第七章 内积空间上的算子

本章主要研究有限维内积空间上的算子

1 自伴算子与正规算子

1.1 伴随

定义 1.1. *伴随(adjoint).T^**

$$\begin{gathered} T \in \mathcal{L}(V, W). \\ \text{伴随} \quad T^*: W \to V, \forall v \in V, \forall w \in W \to \langle Tv, w\rangle = \langle v, T^* w\rangle \end{gathered}$$

这个定义是有意义的。

$$\begin{gathered} T \in \mathcal{L}(V, W). \\ \forall w \in W, V\text{上的线性泛函}\varphi(v) = \langle Tv, w\rangle \\ \text{Reisz} \to \exists s \in V, \varphi(v) = \langle v, s\rangle \\ \text{若}T^* w = s \to \langle v, s\rangle = \langle v, T^* w\rangle = \varphi(v) = \langle Tv, w\rangle \end{gathered}$$

Remark：线性代数中还有一种伴随，和这种伴随没什么关系

例 1.2. $T: R^3 \to R^2, T(x_1, x_2, x_3) = (x_2 + 3x_3, 2x_1)$.求$T^*$

$$\begin{aligned} & T^*: R^2 \to R^3. \\ \langle (x_1, x_2, x_3), T^*(y_1, y_2)\rangle = & \langle T(x_1, x_2, x_3), (y_1, y_2)\rangle \\ = & \langle (x_2 + 3x_3, 2x_1), (y_1, y_2)\rangle \\ = & (x_2 + 3x_3) y_1 + 2x_1 y_2 \\ = & 2x_1 y_2 + x_2 y_1 + 3x_3 y_1 \\ = & \langle (x_1, x_2, x_3), (2y_2, y_1, 3y_1)\rangle \\ \to & T^*(y_1, y_2) = (2y_2, y_1, 3y_1) \end{aligned}$$

例 1.3. $u \in V, x \in W. T \in \mathcal{L}(V, W), \forall v \in V, Tv = \langle v, u\rangle x$.求$T^*$

$$\begin{aligned} & \forall w \in W. \forall v \in V \\ \langle v, T^* w\rangle = & \langle Tv, w\rangle \\ = & \langle \langle v, u\rangle x, w\rangle \\ = & \langle v, u\rangle \langle x, w\rangle \\ = & \langle v, \langle w, x\rangle u\rangle \\ \to & T^* w = \langle w, x\rangle u \end{aligned}$$

定理 1.4. *伴随是线性映射*

$$T \in \mathcal{L}(V, W). T^* \in \mathcal{L}(W, V)$$

证明.

$$T\in\mathcal{L}(V,W).w_1,w_2\in W$$
$$\begin{aligned}
v\in V\to\langle v,T^*(w_1+w_2)\rangle=&\langle Tv,w_1+w_2\rangle\\
=&\langle Tv,w_1\rangle+\langle Tv,w_2\rangle\\
=&\langle v,T^*w_1\rangle+\langle v,T^*w_2\rangle\\
=&\langle v,T^*w_1+T^*w_2\rangle\\
\to&T^*(w_1+w_2)=T^*w_1+T^*w_2
\end{aligned}$$

$$\begin{aligned}
&\forall w\in W,\forall\lambda\in F,\forall v\in V\\
&\to\langle v,T^*(\lambda w)\rangle=\langle Tv,\lambda w\rangle\\
&=\bar\lambda\langle Tv,w\rangle\\
&=\bar\lambda\langle v,T^*w\rangle\\
&=\langle v,\lambda T^*w\rangle\\
&\to T^*(\lambda w)=\lambda T^*(w)
\end{aligned}$$

$$\to T^*\in\mathcal{L}(W,V)$$

□

定理 1.5. *伴随的性质*

1	$\forall S,T\in\mathcal{L}(V,W)$	$(S+T)^*=S^*+T^*$
2	$\forall\lambda\in F,\forall T\in\mathcal{L}(V,W)$	$(\lambda T)^*=\bar\lambda T^*$
3	$\forall T\in\mathcal{L}(V,W)$	$(T^*)^*=T$
4	$I_V\in\mathcal{L}(V)$	$I^*=I$
5	$\forall T\in\mathcal{L}(V,W),\forall S\in\mathcal{L}(W,U).U$是内积空间	$(ST)^*=T^*S^*$

证明.

1 $\quad(S+T)^*=S^*+T^*$
$$\begin{aligned}
&S,T\in\mathcal{L}(V,W).v\in V,w\in W\\
&\langle v,(S+T)^*(w)\rangle=\langle(S+T)v,w\rangle\\
&=\langle Sv+Tv,w\rangle\\
&=\langle Sv,w\rangle+\langle Tv,w\rangle\\
&=\langle v,S^*w\rangle+\langle v,T^*w\rangle\\
&=\langle v,S^*w+T^*w\rangle\\
&\to(S+T)^*=S^*+T^*
\end{aligned}$$

2 $\quad(\lambda T)^*=\bar\lambda T^*$
$$\begin{aligned}
&\forall\lambda\in F,\forall T\in\mathcal{L}(V,W)\\
&\langle v,(\lambda T)^*w\rangle=\langle(\lambda T)v,w\rangle\\
&=\lambda\langle Tv,w\rangle\\
&=\lambda\langle v,T^*w\rangle\\
&=\langle v,\bar\lambda T^*w\rangle\\
&\to(\lambda T)^*=\bar\lambda T^*
\end{aligned}$$

3 $\quad(T^*)^*=T$
$$\begin{aligned}
&T\in\mathcal{L}(V,W)\to T^*\in\mathcal{L}(W,V)\to T^*\in\mathcal{L}(V,W)\\
&\langle w,(T^*)^*v\rangle\\
&=\overline{\langle T^*w,v\rangle}\\
&=\overline{\langle v,T^*w\rangle}\\
&=\overline{\langle Tv,w\rangle}\\
&=\langle w,Tv\rangle\\
&\to(T^*)^*=T
\end{aligned}$$

4 $\quad I^*=I$
$$\begin{aligned}
&\langle v,I^*w\rangle\\
&=\langle Iv,w\rangle\\
&=\langle v,w\rangle\\
&\to I^*=I_V
\end{aligned}$$

$$
\begin{aligned}
5 \quad & (ST)^* = T^*S^* \\
& T \in \mathcal{L}(V,W), S \in \mathcal{L}(W,U) \\
& v \in V \wedge u \in U \\
& \langle v, (ST)^*u \rangle = \langle STv, u \rangle \\
& = \langle Tv, S^*u \rangle \\
& = \langle v, T^*(S^*u) \rangle \\
& \to (ST)^* = T^*S^*
\end{aligned}
$$

□

定理 1.6. *伴随的零空间和值域*

$$
\begin{aligned}
& T \in \mathcal{L}(V,W) \\
1 \quad & \operatorname{null} T^* = (\operatorname{range} T)^\perp \\
2 \quad & \operatorname{range} T^* = (\operatorname{null} T)^\perp \\
3 \quad & \operatorname{null} T = (\operatorname{range} T^*)^\perp \\
4 \quad & \operatorname{range} T = (\operatorname{null} T^*)^\perp
\end{aligned}
$$

证明.

$$
\begin{aligned}
1 \quad & \operatorname{null} T^* = (\operatorname{range} T)^\perp \\
& w \in W \\
& w \in \operatorname{null} T^* \Leftrightarrow T^*w = 0 \\
& \Leftrightarrow \forall v \in V, \langle v, T^*w \rangle = 0 \\
& \Leftrightarrow \forall v \in V, \langle Tv, w \rangle = 0 \\
& \Leftrightarrow w \in (\operatorname{range} T)^\perp \\
& \to \operatorname{null} T^* = (\operatorname{range} T)^\perp
\end{aligned}
$$

$$
\begin{aligned}
4 \quad & \operatorname{range} T = (\operatorname{null} T^*)^\perp \\
& 1 \to \operatorname{null} T^* = (\operatorname{range} T)^\perp \\
& \Leftrightarrow (\operatorname{null} T^*)^\perp = ((\operatorname{range} T)^\perp)^\perp \\
& \Leftrightarrow \operatorname{range} T = (\operatorname{null} T^*)^\perp
\end{aligned}
$$

$$
\begin{aligned}
3 \quad & \operatorname{null} T = (\operatorname{range} T^*)^\perp \\
& 1 \to \operatorname{null} T^* = (\operatorname{range} T)^\perp \\
& \Leftrightarrow \operatorname{null} (T^*)^* = (\operatorname{range} T^*)^\perp \\
& \Leftrightarrow \operatorname{null} T = (\operatorname{range} T^*)^\perp
\end{aligned}
$$

$$
\begin{aligned}
2 \quad & \operatorname{range} T^* = (\operatorname{null} T)^\perp \\
& 4 \to \operatorname{range} T = (\operatorname{null} T^*)^\perp \\
& \Leftrightarrow \operatorname{range} T^* = (\operatorname{null} (T^*)^*)^\perp \\
& \Leftrightarrow \operatorname{range} T^* = (\operatorname{null} T)^\perp
\end{aligned}
$$

□

定义 1.7. *共轭转置(conjugate transpose)*

$$
\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m\times n} \to \begin{pmatrix} \overline{a_{11}} & \cdots & \overline{a_{m1}} \\ \vdots & \ddots & \vdots \\ \overline{a_{1n}} & \cdots & \overline{a_{mn}} \end{pmatrix}_{n\times m}
$$

定理 1.8. *伴随的矩阵*

$$
\begin{gathered}
T \in \mathcal{L}(V,W).\, \boldsymbol{e}\text{是}V\text{的规范正交基}, \boldsymbol{w}\text{是}W\text{的规范正交基} \\
\to \mathcal{M}(T^*, \boldsymbol{w}, \boldsymbol{e})\text{是}\mathcal{M}(T, \boldsymbol{e}, \boldsymbol{w})\text{的共轭转置}
\end{gathered}
$$

证明.

$$
\begin{aligned}
& \forall i, Te_i = \langle Te_1, w_1 \rangle w_1 + \cdots + \langle Te_n, w_n \rangle w_n \\
& \to \mathcal{M}(T, \boldsymbol{e}, \boldsymbol{w})_{i,j} = \langle Te_i, w_j \rangle \\
& T^*w_i = \langle T^*w_1, e_1 \rangle + \cdots + \langle T^*w_m, e_m \rangle e_m \\
& \to \mathcal{M}(T^*, \boldsymbol{w}, \boldsymbol{e})_{i,j} = \langle T^*w_i, e_j \rangle \\
& \langle Te_j, w_i \rangle_{j,i} = \overline{\langle w_i, Te_j \rangle} = \overline{\langle T^* w_i, e_j \rangle} = \overline{\mathcal{M}(T^*, \boldsymbol{w}, \boldsymbol{e})_{i,j}}
\end{aligned}
$$

□

1.2 自伴算子

定义 1.9. *自伴的(self-adjoint)*

$$\begin{array}{ccc} & \text{算子}T\in\mathcal{L}(V) & \\ \text{自伴的} & T=T^* & \Leftrightarrow\langle Tv,w\rangle=\langle v,Tw\rangle \end{array}$$

例 1.10. 求算子的未知数使得算子是自伴的

$$\begin{pmatrix} 2 & b \\ 3 & 7 \end{pmatrix}$$

$\mathcal{M}(T)=\mathcal{M}(T^*)\to\overline{\mathcal{M}'}(T)=\mathcal{M}(T)\to b=3$

定理 1.11. *自伴算子的和是自伴的。实数和自伴算子的积是自伴的*

$$\begin{gathered} T,S\text{是自伴的}\to T+S\text{是自伴的} \\ \lambda\in R,\lambda T\text{是自伴的} \end{gathered}$$

证明.

$$\begin{aligned} \langle(T+W)v,w\rangle &=\langle Tv+Wv,w\rangle \\ &=\langle Tv,w\rangle+\langle Wv,w\rangle \\ &=\langle v,Tw\rangle+\langle v,Ww\rangle \\ &=\langle v,Tw+Ww\rangle \\ &=\langle v,(T+W)w\rangle \end{aligned}$$

□

$$\begin{aligned} \langle\lambda Tv,w\rangle &=\lambda\langle Tv,w\rangle \\ &=\lambda\langle v,Tw\rangle \\ &=\langle v,\bar{\lambda}Tw\rangle \end{aligned}$$

Remark: 算子的伴随算子相当于复数的共轭。如果算子是自伴的，那么它相等于实数。

定理 1.12. *复空间上，自伴算子的本征值是实的*

证明.

$$\begin{gathered} T\text{是}V\text{上的自伴算子},\lambda\text{是}T\text{的本征值，}v\text{是}V\text{中的非零向量}\to Tv=\lambda v \\ \lambda\|v\|^2=\lambda\langle v,v\rangle=\langle\lambda v,v\rangle=\langle Tv,v\rangle=\langle v,Tv\rangle=\langle v,\lambda v\rangle=\bar{\lambda}\langle v,v\rangle=\lambda\|v\|^2 \\ \to\lambda=\bar{\lambda}\to\lambda\in R \end{gathered}$$

□

定理 1.13. *复空间上只有0算子才能使得Tv总是正交于v*

$$V\text{是复内积空间},T\in\mathcal{L}(V).\forall v\in V,\langle Tv,v\rangle=0\to T=\mathbf{0}$$

Remark: 此定理对实内积空间不成立

example:

$$\begin{aligned} &T\in\mathcal{L}(R^2),T(x,y)=(-y,x) \\ &\to\forall v\in V\to\langle Tv,v\rangle=0 \end{aligned}$$

证明.

$$\forall u,w\in V$$
$$\langle Tu,w\rangle = \frac{\langle T(u+w),u+w\rangle - \langle T(u-w),u-w\rangle}{4} + \frac{\langle T(u+iw),u+iw\rangle - \langle T(u-iw),u-iw\rangle}{4} i$$
$$\forall u,w\in V \to \langle Tu,w\rangle = 0$$
$$\to \forall u,w, \langle T(u+w),u+w\rangle = 0 \to T=0$$

□

定理 1.14. *复内积空间上，仅自伴算子才能使得* $\langle Tv,v\rangle \in R$

$$V\text{是复内积空间}, T\in\mathcal{L}(V). T\text{自伴} \Leftrightarrow \forall v\in V, \langle Tv,v\rangle \in R$$

证明.

$$\forall v\in V$$
$$\langle Tv,v\rangle - \overline{\langle Tv,v\rangle} = \langle Tv,v\rangle - \langle v,Tv\rangle = \langle Tv,v\rangle - \langle T^*v,v\rangle = \langle (T-T^*)v,v\rangle$$
$$\forall v\in V, \langle Tv,v\rangle \in R \to \langle Tv,v\rangle - \overline{\langle Tv,v\rangle} = 0$$
$$\to \langle T-T^*\rangle = \mathbf{0}$$
$$\to T\text{是自伴的}$$

□

$$T\text{是自伴的} \to T=T^* \to \langle (T-T^*)v,v\rangle = 0$$
$$\to \langle Tv,v\rangle - \overline{\langle Tv,v\rangle} = 0$$
$$\to \langle Tv,v\rangle \in R$$

定理 1.15. *若算子是自伴的且* $\langle Tv,v\rangle = 0 \to T=\mathbf{0}$

$$T\text{是}V\text{上的自伴算子使得}\forall v\in V, \langle Tv,v\rangle = 0 \to T=0$$

Remark: 在复内积空间已经证明任何算子都满足（不需要自伴）

证明.

$$\text{设}V\text{是实内积空间}. u,w\in V$$
$$\langle Tu,w\rangle = \frac{\langle T(u+w),u+w\rangle - \langle T(u-w),u-w\rangle}{4}$$
$$\langle Tw,u\rangle = \langle w,Tu\rangle = \langle Tu,w\rangle \qquad \text{这里自伴使得第二个等号成立}$$
$$\forall u,w\in V \to \langle Tu,w\rangle = 0 \to \langle T(u+w),u+w\rangle = 0$$
$$\to T=0$$

□

1.3 正规算子

定义 1.16. *正规的(normal)*

$$\text{内积空间上的算子称为正规的。若它和它的伴随可交换}. TT^* = T^*T$$

Remark: 自伴算子都是正规的.$T=T^* \to TT^* = TT = T^*T$

例 1.17. 正规但不自伴的算子

$$\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$$

$$\mathcal{M}(T) \neq \overline{\mathcal{M}(T)} \to T\text{不是自伴的}$$
$$\mathcal{M}(TT^*) = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 2\times2+-3\times-3 & 2\times3-3\times2 \\ 3\times2+2\times-3 & 3\times3+2\times2 \end{pmatrix} = \begin{pmatrix} 13 & 0 \\ 0 & 13 \end{pmatrix}$$
$$\mathcal{M}(T^*T) = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 2\times2+3\times3 & 2\times-3+3\times2 \\ -3\times2+2\times3 & -3\times-3+2\times2 \end{pmatrix} = \begin{pmatrix} 13 & 0 \\ 0 & 13 \end{pmatrix}$$
$$\to T\text{是正规的}$$

定理 1.18. T正规$\Leftrightarrow \forall v \in V, \|Tv\| = \|T^*v\|$

证明.

$$\begin{aligned}
& T \in \mathcal{L}(V). \\
& T\text{是正规的} \Leftrightarrow T^*T = TT^* \\
& \Leftrightarrow \langle (T^*T - TT^*)v, v\rangle = 0 \\
& \Leftrightarrow \langle T^*Tv, v\rangle = \langle TT^*v, v\rangle \\
& \langle T^*Tv, v\rangle = \langle Tv, Tv\rangle \\
& \langle TT^*v, v\rangle = \langle T^*v, T^*v\rangle \\
& \Leftrightarrow \|Tv\|^2 = \|T^*v\|^2
\end{aligned}$$

Remark: 根据此定理 $\to \operatorname{null} T = \operatorname{null} T^*$ □

定理 1.19. *正规算子。 算子和伴随具有相同的本征向量*

$$T \in \mathcal{L}(V)\text{是正规的}, v \in V\text{是}T\text{相应与}\lambda\text{的本征向量} \to v\text{是}T^*\text{相对}\bar{\lambda}\text{的本征向量}$$

证明.

$$\begin{aligned}
& T\text{是正规的}: TT^* = T^*T \\
& I_V\text{是自伴的} \to I^* = I \\
(T - \lambda I)(T - \lambda I)^* &= (T - \lambda I)(T^* - \bar{\lambda} I^*) \\
&= TT^* - \lambda I T^* - T\bar{\lambda} I^* + \lambda I \bar{\lambda} I^* \\
&= T^*T - \bar{\lambda} I^* T - T^* \lambda I + \bar{\lambda} I^* \lambda I \\
&= (T^* - \bar{\lambda} I^*)(T - \lambda I) \\
&= (T - \lambda I)^*(T - \lambda I) \\
& \to T - \lambda I\text{是正规的}
\end{aligned}$$

$$\begin{aligned}
0 = \|(T - \lambda I)v\| = \|(T - \lambda I)^*v\| = \|(T^* - \bar{\lambda} I^*)v\| \\
\to v\text{是}T^*\text{关于本征值}\bar{\lambda}\text{的特征向量}
\end{aligned}$$

□

定理 1.20. *正规算子不同本征值对应的本征向量是正交的*

证明.

$$\begin{aligned}
& \alpha, \beta\text{是}T\text{不同的本征值}. u, v\text{是分别对应的本征向量} \\
& Tu = \alpha u; Tv = \beta v \\
& T^*v = \bar{\beta} v \\
(\alpha - \beta)\langle u, v\rangle &= \langle \alpha u, v\rangle - \langle u, \bar{\beta} v\rangle \\
&= \langle Tu, v\rangle - \langle u, T^*v\rangle \\
&= 0 \\
& \alpha \neq \beta \to \langle u, v\rangle = 0
\end{aligned}$$

□

7.A

2 谱定理

线性算子关于V的某个规范正交基有对角矩阵，那么这个算子是V上性质最好的算子。

谱定理表明：$F = C$.这样的算子为正规算子.$F = R$.这样的算子为自伴算子

2.1 复谱定理

定理 2.1. *复谱定理*

$$\begin{array}{cc} & F=C,T\in\mathcal{L}(V);\text{三条件等价}\\ 1 & T\text{是正规的}\\ 2 & V\text{有一个有本征向量组成的规范正交基}\\ 3 & T\text{关于}V\text{的某个规范正交基有对角阵}\end{array}$$

证明.

$$\begin{gathered} 3\rightarrow 1\\ T\text{在某个规范正交基下有对角矩阵}\mathcal{M}(T)\\ \mathcal{M}(T^*)=(\overline{\mathcal{M}(T)})^t\text{也是对角矩阵}\\ \rightarrow\mathcal{M}(T)\mathcal{M}(T^*)=\mathcal{M}(T^*)\mathcal{M}(T).\text{因为对角矩阵本身可交换}\end{gathered}$$

$$\begin{gathered} 1\rightarrow 3\\ T\text{正规。舒尔定理}\rightarrow T\text{关于某个规范正交基必有上三角矩阵}\\ \mathcal{M}(T,\boldsymbol{e})=\begin{pmatrix} a_{1,1} & \cdots & a_{1,n}\\ 0 & \ddots & \vdots\\ 0 & 0 & a_{n,n}\end{pmatrix}\\ \|Te_1\|^2=\langle a_{1,1}e_1,a_{1,1}e_1\rangle=|a_{1,1}|^2\\ \|T^*e_1\|^2=|a_{1,1}|^2+\cdots+|a_{1,n}|^2\\ T\text{是正规的}\rightarrow\|Te_1\|=\|T^*e_1\|\\ \rightarrow a_{1,2}=\cdots=a_{1,n}=0\\ \text{同样的}\|Te_2\|=|a_{1,2}|^2+|a_{2,2}|^2=|a_{2,2}|^2\\ =\|T^*e_2\|=|a_{2,2}|^2+\cdots+|a_{2,n}|^2\\ \rightarrow a_{2,3}=\cdots=a_{2,n}=0\\ \vdots\\ \mathcal{M}(T,\boldsymbol{e})\text{是对角矩阵}\end{gathered}$$

$$2???$$

□

为了证明实谱定理，需要几个引理

定理 2.2. *可逆的二次式*

$$T\in\mathcal{L}(V)\text{是自伴的},b,c\in R\rightarrow b^2<4c\rightarrow T^2+bT+cI\text{是可逆的}$$

证明.

$$\begin{array}{cc} v\text{是}V\text{中的非零向量},T\text{是自伴的} & \\ \langle(T^2+bT+cI)v,v\rangle=\langle T^2v,v\rangle+b\langle Tv,v\rangle+c\langle v,v\rangle & \\ =\langle Tv,Tv\rangle+b\,\langle Tv,v\rangle+c\,\|v\|^2 & |\langle u,v\rangle|\leqslant\|u\|\,\|v\|\\ \geqslant\|Tv\|^2-|b|\,\|Tv\|\,\|v\|+c\,\|v\|^2 & \text{Cauchy}-\text{Schwarz不等式}\\ =\Big(\|Tv\|-\frac{|b|\,\|v\|}{2}\Big)^2+\Big(c-\frac{b^2}{4}\Big)\|v\|^2 & \\ >0 & \\ \rightarrow v\neq 0\rightarrow(T^2+bT+cI)v\neq 0 & \\ \rightarrow\text{null}\,(T^2+bT+cI)=\{0\} & \\ \rightarrow T^2+bT+cI\text{单}\rightarrow T^2+bT+cI\text{可逆} & \end{array}$$

□

定理 2.3. *实内积空间。 自伴算子都有本征值*

$$V\neq\{0\}.T\in\mathcal{L}(V)\text{是自伴算子}\rightarrow T\text{有本征值}$$

证明.

$$\begin{gathered}
V\text{是实内积空间}.n=\dim V.v\in V\wedge v\neq 0\\
v,Tv,T^2v,\ldots,T^nv\text{不可能线性无关}\\
\rightarrow 0=a_0v+a_1Tv+\cdots+a_nT^nv,a_0,\ldots,a_n\text{不全为}0\\
a_0+a_1x+\cdots+a_nx^n=c(x^2+b_1x+c_1)\cdots(x^2+b_Mx+c_M)(x-\lambda_1)\cdots(x-\lambda_m)\\
b_i,c_i,\lambda_i\in R.\,b_i^2<4c_i.(m+M)\geqslant 1\\
0=a_0v+a_1Tv+\cdots+a_nT^nv\\
=(a_0I+a_1T+\cdots+a_nT^n)v\\
=c(T^2+b_1T+c_1I)\cdots(T^2+b_MT+c_MI)(T-\lambda_1I)\cdots(T-\lambda_mI)v\\
\text{每个}T^2+b_iT+c_iI\text{都可逆}\\
\rightarrow m>0\wedge 0=(T-\lambda_1I)\cdots(T-\lambda_mI)v\\
\rightarrow\text{至少有一个}(T-\lambda_iI)\text{是不单的}\\
\rightarrow T\text{有本征值}
\end{gathered}$$

□

定理 2.4. *自伴算子与不变子空间*

$$\begin{array}{ll}
 & T\in\mathcal{L}(V)\text{是自伴的，}U\text{是}V\text{在}T\text{下的不变子空间}\\
1 & U^{\perp}\text{在}T\text{下不变}\\
2 & T|_U\in\mathcal{L}(U)\text{是自伴的}\\
3 & T|_{U^{\perp}}\in\mathcal{L}(U^{\perp})\text{是自伴的}
\end{array}$$

证明.

$$\begin{array}{l}
1\quad \forall v\in U^{\perp},u\in U\rightarrow\langle v,u\rangle=0\\
\qquad\langle Tv,u\rangle=\langle v,Tu\rangle=0\qquad T\text{自伴且}Tu\in U\\
\qquad\quad\rightarrow Tv\in U^{\perp}
\end{array}$$

$$\begin{array}{l}
2\quad \forall u,v\in U.\,\langle T|_Uu,v\rangle=\langle Tu,v\rangle=\langle u,Tv\rangle=\langle u,(T|_U)v\rangle\\
\qquad\qquad\rightarrow T|_U\text{是自伴的}
\end{array}$$

$$\begin{array}{l}
3\qquad\qquad U^{\perp}\text{在}T\text{下不变}\\
\quad\rightarrow\forall u,v\in U^{\perp}\rightarrow\langle(T|_{U^{\perp}})\,u,v\rangle\\
\ =\langle Tu,v\rangle=\langle u,Tv\rangle=\langle u,(T|_U)\,v\rangle
\end{array}$$

□

定理 2.5. *实谱定理*

$$\begin{array}{ll}
 & \text{实内积空间中，算子}T\in\mathcal{L}(V).\text{三条件等价}\\
1 & T\text{是自伴的}\\
2 & V\text{有一个由}T\text{的本征向量组成的规范正交基}\\
3 & T\text{关于}V\text{的某个规范正交基具有对角矩阵}
\end{array}$$

证明.

$$\begin{array}{ll}
\qquad\qquad 3\rightarrow 1 & \\
T\text{关于}V\text{的规范正交基}\boldsymbol{e},\mathcal{M}(T,\boldsymbol{e})\text{是对角阵} & \\
\quad\rightarrow\mathcal{M}(T^*,\boldsymbol{e})=(\overline{\mathcal{M}(T,\boldsymbol{e})})^t & \\
\qquad=\mathcal{M}(T,\boldsymbol{e})^t & \\
\qquad=\mathcal{M}(T,\boldsymbol{e}) & \text{实对角阵的共轭转置是自身}\\
\qquad\rightarrow T=T^* &
\end{array}$$

$$1\to 2$$
$$\dim V=1.$$
$$Tv=\lambda v\to e_1=\frac{v}{\|v\|}\text{是},e_1\text{是本征向量且是}V\text{的规范正交基}$$
$$\text{设}\dim V=n>1.\dim V<n\text{都有}V\text{的规范正交基}$$
$$T\in\mathcal{L}(V)\text{是自伴的}.u\text{是}T\text{的本征向量且}\|u\|=1 \qquad 2.3\text{保证存在}$$
$$U=\operatorname{span}(u)\to U\text{是}V\text{的一维不变子空间}$$
$$\to T|_{U^\perp}\in\mathcal{L}(U^\perp)\text{是自伴的} \qquad 2.4$$
$$\to U^\perp\text{具有规范正交基}\boldsymbol{u}$$
$$\to\operatorname{length}(\boldsymbol{u},u)=\dim V=n\to(\boldsymbol{u},u)\text{是}V\text{的基}$$

$$2\to 3$$
$$V\text{有一个}T\text{的本征向量组成的规范正交基}\boldsymbol{e}$$
$$\boldsymbol{e}\text{是本征向量}\to\mathcal{M}(T,\boldsymbol{e})\text{是对角阵}$$

□

例 2.6. R^3上的自伴算子T

$$\mathcal{M}(T)=\begin{pmatrix}14&-13&8\\-13&14&8\\8&8&-7\end{pmatrix}$$

$$\text{验证}\frac{(1,-1,0)}{\sqrt{2}},\frac{(1,1,1)}{\sqrt{3}},\frac{(1,1,2)}{\sqrt{6}}\in R^3\text{是}T\text{的本征向量组成的规范正交基，}\mathcal{M}(T,\boldsymbol{e})\text{是对角阵}$$
$$T\text{是自伴的}\to T\text{具有特征值}$$
$$(T-\lambda I)=\begin{pmatrix}14-\lambda&-13&8\\-13&14-\lambda&8\\8&8&-7-\lambda\end{pmatrix}$$
$$T\left(\frac{(1,-1,0)}{\sqrt{2}}\right)=\lambda\left(\frac{(1,-1,0)}{\sqrt{2}}\right)$$
$$\begin{pmatrix}14&-13&8\\-13&14&8\\8&8&-7\end{pmatrix}\begin{pmatrix}1\\-1\\0\end{pmatrix}=(\lambda,-\lambda,0)$$
$$\lambda=14+13+8\times 0;-\lambda=-13\times 1+14\times -1;8\times 1+8\times -1=0$$
$$\lambda=27\text{时满足}\to(1,-1,0)\frac{1}{\sqrt{2}}\text{是特征向量}$$
$$\ldots$$
$$\boldsymbol{e}=\mathcal{M}(T,\boldsymbol{e})\boldsymbol{e}^t$$
$$\left(\frac{1,-1,0}{\sqrt{2}},\frac{1,1,1}{\sqrt{3}},\frac{1,1,2}{\sqrt{6}}\right)=\mathcal{M}(T,\boldsymbol{e})\left(\frac{1,-1,0}{\sqrt{2}},\frac{1,1,1}{\sqrt{3}},\frac{1,1,2}{\sqrt{6}}\right)^t$$
$$a_{1,1}\times\frac{1,-1,0}{\sqrt{2}}+a_{1,2}\times\frac{1,1,1}{\sqrt{3}}+a_{1,3}\times\frac{1,1,2}{\sqrt{6}}=\frac{1,-1,0}{\sqrt{2}}$$
$$\to\frac{a_{1,1}}{\sqrt{2}}+\frac{a_{1,2}}{\sqrt{3}}+\frac{a_{1,3}}{\sqrt{6}}=\frac{1}{\sqrt{2}}$$
$$\frac{-a_{1,1}}{\sqrt{2}}+\frac{a_{1,2}}{\sqrt{3}}+\frac{a_{1,3}}{\sqrt{6}}=\frac{-1}{\sqrt{2}}$$
$$\frac{a_{1,2}}{\sqrt{3}}+\frac{2a_{1,3}}{\sqrt{6}}=0$$
$$\to\frac{\sqrt{2}a_{1,2}}{\sqrt{6}}=\frac{a_{1,3}}{\sqrt{6}}\to\sqrt{2}a_{1,2}=a_{1,3}$$
$$\to\frac{a_{1,1}}{\sqrt{2}}+\frac{a_{1,2}}{\sqrt{3}}+\frac{\sqrt{2}a_{1,2}}{\sqrt{6}}\to\frac{a_{1,1}}{\sqrt{2}}+\frac{2a_{1,2}}{\sqrt{3}}=\frac{1}{\sqrt{2}}$$
$$\to\frac{-a_{1,1}}{\sqrt{2}}+\frac{a_{1,2}}{\sqrt{3}}+\frac{\sqrt{2}a_{1,2}}{\sqrt{6}}\to\frac{-a_{1,1}}{\sqrt{2}}+\frac{2a_{1,2}}{\sqrt{3}}=\frac{-1}{\sqrt{2}}$$
$$\to a_{1,2}=0\wedge a_{1,1}=1$$
$$???\text{这里求}\mathcal{M}(T,\boldsymbol{e})\text{的方法有点问题}$$

Remark: 在复内积空间中，复谱定理给出了V上的正规算子的完全描述。因此可以完全描述V上的自伴算子。(自伴 $\to$ 正规 $\wedge$ 本征值都是实的)

在实内积空间中，实谱定理给出了V上自伴算子的完全描述。在第九章给出V上正规算子的完全描述

7.B

3 正算子与等距同构

3.1 正算子

定义 3.1. *正算子(positive operator)*

$$\text{算子}\mathcal{L}(V)\text{是正的}: T\text{是自伴的} \wedge \forall v \in V, \langle Tv, v\rangle \geqslant 0$$

Remark:在复空间上，自伴条件可以删去.因为 1.14

Remark: 正算子实际上对应于 $[0, +\infty)$ *中的实数*

例 3.2. 正算子

$$\begin{array}{cc} 1 & U\text{是}V\text{的子空间，正交投影}P_U\text{是正算子} \\ & \langle P_U v, v\rangle = \langle P_U(u+w), (u+w)\rangle = \langle u, u+w\rangle = \langle u, u\rangle + \langle u, w\rangle \\ & = \|u\|^2 + 0 \geqslant 0 \\ 2 & T \in \mathcal{L}(V)\text{是自伴的}, b, c \in R \wedge b^2 < 4c. T^2 + bT + cI\text{是正算子} \\ & 2.2 \end{array}$$

定义 3.3. *算子的平方根(square root)*

$$\text{算子}R\text{称为算子}T\text{的平方根}: R^2 = T$$

例 3.4. 有些算子是有平方根的

$$\begin{gathered} T \in \mathcal{L}(F^3), T(z_1, z_2, z_3) = (z_3, 0, 0). \\ R \in \mathcal{L}(F^3), R(z_1, z_2, z_3) = (z_2, z_3, 0) \\ R^2(z_1, z_2, z_3) = R(z_2, z_3, 0) = (z_3, 0, 0) = T \end{gathered}$$

定理 3.5. *正算子的刻画(充要条件)*

$$\begin{array}{cc} & T \in \mathcal{L}(V).\text{五条件等价} \\ 1 & T\text{是正的} \\ 2 & T\text{是自伴的且}T\text{的所有本征值非负} \\ 3 & T\text{有正的平方根} \\ 4 & T\text{有自伴的平方根} \\ 5 & \exists R \in \mathcal{L}(V), T = R^*R \end{array}$$

证明.

$$\begin{gathered} 1 \to 2 \\ T\text{是正的} \to T\text{是自伴的。} \\ \lambda\|v\|^2 = \lambda\langle v, v\rangle = \langle \lambda v, v\rangle = \langle Tv, v\rangle \geqslant 0 \end{gathered}$$

$$\begin{gathered} 2 \to 3 \\ T\text{是自伴的且}T\text{的所有本征值非负} \\ T\text{自伴} \to T\text{有一个本征向量组成的规范正交基}\boldsymbol{e} \\ \boldsymbol{e}\text{中的向量对应的特征值}\lambda_1, \ldots, \lambda_n, \lambda_i \geqslant 0 \\ R \in \mathcal{L}(V), Re_i = \sqrt{\lambda_i} e_i \\ \langle Re_i, e_i\rangle = \langle \sqrt{\lambda_i} e_i, e_i\rangle = \sqrt{\lambda}\|e_i\|^2 \geqslant 0 \\ \to R\text{是正算子} \\ R^2 e_i = R(\sqrt{\lambda_i} e_i) = \sqrt{\lambda_i}(\sqrt{\lambda_i} e_i) = \lambda_i e_i = Te_i \\ \to R^2 = T \\ Rv = \sqrt{\lambda_1}\langle v, e_1\rangle e_1 + \cdots + \sqrt{\lambda_n}\langle v, e_n\rangle e_n \\ \to R\text{是}T\text{的正平方根} \end{gathered}$$

$$3 \to 4$$
$$\langle Rv,v\rangle = \langle v,R^*v\rangle = \left\langle \sum \sqrt{\lambda_i}\langle v,e_i\rangle e_i, \sum \langle v,e_i\rangle e_i\right\rangle$$
$$= \sum \left(\sqrt{\lambda_i}(\langle v,e_i\rangle \times \overline{\langle v,e_i\rangle}) \cdot \langle e_i,e_i\rangle\right)$$
$$= \sum \left(\sqrt{\lambda_i}\|v,e_i\|^2 \cdot \|e_i\|^2\right)$$
$$= \sum \left(\sqrt{\lambda_i}\|v,e_i\|^2\right) = \langle v,R^*v\rangle$$
$$\langle v,Rv\rangle = \left\langle \sum \langle v,e_i\rangle e_i, \sum \sqrt{\lambda_i}\langle v,e_i\rangle e_i\right\rangle$$
$$= \sum \left(\langle v,e_i\rangle \overline{\sqrt{\lambda_i}\langle v,e_i\rangle} \cdot \langle e_i,e_i\rangle\right)$$
$$= \sum \left(\overline{\sqrt{\lambda_i}}\langle v,e_i\rangle \overline{\langle v,e_i\rangle} \cdot \|e_i\|^2\right)$$
$$\lambda_i \in R \to \overline{\sqrt{\lambda_i}} = \sqrt{\lambda_i}$$
$$\to \langle Rv,v\rangle = \langle v,Rv\rangle$$
$$\to R^* = R$$

$$4 \to 5$$
T有自伴的平方根R
$$R = R^* \to R^*R = R^2 = T$$

$$5 \to 1$$
$$R \in \mathcal{L}(V).T = R^*R$$
$$T^* = (R^*R)^* = R^*(R^*)^* = R^*R = T$$
$\to T$是自伴的
$$\langle Tv,v\rangle = \langle R^*Rv,v\rangle = \langle Rv,Rv\rangle = \|Rv\|^2 \geqslant 0$$
$\to T$是正的

□

定理 3.6. *每个正算子都有唯一的正平方根*

证明.

$T \in \mathcal{L}(V)$是正的,v是T的一个本征向量
$$\to \exists\lambda \geqslant 0 \to Tv = \lambda v$$
设R是T的正平方根,需要证明$Rv = \sqrt{\lambda}v$
$\to R$的本征向量是$\sqrt{\lambda_i}$
V有一个T的本征向量构成的基　　　谱定理
$\to R$是唯一的

V有一个R的本征向量组成的规范正交基$\boldsymbol{e}$
R是正算子$\to R$的本征值$\boldsymbol{\lambda}$都是非负的
$$\to Re_i = \sqrt{\lambda_i}e_i$$
$$\to v = \sum a_ie_i$$
$$Rv = R(\sum a_ie_i)$$
$$= \sum R(a_ie_i)$$
$$= \sum a_iR(e_i)$$
$$= \sum a_i\sqrt{\lambda_i}e_i$$
$$R^2v = R(\sum a_i\sqrt{\lambda_i}e_i)$$
$$= \sum R(a_i\sqrt{\lambda_i}e_i)$$
$$= \sum a_i\sqrt{\lambda_i}R(e_i)$$
$$= \sum a_i\lambda_ie_i$$
$$= Tv$$
若v是T的本征向量
$$Tv = \lambda v$$
$$\to \sum a_i\lambda_ie_i = \lambda\sum a_ie_i$$
$$\to a_i(\lambda - \lambda_i)e_i = 0$$
$$\to v = \sum_{\lambda_i=\lambda} a_ie_i$$
$$Rv = \sum_{\lambda_i=\lambda} a_i\sqrt{\lambda}e_i = \sqrt{\lambda}v$$
$\to v$也是R的本征向量,本征值是$\sqrt{\lambda_i}$

□

3.2 等距同构

这里的距离表示范数。因此等距同构表示范数在算子下不变

定义 3.7. *等距同构(isometry)*

$$\begin{gathered}S\in\mathcal{L}(V)\\ \text{等距同构}\quad \forall v\in V,\|Sv\|=\|v\|\end{gathered}$$

例 3.8.

λ_n是绝对值为1的标量，$\boldsymbol{e}$是V的规范正交基.$S\in\mathcal{L}(V),Se_i=\lambda_i e_i.S$是等距同构

$$\begin{aligned}v\in V\\ v=&\langle v,e_1\rangle e_1+\cdots+\langle v,e_n\rangle e_n\\ \|v\|^2=&|\langle v,e_1\rangle|^2+\cdots+|\langle v,e_n\rangle|^2\\ Sv=&\langle v,e_1\rangle Se_1+\cdots+\langle v,e_n\rangle Se_n\\ =&\lambda_1\langle v,e_1\rangle e_1+\cdots+\lambda_n\langle v,e_n\rangle e_n\\ \to\|Sv\|^2=&\|\lambda_1\langle v,e_1\rangle e_1+\cdots+\lambda_n\langle v,e_n\rangle e_n\|^2\\ =&\|\lambda_1\langle v,e_1\rangle e_1\|^2+\cdots+\|\lambda_n\langle v,e_n\rangle e_n\|^2\\ =&|\lambda_1\langle v,e_1\rangle|^2+\cdots+|\lambda_n\langle v,e_n\rangle|^2\\ |\lambda_i|=1\to\|Sv\|^2=&|\langle\lambda,e_1\rangle|^2+\cdots+|\langle\lambda,e_n\rangle|^2=\|v\|^2\\ \to&S\text{是等距同构}\end{aligned}$$

Remark: 实内积空间上的等距算子称为正交算子；复内积空间上的等距算子称为酉算子

定理 3.9. *等距算子的刻画*

$$\begin{array}{ccc} & S\in\mathcal{L}(V).8\text{条件等价} & \\ 1 & S\text{是等距同构} & \\ 2 & \forall u,v\in V & \langle Su,Sv\rangle=\langle u,v\rangle\\ 3 & \boldsymbol{e}\text{是}V\text{的规范正交组} & S\boldsymbol{e}\text{是}V\text{的规范正交组}\\ 4 & \boldsymbol{e}\text{是}V\text{的规范正交基} & S\boldsymbol{e}\text{是}V\text{的规范正交基}\\ 5 & S^*S=I & \\ 6 & SS^*=I & \\ 7 & S^*\text{是等距同构} & \\ 8 & S\text{可逆}\wedge S^{-1}=S^* & \end{array}$$

证明.

$$\begin{aligned}1\to2\colon&\langle Su,Sv\rangle=\langle u,v\rangle\\ S\text{是等距同构}.&S\text{保范数}\to S\text{保内积}\\ \to&\langle Su,Sv\rangle=\langle u,v\rangle\\ \text{若}V\text{是实内积空间},&\forall u,v\in V\\ \langle Su,Sv\rangle=&(\|Su+Sv\|^2-\|Su-Sv\|^2)/4\\ &(\|S(u+v)\|^2-\|S(u-v)\|^2)/4\\ =&(\|u+v\|^2-\|u-v\|^2)/4\\ =&\langle u,v\rangle\end{aligned}$$

$$\begin{aligned}2\to3\colon&\boldsymbol{e}\text{是规范正交组}\to S\boldsymbol{e}\text{是规范正交组}\\ S\text{保内积}\to&\langle Se_i,Se_j\rangle=\langle e_i,e_j\rangle=0\end{aligned}$$

$$\begin{aligned}3\to4\colon&\boldsymbol{e}\text{是规范正交基}\to S\boldsymbol{e}\text{是规范正交基}\\ \text{length}\,S\boldsymbol{e}=\text{length}\,\boldsymbol{e}=\dim V\to&S\boldsymbol{e}\text{是}V\text{的规范正交基}\end{aligned}$$

$$\begin{aligned}4\to5\colon&S\boldsymbol{e}\text{是规范正交基}\to S^*S=I\\ \boldsymbol{e}\text{是}V\text{的规范正交基},&S\boldsymbol{e}\text{是}V\text{的规范正交基}\\ \langle Se_i,Se_j\rangle=&\langle S^*Se_i,e_j\rangle=\langle e_i,e_j\rangle=0\\ \to&S^*S=I\end{aligned}$$

$5\to 6$: $S^*S=I\to SS^*=I$

$S^*S=I\Leftrightarrow SS^*=I$　　第三章习题10 □

$6\to 7$: $SS^*=I\to S^*$是等距同构

$\|S^*v\|^2=\langle S^*v,S^*v\rangle=\langle SS^*v,v\rangle=\langle v,v\rangle=\|v\|^2$

$7\to 8$: S^*是等距同构$\to S^{-1}=S^*$

$\langle S^*v,S^*v\rangle=\langle SS^*v,v\rangle=\langle v,v\rangle$

$\to SS^*=I=S^*S$

$SS^{-1}=S^{-1}S=I$

$\to S^{-1}=S^*$

$8\to 1$: $S^{-1}=S^*\to S$是等距同构

$\langle Sv,Sv\rangle=\langle S^*Sv,v\rangle=\langle Iv,v\rangle=\langle v,v\rangle$

$\to S$是等距同构

Remark: 上述定理表明: 每个等距同构都是正规的$S^*S=SS^*=I$.暗示了等距同构可以用正规算子刻画

定理 3.10. *复内积空间上等距同构的正规算子刻画*

$$V\text{是复内积空间，}S\in\mathcal{L}(V)$$
$$S\text{是等距同构}\Leftrightarrow V\text{有一个}S\text{的本征向量组成的规范正交基且相应的本征值绝对值为}1$$

证明.

S有本征向量组成的规范正交基且本征值的绝对值都是$1\to S$是等距同构 3.8

S是等距同构$\to V$有一个由S的本征向量组成的规范正交基$\boldsymbol{e}$

设λ_i是e_i对应的本征值 □

$|\lambda_i|=\|\lambda_i e_i\|=\|Se_i\|=\|e_i\|=1$

7.C

4 极分解与奇异值分解

4.1 极分解

在复数集和$\mathcal{L}(V)$作类比

$$\begin{array}{ll} z\in C & T\in\mathcal{L}(V)\\ \bar{z}\in C & T^*\in\mathcal{L}(V)\\ z\in R & T\in\mathcal{L}(R)\\ z\geqslant 0 & \text{正算子}T\end{array}$$

C的一个重要子集是单位圆$|z|=1\to z\,\bar{z}=1$。类似的

$$TT^*=I$$

每个非零复数$z=\frac{z}{|z|}|z|=\left(\frac{z}{|z|}\right)\sqrt{\bar{z}z}$.类似的

$$T\in\mathcal{L}(V),T\neq 0.T=\text{等距同构}T_r\cdot\sqrt{T^*T}$$

定义 4.1. *记号:* T是正算子，$\sqrt{T}$记为T的唯一正平方根

定理 4.2. *算子必可极分解*

$$T\in\mathcal{L}(V).\exists 等距同构 S\in\mathcal{L}(V)\to T=S\circ\sqrt{T^*\circ T}$$

证明.

$$\begin{aligned}
&\forall v\in V.\\
\|Tv\|^2=&\langle Tv,Tv\rangle=\langle T^*Tv,v\rangle\\
=&\langle(\sqrt{T^*T}\circ\sqrt{T^*T})v,v\rangle\\
=&\left\langle\sqrt{T^*},Tv,\sqrt{T^*T}v\right\rangle && ???\\
=&\|\sqrt{T^*T}\,v\|^2\\
\to&\|Tv\|=\|\sqrt{T^*T}v\|
\end{aligned}$$

$$\begin{aligned}
S_1\colon \operatorname{range}\sqrt{T^*T}\to\operatorname{range}T, S_1(\sqrt{T^*T}v)=&Tv\\
\forall v_1,v_2\in V,\sqrt{T^*T}v_1=&\sqrt{T^*T}v_2\\
\|Tv_1-Tv_2\|=&\|T(v_1-v_2)\|\\
=&\|\sqrt{T^*T}(v_1-v_2)\|\\
=&\|\sqrt{T^*T}v_1-\sqrt{T^*T}v_2\|\\
=&0\\
\to&S_1 是函数
\end{aligned}$$

$$\begin{aligned}
S_1(\sqrt{T^*T}v_1+\sqrt{T^*T}v_2)=&S_1(\sqrt{T^*T}(v_1+v_2))\\
=&T(v_1+v_2)\\
=&Tv_1+Tv_2\\
=&S_1(\sqrt{T^*T}v_1)+S_2(\sqrt{T^*T}v_2)\\
S_1(\lambda\sqrt{T^*T}v_1)=&S_1(\sqrt{T^*T}(\lambda v))\\
=&T(\lambda v)\\
=&\lambda Tv\\
=&\lambda S_1(\sqrt{T^*T}v)\\
\to&S_1\in\mathcal{L}(\operatorname{range}\sqrt{T^*T},\operatorname{range}T)\\
\to&\forall u\in\operatorname{range}\sqrt{T^*T}\to\|S_1u\|=\|u\|\\
\to&\operatorname{null}S_1=\{0\}\to S_1 单\\
\to&\dim\operatorname{range}\sqrt{T^*T}=\dim\operatorname{range}T\\
\to&\dim(\operatorname{range}\sqrt{T^*T})^\perp=\dim(\operatorname{range}T)^\perp\\
取(\operatorname{range}\sqrt{T^*T})^\perp 和(\operatorname{range}T)^\perp 的&规范正交基\boldsymbol{e}和\boldsymbol{f}\\
S_2\in\mathcal{L}((\operatorname{range}\sqrt{T^*T})^\perp,(\operatorname{range}T)^\perp),S_2(\boldsymbol{ae})=&\boldsymbol{af}\\
\to&\forall w\in(\operatorname{range}\sqrt{T^*T})^\perp\to\|S_2w\|=\|w\|\\
S|_{\operatorname{range}\sqrt{T^*T}}=&S_1;\ S|_{(\operatorname{range}\sqrt{T^*T})^\perp}=S_2\\
\to&S\in\mathcal{L}(V)\\
\forall v\in V,\exists u\in\operatorname{range}\sqrt{T^*T},&\exists w\in(\operatorname{range}\sqrt{T^*T})^\perp\\
Sv=&S_1u+S_2v\\
\|Sv\|^2=\|S_1u+S_2w\|^2=&\|S_1u\|^2+\|S_2w\|^2=\|u\|^2+\|w\|^2=\|v\|^2\\
\to&S 是等距同构
\end{aligned}$$

□

Remark: 极分解定理说的是每个表示V上的每个算子都是一个等距同构和一个正算子的乘积。而这两个算子具有良好的性质。等距同构S关于V的一个规范正交基有对角矩阵， 而正算子$\sqrt{T^*T}$关于V的一个规范正交基有对角矩阵。但这两个对角矩阵不一定相同…否则算子本身就可对角化了。

4.2 奇异值分解

算子的本征值反映了算子的一些性质。而奇异值也是描述算子很有用数。

定义 4.3. *奇异值(singular values)*

$$T\in\mathcal{L}(V).T 的奇异值是\sqrt{T^*T}的本征值$$
$$每个本征值\lambda 都重复\dim E(\lambda,\sqrt{T^*T})=\dim\operatorname{null}(T-\lambda I)次$$

Remark: T的奇异值都是正算子 $\sqrt{T^*T}$ *的本征值，所以它们都非负*

例 4.4. 算子$T\in\mathcal{L}(F^4), T(z_1,z_2,z_3,z_4)=(0,3z_1,2z_2,-3z_4)$. 求T的奇异值

$$\begin{aligned}
\langle Tv,v\rangle=&(0,3z_1,2z_2,-3z_4)(z_1,z_2,z_3,z_4)\\
=&0z_1+3z_1z_2+2z_2z_3-3z_4^2\\
=&3z_1z_2+2z_2z_3-3z_4^2=\langle v,T^*v\rangle\\
=&(z_1,z_2,z_3,z_4)(3z_2,2z_3,0,-3z_4)\\
\to&T^*(z_1,z_2,z_3,z_4)=(3z_2,2z_3,0,-3z_4)\\
(T^*T)(z_1,z_2,z_3,z_4)=&T^*(0,3z_1,2z_2,-3z_4)\\
=&(3(3z_1),2(2z_2),0,-3(-3z_4))\\
=&(9z_1,4z_2,0,9z_4)
\end{aligned}$$

$$\sqrt{T^*T}=\sqrt{\begin{pmatrix}9&&&\\&4&&\\&&0&\\&&&9\end{pmatrix}}=\begin{pmatrix}3&&&\\&2&&\\&&0&\\&&&3\end{pmatrix}$$

$$\dim E(3,\sqrt{T^*T})=\dim\operatorname{null}\begin{pmatrix}3-3=0&&&\\&2-3=-1&&\\&&0-3=-3&\\&&&3-3=0\end{pmatrix}=2$$

$$\dim E(2,\sqrt{T^*T})=\dim\operatorname{null}\begin{pmatrix}3-2=1&&&\\&2-2=0&&\\&&0-2=-2&\\&&&3-2=1\end{pmatrix}=1$$

$$\dim E(0,\sqrt{T^*T})=\dim\operatorname{null}\begin{pmatrix}3&&&\\&2&&\\&&0&\\&&&3\end{pmatrix}=1$$

$$\to T\text{的奇异值为}3,3,2,0$$

$$\mathcal{M}(T)=\begin{pmatrix}0&&&\\3&0&&\\&2&0&\\&&&-3\end{pmatrix}=-3,2$$

根据谱定理：正算子$\sqrt{T^*T}$具有dim V个奇异值。

定理 4.5. *算子的奇异值分解*

$$T\in\mathcal{L}(V).\text{具有奇异值}\boldsymbol{s}.V\text{具有两个规范正交基}\boldsymbol{e},\boldsymbol{f}$$
$$\forall v\in V, Tv=s_1\langle v,e_1\rangle f_1+\cdots+s_n\langle v,e_n\rangle f_n$$

证明.

$$\begin{gathered}
\text{正算子}\sqrt{T^*T}\text{应用谱定理}\\
\exists\text{基}\boldsymbol{e}\in V\to\sqrt{T^*T}e_j=s_je_j\\
v=\langle v,e_1\rangle e_1+\cdots+\langle v,e_n\rangle e_n\\
\sqrt{T^*T}v=\sqrt{T^*T}(\langle v,e_1\rangle e_1+\cdots+\langle v,e_n\rangle e_n)\\
=\langle v,e_1\rangle\sqrt{T^*T}e_1+\cdots+\langle v,e_n\rangle\sqrt{T^*T}e_n\\
=\langle v,e_1\rangle s_1e_1+\cdots+\langle v,e_n\rangle s_ne_n\\
\text{极分解定理:}\exists\text{等距同构}S\in\mathcal{L}(V)\to T=S\sqrt{T^*T}\\
(S\sqrt{T^*T})(v)=S(s_1\langle v,e_1\rangle e_1+\cdots+s_n\langle v,e_n\rangle e_n)\\
=s_1\langle v,e_1\rangle Se_1+\cdots+s_n\langle v,e_n\rangle Se_n\\
S\text{是等距同构}\to S\boldsymbol{e}=\boldsymbol{f},\boldsymbol{f}\text{是规范正交基}\\
\to(S\sqrt{T^*T})(v)=s_1\langle v,e_1\rangle f_1+\cdots+s_n\langle v,e_n\rangle f_n\\
(S\sqrt{T^*T})(v)=Tv=s_1\langle v,e_1\rangle f_1+\cdots+s_n\langle v,e_n\rangle f_n
\end{gathered}$$

□

Remark: 描述算子的时候使用两个不同的基，那么V上的每个算子关于V的某些规范正交基必有对角矩阵

之前都是使用同一个基处理算子即$\mathcal{M}(T,\boldsymbol{e})$.

使用奇异值分解对算子的矩阵使用两个基这样就会有

$$\mathcal{M}(T,\boldsymbol{e},\boldsymbol{f})=\begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_n \end{pmatrix}$$

定理 4.6. *奇异值的计算可以不用对算子开平方就可以描述*

$$T\in\mathcal{L}(V).T\text{的奇异值是算子}T^*T\text{的本征值的非负平方根，本征值重复}\dim E(\lambda,T^*T)\text{次}$$

证明.

$$\begin{aligned}&\text{对算子应用谱定理}\\ \rightarrow&\exists\text{规范正交基}\boldsymbol{e},\text{非负数}\boldsymbol{\lambda},T^*Te_i=\lambda_i e_i\\ &\rightarrow\sqrt{T^*T}e_i=\sqrt{\lambda_i}e_i\end{aligned}$$

□

7.D