Chapter 3 线性映射

F表示$R \vee C.V, W$表示F上的向量空间

1 向量空间的线性映射

定义 1.1. *线性映射(linear map):*

从V到W的线性映射是一个函数$T: V \to W$:

$$\begin{aligned} &\text{加性(additivity)} && \forall u, v \in V \to T(u+v) = T(u) + T(v) \\ &\text{齐性(homogeneity)} && \forall \lambda \in F, \forall v \in V \to T(\lambda v) = \lambda T(v) \end{aligned}$$

$V \to W$的所有线性映射的集合记为$\mathcal{L}(V, W)$

例 1.2. 一些线性映射

1. 零(zero):$\mathbf{0} \in \mathcal{L}(V, W), \mathbf{0}(v) = 0_w$
2. 恒等(identity):$I \in \mathcal{L}(V, V), I(v) = v$
3. 微分(differentiation):$D \in \mathcal{L}(\mathcal{P}(R), \mathcal{P}(R))$: $D(p) = p'$
4. 积分(integration):$T \in \mathcal{L}(\mathcal{P}(R), R), T(p) = \int_0^1 p(x) \mathrm{d}x$
5. 乘以x^2:$T \in \mathcal{L}(\mathcal{P}(R), \mathcal{P}(R)), T(p) = x^2 p(x)$
6. 向后移位:$T \in \mathcal{L}(F^\infty, F^\infty), T(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$
7. $R^3 \to R^2$:$T \in \mathcal{L}(R^3, R^2), T(x, y, z) = (2x - y + 3z, 7x + 5y - 6z)$
8. $F^n \to F^m$: $T \in \mathcal{L}(F^n, F^m), T(\boldsymbol{x}) = (\sum a_{1i} x_i, \sum a_{2i} x_i, \ldots, \sum a_{mi} x_i)$

定理 1.3. *线性映射与定义域的基*

$\boldsymbol{v}$是V的基，$\boldsymbol{w} \in W \to$ 存在唯一线性映射$T: V \to W, T(\boldsymbol{v}) = \boldsymbol{w}$

证明.

$$\begin{gathered} T: V \to W, \forall c_1, \ldots, c_n \in F, T(\textstyle\sum c_i v_i) = \sum (c_i w_i) \qquad \text{构造性证明} \\ \operatorname{span}(\boldsymbol{v}) = V \to \text{定义域满足函数定义} \\ \text{对于每个}c_i = 1, \text{其他}c_j = 0\text{时} \to T(v_i) = w_i \end{gathered}$$

$$\begin{aligned} T(u+v) &= T(\textstyle\sum (a_i + b_i) v_i) \\ &= \textstyle\sum (a_i + b_i) w_i \\ &= \textstyle\sum a_i w_i + \sum b_i w_i \\ &= T(u) + T(v) \end{aligned}$$

$$\begin{aligned} \forall \lambda \in F, T(\lambda v) &= T(\textstyle\sum \lambda a_i v_i) \\ &= \textstyle\sum \lambda a_i w_i \\ &= \lambda a_i w_i \\ &= \lambda T(v) \\ &\to T\text{是线性映射} \end{aligned}$$

$$\begin{gathered} T \in \mathcal{L}(V, W), T(v_i) = w_i, \forall \boldsymbol{c} \in F \qquad \text{唯一性} \\ T(c_i v_i) = c_i T(v_i) \qquad \text{这里指是单值函数} \\ T(\textstyle\sum c_i v_i) = \sum c_i w_i \\ \forall v \in V, v\text{都能被}\textstyle\sum c_i v_i\text{唯一表示} \to T\text{在}v\text{上定义完全} \\ \to \forall v \in V, T(v)\text{唯一} \end{gathered}$$

□

定义 1.4. *$\mathcal{L}(V,W)$上的加法和标量乘法*

$$\begin{array}{cl} \text{加法} & S,T\in\mathcal{L}(V,W),(S+T)(x)=S(x)+T(x) \\ \text{标量乘法} & \forall\lambda\in F,T\in\mathcal{L}(V,W),(\lambda T)(x)=\lambda T(x) \end{array}$$

定理 1.5. *$\mathcal{L}(V,W)$是线性空间:*

$$\begin{aligned} (S+T)(x+y) &= S(x+y)+T(x+y) \\ &= Sx+Sy+Tx+Ty \\ &= Sx+Tx+Sy+Ty \\ &= (S+T)(x)+(S+T)(y) \\ (\lambda T)(ax) &= \lambda T(ax) \\ &= \lambda\,(aT(x)) \\ &= a\,(\lambda T(x)) \\ &= a(\lambda T)(x) \end{aligned}$$

定义 1.6. *$\mathcal{L}(V,W)$上的乘法*

$T\in\mathcal{L}(U,V),S\in\mathcal{L}(V,W)$: 定义$ST,(ST)(x)=S(T(x))$

定理 1.7. $T\in\mathcal{L}(U,V),S\in\mathcal{L}(V,W)\rightarrow ST\in\mathcal{L}(U,W)$

证明.

$$\begin{aligned} (ST)(x+y) &= S(T(x+y)) \\ &= S(T(x)+T(y)) \\ &= S\,(T(x))+S(T(y)) \\ &= (ST)(x)+(ST)(y) \\ ST(\lambda x) &= S(T(\lambda x)) \\ &= S(\lambda T(x)) \\ &= \lambda S(T(x)) \\ &= \lambda(ST)(x) \end{aligned}$$

□

定理 1.8. *线性映射乘法的性质*

$$\begin{array}{cc} \text{结合律(associativity)} & (T_1T_2)T_3=T_1(T_2T_3) \\ \text{单位元(identity)} & \exists I\in\mathcal{L}(U,U)\rightarrow TI=T,\exists I\in\mathcal{L}(V,V): IT=T \\ \text{分配性质(distributive)} & (S_1+S_2)T=S_1T+S_2T \\ T\in\mathcal{L}(U,V),S\in\mathcal{L}(V,W) & S(T_1+T_2)=ST_1+ST_2 \end{array}$$

例 1.9. $D\in\mathcal{L}(\mathcal{P}(R),\mathcal{P}(R)),D(p)=p';T\in\mathcal{L}(\mathcal{P}(R),\mathcal{L}(\mathcal{P}(R)),T(p)=x^2p\rightarrow TD\neq DT$

证明.

$$\begin{array}{c} TD(p)=T(p')=x^2p' \\ DT(p)=D(x^2p)=x^2p'+2xp \end{array}$$

□

定理 1.10. $T\in\mathcal{L}(U,V),T(0)=0$

证明. $T(0)=T(0+0)=T(0)+T(0)\rightarrow 0=T(0)$ □

习题3.A

1. Proof:$b,c\in R.T\colon R^3\to R^2, T(x,y,z)=(2x-4y+3z+b, 6x+cxyz).T\in\mathcal{L}(R^3,R^2)\Leftrightarrow b=c=0$

$$\begin{gathered}T\in\mathcal{L}(R^3,R^2)\\ \to T((a_1,b_1,c_1)+(a_2,b_2,c_2))=T(a_1,b_1,c_1)+T(a_2,b_2,c_2)\\ T(\lambda(a,b,c))=\lambda T(a,b,c)\\ 2a_1-4b_1+3c_1+b+2a_2-4b_2+3c_2+b\\ =2(a_1+a_2)-4(b_1+b_2)+3(c_1+c_2)+b\\ \to 2b=b\to b=0\\ 6a_1+ca_1b_1c_1+6a_2+ca_2b_2c_2\\ =6(a_1+a_2)+c\,(a_1+a_2)(b_1+b_2)(c_1+c_2)\\ =6a_1+6a_2+c\,(\sum a_i\sum b_j\sum c_k)\\ \to c(a_1b_1c_2+a_1b_2c_1+a_2b_1c_1+a_1b_2c_2+a_2b_1c_2+a_2b_2c_1)=0\\ \to c=0\\ 6\lambda a_1+c\lambda a_1\,\lambda b_1\lambda c_1=\lambda(6a_1+ca_1b_1c_1)\\ c\lambda^3=c\lambda\to c=0\\ \to c=0\end{gathered}$$

2. Proof:$b,c\in R.T(\mathcal{P}(R),R^2), T(p)=\left(3p(4)+5p'(6)+bp(1)p(2),\int_{-1}^2 x^3p(x)\mathrm{d}x+c\sin(p(0))\right)$.

   $T\in\mathcal{L}(\mathcal{P}(R),R^2)\Leftrightarrow b=c=0$

$$\begin{gathered}p_1,p_2\in\mathcal{P}(R)\\ T(p_1+p_2)=3(p_1+p_2)(4)+5(p_1+p_2)'(6)+b\,(p_1+p_2)(1)(p_1+p_2)(2)\\ T(p_1)+T(p_2)=3p_1(4)+5p_1'(6)+bp_1(1)p_1(2)+3p_2(4)+5p_2'(6)+bp_2(1)p_2(2)\\ 3(p_1+p_2)(4)=3p_1(4)+3p_2(4)\\ 5(p_1+p_2)'=5(p_1'+p_2')=5p_1'+5p_2'\to 5(p_1+p_2)'(6)=5p_1'(6)+5p_2'(6)\\ b\,(p_1+p_2)(1)(p_1+p_2)(2)=bp_1(1)p_1(2)+bp_2(1)p_2(2)\\ b(p_1(1)+p_2(1))(p_1(2)+p_2(2))=b\,(p_1(1)p_1(2)+p_1(1)p_2(2)+p_2(1)p_1(2)+p_2(1)p_2(2))\\ \to b(p_1(1)p_2(2)+p_2(1)p_1(2))=0\\ \to b=0\end{gathered}$$

$$\begin{gathered}T(ap)=\int_{-1}^2 x^3ap(x)\mathrm{d}x+c\sin(ap(0))\\ aT(p)=a\left(\int_{-1}^2 x^3p(x)\mathrm{d}x+c\sin(p(0))\right)\\ \to c\sin(ap(0))=a\,c\sin(p(0))\\ \to c=0\end{gathered}$$

3. Proof:$T\in\mathcal{L}(F^n,F^m)\to\exists a_{i,j}\in F,\forall(x_1,\ldots,x_n)\in F^n$

   $T(x_1,\ldots,x_n)=(\sum_i a_{1,i}x_i,\ldots,\sum_i a_{n,i}x_i)$

$$T\in\mathcal{L}(F^n,F^m)\to T(x+y)=T(x)+T(y), T(\lambda x)=\lambda T(x)$$

书上的例子大概是证明空间F^n的所有元素都存在一个唯一的线性组合表示

???

4. Proof:$T\in\mathcal{L}(V,W), v_1,\ldots,v_n\in V, Tv_1,\ldots,Tv_n\in W$线性无关$\to v_1,\ldots,v_n$线性无关

$Tv_1,\ldots,Tv_n$线性无关$\to\lambda_1T(v_1)+\cdots+\lambda_nT(v_1)=0\to\lambda_i=0$

$$\begin{gathered}T(\lambda_1v_1)+\cdots+T(\lambda_nv_n)=0\to\lambda_i=0\\ T(\lambda_1v_1+\cdots+\lambda_nv_n)=0\to\lambda_i=0\end{gathered}$$

$T(0)=0\to v_1,\ldots,v_n$有可能线性无关。　　　　差任意$T(x\neq0)\neq0$

$\exists a_i\neq0, T(a_1v_1+\cdots+a_nv_n)=$

$a_1T(v_1)+\cdots+a_nT(v_n)$

$T(\boldsymbol{v})$线性无关$\to T(v_i)\neq0$

$a_i\neq0\to T(a_1v_1+\cdots+a_nv_n)\neq0\to$只有0使得$T(0)=0$

$\to v_1,\ldots,v_n$线性无关

5. Proof: $\mathcal{L}(V,W)$是向量空间

$$\forall f,g\in\mathcal{L}(V,W).f+g=f(x)+g(x)=g(x)+f(x)=g+f \qquad \text{交换律}$$
$$\forall f,g,h\in\mathcal{L}(V,W,X).(f+g)+h=f+(g+h) \qquad \text{结合律}$$
$$\forall\alpha,\beta\in F,(\alpha\beta)f=\alpha(\beta f)$$
$$f(V)\to 0: \qquad \text{加法0}$$
$$\forall x,y\in V\to f(x+y)=0=0+0=f(x)+f(y)$$
$$f(ax)=0=a0=a(f(x))$$
$$\to f\in\mathcal{L}(V,W)$$
$$f+g=0+g(x)=g(x)$$
$$\forall f\in\mathcal{L}(V,W),g=-(f(x)) \qquad \text{加法}-1$$
$$g(x+y)=-f(x+y)=-f(x)-f(y)=g(x)+g(y)$$
$$g(ax)=-f(ax)=a(-f(x))=ag(x)=ag$$
$$\to g\in\mathcal{L}(V,W)$$
$$(f+(-f))(x)=f(x)+-f(x)=0$$
$$I_W(w)=w.I_W\in\mathcal{L}(V,W) \qquad \text{数乘1}$$
$$\forall f\in\mathcal{L}(V,W),(I_Wf)(x)=I_W(f(x))=I_W(w)=w=f(x)=f$$
$$a(f+g)(x)=a(f(x)+g(x))=af(x)+ag(x)=a(f(x))+a(g(x)) \qquad \text{分配律}$$
$$(a+b)f=(a+b)f(x)=af(x)+bf(x)=af+bf$$
$$\to\mathcal{L}(V,W)\text{是向量空间}$$

6. Proof:1.8

$$\forall T_1\in\mathcal{L}(U,V),\forall T_2\in\mathcal{L}(V,W),\forall T_3\in\mathcal{L}(W,S) \quad \text{结合律}$$
$$(T_1T_2)T_3=(T_1(T_2))(T_3(x))$$
$$=T_1(T_2(T_3(x)))$$
$$T_1(T_2T_3)=T_1(T_2(T_3(x))) \qquad \text{左结合}$$

$$\forall T\in\mathcal{L}(U,V),I_U(x)\to x \qquad \text{单位元}$$
$$I_U(x+y)=x+y=I_U(x)+I_U(y)$$
$$I_U(\lambda x)=\lambda x=\lambda(I_U(x))$$
$$\to I_U\in\mathcal{L}(U,U)$$
$$\forall u\in U,UI=U(I(u))=u$$
$$I_V(x)\to x,I_V\in\mathcal{L}(V,V)$$
$$IU(u)=I(U(u))=I(w)=w=U(u)$$

7. Proof:$\dim V=1\wedge T\in\mathcal{L}(V,V),\exists\lambda\in F\to\forall v\in V,Tv=\lambda v$

$$\dim V=1\to\exists b\in V,\forall v\in V,\exists\lambda\to\lambda b=v$$
$$T\in\mathcal{L}(V,V).T(x+y)=T(x)+T(y)$$
$$T(x+y)=T(\lambda_1b+\lambda_2b)=\lambda_1T(b)+\lambda_2T(b)$$
$$=(\lambda_1+\lambda_2)T(b)$$
$$T((\lambda_1+\lambda_2)b)=(\lambda_1+\lambda_2)T(b)$$
$$T(\lambda b)=\lambda T(b)$$
$$T(b)\in V\to T(b)=\mu b$$
???

8. Example:$\varphi:R^2\to R,\forall a\in R,\forall v\in R^2\to\varphi(av)=a\varphi(v)\wedge\varphi\notin\mathcal{L}(R^2,R)$

$$\varphi(\lambda x,\lambda y)=\lambda\varphi(x,y)$$
$$\varphi(x_1+x_2,y_1+y_2)\neq\varphi(x_1,y_1)+\varphi(x_2,y_2)$$
$$\varphi(x,y)=x,x\geqslant 0;-x,x<0.$$

由于标乘
构造不同斜率的区域

$$\varphi(ax,ay)=ax,x\geqslant 0;-\mathrm{ax},x<0$$
$$=\alpha\varphi(x,y)$$
$$\varphi(x_1+x_2,y_1+y_2)=(x_1+x_2),x_1+x_2\geqslant 0;-(x_1,x_2),(x_1+x_2)<0$$
$$x_1=1,x_2=-2\to x_1+x_2=-1<0\to\varphi(x_1+x_2)=1$$
$$\varphi(1,y)+\varphi(-2,y)=1+2=3\neq 1$$

$$\varphi(z)=|z| \qquad \text{复空间}$$

9. Example:$\varphi: C \to C, \forall x, y \in C \to \varphi(x+y) = \varphi(x) + \varphi(y) \wedge \varphi \notin \mathcal{L}(C, C)$

$$\begin{gathered}\varphi(x+y) = \varphi(x) + \varphi(y) \\ \varphi(ax) \neq a\varphi(x) \\ ???\end{gathered}$$

各向异性

Remark: R^R里也有这样的函数。不过目前不足证明它存在... 直觉yyds

10. Proof:$U \subset V \wedge U \neq V. S \in \mathcal{L}(U, W) \wedge S \neq 0$

$$\begin{aligned} T: V \to W \quad T(u) &= S(u) \qquad u \in U \\ &= 0 \quad u \in V \wedge v \notin U \end{aligned}$$

Proof: $T \notin \mathcal{L}(V, W)$

$$\begin{gathered}\forall v \in V, v = au + bw \\ T(v) = T(au + bw), b \neq 0 \to au + bw \notin U \to T(au + bw) = 0 \\ T(au) + T(bw) = T(au). a \neq 0 \to \exists a \to S(au) \neq 0 \\ 0 = T(au + bw) \neq T(au) + T(bw) \neq 0\end{gathered}$$

这a, b, w, u全是向量

11. Proof:$\dim V < \infty, U \subset V, S \in \mathcal{L}(U, W) \to \exists T \in \mathcal{L}(V, W), \forall u \in U, T(u) = S(u)$

$$\begin{gathered}\forall v \in V, v = au + bw \\ T(v) = T(au + bw) \\ = T(au) + T(bw) \\ \forall \mu \in U, \mu = au \to T(au) = S(\mu) \\ \text{若存在一个}W\text{上的线性映射}T_W \\ T = S(au) + T_W(bw), v \notin U \wedge v \notin W \\ T(v) = T(au + bw) = S(au) + T_W(bw). a, b\text{是向量}\end{gathered}$$

definition

$$\begin{gathered}T(v_1 + v_2) = T(a_1 u + a_2 u, b_1 w + b_2 w) = S(a_1 u + a_2 u) + T_W(b_1 w + b_2 w) \\ = S(a_1 u + a_2 u) + T_W(b_1 w + b_2 w) \\ = S(a_1 u) + T_W(b_1 w) + S(a_2 u) + T_W(b_2 w) \\ = T(a_1 u, b_1 w) + T(a_2 u, b_2 w)\end{gathered}$$

$$\begin{gathered}T(\lambda v) = T(\lambda au + \lambda bw) \\ = S(\lambda au) + T_W(\lambda bw) \\ = \lambda S(au) + \lambda T_W(bw) \\ = \lambda(T(au + bw)) \\ \to T \in \mathcal{L}(U, W)\end{gathered}$$

$$\begin{gathered}T_W\text{的存在性：}T_W(a_1 v_1 + \ldots + a_n v_n) = a_1 b_{w1}.(\text{基的第一个元素}) \\ T_W(a_1 v + a_2 v) = (a_1 + a_2) b_{w1} = a_1 b_{w1} + a_w b_{w1} = T_W(a_1 v) + T_W(a_2 w) \\ T_W(\lambda a_1 v) = \lambda a_1 v = \lambda(a_1 v) = \lambda T_W(a_1 v) \\ \to T_W \in \mathcal{L}(\bar{U}, W)\end{gathered}$$

$\dim \bar{U} = 0$的情况是平凡的

12. Proof:$0 < \dim V < \infty, \forall n \in N^+, \dim W > n \to \forall n \in N^+, \dim(\mathcal{L}(V, W)) > n$

$$\begin{gathered}\forall f \in \mathcal{L}(V, W). \\ \text{设}f\text{属于有限维} \to f = \operatorname{span}(\boldsymbol{b}).\text{length}\boldsymbol{b} = n < \infty\end{gathered}$$

???

13. Proof:$\boldsymbol{v} \in V \wedge \boldsymbol{v}$线性相关$, W \neq \{0\}$.Proof: $\exists \boldsymbol{w} \in W \to \forall T \in \mathcal{L}(V, W), \exists i \in 1 \ldots n, T(v_i) \neq w_i$

$$
\begin{gathered}
\text{设}w_i\text{线性无关.这样的向量在}W\text{中是存在的} \\
\text{设}T(\boldsymbol{v})=\boldsymbol{w} \\
\leftarrow T(\sum a_i v_i)=\sum a_i T(v_i)=\sum a_i w_i \\
w_i\text{线性无关}\rightarrow 0=\sum a_i w_i\rightarrow a_i=0 \\
\exists a_i\neq 0\rightarrow \sum a_i v_i=0 \\
T(0)=0\rightarrow T(\sum a_i v_i)=\sum a_i w_i\rightarrow a_i=0\text{矛盾} \\
\rightarrow T(\boldsymbol{v})\neq\boldsymbol{w}
\end{gathered}
$$

14. Proof:$2\leqslant\dim V<\infty$, Proof: $\exists S,T\in\mathcal{L}(V,V)\rightarrow ST\neq TS$

$$
\begin{gathered}
\mathrm{dimV}\geqslant 2\rightarrow\exists\boldsymbol{b},\mathrm{length}\,\boldsymbol{b}\geqslant 2,\mathrm{span}(b)=V. \\
T_1(x_1,x_2,\ldots)=b_1,\ldots,b_1 \\
T_1(x_1+y_1,x_2+y_2,\ldots)=x_1+y_1=T_1(x_1,\ldots)+T_1(x_2,\ldots) \\
T_1(ax_1,\ldots)=ax_1=aT(x_1,\ldots) \\
\rightarrow T_1\in\mathcal{L}(V,V) \\
T_1T_2(1,2,\ldots)=T_1(2,2,\ldots) \\
T_2T_1(1,2,\ldots)=T_2(1,1,\ldots) \\
\rightarrow T_1T_2\neq T_2T_1
\end{gathered}
$$

2 零空间与值域

2.1 零空间与单射性

定义 2.1. *零空间（null space），null T。 核*

$$
\begin{gathered}
T\in\mathcal{L}(V,W).T\text{的零空间值}V\text{中被}T\text{映到}0\in W\text{的向量构成的子集} \\
\mathrm{null}\,T=\{v\colon v\in V\wedge Tv=0\}
\end{gathered}
$$

定理 2.2. *零空间是子空间*

证明.

1. $T\in\mathcal{L}(V,W)\rightarrow T(0)=0\rightarrow 0\in\mathrm{null}\,T$
2. $\forall x,y\in\mathrm{null}\,T.T(x+y)=T(x)+T(y)=0+0=0$
3. $\forall x\in\mathrm{null}\,T,\forall\lambda\in F,T(\lambda x)=\lambda T(x)=\lambda 0=0$

$1.2.3.\rightarrow\mathrm{null}\,T$是子空间 □

例 2.3. 一些零空间

1. $\mathbf{0}\in\mathcal{L}(V,W).\ \mathbf{0}(v)=0\rightarrow\mathrm{null}\,\mathbf{0}=V$
2. $\varphi\in\mathcal{L}(C^3,F).\varphi(z_1,z_2,z_3)=z_1+2z_2+3z_3.\ \mathrm{null}\,\varphi=\{(z_1,z_2,z_3)\in C^3\colon z_1+2z_2+3z_3=0\}$
   $\dim(\mathrm{null}\,\varphi)=2,\mathrm{null}\,\varphi=\mathrm{span}((-2,1,0),(-3,0,1))$
3. $D\in\mathcal{L}(\mathcal{P}(R),\mathcal{P}(R)).Dp=p'.\ \mathrm{null}\,D=\{f(x)=C\}$
4. $T\in\mathcal{L}(\mathcal{P}(R),\mathcal{P}(R)).Tp=x^2p.x\in R\rightarrow x^2p(x)=0\rightarrow p(x)=0\rightarrow\mathrm{null}\,T=\{0\}$
5. $T\in\mathcal{L}(F^\infty,F^\infty).T(x_1,x_2,\ldots)=(x_2,x_3,\ldots).T(x_1,x_2\ldots)=\mathbf{0}\rightarrow x_1\neq 0,x_2=\ldots=0$
   $\rightarrow\mathrm{null}\,T=\{(a,0,0,\ldots)\colon a\in F\}$

定义 2.4. *映射的单性（injective）：$Tu=Tv\rightarrow u=v$. 称T是单的*

定理 2.5. *线性映射. 单性*$\Leftrightarrow\mathrm{null}\,T=\{0\}$

证明.

$$
\begin{gathered}
T \in \mathcal{L}(V, W) \\
T\text{单} \to \operatorname{null} T = \{0\} \\
T\text{单}: T(x) = T(y) \to x = y \\
0 = T(0) = T(x) \to x = 0 \\
\to \operatorname{null} T = 0
\end{gathered}
$$

$$
\begin{gathered}
\operatorname{null} T = \{0\} \to T\text{单} \\
\forall x, y \in V \wedge T(x) = T(y) \\
0 = T(x) - T(y) = T(x - y) \to x - y = 0 \\
\to x = y \\
\to T\text{是单的}
\end{gathered}
$$

□

2.2 值域与满性

定义 2.6. *值域(range). range T; 像*

$T: V \to W. \operatorname{range} T = \{Tv: v \in V\}$

定理 2.7. *线性映射：值域是子空间*

证明.

$$
\begin{gathered}
T \in \mathcal{L}(V, W) \to \operatorname{range} T\text{是}W\text{的子空间} \\
0 \in V \to T(0) \in \operatorname{range} T \to 0 \in \operatorname{range} T \\
\forall T(x), T(y) \in \operatorname{range} T. T(x) + T(y) = T(x + y) \\
x + y \in V \to T(x + y) \in W \\
\forall T(x) \in \operatorname{range} T. \forall \lambda \in F. \lambda T(x) = T(\lambda x) \\
\lambda x \in V \to T(\lambda x) = \lambda T(x) \in \operatorname{range} T \\
\to \operatorname{range} T\text{是}W\text{的子空间}
\end{gathered}
$$

□

例 2.8. 一些线性映射的值域

1. $T \in \mathcal{L}(V, W). T(v) = 0. \operatorname{range} T = \{0\}$
2. $T \in \mathcal{L}(R^2, R^3). T(x, y) = (2x, 5y, x + y). \operatorname{range} T = \{(2x, 5x, x + y): x, y \in R\}$
   $\dim(\operatorname{range} T) = 2. \operatorname{range} T = \operatorname{span}((2, 0, 1), (0, 5, 1))$
3. $D \in \mathcal{L}(\mathcal{P}(R), \mathcal{P}(R)). Dp = p'. Dp \in \mathcal{P}(R) \to \operatorname{range} D = \mathcal{P}(R)$

定义 2.9. *映射的满性(subjective)，映上. $T: V \to W. \operatorname{range} T = W$称$T$为满的*

Remark: *线性映射的满性与W空间有关*

例 2.10.

$$
\begin{gathered}
D \in \mathcal{L}(\mathcal{P}_5(R), \mathcal{P}_5(R)). Dp = p' \\
\operatorname{range} D = \mathcal{P}_4(R) \to D\text{不是满的} \\
S \in \mathcal{L}(\mathcal{P}_5(R), \mathcal{P}_4(R)). Sp = p' \\
\operatorname{range} S = \mathcal{P}_4(R). \text{所以}S\text{是满的}
\end{gathered}
$$

2.3 线性映射基本定理

定理 2.11. *线性映射基本定理*

V是有限维的.$T\in\mathcal{L}(V,W)\rightarrow\operatorname{range}T,\operatorname{null}T$是有限维的

$$\dim V=\dim\operatorname{null}T+\dim\operatorname{range}T$$

证明.

$$\begin{gathered}
\operatorname{null}T\text{是}V\text{的子空间}\rightarrow\operatorname{null}T\text{是有限维的}\\
\rightarrow\boldsymbol{u}\text{是}\operatorname{null}T\text{的基.}\\
\boldsymbol{u}\text{有线性无关组的扩充}\boldsymbol{v}\rightarrow V=\operatorname{span}(\boldsymbol{u},\boldsymbol{v})\qquad\boldsymbol{v}=0\text{是平凡的}\\
\Leftrightarrow T\boldsymbol{v}\text{是}\operatorname{range}T\text{的基}\\
\forall v\in V,v=\boldsymbol{a}\boldsymbol{u}+\boldsymbol{b}\boldsymbol{v}\\
Tv=T(\boldsymbol{a}\boldsymbol{u})+T(\boldsymbol{b}\boldsymbol{v})=T(\boldsymbol{b}\boldsymbol{v})\\
\rightarrow\operatorname{range}T=\operatorname{span}(\boldsymbol{v})\rightarrow\dim\operatorname{range}T<\operatorname{length}\boldsymbol{v}\\
\text{即}\operatorname{range}T\text{是有限维的}
\end{gathered}$$

□

$$\begin{gathered}
0=\boldsymbol{\lambda}T\boldsymbol{v}\rightarrow0=T(\boldsymbol{\lambda}\boldsymbol{v})\\
\boldsymbol{\lambda}\boldsymbol{v}\in\operatorname{null}T\\
\rightarrow\boldsymbol{\lambda}\boldsymbol{v}\in\operatorname{span}\boldsymbol{u}\\
\boldsymbol{u},\boldsymbol{v}\text{线性无关}\rightarrow\boldsymbol{\lambda}=0\\
\rightarrow T\boldsymbol{v}\text{线性无关}\\
\rightarrow\dim\operatorname{range}T=\operatorname{length}v\\
\rightarrow\dim V=\dim\operatorname{null}T+\dim\operatorname{range}T
\end{gathered}$$

定理 2.12. *高维空间向低维空间的线性映射不是单的*

证明.

$$\begin{aligned}
T\in\mathcal{L}(V,W).\dim V>\mathrm{dimW}\\
\dim\operatorname{null}T=\dim V-\dim\operatorname{range}T\\
\geqslant\dim V-\dim W\\
>0
\end{aligned}$$

□

定理 2.13. *低维空间向高维空间的线性映射不是满的*

证明.

$$\begin{aligned}
T\in\mathcal{L}(V,W).\dim V<\dim W\\
\dim\operatorname{range}T=\dim V-\dim\operatorname{null}T\\
\leqslant\dim V\\
<\dim W
\end{aligned}$$

□

例 2.14. 线性映射观点下齐次线性方程组是否有非零解问题

$$\begin{gathered}
m,n>0,\text{齐次线性方程组}\\
\textstyle\sum_{k=1}^n a_{1,k}x_k=0\\
\textstyle\sum_{k=1}^n a_{2,k}x_k=0\\
\vdots\\
\textstyle\sum_{k=1}^n a_{m,k}x_k=0\\
\text{是否具有非0解?}\\
T:F^n\rightarrow F^m:T(x_1,\ldots,x_n)=(\textstyle\sum_k a_{1,k}x_k,\ldots,\sum_k a_{m,k}x_k)\\
T(\boldsymbol{x}+\boldsymbol{y})=(\boldsymbol{a}\boldsymbol{x}+\boldsymbol{a}\boldsymbol{y})=\boldsymbol{a}\boldsymbol{x}+\boldsymbol{a}\boldsymbol{y}=T(\boldsymbol{x})+T(\boldsymbol{y})\\
T(\lambda\boldsymbol{x})=\boldsymbol{a}(\lambda\boldsymbol{x})=\lambda(\boldsymbol{a}\boldsymbol{x})=\lambda T(\boldsymbol{x})\\
\rightarrow T\in\mathcal{L}(F^n,F^m).\\
\text{齐次方程有非0解}\Leftrightarrow\operatorname{null}T\neq\{0\}\\
\operatorname{null}>0\text{的一个条件是}\dim V>\dim W\\
\text{即}n>m.\text{方程数小于未知数数量}
\end{gathered}$$

例 2.15. 线性映射观点下非齐次方程无解问题

$$
\begin{gathered}
m,n>0.\text{非齐次线性方程组}\\
\sum_{k=1}^{n} a_{1,k}x_k=b_1\\
\sum_{k=1}^{n} a_{2,k}x_k=b_2\\
\vdots\\
\sum_{k=1}^{n} a_{m,k}x_k=b_m\\
T(\boldsymbol{x})=\boldsymbol{b}\\
\text{若}\exists\boldsymbol{b}\in F^m,\text{使得}\forall\boldsymbol{x}\in V,T(\boldsymbol{x})\neq\boldsymbol{b}\text{则为矛盾方程组，无解}\\
\text{存在}\boldsymbol{b}\neq 0\text{使得方程组无解}\Leftrightarrow\operatorname{range}T\neq F^m\\
\text{一个条件}:\dim F^n<\dim F^m\text{时},\text{不能映满}F^m\\
\text{即}m>n.\text{方程组数量超过变量数}
\end{gathered}
$$

习题3.B

1. Example: $T\in\mathcal{L}(V,W).\dim\operatorname{null}T=3.\dim\operatorname{range}T=2$

$$
\begin{gathered}
V=R^5.T(x_1,x_2,x_3,x_4,x_5)=(x_1+x_2+x_3,x_4+x_5)\\
\operatorname{null}T=\{(x_1,x_2,x_3,x_4,x_5):x_1+x_2+x_3=0\wedge x_4+x_5=0\}\\
\operatorname{range}T=\{(x,y):x,y\in F\}
\end{gathered}
$$

2. Proof: V是向量空间.$S,T\in\mathcal{L}(V,V)\to\operatorname{range}S\subset\operatorname{null}T$. Proof: $(ST)^2=\mathbf{0}$

$$
\begin{gathered}
(ST)^2=STST\\
T(V)=\operatorname{range}T\\
S(T(V))=S(\operatorname{range}T)\subset\operatorname{range}S\subset\operatorname{null}T\\
T(ST(x))=T(S(T(x)))\subset T(S(\operatorname{range}T))\subset T(\operatorname{range}S)\subset T(\operatorname{null}T)=0\\
S(T(S(T(V))))\subset S(0)=0
\end{gathered}
$$

3. $v_1,\ldots,v_m$是V中的向量组.$T\in\mathcal{L}(F^m,V).T(z_1,\ldots,z_m)=\sum z_iv_i$

   a. T的什么性质$\Leftrightarrow\operatorname{span}(\boldsymbol{v})=V$

   $$\operatorname{span}(\boldsymbol{v})=V\to T(\boldsymbol{v})=V\text{即满性}$$

   b. T的什么性质$\Leftrightarrow v_1,\ldots,v_m$线性无关

   $$
   \begin{gathered}
   v_1,\ldots.v_m\text{线性无关}\to T(\boldsymbol{v})\text{线性无关}\\
   0=\boldsymbol{a}\boldsymbol{v}\to\boldsymbol{a}=0\\
   T(\boldsymbol{a}\boldsymbol{v})=\boldsymbol{a}T(\boldsymbol{v})\to\boldsymbol{a}=0\\
   \to\operatorname{null}T=\{0\}\\
   \to T\text{是单的}
   \end{gathered}
   $$

4. Proof: $\{T\in\mathcal{L}(R^5,R^4):\dim\operatorname{null}T>2\}$不是$\mathcal{L}(R^5,R^4)$的子空间

$$
\begin{gathered}
T_1=(\boldsymbol{x})\to(x_1,x_2,0,0),T_2(\boldsymbol{x})=(0,0,x_3,x_4)\\
T_1(x)=0\to x=(0,0,x,y,z).\dim\operatorname{null}T_1=3\\
T_2(x)=0\to x=(x,y,0,0,z).\dim\operatorname{null}T_2=3\\
(T_1+T_2)(\boldsymbol{x})=(x_1,x_2,x_3,x_4)\\
\operatorname{null}(T_1+T_2)=\{(0,0,0,0,x)\}\\
\dim\operatorname{null}(T_1+T_2)=1\notin T
\end{gathered}
$$

5. Example: $T \in \mathcal{L}(R^4, R^4)$. range T = null T

$$\begin{gathered}\dim R^4 = \dim \operatorname{range} T + \dim \operatorname{null} T \wedge \operatorname{range} T = \operatorname{null} T\\ \rightarrow \dim \operatorname{range} T = \dim \operatorname{null} T = 2\\ \operatorname{range} T = \operatorname{null} T \rightarrow TT(R^4) = T(\operatorname{range} T) = T(\operatorname{null} T) = 0\\ T(\boldsymbol{x}) = (0, 0, x_1, x_2)\\ T(\boldsymbol{x}) = 0 \rightarrow x_1 = x_2 = 0. \operatorname{null} T = (0, 0, x, y)\\ \operatorname{range} T = (0, 0, x, y)\end{gathered}$$

6. Proof: $\forall T \in \mathcal{L}(R^5, R^5)$. Proof: range $T \neq$ null T

$$\begin{gathered}\dim R^5 = \dim \operatorname{range} T + \dim \operatorname{null} T\\ \operatorname{range} T = \operatorname{null} T \rightarrow \dim \operatorname{range} T = \dim \operatorname{null} T\\ \rightarrow \frac{5}{2} = \dim \operatorname{range} T = \dim \operatorname{null} T\end{gathered}$$

这超出了向量空间的定义范围

7. Proof: V, W是有限维的.$2 \leqslant \dim V \leqslant \dim W$. Proof: $\{T \in \mathcal{L}(V, W): T$不单$\}$不是$\mathcal{L}(V, W)$的子空间

$$\begin{gathered}x, y \in V, x \neq y \wedge T(x) = T(y)\\ \forall f \in T, \operatorname{null} f \neq \{0\}\\ f(x, y) = (x, 0). g(x, y) = (0, y). f, g \text{不单}\\ (f + g)(x, y) = (x, y) = I(x, y)\text{是单的.}\\ \rightarrow f + g \notin T \rightarrow T\text{不是子空间}\end{gathered}$$

8. Proof:V, W是有限维的.$\dim V \geqslant \dim W \geqslant 2$. Proof: $\{T \in \mathcal{L}(V, W): T$不满$\}$不是$\mathcal{L}(V, W)$的子空间

$$\begin{gathered}f(x, y) = (x, 0). g(x, y) = (0, y). f, g \text{不满}\\ (f + g)(x, y) = I(x, y)\text{满}\\ \rightarrow f + g \notin T \rightarrow T\text{不是子空间}\end{gathered}$$

9. Proof:$T \in \mathcal{L}(V, W)$是单的.$v_1, \ldots, v_n$在V中线性无关. Proof: $Tv_1, \ldots, Tv_n$在W中线性无关

$$\begin{gathered}0 = \boldsymbol{\lambda} T(\boldsymbol{v})\\ = T(\boldsymbol{\lambda v})\\ T\text{单} \rightarrow \operatorname{null} T = \{0\} \rightarrow 0 = T(\boldsymbol{\lambda v}) \rightarrow \boldsymbol{\lambda} = 0\\ \rightarrow T(\boldsymbol{v})\text{线性无关}\end{gathered}$$

10. Proof:span$(v_1, \ldots, v_n) = V$.$T \in \mathcal{L}(V, W)$. Proof: span$(Tv_1, \ldots, Tv_n)$ = range T

$$\begin{gathered}\forall v \in V. v = \boldsymbol{\lambda v}\\ T(\boldsymbol{\lambda v}) = \boldsymbol{\lambda} T(\boldsymbol{v}) \rightarrow \operatorname{range} T(V) = \operatorname{span} T(\boldsymbol{v})\end{gathered}$$

11. Proof:$S_1, \ldots, S_n$都是单的线性映射.$S_1 S_2 \ldots S_n$有意义. Proof: $S_1 S_2 \ldots S_n$是单的

$$\begin{gathered}\operatorname{null} S = \{0\}, \operatorname{null} T = \{0\}\\ \forall v \neq 0 \rightarrow T(v) \neq 0\\ \forall u \neq 0 \rightarrow S(u) \neq 0\\ ST(v) = S(T(v))\\ \neq S(0) = 0\\ \rightarrow ST\text{是单的.}\\ \rightarrow S_1 S_2 \cdots S_n\text{是单的.}\end{gathered}$$

12. Proof:V是有限维的.$T \in \mathcal{L}(V, W)$. Proof: $\exists U \subset V \rightarrow U \cap \operatorname{null} T = \{0\} \wedge \operatorname{range} T = \{Tu: u \in U\}$

$$U\text{的一个基为}\boldsymbol{u}$$
$$\operatorname{null} T \subset T.\operatorname{span}(\boldsymbol{t}) = \operatorname{null} T$$
$$V = \operatorname{span}(\boldsymbol{u}, \boldsymbol{v}) = \operatorname{span}(\boldsymbol{t}, \boldsymbol{s})$$
$$\forall v \in V.v = \boldsymbol{a}\boldsymbol{u} + \boldsymbol{b}\boldsymbol{v}$$
$$T(\boldsymbol{a}\boldsymbol{u} + \boldsymbol{b}\boldsymbol{v}) = T(\boldsymbol{a}\boldsymbol{u}) + T(\boldsymbol{b}\boldsymbol{v})$$
$$T(\boldsymbol{a}\boldsymbol{u}) = \operatorname{range} T \to \operatorname{span}(T(\boldsymbol{u})) = \operatorname{range} T$$
$$\to \dim U \geqslant \dim \operatorname{range} T \qquad \operatorname{range} T = \operatorname{span}(T(\boldsymbol{u}))$$

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$
$$\dim V \geqslant \dim \operatorname{null} T + \dim U$$
$$\to \dim U \leqslant \dim \operatorname{range} T \qquad V = \operatorname{null} T \oplus U \oplus \text{else}$$

$$\to \dim U = \dim \operatorname{range} T$$
$$V = \operatorname{span}(\boldsymbol{u}, \boldsymbol{v}) = \operatorname{span}(\boldsymbol{t}, \boldsymbol{s})$$
$$\text{length}\, \boldsymbol{u} = \text{length}\, \boldsymbol{s} \wedge \operatorname{span}(\boldsymbol{v}) = \operatorname{span}(\boldsymbol{t}) = \operatorname{null} T$$
$$\to \boldsymbol{u} \in \operatorname{span}(\boldsymbol{s})$$
$$\to U = V - \operatorname{null} T$$

13. Proof:$T \in \mathcal{L}(F^4, F^2)$. $\operatorname{null} T = \{(x_1, x_2, x_3, x_4) : x_1 = 5x_2, x_3 = 7x_4\}$. Proof: T是满的

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$
$$4 = 2 + \dim \operatorname{range} T \to \dim \operatorname{range} T = 2$$
$$\dim F^2 = 2 = \dim \operatorname{range} T$$
$$\to \text{满} \qquad \dim V = \dim S \to V \cong S$$

14. Proof: $U \subset R^8$.$\dim U = 3$.$T \in \mathcal{L}(R^8, R^5)$. $\operatorname{null} T = U$. Proof: T满

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$
$$8 = 3 + \dim \operatorname{range} T$$
$$\dim \operatorname{range} T = 5 = \dim R^5$$
$$\to T\text{满}$$

15. Proof:$\forall T \in \mathcal{L}(F^5, F^2)$. Proof: $\operatorname{null} T \neq \{(x_1, x_2, x_3, x_4, x_5) : x_1 = 3x_2, x_3 = x_4 = x_5\}$

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$
$$5 = 2 + \dim \operatorname{range} T \to \dim \operatorname{range} T = 3$$
$$3 = \dim \operatorname{range} T \geqslant \dim F^2 = 2$$
$$\text{矛盾}$$

16. Proof:$\exists T \in \mathcal{L}(V, W)$. $\dim \operatorname{null} T < \infty, \dim \operatorname{range} T < \infty$. Proof: $\dim V < \infty$

$$\dim \operatorname{null} T = n, \dim \operatorname{range} T = r$$
$$V = \operatorname{null} T \oplus S$$
$$\boldsymbol{s} \in S, T(\boldsymbol{s}) \in \operatorname{range} T \qquad \boldsymbol{s}\text{不是基，}S\text{还不是有限维}$$
$$\to \operatorname{range} T = \operatorname{span}(T(\boldsymbol{s}))$$
$$\dim \operatorname{range} T = r \to \boldsymbol{\lambda} T(\boldsymbol{s}) = 0 \to \boldsymbol{\lambda} = 0$$
$$T(\boldsymbol{\lambda}\boldsymbol{s}) = 0 \to \boldsymbol{\lambda} = 0$$
$$\to \boldsymbol{s}\text{线性无关.}$$

$$\forall u \in \operatorname{range} T.u = T(s)$$
$$\text{若}\exists s \in S \wedge Ts \notin \operatorname{range} T\text{与}\operatorname{range} T\text{矛盾}$$
$$\to \operatorname{range} T\text{的基}\boldsymbol{r}\text{的原像构成}S\text{的一个基} \qquad \text{只论存在性}$$
$$\to \dim S = \dim \operatorname{range} T \to S\text{有限维}$$
$$V = \operatorname{span}(\boldsymbol{s}, \boldsymbol{t}) = n + r < \infty$$

17. Proof:$\dim V<\infty, \dim W<\infty$. Proof: $\exists T\in\mathcal{L}(V,W)\wedge T$单$\Leftrightarrow \dim V\leqslant \dim W$

$$\begin{gathered}
T\text{单}\rightarrow \operatorname{null} T=\{0\}\rightarrow \dim \operatorname{null} T=0\\
\dim V=\dim \operatorname{null} T+\dim \operatorname{range} T\\
\rightarrow \dim V=\dim \operatorname{range} T\\
\operatorname{range} T\subset W\rightarrow \dim \operatorname{range} T\leqslant \dim W\\
\rightarrow \dim V\leqslant \mathrm{dimW}
\end{gathered}$$

$$\begin{gathered}
\dim V\leqslant \dim W:\\
V=\operatorname{span}(\boldsymbol{v}). W=\operatorname{span}(\boldsymbol{w})\\
i\leqslant \dim V: T(v_i)=w_i\\
\forall v\in V, T(v)=T(\boldsymbol{a}\boldsymbol{v})=\boldsymbol{a}T(\boldsymbol{v})\\
\rightarrow T\in\mathcal{L}(V,W)\\
\forall T(v)\neq T(u)\\
\rightarrow T(v)-T(u)=T(v-u)\neq 0\\
\rightarrow u\neq u\\
\rightarrow T\text{单}
\end{gathered}$$

这里需要的条件应该更强一点

$\|v_i\|=\lambda\|w_i\|$.否则不构成线性映射

18. Proof:$\dim V<\infty, \dim W<\infty$. Proof: $\exists T\in\mathcal{L}(V,W)\wedge T$满$\Leftrightarrow \dim V\geqslant \dim W$

$$\begin{gathered}
T\text{满}\rightarrow \operatorname{range} T=W\rightarrow \dim \operatorname{range} T=\dim W\\
\dim V=\dim \operatorname{null} T+\dim \operatorname{range} T\\
=\dim \operatorname{null} T+\dim W\\
\rightarrow \dim V\geqslant \dim W
\end{gathered}$$

$$\begin{gathered}
\dim V\geqslant \dim W:\\
i\leqslant \dim W: T(v_i)=w_i\\
\dim V\geqslant i>\dim W: T(v_i)=0\\
T\in\mathcal{L}(V,W).\\
\boldsymbol{v}\in V. \operatorname{span}(T(\boldsymbol{v}))=W\\
\rightarrow T\text{满}
\end{gathered}$$

19. Proof:V,W有限维.$U\subset V$. Proof: $\exists T\in\mathcal{L}(V,W), \operatorname{null} T=U\Leftrightarrow \dim U\geqslant \dim V-\dim W$

$$\begin{gathered}
T\in\mathcal{L}(V,W). \operatorname{null} T=U.\\
\dim V=\dim \operatorname{null} T+\dim \operatorname{range} T\\
=\dim U+\dim \operatorname{range} T\\
\leqslant \dim U+\dim W\\
\rightarrow \dim V-\dim W\leqslant \dim U
\end{gathered}$$

$$\begin{gathered}
\dim V-\dim W\leqslant \dim U\\
\rightarrow \dim V\leqslant \dim W+\dim U\\
\dim V=\dim \operatorname{range} T+\dim \operatorname{null} T\\
U\subset V\rightarrow \dim U\leqslant \dim V\\
U=\operatorname{span}(\boldsymbol{u}). V=\operatorname{span}(\boldsymbol{u},\boldsymbol{v})\\
\forall\boldsymbol{\lambda}, T(\boldsymbol{\lambda}\boldsymbol{u})=0\rightarrow \operatorname{span}(T(\boldsymbol{u}))\subset \operatorname{null} T\\
\dim V-\dim U\leqslant \dim W\\
\rightarrow \exists T(V-U)\rightarrow W\text{的单射}\\
\rightarrow \forall t\in V-U, t\neq 0\rightarrow T(t)\neq 0\\
\rightarrow V-U\not\subset \operatorname{null} T\\
V=U\cup(V-U)\rightarrow \operatorname{null} T=U
\end{gathered}$$

20. Proof:$\dim W<\infty$.$T\in\mathcal{L}(V,W)$. Proof: T单$\Leftrightarrow\exists S\in\mathcal{L}(W,V)\rightarrow ST=I_V$

$$
\begin{array}{cc}
T\text{单}: \operatorname{null} T = \{0\} & \\
\forall T(x) = T(y) \to x = y & \\
S: W \to V . S(T(x)) = x & T\text{单}: x\text{确定} \to T(x)\text{确定} \\
S(T(x) + T(y)) = S(T(x + y)) = x + y = S(T(x)) + S(T(y)) & S(T(x)) \to x . S\text{合理} \\
S(\lambda T(x)) = S\,(T(\lambda x)) = \lambda x = \lambda S(T(x)) & \\
\to S \in \mathcal{L}(W, V) & \\
ST(x) = S(T(x)) = x &
\end{array}
$$

$$
\begin{array}{cc}
\exists S \in \mathcal{L}(W, V). ST = I_V: & \\
T(x) = T(y). S(T(x)) = S(T(y)) \to I_V(x) = I_V(y) & S\text{无法分辨}T(x), T\,(y) \\
\to T\text{是单的}. & \text{都映射到同一个元素}
\end{array}
$$

21. Proof:$\dim V < \infty . T \in \mathcal{L}(V, W)$. Proof: T满 $\Leftrightarrow \exists S \in \mathcal{L}(W, V) \to TS = I_W$

$$
\begin{array}{cc}
\forall x \in W, TS(x) = x \to \forall x, y \in W, x \neq y & \text{先证明}S\text{是单的} \\
\to S(x) \neq S(y).\text{否则}T(S(x)) = T(S(y)) = x = y\text{矛盾} & \text{前后都需要使用}S\text{单性} \\
\to S\text{是单的} & \text{逆否命题}
\end{array}
$$

$$
\begin{array}{cc}
T\text{满}: \operatorname{range} T = W & \\
S: W \to V, T(S(x)) = x & \\
S\text{单} \to \forall x \neq y \in W, S(x) \neq S(y) & \\
T(S(x + y)) = x + y = T(S(x)) + T(S(y)) = T(S(x) + S\,(y)) & \text{单性}: TS(ax + by) = ax + by \\
T(\lambda S(x)) = T(S(\lambda x)) = \lambda x = \lambda T(S(x)) & \text{保证}T\text{的定义合理} \\
\to S \in \mathcal{L}(W, V) & \text{与20中单性的作用一致}
\end{array}
$$

$$
\begin{array}{cc}
TS = I_W: & \\
\exists T \in \mathcal{L}(V, W) & \\
W = \operatorname{span}(\boldsymbol{w}) & \\
\operatorname{range} S = \operatorname{span}\,(S(\boldsymbol{w})) \subset V & \\
T: T(S(w)) = w;\ T(V - \operatorname{range} S) = 0 & \text{由于}S\text{单, 这样的}T\text{是存在的} \\
T(S(x) + S(y)) = T(S(x + y)) = x + y = T(S(x)) + T(S(y)) & \\
T(\lambda S(x)) = T(S(\lambda x)) = \lambda x = \lambda T\,(S(x)) & \\
\to T \in \mathcal{L}(V, W) & \\
\operatorname{range} T = \operatorname{span} \boldsymbol{w} = W \to T\text{满} &
\end{array}
$$

22. Proof:U, V有限.$S \in \mathcal{L}(V, W). T \in \mathcal{L}(U, V)$. Proof: $\dim \operatorname{null} ST \leqslant \dim \operatorname{null} S + \dim \operatorname{null} T$

$$
\begin{array}{c}
\operatorname{null} ST: \{x \in U: ST(x) = 0\} \\
\operatorname{null} S: \{x \in V: S(x) = 0\} \\
\operatorname{null} T: \{x \in U: T(x) = 0\}
\end{array}
$$

$$
\begin{array}{c}
\forall x \in \operatorname{null} ST \to S(T(x)) = 0. \\
\to T(x) \in \operatorname{null} S \vee x \in \operatorname{null} T \\
\to \dim \operatorname{null} ST \leqslant \dim \operatorname{null} S + \dim \operatorname{null} T \\
\text{???这里差把或关系转成加法}
\end{array}
$$

23. Proof:U, V有限.$S \in \mathcal{L}(V, W). T \in \mathcal{L}(U, V)$. Proof: $\dim \operatorname{range} ST \leqslant \min\,(\dim \operatorname{range} S, \dim \operatorname{range} T)$

$$
\begin{array}{c}
\forall x \in \operatorname{range} ST. \\
\text{let}: x \neq 0: x = ST(u) \\
\to u \neq 0 \wedge T(u) \neq 0 \wedge T(u) \notin \operatorname{null} S \\
\to u \notin \operatorname{null} T \wedge T(u) \notin \operatorname{null} S \\
\to u \in \operatorname{range} T \wedge T(u) \in \operatorname{range} S \\
\dim ST \leqslant \dim \operatorname{range} T;\ \dim T(u) = \dim \operatorname{range} T \\
\dim \operatorname{range} T \leqslant \dim \operatorname{range} S \\
\dim ST \leqslant \min\,(\dim \operatorname{range} T, \dim \operatorname{range} S) \\
\text{???需要把且关系转换成最小值}
\end{array}
$$

24. Proof:W有限.$T_1, T_2 \in \mathcal{L}(V, W)$. Proof: $\operatorname{null} T_1 \subset \operatorname{null} T_2 \Leftrightarrow \exists S \in \mathcal{L}(W, W) \to T_2 = ST_1$

$$\text{null}\,T_1 \subset \text{null}\,T_2 \rightarrow \dim \text{range}\,T_1 \geqslant \dim \text{range}\,T_2$$
$$\text{range}\,T_1 = \text{span}(\boldsymbol{x}), \text{range}\,T_2 = \text{span}(\boldsymbol{y})$$
$$S\colon W \rightarrow W.T(x_i) = T_2(y_i), i \leqslant \dim \boldsymbol{y}; T(x_i) = 0, i > \dim \boldsymbol{y}$$
$$S \in \mathcal{L}(W, W).$$
$$\forall x \in V.S(T_1(x)) = T_2(y)$$
对基进行相互映射，保证了张成空间的一致性。

25. Proof:W有限.$T_1, T_2 \in \mathcal{L}(V, W)$. Proof: $\text{range}\,T_1 \subset \text{range}\,T_2 \Leftrightarrow \exists S \in \mathcal{L}(V, V) \rightarrow T_1 = T_2 S$

$$\text{range}\,T_1 \subset \text{range}\,T_2$$
$$\text{range}\,T_1 = \text{span}\,(\boldsymbol{x}), \text{range}\,T_2 = \text{span}(\boldsymbol{x}, \boldsymbol{y})$$
$$S\colon S(x_i) = S(x_i), i \leqslant \dim \text{range}\,T_1$$
$$S(x_i) = 0, i > \dim \text{range}\,T_1$$
$$\rightarrow S \in \mathcal{L}(V, V).$$
$$\forall x \in V, T_1(\boldsymbol{a}\boldsymbol{x} + \boldsymbol{b}\boldsymbol{y}) = T_2(S(\boldsymbol{a}\boldsymbol{x} + \boldsymbol{b}\boldsymbol{y})) = T_2(\boldsymbol{a}\boldsymbol{x})$$

26. Proof:$D \in \mathcal{L}(\mathcal{P}(R), \mathcal{P}(R))$. $\deg(Dp) = (\deg p) - 1$. Proof: D是满的

$$\forall y \in \mathcal{P}(R).$$
$$y = a_{\,n} p^n + \cdots + a_1 p + a_0$$
$$Dp = a_n p^n + \cdots + a_0 \rightarrow p = b_{n+1} p^{n+1} + \cdots + a_0$$
$$D^{-1}(y) = x\text{是存在的.}$$
$$\text{现证}f^{-1}\text{是单的}$$
$$\text{null}\,D^{-1} = \{x\colon D^{-1}(x) = 0\}$$
$$\forall p \in \mathcal{P}(R).f^{-1}(p) = \deg p + 1 \rightarrow \deg f^{-1}(p) \geqslant 1$$
$$\rightarrow D^{-1}(p) = \{0\}$$
$$\rightarrow D^{-1}(p)\text{是单的}$$
$$\rightarrow D\text{是满的}$$

27. Proof: $\forall p \in \mathcal{P}(R)$. Proof: $\exists q \in \mathcal{P}(R) \rightarrow p = 5q'' + 3q'$

$$\forall p \in \mathcal{P}(R).\, p = \boldsymbol{a}\boldsymbol{x}.\boldsymbol{a} = (a_0, a_1, \ldots).\boldsymbol{x} = (1, x, x^2, x^3, \ldots)$$
$$\text{由于}D \in \mathcal{L}(\mathcal{P}(R), \mathcal{P}(R)).D\text{满}$$
$$D(q), D(D(q))\text{满} \rightarrow Dq\text{与}D(Dq)\text{都是满射}$$
$$\rightarrow 5\,D(D(q))\text{是满射}, 3\,D(q)\text{也是满射} \qquad D(D)\text{是满射.数乘是满射}$$
$$\rightarrow 5D(D(q)) + 3D(q)\text{是满射} \qquad \text{加法是满射}$$
$$\rightarrow \text{range}(5D^2, 3D) = \mathcal{P}(R)$$

28. Proof: $T \in \mathcal{L}(V, W).\boldsymbol{w}$是$\text{range}\,T$的基. Proof: $\exists \varphi_1, \ldots, \varphi_n \in \mathcal{L}(V, F) \rightarrow \forall v \in V, Tv = \sum_i \varphi_i(v) w_i$

$$\text{range}\,T = \text{span}(\boldsymbol{w})$$
$$\forall v \in V, Tv = \boldsymbol{\lambda}\boldsymbol{w}$$
$$\text{现证: }\boldsymbol{\lambda}\text{线性变化} \rightarrow v\text{线性变化}$$
$$(\boldsymbol{m} + \boldsymbol{n})\boldsymbol{w} = \boldsymbol{m}\boldsymbol{w} + \boldsymbol{n}\boldsymbol{w} = T(m) + T(n) = T(m + n)$$
$$\alpha\boldsymbol{\lambda}\boldsymbol{w} = \alpha T(\lambda) = T(\alpha\lambda)$$
$$\rightarrow \boldsymbol{w}\text{固定时系数是线性变换}$$
$$\rightarrow T(v) = \boldsymbol{w}\boldsymbol{\lambda}$$
$$\text{let: } \varphi_i(v) = \lambda_i \rightarrow \varphi_i \in \mathcal{L}(V, F)$$
$$Tv = \boldsymbol{\lambda}\boldsymbol{w} = \sum \lambda_i w_i = \sum \varphi_i(v) w_i$$

29. Proof:$\varphi \in \mathcal{L}(V, F).u \in V \wedge u \notin \text{null}\,\varphi$. Proof: $V = \text{null}\,\varphi \oplus \{\lambda u\colon \lambda \in F\}$

$$\dim V = \dim \text{null}\,T + \dim \text{range}\,T$$
$$\dim \text{range}\,T \leqslant \dim F = 1$$

$$\dim \text{range}\,T = 0 \rightarrow \forall v \notin \text{null}\,T.v = \{0\}$$
$$\rightarrow V = \text{null}\,T$$
$$\dim \text{range}\,T = 1 \rightarrow \text{range}\,T = \text{span}\,(v).\ \text{range}\,T = \{\lambda u\colon \lambda \in F\}$$

30. Proof: $\varphi_1, \varphi_2 \in \mathcal{L}(V, F)$.null $\varphi_1 = \text{null}\, \varphi_2$. Proof: $\exists \lambda \in F \rightarrow \varphi_1 = \lambda \varphi_2$

$$\dim V = \dim \text{null}\, \varphi + \dim \text{range}\, \varphi$$
$$\dim \text{range}\, \varphi \leqslant \dim F = 1$$
$$\dim \text{range}\, \varphi_1 = \dim \text{range}\, \varphi_2 = 0:$$
$$\rightarrow \forall \lambda \in F, \varphi_1(x) = 0 = \lambda 0 = \lambda \varphi_2(x)$$

$$\dim \text{range}\, \varphi_1 = \dim \text{range}\, \varphi_2 = 1$$
$$\text{null}\, \varphi_1 = \text{null}\, \varphi_2 \rightarrow \text{null}\, \varphi_1 = \text{span}\, (\boldsymbol{v}) = \text{null}\, \varphi_2$$
$$V = \text{span}\, (\boldsymbol{v}, \boldsymbol{w})$$
$$\text{length}\, \boldsymbol{w} = 1$$
$$\rightarrow \dim \text{range}\, V = \dim V - \dim \text{null}\, V = 1$$
$$\forall x = \lambda w. \varphi_1 \in \mathcal{L}(V, F)$$
$$\rightarrow \varphi_1(w) = \lambda_1 w; \varphi_2(w) = \lambda_2 w$$
$$\rightarrow \varphi_1 = \frac{\lambda_2}{\lambda_1} w$$

31. Example: $T_1, T_2 \in \mathcal{L}(R^5, R^2)$. null $T_1 = \text{null}\, T_2 \wedge T_1 \neq \lambda T_2$

$$T_1(\boldsymbol{x}) = (x_4, x_5); T_2(\boldsymbol{x}) = (x_5, x_4)$$
$$\text{null}\, T_1 = (x, y, z, 0, 0). \text{null}\, T_2 = (x, y, z, 0, 0)$$
$$T_1(0,0,0,1,2) = (1,2); T_2(0,0,0,1,2) = (2,1)$$
$(1,2) = \lambda(2,1)$是不可能的

3 矩阵

3.1 矩阵定义

定义 3.1. *矩阵(matrix)*

$m, n \in N^+ . m \times n$矩阵$A$是由$F$的元素构成的$m$行$n$列的数字组合

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix}$$

$a_{i,j} \in F$.

定义 3.2. *线性映射的矩阵(matrix of a linear map),* $\mathcal{M}(T)$

$T \in \mathcal{L}(V, M)$. $\boldsymbol{v}$是V的基. $\boldsymbol{w}$是W的基. T关于这些基的矩阵为$m \times n$矩阵$\mathcal{M}(T)$.

$$a_{i,j} = T(v_k) = a_{1,k} w_1 + a_{2,k} w_2 + \cdots + a_{m,k} w_m$$
需要强调具体的基则使用记号$\mathcal{M}(T, (v_1, \ldots, v_n), (w_1, \ldots, w_m))$

例 3.3. 一些线性空间之间基的矩阵

1. $T \in \mathcal{L}(F^2, F^3)$. $T(x, y) = (x + 3y, 2x + 5y, 7x + 9y)$
$$\mathcal{M}(T) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}$$
2. $D \in \mathcal{L}(\mathcal{P}_3(R), \mathcal{P}_2(R))$. $D(p) = p'$.关于标准基的矩阵
$$(x^n)' = n x^{n-1}. p' = a x^0 \rightarrow p = a x + b.$$
$$\mathcal{M}(T) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

3.2 矩阵运算，加法与标量乘法

定义 3.4. *矩阵加法(matrix addition)*

$m \times n$阶的两个矩阵的和定义为矩阵中对应位置元素之和

$$A+B=\begin{pmatrix} a_{1,1}+b_{1,1} & \cdots & a_{1,n}+b_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1}+b_{m,1} & \cdots & a_{m,n}+b_{m,n} \end{pmatrix}$$

定理 3.5. *线性映射和的矩阵等于各自矩阵的和*

$$S,T\in\mathcal{L}(V,W).\mathcal{M}(S+T)=\mathcal{M}(S)+\mathcal{M}(T)$$

证明. 换成两个基的和，再拆开。显然 □

定义 3.6. *矩阵的标量乘法(scalar multiplication of a matrix)*

$$\lambda\begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix}=\begin{pmatrix} \lambda a_{1,1} & \cdots & \lambda a_{1,n} \\ \vdots & \ddots & \vdots \\ \lambda a_{m,1} & \cdots & \lambda a_{m,n} \end{pmatrix}$$

定理 3.7. *线性映射的标量乘法的矩阵等于线性映射的矩阵的标量乘法*

$$\lambda\in F.T\in\mathcal{L}(V,W).\mathcal{M}(\lambda T)=\lambda\mathcal{M}(T)$$

证明. 换成各自基的标乘,再合并到矩阵.显然 □

定义 3.8. *矩阵在构成的向量空间 $F^{m,n}$*

$$m,n\in N^{+}.\text{所有取自}F\text{的}m\times n\text{矩阵的集合记作}F^{m,n}$$

$$\dim F^{m,n}=mn$$

3.3 矩阵乘法

定义 3.9. *矩阵乘法(matrix multiplication)*

$A\in F^{m,n},B\in F^{n,p}.AB\in F^{m,p}$

$$(ab)_{i,j}=\sum_{r=1}^{n}a_{i,r}\cdot b_{r,j}$$

把A的i行与B的j列对应元素相乘再相加

定理 3.10. *在U,V,W的公共基下。两个线性映射复合的矩阵等于各自矩阵的乘积矩阵*

$$S\in\mathcal{L}(V,W),T\in\mathcal{L}(U,V).\mathcal{M}(ST)=\mathcal{M}(S)\cdot\mathcal{M}(T)$$

证明.

$$\begin{aligned}\mathcal{M}(S)=A.\mathcal{M}(T)=B\\ \forall u\in U.(ST)(u)=S(\textstyle\sum_{r=1}^{n}B_{r,k}v_r)\\ =\textstyle\sum_{r=1}^{n}B_{r,k}\cdot S(v_r)\\ =\textstyle\sum_{r=1}^{n}B_{r,k}\cdot(\sum_{j=1}^{m}A_{j,r}\cdot w_j)\\ =\textstyle\sum_{j=1}^{m}(\sum_{r=1}^{n}A_{j,r}\cdot B_{r,k})w_j\\ \to\mathcal{M}(ST)\text{是}w_j\text{的系数为}\textstyle\sum_{j=1}^{m}(\sum_{r=1}^{n}A_{j,r}\cdot B_{r,k})\text{的线性组合}\end{aligned}$$

□

例 3.11.

$$\begin{pmatrix}1&2\\3&4\\5&6\end{pmatrix}\begin{pmatrix}6&5&4&3\\2&1&0&-1\end{pmatrix}=\begin{pmatrix}10&7&4&1\\26&19&12&5\\42&31&20&9\end{pmatrix}$$

定义 3.12. *矩阵的行、矩阵的列*

$$\begin{gathered}A_{j,\cdot}\text{表示矩阵的}j\text{行组成的矩阵}\\ A_{\cdot,i}\text{表示矩阵的第}i\text{列组成的矩阵}\end{gathered}$$

例 3.13.

$$A=\begin{pmatrix}8&4&5\\1&9&7\end{pmatrix}.A_{2,\cdot}=(1,9,7).A_{\cdot,2}=\begin{pmatrix}4\\9\end{pmatrix}$$

定理 3.14. *矩阵的乘积元素等于行乘以列*

$$(AB)_{i,j}=A_{i,\cdot}\cdot B_{\cdot,j}$$

定理 3.15. *矩阵乘积的列元素等于矩阵乘以列*

$$(AB)_{\cdot,j}=AB_{\cdot,j}$$

定理 3.16. *矩阵可以理解为列的线性组合*

$$\begin{aligned}A\in F^{m,n},c\in F^{n,1}\\ Ac=c_1A_{\cdot,1}+c_2A_{\cdot,2}+\cdots+c_nA_{\cdot,n}\end{aligned}$$

习题3.C

1. Proof: $\dim V,\dim W<\infty.T\in\mathcal{L}(V,W).\text{Proof:}\forall\boldsymbol{v},\boldsymbol{w}.\dim\mathcal{M}(T)\geqslant\dim\operatorname{range}T$

$$\begin{aligned}\text{Assume:}\dim\mathcal{M}(T)<\dim\operatorname{range}T\\ \boldsymbol{w}=\mathcal{M}(T)\cdot\boldsymbol{v}\\ \to\text{length}\,\boldsymbol{w}<\operatorname{range}T.\text{这是不可能的}\\ \to\dim(T)>\dim\operatorname{range}T\end{aligned}$$

2. Compute: $D\in\mathcal{L}(\mathcal{P}_3(R),\mathcal{P}_2(R)).D(p)=p'.$求$\mathcal{P}_3$的基和$\mathcal{P}_2$的基使得$D$关于这些基的矩阵为单位阵

$$\begin{aligned}\mathcal{P}_2(R)=\operatorname{span}(1,x,x^2);\mathcal{P}_3(R)=\operatorname{span}\left(x,\tfrac{1}{2}x^2,\tfrac{1}{3}x^3,1\right)\\ \mathcal{M}(T,\boldsymbol{v},\boldsymbol{w})=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}\end{aligned}$$

3. Proof: $\dim V,W<\infty.T\in\mathcal{L}(V,W).\text{Proof:}\exists$基$\boldsymbol{v},\boldsymbol{w}.\mathcal{M}(T,\boldsymbol{v},\boldsymbol{w})$是对角的且有$\operatorname{range}T$个1

$$\mathcal{M}(T)=\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}\to T(u_i)=w_i$$

$\forall v\in V.v=\boldsymbol{a}\boldsymbol{v}.T(\boldsymbol{a}\boldsymbol{u})=\boldsymbol{a}\boldsymbol{w}.$ 有点高阶函数的味道了

这样的T的线性映射$\mathcal{M}(T)$是这样的矩阵

4. Proof: $\boldsymbol{v}\in V$是基.$\dim W<\infty.T\in\mathcal{L}(V,W)$. Proof: $\exists\boldsymbol{w}\in W.\mathcal{M}(T,\boldsymbol{v},\boldsymbol{w})_{1,\cdot}$只有第一个不为0

$$\begin{gathered}\mathcal{M}(T,\boldsymbol{v},\boldsymbol{w})_{1,\cdot}=(1,0,0,\dots)\\ \to T(v_1)=w_1.T(v_2,v_3,\dots)\in\operatorname{span}(w_2,w_3,\dots)\\ T_1(v)=T(av_1+\boldsymbol{\lambda}\boldsymbol{v}^-)=aT(v_1).T_1\in\mathcal{L}(V,W)\\ T_-(v)=T_-(av_1+\boldsymbol{\lambda}\boldsymbol{v}^-)=\boldsymbol{\lambda}T(\boldsymbol{v}^-).\\ T_-(x+y)=T_-((a+b)v_1+(\boldsymbol{a}+\boldsymbol{b})\boldsymbol{v}^-)=T((\boldsymbol{a}+\boldsymbol{b})\boldsymbol{v}^-)\\ =T(\boldsymbol{a}\boldsymbol{v}^-)+T(\boldsymbol{b}\boldsymbol{v}^-)=T_-(av_1+\boldsymbol{a}\boldsymbol{v}^-)+T_-(bv_1+\boldsymbol{b}\boldsymbol{v}^-)\\ T_-(\lambda x)=T_-(\lambda av_1+\lambda\boldsymbol{a}\boldsymbol{v}^-)=T(\lambda\boldsymbol{a}\boldsymbol{v}^-)=\lambda T(\boldsymbol{a}\boldsymbol{v}^{-1})=\lambda T_-(av_1+\boldsymbol{a}\boldsymbol{v}^{-1})\\ \to T_-\in\mathcal{L}(V,W)\\ \mathcal{M}(T_-)_{1,\cdot}=(0,\dots,0)\\ T_1+T_-\in\mathcal{L}(V,W)\\ \mathcal{M}(T_-+T_1)_{1,\cdot}=(1,0,\dots,0)+(0,0,\dots,0)=(1,0,\dots,0)\end{gathered}$$

线性映射用基
分割的映射是线性的

5. Proof: $\boldsymbol{w}$是W的基.V有限维.$T\in\mathcal{L}(V,W)$.Proof: $\exists V$的基$\boldsymbol{v}\to\mathcal{M}(T,\boldsymbol{v},\boldsymbol{w})_{1,\cdot}$只有第一个不为0

   同上

6. Proof: V,W有限.$T\in\mathcal{L}(V,W)$.Proof: $\dim\operatorname{range}T=1\Leftrightarrow\exists$基$\boldsymbol{v},\boldsymbol{w}\to\mathcal{M}(T)_{i,j}=1$

$$\begin{gathered}\dim\operatorname{range}T=1\to\exists\boldsymbol{v},\boldsymbol{w}\to\mathcal{M}(T)_{i,j}=1\\ \operatorname{range}T\in\operatorname{span}(\boldsymbol{w}).\\ \operatorname{range}T=k\boldsymbol{\lambda}\boldsymbol{w}.\text{选取}\boldsymbol{\lambda}=(1,\dots,1)\\ w_i=T(v_1)+\dots+T(v_n)\\ \to T(\boldsymbol{a}\boldsymbol{v})=\sum\boldsymbol{a}\\ T(x+y)=T(x)+T(y).T(\lambda x)=\lambda T(x)\to T\in\mathcal{L}(V,W)\end{gathered}$$

方向错了，证成存在一个T了

7.

8.

9.

10. $A\in F^{m,n},B\in F^{n,p}$. Proof: $(AB)_{i,\cdot}=A_{i,\cdot}B$ 积矩阵的行等于行矩阵乘矩阵

11. $a=\begin{pmatrix}a_1 & \dots & a_n\end{pmatrix}\in F^{1,n}.C\in F^{n,p}$.Proof: $aC=a_1C_{1,\cdot}+\dots+a_nC_{n,\cdot}$

12. Example: $A,B\in F^{2,2}.AB\neq BA$

   $\begin{pmatrix}1&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\1&0\end{pmatrix}=\begin{pmatrix}2&0\\0&0\end{pmatrix};\begin{pmatrix}1&0\\1&0\end{pmatrix}\begin{pmatrix}1&1\\0&0\end{pmatrix}=\begin{pmatrix}1&1\\1&1\end{pmatrix}$

13. Proof: 矩阵乘法具有分配律

   $A(B+C)=AB+AC$
   $(A+B)C=AC+BC$

14. Proof: 矩阵乘法有结合律$(AB)C=A(BC)$

15. Compute: $A\in F^{n,n}.(A^3)_{i,j}=\sum_{p=1}^n\sum_{q=1}^n A_{i,p}A_{p,q}A_{q,j}$

4 可逆性与同构的向量空间

定义 4.1. *线性映射. 可逆(invertible), 逆(inverse)*

$$\begin{array}{ll} T\in\mathcal{L}(V,W).\exists S\in\mathcal{L}(W,V).ST=I_V.TS=I_W & \text{可逆} \\ S\in\mathcal{L}(W,V)\text{称为}T\text{的逆}. & \text{逆} \end{array}$$

定理 4.2. *可逆线性映射的逆是唯一的.T^{-1}*

证明.

$$\begin{array}{c} T\in\mathcal{L}(V,W)\text{可逆}, S_1\neq S_2\in\mathcal{L}(W,V)\text{是}T\text{的逆} \\ S_1=S_1I_W=S_1(TS_2)=(S_1T)S_2=I_VS_2=S_2 \end{array}$$

□

定义 4.3. *线性映射。*可逆 ⇔ 即单又满

证明.

$$\begin{array}{cc} T\in\mathcal{L}(V,W).T\text{可逆}\rightarrow T\text{即单又满} & \\ \forall u,v\in V.Tu=Tv & \\ \rightarrow T^{-1}(T(u))=T^{-1}(T(v)) & T^{-1}\text{在同一结果} \\ \rightarrow u=v & \\ \rightarrow T\text{是单的} & \\ & \\ \forall w\in W.w=I_w(w)=T(T^{-1}(w)) & \\ \rightarrow w\in\operatorname{range}T\rightarrow\operatorname{range}T=W & \\ \rightarrow T\text{满} & \\ & \\ T\text{即单又满}\rightarrow T\text{可逆} & \\ T\text{单}\rightarrow\dim V=\dim\operatorname{range}T & \\ T\text{满}\ \dim\operatorname{range}T=\dim W & \\ \rightarrow\dim V=\dim W & \\ & \\ T\text{单}.T(x)=T(y)\rightarrow x=y & \\ T\text{满}.\forall w\in W.\exists x\in V\rightarrow T(x)=w & \\ \text{let}: S(w)\in V\rightarrow T(S(w))=w.TS=I_W & TS=I_W \\ \rightarrow S(w)\text{是必然存在于}V\text{的}.S(w)\text{必然是唯一的} & \\ T(S(x+y))=x+y=TS(x)+TS(y) & \\ TS(\lambda x)=\lambda x=\lambda T(x) & \\ \rightarrow S\in\mathcal{L}(W,V) & S\in\mathcal{L}(W,V) \\ & \\ \forall v\in V.T(S(T(v)))=(TS)(T(v))=I_WT(v)=T(v) & \\ T\text{满}\rightarrow\operatorname{range}T=W & \\ \rightarrow\operatorname{range}TS=W & \\ T\text{单}\rightarrow T(ST(v))=T(v)\rightarrow ST=I_V & ST=I_V \end{array}$$

□

例 4.4. 不可逆的线性映射

1. $T\in\mathcal{L}(\mathcal{P}(R),\mathcal{P}(R)).T(p)=x^2p$ $\quad 1\notin\operatorname{range}T$
2. $T\in\mathcal{L}(F^\infty,F^\infty).T(x_1,x_2,\ldots)=(x_2,x_3,\ldots)$ $\quad T(x,x_2,\ldots)=T(y,x_2,\ldots)$

4.1 同构的向量空间

定义 4.5. *两个向量空间的同构(isomorphism).同构的(isomorphic)*

$$\begin{array}{ll} \text{从向量空间}A\text{到向量空间}B\text{的可逆线性映射} & \text{同构} \\ \text{若}A,B\text{存在同构} & \text{同构的} \end{array}$$

定理 4.6. *两个向量空间。*维数相同 $\Leftrightarrow$ 同构

证明.

$$V,W\text{同构}\to\text{维数相同}$$
$$\exists T\in\mathcal{L}(V,W).T\text{可逆}$$
$$\operatorname{null}T=\{0\}.\operatorname{range}T=W.$$
$$\dim V=\dim\operatorname{null}T+\dim\operatorname{range}T$$
$$\to\dim V=\dim W$$

$$V,W\text{维数相同}\to V,W\text{存在同构}$$
$$\dim V=\dim W=d$$
$$V=\operatorname{span}(\boldsymbol{v}).W=\operatorname{span}(\boldsymbol{w})$$
$$T(\boldsymbol{av})=\boldsymbol{aw}$$
$$T(x+y)=(\boldsymbol{a}+\boldsymbol{b})\boldsymbol{w}=\boldsymbol{aw}+\boldsymbol{bw}=T(x)+T(y)$$
$$T(\lambda x)=\lambda\boldsymbol{aw}=\lambda(T(x))$$
$$\to T\in\mathcal{L}(V,W)$$
$$\boldsymbol{w}\text{线性无关}\to T(x)=0\to x=0\to\operatorname{null}T=\{0\}\to T\text{单}$$
$$\operatorname{span}(\boldsymbol{w})=W\to\operatorname{range}T=W\to T\text{满}$$
$$\to T\text{是同构}$$

□

定理 4.7. $\mathcal{L}(V,W)\cong F^{m,n}$

$\boldsymbol{v}$是V的基.$\boldsymbol{w}$是W的基.$\mathcal{M}$是$\mathcal{L}(V,W)$和$F^{m,n}$的同构

证明.

$$\mathcal{M}(T_1+T_2)=\mathcal{M}(T_1)+\mathcal{M}(T_2)\qquad\text{已证}$$
$$\mathcal{M}(\lambda T_1)=\lambda\mathcal{M}(T_1)\qquad\text{已证}$$
$$\to\mathcal{M}\in\mathcal{L}(\mathcal{L}(V,W),F^{m,n})$$

$$\mathcal{M}(T)=0_{m,n}\to T(v_i)=0$$
$$\to\operatorname{span}(v_i)=\{0\}\to\dim\operatorname{null}\mathcal{M}=0_{V,W}$$
$$\to\mathcal{M}\text{是单的}$$
$$Tv_i=\sum_{j=1}^{m}A_{j,i}\cdot w_j$$
$$\mathcal{M}(T)=A.\text{由于}A_{j,i}\text{取值的任意性}$$
$$\to\mathcal{M}\text{是满的}$$
$$\to\mathcal{M}\text{是}\mathcal{L}(V,W),F^{m,n}\text{的同构}$$

□

定理 4.8. $\dim\mathcal{L}(V,W)=\dim V\cdot\dim W$

4.2 线性映射等价于矩阵乘

定义 4.9. *向量的矩阵(matrix of a vector),* $\mathcal{M}(v)$

$v\in V.\boldsymbol{v}$是V的基.v关于$\boldsymbol{v}$的矩阵是$n\times 1$矩阵

$$\mathcal{M}(v,\boldsymbol{v})=(a_1,a_2,\ldots,a_n)^T$$
$$v=a_1v_1+\cdots+a_nv_n$$

例 4.10. 一些向量的矩阵

1. $\mathcal{M}(2-7x+5x^3,(1,x,x^2,x^3))=(2,-7,0,5)^T$
2. $x\in F^n$的标准基是x的各个坐标元素的$n\times 1$矩阵.$\mathcal{M}(x)=(x_1,\ldots x_n)^T$

定理 4.11. $\mathcal{M}(T)_{\cdot,k}=\mathcal{M}(Tv_k)$

$\mathcal{M}(T)_{\cdot,k}=A_{\cdot,k}.\mathcal{M}(Tv_k)=\mathcal{M}(w_k)=A_{\cdot,k}$

定理 4.12. *线性映射等价于矩阵乘*

$$T \in \mathcal{L}(V, W). v \in V. \boldsymbol{v}\text{是}V\text{的基}.\boldsymbol{w}\text{是}W\text{的基}. \to \mathcal{M}(T(v)) = \mathcal{M}(T)\,\mathcal{M}(v)$$

证明.

$$\begin{aligned}
T(v) &= T(a_1 v_1) + \cdots + T(a_n v_n) \\
\mathcal{M}(T(v)) &= a_1 \mathcal{M}(T(v_1)) + \cdots + a_n \mathcal{M}(T(v_n)) \\
&= a_1 \mathcal{M}(T)_{\cdot,1} + \cdots + a_n \mathcal{M}(T)_{\cdot,n} && 4.11 \\
&= \mathcal{M}(T)\mathcal{M}(v) && 3.16
\end{aligned}$$

□

4.3 算子

定义 4.13. *算子(operator),* $\mathcal{L}(V)$

$$\text{向量空间到自身的映射称为算子. 记号}\mathcal{L}(V) = \mathcal{L}(V, V)$$

定理 4.14. *无限维空间中。单性不能得可逆；满性不能得可逆*

$$\begin{gathered}
T \in \mathcal{L}(\mathcal{P}(R)).T(p) = x^2 p. 1 \notin \operatorname{range} T \to T\text{不满} \\
\forall x \neq y \in \operatorname{range} T. x = x^2 p = x^2 q = y \to p = q \to T\text{单} \\
T\text{不可逆}.???\text{缺证明}
\end{gathered}$$

$$\begin{gathered}
T \in \mathcal{L}(F^\infty). T(x_1, x_2, \ldots) = T(x_2, x_3, \ldots) \\
\forall \boldsymbol{x} \in F^\infty. (0, \boldsymbol{x}) \in F^\infty. T(0, \boldsymbol{x}) = \boldsymbol{x} \to T\text{满} \\
T(1, 1, 1, \ldots) = T(0, 1, 1, \ldots). \to T\text{不单} \\
T\text{不可逆，因为无法选取唯一的元素使得}T^{-1}\text{是映射}
\end{gathered}$$

定理 4.15. *有限维向量空间的算子。*单性 ⇔ 满性 ⇔ 可逆

证明.

$$\begin{gathered}
\text{单} \to \text{满} \\
\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T \\
\operatorname{null} T = \{0\} \to \dim \operatorname{range} T = \dim V \\
\to \text{满}
\end{gathered}$$

$$\begin{gathered}
\text{满} \to \text{单} \\
\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T \\
\to \dim \operatorname{range} T = \dim V \\
\to \dim \operatorname{null} T = 0 \to \operatorname{null} T = \{0\} \\
\to T\text{单}
\end{gathered}$$

$$(\text{单} \to \text{满} \vee \text{满} \to \text{单}) \to \text{可逆}$$

□

例 4.16. $\forall q \in \mathcal{P}(R), \exists p \in \mathcal{P}(R) \to ((x^2 + 5x + 7)p)'' = q.$

证明.

$$\begin{aligned}
&\forall p \in \mathcal{P}(R). \deg p = m \to p \in \mathcal{P}_m(R) \\
&q \in \mathcal{P}_m(R). T\colon \mathcal{P}_m(R) \to \mathcal{P}_m(R), T(p) = ((x^2 + 5x + 7)p)'' \\
&\deg((x^2 + 5x + 7)p)'' = \deg((x^2 + 5x + 7)p) - 2 = m + 2 - 2 = m \\
&\to T \in \mathcal{L}(\mathcal{P}_m(R), \mathcal{P}_m(R)) \\
&T(p) = 0 \to ((x^2 + 5x + 7)p)'' = 0 \\
&\to p = 0 \to \operatorname{null} T = \{0\} \\
&\to T\text{是单的} \\
&\to T\text{是满的} && 4.15 \\
&\to \exists p \in \mathcal{P}_m(R) \to T(p) = q
\end{aligned}$$

□

习题3.D

1.Pf $T\in\mathcal{L}(U,V),S\in\mathcal{L}(V,W).T,S$可逆. Proof: ST可逆$\wedge(ST)^{-1}=T^{-1}S^{-1}$
T,S可逆$\to\dim U=\dim V=\dim W$
$ST(x)=S(T(y))\to T(x)=T(y)\to x=y\to ST$单
T满$\to\forall v\in V,\exists x\in V,T(x)=v.\to\forall w\in V,\exists T(x)\in V.S(T(x))=w\to ST$满
$\to ST$可逆

$T^{-1}S^{-1}ST=T^{-1}(S^{-1}S)T=T^{-1}I_VT=T^{-1}T=I_V$
$STT^{-1}S^{-1}=S(TT^{-1})S^{-1}=SI_VS^{-1}=SS^{-1}=I_V$
def $\to(ST)^{-1}=T^{-1}S^{-1}$.

2.Pf V有限维.$\dim V>1$. Proof: V上的不可逆算子构成的集合不是$\mathcal{L}(V)$的子空间
$T_1(x,y)=(x,0).T_2(x,y)=(0,y).\operatorname{null}T_1=(0,x),\operatorname{null}T_2=(x,0)\to T_1,T_2$不可逆
$\to(T_1+T_2)(x,y)=(x,0)+(0,y)=(x,y)=I_V(x,y)$可逆

3.Pf 有限维$V.U$是V的子空间.$S\in\mathcal{L}(U,V)$. Proof: $(\exists$可逆$T\in\mathcal{L}(V)\to\forall u\in U,Tu=Su)\Leftrightarrow S$单
可逆$T\in\mathcal{L}(V),\forall u\in U,Tu=Su\to S$单
$\forall S(x)=S(y).S(x)=T(x)=T(y)=S(y)\to x=y\to S$单

S单$\to\exists$可逆$T\in\mathcal{L}(V),\forall u\in U,Tu=Su$
$\dim\operatorname{null}S=0$
$\dim U=\dim\operatorname{null}S+\dim\operatorname{range}S$
$\to\dim\operatorname{range}S=\dim U-\dim\operatorname{null}S$
$\to\dim\operatorname{range}S=\dim U\leqslant\dim V$
$\to\exists W\subset V\wedge W\cap U=\{0\}.\dim W=\dim V-\dim U$
$T=S+T'.T'\in\mathcal{L}(W,V).T(\boldsymbol{au}+\boldsymbol{bw})=\boldsymbol{bw}$
$S(\boldsymbol{au}+\boldsymbol{bv})=S(\boldsymbol{au})$
$T\in\mathcal{L}(V).Tu=u$

4.Pf 有限维$W.T_1,T_2\in\mathcal{L}(V,W)$. Proof: $\operatorname{null}T_1=\operatorname{null}T_2\Leftrightarrow\exists$可逆$S\in\mathcal{L}(W)\to T_1=ST_2$

V有限维
$\operatorname{null}T_1=\operatorname{null}T_2\to\dim\operatorname{range}T_1=\dim\operatorname{range}T_2$
$\to\operatorname{range}T_1\cong\operatorname{range}T_2$
$\to\exists$同构映射$S\in\mathcal{L}(\operatorname{range}T_2,\operatorname{range}T_1).T_1(x)=ST_2(x)$

5.Pf 有限维$V.T_1,T_2\in\mathcal{L}(V,W)$. Proof: $\operatorname{range}T_1=\operatorname{range}T_2\Leftrightarrow\exists$可逆$S\in\mathcal{L}(V)\to T_1=T_2S$
$\dim V=\dim\operatorname{null}T_1+\dim\operatorname{range}T\to\operatorname{null}T_1$和$\dim\operatorname{range}T$都是有限维
$\operatorname{range}T_1=\operatorname{range}T_2\to\dim\operatorname{range}T_1=\dim\operatorname{range}T_2$.
$\to\exists S\in\mathcal{L}(\operatorname{range}T_1,\operatorname{range}T_2).T_2=ST_1.\mathcal{M}(S)=S\boldsymbol{v}=\boldsymbol{v}\to T_1=S^{-1}ST_1=S^{-1}T_2$
$\to T_1=S^{-1}T_2$

6.Pf 有限维$V,W.T_1,T_2\in\mathcal{L}(V,W)$. Proof: $\exists$可逆$R\in\mathcal{L}(V),\exists$可逆$S\in\mathcal{L}(W)\to T_1=ST_2R$
$\Leftrightarrow\dim\operatorname{null}T_1=\dim\operatorname{null}T_2$
$T_1=ST_2R\to\operatorname{range}\dim T_1=\dim\operatorname{range}ST_2R$
R满$\to\forall S,\dim\operatorname{range}SR=\dim\operatorname{range}S$
$\to\dim\operatorname{range}ST_2R=\dim\operatorname{range}ST_2$
S单$\to\forall T,\dim\operatorname{range}ST=\dim\operatorname{range}T$
$\to\dim T_1=\dim\operatorname{range}ST_2=\dim\operatorname{range}T_2$

$\dim\operatorname{null}T_1=\dim\operatorname{null}T_2\to\exists R,S\to T_1=ST_2R$
$\dim\operatorname{null}T_1=\dim\operatorname{null}T_2\to\dim\operatorname{range}T_1=\dim\operatorname{range}T_2$
$\to\exists S\in\mathcal{L}(W,W)\to T_1=ST_2$
$I_V\in\mathcal{L}(V,V)\to T_1=ST_2=ST_2I_V$

7. 有限维$V,W.v\in V.E=\{T\in\mathcal{L}(V,W):Tv=0\}$

Pf E是$\mathcal{L}(V,W)$的子空间

$\dim V=\operatorname{span}(v,\boldsymbol{v})$.

$\mathbf{0}(v)=0\rightarrow\mathbf{0}\in E$

$T(av)+T(\boldsymbol{av})=T(\boldsymbol{av})=T(av+\boldsymbol{a}\,\boldsymbol{v})\in E$

$T(av+\boldsymbol{av})+T(bv+\boldsymbol{bv})=T(\boldsymbol{av})+T(\boldsymbol{bv})=T((av+\boldsymbol{av})+(bv+\boldsymbol{bv}))\in E$

$\lambda T((av+\boldsymbol{av}))=\lambda T(\boldsymbol{av})=T(\lambda av+\lambda\boldsymbol{av})\in E$

$\rightarrow E$是子空间

Cp $v\neq 0$. Compute $\dim E$

$v\neq 0\rightarrow Tv=0$

$\rightarrow\forall x\in V,x=av+\boldsymbol{av}$

$T(ax+\boldsymbol{av})=T(\boldsymbol{ax})$

$\rightarrow\mathcal{M}(T)_{\cdot,1}=(0,0,\ldots,0)$

$\rightarrow\dim\mathcal{M}(T)=(\dim V-1)\times\dim W$

$\mathcal{M}(T)$，T同构$\rightarrow\dim T=\dim\mathcal{M}(T)=(\dim V-1)\times\dim W$

8.Pf 有限维$V.T\in\mathcal{L}(V,W)$, 且T满. Proof: $\exists U\subset V\rightarrow T|_U$是$U,W$的同构.$T|_U$是$T$在$U$上的限制

T满$\rightarrow\dim\operatorname{range}T=\dim W\rightarrow W$有限维

$\dim V\geqslant\dim W$

$\operatorname{span}(\boldsymbol{v},\boldsymbol{s})=V\rightarrow\operatorname{length}\boldsymbol{v}=\dim W$

$\rightarrow\dim\operatorname{span}(\boldsymbol{v})=\dim W$

$\rightarrow\exists$同构$S.\ T=RS.$

9.Pf 有限维$V.S,T\in\mathcal{L}(V)$. Proof: ST可逆$\Leftrightarrow S$可逆$\wedge T$可逆

ST可逆$\rightarrow ST$单

$\forall x\neq y\in V.ST(x)\neq ST(y)$

设T不单$\rightarrow\exists x\neq y\in V.T(x)=T(y)\rightarrow ST(x)=ST(y)$

与ST的单性矛盾

同理S单

$\rightarrow S,T$都满

$\rightarrow S,T$可逆

Pf2

$\dim V=\dim\operatorname{null}ST+\dim\operatorname{range}ST$

$\dim V=\dim\operatorname{range}ST$

$\dim V=\dim\operatorname{null}S+\dim\operatorname{range}S$

$\dim V=\dim\operatorname{null}T+\dim\operatorname{range}T$

$\dim\operatorname{range}X\leqslant\dim\operatorname{range}XY$

$\rightarrow\dim V=\dim\operatorname{range}ST=\dim\operatorname{range}S=\dim\operatorname{range}T$

$\rightarrow S$可逆，T可逆

S,T可逆$\rightarrow ST$可逆

$\forall x\in V$

$\rightarrow(T^{-1}S^{-1})(ST)(x)=T^{-1}(S^{-1}S)T(x)$

$=T^{-1}I_VT(x)=(T^{-1}T)(x)=I_V(x)=x$

$\rightarrow(ST)(T^{-1}S^{-1})(x)=S(TT^{-1})S^{-1}(x)$

$=SI_VS^{-1}(x)=(SS^{-1})(x)=x$

$\rightarrow T^{-1}S^{-1}$是ST的逆

$\rightarrow ST$可逆

10.Pf 有限维$V.S,T\in\mathcal{L}(V)$.Proof: $ST=I_V\Leftrightarrow TS=I_V$

$$I_V可逆\to S,T可逆\to TS可逆$$
$$I_V=(T^{-1}S^{-1})(ST)\to I_V=T^{-1}S^{-1}$$
$$I_V=STST\to TS=S^{-1}T^{-1}=(TS)^{-1}$$
$$I_V=TS(TS)^{-1}=TSTS=TS$$

11.Pf 有限维$V.S,T,U\in\mathcal{L}(V).STU=I_V.$ Proof: T可逆$\wedge T^{-1}=US$

$$STU=I_V\rightarrow(ST)U=I_V\rightarrow U(ST)=I_V$$
$$(US)T=I_V$$
$$I_V\text{可逆}\rightarrow US\text{可逆}\wedge T\text{可逆}$$
$$(US)T=I_V\rightarrow T(US)=I_V$$

Df $\rightarrow T^{-1}=US$

12.Eg 无限维$V.$上述结论不成立

$$I_V\text{可逆}f(x)=x.\sigma(p(x))=p'(x).\varphi(p(x))=\int p(x)$$
$$\sigma\varphi(p)=I_P(p)=p$$
$$\sigma I_P\varphi(p)=p$$
$$\varphi I_P\sigma(p)=\{p+c\times 1\colon p\}$$

13.Pf 有限维$V.R,S,T\in\mathcal{L}(V),RST$满. Proof: S单

$$\text{有限}\rightarrow RST\text{满}\rightarrow RST\text{单}$$
$$\rightarrow(RST)^{-1}\text{存在}$$
$$\rightarrow R^{-1},S^{-1},T^{-1}\text{存在}$$
$$\rightarrow R,S,T\text{即单又满}$$

14.Pf $\boldsymbol{v}$是V的基.$T\colon V\rightarrow F^{n,1},Tv=\mathcal{M}(v).$ Proof: T是$V,F^{n,1}$的同构

$$\forall T(x)\neq T(y).x\neq y\rightarrow x=\boldsymbol{av},y=\boldsymbol{bv}\wedge\boldsymbol{a}\neq\boldsymbol{b}$$
$$\rightarrow\mathcal{M}(x)=\boldsymbol{a}'\neq\boldsymbol{b}'=\mathcal{M}(y)$$
$$\rightarrow T\text{是单的}$$
$$V\text{是有限维的}\wedge F^{n,1}\text{是有限维的}\rightarrow T\text{是满的}$$
$$\rightarrow T\text{是同构}$$

15.Pf $T\in\mathcal{L}(F^{n,1},F^{m,1}),\exists m,n$矩阵$A\rightarrow\forall x\in F^{n,1},Tx=Ax$

$$\forall\boldsymbol{x}\in F^{n,1},\boldsymbol{x}=\boldsymbol{av}.$$
$$T(\boldsymbol{x})=\boldsymbol{T}(\boldsymbol{av})=\boldsymbol{a}T(\boldsymbol{v})$$
$$\mathcal{M}(T)=\begin{pmatrix}T(v_1)&\cdots&T(v_n)\\\vdots&\ddots&\vdots\\T(v_1)&\cdots&T(v_n)\end{pmatrix}$$
$$\mathcal{M}(T)\,\boldsymbol{x}=\begin{pmatrix}T(v_1)&\ldots&T(v_n)\end{pmatrix}\begin{pmatrix}a_{1,1}&\cdots&a_{1,n}\\\vdots&\ddots&\vdots\\a_{n,1}&\cdots&a_{n,n}\end{pmatrix}\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}=\boldsymbol{a}T(\boldsymbol{v})$$

???

16.Pf 有限维$V.T\in\mathcal{L}(V).$ Proof: $Tx=(\lambda I_V)x\Leftrightarrow\forall S\in\mathcal{L}(V),ST=TS$

$$Tx=\lambda I_V(x).\forall S\in\mathcal{L}(V).ST=S(\lambda I_V(x))=S(\lambda x)=\lambda S(x)=\lambda I_V(S)=TS$$
$$\forall S\in\mathcal{L}(V),ST=TS$$
$$\rightarrow STST=(ST)(ST)=(TS)(TS)=S^2T^2=T^2S^2$$

???

17.Pf 有限维$V.\mathcal{E}$是$\mathcal{L}(V)$的子空间.$\forall S\in\mathcal{L}(V),\forall T\in\mathcal{E},ST\in\mathcal{E}\wedge TS\in\mathcal{E}.$Proof: $\mathcal{E}=\{0\}\vee\mathcal{E}=\mathcal{L}(V)$

$$\mathcal{E}=\{0\}\rightarrow\forall S\in\mathcal{L}(V).ST=S\boldsymbol{0}=\boldsymbol{0}S=\boldsymbol{0}\in\mathcal{E}\rightarrow\mathcal{E}=\{0\}\text{时上述结论成立}$$

轮换 设$\mathcal{E}\neq\{0\}\wedge\mathcal{E}\neq\mathcal{L}(V)$

$$\forall U\subseteq V.\mathcal{E}=\mathcal{L}(U).$$
$$TS\in\mathcal{E}\rightarrow\text{任意轮换线性变换}S(x_1,\ldots,x_n)=(x_2,x_3,\ldots,x_n,x_1)$$
$$S^2\ldots S^n\in\mathcal{L}(V).$$
$$\forall ST\in\mathcal{E},\text{这样每个被}T\text{变换掉的变量会重新出现}$$
$$\rightarrow ST\subset\mathcal{L}(U).TS\in\mathcal{L}(V)$$
$$\rightarrow\mathcal{E}=\mathcal{L}(V)$$

$$
\begin{aligned}
18.\mathrm{Pf}\quad & V\text{和}\mathcal{L}(F,V)\text{是同构的向量空间}\\
& \forall \boldsymbol{t}\in V.T(\boldsymbol{t})=\Phi(\boldsymbol{t},\boldsymbol{x})\\
& \Phi(\boldsymbol{t},\boldsymbol{x})=\boldsymbol{t}\cdot\boldsymbol{x}\\
& \Phi(\boldsymbol{t},\boldsymbol{x}+\boldsymbol{y})=\boldsymbol{t}\cdot(\boldsymbol{x}+\boldsymbol{y})\\
& =\boldsymbol{t}\cdot\boldsymbol{x}+\boldsymbol{t}\cdot\boldsymbol{y}\\
& =\Phi(\boldsymbol{t},\boldsymbol{x})+\Phi(\boldsymbol{t},\boldsymbol{y})\\
& \Phi(\boldsymbol{t},\lambda\boldsymbol{x})=\boldsymbol{t}\cdot\lambda\boldsymbol{x}\\
& =\lambda(\boldsymbol{t}\cdot\boldsymbol{x})\\
& =\lambda\Phi(\boldsymbol{t}\cdot\boldsymbol{x})\\
& \to\Phi(\boldsymbol{t},\boldsymbol{x})\in\mathcal{L}(V,F)\\
& \to T\text{是}V\text{和}\mathcal{L}(F,V)\text{的同构}\\
& \mathcal{L}(F,V)\cong\mathcal{L}(V,F)\\
& \to V\text{和}\mathcal{L}(F,V)\text{存在同构}
\end{aligned}
$$

$$
\begin{aligned}
19.\quad & T\in\mathcal{L}(\mathcal{P}(R))\text{单}.\forall p\in\mathcal{P}(R)\wedge p\neq 0.\deg Tp\leqslant\deg p.\\
\mathrm{Pf}\quad & T\text{满}\\
& \deg p=0\to\deg T(p)\leqslant 0\to Tp=c\\
& \deg p=1\to\deg T(p)\leqslant 1\to Tp=ax+b\\
& \ldots\\
& \to\deg p=n\to\deg T(p)\leqslant n\to\deg Tp=n\\
& \to\operatorname{span}(x^0,x^1,\ldots,x^n,\ldots)=\mathcal{P}(R)\\
& \to T\text{满}\\
\\
\mathrm{Pf}\quad & \forall p\in\mathcal{P}(R)\wedge p\neq 0.\deg Tp=\deg p
\end{aligned}
$$

$$
\begin{aligned}
20.\mathrm{Pf}\quad & A\text{是方阵. 下面两命题等价}\\
1.\quad & A\boldsymbol{x}=\boldsymbol{0}\text{的唯一解}\boldsymbol{0}\\
2.\quad & \forall\boldsymbol{b},A\boldsymbol{x}=\boldsymbol{b}\text{有解}\\
\\
1\to 2\quad & A\boldsymbol{x}=0\to\boldsymbol{x}=0\to\operatorname{null}A=0\\
& \to\dim\operatorname{range}A=\dim V\\
& \to\forall\boldsymbol{b}\in V.A\boldsymbol{x}=\boldsymbol{b}\text{有解}\\
\\
2\to 1\quad & \forall\boldsymbol{b}\in V,A\boldsymbol{x}=\boldsymbol{b}\text{有解}\\
& \to\dim\operatorname{range}A=\dim V\\
& \to\dim\operatorname{null}A=\{0\}\\
& \to Ax=0\to\boldsymbol{x}=0
\end{aligned}
$$

5 向量空间的积与商

通常处理多个向量空间时，这些向量空间都应该定义在同一个域上。

5.1 向量空间的积

定义 5.1. *向量空间的有限积(product of vector space)*

$$
\begin{aligned}
& V_1,\ldots,V_n\text{是}F\text{上的向量空间}\\
V_1\times V_1\times\cdots\times V_n=\{&(v_1,\ldots,v_n)\colon v_1\in V_1,\ldots,v_n\in V_n\}
\end{aligned}
$$

$$\forall x,y\in\prod V_i.x+y=(x_1+y_1,\ldots,x_n+y_n)\quad\text{加法}$$
$$\forall x\in\prod V_i.\forall\lambda\in F.\lambda x=(\lambda x_1,\ldots,\lambda x_n)\quad\text{标乘}$$

例 5.2. $A=\mathcal{P}_2(R)\times R^3.\forall x\in A.\operatorname{length}x=2.(x^2+x+1,(1,1,1))\in A$

定理 5.3. *向量空间的有限积是向量空间*

1. $\forall x,y\in S.x+y=y+x$
2. $\forall x,y,z\in S.(x+y)+z=x+(y+z)$
3. $\exists 0\in S.0=(0,\ldots,0).\forall x\in S.x+0=x$
4. $\forall x\in S.\exists -x=(-x_1,\ldots,-x_n).x+-x=0$
5. $1\in F.\forall x\in S.1\cdot x=x$
6. $\forall\lambda\in F,\forall x,y\in S.\lambda(x+y)=\lambda x+\lambda y.$
   $\forall a,b\in F,\forall x\in S.(a+b)x=ax+bx.$

$\rightarrow$任意有限个向量空间的积空间S是向量空间

定理 5.4. *有限维向量空间的有限积。积空间的维数等于各个向量空间维数的和*

$$\dim V_1\times V_2=\dim V_1+V_2$$

证明.

$$\begin{aligned}&V_1=\operatorname{span}(\boldsymbol{v}_1).V_2=\operatorname{span}(\boldsymbol{v}_2)\\ \rightarrow&V_1\times V_2=(\boldsymbol{a}\boldsymbol{v}_1,\boldsymbol{b}\boldsymbol{v}_2)=\operatorname{span}(\boldsymbol{v}_1,\boldsymbol{v}_2)\\ \rightarrow&\dim V_1\times V_2=\dim V_1+\dim V_2\end{aligned}$$ □

定理 5.5. *积与直和的关系*

$U_1,\ldots,U_m$为V的子空间.线性映射$\Gamma\colon U_1\times\cdots\times U_m\rightarrow U_1+\cdots+U_m$
$$\Gamma(u_1,\ldots,u_m)=u_1+\cdots+u_m.$$
$$U_1+\cdots+U_m\text{是直和}\Leftrightarrow\Gamma\text{是单的}$$

Remark: *有限维空间中.*U_i是有限维的$\rightarrow\Gamma$是满的

证明.

$$\begin{aligned}&\Gamma\text{单}\Leftrightarrow\operatorname{null}\Gamma=\{0\}\\ \rightarrow&\Gamma(x)=0=u_1+\cdots+u_m\Leftrightarrow u_1=u_2=\cdots=u_m=0\end{aligned}$$ □

定理 5.6. *线性空间的和为直和*$\Leftrightarrow$*和空间维数为个空间维数之和*

V有限维.$U_1,\ldots,U_m$是V的子空间.$U_1+\cdots+U_m$是直和$\Leftrightarrow\dim(U_1+\cdots+U_m)=\dim U_1+\cdots+\dim U_m$

证明.

$$\begin{aligned}&\Gamma\colon U_1\times\cdots\times U_m\rightarrow U_1+\cdots+U_m.\\ &\dim(U_1+\cdots+U_m)=\dim(U_1\times\cdots\times U_m)\\ &\dim(U_1\times\cdots\times U_m)=\dim(U_1+\cdots+U_m)\Leftrightarrow\Gamma\text{单}\\ &\Gamma\text{单}\Leftrightarrow U_1\oplus\cdots\oplus U_m\end{aligned}$$ □

5.2 向量空间的商

定义 5.7. *向量与子空间的和*

$$v\in V.U\text{是}V\text{的子空间}.v+U\subset V.v+U=\{v+u\colon u\in U\}$$

定义 5.8. *仿射子集(affine subset).平行(parallel)*

$$\begin{array}{cc}\text{仿射子集} & \forall v\in V.U\text{是}V\text{的子空间}.v+U\\ \text{平行} & \forall v\in V.U\text{是}V\text{的子空间}.\text{仿射子集}v+U\text{平行于}U\end{array}$$

Remark: 这里平行的概念与几何上的平行不同。R^3的直线不平行与R^3中的平面

定义 5.9. *商空间(quotient space), V/U*

$$\begin{array}{l}U\text{是}V\text{的子空间}.\text{商空间}V/U\text{指所有平行与}U\text{的仿射子集的集合}\\ \qquad V/U=\{v+U\colon v\in V\}\end{array}$$

例 5.10. 商空间

$$\begin{array}{cc}1. & U=\{x,2x\}\in R^2.R^2/U\text{是}R^2\text{中所有斜率为2的直线的集合}\\ 2. & U=\{(x,y,z)\colon A_1x+B_1y+C_1z=0;A_2x+B_2y+C_2z=0\}.\\ & R^3/U=\{(x,y,z)\colon A_1x+B_1y+C_1z+D_1=0;A_2x+B_2y+C_2z+D_2=0.D_1,D_2\text{是随机的}\}\\ 3. & U=\{(x,y,z)\colon Ax+By+Cz=0\}.R/U=\{(x,y,z)\colon Ax+By+Cz+D=0.D\text{是随机的}\}\end{array}$$

定理 5.11. *平行于U的两个仿射子集相等或不相交*

$$\begin{array}{cc}& U\text{是}V\text{的子空间}.v,w\in V.\text{以下命题等价}\\ 1. & v-w\in U\\ 2. & v+U=w+U\\ 3. & (v+U)\cap(w+U)\neq\varnothing\end{array}$$

证明.

$$\begin{array}{cc}1\rightarrow 2 & v-w\in U.\forall u\in U.v+u=w+((v-w)+u)\in w+U\\ & \rightarrow u+U\in w+U\\ & w+u=v-(v-w)+u\in v+U\\ & \rightarrow w+U\in v+U\\ & \rightarrow v+U=w+U\end{array}$$

$$\begin{array}{cc}2\rightarrow 3 & v+U=w+U\rightarrow v+U\cap w+U=v+U=w+U\neq\varnothing\end{array}$$

$$\begin{array}{cc}3\rightarrow 1 & (v+U)\cap(w+U)\neq\varnothing\\ & \rightarrow\exists u_1,u_2\in U\rightarrow v+u_1=w+u_2\\ & v-w=u_2-u_1\in U\end{array}$$

□

定义 5.12. *商空间商的加法和标量乘法(addition and scalar multiplication on V/U)*

$$\begin{array}{cc}& U\text{是}V\text{的子空间}.V/U\text{上定义运算}\\ \text{加法} & (v+U)+(w+U)=(v+w)+U\\ \text{标乘} & \lambda(v+U)=\lambda v+U\end{array}$$

定理 5.13. *商空间上定义的加法和标乘使得商空间构成向量空间*

证明.

$$
\begin{gathered}
v,w\in V.\hat{v},\hat{w}\in V\rightarrow v+U=\hat{v}+U.w+U=\hat{w}+U\\
v+U=\hat{v}+U\rightarrow v-\hat{v}\in U,w-\hat{w}\in U\\
\rightarrow(v-\hat{v})-(w-\hat{w})\in U\\
\rightarrow(v+w)-(\hat{v}+\hat{w})\in U\\
\rightarrow(v+w)+U=(\hat{v}+\hat{w})\in U\\
\text{表示不同的相同元素的加法结果是同一和}\\
\rightarrow\text{加法映射是合理的}
\end{gathered}
$$

$$
\begin{gathered}
v+U=\hat{v}+U\rightarrow v-\hat{v}\in U\\
\rightarrow\lambda(v-\hat{v})\in U\\
\rightarrow\lambda v-\lambda\hat{v}\in U\\
\rightarrow\lambda v+U=\lambda\hat{v}+U\\
\rightarrow\text{相同元素的标量乘法表示的是同一个元素}\\
\rightarrow\text{标乘映射是合理的}
\end{gathered}
$$

□

1. $\forall x,y\in V.(x+U)+(y+U)=(x+y)+U=(y+x)+U=(y+U)+(x+U)$
2. $\forall x,y,z\in V.(x+y)+U+(z+U)=(x+y+z)+U=x+U+(y+z)+U$
3. $0\in V.\forall x\in V.(x+U)+(0+U)=(x+0)+U=x+U$
4. $\forall x\in V.-x\in V.(x+U)+(-x+U)=(x+-x)+U=0+U$
5. $1\in F.\forall x\in V.1(x+U)=(1x)+U=x+U$
6. $$
\begin{gathered}
\lambda((x+U)+(y+U))=(\lambda x+\lambda y)+U\\
=(\lambda x+U)+(\lambda y+U)\\
=\lambda(x+U)+\lambda(y+U)\\
(a+b)(x+U)=((a+b)x)+U\\
=(ax+bx)+U\\
=(ax+U)+(bx+U)\\
=a(x+U)+b(x+U)\\
\rightarrow\text{向量空间对子空间的商空间是向量空间}
\end{gathered}
$$

定义 5.14. *商映射(quotient map)*

U是V的子空间.商映射π是映射$\pi\colon V\rightarrow V/U$

$$\pi(v)=v+U$$

定理 5.15. *商映射对变量v是线性映射*

证明.

$$
\begin{aligned}
\forall x,y\in V.\pi(x+y)&=(x+y)+U\\
&=(x+U)+(y+U)\\
&=\pi(x)+\pi(y)\\
\forall x\in V,\forall\lambda\in F.\pi(\lambda x)&=(\lambda x)+U\\
&=\lambda(x+U)\\
&=\lambda\pi(x)
\end{aligned}
$$

□

定理 5.16. *商空间的维数*

$$V\text{是有限维的}.U\text{是}V\text{的子空间}.\dim V/U=\dim V-\dim U$$

证明.

$$\begin{aligned}
&\pi\colon V\to V/U.\\
&\operatorname{null}\pi=U.\operatorname{range}\pi=V/U\\
\to&\dim V=\dim\operatorname{null}\pi+\dim\operatorname{range}\pi\\
=&\dim U+\dim V/U\\
\to&\dim V/U=\dim V-\dim U
\end{aligned}$$

□

定义 5.17. *线性映射诱导的商空间映射*

$$T\in\mathcal{L}(V,W).\tilde{T}\colon V/(\operatorname{null}T)\to W$$
$$\tilde{T}(v+\operatorname{null}T)=T(v)$$

验证它是个映射 $\to\forall u,v\in V\to u+\operatorname{null}T=v+\operatorname{null}T$

$\to u-v\in\operatorname{null}T$

$T(u-v)=T(u)-T(v)=0\to T(u)=T(v)$

$\to$若u，v诱导出同一个商空间则它们的线性映射相同

定理 5.18. $\tilde{T}$的零空间与值域

$$T\in\mathcal{L}(V,W)$$

1. $\tilde{T}$是$V/\operatorname{null}T\to W$的线性映射
2. $\tilde{T}$单
3. $\operatorname{range}\tilde{T}=\operatorname{range}T$
4. $V/\operatorname{null}T$与$\operatorname{range}T$同构

证明.

1. $\forall x,y\in V/\operatorname{null}T.\tilde{T}(x+y)=T((x+y)+\operatorname{null}T)=T(x+y)$
$=T(x)+T(y)=\tilde{T}(x+\operatorname{null}T)+\tilde{T}(y+\operatorname{null}T)$
$\forall x\in V/\operatorname{null}T,\lambda\in F.\tilde{T}(\lambda x)=\tilde{T}(\lambda x+\operatorname{null}T)$
$=T(\lambda x)=\lambda T(x)=\lambda(\tilde{T}(x+\operatorname{null}T))$
$\to\tilde{T}\in\mathcal{L}(V/\operatorname{null}T,W)$

2. $\forall v\in V,\tilde{T}(v+\operatorname{null}T)=T(v)=0\to v\in\operatorname{null}T.$
$\to v+\operatorname{null}T=0+\operatorname{null}T$
$\to\operatorname{null}\tilde{T}=\{0\}$
$\to\tilde{T}$单

□

3. $\forall v\in V.\tilde{T}(v+\operatorname{null}T)=T(v)$
$\to\operatorname{range}\tilde{T}(v+\operatorname{null}T)=\operatorname{range}T(v)$

4. $\tilde{T}$即单又满$\to\tilde{T}$可逆$\to\tilde{T}$是$V/\operatorname{null}T$和$\operatorname{range}T$的同构

习题3.E

1. Proof: $T\colon V\to W.T$的图是$V\times W$的的子集T的图$=\{(v,Tv)\in V\times W\colon v\in V\}$.

Proof: T是线性映射$\Leftrightarrow T$的图是$V\times W$的子空间

$$
\begin{gathered}
T\text{是线性映射}\rightarrow T\text{的图是}V\times W\text{的子空间}\\
0\in V,T(0)=0\in W\rightarrow(0,T(0))\in G(T)\\
\forall x,y\in G(T).x+y=(x,T(x))+(y,T(y))\\
=(x+y,T(x)+T(y))\\
=(x+y,T(x+y))\\
x,y\in V\rightarrow T(x+y)\in\operatorname{range}T\\
\rightarrow x+y\in G(T)
\end{gathered}
$$

$$
\begin{gathered}
\forall x\in G(T).\lambda\in F.\lambda x=(\lambda x,\lambda T(x))\\
=(\lambda x,T(\lambda x))\\
\lambda x\in V\rightarrow T(\lambda x)\in\operatorname{range}T\\
\rightarrow(\lambda x,T(\lambda x))\in G(T)\\
\rightarrow G(T)\text{是子空间}
\end{gathered}
$$

$$
\begin{gathered}
G(T)\text{是子空间}\rightarrow T\text{是线性映射}\\
\forall(x,Tx),(y,Ty)\in G(T)\rightarrow(x+y,Tx+Ty)\in G(T)\\
(x+y,T(x+y))\in G(T)\rightarrow T(x+y)\\
\forall u\in V,T(u)=T(u)\rightarrow T(x)+T(y)=T(x+y)
\end{gathered}
$$

$$
\begin{gathered}
\forall(x,Tx)\in G(T).\lambda(x,Tx)=(\lambda x,\lambda Tx)\in G(T)\\
(\lambda x,T(\lambda x))\in G(T)\rightarrow\lambda Tx=T(\lambda x)\\
\rightarrow T\in\mathcal{L}(V,W)
\end{gathered}
$$

2. Proof: $V_1,\ldots,V_m$都是向量空间使得$V_1\times\cdots\times V_m$是有限维的.Proof: 每个$V$都是有限维的

$$
\begin{gathered}
\dim(V_1\times\cdots\times V_m)=\dim V_1+\cdots+\dim V_m\\
\text{若}V_i\text{无限}\rightarrow\dim(V_1\times\cdots\times V_m)\geqslant\dim V_i\text{矛盾}\\
\rightarrow V_i\text{有限维}
\end{gathered}
$$

3. Example: 给出向量空间和两个子空间U_1,U_2的例子。$U_1\times U_2$同构于U_1+U_2，但U_1+U_2不是直和

$$
\begin{gathered}
V=R^{\infty}.U_1=U_{\text{odd}};U_2=U_{\text{even}}+\operatorname{span}((x,0\ldots)).U_1\times U_2\cong U_1+U_2.U_1\cap U_2=\operatorname{span}((x,0,\ldots))\\
\rightarrow U_1+U_2\text{不是直和}
\end{gathered}
$$

4. Proof: $V_1,\ldots,V_m$均为向量空间. Proof: $\mathcal{L}(V_1\times\cdots\times V_m,W)$和$\mathcal{L}(V_1,W)\times\cdots\times\mathcal{L}(V_m,W)$同构

$$
\begin{gathered}
\forall f\in\mathcal{L}(V_1\times\cdots\times V_m,W).\\
f_i\colon V^m\rightarrow V_i.f_i\in\mathcal{L}(V_i,W)\\
f=\sum f_i\\
\operatorname{null}f\rightarrow f=\mathbf{0}\Leftrightarrow f_i=\mathbf{0}\\
\varphi\colon f\rightarrow(f_i)\\
\varphi\colon\mathcal{L}(V_1\times\cdots\times V_m,W)\rightarrow\mathcal{L}(V_1,W)\times\cdots\times\mathcal{L}(V_m,W)\\
\varphi\in\mathcal{L}(\mathcal{L}(V_1\times\cdots\times V_m,W),\mathcal{L}(V_1,W)\times\cdots\times\mathcal{L}(V_m,W))\\
\operatorname{null}\varphi=\{f\colon\varphi(f)=0\}\\
\varphi(f)=0\rightarrow f_i=\mathbf{0}\rightarrow f=\mathbf{0}\\
\rightarrow\varphi\text{是单的}\\
\forall f\in\mathcal{L}(V_1,W)\times\cdots\times\mathcal{L}(V_m,W)\\
f=(f_i),f_i\in\mathcal{L}(V_i,W)\\
\sum f_i=f\in\mathcal{L}(V_1\times\cdots\times V_m,W)\\
\varphi(f)=(f_i)\\
\rightarrow\varphi\text{满}\\
\rightarrow\varphi\text{是同构}
\end{gathered}
$$

5. Proof: $W_1,\ldots,W_m$为向量空间.Proof: $\mathcal{L}(V,W_1\times\cdots\times W_m)$和$\mathcal{L}(V,W_1)\times\cdots\times\mathcal{L}(V,W_m)$同构

$$\varphi_i\colon \mathcal{L}(V,W_1\times\cdots\times W_n)\to\mathcal{L}(V,W_i)$$
$$\varphi_i(f(V,W_1\times\cdots\times W_n))=f_i(V,W_i)$$
$$f\in\mathcal{L}(V,W_1\times\cdots\times W_n)\to f(v,w)=f(v,\boldsymbol{aw})$$
$$\to f=\sum f_i$$
$$f_i\in\mathcal{L}(V,W_i)$$
$$\operatorname{null}\varphi\to\varphi(f)=\mathbf{0}\to\sum f_i=\mathbf{0}$$
$$\to f_i=\mathbf{0}\to f=\mathbf{0}$$
$\to\varphi$是单射

$$\forall f\in\mathcal{L}(V,W_1)\times\cdots\times\mathcal{L}(V,W_n)$$
$$\sum(f_i)-f\in\mathcal{L}(V,W_1\times\cdots\times W_n)$$
φ是满射

$\to\varphi$是$\mathcal{L}(V,W_1\times\cdots\times W_n)$和$\mathcal{L}(V,W_1)\times\cdots\times\mathcal{L}(V,W_n)$

6. Proof: $n\in N^+,V^n=V\times\ldots\times V$. Proof: $V^n\cong\mathcal{L}(F^n,V)$

只证了有限维的

$$\forall f\in\mathcal{L}(V,F),f=\boldsymbol{av}$$
???
$$\operatorname{length}\boldsymbol{a}=\operatorname{length}\boldsymbol{v}=\dim V$$
$$\to\mathcal{L}(V,F)\cong V$$
$$4,5$$
$$\to V\cong\mathcal{L}(F,V)$$
???

有限维貌似也不会

7. Proof: $\exists x,y\in V.U,W$是V的子空间$,x+U=y+W$. Proof: $U=W$

$$\forall u\in U,\exists w\in W,x+u=y+w$$
$$\to(x-y)-w=u\in U$$
$$x\in x+U,y\in y+W\to x-(y+w)=u\in U$$
$$x-y\in U-W$$
$$\to\forall w\in W,(x-y)-u=w\in W$$
$$\to U\subset W$$
$$U-W-W\in U\to U-W\in U$$
这里的$-$表示各个元素进行$-$运算
$$W-U\subset U$$
$$W\subset U$$

8. Proof:V的非空子集A是V的仿射子集$\Leftrightarrow\forall x,y\in A,\forall\lambda\in F\to\lambda x+(1-\lambda)y\in A$

A是V的仿射子集$\to\forall x,y\in A,\forall\lambda\in F\to\lambda v+(1-\lambda)w\in A$
$$A=v+U.\forall x,y\in A,x=v+u_1+v+u_2$$
$$\lambda(v+u_1)+(1-\lambda)(v+u_2)$$
$$=\lambda v+\lambda u_1+(1-\lambda)v+(1-\lambda)u_2$$
$$=v+\lambda u_1+(1-\lambda)u_2$$
$$\in v+U=A$$

$\forall x,y\in A,\forall\lambda\in F\to\lambda x+(1-\lambda)y\in A\to$A是仿射子集
$$\lambda x+(1-\lambda)y\in A\to\lambda x+y-\lambda y\in A$$
$$\lambda(x-y)+y\in A$$
$$\to y+S\in A$$
$$S=\lambda(x-y)$$
反向证明$\forall y,(\forall x\in A,x=y+\lambda(x-y))$
$$\forall x\in A,x+(\lambda(x-y)+y)\in A$$
$$x+\lambda x+y-\lambda y\in A$$
$$x=y+(x-y).\lambda=1$$
$$\to A\in y+S$$
$$\to A=y+\lambda(x-y)$$
由λ的任意性$,S=\lambda(x-y)$对任意固定的x,y都是子空间

$\to A=y+S$是仿射子集

9. Proof: A_1和A_2均为V的仿射子集. Proof: $A_1 \cap A_2$是V的仿射子集或空集

$$\begin{gathered}\forall x \in A_1, x = a_1 + U_1. \forall y \in A_2, y = a_2 + U_2\\ \forall t \in A_1 \cap A_2. t = a_1 + u_1 = a_2 + u_2\\ \to a_1 - a_2 = u_2 - u_1\end{gathered}$$

$$\begin{gathered}\text{设} U_1 \cap U_2 = \varnothing\\ a_1 + u_1 = a_2 + u_2\\ \to a_1 - a_2 = u_2 - u_1\\ \text{但} u_2 \text{与} u_1 \text{是不同空间中的不同向量} \to a_1 - a_2 \text{是不确定的}\\ \to A_1 \cap A_2 = \varnothing\end{gathered}$$

$$\begin{gathered}\text{设} U_1 \cap U_2 \neq \varnothing\\ \to U_1 \cap U_2 \text{是子空间}\\ a_1 + u_1 = a_2 + u_2\\ \to a_1 - a_2 = u_2 - u_1 \subset U_1 \cap U_2\\ \to a_1 - a_2 \subset U_1 \cap U_2\\ \forall x \in A_1 \cap A_2. x = a_1 - a_2 + U_1 \cap U_2 \text{是仿射子集}\end{gathered}$$

10. Proof: V的任意一组仿射子集的交是V的仿射子集或空集

$$\begin{gathered}\text{任意两个仿射子集之交是仿射子集或空集}\\ \to \text{任意有限个仿射子集之交是仿射子集是仿射自己或空集}\\ U_a \subset V. \forall x, y \in \bigcap U_a. \exists a \in E. x \in U_a, y \in U_b\\ x + y \in U_a \cap U_b \vee U_a \cap U_b = \varnothing\\ \forall x \in \bigcap U_a. \exists a \in E, x \in U_a. \lambda x \in U_a\\ \to \bigcap U_a \text{是子空间或空集}\\ \to a_1 + U_1 = a_2 + U_2 = \cdots = a_\alpha + U_\alpha\\ \text{若} \bigcap_a U_a = \varnothing \to u_a - u_{a-1} \text{必至少有一个不确定}\\ \to a_a - a_{a-1} \text{是向量空间这与} a_\alpha \text{是确定的矛盾}\\ ???\end{gathered}$$

11. $v_1, \ldots v_m \in V$.

$$A = \{\lambda_1 v_1 + \cdots + \lambda_m v_m : \lambda_1, \ldots, \lambda_m \in F \wedge \lambda_1 + \cdots + \lambda_m = 1\}$$

a. Proof: A是V的仿射子集

$$\begin{gathered}\forall x \in A. x = \sum \lambda_i v_i. \sum \lambda_i = 1\\ x = \sum \lambda_i v_i \to \lambda_1 v_1 = x - \sum_{i=2}^n \lambda_i v_i\\ v_1 = \frac{x - \sum_{i=2}^n \lambda_i v_i}{\lambda_1}\end{gathered}$$

b. Proof: V的每个包含$v_1, \ldots, v_m$的仿射子集均包含A

c. Proof: $\exists v \in V, \exists U$是$V$的子空间$\to A = v + U \wedge \dim U \leqslant m - 1$

12. Proof: U是V的子空间$\wedge V/U$是有限维的. Proof: $V \cong U \times (V/U)$.

$$\begin{gathered}U \text{是} V \text{的子空间} \to V/U \text{是有限维的}\\ V/U = \{v + U : v \in V\}\\ ???\end{gathered}$$

13. Proof: U是V的子空间, $v_1 + U, \ldots, v_m + U$是V/U的基. $u_1, \ldots, u_n$是U的基. Proof: $v_1, \ldots, v_m$, $u_1, \ldots, u_n$是V的基

$$v_1+U,\ldots v_m+U\text{是}V/U\text{的基}$$
$$\to\operatorname{span}(v_1+U,\ldots,v_m+U)=V/U\to\dim V/U=m$$
$$u_1,\ldots,u_n\text{是}U\text{的基}\to\operatorname{span}(u_1,\ldots,u_n)=U\to\dim U=n$$
$$v_1+U\ldots v_m+U\text{是基}\to v_1,\ldots,v_m\notin U$$
$$\dim V/U=\dim V-\dim U$$
$$\to\dim V=\dim V/U+\dim U$$
$$v_1,\ldots,v_m\notin U\to u_1,\ldots,u_n\text{与}v_i\text{线性无关}$$
$$\to u_1,\ldots,u_n,v_1,\ldots,v_m\text{线性无关}$$
$$\text{而}\dim V=\dim V/U+\dim U=\operatorname{length}(\boldsymbol{u})+\operatorname{length}(\boldsymbol{v})$$
$$\to\boldsymbol{u},\boldsymbol{v}\text{是}V\text{的基}$$

14. $U=\{(x_1,\ldots)\in F^\infty\colon$ 只有至多有限个$j,x_j\neq 0\}$

   a. Proof:U是F^∞的子空间

$$(0,0,\ldots)\text{有}0\text{个}x_i\neq 0\to(0,0,\ldots)\in U$$
$$\forall x,y\in U.x\text{有}M\text{个，}y\text{有}N\text{个}.M,N\in N.\operatorname{count}(x+y)\leqslant M+N$$
$$\to x+y\in U$$
$$\forall x\in U.\forall\lambda\in F.\text{count }\lambda x=M\vee 0x=0\in U$$
$$\to\lambda x\in U$$
$$\to U\text{是}F^\infty\text{的子空间}$$

   b. Proof: F^∞/U是无限维的

$$F^\infty/U\colon\text{至少有无穷个}x_i\neq 0$$
$$\to F^\infty/U\cong F^\infty$$
$$\to F^\infty/U\text{是无穷维的}$$

15. Proof: $\varphi\in\mathcal{L}(V,F),\varphi\neq 0.$ Proof: $\dim(V/\operatorname{null}\varphi)=1.$

$$\varphi\in\mathcal{L}(V,F).\varphi\neq 0$$
$$\operatorname{null}\varphi=\{v\colon\varphi(v)=0\}$$
$$\text{设}\varphi(v)=t_0.\varphi(\lambda v)=\lambda\varphi(v)=\lambda t_0$$
$$\to\dim\operatorname{range}\varphi=1$$
$$\dim V=\dim\operatorname{range}\varphi+\dim\operatorname{null}\varphi$$
$$\dim V/\operatorname{null}\varphi=\dim V-\dim\operatorname{range}\varphi=1$$

16. Proof: U是V的子空间$\wedge\dim(V/U)=1.$ Proof: $\exists\varphi\in\mathcal{L}(V,F)\to\operatorname{null}\varphi=U$

$$\varphi\in\mathcal{L}(V,F).$$
$$\dim V/U=1\to V/U=\operatorname{span}w$$
$$\varphi(\boldsymbol{au}+\boldsymbol{bv})=bw$$
$$\text{易证}\varphi(u)\in\mathcal{L}(V,F)$$
$$\forall u\in U.\varphi(u)=0$$
$$\forall w\in V/U\to\varphi(w+U)=\varphi(w)=w$$
$$U=\operatorname{span}(u,w)$$
$$\to\operatorname{null}\varphi=U$$

17. Proof: U是V的子空间，V/U是有限维的.Proof: $\exists W$是V的子空间$\to\dim W=\dim V/U\wedge V=U\oplus W$

$$V/U\text{是有限维的}\to V/U=\operatorname{span}(\boldsymbol{v}_u)$$
$$\boldsymbol{v}_u=v_i+U$$
$$W=\operatorname{span}(\boldsymbol{v})$$
$$\dim W=\dim V/U.$$
$$\forall x\in U\cap W.x=\boldsymbol{av}=u$$
$$\text{但这是不可能的},v_i\in U.V/U=\{0\}\to V=U$$

18. Proof: $T\in\mathcal{L}(V,W), U$是V的子空间.商映射$\pi: V\to V/U$. Proof: $\exists S\in\mathcal{L}(V/U,W)\to T=S\circ\pi\Leftrightarrow U\subset\operatorname{null}T$

$$\begin{gathered}S\in\mathcal{L}(V/U,W).T=S\circ\pi\to U\subset\operatorname{null}T\\ T=S\circ\pi.\operatorname{null}T=\operatorname{null}S\circ\pi\\ S\circ\pi(v)=S(v+U)\in\mathcal{L}(V/U,W)\\ u_1,u_2\in U\wedge u_1\neq u_2\to S(v+u_1)=S(v+u_2)\\ \to T(v+u_1)=T(v+u_2)\\ \to T(v)+T(u_1)=T(v)+T(u_2)\\ \to T(u_1)=T(u_2)\\ \text{设}T(u)\neq 0\to T(\lambda u)=\lambda T(u)\\ \text{let: }u_1=\lambda u_2\\ T(u_1)=\lambda T(u_2)=T(u_2)\to T(u_2)=0\\ \to T(u)=0\\ \to U\subset\operatorname{null}T\end{gathered}$$

$$\begin{gathered}U\subset\operatorname{null}T.\forall v\in V.S\circ\pi(v)=S(v+U)=S(v)+S(U)\\ \text{let: }S(\boldsymbol{av}+\boldsymbol{bu})=\boldsymbol{av}+U\\ S\circ\pi(v)=S(\pi(\boldsymbol{av}+\boldsymbol{bu}))=S(\boldsymbol{av}+U)=\boldsymbol{av}\\ \forall x,y\in V.S\circ\pi(x+y)=S\circ\pi(\boldsymbol{av}+\boldsymbol{bu}+\boldsymbol{cv}+\boldsymbol{du})\\ =S\circ\pi((\boldsymbol{a}+\boldsymbol{c})\boldsymbol{v}+U)\\ =S((\boldsymbol{a}+\boldsymbol{c})\boldsymbol{v}+U)=(\boldsymbol{a}+\boldsymbol{c})\boldsymbol{v}\\ =\boldsymbol{av}+\boldsymbol{bv}\\ =S(\boldsymbol{av}+U)+S(\boldsymbol{bv}+U)\\ \forall x\in V.\forall\lambda\in F.S\circ\pi(\lambda x)=S\circ\pi(\lambda\boldsymbol{av}+\lambda\boldsymbol{bu})\\ =S(\lambda\boldsymbol{av}+U)\\ =\lambda\boldsymbol{av}\\ =\lambda S(\boldsymbol{av}+U)\\ \to S\in\mathcal{L}(V/U,W)\end{gathered}$$

19. Example: 有限集给出一个类比于(和是直和$\Leftrightarrow$和的维数是维数的和)的命题。

    集合的并类比于子空间的和，不交并类比于直和

    $\operatorname{card}A,\operatorname{card}B<\infty.A\cap B=\varnothing\leftrightarrow\operatorname{card}A\cup B=\operatorname{card}A+\operatorname{card}B$

20. U是V的子空间.$\Gamma:\mathcal{L}(V/U,W)\to\mathcal{L}(V,W).\Gamma(S)=S\circ\pi$.

    a. Proof: Γ是线性映射

    $$\begin{gathered}\Gamma(S)=S\circ\pi\\ \forall X,Y\in\mathcal{L}(V/U,W)\\ \Gamma(X+Y)=(X+Y)\circ\pi\\ =X\circ\pi+Y\circ\pi\\ =\Gamma(X)+\Gamma(Y)\\ \Gamma(\lambda X)=\lambda X\circ\pi\\ =\lambda(\Gamma(X))\\ \to\Gamma\text{是线性映射}\end{gathered}$$

    b. Proof: Γ单

    $$\begin{gathered}\forall X\neq Y\in\mathcal{L}(V/U,W)\\ \Gamma(X)=X\circ\pi.\Gamma(Y)=Y\circ\pi\\ \forall v\in V.\text{let: }X\circ\pi(v)=Y\circ\pi(v)\\ \to X(v+U)=Y(v+U)\\ \to X(v)=Y(v)\\ \text{而}\operatorname{span}(v)=V/U\wedge v\notin\operatorname{span}(\boldsymbol{u})\\ \to X(v)\neq Y(v)\\ \to\Gamma\text{单}\end{gathered}$$

c. Proof: $\text{range}\,\Gamma = \{T \in \mathcal{L}(V, W): U \subset \text{null}\,T\}$

$$\begin{aligned} T(u) = 0.\Gamma(S) &= T = S \circ \pi \\ T(u) = S(u+U) &\to T(\boldsymbol{bu}) = S(\boldsymbol{bu}+U) \\ &= S(U) = 0 \\ &\to U \subset \text{null}\,T \\ &\to \{T \in \mathcal{L}(V, W): U \subset \text{null}\,T\} \subset \text{range}\,\Gamma \end{aligned}$$

$$\begin{aligned} &\forall T \wedge \exists u \notin U, T(u) \neq 0. \\ T(u) &= T(\boldsymbol{bu}) = S(\boldsymbol{bu}+U) \\ &= S(U) \\ &= T(0) = 0 \\ &\to u \in \text{null}\,T \\ &\to \{T \in \mathcal{L}(V, W): U \subset \text{null}\,T\} = \text{range}\,T \end{aligned}$$

6 对偶

6.1 对偶空间与对偶映射

定义 6.1. *线性泛函(linear functional)*

$$V\text{上的线性泛函} = \{\mathcal{L}(V, F)\}.$$

例 6.2. 一些线性泛函

1. $\varphi: R^3 \to R. \varphi(x, y, z) = 4x - 5y + 2z. \varphi$是$R^3$上的线性泛函

2. $(c_1, \ldots. c_n) \in F^n. \varphi: F^n \to F. \varphi(x_1, \ldots, x_n) = \sum_{i=1}^n c_i x_i. \varphi$是$F^n$上的线性泛函

3. $\varphi: \mathcal{P}(R) \to R. \varphi(p) = 3p''(5) + 7p(4). \varphi$是$\mathcal{P}(R)$上的线性泛函

4. $\varphi: \mathcal{P}(R) \to R. \varphi(p) = \int_0^1 p(x)\mathrm{d}x. \varphi$是$\mathcal{P}(R)$上的线性泛函

定义 6.3. *对偶空间(dual space), V'*

$$V\text{上的所有线性泛函构成的向量空间成为}V\text{的对偶空间, 记为}V'. V' = \mathcal{L}(V, F)$$

定理 6.4. $\dim V' = \dim V$

证明. let: $S_i(V) = a_i v_i. \forall \varphi \in V'. \varphi = \sum S_i \to \dim V' = \dim \sum c_i S_i = i \dim S_i = n$ □

定义 6.5. *对偶基(dual basis)*

$$\boldsymbol{v}\text{是}V\text{的基}. \boldsymbol{v}\text{的对偶基是}V'\text{中的元素组}\boldsymbol{\varphi}. \boldsymbol{\varphi}_i(v_k) = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$$

例 6.6. F^∞的标准基$\boldsymbol{e}$的对偶基

$\varphi_i(x_1, \ldots, x_j) = x_i.$

$\varphi_i(e_k) = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$

定理 6.7. *有限维空间中。基的对偶基是对偶空间的基*

证明.

$$\begin{aligned}
&\boldsymbol{v}\text{是}V\text{的基。}\boldsymbol{\varphi}\text{是}\boldsymbol{v}\text{的对偶基}\\
&\text{let}: 0=\boldsymbol{a}\boldsymbol{\varphi}\\
&\rightarrow\forall v\in V.\boldsymbol{a}\boldsymbol{\varphi}(v)=0\\
\text{let}: v=v_i\rightarrow \boldsymbol{a}\boldsymbol{\varphi}(v)&=\boldsymbol{a}\boldsymbol{\varphi}(v_i)=0\\
&=a_i=0\\
&\rightarrow\boldsymbol{a}=\boldsymbol{0}\\
&\rightarrow\boldsymbol{\varphi}\text{线性无关}
\end{aligned}$$

$\dim V=\dim V'=\operatorname{length}\boldsymbol{\varphi}\rightarrow\boldsymbol{\varphi}$是$V'$的基 □

定义 6.8. *对偶映射(dual map).* T关于线性泛函空间W'的对偶映射T'.

$$T\in\mathcal{L}(V,W), T\text{的对偶线性映射}T'\in\mathcal{L}(W',V'): \forall\varphi\in W', T'(\varphi)=\varphi\circ T$$

Remark: $T'(\varphi)$是一个线性泛函，T'是在W的对偶空间W'上的所有线性泛函组成的空间

由于$\varphi\in W'=\mathcal{L}(W,F).T\in\mathcal{L}(V,W)\rightarrow\varphi\circ T\in\mathcal{L}(V,F)\rightarrow T'(\varphi)\in V'$

$$\begin{aligned}
\forall\varphi,\phi\in W', T'(\varphi+\phi)=(\varphi+\phi)\circ T=\varphi\circ T+\phi\circ T=T'(\varphi)+T'(\phi)\\
\forall\lambda\in F.\varphi\in W'.T'(\lambda\varphi)=(\lambda\varphi)\circ T=\lambda(\varphi\circ T)=\lambda T'(\varphi)\\
\rightarrow\forall T'\in\mathcal{L}(W',V')
\end{aligned}$$

例 6.9. $D:\mathcal{P}(R)\rightarrow\mathcal{P}(R).Dp=p'$

1. $\varphi\in\mathcal{L}(\mathcal{P}(R),F).\varphi(p)=p(3).D'(\varphi)=(D'(\varphi))(p)=(\varphi\circ D)(p)=\varphi(D(p))=p'(3).$

   D'是$\mathcal{P}(R)$上将p变为$p'(3)$的线性泛函

2. $\varphi\in\mathcal{L}(\mathcal{P}(R),F).\varphi=\int_0^1 p(x)\mathrm{d}x.D'(\varphi)=(D'(\varphi))(p)=(\varphi\circ D)(p)=\varphi(Dp)=\varphi(p')=\int_0^1 p'(x)\mathrm{d}x$ $p(1)-p(0).$

   $D'(\varphi)$是$\mathcal{P}(R)$上将p变为$p(1)\rightarrow p(0)$的线性泛函

定理 6.10. *对偶映射的代数性质*

1. $\forall S,T\in\mathcal{L}(V,W)\rightarrow(S+T)'=S'+T'$
2. $\forall\lambda\in F,\forall T\in\mathcal{L}(V,W)\rightarrow(\lambda T)'=\lambda T'$
3. $\forall T\in\mathcal{L}(U,V),\forall S\in\mathcal{L}(V,W)\rightarrow(ST)'=T'S'$

证明.

1. $(S+T)'(\varphi)=\varphi\circ(S+T)=\varphi\circ S+\varphi\circ T=S'(\varphi)+T'(\varphi)$
2. $(\lambda T')(\varphi)=\varphi\circ\lambda T=\lambda(\varphi\circ T)=\lambda T'(\varphi)$
3. $(ST)'(\varphi)=\varphi\circ(ST)=(\varphi\circ S)\circ T=T'(\varphi\circ S)=T'(S'(\varphi))=(T'S')(\varphi)$

□

6.2 线性映射的对偶空间的零空间与值域

定义 6.11. *零化子(annihilator).* V是向量空间，U是V的子空间。U^0

$$U\subset V.U\text{的零化子}U^0=\{\varphi\in V': U\subset\operatorname{null}\varphi\}$$

Explanation: φ是V上的线性泛函.所有$\varphi(U)=0$的泛函是U的零化子

例 6.12.

设U是$\mathcal{P}(R)$的用x^2乘以多项式得到的子空间.若φ是$\mathcal{P}(R)$上由$\varphi(p)=p'(0)$定义的线性泛函.$\varphi\in U^0$

$$\forall u\in U.u=x^2p.$$
$$\varphi\in V'\rightarrow\varphi\in\mathcal{L}(\mathcal{P}(R),F)$$
$$\varphi(p)=p'(0).\varphi(u)=(x^2p)'(0)=(2xp+x^2p')(0)=0$$
$$\rightarrow U\subset\operatorname{null}\varphi$$
$$\rightarrow\varphi\in U^0$$

例 6.13.

$\boldsymbol{e}$是R^5的标准基，$\boldsymbol{\varphi}$表示$(R^5)'$的对偶基

$$U=\operatorname{span}(e_1,e_2)=\{(x_1,x_2,0,0,0)\in R^5\colon x_1,x_2\in R\}.$$
$$\varphi_i(\boldsymbol{x})=x_i$$
$$\forall\varphi\in\operatorname{span}(\varphi_3,\varphi_4,\varphi_5).\varphi(x_1,x_2,0,0,0)=0$$
$$\rightarrow U\subset\operatorname{null}\varphi$$
$$\rightarrow\varphi\in U^0$$
$$\rightarrow\operatorname{span}(\varphi_3,\varphi_4,\varphi_5)\subset U^0$$

$$\forall\varphi\in U^0.\text{对偶基}\operatorname{span}\boldsymbol{\varphi}=(R^5)'.$$
$$\rightarrow\forall\varphi\in(R^5)'=\textstyle\sum_1^5 c_i\varphi_i$$
$$e_1\in U\wedge\varphi\in U^0.\varphi(e)=0$$
$$\rightarrow 0=\varphi(e_1)=(\textstyle\sum_1^5 c_i\varphi_i)(e_1)=c_1$$
$$\text{同}\rightarrow 0=c_2$$
$$\rightarrow\forall\varphi\in U^0.\varphi=\textstyle\sum_3^5 c_i\varphi_i$$
$$\rightarrow\varphi\in\operatorname{span}(\varphi_3,\varphi_4,\varphi_5)$$
$$\rightarrow U^0\subset\operatorname{span}(\varphi_3,\varphi_4,\varphi_5)$$

$$\rightarrow U^0=\operatorname{span}(\varphi_3,\varphi_4,\varphi_5)$$

定理 6.14. *零化子是子空间*

$$U\subset V.U^0\text{是}V'\text{的子空间}$$

证明.

$$U^0=\{\varphi\in V'\colon U\subset\operatorname{null}\varphi\}$$
$$\varphi\in\mathcal{L}(V,F)\rightarrow\varphi(\mathbf{0})=0\rightarrow U\subset V=\operatorname{null}\mathbf{0}\rightarrow\mathbf{0}\in U^0$$
$$\forall\varphi,\phi\in U^0.\forall u\in U.(\varphi+\phi)(u)=\varphi(u)+\phi(u)=0+0=0\quad\text{逐点加}$$
$$\rightarrow U\subset\operatorname{null}(\varphi+\phi)\rightarrow\varphi+\phi\in U^0$$
$$\forall\lambda\in F,\forall\varphi\in U^0.\forall u\in U,(\lambda\varphi)(u)=\lambda(\varphi(u))=\lambda 0=0\qquad\text{标量乘}$$
$$\rightarrow U\subset\operatorname{null}(\lambda\varphi)\rightarrow\lambda\varphi\in U^0$$

$$\rightarrow U^0\text{是子空间}$$

□

定理 6.15. *零化子的维数*

$$\dim U + \dim U^0 = \dim V$$

证明.

$$\begin{aligned}
&\forall i \in \mathcal{L}(U, V). \forall u \in U, i(u) = u. i' \in \mathcal{L}(V', U') \\
&\dim \operatorname{range} i' + \dim \operatorname{null} i' = \dim V' \\
&\operatorname{null} i' = \{\varphi \in V' : i'(\varphi) = \mathbf{0} \in U'\} = U^0 \qquad \text{定义} \\
&\dim V = \dim V' \\
&\to \dim V = \dim U^0 + \dim \operatorname{range} i' \\
&\boldsymbol{\varphi}\text{是}U'\text{的基} \\
&\forall \varphi \in \boldsymbol{\varphi}, \varphi\text{可以扩张成}V\text{上的线性泛函组}\boldsymbol{\psi} \\
&\to i'(\boldsymbol{\psi}_i) = \varphi_i \to \varphi_i \in \operatorname{range} i' \\
&\to \operatorname{range} i' = U' \\
&\to \dim V = \dim U^0 + \dim U' \\
&\to \dim V = \dim U^0 + \dim U
\end{aligned}$$

□

定理 6.16. T的对偶映射T'的零空间

V, W是有限维, $T \in \mathcal{L}(V, W)$

1. $\operatorname{null} T' = (\operatorname{range} T)^0$
2. $\dim \operatorname{null} T' = \dim \operatorname{null} T + \dim W - \dim V$

证明.

1.
$$\begin{aligned}
&\forall \varphi \in \operatorname{null} T'. \mathbf{0} = T'(\varphi) = \varphi \circ T \\
&\forall v \in V \to 0 = (\varphi \circ T)(v) = \varphi(T(v)) \\
&\to \varphi \in (\operatorname{range} T)^0 \qquad \operatorname{range} T \to 0 \\
&\to \operatorname{null} T' \subset (\operatorname{range} T)^0 \\
\\
&\forall \varphi \in (\operatorname{range} T)^0. \\
&\forall v \in V. \varphi(T(v)) = 0 \\
&0 = \varphi \circ T = T'(\varphi) \\
&\to \varphi \in \operatorname{null} T' \\
&\to (\operatorname{range} T)^0 \subset \operatorname{null} T' \\
\\
&\to \operatorname{null} T' = (\operatorname{range} T)^0
\end{aligned}$$

2.
$$\begin{aligned}
\dim \operatorname{null} T' &= \dim (\operatorname{range} T)^0 & \dim (\operatorname{range} T)^0 = \dim W - \dim \operatorname{range} T \\
&= \dim W - \dim \operatorname{range} T & \dim W = \dim \operatorname{range} T + \dim (\operatorname{range} T)^0 \\
&= \dim W - (\dim V - \dim \operatorname{null} T) \\
&= \dim \operatorname{null} T + \dim W - \dim V
\end{aligned}$$

□

定理 6.17. T是满的$\Leftrightarrow T'$是单的

$$V, W\text{有限维}. T \in \mathcal{L}(V, W). T\text{满} \Leftrightarrow T'\text{单}$$

证明.

$T \in \mathcal{L}(V, W)$满$\Leftrightarrow \operatorname{range} T = W \Leftrightarrow (\operatorname{range} T)^0 = \{0\} \Leftrightarrow \operatorname{null} T' = \{0\} \Leftrightarrow T'$单 □

定理 6.18. T'的值域. V, W是有限维的，$T \in \mathcal{L}(V, W)$.

$$\begin{aligned}
&\dim \operatorname{range} T' = \dim \operatorname{range} T \\
&\operatorname{range} T' = (\operatorname{null} T)^0
\end{aligned}$$

证明.

1. $$\begin{aligned}\dim \operatorname{range} T' &= \dim W' - \dim \operatorname{null} T' \\ &= \dim W' - \dim(\operatorname{range} T)^0 \\ &= \dim \operatorname{range} T\end{aligned}$$

2. $$\varphi \in \operatorname{range} T'. \exists \psi \in W' \to T'(\psi) = \varphi.$$
$$\forall v \in \operatorname{null} T, \varphi(v) = (T'(\psi))(v) = (\psi \circ T)(v) = \psi(T(v)) = \psi(0) = 0$$
$$\to \varphi \in (\operatorname{null} T)^0 \to \operatorname{range} T' \subset (\operatorname{null} T)^0$$

□

$$\begin{aligned}\dim \operatorname{range} T' &= \dim \operatorname{range} T \\ &= \dim V - \dim \operatorname{null} T \\ &\dim (\operatorname{null} T)^0\end{aligned}$$

$$\begin{aligned}\dim \operatorname{range} T' = \dim (\operatorname{null} T)^0 \wedge \operatorname{range} T' \subset \operatorname{null} T^0 \\ \to \operatorname{range} T' = (\operatorname{null} T)^0\end{aligned}$$

定理 6.19. T单$\Leftrightarrow T'$满

证明.

$T \in \mathcal{L}(V, W)$单$\Leftrightarrow \operatorname{null} T = \{0\} \Leftrightarrow (\operatorname{null} T)^0 = V' \Leftrightarrow \operatorname{range} T' = V' \Leftrightarrow T'$满 □

6.3 对偶映射的矩阵，转置

定义 6.20. *矩阵的转置(transpose).A'*

$$矩阵A \in F^{m,n}的转置. A^t \in F^{n,m}; 元素A^t_{(i,j)} = A_{j,i}$$

例 6.21. 矩阵的转置

$$A = \begin{pmatrix} 5 & -7 \\ 3 & 8 \\ -4 & 2 \end{pmatrix} \Leftrightarrow A^t = \begin{pmatrix} 5 & 3 & -4 \\ -7 & 8 & 2 \end{pmatrix}$$

定理 6.22. *转置对加法和标乘不变*

$$\begin{gathered}\forall A, B \in \mathcal{M}(m, n), \forall \lambda \in F \\ (A + B)^t = A^t + B^t. \\ (\lambda A)^t = \lambda(A^t)\end{gathered}$$

定理 6.23. *矩阵乘积的转置*

$$A \in F^{m,n}, B \in F^{n,p} \to (AB)^t = B^t A^t$$

证明.

$$\begin{aligned}(AB)^t_{k,j} &= (AB)_{j,k} \\ &= \sum_{r=1}^{n} A_{j,r} \cdot B_{r,k} \\ &= \sum_{r=1}^{n} (B^t)_{k,r} \cdot (A^t)_{r,j} \\ &= (B^t A^t)_{k,j}\end{aligned}$$

$\square$

定理 6.24. *对偶映射的矩阵是原映射矩阵的转置*

$$T\in\mathcal{L}(V,W).\mathcal{M}(T')=(\mathcal{M}(T))^t$$

证明.

$$A=\mathcal{M}(T).C=\mathcal{M}(T').1\leqslant j\leqslant m,1\leqslant k\leqslant n$$
$$T'(\psi_j)=\sum_{r=1}^{n}C_{r,j}\cdot\varphi_r$$
$$T'(\psi_j)=\psi_j\circ T\to(\psi_j\circ T)(v_k)=\sum_{r=1}^{n}C_{r,j}\cdot\varphi_r(v_k)=C_{k,j}$$

$$\begin{aligned}(\psi_j\circ T)(v_k)&=\psi_j(T(v_k))\\&=\psi_j\Big(\sum_{r=1}^{m}A_{r,k}\cdot w_r\Big)\\&=\sum_{r=1}^{m}A_{r,k}\cdot\psi_j(w_r)\\&=A_{j,k}\end{aligned}$$

$$\to C_{k,j}=A_{j,k}$$

$$\to C=A^t$$

$\square$

定义 6.25. *矩阵的行秩、列秩*

$$\begin{array}{ll} & A\in F^{m,n}.\\ \text{行秩} & \dim\operatorname{span}(A_{\cdot,i})\\ \text{列秩} & \dim\operatorname{span}(A_{j,\cdot})\end{array}$$

例 6.26. 矩阵的秩

$$A=\begin{pmatrix}4&7&1&8\\3&5&2&9\end{pmatrix}.$$
$$\dim\operatorname{span}((4,7,1,8),(3,5,2,9))=2$$
$$\dim\operatorname{span}\left(\begin{pmatrix}4\\3\end{pmatrix},\begin{pmatrix}7\\5\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}8\\9\end{pmatrix}\right)\leqslant2\wedge\forall\lambda\in R.\lambda\begin{pmatrix}4\\3\end{pmatrix}\neq\begin{pmatrix}7\\5\end{pmatrix}.\to\dim\operatorname{span}\sim=2$$

定理 6.27. $\dim\operatorname{range}T=\mathcal{M}(T)$的列秩

$$V,W\text{有限维}.\forall T\in\mathcal{L}(V,W).\operatorname{range}T=\mathcal{M}(T)\text{的列秩}$$

证明.

$$v_1,\ldots,v_n\text{是}V\text{的基},w_1,\ldots,w_n\text{是}W\text{的基}.$$
$$w\in\operatorname{span}(Tv_1,\ldots,Tv_n)\to\mathcal{M}(w)\text{的函数是}\operatorname{span}(T\boldsymbol{v})\text{和}\operatorname{span}(\boldsymbol{w})\text{的同构}$$
$$\to\dim\operatorname{span}(T\boldsymbol{v})=\dim\operatorname{span}(\mathcal{M}(Tv_1),\ldots,\mathcal{M}(Tv_n)).$$
$$\text{而}\mathcal{M}(Tv_i)\text{是}\mathcal{M}(T)\text{的列}$$
$$\operatorname{range}T=\operatorname{span}(Tv_1,\ldots,Tv_n)$$
$$\to\dim\operatorname{range}T=\mathcal{M}(T)\text{的列秩}$$

$\square$

定理 6.28. *行秩等于列秩*

$$\forall A\in F^{m,n}.A\text{的行秩}=A\text{的列秩}$$

证明.

$$
\begin{aligned}
T\colon F^{n,1}\to F^{m,1}.Tx=Ax.\mathcal{M}(T)=A.\\
A\text{的列秩}=\mathcal{M}(T)\text{的列秩}\\
=\dim\operatorname{range}T\\
=\dim\operatorname{range}T'\\
=\mathcal{M}(T')\text{的列秩}\\
=A^t\text{的列秩}\\
=A\text{的行秩}
\end{aligned}
$$

□

定义 6.29. *矩阵的秩(rank)*

$$\forall A\in F^{m,n}.\operatorname{rank}A=A\text{的列秩}$$

习题3.F

1. Explanation: 线性泛函是满的或零

$$
\begin{aligned}
\forall\varphi\in\mathcal{L}(V,F).\varphi=\mathbf{0}\to\varphi\in\mathcal{L}(V,F).\\
\varphi\neq 0.\exists v\in V\to\varphi(v)\neq 0.\operatorname{let}\varphi(v)=t\\
\forall\lambda\in F,\varphi(\lambda v)=\lambda t.F=\operatorname{span}(t)\\
\to\varphi\text{满}
\end{aligned}
$$

2. Exapmle: $R^{[0,1]}$上的三个不同的线性泛函

$$
\begin{aligned}
&\forall f\in R^{[0,1]},f\colon[0,1]\to R\\
1\quad&\varphi_1(f)=f\left(\tfrac{1}{2}\right)\\
2\quad&\varphi_2(f)=f(0)\\
3\quad&\varphi_3(f)=f(1)\\
&\text{没有可微、可积条件}\\
&\varphi(f)=\textstyle\int_0^1 f(x)\mathrm{d}x\\
&\varphi(f)=f'\left(\tfrac{1}{2}\right)
\end{aligned}
$$

3. Proof: V有限维，$v\in V\wedge v\neq 0.$ Proof: $\exists\varphi\in V'\to\varphi(v)=1$

$$
\begin{aligned}
\text{let: }\varphi(v)=1..\\
\forall\lambda\in F,\varphi(\lambda v)=\lambda\varphi(v),\text{给出}v\text{的所有倍数的定义}\\
\operatorname{span}v\text{是}V\text{的一维子空间}\\
\varphi(U+v)=\varphi(U).\forall x\in V=au+\boldsymbol{b}\boldsymbol{v}\qquad\text{基的扩张}\\
\varphi(au+\boldsymbol{b}\boldsymbol{v})=a\in F.\\
\varphi\in\mathcal{L}(V,F)\to\varphi\in V'
\end{aligned}
$$

4. Proof: V有限维,U是V的子空间$\wedge U\neq V.$ Proof: $\exists\varphi\in V'\to\forall u\in U,\varphi(u)=0\wedge\varphi\neq\mathbf{0}$

$$
\begin{aligned}
U=\operatorname{span}\boldsymbol{u}.V=\operatorname{span}(\boldsymbol{u},\boldsymbol{v}).\\
\varphi(v)=\varphi(\boldsymbol{a}\boldsymbol{u}+\boldsymbol{b}\boldsymbol{v})=\textstyle\sum\boldsymbol{b}\in F\\
\varphi(x+y)=\varphi(\boldsymbol{a}\boldsymbol{u}+\boldsymbol{b}\boldsymbol{v}+\boldsymbol{c}\boldsymbol{u}+\boldsymbol{d}\boldsymbol{v})\\
=\varphi((\boldsymbol{a}+\boldsymbol{c})\boldsymbol{u}+(\boldsymbol{b}+\boldsymbol{d})\boldsymbol{v})\\
=\textstyle\sum(\boldsymbol{b}+\boldsymbol{d})\\
=\textstyle\sum\boldsymbol{b}+\sum\boldsymbol{d}\\
=\varphi(\boldsymbol{a}\boldsymbol{u}+\boldsymbol{b}\boldsymbol{v})+\varphi(\boldsymbol{c}\boldsymbol{u}+\boldsymbol{d}\boldsymbol{v})\\
=\varphi(x)+\varphi(y)\\
\varphi(\lambda x)=\lambda\varphi(x)\\
\to\varphi\in V'\\
v=(1,\ldots,1)\cdot\boldsymbol{v}.v\notin U,\varphi(v)=\varphi((1,\ldots,1)\cdot\boldsymbol{v})=\dim U^0\neq 0
\end{aligned}
$$

5. Proof: $V_1,\ldots,V_m$为向量空间. Proof: $(V_1\times\cdots\times V_m)'\cong V_1'\times\cdots\times V_m'$

$$
\begin{gathered}
(V_1\times\cdots\times V_m)'=\mathcal{L}(V_1\times\cdots\times V_m,F)\\
V_1\cong V_1',\ldots,V_m\cong V_m'\\
\rightarrow\mathcal{L}(V_1,F)\cong\mathcal{L}(V_1',F)\\
\rightarrow\mathcal{L}(V_1\times\cdots\times V_m,F)\cong\mathcal{L}(V_1'\times\cdots\times V_m',F)\\
\rightarrow V_1\times\cdots\times V_m\cong V_1'\times\cdots\times V_m'\\
V_1\times\cdots\times V_m\cong(V_1\times\cdots\times V_m)'\\
\rightarrow(V_1\times\cdots\times V_m)'\cong V_1'\times\cdots\times V_m'
\end{gathered}
$$

6. V有限维, $v_1,\ldots,v_m\in V$. $\Gamma: V'\rightarrow F^m$, $\Gamma(\varphi)=(\varphi(v_1),\ldots,\varphi(v_m))$

   a. Proof: $\operatorname{span}(\boldsymbol{v})=V\Leftrightarrow\Gamma$单

$$
\begin{gathered}
\operatorname{span}(\boldsymbol{v})=V\rightarrow\Gamma\text{单}\\
\operatorname{span}(\boldsymbol{v})=V.\Gamma(\varphi)=(\varphi(v_1),\ldots,\varphi(v_m))\\
\varphi_1,\varphi_2\in V'.\varphi_1\neq\varphi_2.\\
\Gamma(\varphi_1)=(\varphi_1(v_1),\ldots,\varphi_1(v_m))\\
\Gamma(\varphi_2)=(\varphi_2(v_1),\ldots,\varphi_2(v_m))\\
\text{Assume: }\Gamma(\varphi_1)=\Gamma(\varphi_2)\rightarrow\varphi_1(v_i)=\varphi_2(v_i)\\
\operatorname{span}(\boldsymbol{v})=V\rightarrow\varphi_1(\boldsymbol{av})=\varphi_2(\boldsymbol{av})\\
\forall v\in V.v=\boldsymbol{av}.\\
\rightarrow\boldsymbol{a}\varphi_1(\boldsymbol{v})=\boldsymbol{a}\varphi_2(\boldsymbol{v})\\
\rightarrow\varphi_1(\boldsymbol{v})=\varphi_2(\boldsymbol{v})\\
\rightarrow\varphi_1=\varphi_2\text{这是不可能的}\\
\rightarrow\Gamma(\varphi_1)\neq\Gamma(\varphi_2)\\
\rightarrow\Gamma\text{单}
\end{gathered}
$$

$$
\begin{gathered}
\Gamma\text{单}\rightarrow\operatorname{span}(\boldsymbol{v})=V\\
\Gamma(\varphi_1)=\Gamma(\varphi_2)\rightarrow\varphi_1=\varphi_2\\
\Gamma(\varphi_1)=(\varphi_1(v_1),\ldots,\varphi_1(v_m))=(\varphi_2(v_1),\ldots,\varphi_2(v_m))=\Gamma(\varphi_2)\\
\varphi_1(v_1)=\varphi_2(v_1),\ldots,\varphi_1(v_m)=\varphi_2(v_m)\rightarrow\varphi_1=\varphi_2\\
\text{设}\operatorname{span}(\boldsymbol{v})\subseteq V.\operatorname{span}(\boldsymbol{v},\boldsymbol{u})=V.\forall i\in1\ldots m.\varphi_1(v_i)=\varphi_2(v_i)\\
\text{let: }\varphi_1(u)=0.\varphi_2(u)\neq0,\text{这样的}\varphi_1,\varphi_2\text{是存在的}\\
\rightarrow\text{矛盾}\\
\rightarrow\operatorname{span}(\boldsymbol{v})=V
\end{gathered}
$$

   b. Proof: $\boldsymbol{v}$线性无关$\Leftrightarrow\Gamma$满

$$
\begin{gathered}
\boldsymbol{v}\text{线性无关}\rightarrow\Gamma\text{满}\\
\forall x\in F^m,\Gamma(\varphi)=(\varphi(v_1),\ldots,\varphi(v_m))\\
\varphi\in V'.\varphi(\lambda v_1)=\lambda\varphi(v_1)\\
\boldsymbol{v}\text{线性无关}\rightarrow\varphi(\boldsymbol{v})\text{是线性无关的}.\\
0=(\varphi(v_1),\ldots,\varphi(v_m)).v_1,\ldots,v_m\neq0\\
\text{这是不可能的，除非}\varphi=\boldsymbol{0}\\
\rightarrow\varphi(\boldsymbol{v})\text{也线性无关}\\
\rightarrow\exists\varphi\in V'.\Gamma(\varphi)=x\\
\rightarrow\Gamma\text{满}
\end{gathered}
$$

$$
\begin{gathered}
\Gamma\text{满}\rightarrow\boldsymbol{v}\text{线性无关}\\
\forall x\in F^m,\exists\varphi\in V'\rightarrow\Gamma(\varphi)=x.\\
\text{Assume: }\boldsymbol{v}\text{线性相关}\rightarrow v_i=\sum c_iv_i\\
\varphi(v_i)=\varphi(\sum c_iv_i)\\
\varphi(v_i)=\boldsymbol{c}\varphi(\boldsymbol{v})\\
\varphi(v_i)=c_i\varphi(\boldsymbol{v})\text{这样的}\dim\varphi(\boldsymbol{v})=\dim F^m-1\\
\rightarrow\Gamma\text{不满}\\
\rightarrow\text{矛盾}\\
\rightarrow\boldsymbol{v}\text{线性无关}
\end{gathered}
$$

7. Proof: $m\in N^+$. Proof: $\mathcal{P}_m(R)$的基$x^0,\dots,x^m$的对偶基$\varphi_0,\dots,\varphi_m$. $\varphi_n(p)=\frac{p^{(n)}(0)}{n!}$. $p^{(n)}$为n阶导数

$$\begin{gathered}
\text{根据对偶基的定义:}\\
\varphi_i(x^0)=0,\dots,\varphi_i(x^i)=1,\dots,\varphi_i(x^m)=0\\
\varphi\in\mathcal{P}_m(R)\to\varphi_i(p)=\varphi_i(\textstyle\sum a_ix^i)=a_i\\
p=a_ix^i+\cdots+a_1x+0!a_0x^0\\
p'=i\,a_ix^{i-1}+(i-1)a_{i-1}x^{i-2}+\cdots+1!a_1x^0\\
p''=i(i-1)a_ix^{i-2}+(i-1)(i-2)a_{i-1}x^{i-2}+\cdots+2!\cdot a_2x^0\\
\dots\\
p^{(n)}=\tfrac{i!}{(i-n)!}a_ix^{i-n}+\tfrac{(i-1)!}{(i-n-1)!}a_{i-1}x^{i-n-1}+\cdots+n!\cdot a_nx^0\\
p^{(n)}(0)=n!a_nx^0\\
a_n=\tfrac{p^{(n)}(0)}{n!}\\
\to\varphi_i(p)=\tfrac{p^{(n)}(0)}{n!}
\end{gathered}$$

8. $m\in N^+$.

   a. Proof: $(x-5)^0,\dots,(x-5)^m$是$\mathcal{P}_m(R)$的基

$$\begin{gathered}
\forall i\neq j,i,j\in0\dots m.\,(x-5)^i\neq\lambda(x-5)^j\\
(x-5)^m=\textstyle\sum c^i(x-5)^i\to c^i=1\\
(x-5)^m=\textstyle\sum(x-5)^i\text{这是不可能的}\\
\to(x-5)^i\text{在}\mathcal{P}_m(R)\text{中线性无关}\\
\text{length}((x-5)^m)=m+1=\dim\mathcal{P}_m(R)\\
\to(x-5)^0,(x-5)^1,\dots,(x-5)^m\text{是}\mathcal{P}_m(R)\text{的基}
\end{gathered}$$

   b. Compute: $(x-5)^n$的对偶基

$$\begin{gathered}
\text{根据对偶基定义:}\\
\varphi_i((x-5)^0)=0,\dots,\varphi_i((x-5)^i)=1,\dots,\varphi_i((x-5)^m)=0\\
\forall p\in\mathcal{P}_m(R).\,p=\textstyle\sum_0^m c_i(x-5)^m\\
p=c_m(x-5)^m+c_{m-1}(x-5)^{m-1}+\cdots+0!\cdot c_0(x-5)^0\\
p'=c_m\cdot m(x-5)^{m-1}+c_{m-1}\cdot(m-1)(x-5)^{m-1}+\cdots+1!\cdot c_1(x-5)^0\\
p^{(n)}=c_m\tfrac{m!}{(m-n)!}(x-5)^{m-n}+c_{m-1}\cdot\tfrac{(m-1)!}{(m-n-1)!}(x-5)^{m-n-1}+\cdots+c_n\cdot n!(x-5)^0\\
\to c_n=\tfrac{p^{(n)}(5)}{n!}=\tfrac{0+0+\cdots+c_n\cdot n!}{n!}\\
\to\varphi_n=\tfrac{p^{(n)}(5)}{n!}
\end{gathered}$$

9. Proof: $\boldsymbol{v}$是V的基, $\boldsymbol{\varphi}$是V'的对应基. $\psi\in V'$. Proof: $\psi=\psi(v_1)\varphi_1+\cdots+\psi(v_n)\varphi_n$

$$\begin{gathered}
\forall\psi\in V',\boldsymbol{\varphi}\text{是}V'\text{的基}\to\psi=\textstyle\sum c_i\varphi_i\\
\leftarrow\psi(v_i)=c_i\\
\forall v\in V,v=\boldsymbol{av}.\psi(v)=\psi(\boldsymbol{av})=\boldsymbol{a}\psi(\boldsymbol{v})\\
=\textstyle\sum a_i\psi(v_i)\\
\leftarrow\psi(v)=\textstyle\sum\psi(v_i)\varphi_i(v)\\
=\textstyle\sum\psi(v_i)\varphi_i(\boldsymbol{av})\\
=\textstyle\sum\psi(v_i)(\sum a_i\varphi(v_i))\\
???
\end{gathered}$$

10.

11. $$\begin{gathered}A\in F^{m,n}\wedge A\neq0.\text{ Proof:}\\ \text{rank}\,A=1\Leftrightarrow\exists\boldsymbol{c}\in F^m,\exists\boldsymbol{d}\in F^n\to\forall j\in1\dots m,\forall k\in1\dots n\to A_{j,k}=c_jd_k\end{gathered}$$

$$
\begin{gathered}
\operatorname{rank} A = 1 \rightarrow \exists \boldsymbol{c} \in F^m, \exists \boldsymbol{d} \in F^n, A_{i,j} = c_i\, d_j \\
\operatorname{rank} A = 1 \Leftrightarrow \dim(\operatorname{span}(A_{\cdot,i})) = 1 \\
\rightarrow \operatorname{span}(A_{\cdot,i}) = \{\lambda A_{\cdot,i}, \lambda \in F\} \\
\text{let: } \boldsymbol{d} = A_{\cdot,i} \neq 0.\, A \neq 0 \rightarrow \text{这样的}\boldsymbol{d}\text{是存在的} \\
A_{\cdot,i} = \lambda_i \boldsymbol{d} \\
\text{let: } c_i = \lambda_i \\
A_{i,j} = \lambda_i d_j
\end{gathered}
$$

$$
\begin{gathered}
\exists \boldsymbol{c} \in F^m, \exists \boldsymbol{d} \in F^n, A_{i,j} = c_i d_j \rightarrow \operatorname{rank} A = 1 \\
A_{\cdot,i} = c_i \boldsymbol{d} \rightarrow A_{\cdot,i} = \frac{c_i}{c_j} \boldsymbol{d} \\
\operatorname{rank} A = \operatorname{span}\left(\frac{c_i}{c_j} \boldsymbol{d}\right) = 1.\text{除非}\boldsymbol{c} = 0 \wedge \boldsymbol{d} = 0\text{此时}\operatorname{rank} A = 0
\end{gathered}
$$

12. Proof: I_V的对偶映射是$I_{V'}$

$$
\begin{gathered}
\forall v \in V, I_V(v) = v. \\
\forall \varphi \in V', (I_V)'(\varphi) = \varphi \circ I_V \\
(I_V)'(\varphi)(v) = \varphi \circ I_V(v) = \varphi(v) \\
\rightarrow (I_V)'(\varphi) = \varphi \\
\rightarrow (I_V)' = I_{V'}
\end{gathered}
$$

13. T: $R^3 \rightarrow R^2$, $T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z)$. φ_1, φ_2是R^2标准基的对偶基, ψ_1, ψ_2, ψ_3是R^3标准基的对偶基.

    a. 描述: $T'(\varphi_1), T'(\varphi_2)$

    $$
    \begin{gathered}
    R^2\text{的标准基}: (1,0), (0,1) \\
    \varphi_1((1,0)) = 1, \varphi_1((0,1)) = 0 \\
    \varphi_2((1,0)) = 0, \varphi_2((0,1)) = 1 \\
    \rightarrow \varphi_1(x,y) = x.\, \varphi_2(x,y) = y. \\
    \psi_1(x,y,z) = x; \psi_2(x,y,z) = y; \psi_3(x,y,z) = z \\
    T'(\varphi_1) = \varphi_1 \circ T = 4x + 5y + 6z \\
    T'(\varphi_2) = \varphi_2 \circ T = 7x + 8y + 9z
    \end{gathered}
    $$

    b. 计算: $T'(\varphi_1), T'(\varphi_2)$的$\psi_1$, ψ_2, ψ_3的线性组合

    $$
    \begin{gathered}
    \forall (x,y,z) \in R^3.\, T'(\varphi_1)(x,y,z) = (\varphi_1 \circ T)(x,y,z) = (4x + 5y + 6z)(x,y,z) \\
    \forall (x,y,z) \in R^3.\, T'(\varphi_2)(x,y,z) = (\varphi_2 \circ T)(x,y,z) = (7x + 8y + 9z)(x,y,z) \\
    T'(\varphi_1)(x,y,z) = 4\psi_1(x,y,z) + 5\psi_2(x,y,z) + 6\psi_3(x,y,z) \\
    T'(\varphi_2)(x,y,z) = 7\psi_1(x,y,z) = 8\psi_2(x,y,z) + 9\psi_3(x,y,z)
    \end{gathered}
    $$

14. T: $\mathcal{P}(R) \rightarrow \mathcal{P}(R), T(p)(x) = (x^2 p)(x) + p''(x)$

    a. $\varphi \in \mathcal{P}(R)', \varphi(p) = p'(4)$.描述$\mathcal{P}(R)$上的线性泛函$T'(\varphi)$

    b. $\varphi \in \mathcal{P}(R)', \varphi(p) = \int_0^1 p(x) \mathrm{d}x$.求: $(T'(\varphi))(x^3)$

15. Proof: W有限维.$T \in \mathcal{L}(V, W)$. Proof: $T' = \mathbf{0} \Leftrightarrow T = \mathbf{0}$

16. Proof: V, W有限维.Proof: $T \in \mathcal{L}(V, W) \rightarrow T' \in \mathcal{L}(W', V')$的映射是$\mathcal{L}(V, W)$和$\mathcal{L}(W', V')$的同构

17. Explanation: $U \subset V$. Explanation: $U^0 = \{\varphi \in V' : U \subset \operatorname{null} \varphi\}$

18. Proof: V有限维.$U \subset V$. Proof: $U = \{0\} \Leftrightarrow U^0 = V'$

19. Proof: V有限维.U是V的子空间. Proof: $U = V \Leftrightarrow U^0 = \{0\}$

20. Proof: $U \subset V, W \subset V. U \subset W$. Proof: $W^0 \subset U^0$

21. Proof: V有限维,U, W是V的子空间且$W^0 \subset U^0$. Proof: $U \subset W$.

22. Proof: U, W是V的子空间. Proof: $(U+W)^0 = U^0 \cap W^0$

23. Proof: V是有限维的.U, W是V的子空间. Proof: $(U \cap W)^0 = U^0 + W^0$

24.

25. Proof: V是有限维的.U是V的子空间. Proof: $U = \{v \in V : \forall \varphi \in U_0 \to \varphi(v) = 0\}$.

26. Proof: V是有限维的.Γ是V'的子空间. Proof: $\Gamma = \{v \in V : \forall \varphi \in \Gamma \to \varphi(v) = 0\}^0$

27. Proof: $T \in \mathcal{L}(\mathcal{P}_5(R), \mathcal{P}_5(R)) \wedge \operatorname{null} T' = \operatorname{span}(\varphi)$. φ是$\mathcal{P}_5(R)$上的$\varphi(p) = p(8)$定义的线性泛函.
Proof: $\operatorname{range} T = \{p \in \mathcal{P}_5(R) : p(8) = 0\}$

28. Proof: V, W是有限维的.$T \in \mathcal{L}(V, W)$.$\exists \varphi \in W' \to \operatorname{null} T' = \operatorname{span}(\varphi)$. Proof: $\operatorname{range} T = \operatorname{null} \varphi$

29. Proof: V, W是有限维的.$T \in \mathcal{L}(V, W)$.$\exists \varphi \in V' \to \operatorname{range} T' = \operatorname{span}(\varphi)$. Proof: $\operatorname{null} T = \operatorname{null} \varphi$

30. Proof: V是有限维的.$\varphi_1, ..., \varphi_m$是$V'$中的一个线性无关组. Proof: $\dim((\operatorname{null} \varphi_1) \cap \cdots \cap (\operatorname{null} \varphi_m)) = (\dim V) - m$

31. Proof: V是有限维的.$\varphi_1, \ldots, \varphi_m$是$V'$的基. Proof: $\exists V$的基使得其对偶基是$\varphi_1, \ldots, \varphi_m$

32. $T \in \mathcal{L}(V)$.$u_1, \ldots, u_n$是V的基.$v_1, \ldots, v_n$是V的基. Proof: 下列命题等价

$\mathcal{M}(T) = \mathcal{M}(T, u, v)$

1. T可逆
2. $\mathcal{M}(T)$的列在$F^{n,1}$中是线性无关的
3. $\operatorname{span}(\mathcal{M}(T)_{\cdot,i}) = F^{n,1}$
4. $\mathcal{M}(T)$的行在$F^{1,n}$中线性无关
5. $\operatorname{span}(\mathcal{M}(T)_{i,\cdot}) = F^{1,n}$

33. Proof: $m, n \in N^+$. Proof: $\varphi: A \to A^t \in \mathcal{L}(F^{m,n}, F^{n,m}) \wedge \varphi$可逆

$$\begin{array}{ll} \forall A, B \in F^{m,n} & \varphi(A+B) = (A+B)^t = A^t + B^t = \varphi(A) + \varphi(B) \\ \forall A \in F^{m,n}, \lambda \in F & \varphi(\lambda A) = (\lambda A)^t = \lambda A^t = \lambda \varphi(A) \\ & \to \varphi \in \mathcal{L}(F^{m,n}, F^{n,m}) \end{array}$$

$$\begin{gathered} \forall A \neq B \to \exists i, j \in \ldots \to A_{i,j} \neq B_{i,j} \\ \varphi(A)_{j,i} \neq \varphi(B)_{j,i} \to \varphi(A) \neq \varphi(B) \\ \to \varphi\text{单} \\ \forall X \in F^{n,m}, X^t \in F^{m,n}. \varphi(X^t) = (X^t)^t = X \\ \to \varphi\text{满} \end{gathered}$$

$$\to \varphi\text{可逆}$$

34. 定义V的二次对偶空间(V'')是V'的对偶空间.$V'' = (V')'$
$\Lambda: V \to V''$,$\forall v \in V, \varphi \in V'$.$\Lambda(v)(\varphi) = \varphi(v)$
    1. Proof: $\Lambda \in \mathcal{L}(V, V'')$
    2. Proof: $T \in \mathcal{L}(V) \to T'' \circ \Lambda = \Lambda \circ T$
    3. Proof: V是有限维的,Λ是V和V''的同构

35. Proof: $(\mathcal{P}(R))' \cong R^\infty$

36. U是V的子空间.设$i: U \to V, i(u) = u. i' \in \mathcal{L}(V', U')$.

a. Proof: $\operatorname{null} i' = U^0$

b. Proof: V是有限维的$\to \operatorname{range} i' = U'$

c. Proof: V是有限维的$\to \tilde{i'}$是V' / U^0和U'的同构

37. U是V的子空间,$\pi: V \to V / U$是商映射,$\pi' \in \mathcal{L}((V / U)', V')$.

a. Proof: π'单

b. Proof: $\operatorname{range} \pi' = U^0$

c. Proof: π'是$(V / U)'$和U^0的同构