i=1}^n n a_i + \sum_{i=n}^m (i-n+1) a_i \\
&\quad \lim n a_i = 0 \\
&\quad ???
\end{aligned}$$

e. Proof:  $M < \infty, \forall n \in N^+, |n a_n| \leq M, \lim \sigma_n = \sigma$ . Proof:  $\lim s_n = \sigma$

15. 推广各个定理到  $R^k$

16.  $\alpha, x_1 > \sqrt{\alpha}, x_{n+1} = \frac{1}{2} \left( x_n + \frac{\alpha}{x_n} \right)$ .

a. Proof:  $x_n$  单调减.  $\lim x_n = \sqrt{\alpha}$

$$\begin{aligned}
\frac{x_{n+1}}{x_n} &= \frac{\left( x_n + \frac{\alpha}{x_n} \right)}{2x_n} = \frac{1}{2} + \frac{\alpha}{2x_n^2} \\
x_1 > \sqrt{\alpha} &\rightarrow x_1^2 > \alpha \rightarrow \frac{\alpha}{x_1^2} < 1 \\
\rightarrow \frac{x_{n+1}}{x_n} &< 1 \rightarrow x_{n+1} < x_n \\
&\rightarrow x_n \text{ 单调减}
\end{aligned}$$

$$\begin{aligned}
x_{n+1} &= \frac{1}{2} \left( x_n + \frac{\alpha}{x_n} \right) = \frac{1}{2} x_n + \frac{1}{2} \frac{\alpha}{x_n} \\
&\geq \frac{1}{2} (x_n + \sqrt{\alpha}) \\
x_n x_{n+1} &= x_n \frac{1}{2} \left( x_n + \frac{\alpha}{x_n} \right) = \frac{1}{2} (x_n^2 + \alpha)
\end{aligned}$$

$$\begin{aligned}
x_n > \alpha &\rightarrow x_{n+1} = \frac{1}{2} \left( x_n + \frac{\alpha}{x_n} \right) \\
&\geq \frac{1}{2} (2\sqrt{\alpha}) = \sqrt{\alpha} \\
&\rightarrow x_n \geq \sqrt{\alpha} \\
&\rightarrow x_n \text{ 有极限}
\end{aligned}$$

$$\begin{aligned}
x_n \text{ 收敛} &\rightarrow \lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = \frac{\frac{1}{2} \left( x_n + \frac{\alpha}{x_n} \right)}{x_n} = \frac{x_n^2 + \alpha}{2x_n^2} = 1 \\
&\rightarrow \lim (x_n^2 + \alpha) = \lim 2x_n^2 \\
&\rightarrow \lim x_n^2 + \alpha = 2 \lim x_n^2 \\
&\rightarrow \alpha = \lim x_n^2 \\
&\rightarrow \lim x_n = \sqrt{\alpha}
\end{aligned}$$

b. Proof:  $\varepsilon_n = x_n - \sqrt{\alpha}$ . Proof:  $\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}, \beta = 2\sqrt{\alpha} \rightarrow \varepsilon_{n+1} < \beta \left( \frac{\varepsilon_1}{\beta} \right)^{2^n}$

$$\begin{aligned}
x_{n+1} - \sqrt{\alpha} &= \frac{1}{2} \left( x_n + \frac{\alpha}{x_n} \right) - \sqrt{\alpha} \\
\varepsilon_n &= x_n - \sqrt{\alpha} \\
\varepsilon_{n+1} &= \frac{1}{2} \left( x_n + \frac{\alpha}{x_n} - 2\sqrt{\alpha} \right) \\
&= \frac{1}{2} \left( \frac{x_n^2 + \alpha - 2x_n\sqrt{\alpha}}{x_n} \right) \\
&= \frac{1}{2} \left( \frac{(x_n - \sqrt{\alpha})^2}{x_n} \right) \\
&= \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}} \quad x_n > \sqrt{\alpha} \\
\varepsilon_{n+1} &< \frac{\varepsilon_n^2}{2\sqrt{\alpha}} \rightarrow \varepsilon_{n+1} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}} < \frac{\left( \frac{\varepsilon_{n-1}^2}{2\sqrt{\alpha}} \right)^2}{2\sqrt{\alpha}} \\
&< \frac{\varepsilon_1^{2^n}}{(2\sqrt{\alpha})^{2^0+2^1+\dots+2^{n-1}}} = 2\sqrt{\alpha} \left( \frac{\varepsilon_1}{2\sqrt{\alpha}} \right)^{2^n}
\end{aligned}$$

c. Proof:  $\alpha = 3, x_1 = 2$ . Proof:  $\frac{\varepsilon_1}{\beta} < \frac{1}{10} \rightarrow \varepsilon_5 < 4 \cdot 10^{-16}, \varepsilon_6 < 4 \cdot 10^{-32}$

$$\begin{aligned}
\alpha &= 3, x_1 = 2. \\
\frac{\varepsilon_1}{\beta} &= \frac{1}{2} \left( 2 + \frac{3}{2} \right) = \frac{7}{4} \\
\frac{\varepsilon_1}{\beta} &< \left( \frac{2 - \sqrt{3}}{2\sqrt{3}} \right) = 0.07735 \dots < \frac{1}{10} \\
\frac{\varepsilon^5}{\beta} &< \left( \frac{\varepsilon_1}{\beta} \right)^{2^4} = 10^{-16} \\
\varepsilon^5 &< \beta \cdot 10^{-16}, \beta = 2\sqrt{3} < 4 \\
&\rightarrow \varepsilon^5 < 4 \cdot 10^{-16}
\end{aligned}$$

17.  $\alpha > 1, x_1 > \sqrt{\alpha}, x_{n+1} = \frac{\alpha + x_n}{1 + x_n} = x_n + \frac{\alpha - x_n^2}{1 + x_n}$

a. Proof:  $x_1 > x_3 > x_5 > \dots$

$$\begin{aligned}
&x_{2n+1} - x_{2n-1} \\
&= \frac{\alpha + x_{2n}}{1 + x_{2n}} - x_{2n-1} \\
&= \frac{\alpha + \frac{\alpha + x_{2n-1}}{1 + x_{2n-1}}}{1 + \frac{\alpha + x_{2n-1}}{1 + x_{2n-1}}} - x_{2n-1} \\
&= \frac{\frac{\alpha(1 + x_{2n-1}) + \alpha + x_{2n-1}}{1 + x_{2n-1}}}{\frac{1 + x_{2n-1} + \alpha + x_{2n-1}}{1 + x_{2n-1}}} - x_{2n-1} \\
&= \frac{2\alpha + (\alpha + 1)x_{2n-1}}{1 + \alpha + 2x_{2n-1}} - x_{2n-1} \\
x_1 &> \sqrt{\alpha} > 1 \rightarrow 2\alpha + \alpha + 1x_{2n-1} > 1 + \alpha + 2x_{2n-1} \\
&\rightarrow x_{2n+1} - x_{2n-1} > 0 \\
&\rightarrow x_{2n+1} > x_{2n-1}
\end{aligned}$$

b. Proof:  $x_2 < x_4 < x_6 < \dots$

$$\begin{aligned}
x_2 &= \frac{\alpha + x_n}{1 + x_n} \\
\frac{x_{2(n+1)}}{x_{2n}} &= \frac{x_{2n+1}}{x_{2n}}
\end{aligned}$$

c. Proof:  $\lim x_n = \sqrt{\alpha}$

d. Compute: 估计  $x_n$  的收敛速度

18.  $\alpha > 0, x_1 > \sqrt{\alpha}, p > 0, x_{n+1} = \frac{p-1}{p}x_n + \frac{\alpha}{p}x_n^{-p+1}$ . 计算是否收敛, 估计收敛速度

19.  $\forall a = \{a_n; a_n = 0 \vee 2\}, x(a) = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ . Proof:  $\{x(a)\}$  是 Cantor 集

20. Proof:  $\{x_n\}$ 是度量空间 $X$ 中的Cauchy序列,  $\exists \{x_{n_i}\} \rightarrow \lim x_{n_i} = x$ . Proof:  $\lim x_n = x$

$$\begin{aligned} \lim x_{n_i} = x &\rightarrow \forall \varepsilon > 0, \exists N_1 \in \mathbb{N}^+, n_i > N \rightarrow d(x_{n_i}, x) < \varepsilon \\ \forall \varepsilon > 0, \exists N_2 \in \mathbb{N}^+, i, j > N &\rightarrow d(x_i, x_j) < \varepsilon \\ \text{let } N = \max(N_1, N_2). &d(x_{n_i}, x_j) < \varepsilon \\ j \text{ 的任意性} &\rightarrow n > n_i \rightarrow d(x_n, x_j) < \varepsilon \\ d(x_n, x) &\leq d(x_n, x_{n_i}) + d(x_{n_i}, x) = 2\varepsilon \\ &\rightarrow \lim x_n = x \end{aligned}$$

21. Proof:  $\{E_n: E_n \in \text{度量空间 } X, E_n \text{ 有界} \wedge \text{闭}\}. E_n \supset E_{n+1} \wedge \lim \text{diam } E_n = 0 \rightarrow \text{card} \bigcap_{n=1}^{\infty} E_n = 1$

22. Proof: Baire.  $X$ 是完备度量空间,  $\{G_n: G_n \text{ 是 } X \text{ 的稠密开子集}\}$ . Proof:  $\bigcap_{n=1}^{\infty} G_n \neq \emptyset$

23. Proof:  $\{p_n\}, \{q_n\}$ 是度量空间 $X$ 中的Cauchy序列. Proof:  $\{d_n = d(p_n, q_n)\}$ 收敛.

24.  $X$ 是度量空间

a. Proof: Cauchy序列 $\{p_n\}, \{q_n\}$ 的关系 $\sim$ .  $\lim d(p_n, q_n) = 0 \rightarrow p_n, q_n \in \sim$ . Proof:  $\sim$ 是等价关系

$$\begin{aligned} \forall p_n, q_n. \lim d(p_n, q_n) &= 0 \\ \lim d(p_n, q_n) &= \lim 0 = 0 &\rightarrow (p_n, p_n) \in \sim \\ \lim d(p_n, q_n) = 0 \rightarrow \lim d(q_n, p_n) &= 0 &\rightarrow (p_n, q_n) \in \sim \rightarrow (q_n, p_n) \in \sim \\ \lim d(x_n, y_n) = 0, \lim d(y_n, z_n) &= 0 \\ \rightarrow d(x_n, z_n) &\leq d(x_n, y_n) + d(y_n, z_n) \\ &= 0 + 0 = 0 &\rightarrow (x_n, y_n) \in \sim \wedge (y_n, z_n) \in \sim \rightarrow (x_n, z_n) \in \sim \\ &\rightarrow \sim \text{是等价关系} \end{aligned}$$

b. Proof:  $X^*$ 是上述等价类的集.  $P \in X^*, Q \in X^*, \{p_n\} \in P, \{q_n\} \in Q. \Delta(P, Q) = \lim d(p_n, q_n)$ . Proof:  $\Delta$ 是度量

$$\begin{aligned} \Delta(x, x) &= \lim d(x_{1n}, x_{2n}) = d(x, x) = 0 \\ \Delta(x, y) &= \lim d(x_{1n}, x_{2n}) = d(x_1, x_2) = d(x_2, x_1) = \lim d(x_{2n}, x_{1n}) = \Delta(y, x) \\ \Delta(x, y) + \Delta(y, z) &= \lim d(x_n, y_n) + \lim d(y_n, z_n) \\ &= d(x, y) + d(y, z) \geq d(x, z) = \lim d(x_n, z_n) = \Delta(x, z) \\ &\rightarrow \Delta \text{是度量} \end{aligned}$$

c. Proof:  $X^*$ 是完备的

$$\begin{aligned} \forall x_n \in X^*. \forall \varepsilon > 0, \exists N \in \mathbb{N}^+, \forall i, j > N &\rightarrow \Delta(x_i, x_j) < \varepsilon \\ x_n: \lim a_n = x_i, \lim b_n = x_j, \dots & \\ \Delta(x_i, x_j) < \varepsilon \rightarrow \lim d(a_n, b_n) < \varepsilon & \\ \rightarrow \lim a_n = \lim b_n = x & \\ \rightarrow \Delta(x_n, x) \leq \Delta(x_n, a_n) + \Delta(a_n, x) & \\ &= 2\varepsilon \\ \rightarrow \lim x_n = x \in X^* & \\ \rightarrow X^* \text{是完备的} & \end{aligned}$$

d. Proof:  $\forall x \in X. \exists \{p\} \in X; P_p \in X^* \wedge \{p\} \in P_p$ . Proof:  $\Delta(P_p, P_q) = d(p, q)$

$$\begin{aligned} \Delta(P_p, P_q) &= \lim d(P_p, P_q) \\ &= \lim d(P_p, P_q) \\ &\leq \lim d(P_p, p) + \lim d(P_q, q) + \lim d(p, q) \\ p \text{ 是Cauchy的} &\rightarrow p \text{ 是收敛的} \rightarrow d(P_p, p) < \varepsilon \\ &= 2\varepsilon + \lim d(p, q) = d(p, q) \\ &\rightarrow \Delta(P_p, P_q) \leq \Delta(p, q) \end{aligned}$$

???需要反向证明 $\Delta(p, q) \leq \Delta(P_p, P_q)$



e. Proof:  $\varphi: X \rightarrow X^*$ .  $\varphi(p) = \{p\}$ .  $\varphi(X)$ 在 $X^*$ 中稠密;  $X$ 完备  $\rightarrow X = X^*$

25. Construct:  $X$ 是度量空间.  $X \subset Q$ .  $d(x, y) = |x - y|$ . 求 $X$ 的完备化

$$X^* = R.$$

这是实数的Cauchy定义