Principles of Mathematical Analysis

BY WALTER RUDIN

2021-4-15

第一章 实数系和复数系

1 导引

分析学的主要概念（收敛、连续、微分法、积分法），都依赖于精确定义的实数。

例如：有理数中$\forall x\in Q, x^2\neq 2$.有理数序列$X=\{x_n\colon n\in N^+, \lim_{n\to\infty}x_n=\sqrt{2}\}$中,如果不定义$\sqrt{2}$，那么无法确定序列收敛于什么...

例 1.1. 证明方程$p^2=2$在有理数中不成立

$p=\frac{m}{n}\to p^2=\frac{m^2}{n^2}\to m^2=2n^2, m=2\to 2\times 2=2\times n\times n\to 2=n\times n$

不可能成立

$m\neq 2$,那么$m\times m$不能被2整除，所以也不可能成立。（整数的质数分解）

$A=\{x\colon x^2<2\}, B=\{x\colon x^2>2\}\to$A中没有最大元素，B中没有最小元素。

$$q=p-\frac{p^2-2}{p+2}=\frac{2p+2}{p+2}, q^2-2=\frac{2(p^2-2)}{(p+2)^2}$$

$$\begin{aligned}\forall p>1\in A, p^2-2<0, q=p-\frac{p^2-2}{p+2}>p\\ \to q>p\\ q^2-2=\frac{2(p^2-2)}{(p+2)^2}<0\\ \to q\in A\end{aligned}$$

$$\begin{aligned}\forall p\in B, p^2-2>0, q=p-\frac{p^2-2}{p+2}<p\\ \to q<p\\ q^2-2=\frac{2(p^2-2)}{(p+2)^2}>0\\ \to q\in B\end{aligned}$$

以上结论表示在序结构中两个元素之间有些元素不够完善。因此构造某种数填补这些空隙，实数填补了这种空隙。

定义 1.2. *若A是集合，$x\in A, x\notin A$*

$$\begin{gathered}\forall x, x\notin A\to A=\varnothing\\ \forall x\in A, x\in B\to A\subset B\\ \forall x\in A, x\in B\wedge\exists y\in B\wedge y\notin A\to A\subseteq B\\ A\subset B\wedge B\subset A\to A=B\end{gathered}$$

2 有序集

定义 2.1. *有序集*

S是一个集合。S上的关系$<$满足性质

1. *唯一性：$\forall x,y\in S\rightarrow x<y,x=y,y<x$有且只有一个成立*
2. *传递性：$\forall x,y,z\in S,x<y\wedge y<z\rightarrow x<z$*

定义 2.2. *S上定义了一种序关系，称为有序集*

eg：$\mathbb{Q}$

定义 2.3. *上有界：有序集$S,E\subset S,\exists b\in S\rightarrow\forall x\in E,x<b$,称$E$上有界，$b$为$E$的一个上界*

定义 2.4. *上确界：有序集$S,E\subset S,E$上有界.若$\exists a\in S,\rightarrow$*

1. *$\forall x\in E\rightarrow x<a$ $\quad$ a是上界*
2. *$\forall b<a,\exists x\in E\rightarrow x>a$ $\quad$ 比a小的都不是上界*

记作$a=\sup(E)$，类似的下确界$a=\inf(E)$: $\forall x\in E,x>a,\forall b>a,\exists x\in E\rightarrow x<a$

Remark:

据例1.1,Q的一些子集不具有上确界和下确界

$a=\mathrm{supE}$存在，$a\in E\vee a\nless E$

$S=\left\{\frac{1}{n}:n\in N^{+}\right\},\mathrm{supS}=1\in S,\mathrm{infS}=0\notin S$

定义 2.5. *最小上界性：有序集$S.\forall E\neq\varnothing\subset S,\mathrm{supE}\in S.$称$S$具有最小上界性*

定理 2.6. *最小上界性$\Leftrightarrow$最大下界性*

最小上界性的有序集$S,B\subset S\wedge B\neq\varnothing\wedge B$下有界，$L=\{x:x$是$b$的下界$\},a=\mathrm{supL}=\mathrm{infB}$

证明.

$\forall x\in B,\mathrm{supL}\leqslant x,\forall a>\mathrm{supL},\exists b\in B\rightarrow b<a$

B下有界,$\forall x\in B,\exists b\in S\rightarrow b\leqslant x\rightarrow L\neq\varnothing$

$\forall x\in B,\exists b\in S\rightarrow b\leqslant x\rightarrow x$是$L$的上界

$\rightarrow L$具有最小上界a

$\forall x<a\rightarrow\exists y\in L\rightarrow y>x\rightarrow a$是$L$的上界 $\quad\square$

$\rightarrow x\notin B$ $\quad$ x是比supL更小的数，x是B的下界，否则

$x\in B\wedge x\in L\rightarrow x=\mathrm{infB}$, 但$x<\mathrm{infB}$

$\rightarrow\forall b\in B,a\leqslant b\rightarrow a\in L$ $\quad$ $\mathrm{supL}\leqslant x\in B$

$\forall\beta>a\rightarrow\beta\notin L$

$\rightarrow a=\mathrm{infL}$

3 域

定义 3.1. *域*

集合F和运算加法和乘法。满足AMD公理:

A:
$$\begin{gathered}\forall x,y\in F\to x+y\in F\\ \forall x,y\in F\to x+y=y+x\\ \forall x,y,z\in F\to(x+y)+z=x+(y+z)\\ \exists 0\in F,\forall x\in F\to 0+x=x\\ \forall x\in F,\exists y\in F\to x+y=0\end{gathered}$$

M:
$$\begin{gathered}\forall x,y\in F\to xy\in F\\ \forall x,y\in F\to xy=yx\\ \forall x,y,z\in F\to(xy)z=x(yz)\\ \exists 1\neq 0\in F,\forall x<F\to 1x=x\\ \forall x\neq 0\in F,\exists y\in F\to xy=1\end{gathered}$$

D: $\forall x,y,z\in F\to x(y+z)=xy+xz$

eg:Q是一个域

命题 3.2. *A的性质*

$$\begin{array}{cc} x+y=x+z\to y=z & y=(-x+x)+y=-x+(x+y)=(-x+x)+z=z\\ x+y=x\to y=0 & x+y=x+0\to y=0\\ x+y=0\to y=-x & x+y=x+-x\to y=-x\\ -(-x)=x & x+-x=0\to -x+-(-x)=0\to -(-x)=x\end{array}$$

命题 3.3. *M的性质*

$$\begin{array}{lcc} x\neq 0 & xy=xz\to y=z & y=\frac{1}{x}xy=\frac{1}{x}xz=z\\ x\neq 0 & xy=x\to y=1 & xy=x1\to y=1\\ x\neq 0 & xy=1\to y=\frac{1}{x} & xy=x\frac{1}{x}\to y=\frac{1}{x}\\ x\neq 0 & \frac{1}{1/x}=x & x\frac{1}{x}=1\to\frac{1}{x}\frac{1}{1/x}=\frac{1}{1}=1\to\frac{1}{1/x}=x\end{array}$$

命题 3.4. *AMD的性质*

$$\begin{array}{cc} 0x=0 & 0x+0x=(0+0)x=0x\to 0x=0\\ x\neq 0,y\neq 0\to xy\neq 0 & 1=\frac{1}{x}\frac{1}{y}yx=\frac{1}{x}\frac{1}{y}0=0,\text{但}1\neq 0\\ (-x)y=x(-y)=-(xy) & (-x)y+xy=(-x+x)y=0y=0,\\ & x(-y)+xy=x(-y+y)=0x=0\\ & (-x)y=x(-y)=-(xy)=-xy\end{array}$$

$$(-x)(-y)=xy \qquad (-x)(-y)=-(x)(-y)=-(-(xy))=xy$$

定义 3.5. *有序域:域F满足*

1. $\forall x,y,z\in F,y<z\to x+y<x+z$

2. $\forall x,y\in F, x>0\wedge y>0\rightarrow xy>0$

命题 3.6. *有序域的性质:*

$$\begin{array}{cc}
x>0\Leftrightarrow -x<0, x<0\Leftrightarrow -x>0 & x>0, 0=x+-x>-x+0=-x \\
x>0, y<z\rightarrow xy<xz & z>y\rightarrow z-y>y-y\rightarrow x(z-y)>0 \\
 & \rightarrow xz=x(z-y)+xy>0+xy=xy \\
x<0, y<z\rightarrow xy>xz & x(z-y)<0\rightarrow xz=x(z-y)+xy<0+xy=xy \\
x\neq 0\rightarrow x^2>0.\ 1>0 & x>0\rightarrow xx>0(\text{定义}), x<0\rightarrow xx=--(xx)>0 \\
0<x<y\rightarrow 0<\frac{1}{y}<\frac{1}{x} & x,y>0\rightarrow \frac{1}{x},\frac{1}{y}>0, x<y\rightarrow \frac{1}{x}\frac{1}{y}x<\frac{1}{x}\frac{1}{y}y\rightarrow \frac{1}{y}<\frac{1}{x}
\end{array}$$

4 实数域

定理 4.1. *具有最小上界性的有序域存在。R*

一般通过Dedekind分割或者Cauchy序列的等价类构造性证明。

证明. *Dedekind:*

1. 定义R的元素: 分划: 集合a满足:

$$\begin{array}{cc}
\alpha\neq\varnothing,\ \alpha\neq Q & \\
p\in\alpha, q\in Q\wedge q<p\rightarrow q\in\alpha & p\in\alpha, \text{比}p\text{小的}Q\text{都在}\alpha\text{中} \\
p\in\alpha, \exists r\in\alpha\rightarrow p<r & \alpha\text{中没有最大元}
\end{array}$$

$p\in\alpha, q\notin\alpha\rightarrow q>p,\ r\notin\alpha, r<s\rightarrow s\notin\alpha$

2. 定义序关系: $\alpha<\beta\Leftrightarrow\alpha\subseteq\beta$

$$\begin{array}{c}
\text{验证?:}\ \alpha<\beta, \alpha=\beta, \alpha>\beta\rightarrow\alpha\subseteq\beta, \alpha=\beta, \alpha\supseteq\beta\text{有且只有一个成立} \\
\alpha<\beta, \beta<\gamma\rightarrow\alpha<\gamma: \alpha\subseteq\beta, \beta\subseteq\gamma\rightarrow a\subseteq\gamma \\
\rightarrow\text{序关系符合定义1.3}
\end{array}$$

R是有序集。

3. *R具有最小上界性*

$$\begin{array}{cc}
A\neq\varnothing\subset R, b\text{是}A\text{的上界}, & \\
\gamma=\bigcup\{\alpha\in A\} & p\in\gamma\Leftrightarrow p\in\alpha, \alpha\in A \\
\text{要证明:}\ \gamma\in R\wedge\gamma=\text{supA} & \\
A\neq\varnothing\rightarrow\exists\alpha_0\in A\rightarrow\bigcup\{\alpha\in A\}\neq\varnothing & \gamma\neq\varnothing \\
\forall x\in\gamma, x\in\alpha\in A\rightarrow\alpha\subset\gamma\rightarrow\forall y<x\rightarrow y\in\alpha\rightarrow y\in\gamma & y<x, y\in Q\rightarrow y<\gamma \\
\forall x\in\gamma, x\in\alpha\in A\rightarrow\exists y\in\alpha, y>x\rightarrow y\in\gamma & \forall x\rightarrow\exists y>x, y\in\gamma \\
\rightarrow\gamma\in R & \gamma\text{是分划} \\
 & \\
\forall\alpha\in A\rightarrow\alpha\in\gamma\rightarrow\alpha\leqslant\gamma & \gamma\text{是}A\text{的上界} \\
\forall\delta<\gamma\rightarrow\exists s\in\gamma\wedge s\notin\sigma & \\
s\in\gamma\rightarrow\exists\beta, s\in\beta\wedge\beta\in A\rightarrow\delta<\beta & \forall\delta<\gamma, \text{存在}\beta>\delta\rightarrow\delta\text{不是上界} \\
\rightarrow\gamma=\sup(A) &
\end{array}$$

这个证明过程说明了是先定义了可数 ∞ 才能定义出R,$\bigcup$ 需要在Q上执行可数并

4. R上的加法(并验证A公理)

$$\alpha,\beta\in R,\alpha+\beta=\{(a+b)\colon a\in\alpha,b\in\beta\} \qquad \text{definition}$$

$$\alpha+\beta\in R$$

$$\alpha\neq\varnothing,\beta\neq\varnothing\rightarrow\alpha+\beta\neq\varnothing \qquad \alpha+\beta\neq\varnothing \qquad A1$$

$$\alpha\neq Q,\beta\neq Q\rightarrow\exists a\notin\alpha,b\notin\beta$$

$$\rightarrow\forall x\in\alpha,y\in\beta\rightarrow a>x,b>y\rightarrow a+b>x+y$$

$$\rightarrow a+b\notin\alpha+\beta\rightarrow\alpha+\beta\neq Q \qquad \alpha+\beta\neq Q$$

$$\forall s\in\alpha+\beta\rightarrow s=a+b.\forall q\in Q,q<s\rightarrow q-b<a\rightarrow q-b\in\alpha$$

$$q=(q-b)+b\in\alpha+\beta$$

$$\forall s\in\alpha+\beta\rightarrow s=a+b$$

$$a\in\alpha\rightarrow\exists t>a\in\alpha$$

$$t+b>a+b\in\alpha+\beta \qquad \forall x\in\alpha,\exists y>x\in\alpha$$

$$\rightarrow\alpha+\beta\in R$$

$$\alpha+\beta=\{(a+b)\colon a\in\alpha\wedge b\in\beta\}=\{(b+a)\colon b\in\beta\wedge a\in\alpha\} \qquad \alpha+\beta=\beta+\alpha \qquad A2$$

$$\alpha+\beta+\gamma=\{((a+b)+c)\colon a\in\alpha\wedge b\in\beta\wedge c\in\gamma\}$$

$$\{(a+(b+c))\colon a\in\alpha\wedge b\in\beta\wedge c\in\gamma\} \qquad (\alpha+\beta)+\gamma=a+(\beta+\gamma) \qquad A3$$

$$0^R=\{x\in Q\colon x<0\},0^R\in R \qquad A4$$

$$\forall\alpha\in R,\alpha+0^R=a+b<a\rightarrow\alpha+0^R\subset\alpha$$

$$\forall a\in\alpha,\exists b\in\alpha\wedge b>a\rightarrow a-b<0\rightarrow a-b\in 0^R$$

$$\rightarrow a=(a-b)+b\rightarrow a\in(0^R+\alpha)$$

$$\rightarrow\alpha\in(0^R+\alpha)$$

$$\rightarrow\alpha+0^R=\alpha \qquad \exists 0\in F,\forall x\in F\rightarrow x+0=x$$

$$\forall\alpha\in R,\beta=\{b\in Q\colon b=\exists r>0\rightarrow -b-r\notin\alpha\} \qquad \forall x\in R,\exists -x\in R\rightarrow x+-x=0 \qquad A5$$

$$\beta\in R$$

$$\forall a\notin\alpha,b=-a-2,-b=a+2\notin a,r=1$$

$$\rightarrow -b-r=a+2-1=a+1\notin\alpha\rightarrow b\neq\varnothing \qquad \beta\neq\varnothing$$

$$\forall a\in\alpha,\exists r>a\in\alpha\rightarrow b=-a$$

$$-b=a\rightarrow\forall q\in Q,-b-r\in\alpha\rightarrow\beta\neq Q \qquad \beta\neq Q$$

$$\forall b\in\beta,r>0,-p-r\notin\alpha.\forall q\in Q\wedge q<b$$

$$\rightarrow -q-r>-b-r\rightarrow -q-r\notin a\rightarrow q\in\beta \qquad \forall b\in\beta,\forall q\in Q\wedge q<b\rightarrow q\in\beta$$

$$t=b+\frac{r}{2}\rightarrow t>b$$

$$-t-\frac{r}{2}=-p-r\notin a\rightarrow t\in\beta \qquad \forall b\in\beta,\exists t>b\wedge t\in\beta$$

$$\rightarrow\beta\in R$$

$$a\in\alpha,b\in\beta\rightarrow\forall s\in\alpha+\beta,s=a+b$$

$$-b\notin\alpha\rightarrow a<-b,a+b<0\rightarrow\alpha+\beta\subset 0^R$$

$$\forall v\in 0^R,w=-\frac{v}{2}\rightarrow w>0$$

$$\rightarrow\exists n\rightarrow nw\in\alpha\wedge(n+1)w\notin\alpha$$

$$p=-(n+2)w$$

$$\rightarrow -p-w\notin\alpha\rightarrow p\in\beta$$

$$\rightarrow v=nw+p\in\alpha+\beta$$

$$\rightarrow 0^R\subset\alpha+\beta$$

$$\rightarrow 0^R=\alpha+\beta$$

Q具有阿基米德性.
(但是这能从最小上界性推出来)
$\rightarrow$阿基米德性弱于最小上界性

5. R^+上的乘法

$$
\begin{aligned}
& R^+ = \{x \in R : x > 0^R\} \\
& \forall \alpha, \beta \in R^+, \alpha\beta = \{p \in Q : p \leqslant ab, a > 0 \in \alpha \wedge b > 0 \in \beta\} && \text{definition} \\
& a \in \alpha, b \in \beta \to ab = p \in \alpha\beta \to \alpha\beta \neq \varnothing && && M1 \\
& \exists a \notin \alpha, \exists b \notin \beta \to ab \notin \alpha\beta \to \alpha\beta \neq Q \\
& \forall s \in \alpha\beta, \forall q \in Q \wedge q < s \to s = ab, q < ab \to q \in \beta && \forall q \in Q \wedge q < s \to q \in \beta \\
& \forall s \in \alpha\beta, s = ab, \exists r \in \alpha \wedge r > a \\
& \to rb \in \beta \wedge rb > ab && \forall x \in \alpha\beta, \exists y > x, y \in \alpha\beta \\
& \to \alpha\beta \in R
\end{aligned}
$$

$$
\begin{aligned}
& \forall \alpha, \beta \in R^+, \alpha\beta = \{p \in Q^+ : p \leqslant ab\} = \{p \in Q : p \leqslant ba\} = \beta\alpha && \alpha\beta = \beta\alpha && M2 \\
& \forall \alpha, \beta, \gamma \in R^+, (\alpha\beta)\gamma = \{p \in Q : p \leqslant (ab)r\} && && M3 \\
& = \{p \in Q : p \leqslant a(br)\} = \alpha(\beta\gamma) && (\alpha\beta)\gamma = \alpha(\beta\gamma) \\
& \forall \alpha \in R^+, \alpha 1^R = \{p \in Q : p \leqslant ab\} && && M4 \\
& \forall \alpha \in R^+, \alpha 1^R < \alpha \to \alpha 1^R \subset \alpha \\
& \forall a \in \alpha, \exists b > a \wedge b \in \alpha \to \frac{a}{b} < 1 \to \frac{a}{b} \in 1^R \\
& a = \frac{a}{b} b = 1^R \alpha \\
& \to 1^R \alpha \subset \alpha \\
& \to \alpha = 1^R \alpha && 1\alpha = \alpha
\end{aligned}
$$

$$
\begin{aligned}
& \forall \alpha \in R^+, \beta = \left\{p \in Q : p \leqslant \frac{1}{a}\right\} && && M5 \\
& \beta \in R \text{易证} \\
& \alpha\beta = \left\{p \in Q : p \leqslant a \frac{1}{a}\right\} = \{p \in Q : p \leqslant 1\} = 1^R
\end{aligned}
$$

6. *R满足有序域公理 3.5*

$$
\begin{aligned}
& \forall \alpha, \beta, \gamma \in R, \beta < \gamma \to \alpha + \beta < \alpha + \gamma \\
& \beta < \gamma \to \beta \subseteq \gamma \\
& \forall x \in \alpha + \beta = a + b < a + r \in \alpha + \gamma \to \alpha + \beta \subseteq \alpha + \gamma
\end{aligned}
$$

$$
\begin{aligned}
& \forall \alpha > 0, \beta > 0 \in R^+ \to \exists a > 0, b > 0, a \in \alpha, b \in \beta \\
& \alpha\beta = \{p : p \leqslant ab\}, ab > 0 \to 0^R < \alpha\beta
\end{aligned}
$$

7. *R上的乘法*

$$
\begin{aligned}
& \forall \alpha, \beta \in R \\
& \alpha\beta && = && \alpha\beta && \alpha > 0, \beta > 0 && \text{definition} \\
& && = && -[(-\alpha)\beta] && \alpha < 0, \beta > 0 \\
& && = && -[\alpha(-\beta)] && a > 0, \beta < 0 \\
& && = && (-\alpha)(-\beta) && \alpha < 0, \beta < 0
\end{aligned}
$$

使用 $-(-\alpha) = \alpha$结合R^+上的乘法定义易证

分配律需要分情况讨论

$$
\begin{gathered}
\alpha(\beta+\gamma)=\alpha\beta+\alpha\gamma\\
\alpha>0,\beta<0,\beta+\gamma>0\\
\gamma=(\beta+\gamma)+-\beta\\
\alpha\gamma=\alpha(\beta+\gamma)+\alpha(-\beta)\\
\alpha(-\beta)=-(\alpha\beta)\rightarrow\alpha\beta+\alpha\gamma=\alpha(\beta+\gamma)
\end{gathered}
$$

R是具有最小上界性的有序域。

8. *Q^R与Q的性质对应关系*

$$
\begin{aligned}
&Q^R\subset R, Q^R=\{x\colon x<q, q\in Q\}\quad \text{definition}\\
&\qquad \forall x\in Q^R\text{是分划易证}
\end{aligned}
$$

$$
\begin{array}{ll}
Q^R\text{的元素具有性质:} & \\
\forall a^R+b^R=(a+b)^R & \alpha=\{x\colon x<a\},\beta=\{x\colon x<b\},\alpha+\beta=\{x\colon x<a+b\}\\
a^Rb^R=(ab)^R & \alpha=\{x\colon x<a\},\beta=\{x\colon x<b\},\alpha\beta=\{x\colon x<ab\}\\
a^R<b^R\Leftrightarrow a<b & a<b\rightarrow a\in b^R\wedge a\notin a^R\rightarrow a^R<b^R\\
 & a^R<b^R\rightarrow\exists p, a\leqslant p<b\rightarrow a<b
\end{array}
$$

9. *域$Q\subseteq R$与Q同构*

根据8,Q中的运算和Q^R中的运算可以构成同构 □

定理 4.2. *任何具有最小上界性的有序域同构*

证明. *略* □

定理 4.3. *R的阿基米德性和稠密性*

$$
\begin{array}{ll}
\text{阿基米德} & \forall x,y\in R\wedge x>0,\exists n\in N^+\rightarrow nx>y\\
\text{稠密性} & \forall x,y\in R\wedge x<y,\exists p\in Q\rightarrow x<p<y
\end{array}
$$

证明.

$$
\begin{gathered}
A=\{nx\colon n\in N^+\}\\
\text{若阿基米德性不成立}\rightarrow\exists y\in R\rightarrow y\geqslant \text{supA}, a=\text{supA}\\
x>0\rightarrow a-x<a, a-x\text{不是}A\text{的上界}\rightarrow\exists nx\in A\rightarrow nx>a-x\\
a<(n+1)x\in A\text{与}a=\text{supA}\text{矛盾}\\
\rightarrow\exists n\in N^+\rightarrow nx>y
\end{gathered}
$$

$$
\begin{gathered}
x<y\rightarrow y-x>0\rightarrow n(y-x)>1\\
\exists m_1\in N^+\rightarrow m_1>-nx,\exists m_2\in N^+\rightarrow nx<m_2\\
\rightarrow m_1<nx<m_2\\
\rightarrow\exists m\rightarrow m-1\leqslant nx<m \qquad\qquad ???\\
\rightarrow nx<m\leqslant 1+nx<ny\\
\rightarrow x<\frac{m}{n}<y
\end{gathered}
$$

□

定理 4.4. *任意整数次根。$\forall n\in N^+,\sqrt[n]{x}\in R.\forall x>0,\forall n\in N^+,\exists y\in R\rightarrow y^n=x\wedge\forall z\neq y, z^n\neq x$*

证明.

$$
\begin{array}{cc}
E=\{t:t^n<x\} & \text{构造结果} \\
t=\frac{x}{1+x}\rightarrow & \\
0<t<1\rightarrow t^n<t<x\rightarrow E\neq\varnothing & \\
t>1+x\rightarrow t>x,t^n>x\rightarrow t\notin E\rightarrow E\neq Q\rightarrow 1+x\geqslant \mathrm{supE} & \\
\exists y=\mathrm{supE}\in R & y=\mathrm{supE} \\
y^n=x\rightarrow y^n\nless x\wedge y^n\ngtr x & \\
0<a<b,b^n-a^n=(b-a)(b^{n-1}+b^{n-2}a+\cdots+a^{n-1}) & \\
\rightarrow b^n-a^n<(b-a)nb^{n-1} & \\
y^n<x,\exists h,0<h<1 & \\
h<\frac{x-y^n}{n(y+1)^{n-1}} & \\
\rightarrow(y+h)^n-y^n<hn(y+h)^{n-1}<hn(y+1)^{n-1}<x-y^n & \\
\rightarrow(y+h)^n<x\wedge y+h\in E,\text{但}y+h>y=\mathrm{supE} & \\
\rightarrow y^n\nless x & y^n\nless x
\end{array}
$$

□

$$
\begin{array}{cc}
y^n>x & \\
k=\frac{y^n-x}{ny^{n-1}}\rightarrow 0<k<y,\text{令}t\geqslant y-k & \\
\rightarrow y^n-t^n\leqslant y^n-(y-k)^n<kny^{n-1}=y^n-x & \\
\rightarrow t^n>x\rightarrow t\notin E & \\
\rightarrow(y-k)^n>t^n\rightarrow y-k>t & \\
\rightarrow y-k\geqslant \mathrm{supE}=y & \\
\rightarrow y^n\ngtr x & y^n\ngtr x \\
\rightarrow y=x &
\end{array}
$$

$$
\begin{array}{cc}
\exists y_0\neq y\rightarrow y_0>y\vee y_0<y & \\
y_0>y\rightarrow y_0^n>y^n=x & \\
y_0<y\rightarrow y_0^n<y^n=x & \\
\rightarrow y_0=y & y\text{唯一}
\end{array}
$$

例 4.5. $\forall a,b\in R,\forall n\in N^+\rightarrow(ab)^{\frac{1}{n}}=a^{\frac{1}{n}}b^{\frac{1}{n}}$

证明.

$$
\begin{array}{rl}
\alpha=a^{\frac{1}{n}},\beta=b^{\frac{1}{n}}\rightarrow ab=\alpha^n\beta^n=(\alpha\beta)^n & \text{交换律} \\
\alpha\beta=((\alpha\beta)^n)^{\frac{1}{n}}=(ab)^{\frac{1}{n}} & \text{唯一性}
\end{array}
$$

□

定义 4.6. *R的十进制表示*

$$
\begin{array}{cl}
\forall x\in R\wedge x>0 & \\
\rightarrow\exists n_0\rightarrow n_0\leqslant x,\forall n>n_0>x & \text{阿基米德性} \\
n_0+\frac{n_1}{10}+\cdots+\frac{n_m}{10^m}\leqslant x & \text{阿基米德性} \\
E=\left\{y\in R: y=\sum_{i=0}^{\infty}\frac{n_i}{10^i}\leqslant x\right\} & \\
\text{根据定义},x=\mathrm{supE} &
\end{array}
$$

5 广义实数系

定义 5.1. *广义实数系$R\cup\{-\infty,+\infty\}$,规定序关系及一些运算*

1. $\forall x\in R,-\infty<x<+\infty$

2. $\forall x \in R, x + \infty = \infty, x - \infty = -\infty, \frac{x}{+\infty} = \frac{x}{-\infty} = 0$

3. $\forall x \in R \wedge x > 0, x \times \infty = \infty, x \times -\infty = -\infty$

4. $\forall x \in R \wedge x < 0, x \times \infty = -\infty, x \times -\infty = \infty$

Remark:广义实数系不是域.$+\infty, -\infty$不构成逆元

6 复数域

定义 6.1. *复数: $\forall a, b \in R$, 有序对(a, b)称为复数，并定义其上的运算:*

加法	$\forall x, y \in C, x + y = (a, b) + (c, d) = (a + c, b + d)$
乘法	$\forall x, y \in C, xy = (ac - bd, ad + bc)$

定理 6.2. *复数和复数上的加法和乘法构成了复数域*

证明.

$$
\begin{aligned}
&x = (a, b), y = (c, d), z = (e, f) \\
&x + y = (a + c, b + d) \in C && A1 \\
&x + y = (a + c, b + d) = (c + a, d + b) = y + x && A2 \\
&(x + y) + z = (a + c + e, b + d + f) && A3 \\
&= (a + (c + e), b + (d + f)) \\
&= x + (y + z) \\
&x + 0 = (a, b) + (0, 0) = (a, b) = x && A4 \\
&x + -x = (a, b) + (-a, -b) = (0, 0) = 0 && A5 \\
\\
&xy = (ac - bd, ad + bc) \in C && M1 \\
&xy = (ac - bd, ad + bc) && M2 \\
&= (ca - db, da + cb) = yx \\
&xyz = (ac - bd, ad + bc)(e, f) && M3 \\
&= ((ac - bd)e - (ad + bc)f, (ac - bd)f + (ad + bc)e) \\
&= (ace - bde - adf + bcf, acf - bdf + ade + bce) \\
&= (a(ce - df) - b(de - cf), a(cf + de) + b(ce - df)) \\
&= (a, b)(ce - df, cf + de) \\
&= x(yz) \\
&x \times 1 = (a, b)(1, 0) = (a1 - b0, a0 + b1) && M4 \\
&= (a, b) = x \\
&x \neq 0 \rightarrow (a, b) \neq (0, 0) && M5 \\
&\rightarrow \frac{1}{x} = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right) \\
&x\frac{1}{x} = (a, b)\left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right) \\
&= \left(\frac{a^2}{a^2 + b^2} - \frac{-b^2}{a^2 + b^2}, \frac{-ab}{a^2 + b^2} + \frac{ab}{a^2 + b^2}\right) = (1, 0) \\
\\
&x(y + z) = (a, b)(c + e, d + f) && D \\
&= (a(c + e) - b(d + f), a(d + f) + b(c + e)) \\
&= (ac + ae - bd - bf, ad + af + bc + be) \\
&= (ac - bd, ad + bc) + (ae - bf, af + be) \\
&= xy + xz
\end{aligned}
$$

□

定理 6.3. $R_C = \{(a, 0) \in C\} \subseteq C$与$R$域同构

$\forall x, y = \{(a, 0)\} \in C, x + y \in \{(a, 0)\}, xy \in \{(a, 0)\}$

$$\begin{aligned} x + y &= (a, 0) + (b, 0) = (a + b, 0) \in \{(\lambda, 0)\} \\ xy &= (a, 0)(b, 0) = (ab, 0) \in \{(\lambda, 0)\} \end{aligned}$$

定义 6.4. $i = (0, 1)$

定理 6.5. $i^2 = -1$: $i^2 = (0, 1)(0, 1) = (0 \times 0 - 1 \times 1, 0 \times 1 + 0 \times 1) = (-1, 0)$

定理 6.6. $(a, b) \in C = a + bi$

$a + bi = (a, 0) + (b, 0)(0, 1) = (a, 0) + (0b - 0 \times 1, 1b + 0 \times 0) = (a, 0) + (0, b) = (a, b)$

定义 6.7. $\forall z \in C, z = a + bi, \bar{z} = a - bi$记$\bar{z}$为$z$的共轭。$\mathrm{Re}(z) = a, \mathrm{Im}(z) = b$

定理 6.8. *共轭复数的性质*

1. $\overline{x + y} = \bar{x} + \bar{y}$.
2. $\overline{xy} = \bar{x}\bar{y}$
3. $x + \bar{x} = 2\mathrm{Re}(x) = 2\mathrm{Re}(\bar{x}), x - \bar{x} = 2i\mathrm{Im}(x)$
4. $\mathrm{Im}(x\bar{x}) = 0 \wedge \mathrm{Re}(x\bar{x}) \geqslant 0. x = 0 \rightarrow \mathrm{Re}(x\bar{x}) = 0$

证明.

$$\begin{aligned} \overline{x + y} &= \overline{(a + c, b + d)} = (a + c, -b - d) = \bar{x} + \bar{y} && 1 \\ \overline{xy} &= \overline{(ac - bd)(ad + bc)} = (ac - bd, -ad - bc) && 2 \\ \bar{x}\bar{y} &= (a, -b)(c, -d) = (ac - bd, a(-d) + (-b)c) && \\ & && 3 \\ z\bar{z} &= (a, b)(a, -b) = (a^2 + b^2, -ab + \mathrm{ab}) && 4 \\ &= (a^2 + b^2, 0) && \\ \mathrm{Im}(a^2 + b^2, 0) &= 0, \mathrm{Re}(a^2 + b^2, 0) = a^2 + b^2 \geqslant 0 && \end{aligned}$$

$\square$

定义 6.9. *复数的绝对值* $|z| = \sqrt{\mathrm{Re}(z\bar{z})} \in R$. *6.8,4.4*

$z \in R_C, |z| = \sqrt{a^2} \rightarrow |z| = |x|$

定理 6.10. *复数绝对值的一些性质*

1. $|z| = 0 \Leftrightarrow z = 0, |0| = 0$
2. $|z| = |\bar{z}|$
3. $|zw| = |z| \times |w|$
4. $|\mathrm{Re}(z)| \leqslant |z|, |\mathrm{Im}(z)| \leqslant |z|$
5. $|z + w| \leqslant |z| + |w|$

证明.

$$|z|=0\rightarrow\sqrt{z\bar{z}}=\sqrt{(a^2+b^2)}=0\rightarrow a=b=0\rightarrow z=0 \quad 1$$

$$|z|=\sqrt{z\bar{z}}=\sqrt{\bar{z}z}=|\bar{z}| \quad 2$$

$$\begin{aligned}|zw|&=|(a,b)(c,d)|=|(ac-bd,ad+bc)| \quad 3\\&=\sqrt{(ac-bd)^2+(ad+bc)^2}\\&=a^2c^2+b^2d^2-2abcd+a^2d^2+b^2c^2+2abcd\\&=a^2c^2+b^2d^2+a^2d^2+b^2c^2\\|z|\times|w|&=\sqrt{a^2+b^2}\sqrt{c^2+d^2}\\&=a^2c^2+a^2d^2+b^2c^2+b^2d^2\end{aligned}$$

□

$$\begin{aligned}|\mathrm{Re}(z)|&=\sqrt{a^2}\leqslant\sqrt{a^2+b^2} \quad 4\\|\mathrm{Im}(z)|&=\sqrt{b^2}\leqslant\sqrt{a^2+b^2}\end{aligned}$$

$$\begin{aligned}|z+w|^2&=(z+w)(\bar{z}+\bar{w})=z\bar{z}+z\bar{w}+\bar{z}w+w\bar{w} \quad 5\\&=|z|^2+2\mathrm{Re}(z\bar{w})+|w|^2\\&\leqslant|z|^2+2|z\bar{w}|+|w|^2\\&=|z|^2+2|z|\times|w|+|w|^2\\&=(|z|+|w|)^2\\&\rightarrow|z+w|\leqslant|z|+|w|\end{aligned}$$

定理 6.11. *Schwarz不等式*

$$\left|\sum_{i=1}^n a_i\bar{b_i}\right|^2\leqslant\sum_{i=1}^n|a_i|^2\sum_{i=1}^n|b_i|^2$$

证明.

$$\begin{aligned}&A=\sum|a_i|^2,B=\sum|b_i|^2,C=\sum a_i\bar{b_i}\\&B=0\rightarrow b_i=0\rightarrow 0\leqslant\sum|a_i|^2 \quad \text{成立}\\&B>0\rightarrow\\&\sum|Ba_i-Cb_i|^2=\sum(Ba_i-Cb_i)(B\bar{a_i}-\overline{Cb_i})\\&=B^2\sum|a_i|^2-B\bar{C}\sum a_i\bar{b_i}-BC\sum b_i\bar{a_i}+|C|^2\sum|b_i|^2\\&=B^2A-B\bar{C}C-BC\bar{C}+|C|^2B\\&=B^2A-B|C|^2\\&=B(AB-|C|^2)\\&\rightarrow B(AB-|C|^2)\geqslant 0\wedge B\geqslant 0\\&\rightarrow AB-|C|^2\geqslant 0\\&\sum|a_i|^2\sum|b_i|^2-|\sum a_i\bar{b_i}|^2\geqslant 0\\&\rightarrow|\sum a_i\bar{b_i}|^2\leqslant\sum|a_i|^2\sum|b_i|^2\end{aligned}$$

□

7 欧氏空间

定义 7.1. *向量，向量加法和标量乘法，向量内积,向量范数.欧氏空间*

$n\in N^+$, n个有序实数构成的元素$\boldsymbol{x}=(x_1,\ldots,x_n)$叫向量	definition	向量
$\forall k\in N^+, R^k=\{\boldsymbol{x}: \boldsymbol{x}=(x_1,\ldots,x_k), x_i\in R\}$	definition	向量空间
$\forall\boldsymbol{x},\boldsymbol{y}\in R^n: \boldsymbol{x}+\boldsymbol{y}=(x_1+y_1,\ldots,x_n+y_n)\in R^n$	definition	向量加法
$\forall\boldsymbol{x}\in R^n,\lambda\in R: \lambda\boldsymbol{x}=(\lambda x_1,\ldots,\lambda x_n)\in R^n$	definition	向量数乘
$\forall\boldsymbol{x},\boldsymbol{y}\in R^n, \boldsymbol{x}\cdot\boldsymbol{y}=\sum x_iy_i$	definition	向量内积
$\forall x\in R^n, \lvert\boldsymbol{x}\rvert=\sqrt{(\boldsymbol{x}\cdot\boldsymbol{x})}=\sqrt{\sum x_i^2}$	definition	向量范数
具有内积和范数的R上的n维向量空间叫欧氏空间	definition	欧氏空间

定理 7.2. *欧氏空间的一些性质*

1. $\forall x \in R^k, |x| \geqslant 0$
2. $\forall x \in R^k, |x| = 0 \Leftrightarrow x = 0$
3. $\forall \boldsymbol{x} \in R^k, \forall a \in R, |a\boldsymbol{x}| = |a||\boldsymbol{x}|$
4. $\forall x, y \in R^k, |x \cdot y| \leqslant |x||y|$
5. $\forall x, y \in R^k, |x + y| \leqslant |x| + |y|$
6. $\forall x, y, z \in R^k, |x - z| \leqslant |x - y| + |y - z|$

证明.

$$
\begin{array}{cc}
\forall x \in R^k, |x| = \sqrt{\sum x_i^2} \geqslant 0 & 1 \\
|x| = 0 \rightarrow \sqrt{\sum x_i^2} = 0 \rightarrow x_i = 0 \rightarrow x = 0 & 2 \\
|a\boldsymbol{x}| = \sqrt{\sum a^2 x_i^2} = |a| \sqrt{\sum x_i^2} = |a||x| & 3 \\
\text{Schwarz不等式} \rightarrow |x \cdot y|^2 \leqslant |x|^2 |y|^2 & 4 \\
\rightarrow |x \cdot y| \leqslant |x||y| & \\
 & \\
|x + y|^2 = (x + y)(x + y) & 5 \\
= x \cdot x + y \cdot y + x \cdot y + y \cdot x & \\
= |x|^2 + 2x \cdot y + |y|^2 & \\
\leqslant |x|^2 + 2|x| \times |y| + |y|^2 & \\
= (|x| + |y|)^2 & \\
\rightarrow |x + y| \leqslant |x| + |y| & \\
 & \\
|x - z| = |x - y + y - z| & 6 \\
\leqslant |x - y| + |y - z| &
\end{array}
$$

□

定义 7.3. *R^n上的度量:$d(x, y) = |x - y|$*

满足度量公理：R^n是可度量化空间（Nanata-Smirnov）

$$
\begin{array}{cc}
\forall x, y \in R^k, d(x, y) \geqslant 0, 0 = d(x, y) \Leftrightarrow x = y & \text{正性} \\
\forall x, y \in R^k, d(x, y) = d(y, x) & \text{交换律} \\
\forall x, y, z \in R^k, |x - y| \leqslant |x - y| + |y - z| & \text{三角不等式}
\end{array}
$$

度量的定义应该只有正性和三角不等式.???

Remark:至此，R^n成为了度量空间

习题

1. Proof: $r \neq 0 \in Q, x \notin Q \rightarrow r + x \notin Q . rx \notin Q$

$$
\begin{array}{c}
r + x \in Q \rightarrow r + x = \frac{m}{n} \rightarrow x = \frac{m}{n} - \frac{p}{q} \in Q \rightarrow x \notin Q \\
rx \in Q \rightarrow rx = \frac{m}{n} \rightarrow \frac{p}{q}x = \frac{m}{n} \rightarrow x = \frac{mq}{np} \in Q \rightarrow x \notin Q
\end{array}
$$

2. Proof: $\forall x \in Q, x^2 \neq 12$

$$
x^2 = 12 \rightarrow \frac{p^2}{q^2} = 12 \rightarrow p^2 = 12q^2 \rightarrow p^2 = 3 \times 2 \times 2 \times q^2
$$

但是p^2中的质因数分解有偶数个3与$3 \times (2q)^2$中奇数个3矛盾$\rightarrow x^2 \neq 12$

3.

4. Proof: E为有序集的非空子集。a为E的下界，b为E的上界.Proof: $a \leqslant b$

$$\forall x \in E, x \geqslant a, x \leqslant b \to a \leqslant x \leqslant b \to a \leqslant b$$

5. Proof: A为非空实数集，下有界。$-A = \{-x : x \in A\}$.Proof: $\mathrm{infA} = -\sup(-A)$

$$\begin{aligned}
&\forall x \in A, x \geqslant \inf A \to \forall x \in -A, x \leqslant -\inf A\\
&\to -\inf(A)\text{是}-A\text{的上界} \to \sup(-A)\text{存在}\\
&\forall t > \inf A, \exists y \in A \to y < t\\
&\to \forall -t > \inf(A), \exists y \in -A \to y < -t \qquad \text{变量代换,}\\
&\forall t < \inf(-A), \exists y \in -A \to y > t\\
&\to -\inf(A) = \sup(-A)\\
&\to \inf(A) = -\sup(-A)
\end{aligned}$$

6. $b > 1, b \in R$

a. $m, n, p, q \in Z, n > 0, q > 0, r = \frac{m}{n} = \frac{p}{q}$, Proof: $(b^m)^{1/n} = (b^p)^{1/q}$.因此$b^r = (b^m)^{1/n}$合理

$$\begin{aligned}
&\frac{m}{n} = \frac{p}{q}, m > p \to m = kp, n = kq \to \frac{kp}{kq} = \frac{p}{q}\\
&(b^{kp})^{1/kq} \in R, (b^p)^{1/q} \in R\\
&b^{kp} = x, b^p = y\\
&\to x = y^k \qquad \text{整数上有这种定义}\\
&\to y = x^{1/k} \qquad 4.4\\
&\to b^p = (b^{kp})^{1/k}\\
&\to (b^p)^{1/q} = ((b^{kp})^{1/k})^{1/q} \qquad b^p \in R \wedge (b^{kp})^{1/k} \in R\\
&\to (b^p)^{1/q} = (b^m)^{1/n}\\
&\to \text{任意正实数的有理数次指数唯一}
\end{aligned}$$

b. $r, s \in Q$, Proof: $b^{r+s} = b^r \cdot b^s$

$$\begin{aligned}
&r = \frac{p}{q}, s = \frac{m}{n}\\
&\to b^{r+s} = b^{\frac{p}{q} + \frac{m}{n}} = b^{\frac{pn+mq}{qn}}\\
&b^{pn+mq} = t.t = b^{pn} b^{mq} \qquad Z\text{上的操作}\\
&\to b^{r+s} = t^{1/qn} \qquad a\text{提供了有理数指数的定义}\\
&b^{\frac{p}{q}} = (b^p)^{1/q}, b^{\frac{m}{n}} = (b^m)^{1/n}\\
&\to ((b^p)^{1/q} (b^m)^{1/n})^{qn} = b^{pn} b^{mq}\\
&\to b^{\frac{p}{q}} b^{\frac{m}{n}} = (b^{pn} b^{mq})^{1/qn} = t^{1/qn}\\
&\to b^{r+s} = b^r b^s \qquad x^{1/n}\text{的唯一性}4.4
\end{aligned}$$

c. $x \in R. B(x) = \{b^t : t \leqslant x \wedge t \in Q\}$.Proof: $b^r = \sup B(r)$.因此$b^x = \sup B(x)$合理

$$\begin{gathered}
b \in R \to b < \infty . t \leqslant x . nt > x \to B(x)\text{有上界}\\
\gamma = \sup(B(x))\\
\text{设}b^x < \sup(B(x)) \to \exists a > x \to b^a \in B(x) \to b^a > b^x\\
\text{但}b^x = \sup(B(x))\text{矛盾}\\
\to b^x \not< \sup(B(x))\\
\text{设}b^x > \sup(B(x)) \to \forall a \in B(x), b^a < b^x\\
\exists a \notin B(x), b^a > b^x.\text{否则}: \forall a \notin B(x). b^a \leqslant b^x \to b^a < \sup(B)\\
\text{但}\forall u \in A, b^u < b^a \to b^u\text{即}b^a \geqslant \sup(B).\text{这与}b^a < \sup(B)\text{矛盾}\\
\to \exists a \notin B(x), b^a > b^x\text{与}B\text{的定义}b^t < x\text{矛盾}\\
\to b^x \not> \sup(B(x))\\
\to b^x = \sup(B(x))
\end{gathered}$$

d. $\forall x, y \in R. b^{x+y} = b^x \cdot b^y$

$$
\begin{aligned}
b^{x+y} &= \sup(\{b^t : t \leqslant x+y\}) \\
b^x b^y &= \sup\{b^p b^q : p \leqslant x, q \leqslant y\} \\
&= \sup\{b^{p+q} : p \leqslant x, q \leqslant y\} && b\text{给出了有理数的指数定义} \\
&= \sup\{b^{p+q} : p+q \leqslant x+y\} && p \leqslant x, q \leqslant y \to p+q \leqslant x+y \\
\sup\{b^t : t \leqslant x+y\} &= \sup\{b^{p+q} : p+q \leqslant x+y\}
\end{aligned}
$$

7. $b > 1, y > 1, b, y \in R$. Proof: $\exists x \to b^x = y. (x = \log_b(y))$

a. $\forall n \in N^+, b^n - 1 \geqslant n(b-1)$

$$
\begin{aligned}
b^n - 1 &= b^n - 1^n \\
&= (b-1)(b^{n-1} + \cdots + 1^{n-1}) \\
b > 1 &\to b^k > 1 \\
&\to > (b-1)n
\end{aligned}
$$

b. $b - 1 \geqslant n(b^{1/n} - 1)$

$$
\begin{aligned}
b^n - 1 &\geqslant n(b-1) \\
b = b^n &\to b - 1 \geqslant n(b^{1/n} - 1)
\end{aligned}
$$

c. $t > 1, n > \frac{b-1}{t-1}$. Proof: $b^{1/n} < t$

$$
\begin{aligned}
&\to n(t-1) > b - 1 \geqslant n(b^{1/n} - 1) \\
&n(t-1) > n(b^{1/n} - 1) \\
&\to t > b^{1/n}
\end{aligned}
$$

d. $w \to b^w < y$.Proof: $\exists n \in N^+, b^{w+1/n} < y$.　用c令$t = yb^{-w}$

$$
\begin{aligned}
&\forall w \to b^w < y \\
&\to yb^{-w} > 1 \\
&\to \exists n < N^+ \to yb^{-w} > b^{1/n} \\
&\to y > b^{w+1/n}
\end{aligned}
$$

e. $b^w > y, \exists n \in N^+, b^{w-1/n} > y$

$$
\begin{aligned}
&b^w > y. b^{w-1/n} = b^w / b^{1/n} = \\
&y > 1 \to \exists n \in N^+ \to y > b^{1/n} \\
&yb^{-w} < 1 \to y^{-1} b^w > 1 \\
&\to \exists n \in N^+ \to y^{-1} b^w > b^{1/n} \\
&\to y^{-1} > b^{1/n - w} \\
&y < b^{w-1/n}
\end{aligned}
$$

f. $A = \{w : b^w < y\}$. Proof: $x = \sup A \to b^x = y$

$$
\begin{gathered}
d, e \to \\
\text{设} x \neq \sup A \\
x < \sup A : d \to \forall w \in A \to \exists n \in N^+, b^{w+1/n} \in A \\
\to x \not< \sup A \\
x > \sup A : e \to \forall w \in Q \wedge w > \sup A \\
\to \exists n \in N^+, b^{w-1/n} > x \\
\to b^w > b^{w-1/n} \to b^{w-1/n} > \sup A \\
\text{但} w - 1/n < w \to w \in A \leqslant \sup A \\
\text{矛盾} \\
\to b^w = \sup A
\end{gathered}
$$

g. $\forall x\neq y, b^x\neq b^y$

$$\begin{aligned} y\neq x &\rightarrow y>x\vee y<x\\ y>x&\rightarrow \exists t\in Q, x<t<y\\ &\rightarrow b^x<b^t<b^y \end{aligned}$$ 稠密性决定在有理数上有各种内插

8. Proof: C不能定义序关系成为有序域

$$\begin{aligned} &\forall x,y>0\rightarrow xy>0\\ &i\neq 0\rightarrow i>0\rightarrow i^2=-1<0\\ &i<0\rightarrow i^2=(-i)^2=-1<0\\ &\rightarrow \text{不满足有序域定义} \end{aligned}$$

9. Proof: $z=a+bi, w=c+di. z<w$: $a<c\vee(a=c\wedge b<d)$.Proof: 这种序关系使复数构成有序集

Proof or Disproof: 这种序关系下复数集具有最小上界性

$$\forall z,w\in C, z\neq w \qquad \text{验证2.1}$$

$$\begin{aligned} z<w&\rightarrow a<c\vee(a=c\wedge b<d) \qquad 1\\ &\rightarrow c\nleqslant a\vee(c=a\wedge d\nleqslant b)\\ &\rightarrow c\nless a\vee c=a\wedge d\nless b\\ &\rightarrow w\nless z \end{aligned}$$

$$\begin{gathered} \forall z,w,c\in C, z<w, w<c\rightarrow \qquad 2\\ (a<c\vee a=c\wedge b<d)\wedge(c<e\vee c=e\wedge d<f)\\ a<c\wedge c<e\rightarrow a<e\\ (a=c\wedge b<d)\wedge(c=e\wedge d<f)\rightarrow a=e\wedge b<f\\ \rightarrow a<e\vee a=e\wedge b<f\\ \rightarrow z<c \end{gathered}$$

10. Proof: $z=a+bi, w=u+vi$

$$a=\sqrt{\frac{|w|+u}{2}}, b=\sqrt{\frac{|w|-u}{2}}$$

Proof: $v\geqslant 0\rightarrow z^2=w. v\leqslant 0\rightarrow(\bar{z})^2=w. \forall z\neq 0\in C, \exists x\neq y\rightarrow x^2=y^2=z$.

$$\begin{gathered} z^2=\left(\sqrt{\tfrac{|w|+u}{2}}, \sqrt{\tfrac{|w|-u}{2}}\right)\left(\sqrt{\tfrac{|w|+u}{2}}, \sqrt{\tfrac{|w|-u}{2}}\right)\\ =\left(\sqrt{\tfrac{|w|+u}{2}}\sqrt{\tfrac{|w|+u}{2}}-\sqrt{\tfrac{|w|-u}{2}}\sqrt{\tfrac{|w|-u}{2}}, 2\sqrt{\tfrac{|w|+u}{2}}\sqrt{\tfrac{|w|-u}{2}}\right)\\ =\left(\tfrac{|w|+u}{2}-\tfrac{|w|-u}{2}, \sqrt{|w|^2-u^2}\right)\\ =\left(u, \sqrt{|w|^2-u^2}\right)\\ \rightarrow\sqrt{|w|^2-u^2}\pm v\\ \sqrt{|w|^2-u^2}=\sqrt{u^2+v^2-u^2}=\sqrt{v^2}\\ v\geqslant 0\rightarrow\sqrt{v^2}=v\rightarrow z^2=w\\ v\leqslant 0\rightarrow\sqrt{v^2}=-v\rightarrow\bar{z}^2=w\\ \rightarrow\exists w\in Z, z^2=w\rightarrow z=a+bi, a-bi \end{gathered}$$

11. Proof: $z\in C$. Proof: $\exists r\geqslant 0, \exists w\wedge|w|=1\rightarrow z=rw$.

Proof or Disproof: $\forall z\in C, z=rw=\lambda v\rightarrow r=\lambda\wedge w=v$

$$\forall z = C, z = 0 \to r = 0 \vee w = 0. \, |w| = 1 \to w \neq 0 \to r = 0$$

$$z \neq 0 \to z = a + bi, z = \left(\sqrt{a^2+b^2}\right)\left(\tfrac{a}{\sqrt{a^2+b^2}} + \tfrac{bi}{\sqrt{a^2+b^2}}\right)$$

$$r = \sqrt{a^2+b^2}, w = \tfrac{a}{a^2+b^2} + \tfrac{bi}{a^2+b^2}$$

$$|w| = \frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}} = 1$$

$$z = 0 \to r = 0, w_1 = 1 + 0i, w_2 = 0 + 1i$$

$$rw_1 = 0 = rw_2 \wedge w_1 \neq w_2$$

12. Proof: $z_1, \ldots, z_n \in C$. Proof: $|z_1 + \cdots + z_n| \leqslant |z_1| + \ldots + |z_n|$

$$|z_1 + z_2| \leqslant |z_1| + |z_2|$$

$$\forall s_i = \textstyle\sum_{i=1}^k z_i.\text{假定}|s_i + z_{i+1}| \leqslant |s_i| + |z_{i+1}|$$

$$|s_{i+1} + z_{i+2}| = |s_i + z_{i+1} + z_{i+2}| \leqslant |s_i + z_{i+1}| + |z_{i+2}|$$

$$\to \forall n \in N^+, |\textstyle\sum_n z_i| \leqslant \sum_n |z_i|$$

13. Proof: $\forall x, y \in C$. Proof: $||x| - |y|| \leqslant |x - y|$

$$\begin{aligned} ||x| - |y||^2 &= (|x| - |y|)^2 \\ &= |x|^2 + |y|^2 - 2|x||y| \\ |x - y|^2 &= (x - y)\overline{(x - y)} \\ &= (x - y)(\bar{x} - \bar{y}) \\ &= |x|^2 + |y|^2 - x\bar{y} - \bar{x}y \\ &\leftarrow 2|xy| \geqslant x\bar{y} + \bar{x}y \\ &= x\bar{y} + \overline{x\bar{y}} = 2\mathrm{Re}\,(x\bar{y}) \\ \leftarrow |xy| = |x||y| &= |x||\bar{y}| = |x\bar{y}| \geqslant \mathrm{Re}(x\bar{y}) \end{aligned}$$

14. Compute: $z \in C \wedge |z| = 1.(z\bar{z} = 1)$. Compute: $|1 + z|^2 + |1 - z|^2$

$$\begin{aligned} |1 + z|^2 + |1 - z|^2 &= (1 + z)\overline{(1 + z)} + (1 - z)\overline{(1 - z)} \\ &= (1 + z)(1 + \bar{z}) + (1 - z)(1 - \bar{z}) \\ &= 1 + z + \bar{z} + |z|^2 + 1 - z - \bar{z} + |z|^2 \\ &= 2 + 2|z|^2 \end{aligned}$$

15. Special Value: Schwarz不等式中等号成立的条件

$$|\textstyle\sum a_i\bar{b_i}|^2 = \sum |a_i|^2 \cdot \sum |b_i|^2$$

$$\text{若}a_i = 0\text{或}b_i = 0\text{等号显然成立}$$

$$a_i \neq 0 \wedge b_i \neq 0$$

$$(\textstyle\sum a_i\bar{b_i})\overline{(\sum a_i\bar{b_i})} = (\sum (a_i\bar{a_i}))(\sum b_i\bar{b_i})$$

$$\tfrac{(\sum a_i\bar{b_i})(\sum \bar{a_i}b_i)}{\sum a_i\bar{a_i}} = \textstyle\sum b_i\bar{b_i}$$

$$a_i = \lambda b_i$$

$$\to \tfrac{\lambda \sum a_i\bar{a_i} \times \lambda \sum \bar{a_i}a_i}{\sum a_i\bar{a_i}} = \lambda^2 \textstyle\sum a_i\bar{a_i}$$

$$\text{成立}$$

$$\text{对于}n = 2\text{时}$$

$$\text{若}a_i \neq \lambda b_i, \text{设}b_1 = \mu a_1, b_2 = \varepsilon a_2$$

$$\to \tfrac{(\mu a_1\bar{a}_1 + \varepsilon a_2\bar{a}_2)(\mu\bar{a}_1a_1 + \varepsilon\bar{a}_2a_2)}{a_1\bar{a}_1 + a_2\bar{a}_2}$$

$$\frac{(\mu a_1\bar{a}_1 + \varepsilon a_2\bar{a}_2)(\mu\bar{a}_1a_1 + \varepsilon\bar{a}_2a_2)}{a_1\bar{a}_1 + a_2\bar{a}_2} = \frac{(\mu|a_1|^2 + \varepsilon|a_2|^2)^2}{|a_1|^2 + |a_2|^2}$$

$$
\begin{aligned}
b_1\bar{b_1}+b_2\bar{b_2} &= \mu^2|a_1|^2+\varepsilon^2|a_2|^2\\
\rightarrow\frac{(\mu|a_1|^2+\varepsilon|a_2|^2)^2}{\mu^2|a_1|^2+\varepsilon^2|a_2|^2} &= |a_1|^2+|a_2|^2\\
\frac{\mu^2|a_1|^4+\varepsilon^2|a_2|^4+2|a_1|^2|a_2|^2}{|a_1|^2+|a_2|^2} &= \mu^2|a_1|^2+\varepsilon^2|a_2|^2\\
(|a_1|^2+|a_2|^2)(\mu^2|a_1|^2+\varepsilon^2|a_2|^2) &= \mu^2|a_1|^4+\varepsilon^2|a_2|^4\\
&+ \mu^2|a_1|^2|a_2|^2+\varepsilon^2|a_1|^2|a_2|^2\\
\rightarrow 2|a_1|^2|a_2|^2 &= (\mu^2+\varepsilon^2)|a_1|^2|a_2|^2\\
&\rightarrow \mu^2+\varepsilon^2=2
\end{aligned}
$$

$$
\begin{aligned}
\text{再使用一手数学归纳法} & \quad n\text{个项平方一下就会有}\frac{n(n+1)}{2}\text{个}2ab\text{项}\\
n=3\text{时}\rightarrow & \\
2|a_1|^2|a_2|^2+2|a_1|^2|a_3|^2+2|a_2|^2|a_3|^2 &= (\lambda_1^2|a_1|^2+\lambda_2^2|a_2|^2+\lambda_3^2|a_3|^2)(|a_1|^2+|a_2|^2+\\
&\quad |a_3^2|^2)\\
&= \lambda_1^2|a_1|^2|a_2|^2+\lambda_1^2|a_1|^2|a_3|^2\\
&+ \lambda_2^2|a_2|^2|a_1|^2+\lambda_2^2|a_2|^2|a_3|^2\\
&+ \lambda_3^2|a_3|^2|a_2|^2+\lambda_3^2|a_3|^2|a_1|^2\\
&= (\lambda_1^2+\lambda_2^2)|a_1|^2|a_2|^2\\
&+ (\lambda_1^2+\lambda_3^2)|a_1|^2|a_3|^2\\
&+ (\lambda_2^2+\lambda_3^2)|a_2|^2|a_3|^2\\
\leftarrow 2=(\lambda_i^2+\lambda_j^2)\text{等式成立} & \\
\rightarrow\text{等号成立需要任意两个复数之比的平方和} &= 2\\
??? & \quad \text{但是还没有讨论完。。。因为这tm都是实数}
\end{aligned}
$$

16. Proof: $k\geqslant 3, \boldsymbol{x},\boldsymbol{y}\in R^k$, $|\boldsymbol{x}-\boldsymbol{y}|=d>0\wedge r>0.$ Proof:

   a. $2r>d\rightarrow\exists\boldsymbol{z}\in R^k\rightarrow|\boldsymbol{z}-\boldsymbol{x}|=|\boldsymbol{z}-\boldsymbol{y}|=r.$这样的$\boldsymbol{z}$有无穷多个

   $$
   \begin{aligned}
   2r=|z-x|+|z-y|&\geqslant|z-x-z+y|\\
   =|y-x|&=|x-y|=d\\
   ???\text{没用}k\geqslant 3\text{的条件。。。}&
   \end{aligned}
   $$

   b. $2r=d\rightarrow$ 只存在一个$\boldsymbol{z}$

   $$
   \begin{gathered}
   2r=d\\
   |z-x|+|z-y|=|x-y|\\
   \rightarrow x\cdot y=|x||y|\rightarrow y=\lambda x,\lambda>0\\
   \rightarrow z=\tfrac{x+y}{2}\\
   \rightarrow|z-x|+|z-y|=\left|\tfrac{y-x}{2}\right|+\left|\tfrac{x-y}{2}\right|=|x-y|
   \end{gathered}
   $$

   c. $2r<d\rightarrow\forall\boldsymbol{z}\in R^k, |\boldsymbol{z}-\boldsymbol{x}|=|\boldsymbol{z}-\boldsymbol{y}|\neq r$

   $$
   \begin{gathered}
   2r=|z-x|+|z-y|\geqslant|z-x-z+y|=|y-x|=d\\
   \rightarrow 2r\geqslant d\text{矛盾}\\
   \rightarrow r\text{不存在}
   \end{gathered}
   $$

k=1,2时上述命题如何?

$$k=1\colon |z-x|=|z-y| \to z=\frac{x+y}{2}$$
$$\to \text{故这样的}z\text{只有一个}$$
$$k=2\colon\ |z-x|=|z-y| \to z=\frac{x+y}{2}=\left(\frac{x_1+y_1}{2},\frac{x_2+y_2}{2}\right)$$
$$\text{此时平面上}|z-x|=|z-y|\text{只有两个点}$$
$$???$$

17. Proof: $x\in R^k, y\in R^k$. Proof: $|x+y|^2+|x-y|^2=2|x|^2+2|y|^2$

Explanation: 几何上的平行四边形中的命题

$$\begin{gathered}|x+y|^2+|x-y|^2\\=(x+y)(\bar{x}+\bar{y})+(x-y)(\bar{x}-\bar{y})\\=|x|^2+|y|^2+x\bar{y}+\bar{x}y+|x|^2+|y|^2-x\bar{y}-\bar{x}y\\=2|x|^2+2|y|^2\end{gathered}$$

18. Proof or Disproof: $k\geqslant 2, x\in R^k$. Proof: $\exists y\in R^k\wedge y\neq 0\wedge x\cdot y=0$.

$k=1???$

$$\begin{gathered}x=(x_1,\ldots,x_n), y=(y_1,\ldots,y_n)\\x\cdot y=\sum x_i y_i=0\\\text{若}x_j=0\to y_j\neq 0\wedge y_{\neg j}=0\to x\cdot y=0\\\text{若}\forall x_i\neq 0,\to x_j y_j=\textstyle\sum_{\neg j} x_i y_i=x_j y_j\\\text{设}x_j=x_k, y_j=-y_k=1, y_{\neg j,\neg k}=0\to x\cdot y=0\\\forall x_i\neq x_j\to\forall t\in R\to t\in\operatorname{span}(x_1)\to\exists y_1,y_2\in R\to x_1 y_1+y_1 y_2=0\end{gathered}$$

$$k=1\to x\cdot y=0\to x=0\vee y=0\to y=0\text{故不成立.无法张成任意实数}$$

19. Compute: $a,b\in R^k$. Compute: $c\in R^k, r>0\to |x-a|=2|x-b|\Leftrightarrow |x-c|=r$

$$\begin{gathered}k=1\to |x-a|=2|x-b|\\\to(x-a)^2=4(x-b)^2\\\to x^2+a^2-2ax=4x^2+4b^2-8xb\\3x^3+(2a-8b)x+4b^2-a^2=0\\x=-a+2b, x=\frac{a+2b}{3}\end{gathered}$$

$$\begin{gathered}(x-c)^2=r^2\\\to x^2-2cx+c^2-r^2=0\\x=c-r, x=c+r\\c=x+r, x-r\to\\c=-a+2b+r\to |x-c|=r\\c=\frac{a+2b}{3}+r\to |x-c|=r\\c=-a+2b-r\to |x-c|=r\\c=\frac{a+2b}{3}-r\to |x-c|=r\end{gathered}$$

$$\begin{gathered}|x-a|=2|x-b|\Leftrightarrow |x-c|=r\\(x-a)\cdot(x-a)=4(x-b)\cdot(x-b).\ (x-c)(x-c)=r^2\\\sum (x_i-a_i)^2=4\sum (x_i-b_i)^2.\ \sum (x_i-c_i)^2=r^2\\\sum x_i^2+a_i^2-2a_i x_i=4\sum x_i^2+b_i^2-2b_i x_i\\\sum x_i^2+c_i^2-2c_i x_i=r^2\\\sum x_i^2=r^2-\sum (c_i^2+2c_i x_i)\\\sum a_i^2-2a_i x_i=3\sum x_i^2+4\sum b_i^2-2b_i x_i\\\sum a_i^2-2a_i x_i=3(r^2-\sum (c_i^2+2c_i x_i))+4\sum b_i^2-2b_i x_i\end{gathered}$$

20. Proof: R存在性定理4.1中，第一步定义分划去掉第三条没有最大元的性质。即可以取有理数为最大元。Proof: 满足A公理1-4。但不满足A5

$$\forall A \subset R^p, B \subset R^p \wedge A \subseteq B$$
$$T = \bigcup \alpha, \alpha \in A$$
$$A \neq \varnothing \to \exists \alpha \in A, \wedge \alpha \neq \varnothing \to T \neq \varnothing$$
$$\forall a \in T, a \in B \to T \neq Q$$
$$\to T \in R^p$$

$$\forall x \in A, x \subset T \to x \leqslant T \to T\text{是上界}$$
$$\forall \alpha < T \to \alpha \subseteq T, \exists p \in T \wedge p \notin \alpha$$
$$\to \exists \beta \in A, p \in \beta \wedge \alpha < \beta \to \alpha\text{不是}A\text{的上界}$$
$$\to T = \sup(A)$$

$$\alpha, \beta \in R^p, \alpha + \beta = \{a + b : a \in \alpha, b \in \beta\} \qquad \text{definitation}$$
$$\alpha \neq \varnothing, \beta \neq \varnothing \to \alpha + \beta \neq \varnothing \qquad A1$$
$$\forall a \notin \alpha, b \notin \beta, a + b \notin \alpha + \beta \to \alpha + \beta \neq Q$$
$$\forall x \in \alpha + \beta, x = (a + b)$$
$$\forall p < x, t = x - p, p = (a + b) - t = a + (b - p)$$
$$b - p < b \to b - p \in \beta$$
$$\to p \in \alpha + \beta$$
$$\to \alpha + \beta \in R^p$$

$$\alpha + \beta = \{a + b : a \in \alpha, b \in \beta\} = \{b + a : a \in \alpha, b \in \beta\} = \beta + \alpha \qquad A2$$
$$(\alpha + \beta) + \gamma = \cdots = \alpha + (\beta + \gamma) \qquad A3$$
$$0^p = \{x \in Q : x \leqslant 0\} \qquad A4$$
$$\forall \alpha \in R^p, \forall a + p \leqslant a + 0 = a \to \alpha + 0^p \subset \alpha$$
$$\forall a \in \alpha, a = a + 0 \to \alpha \subset \alpha + 0^p$$
$$\to \alpha + 0^p = \alpha$$
$$\forall \alpha \in R^p, \alpha = \{x \in Q : x < A\}.\alpha\text{满足上述性质} \qquad A5$$
$$\alpha + -\alpha = \{x \in Q : x < 0\} \neq 0^p$$
$$\to A5\text{不成立}$$