第十章 迹与行列式

1 迹

研究基变化时算子的矩阵的变化

1.1 基的变更

定义 1.1. *单位矩阵(identity matrix), I*

$$I=\begin{pmatrix}1 & & \\ & \ddots & \\ & & 1\end{pmatrix}_{n\times n}$$

Remark:$\forall A\in F^{n,n}, AI=IA$

定义 1.2. *可逆的(invertible), 逆(inverse),* A^{-1}

$$A\text{可逆}:=\exists B\rightarrow AB=BA=I.A^{-1}=B.$$

定理 1.3. *线性算子复合的矩阵*

$$\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\text{是}V\text{的基}, S,T\in\mathcal{L}(V)\rightarrow\mathcal{M}(ST,\boldsymbol{u},\boldsymbol{w})=\mathcal{M}(S,\boldsymbol{v},\boldsymbol{w})\cdot\mathcal{M}(T,\boldsymbol{u},\boldsymbol{v})$$

定理 1.4. *恒等算子关于两个基的矩阵*

$$\boldsymbol{u},\boldsymbol{v}\text{是}V\text{的基}.\mathcal{M}(I,\boldsymbol{u},\boldsymbol{v})\text{和}\mathcal{M}(I,\boldsymbol{v},\boldsymbol{u})\text{都是可逆的，且互为逆}$$

证明.

$$\begin{array}{l} I=\mathcal{M}(I,\boldsymbol{v},\boldsymbol{u})\mathcal{M}(I,\boldsymbol{u},\boldsymbol{v})=\mathcal{M}(I,\boldsymbol{u},\boldsymbol{u})\\ I=\mathcal{M}(I,\boldsymbol{u},\boldsymbol{v})\mathcal{M}(I,\boldsymbol{v},\boldsymbol{u})=\mathcal{M}(I,\boldsymbol{v},\boldsymbol{v})\end{array}$$

□

例 1.5. F^2的基$(4,2),(5,3)$和$(1,0),(0,1)$

$$\begin{gathered}
\mathcal{M}(I,((4,2),(5,3)),((1,0),(0,1)))=\begin{pmatrix}4 & 5\\ 2 & 3\end{pmatrix}\\
\mathcal{M}(I,((1,0),(0,1)),((4,2),(5,3))\\
I(1,0)=x_{1,1}(4,2)+x_{2,1}(5,3)\\
\begin{cases}1=4x_{1,1}+5x_{2,1}\\ 0=2x_{1,1}+3x_{2,1}\end{cases}\rightarrow\begin{cases}x_{2,1}=-1\\ x_{1,1}=\frac{3}{2}\end{cases}\\
I(0,1)=x_{1,2}(4,2)+x_{2,2}(5,3)\\
\begin{cases}0=4x_{1,2}+5x_{2,2}\\ 1=2x_{1,2}+3x_{2,2}\end{cases}\rightarrow\begin{cases}x_{2,2}=2\\ x_{1,2}=-\frac{5}{2}\end{cases}\\
\rightarrow\mathcal{M}(I,((1,0),(0,1)),((4,2),(5,3)))=\begin{pmatrix}\frac{3}{2} & -\frac{5}{2}\\ -1 & 2\end{pmatrix}\\
\begin{pmatrix}4 & 5\\ 2 & 3\end{pmatrix}\begin{pmatrix}\frac{3}{2} & -\frac{5}{2}\\ -1 & 2\end{pmatrix}=\begin{pmatrix}4\times\frac{3}{2}-1\times 5 & 4\times-\frac{5}{2}+5\times 2\\ 2\times\frac{3}{2}-1\times 3 & 2\times-\frac{5}{2}+3\times 2\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\
\begin{pmatrix}\frac{3}{2} & -\frac{5}{2}\\ -1 & 2\end{pmatrix}\begin{pmatrix}4 & 5\\ 2 & 3\end{pmatrix}=\begin{pmatrix}\frac{3}{2}\times 4+-\frac{5}{2}\times 2 & \frac{3}{2}\times 5-\frac{5}{2}\times 3\\ -1\times 4+2\times 2 & -1\times 5+2\times 3\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\
\rightarrow\begin{pmatrix}4 & 5\\ 2 & 3\end{pmatrix}^{-1}=\begin{pmatrix}\frac{3}{2} & -\frac{5}{2}\\ -1 & 2\end{pmatrix}
\end{gathered}$$

定理 1.6. *基变更公式*

$$\begin{gathered}T\in\mathcal{L}(V).\boldsymbol{u},\boldsymbol{v}\text{是}V\text{的基}.A=\mathcal{M}(I,\boldsymbol{u},\boldsymbol{v})\\ \mathcal{M}(T,\boldsymbol{u})=A^{-1}\mathcal{M}(T,\boldsymbol{v})A\end{gathered}$$

证明.

$$\begin{aligned}\mathcal{M}(T,\boldsymbol{u}) &= \mathcal{M}(T,\boldsymbol{v},\boldsymbol{u})\mathcal{M}(T,\boldsymbol{u},\boldsymbol{v}) = A^{-1}\mathcal{M}(T,\boldsymbol{u},\boldsymbol{v})\\ \mathcal{M}(T,\boldsymbol{u},\boldsymbol{v}) &= \mathcal{M}(T,\boldsymbol{v})A\\ \to \mathcal{M}(T,\boldsymbol{u}) &= A^{-1}\mathcal{M}(T,\boldsymbol{u},\boldsymbol{v}) = A^{-1}\mathcal{M}(T,\boldsymbol{v})A\end{aligned}$$

□

1.2 迹：算子与矩阵之间的联系

$$\begin{gathered}T\in\mathcal{L}(V).\lambda\text{是}T\text{的本征值}.n=\dim V\\ \lambda\text{的重数}:=\dim G(\lambda,T)=\dim\operatorname{null}(T-\lambda I)^n\\ V=\sum\oplus G(\lambda_i,T)\\ \to\dim V=\sum\dim G(\lambda,T)\\ \text{按重数重复的全体本征值之和}:=\boldsymbol{\lambda}\text{是}T\text{的本征值},\boldsymbol{d}\text{是重数}\\ \sum\lambda_i d_i\end{gathered}$$

定义 1.7. *算子的迹(trace of an operator)*

$$\begin{aligned}&T\in\mathcal{L}(V).\\ F=C\qquad &T\text{的迹}=\sum d_i\lambda_i\\ F=R\quad &T\text{的迹}=T_C\text{的所有本征值}\sum d_i\lambda_i\end{aligned}$$

例 1.8. $T\in\mathcal{L}(C^3)$

$$\mathcal{M}(T)=\begin{pmatrix}3&-1&-2\\3&2&-3\\1&2&0\end{pmatrix}$$

$$\begin{gathered}T\text{不是自伴的}\to T\text{没有足够多的本征值}\\ \to T_C\text{的本征值；}\det\begin{pmatrix}3-\lambda&-1&2\\3&2-\lambda&-3\\1&2&-\lambda\end{pmatrix}=-\lambda^3+5\lambda^2-17\lambda+13\\ \lambda=1;\lambda=2-3i;\lambda=2+3i\\ \to\operatorname{trace}T=1+2-3i+2+3i=5\end{gathered}$$

Remark: 迹和特征多项式的联系

$$\begin{gathered}\boldsymbol{\lambda}\text{是}T\text{的本征值或}T_C\text{的本征值}\\ T\text{的特征多项式}(z-\lambda_1)\ldots(z-\lambda_n)\\ (-1)^0z^n+(-1)^1(\lambda_1+\cdots+\lambda_n)z^{n-1}+\cdots+(-1)^n(\lambda_1\lambda_2\cdots\lambda_n)\end{gathered}$$

定理 1.9. *迹和特征多项式*

$$T\in\mathcal{L}(V).\dim V=n.\operatorname{trace}T=\text{特征多项式中}z^{n-1}\text{的系数的相反数}$$

定义 1.10. *矩阵的迹(trace of matrix)*

$$\operatorname{trace}A=\sum_{i=1}^n a_{i,i}$$

定理 1.11. *矩阵的迹运算可交换*

$$A,B\in F^{n,n}.\operatorname{trace}AB=\operatorname{trace}BA$$

证明.

$$\begin{aligned}\operatorname{trace}AB &= \textstyle\sum_{j=1}^n\sum_{k=1}^n A_{j,k}B_{k,j}\\ &= \textstyle\sum_{k=1}^n\sum_{j=1}^n B_{k,j}A_{j,k}\\ &\operatorname{trace}BA\end{aligned}$$

□

定理 1.12. *算子的矩阵的迹与基的选取无关*

$$T \in \mathcal{L}(V).\boldsymbol{u}, \boldsymbol{v}\text{是}V\text{的基}$$
$$\operatorname{trace}\mathcal{M}(T, \boldsymbol{u}) = \operatorname{trace}\mathcal{M}(T, \boldsymbol{v})$$

证明.

$$\begin{aligned} A &= \mathcal{M}(I, \boldsymbol{u}, \boldsymbol{v}) \\ \operatorname{trace}\mathcal{M}(T, \boldsymbol{u}) &= \operatorname{trace}(A^{-1}\mathcal{M}(T, \boldsymbol{v})A) \\ &= \operatorname{trace}(\mathcal{M}(T, \boldsymbol{v})A^{-1}A) \\ &= \operatorname{trace}(\mathcal{M}(T, \boldsymbol{v})) \end{aligned}$$

□

定理 1.13. *算子的迹等于算子矩阵的迹*

$$\operatorname{trace} T = \operatorname{trace}\mathcal{M}(T)$$

证明.

$$\begin{gathered} \operatorname{trace}\mathcal{M}(T)\text{和基的选取无关} \to \text{对某个基相等即可} \\ \text{对于复算子，根据舒尔定理有上三角矩阵} \to \operatorname{trace} T = \operatorname{trace}\mathcal{M}(T) \\ \text{对于实算子}, T_C\text{同样使用舒尔定理} \to \operatorname{trace} T = \operatorname{trace}\mathcal{M}(T) \end{gathered}$$

□

例 1.14. $T \in \mathcal{L}(C^5)$

$$\begin{pmatrix} & & & & -3 \\ 1 & & & & 6 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \end{pmatrix}$$

虽然无法计算算子的精确本征值，但本征值的和=$\operatorname{trace}\mathcal{M}(T) = 0$

定理 1.15. *迹具有可加性*

$$S, T \in \mathcal{L}(V).\operatorname{trace}(S + T) = \operatorname{trace} S + \operatorname{trace} T$$

证明.

$$\begin{aligned} \operatorname{trace}(S + T) &= \operatorname{trace}(\mathcal{M}(S + T)) \\ &= \operatorname{trace}(\mathcal{M}(S) + \mathcal{M}(T)) \\ &= \operatorname{trace}(\mathcal{M}(S)) + \operatorname{trace}(\mathcal{M}(T)) \\ &= \operatorname{trace} S + \operatorname{trace} T \end{aligned}$$

□

定理 1.16. *恒等算子不是两个交换积之差*

$$\forall S, T \in \mathcal{L}(V).ST - TS \neq I$$

证明.

$$\begin{aligned} \operatorname{trace}(ST - TS) &= \operatorname{trace}(ST) - \operatorname{trace}(TS) \\ &= \operatorname{trace}(\mathcal{M}(ST)) - \operatorname{trace}(\mathcal{M}(TS)) \\ &= 0 \neq \operatorname{trace} I = n \end{aligned}$$

□

10.A

2 行列式

2.1 算子的行列式

仿照算子迹的定义

迹:=本征值和重数乘积之和

行列式:=本征值的重数次方之积

定义 2.1. *算子的行列式(determinant of an operator), det T*

$$T \in \mathcal{L}(V)$$

$$F = C \qquad \det T := \prod_{i=1}^{m} \lambda_i^{d_i}$$

$$F = R \quad \det T := \det T_C = \prod_{i=1}^{m} \lambda_i^{d_i}$$

例 2.2. 算子$T \in \mathcal{L}(C^3)$

$$\mathcal{M}(T) = \begin{pmatrix} 3 & -1 & -2 \\ 3 & 2 & -3 \\ 1 & 2 & 0 \end{pmatrix}$$

T的本征值为$1; 2+3i; 2-3i;$
$\det T = 1 \times (2+3i) \times (2-3i) = 13$

定理 2.3. *行列式和特征多项式*

$$T \in \mathcal{L}(V). n = \dim V. \det T = (-1)^n \text{乘常数项}$$

定理 2.4. *特征多项式、迹、行列式*

$$T \in \mathcal{L}(V). T\text{的特征多项式} = z^n - \operatorname{trace}(T) z^{n-1} + \cdots + (-1)^n (\det T)$$

定理 2.5. *算子。* 可逆$\Leftrightarrow$行列式$\neq 0$

证明.

设V是复向量空间，$T \in \mathcal{L}(V)$
T可逆$\Leftrightarrow 0$不是T的本征值
$\Leftrightarrow \det T \neq 0$

□

设T是实向量空间，$T \in \mathcal{L}(V)$
T可逆$\Leftrightarrow T_C$可逆$\Leftrightarrow \det T_C \neq 0$
$\Leftrightarrow \det T \neq 0$

定理 2.6. *T的特征多项式*$=\det(zI - T)$

$$T \in \mathcal{L}(V). T\text{的特征多项式} = \det(zI - T)$$

证明.

V是复向量空间，$\lambda, z \in C.$
$$-(T - \lambda I) = (zI - T) - (z - \lambda) I$$
$$(-1)^n (T - \lambda I)^n = ((zI - T) - (z - \lambda) I)^n$$
$$\operatorname{null} (T - \lambda I)^n = \operatorname{null}((zI - T) - (z - \lambda) I)^n$$

λ是T的本征值$\Leftrightarrow (z - \lambda)$是$(zI - T)$的本征值
λ作为T的本征值的重数等于$(z - \lambda)$作为$(zI - T)$的本征值的重数
λ_i为T的全体本征值
$\forall z \in C, (zI - T)$的全体本征值为$z - \lambda_i$
$\det(zI - T) = (z - \lambda_1) \cdots (z - \lambda_n)$
满足特征多项式的定义
$\rightarrow \det(zI - T)$是T的特征多项式

□

V是实向量空间，对T_C使用复空间的定义得到同样的结果

2.2 矩阵的行列式

从算子的矩阵计算算子行列式的方法。

例 2.7. $\boldsymbol{a}\in F$

$$A=\begin{pmatrix} 0 & & & a_n \\ a_1 & 0 & & \\ & \ddots & \ddots & \\ & & a_{n-1} & 0 \end{pmatrix}, a_i\neq 0$$

$$\begin{gathered} T^0v_1=v_1 \\ Tv_1=a_1v_2 \\ T^2v_1=Ta_1v_2=a_1Tv_2=(a_1a_2)v_3 \\ \cdots \\ T^{n-1}v_1=a_1T^{n-2}v_2=\cdots=(a_1\cdots a_{n-1})v_n \\ \to v_1,Tv_1,\ldots,T^{n-1}v_1\text{是线性无关的}(v_i\text{线性无关}\wedge a_i\neq 0) \\ \to T\text{的极小多项式的次数为}n \\ \forall i, T^nv_i=(a_1\cdots a_n)v_i\to T^n=(a_1\cdots a_n)I \\ \to z^n-a_1\cdots a_n\text{是}T\text{的极小多项式} \\ \to z^n-a_1\cdots a_n\text{是}T\text{的特征多项式} \\ \to \det T=(-1)^{n-1}a_1\cdots a_n \end{gathered}$$

定义 2.8. *排列(permutation), perm n*

$$\begin{gathered} (1,\ldots,n)\text{中的一个排列指}(m_1,\ldots,m_n)\text{中的每个数在其中抢号出现一次} \\ \operatorname{prem} n=\{(1,\ldots,n)\text{的所有排列}\} \end{gathered}$$

例 2.9. $\operatorname{prem} 2=\{(1,2),(2,1)\}.\operatorname{prem} 3=\{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$

例 2.10. $T\in\mathcal{L}(V),\forall k\in 1\ldots n, Tv_k=a_kv_{p_k}$.求$\det T$

$$\begin{gathered} T\text{关于基}v_1,\ldots,v_n\text{的矩阵是分块对角矩阵} \\ A=\begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_M \end{pmatrix}.A_i=\begin{pmatrix} 0 & & & a_n \\ a_1 & \ddots & & \\ & \ddots & \ddots & \\ & & a_{n-1} & 0 \end{pmatrix} \\ V=V_1\oplus\cdots\oplus V_M \\ \det T=(\det T|_{V_1})\times\cdots\times(\det T|_{V_M}) \\ \to\det T=(-1)^{n_1-1}\cdots(-1)^{n_M-1}a_1\cdots a_n \end{gathered}$$

定义 2.11. *排列的符号(sign of a permutation)*

$$\begin{gathered} \text{组}(m_1,\ldots,m_n)\text{中使得下标}i<j\text{的整数对的个数是偶数，符号为}1 \\ \text{下标}i<j\text{的整数对的个数是奇数，符号为}-1 \end{gathered}$$

Remark: 这个是瞎写的，但意思是自明的

例 2.12.

$$\begin{gathered} (2,1,3,4).2<1\to\operatorname{sign}(2,1,3,4)=-1 \\ (2,3,\ldots,n,1).2>1;3>1\ldots n>1\to\operatorname{sign}(2,3,\ldots,n,1)=(-1)^{n-1} \end{gathered}$$

定理 2.13. *交换排列中的两个元素，排列变号*

证明.

$$\begin{aligned} &\text{交换两个元素必然改变这两个元素的序改变} && -1\\ (B,A)\to(A,B) &A\text{前面的逆序全改变，}B\text{后面的逆序全改变} && 0\\ &A,B\text{中间元素的逆序数改变了} && 2\wedge 0\\ &\to\text{排列变号} \end{aligned}$$

这里书上的不计算逆序数的证法。。。 □

定义 2.14. *矩阵的行列式(determinant of a matrix). det A*

$$A=\begin{pmatrix} a_{1,1} & \cdots & a_{1,n}\\ \vdots & \ddots & \vdots\\ a_{n,1} & \cdots & a_{n,n}\end{pmatrix}_{n\times n}$$

$$\det A=\sum_{m_1,\ldots,m_n\in\mathrm{prem}\,n}(\mathrm{sign}(m_1,\ldots,m_n))A_{m_1,1}\times\cdots\times A_{m_n,n}$$

例 2.15.

$$\begin{aligned} A_{1\times1}.\ \mathrm{prem}\,1=1\to\det A=\mathrm{sign}(1)a_{1,1}=a_{1,1}\\ A_{2\times2}.\ \mathrm{prem}\,2=(1,2),(2,1)\to\det A=(\mathrm{sign}(1,2))A_{1,1},A_{2,1}+(\mathrm{sign}(2,1))A_{2,1}A_{1,2}\\ =A_{1,1}A_{2,1}-A_{2,1}A_{1,2}\\ A_{3\times3}.\ \mathrm{prem}\,3=(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\\ \det A=\mathrm{sign}(1,2,3)A_{1,1}A_{2,2}A_{3,3}+\mathrm{sign}(1,3,2)A_{1,1}A_{3,2}A_{2,3}\\ +\mathrm{sign}(2,1,3)A_{2,1}A_{1,2}A_{3,3}+\mathrm{sign}(2,3,1)A_{2,1}A_{3,2}A_{1,3}\\ +\mathrm{sign}(3,1,2)A_{3,1}A_{1,2}A_{2,3}+\mathrm{sign}(3,2,1)A_{3,1}A_{2,2}A_{1,3}\\ =A_{1,1}A_{2,2}A_{3,3}-A_{1,1}A_{3,2}A_{2,3}\\ +A_{3,1}A_{1,2}A_{2,3}-A_{3,1}A_{2,2}A_{1,3}\\ +A_{3,1}A_{2,2}A_{3,3}-A_{3,1}A_{2,2}A_{1,3}\end{aligned}$$

$$\begin{aligned}&\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3}\\ a_{2,1} & a_{2,2} & a_{2,3}\\ a_{3,1} & a_{3,2} & a_{3,3}\end{vmatrix}\\ =&a_{1,1}a_{2,2}a_{3,3}+a_{1,2}a_{2,3}a_{3,1}+a_{1,3}a_{2,1}a_{3,2}\\ -&a_{3,1}a_{2,2}a_{1,3}-a_{2,1}a_{1,2}a_{3,3}-a_{1,1}a_{3,2}a_{2,3}\end{aligned}$$

例 2.16. 计算上三角矩阵

$$A=\begin{pmatrix} A_{1,1} & & *\\ & \ddots & \\ 0 & & A_{n,n}\end{pmatrix}$$

$\det A=A_{1,1}\cdots A_{n,n}$

定理 2.17. *交换矩阵的两列，行列式变为相反数*

$$A\in F^{n,n}.\ B\text{是}A\text{交换两列的矩阵}\to\det A=-\det B$$

证明.

$$\begin{aligned}&\mathrm{sign}(m_1,\ldots,m_n)a_{m_1,1},\ldots,a_{m_n,n}\\ =&-1\mathrm{sign}(m_1,\ldots,m_n)a_{m_1,1},\ldots,a_{m_n,n}\qquad\text{这里没写交换了两个坐标}\end{aligned}$$

根本上使用了交换两个元素的排列会导致逆序数 $\times -1$ □

定理 2.18. *有两列元素相等的矩阵行列式为0*

证明.

$$\begin{aligned}\det A =& -\det B = -\det A\\ \rightarrow& \det A = 0\end{aligned}$$

□

定理 2.19. *重拍列的矩阵的行列式等于重拍的逆序数乘以原行列式*

$$A = (A_{\cdot,1}, \ldots, A_{\cdot,n})\text{是}n \times n\text{矩阵}. \det(A_{\cdot,m1}, \ldots, A_{\cdot,m_n}) = (\operatorname{sign}(m_1, \ldots, m_n))\det A$$

证明.

显然。 □

定理 2.20. *行列式关于每一列都是线性的*

$$\det(A_{\cdot,1}, \ldots, A_{\cdot,k}, \ldots, A_{\cdot,n})\text{是}A_{\cdot,k}\text{的线性函数}$$

证明.

$$\begin{aligned}\det A =& \sum \operatorname{sign}(m_1, \ldots, m_n) a_{m_1,1} \cdots a_{m_n,1}\\ \det(A + A_{\cdot,k}) =& \sum \operatorname{sign}(m_1, \ldots, m_n) a_{m_1,1} \cdots (a_{m_k,k} + b) \cdots a_{m_n,n}\\ =& \sum \operatorname{sign}(m_1, \ldots, m_n) a_{m_1,1} \ldots a_{m_k,k} \cdots a_{m_n,n} + \sum \operatorname{sign}(m_1, \ldots, m_n) a_{m_1,1} \ldots b \ldots a_{m_n,n}\\ =& \det A + \det A^{\cdot}\\ \det(A, \lambda A_{\cdot,k}) =& \sum \operatorname{sign}(m_1, \ldots, m_n) a_{m_1,1} \ldots \lambda a_{m_k,k} \ldots a_{m_n,n}\\ =& \lambda \sum \operatorname{sign}(m_1, \ldots, m_n) a_{m_1,1} \ldots a_{m_k,k} \cdots a_{m_n,n}\\ =& \lambda \det A\end{aligned}$$

□

定理 2.21. *矩阵的行列式是可乘的*

$$\det(AB) = (\det A) \times (\det B)$$

证明.

$$\begin{aligned}A =& (A_{\cdot,1}, \ldots, A_{\cdot,n}).B = (B_{\cdot,1}, \ldots, B_{\cdot,n})\\ \det(AB) =& \det(AB_{\cdot,1}, \ldots, AB_{\cdot,n})\\ =& \det(A(\textstyle\sum_{m_1=1}^{n} B_{m_1,1} e_{m_1}) \cdots A(\sum_{m_n=1}^{n} B_{m_n,n} e_{m_n}))\\ =& \det(\textstyle\sum_{m_1=1}^{n} B_{m_1,1} A e_{m_1} \cdots \sum_{m_n=1}^{n} B_{m_n,n} A e_{m_n})\\ =& \textstyle\sum_{m_1=1}^{n} \cdots \sum_{m_n=1}^{n} B_{m_1,1} \cdots B_{m_n,n} \det(A e_{m_1} \cdots A e_{m_n})\\ \det(AB) =& \textstyle\sum_{(m_1, \ldots, m_n) \in \operatorname{prem} n} B_{m_1,1} \cdots B_{m_n,n} \det(A e_{m_1} \cdots A e_{m_n})\\ =& \textstyle\sum_{(m_1, \ldots, m_n) \in \operatorname{prem} n} B_{m_1,1} \cdots B_{m_n,n} (\operatorname{sign}(m_1, \ldots, m_n)) \det A\\ =& \det A \textstyle\sum_{(m_1, \ldots, m_n) \in \operatorname{prem} n} (\operatorname{sign}(m_1, \ldots, m_n)) B_{m_1,1} \cdots B_{m_n,n}\\ =& (\det A)(\det B)\\ \det(AB) =& (\det A)(\det B) = (\det B)(\det A) = \det(BA)\end{aligned}$$

□

定理 2.22. *算子矩阵的行列式与基的选择无关*

$$\begin{aligned}T \in \mathcal{L}(V).\boldsymbol{u}, \boldsymbol{v}\text{是}V\text{的基}\\ \rightarrow \det \mathcal{M}(T, \boldsymbol{u}) = \det \mathcal{M}(T, \boldsymbol{v})\end{aligned}$$

证明.

$$
\begin{aligned}
A=&\mathcal{M}(T,\boldsymbol{u},\boldsymbol{v})\\
\det\mathcal{M}(T,\boldsymbol{u})=&\det(\mathcal{M}(T,\boldsymbol{u},\boldsymbol{v})^{-1}\mathcal{M}(T,\boldsymbol{v})\mathcal{M}(T,\boldsymbol{u},\boldsymbol{v}))\\
=&\det(\mathcal{M}(T,\boldsymbol{u},\boldsymbol{v})^{-1})\det\mathcal{M}(T,\boldsymbol{u})\det\mathcal{M}(T,\boldsymbol{u},\boldsymbol{v})\\
=&\det\mathcal{M}(T,\boldsymbol{u})\det\mathcal{M}(T,\boldsymbol{u},\boldsymbol{v}^{-1})\det\mathcal{M}(T,\boldsymbol{u},\boldsymbol{v})\\
=&\det\mathcal{M}(T,\boldsymbol{u})\det(\mathcal{M}(T,\boldsymbol{u},\boldsymbol{v})^{-1}\mathcal{M}(T,\boldsymbol{u},\boldsymbol{v}))\\
=&\det\mathcal{M}(T,\boldsymbol{u})\det(I_V)\\
=&\det\mathcal{M}(T,\boldsymbol{u})
\end{aligned}
$$
□

定理 2.23. *算子的行列式等于矩阵的行列式*

证明.

$$
\begin{aligned}
&V\text{是复空间，舒尔定理}\to T\text{有上三角阵}\\
&\to\det T=\lambda_1\cdots\lambda_n\text{成立}=\det\mathcal{M}(T)\\
&V\text{是实空间，}T_C\text{使用舒尔定理}\to T_C\text{有上三角阵}\\
&\to\det T=\det T_C=\lambda_1\cdots\lambda_n=\det\mathcal{M}(T_C)=\det\mathcal{M}(T)
\end{aligned}
$$
□

例 2.24. $T\in\mathcal{L}(C^5)$

$$
\mathcal{M}(T)=\begin{pmatrix} &&&&-3\\ 1&&&&6\\ &1&&&\\ &&1&&\\ &&&1& \end{pmatrix}
$$

$\det T=\det\mathcal{M}(T)=-3\to\lambda_1\cdots\lambda_n=-3$

定理 2.25. *算子的行列式是可乘的*

$$
S,T\in\mathcal{L}(V).\det(S\circ T)=\det S\times\det T
$$

证明.

$$
\begin{aligned}
\det(S\circ T)=&\det(\mathcal{M}(S\circ T))\\
=&\det(\mathcal{M}(S)\times\mathcal{M}(T))\\
=&\det(\mathcal{M}(S))\times\det(\mathcal{M}(T))\\
=&\det(S)\times\det(T)
\end{aligned}
$$
□

2.3 行列式的符号

定理 2.26. *等距同构的行列式绝对值为1*

$$
V\text{是内积空间},S\in\mathcal{L}(V)\text{是等距同构}\to|\det S|=1
$$

证明.

$$
\begin{aligned}
&V\text{是复空间}\to S\text{等距同构}\to|\lambda_i|=1\\
&\to|\det S|=|\lambda_1\times\cdots\times\lambda_n|=1\\
&V\text{是实空间}\\
1\quad&\text{考虑}S_C,\det S_C=\det S\to|\det S_C|=1\\
2\quad&\text{实空间上的矩阵必可分解为分块对角阵}\\
&\det\mathcal{M}(S)=\det(\lambda_1)\cdots\det(\lambda_n)\det(A_1)\ldots\det(A_M)\\
&A_i=\begin{pmatrix}\cos\theta&-\sin\theta\\ \sin\theta&\cos\theta\end{pmatrix}\to\det A_i=\cos^2\theta+\sin^2\theta=1\\
&\to\det\mathcal{M}(S)=1
\end{aligned}
$$
□

行列式与极分解的关系

例 2.27. V是实内积空间，$T\in\mathcal{L}(V)$是可逆的。$\det T$的几何解释

$$
\begin{gathered}
T = S\circ\sqrt{T^*T}\\
S\text{是等距同构}\rightarrow|\det S| = 1\\
\text{如果}v\in V, Sv = -v\rightarrow -1\text{是}S\text{的一个本征值}\\
S\text{的此不变子空间是原空间的方向相反}\\
\det T = \det(S)\det(\sqrt{T^*T}).\ \sqrt{T^*T}\text{是正算子}\rightarrow\lambda_i > 0\\
\rightarrow\det\sqrt{T^*T} > 0
\end{gathered}
$$

$\rightarrow$det 的大小根据$\sqrt{T^*T}$决定，det的符号根据S反向的空间次数决定

2.4 体积

定理 2.28. $|\det T| = \det\sqrt{T^*T}$

定义 2.29. *向量空间中的长方体(box)*

$$
\begin{gathered}
\{(y_1,\ldots,y_n)\in R^n : x_i < y_i < x_i + r_i\}\\
r_i\text{称为长方体的边长}
\end{gathered}
$$

定义 2.30. *向量空间中长方体的体积(volume of a box)*

$$\operatorname{volume} B = r_1\cdots r_n$$

定义 2.31. *向量空间中的体积(volume)*

$$
\begin{gathered}
\text{集合}\Omega\in R^n.\\
\operatorname{volume}\Omega = \sup\sum_{B_i\in\Omega}\operatorname{volume} B_i
\end{gathered}
$$

定义 2.32. *集合上的函数*

$$T(\Omega) = \{Tx : x\in\Omega\}$$

定理 2.33. *正算子使得体积改变了det T倍*

$$\text{正算子}T\in\mathcal{L}(R^n), \operatorname{volume} T(\Omega) = (\det T)(\operatorname{volume}\Omega)$$

证明.

$$
\begin{gathered}
\text{考虑不带旋转的算子}T\\
T(x_1,\ldots,x_n) = (\lambda_1 x_1,\ldots,\lambda_n x_n)\\
= (\lambda_1\cdots\lambda_n)(x_1\cdots x_n)\\
= \det T\operatorname{volume}\Omega
\end{gathered}
$$

□

任意正算子T，根据谱定理$\rightarrow T$具有对角阵

也成立上述结论

???这里瞎证明的???

定理 2.34. *等距同构不改变体积*

证明.

$$
\begin{aligned}
&\|Sx-Sy\| = \|S(x-y)\| = \|x-y\| \\
&\quad \to \text{对于不变子空间}, |\lambda_i| = 1 \\
&\quad \to \lambda_1 \ldots \lambda_n x_1 \ldots x_n = x_1 \cdots x_n
\end{aligned}
$$

□

定理 2.35. T使体积改变$|\det T|$倍

证明.

$$
\begin{aligned}
T &= S \circ \sqrt{T^* \circ T} \\
\operatorname{volume} T(\Omega) &= \operatorname{volume}(S\sqrt{T^*T}(\Omega)) \\
&= \operatorname{volume}(\sqrt{T^*T}(\Omega)) \\
&= (\det\sqrt{T^*T})(\operatorname{volume}\Omega) \\
&= |\det T|(\operatorname{volume}\Omega)
\end{aligned}
$$

□

定义 2.36. *积分(integral),* $\int_\Omega f$

$$
\Omega \in R^n. f\text{是}\Omega\text{上的实函数}
$$
$$
\int_\Omega f = \sup U(P, f)
$$

定理 2.37. *可微(differentible),导数(derivative),* $\sigma'(x)$

$$
\Omega\text{是}R^n\text{的开子集，}\sigma\text{是}\Omega \to R^n\text{的函数}
$$
$$
\sigma\text{在}x\text{可微}\quad \exists T \in \mathcal{L}(R^n) \to \lim_{y\to 0} \frac{\|\sigma(x+y) - \sigma(x) - Ty\|}{\|y\|} = 0
$$

$$
\text{在}x\text{可微} \to x\text{处所有偏导数都存在}
$$
$$
\mathcal{M}(\sigma'(x)) = \begin{pmatrix} D_1\sigma_1(x) & \cdots & D_n\sigma_1(x) \\ \vdots & \ddots & \vdots \\ D_1\sigma_n(x) & \cdots & D_n\sigma_n(x) \end{pmatrix}
$$

记法 2.38. *积分的变量替换公式*

$$
\Omega\text{是}R^n\text{的开子集，}\sigma\colon \Omega \to R^n\text{在}\Omega\text{上可微，}f\text{是}\sigma(\Omega)\text{上的实函数}
$$
$$
\int_{\sigma(\Omega)} f(y)\mathrm{d}y = \int_\Omega f(\sigma(x))|\det\sigma'(x)|\mathrm{d}x
$$

证明.

这里需要的是行列式对每个矩阵元素的连续性吧。 □

例 2.39. 极坐标积分变换

$$
\sigma\colon R^2 \to R^2, \sigma(r,\theta) = (r\cos\theta, r\sin\theta)
$$
$$
\mathcal{M}(\sigma') = \begin{pmatrix} \frac{\partial(r\cos\theta)}{\partial r} & \frac{\partial(r\cos\theta)}{\partial\theta} \\ \frac{\partial(r\sin\theta)}{\partial r} & \frac{\partial(r\sin\theta)}{\partial\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}
$$
$$
\det\mathcal{M}(\sigma') = r\cos^2\theta + r\sin^2\theta = r
$$
$$
\to \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\mathrm{d}y\mathrm{d}y = \int_0^{2\pi}\int_0^1 f(r\cos\theta, r\sin\theta) r\mathrm{d}r\mathrm{d}\theta
$$

例 2.40. 球坐标

$$\sigma: R^3 \rightarrow R^3, \sigma(\rho, \varphi, \theta) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)$$

$$\mathcal{M}(\sigma'(\rho, \varphi, \theta)) = \begin{pmatrix} \frac{\partial(\rho\sin\varphi\cos\theta)}{\partial\rho} & \frac{\partial(\rho\sin\varphi\cos\theta)}{\partial\varphi} & \frac{\partial(\rho\sin\varphi\cos\theta)}{\partial\theta} \\ \frac{\partial(\rho\sin\varphi\sin\theta)}{\partial\rho} & \frac{\partial(\rho\sin\varphi\sin\theta)}{\partial\varphi} & \frac{\partial(\rho\sin\varphi\sin\theta)}{\partial\theta} \\ \frac{\partial(\rho\cos\varphi)}{\partial\rho} & \frac{\partial(\rho\cos\varphi)}{\partial\varphi} & \frac{\partial(\rho\cos\varphi)}{\partial\theta} \end{pmatrix}$$

$$= \begin{pmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\theta\cos\theta \\ \cos\theta & -\rho\sin\varphi & 0 \end{pmatrix}$$

$$\det\mathcal{M}(\sigma'(\rho, \varphi, \theta)) = \rho^2\sin\varphi$$

$$\rightarrow \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f(x, y, z)\mathrm{d}z\mathrm{d}y\mathrm{d}x$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2\sin\varphi\,\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta$$

10.B

1. Pr: V是实向量空间，$T \in \mathcal{L}(V)$没有本征值. Pr: $\det T > 0$

$$T\text{没有本征值} \rightarrow \mathcal{M}(T) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

$$\det T = a^2 + b^2 > 0$$

$$\rightarrow \det T = \det T|_{U_1} \times \cdots \times \det T|_{U_n} > 0$$

2.