第三章 数列与级数

1 收敛序列

定义 1.1. *度量空间中的收敛序列。极限*

度量空间中的序列$\{x_n\}$: $\exists x\in X\to(\forall\varepsilon>0,\exists N\in N^+,n>N\to d(x_n,x)<\varepsilon)$.

$\{x_n\}$称为收敛序列	序列收敛
若$\{x_n\}$不收敛称序列发散	序列发散
收敛序列$\{x_n\}$与x定义极限: $\lim\limits_{n\to\infty}x_n=x$	极限
若$\{x_n\}$有界,称序列有界	有界

Remark: 收敛序列的定义同时依赖于序列 $\{x_n\}$ 和空间X.

Eg: $\{1/n\}$在R中收敛于0，但在R^+中不收敛

定理 1.2. *收敛序列的性质*

1.	$\{x_n\}$收敛于$x\in X\Leftrightarrow\forall U_x,\mathrm{card}(x_n-U_x)<\omega.\,x_n$至多有限项在$U_x$外	致密
2.	$\forall x,y\in X,\lim_{n\to\infty}x_n=x,\lim_{n\to\infty}x_n=y\to x=y$	唯一性
3.	$\{x_n\}$收敛$\to\{x_n\}$有界	
4.	$E\subset X,p$是E的极限点$\to\exists\{p_n\}\in E,\lim_{n\to\infty}p_n=p$	

证明.

1.
$$\lim_{n\to\infty}x_n=x\to\forall U_x,x_n\text{至多有限项在}U_x\text{外}$$
$$\lim_{n\to\infty}x_n=x:\forall\varepsilon>0,\exists N\in N^+,n>N\to d(x_n,x)<\varepsilon$$
$$\to n>N,x_n\in U_x(\varepsilon)$$
$$\text{由}\varepsilon\text{的任意性},U_x(\varepsilon)\text{是任意邻域.在}U_x(\varepsilon)\text{外的只有}N\text{项}$$

$$\forall U_x,x_n\text{至多有限项在}U_x\text{外}\to\lim_{n\to\infty}x_n=x$$
$$\text{至多有限项设最大的为}x_N,n>N\to x_n\in U_x$$
$$\to d(x_n,x)<r$$
$$\text{由于}r\text{的任意性}\to\lim_{n\to\infty}x_n=x$$

2.
$$\text{Assume: }x\neq y:x-y=r$$
$$U_x\left(\frac{r}{2}\right)\cap U_y\left(\frac{r}{2}\right)=\varnothing$$
$$\exists N\in N^+,n>N\to d(x_n,x)<\frac{r}{2},d(x_n,y)<\frac{r}{2}$$
$$\text{这与}x_n-y\geqslant\frac{r}{2}\text{矛盾}$$
$$\to x=y$$

□

3. $\{x_n\}$收敛$\to\exists x\in X,\forall\varepsilon>0,\exists N\in N^+,n>N\to d(x_n,x)<\varepsilon$
$$x_1,\ldots,x_N\text{有界}$$
$$n>N,d(x_n,x)<\varepsilon\to x_n\in U_x(\varepsilon)\to x_n\text{有界}$$
$$\to x_n\text{有界}$$

$$\{x_n\}\text{收敛},\exists N,n>N\to d(x_n,x)<1$$
$$r=\max\{1,d(x_1,x),d(x_2,x),\ldots\},\text{此集合有界且必有}r\in R$$
$$\forall x_n,d(x_n,x)\leqslant r$$
$$\to\{x_n\}\text{有界}$$

4.
$$E\subset X,p\text{是}E\text{的极限点}\to\exists\{p_n\}\in E,\lim_{n\to\infty}p_n=p$$
$$\text{用选择公理。}$$
$$p\text{是}E\text{的极限点}\to\forall U_p^0\cap E\neq\varnothing$$
$$\text{构造序列：}\varepsilon=\frac{1}{n},\text{每次取}U_p^0\text{中的点}p_n$$
$$\forall\varepsilon>0,\exists N,n>N\to\frac{1}{n}<\varepsilon\to d(p_n,p)<\frac{1}{n}<\varepsilon$$
$$\to\lim_{n\to\infty}p_n=p$$

定理 1.3. *序列收敛性与代数运算的关系*

$\{x_n\},\{y_n\}$都是复数序列，$\lim_{n\to\infty}x_n=x,\lim_{n\to\infty}y_n=y$

1. $$\lim(x_n+y_n)=x+y$$
2. $$\forall\lambda\in F,\lim\lambda x_n=\lambda x;\lim(\lambda+x_n)=\lambda+x$$
3. $$\lim x_n\cdot y_n=x\cdot y$$
4. $$s_n\neq 0\wedge s\neq 0\to\lim\frac{1}{s_n}=\frac{1}{s}$$

证明.

1. $$\begin{gathered}\lim(x_n+y_n)=x+y\\ \forall\varepsilon>0,\exists N_x,n>N_x\to d(x_n,x)<\varepsilon;n>N_y\to d(y_n,y)<\varepsilon\\ \to n>\max(N_x,N_y)\to d(x_n,x)+d(y_n,y)<2\varepsilon\\ \to\{x_n+y_n\}\text{收敛于}x+y\end{gathered}$$

2. $$\begin{gathered}\lim\lambda x_n=\lambda x\\ \forall\varepsilon>0,n>N\to d(x_n,x)<\varepsilon\\ |\lambda x_n-\lambda x|=|\lambda|\cdot|x_n-x|<|\lambda|\varepsilon\\ |\lambda+x_n-\lambda-x|=|x_n-x|<\varepsilon\end{gathered}$$

这里对C的所有度量有点困难

3. $$\begin{gathered}\lim x_n y_n=xy\\ x_n y_n-xy=(x_n-x)(y_n-y)+x(y_n-y)+y(x_n-x)\\ \forall\varepsilon>0,\exists N_x,N_y\to n>N_x\to d(x_n,x)<\varepsilon,d(y_n,y)<\varepsilon\\ n>\max(N_x,N_y)\to|(x_n-x)(y_n-y)|<\varepsilon^2\\ \to\lim(x_n-x)(y_n-y)=0\\ \to\lim x_n y_n=\lim(x_n-x)(y_n-y)+x(y_n-y)+y(x_n-x)+xy\\ =0+xy=xy\end{gathered}$$

□

4. $$\begin{gathered}x_n\neq 0\wedge x\neq 0\to\lim\frac{1}{x_n}=\frac{1}{x}\\ \forall\varepsilon>0,\exists N,n>N\to d(x_n,x)<\varepsilon\\ \left|\frac{1}{x_n}-\frac{1}{x}\right|=\left|\frac{x_n-x}{x_n x}\right|=\frac{|x_n-x|}{|x_n x|}<\frac{1}{|x_n x|}\varepsilon\end{gathered}$$

$x_n x$是有界量

定理 1.4. *R^k上的序列收敛条件，性质*

1. $x_n\in R^k.x_n$收敛与$x\Leftrightarrow\forall i\in 1\ldots k,\lim_n x_{n,i}\to x_i$
2. $\{x_n\},\{y_n\}$是R^k中的收敛序列,$\{\lambda_n\}$是R中的收敛序列
$$\begin{gathered}\lim(x_n+y_n)=x+y\\ \lim\boldsymbol{x}_n\cdot\boldsymbol{y}_n=\boldsymbol{x}\cdot\boldsymbol{y}\\ \lim\lambda_n\boldsymbol{x}_n=\lambda\boldsymbol{x}\end{gathered}$$

Remark: 这里的R^k向量的度量为范数诱导的度量。

证明.

1. $$\begin{gathered}\boldsymbol{x}_n\to\boldsymbol{x}:|x_{n,i}-x_i|\leqslant|\boldsymbol{x}_n-\boldsymbol{x}|=(\textstyle\sum_i(x_{n,i}-x_i)^2)^{1/2}<\varepsilon\\ \forall i\in 1\ldots k,|x_{n,i}-x_i|<\varepsilon\to|\boldsymbol{x}_n-\boldsymbol{x}|=(\textstyle\sum_i(x_{n,i}-x_i)^2)^{1/2}\\ <(k\varepsilon^2)^{1/2}=\sqrt{k}\varepsilon\\ \to d(\boldsymbol{x}_n,\boldsymbol{x})<\sqrt{k}\varepsilon\to\boldsymbol{x}_n\text{收敛}\end{gathered}$$

□

2. $$\begin{gathered}\lim(\boldsymbol{x}_n+\boldsymbol{y}_n)\Leftrightarrow\lim(x_{i,n}+y_{i,n})=x+y\\ \ldots\end{gathered}$$

2 子序列

定义 2.1. *子序列,部分极限*

$$序列\{x_n\},取正整数序列\{n_i\},n_i<n_{i+1}.称\{x_{n_i}\}为\{x_n\}的子序列 \quad 子序列$$
$$若\{x_{n_i}\}收敛，称为\{x_n\}的部分极限 \quad 部分极限$$

定理 2.2. *序列收敛与x⇔任意子序列收敛与x*

$$\begin{gathered}\lim x_n=x\to\lim x_{n_i}=x\\ \lim x_n=x,\forall\varepsilon>0,\exists N,n>N\to d(x_n,x)<\varepsilon\\ \forall\{x_{n_i}\},\exists n_i>N,d(x_{n_i},x)<\varepsilon \qquad nx>r\\ \to\lim x_{n_i}=x\end{gathered}$$

$$\begin{gathered}\lim x_{n_i}=x\to\lim x_n=x\\ 由n_i的任意性令n_i=i\\ \to\lim x_i=x\end{gathered}$$

定理 2.3. *紧度量空间中的任意序列存在收敛于紧度量空间内部的子序列.*

证明.

$$\begin{gathered}\{p_n\}的值域有限\to\exists\{p_{n_i}\}=p. \qquad 否则与无限性矛盾\\ \{p_n\}的值域E无限\to E在X中有极限点x \quad 紧集子集的极限点在紧集中.第二章\\ 使用选择公理可以构造序列d(x_n,x)<\frac{1}{n}\\ \to\{x_n\}\subset\{p_n\}\\ \to\{p_{n_i}\}\to x\end{gathered}$$

□

推论 2.4. *（致密性定理）R^k中的有界序列必有收敛子列.（参考第二章Weierstrass定理）*

定理 2.5. *度量空间X里，序列$\{x_n\}$的所有部分极限$\{x_p\}$是X中的闭集*

证明.

$$\begin{gathered}E=\{p\colon p是\{x_n\}的部分极限\}\\ \Leftrightarrow q是E的极限点\to q\in E\\ q是E的极限点:\forall U_q^0(r)\cap E\neq\varnothing\\ 选择公理:\exists\{q_n\}\in E,\lim q_n=q\\ \{q_n\}\subset\{x_n\}\\ 极限点定义:n>N_d\to d(q_n,q)<\varepsilon\\ E的定义,部分极限:\forall q_n\in E,\exists x_{n_i}\subset x_n,i>N_p\to d(x_{n_i},q_n)<\varepsilon\\ \to n>\max(N_p,N_d)\to d(x_{n_i},q)\leqslant d(x_{n_i},q_n)+d(q_n,q)<2\varepsilon\\ \to x_{n_i}\to q \qquad 收敛定义\\ \to q\in E\end{gathered}$$

证明是两步走：1. 极限点则必然距离小于任意正数；2. 序列中每个值必然是x_n的部分序列;

进而任意正数都有一个部分序列和极限点p的距离小于这个正数 □

3 Cauchy序列

定义 3.1. *Cauchy序列*

$$度量空间X中的序列\{x_n\}\colon\forall\varepsilon>0,\exists N\in N^+,\forall n\geqslant N,m\geqslant N\to d(x_n,x_m)<\varepsilon \quad 称为Cauchy序列$$

定义 3.2. *度量空间X的子集E的直径。* $\operatorname{diam}E=\sup\{d(x,y)\colon x,y\in E\}$

定理 3.3. *X中的序列是Cauchy序列*$\Leftrightarrow\lim_{N\to\infty}\operatorname{diam}\{x_N,x_{N+1},\dots\}=0$

定理 3.4. *集合直径的性质*

1.
$$E\subset X,\operatorname{diam}E=\operatorname{diam}\bar{E}$$
$$E\subset\bar{E}\rightarrow\operatorname{diam}E\leqslant\operatorname{diam}\bar{E}$$
$$\bar{E}=E\cup E'$$
$$\text{若}\operatorname{diam}\bar{E}>\operatorname{diam}E,\text{那么至少有一个点在}E'\text{里}$$
$$x\in E'\wedge y\in E:$$
$$d(x,y)\leqslant d(x,x_n)+d(x_n,y)\leqslant\varepsilon+\operatorname{diam}E=\operatorname{diam}E$$
$$x\in E'\wedge y\in E'$$
$$d(x,y)\leqslant d(x_n,x)+d(y_n,y)+d(x_n,y_n)=2\varepsilon+\operatorname{diam}E=\operatorname{diam}E$$
$$\rightarrow\operatorname{diam}E=\operatorname{diam}\bar{E}$$

2. $\{K_n\}$是X中紧集的序列$\wedge K_{n+1}\subset K_n\wedge\lim\operatorname{diam}K_n=0\rightarrow\operatorname{card}\bigcap_1^\infty K_n=1$
$$\text{let: }K=\bigcap_1^\infty K_n.\text{若}\exists K_n=\varnothing\text{那么结果不是重要的}$$
$$\text{令}K_n\neq\varnothing\rightarrow K\neq\varnothing$$
$$\exists x\neq y\in K\rightarrow\operatorname{diam}K\geqslant d(x,y)>0.\text{矛盾}$$
$$\rightarrow\operatorname{card}K=1$$

定理 3.5. *Cauchy序列的性质*

1.
$$\text{度量空间中: 收敛序列是Cauchy序列}$$
$$\lim x_n=x\rightarrow\forall\varepsilon>0,\exists N,n>N\rightarrow d(x_n,x)<\varepsilon$$
$$i,j>N\rightarrow d(x_i,x_j)\leqslant d(x_i,x)+d(x,x_j)<2\varepsilon$$
$$\rightarrow x_n\text{是柯西序列}$$

2.
$$\text{紧度量空间}X\text{中的Cauchy序列}\{x_n\}\text{收敛于}X\text{的内部}$$
$$\{x_n\}\text{是紧度量空间中的Cauchy序列: }n>N\rightarrow d(x_i,x_j)<\varepsilon$$
$$E_N=\{x_N,x_{N+1},\dots\}$$
$$\lim_N(\operatorname{diam}\overline{E_N})=0$$
$$\overline{E_N}\text{闭}\wedge\overline{E_N}\text{在紧空间中}\rightarrow\overline{E_N}\text{也是紧集}$$
$$E_{N+1}\subset E_N\rightarrow\overline{E_{N+1}}\subset\overline{E_N}$$
$$\rightarrow\operatorname{card}(\bigcap_1^\infty\overline{E_N})=1\qquad 3.4-2$$
$$\forall\varepsilon>0,\exists N_0\in N^+.n>N_0\rightarrow\operatorname{diam}\overline{E_N}<\varepsilon$$
$$\text{let }x\in\bigcap_1^\infty\overline{E_N},x\in\overline{E_N},\forall y\in\overline{E_N},d(x,y)<\varepsilon$$
$$\rightarrow d(x_n,x)<\varepsilon$$
$$\rightarrow\lim_{n\rightarrow\infty}x_n=x,x\in\overline{E_N}$$

3.
$$R^k\text{中的所有Cauchy序列收敛}$$
$$\{\boldsymbol{x}_n\}\subset R^k\text{是Cauchy序列: }i,j>N\rightarrow d(\boldsymbol{x}_i,\boldsymbol{x}_j)<\varepsilon$$
$$\text{let: }\varepsilon=1\rightarrow d(x_i,x_j)<1$$
$$\operatorname{range}\{\boldsymbol{x}_n\}\subset\operatorname{range}\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_N\}\cup U_{\boldsymbol{x}_{N+1}}(1)$$
$$\rightarrow\{\boldsymbol{x}_n\}\text{有界}$$
$$\rightarrow\exists F\subset R^k\wedge F\text{有界}\rightarrow\{\boldsymbol{x}_n\}\subset F,F\text{紧}$$
$$\rightarrow\{\boldsymbol{x}_n\}\text{在}R^k\text{有收敛点}$$

定义 3.6. *空间的完备性。每个Cauchy序列都在X中收敛*

Remark: 所有紧度量空间是完备的；欧氏空间是完备的；完备度量空间的闭子集是完备的;

Q不是完备的;

R^k中的有界序列不一定收敛;R中的单调有界序列收敛;

定义 3.7. *单调序列.*

实数序列 $\{x_n\}$

单调增	$x_n\leqslant x_{n+1}$	$n\in N^+$
单调减	$x_n\geqslant x_{n+1}$	

定理 3.8. *单调序列. 收敛 $\Leftrightarrow$ 有界*

证明.

$$\begin{aligned}
&\text{单调增序列}\{x_n\}.x_n \leqslant x_{n+1} \to \exists x \in X, \lim x_n = x \\
&\text{range}\ \{x_n\}\text{有界}.x = \sup \text{range}\{x_n\} \\
&\forall n \in N^+, x_n \leqslant s \\
&\forall \varepsilon > 0, \exists N \in N^+, x - \varepsilon < x_n \leqslant x \\
&x_n \leqslant x_{n+1} \to n \geqslant N, x - \varepsilon < x_n \leqslant x \\
&\to \lim x_n = x
\end{aligned}$$

$$\lim x_n = x \to \text{range}\ \{x_n\}\text{有界}. \qquad 1.2$$

□

4 上极限和下极限

定义 4.1. *特殊的发散.收敛于 $-\infty, +\infty$*

$$\begin{array}{lll}
+\infty & \forall M \in R, \exists N \in N^+, n > N \to x_n \geqslant M & \lim x_n = +\infty \\
-\infty & \forall M \in R, \exists N \in N^+, n > N \to x_n \leqslant M & \lim x_n = -\infty
\end{array}$$

定义 4.2. *上极限，下极限*

$$\begin{gathered}
\{x_n\} \in R, E\text{是所有子序列}\{x_{n_i}\}\text{的极限}x\text{的集合}.E\text{还包括} -\infty, +\infty \\
\text{上极限}: x^* = \sup E. \lim_{n\to\infty} \sup x_n = x^* \\
\text{下极限}: x_* = \inf E. \lim_{n\to\infty} \inf x_n = x_*
\end{gathered}$$

定理 4.3. *上下极限的性质*

$$\begin{array}{ll}
1. & x^*, x_* \in E \\
2. & y > x^*, \exists N \in N^+, n > N \to x_n < y \\
& x^*\text{是唯一满足上述两个性质的数}
\end{array}$$

证明.

$$\begin{gathered}
x^* = +\infty: E\text{无上界} \to \exists x_{n_i}, \lim_i x_{n_i} = +\infty \\
+\infty \in E
\end{gathered}$$

$$\begin{gathered}
x^* \in R: \\
E\text{闭} \qquad 2.5 \\
E\text{上有界}, \sup E \in \bar{E} \qquad \text{第二章定理2.28} \\
\to x^* \in E = \bar{E}
\end{gathered}$$

$$\begin{gathered}
x^* = -\infty: E = \{-\infty\} \to E\text{没有部分极限} \\
\to \forall M \in R, x_n\text{中只有有限个} > M \\
\to \lim x_n \to -\infty \\
\to x^* \in E
\end{gathered}$$

$$\begin{gathered}
\text{设有无限多}x_n > y > x^* \\
\exists x_{n_i} \subset x_n \to \lim x_{n_i} > y \to \exists t \in R \cup \{-\infty, +\infty\} \to t > x^* \\
t \in E\text{与}x^* = \sup E\text{矛盾}
\end{gathered}$$

唯一性

$$\begin{gathered}
p \neq q \wedge p, q\text{满足}1.;2. \\
p < q \to q = \sup E, \forall p < q \to p \neq \sup E \quad \text{依赖广义实数系上的sup 唯一性} \\
\text{矛盾} \\
\to p\text{唯一}
\end{gathered}$$

□

定理 4.4.

$x_n, y_n . n > N, x_n \leqslant y_n . \to \liminf x_n \leqslant \liminf y_n \wedge \limsup x_n \leqslant \limsup y_n$

例 4.5. 一些序列的上下极限

1. $\operatorname{range}\{x_n\} = Q$. 那么 $E = R \cup \{-\infty, +\infty\}$. $\limsup x_n = +\infty, \liminf x_n = -\infty$
2. $$x_n = \frac{(-1)^n}{1+1/n}. \limsup x_n = 1. \liminf x_n = 1$$
3. $$\operatorname{range}\{x_n\} \in R. \lim x_n = x \Leftrightarrow \limsup x_n = \liminf x_n = x$$

5 一些特殊序列

定理 5.1. *一些特殊序列的极限*

1. $$p > 0. \lim_n \frac{1}{n^p} = 0 \qquad \text{幂函数的倒数极限为0}$$
$$\forall \varepsilon > 0, n > \left(\frac{1}{\varepsilon}\right)^{1/p} \to x_n < \left(\left(\left(\frac{1}{\varepsilon}\right)^{1/p}\right)^p\right)^{-1} = \varepsilon$$

2. $$\begin{gathered} p > 0. \lim_n \sqrt[n]{p} = 1 \\ p = 1 \colon \lim_n 1 = 1 \\ p > 1 \colon x_n = \sqrt[n]{p} - 1, x_n > 0 \\ 1 + n x_n \leqslant (1 + x_n)^n = p \\ \to 0 < x_n \leqslant \frac{p-1}{n} \\ \to x_n \to 0 \end{gathered} \qquad \text{夹逼准则.需要额外证一下，没啥难度}$$

3. $$\begin{gathered} \lim_n \sqrt[n]{n} = 1 \\ x_n = \sqrt[n]{n} - 1 \to x_n > 0 \\ n = (1 + x_n)^n = \sum_{i=0}^n \mathrm{C}_n^i 1^{n-i} x_n^i \\ \geqslant \mathrm{C}_n^2 x_n^2 = \frac{n(n-1)}{1 \cdot 2} x_n^2 \\ 1 \geqslant \frac{n-1}{2} x_n^2 \\ \to 0 < x_n \leqslant \sqrt{\frac{2}{n-1}} \\ \to x_n \to 0 \end{gathered}$$

4. $$p > 0, \alpha \in R. \lim_n \frac{n^\alpha}{(1+p)^n} = 0 \qquad \text{任意幂函数相对任意增指数函数为0}$$
$$\exists k \in N^+, k > \alpha . n > 2k$$
$$\to (1+p)^n > \mathrm{C}_n^k p^k = \frac{n(n-1)\dots(n-k+1)}{k!} p^k > \frac{n^k p^k}{2^k k!} \qquad \frac{n}{1} \cdot \frac{n-1}{1} \dots \frac{n-k+1}{1} > \frac{n}{2} = k$$
$$\begin{gathered} (1+p)^n > \frac{n^k}{2^k} \frac{p^k}{k!} \\ \to \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha - k} \\ \alpha < k \to \lim_n n^{\alpha - k} = 0 \end{gathered} \qquad 1.$$
$$???\left(\frac{2}{p}\right)^k \text{可以超过} n^k \text{的。循环论证}$$

5. $$\begin{gathered} |x| < 1. \lim x^n = 0 \\ |x| < 1. t = \frac{1}{x}. |t| > 1 \\ t_n = t^n - 1 \\ \lim t_n = 0 \to |t^n| = \infty \\ \lim x^n = 0 \\ ??? \end{gathered}$$

6 级数

这里指复级数和有限维向量空间中的级数

定义 6.1. *无穷级数,级数*

$$\begin{array}{cc} & \text{序列}\{a_n\} \\ \text{序列的部分和} & s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n \\ \text{无穷级数} & \sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots \\ \text{级数的收敛} & \text{部分和序列}s_n\text{收敛.记作}\sum_{i=1}^{n} a_n = s \\ \text{级数的发散} & \text{部分和序列}s_n\text{不收敛} \end{array}$$

Remark: 级数的和是由加法和极限两个运算定义的。

定理 6.2. *级数的Cauchy准则*

$\sum a_n$收敛$\Leftrightarrow \forall \varepsilon > 0, \exists N \in N^+, m \geqslant n \geqslant N \rightarrow |\sum_{i=n}^{m} a_n| < \varepsilon$

Remark: 这里Cauchy序列一定收敛表明级数的空间已经被限定在有限维空间中

推论 6.3. $\{a_n\}$是Cauchy序列: $m = n \rightarrow |a_n| < \varepsilon \rightarrow \lim a_n = 0$

Remark: 逆命题不一定成立. $s_n = \sum \frac{1}{n}$

定理 6.4. $a_n \geqslant 0 \wedge s_n$收敛$\Leftrightarrow s_n$有界

定理 6.5. *比较验敛法*

1. $\exists N \in N^+ . \forall n > N, |a_n| \leqslant b_n \wedge \sum b_n$收敛$\rightarrow \sum a_n$收敛
2. $\exists N \in N^+ . \forall n > N, a_n \geqslant b_n \geqslant 0 \wedge \sum b_n$发散$\rightarrow \sum a_n$发散 a_n是正项级数

证明.

1. $$\begin{array}{c} \sum b_n\text{收敛} \rightarrow \sum b_n\text{是Cauchy序列} \\ \rightarrow |a_n| < b_n \rightarrow |\sum_{i=n}^{m} a_n| \leqslant \sum_{i=n}^{m} |a_n| \leqslant \sum_{i=n}^{m} b_n < \varepsilon \\ \rightarrow |\sum_{i=n}^{m} a_n| < \varepsilon \\ \rightarrow a_n\text{是Cauchy序列} \rightarrow a_n\text{收敛} \end{array}$$ □

2. $$\begin{array}{cc} \text{若}a_n\text{收敛} \rightarrow b_n\text{收敛} & 1 \\ b_n\text{不收敛} \rightarrow a_n\text{不收敛} & \text{逆否} \end{array}$$

Remark: 应用比较验敛法需要记忆常用的收敛或发散的非负项级数

7 非负项级数

定理 7.1. *几何级数的收敛性*

$$\begin{array}{lcc} \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} & & 0^0 = 0.x^0 = 1 \\ & 0 \leqslant x < 1 & \text{收敛} \\ & x \geqslant 1 & \text{发散} \end{array}$$

证明.

$$\begin{array}{lccc} x \neq 1 & \sum_{n=0}^{k} x^n = \frac{1-x^{k+1}}{1-x} & & \\ & \lim_{k\rightarrow\infty} \frac{1-x^{k+1}}{1-x} & & \\ & 0 \leqslant x < 1 & \lim \frac{1-\lim x^{k+1}}{1-x} = \frac{1-0}{1-x} = \frac{1}{1-x} & \text{收敛} \\ & x > 1 & \lim \frac{1-\lim x^{k+1}}{1-x} = \frac{1-\infty}{1-x} = \frac{-\infty}{-x} = \infty & \text{发散} \\ & x = 1 & \sum x^n = 1 + 1 + \cdots = \infty & \text{发散} \end{array}$$ □

定理 7.2. *Cauchy.* 正项级数$\sum a_n . a_{n+1} \leqslant a_n . \sum_1^{\infty} a_n$收敛$\Leftrightarrow \sum_0^{\infty} 2^n a_{2^n}$收敛

证明.

$$\sum 2^n a_{2^n}\text{收敛} \to \sum a_n\text{收敛}$$
$$s_n = \sum_{i=1}^n a_i; t_n = \sum_{i=0}^n 2^i a_{2^i}$$
$$n < 2^k: s_n \leqslant a_1 + (a_2 + a_3) + \cdots + (a_{2^i} + \cdots + a_{2^{i+1}-1})$$
$$\leqslant a_1 + 2a_2 + \ldots + 2^k a_{2^k} = t_k.$$
$$t_k\text{收敛} \to \lim_k t_k = t \to t_k < t \to s_n\text{有界} \to s_n\text{收敛}$$

□

$$\sum a_n\text{收敛} \to \sum 2^n a_{2^n}\text{收敛}$$
$$n > 2^k: t_k = a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}$$
$$\leqslant a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots +$$
$$= s_{2^{k+1}} \leqslant 2s_n$$
$$\to t_k\text{有界} \to t_k\text{收敛}$$

定理 7.3. p级数的敛散性.

p级数: $\sum_n \frac{1}{n^p}$

$$p > 1 \quad \text{收敛}$$
$$p \leqslant 1 \quad \text{发散}$$
$$p \leqslant 0 \quad \text{发散}$$

证明.

$$p = 0: \to \lim_{n\to\infty} n^p = 1 \to \lim_{n\to\infty} \frac{1}{n^p} = \frac{1}{1} = 1 \quad a_n, a\text{都存在不为}0 \to a_n^{-1} \to a^{-1}$$
$$\to \sum \frac{1}{n^p}\text{发散} \qquad 6.3$$
$$p < 0: -p > 0 \to \frac{1}{n^p} = n^{-p}.$$
$$\lim_{n\to\infty} n^{-p} = \infty \to \sum n^{-p}\text{发散} \qquad 6.3$$

$$\sum n^{-p}\text{与}\sum_{k=0}^{\infty} 2^k \frac{1}{2^{kp}} = \sum_0^{\infty} 2^{k(1-p)}\text{同敛散} \qquad 7.2$$
$$\to\text{若收敛} \to \lim_k 2^{k(1-p)} = 0$$
$$2^{k(1-p)} = (2^{1-p})^k \to \text{后者是几何级数}$$
$$0 \leqslant 2^{1-p} < 1\text{时收敛}$$
$$\to 1 - p < 0 \to p > 1$$
$$\text{其它都发散}$$
$$p > 1: \text{收敛}$$
$$p < 1: \text{发散}$$

□

定理 7.4. *级数* $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ *的敛散性*

$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$

$$p > 1 \quad \text{收敛}$$
$$p \leqslant 1 \quad \text{发散}$$

Remark: 这里使用了 $\forall x > 1, \lim_{n\to\infty} \frac{\log n}{x^n} = 0$*.对数函数增长速度低于幂函数.*

证明.

$$n \log n\text{单调增} \to \frac{1}{n \log n}\text{单调减}$$
$$\sum_{n=2}^{\infty} \frac{1}{n\,(\log n)^p}\text{等价于}\sum_{k=1}^{\infty} 2^k \frac{1}{2^k(\log 2^k)^p}\text{的敛散性}$$
$$\sum_{k=1}^{\infty} 2^k \frac{1}{2^k(\log 2^k)^p} = \sum_{k=1}^{\infty} \frac{1}{(k \log 2)^p} = \sum_{k=1}^{\infty} \frac{1}{(\log 2)^p} \cdot \frac{1}{k^p}$$
$$= \frac{1}{(\log 2)^p} \sum_{k=1}^{\infty} \frac{1}{k^p}$$
$$\frac{1}{(\log 2)^p} \in R \to \text{级数等价于}p\text{级数}$$
$$\to p > 1: \text{收敛}$$
$$p \leqslant 1: \text{发散}$$

□

这种级数构造法可以不断构造新的收敛速度更慢的级数.

$\sum_{n=3}^{\infty} \frac{1}{n\,(\log n)\,(\log(\log(n)))^1}$ 发散

$\sum_{n=3}^{\infty} \frac{1}{n\,(\log n)(\log(\log(n)))^2}$ 收敛

但是收敛级数和发散级数之间没有明确的界限。参考习题11，12

Reference: Theory and Application of Infinite Series. Knopp. Chapter IX.

8 数e

定义 8.1. *e*

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

证明. 上述级数收敛

$$\begin{aligned} n>4 \rightarrow n! &= 1\times 2\times 3\times 4\times \cdots \\ &> 1\times 2\times 2\times 2^2\times 2\ldots = 2^n \\ 0! &= 2^0 = 1 \\ 1! &= 1 < 2^1 = 2 \\ 2! &= 2 < 2^2 = 4 \\ 3! &= 6 < 2^3 = 8 \end{aligned}$$

$$\begin{aligned} \textstyle\sum_{n=1}^{k} x^n = s.\, x\, s &= \textstyle\sum_{n=2}^{k+1} x^n \\ x\, s - s &= \textstyle\sum_{n=2}^{k+1} x^n - \sum_{n=1}^{k} x^n = x^{k+1} - x^1 \\ s &= \tfrac{x^{k+1} - x^1}{x-1} \end{aligned}$$

$$\begin{aligned} \textstyle\sum_{n=0}^{\infty} \frac{1}{n!} &= \tfrac{1}{0!} + \tfrac{1}{1!} + \tfrac{1}{2!} + \cdots \\ &\leqslant 1 + \tfrac{1}{1} + \tfrac{1}{2^1} + \tfrac{1}{2^2} + \tfrac{1}{2^3} + \cdots \\ &= 1 + 1 + \tfrac{2^{-n-1} - 2^{-1}}{2^{-1} - 1}.\ \{= 2(2^{-1} - 2^{-n-1}) = 1 - 2^{-n}\} \\ &\leqslant 1 + 1 + 1 = 3 && \text{有界} \\ \tfrac{1}{n!} > 0 &\rightarrow \textstyle\sum_0^{\infty} \frac{1}{n!}\text{收敛} && \text{单调} \end{aligned}$$

□

推论 8.2. *$\sum_{i=0}^{\infty} \frac{1}{n!}$的收敛速度*

$$\begin{aligned} e - s_n &= \tfrac{1}{(n+1)!} + \tfrac{1}{(n+2)!} + \cdots \\ &< \tfrac{1}{(n+1)!}\left(1 + \tfrac{1}{n+1} + \tfrac{1}{(n+1)^2} + \cdots\right) \\ &= \tfrac{1}{(n+1)!} \cdot \lim_{k\rightarrow\infty} \tfrac{(n+1)^{-k-1} - (n+1)^{-1}}{(n+1)^{-1} - 1} \\ &= \tfrac{1}{n!n} \end{aligned}$$

定理 8.3. *e是无理数*

证明.

$$\begin{aligned} &\text{反证}: e = p/q \\ &\rightarrow e - s_n < \tfrac{1}{n!n} \\ &0 < q!(e - s_n) < \tfrac{1}{q} \\ &q!(e - s_n) = q!\, e - q!\, s_n \in N^+ \\ &\text{但没有正整数在区间}\left(0, \tfrac{1}{q}\right)\text{内} \\ &\rightarrow e \neq \tfrac{p}{q} \end{aligned}$$

Remark: e不是代数数.

Reference:

Irrational Numbers, Carus Mathematical Monograph No. 11. Niven, I.M.

Topics in Algebra. Herstein, I. N. □

定理 8.4. $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$

证明.

$$
\begin{aligned}
s_n = \sum_{k=0}^{n}\frac{1}{k!},\ t_n = (1+\tfrac{1}{n})^n \\
1+\frac{1}{n} = \frac{n+1}{n} \\
t_n = \sum_{i=0}^{n+1}\mathrm{C}_n^i 1^i \frac{1}{n^{n+1-i}} \\
=\sum_{i=0}^{n+1}\frac{(n+1)!}{i!(n+1-i)!}\cdot\frac{1}{n^{n+1-i}} = \frac{n-1}{n}\cdot\frac{n-2}{n}\cdots \\
=1+1+\frac{1}{2!}(1-\frac{1}{n})+\frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n})+\cdots+\frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})\ldots(1-\frac{n-1}{n}) \\
\to t_n \leqslant s_n \\
\lim_{n\to\infty}\sup t_n \leqslant e \qquad 4.4
\end{aligned}
$$

□

$$
\begin{aligned}
n \geqslant m \to t_{n,m} \geqslant 1+1+\frac{1}{2!}(1-\frac{1}{n})+\cdots+\frac{1}{m!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{m-1}{n}) \\
\lim_{n\to\infty}\inf t_n \geqslant 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{m!} \\
\to s_m \leqslant \lim_{n\to\infty}\inf t_{n,m} \\
\to e \leqslant \lim_{m\to\infty}, \lim_{n\to\infty}\inf t_{n,m} \\
\to \lim t_n = \lim s_n = e
\end{aligned}
$$

9 根值验敛法与比率验敛法

定理 9.1. *根值验敛法*

$$
\begin{aligned}
\text{级数}\sum a_n.\alpha = \lim_{n\to\infty}\sup\sqrt[n]{|a_n|} \\
\alpha < 1\text{时}, \sum a_n\text{收敛} \\
\alpha > 1\text{时}, \sum a_n\text{发散} \\
\alpha = 1\text{时}, \sum a_n\text{无法判断}
\end{aligned}
$$

证明.

$$
\begin{aligned}
\alpha < 1 \to \forall 0<\beta<1, \exists N \in N^+, n \geqslant N \to \sqrt[n]{|a_n|} < \beta \\
\to |a_n| < \beta^n \\
0 \leqslant \beta < 1 \to \sum \beta^n\text{收敛} \\
\to \sum a_n\text{收敛} \qquad 6.5\text{比较验敛法}
\end{aligned}
$$

$$
\begin{aligned}
\alpha > 1 \to \lim_{n\to\infty}\sup\sqrt[n]{|a_n|} > 1 \\
\to \exists n_i \to |a_{n_i}| > 1 \\
\to \sum_{n=1}^{\infty} a_n\text{发散} \qquad 6.3
\end{aligned}
$$

□

$$
\begin{aligned}
\alpha = 1: \sum\frac{1}{n}\text{发散}, \sum\frac{1}{n^2}\text{收敛} \\
\lim_{n\to\infty}\frac{1}{\sqrt[n]{n}} = 1; \lim_{n\to\infty}\frac{1}{\sqrt[n]{n^2}} = 1
\end{aligned}
$$

定理 9.2. *比率验敛法*

级数$\sum a_n$

$$\lim_{n\to\infty}\sup\left|\frac{a_{n+1}}{a_n}\right|<1\text{: }\sum a_n\text{收敛}$$
$$\exists N\in N^+,n>N\to\left|\frac{a_{n+1}}{a_n}\right|\geqslant 1.\sum a_n\text{发散.}\left(\lim\inf\left|\frac{a_{n+1}}{a_n}\right|\geqslant 1\right)$$
$$\text{不满足上述条件的一切情况都无法判断}$$

证明.

$$\alpha<1\to\exists\alpha<\beta<1,n>N\to\left|\frac{a_{n+1}}{a_n}\right|<\beta$$
$$|a_{n+1}|<\beta|a_n|$$
$$\to|a_n|<\beta^{-N}|a_N|\cdot\beta^n$$
$$\sum\beta^n\text{收敛}\to|a_n|\text{收敛}\qquad 6.5$$

$$\alpha>1\to\exists\alpha>\beta>1,n>N\to\left|\frac{a_{n+1}}{a_n}\right|>\beta$$
$$\to|a_{n+1}|>|a_n|.\exists a_n>0\to a_{n+1}>0$$
$$\to\lim a_n>0$$
$$\to\sum a_n\text{不收敛}\qquad 6.3$$

$$\alpha=1\to\sum\frac{1}{n}\text{发散},\sum\frac{1}{n^2}\text{收敛}$$
$$\lim\sup\frac{1/(n+1)}{1/n}=\lim\sup\frac{n}{n+1}=1$$
$$\lim\sup\frac{1/(n+1)^2}{1/n^2}=\lim\sup\frac{n^2}{(1+n^2)}=1$$

□

定理 9.3. *正项级数. 比率验敛法能判断出级数收敛，根值验敛法一定能判断出收敛性*

$$\forall\{c_n: c_n>0\}$$

1. $\lim\inf\frac{c_{n+1}}{c_n}\leqslant\lim\inf\sqrt[n]{c_n}$
2. $\lim\sup\sqrt[n]{c_n}\leqslant\lim\sup\frac{c_{n+1}}{c_n}$

证明.

2. $$\alpha=\lim_{n\to\infty}\sup\frac{c_{n+1}}{c_n}.\ \alpha\geqslant 0$$
$$\alpha=+\infty\to\lim\sup\sqrt[n]{c_n}\leqslant+\infty$$
$$\alpha\neq+\infty.\forall\alpha<\beta,n\geqslant N\to\frac{c_{n+1}}{c_n}<\beta$$
$$c_{N+k+1}\leqslant\beta c_{N+k}$$
$$\to c_{N+p}\leqslant\beta^p c_N$$
$$c_n\leqslant c_N\beta^{-N}\cdot\beta^n$$
$$\to\sqrt[n]{c_n}\leqslant\sqrt[n]{c_N\beta^{-N}}\cdot\beta$$
$$\to\lim_{n\to\infty}\sup\sqrt[n]{c_n}\leqslant\beta$$
$$\beta\text{的任意性}\to\lim_{n\to\infty}\sup\sqrt[n]{c_n}\leqslant\alpha$$

1. 与2.类似

□

Remark: 此定理表明比率验敛法能判断收敛的级数，根值验敛法一定也能判断出其收敛

但这两个验敛法都不能判断级数发散。总是从$n\to\infty,a_n\neq 0$来判断发散性。

例 9.4. 比率验敛法失效，根式验敛法有效的例子

1. $\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^3}+\cdots=\sum\left(\frac{1}{2^n}+\frac{1}{3^n}\right)$

$$\begin{aligned}
\liminf \tfrac{a_{n+1}}{a_n} = \lim \left(\tfrac{2}{3}\right)^n = 0 & \quad \text{没用} \\
\liminf \sqrt[n]{a_n} = \lim \sqrt[2n]{\tfrac{1}{3^n}} = \tfrac{1}{\sqrt{3}} & \quad \text{没用} \\
\limsup \tfrac{a_{n+1}}{a_n} = \lim \left(\tfrac{3}{2}\right)^n = +\infty & \quad \text{无法判断} \\
\limsup \sqrt[n]{a_n} = \sqrt[2n]{\tfrac{1}{2^n}} = \tfrac{1}{\sqrt{2}} & \quad \text{收敛}
\end{aligned}$$

2. $\frac{1}{2}+1+\frac{1}{8}+\frac{1}{4}+\frac{1}{32}+\frac{1}{16}+\frac{1}{128}+\frac{1}{64}+\cdots=\sum_0^\infty\left(\frac{1}{2^{2n+1}}+\frac{1}{2^{2n}}\right)$

$$\begin{aligned}
\liminf \tfrac{a_{n+1}}{a_n} &= \tfrac{1}{8} \\
\limsup \tfrac{a_{n+1}}{a_n} &= 2 \\
\limsup \sqrt[n]{a_n} &:
\end{aligned}$$

$$\begin{aligned}
\limsup \sqrt[n]{\tfrac{1}{2^{n+1}}} &= 2^{\frac{-n-1}{n}} = 2^{-1} \\
\limsup \sqrt[n]{\tfrac{1}{2^n}} &= 2^{\frac{-n}{n}} = 2^{-1} \\
\rightarrow \qquad \limsup \sqrt[n]{a_n} &= \tfrac{1}{2}
\end{aligned}$$

10 幂级数

定义 10.1. *幂级数*

$$\{z_n : z \in C\}: \ \sum_{n=0}^{\infty} c_n z^n$$

称为幂级数。c_n称为系数

定理 10.2. *幂级数的收敛性*

幂级数$\sum c_n z^n$. $\alpha = \lim_{n\to\infty} \sup \sqrt[n]{|c_n|}, R = \frac{1}{\alpha}$ $\quad \alpha = 0 \rightarrow R = \infty, \alpha = +\infty, R = 0$

$\lvert z\rvert < R$	收敛
$\lvert z\rvert > R$	发散
$\lvert z\rvert = R$	具体判断

Remark: R叫级数的收敛半径

证明.

$a_n = c_n z^n$.根值验敛法

$\lim_{n\to\infty} \sup \sqrt[n]{|c_n z^n|} = |z| \lim_{n\to\infty} \sup \sqrt[n]{|c_n|} = \frac{|z|}{R}$ □

例 10.3. 一些幂级数的收敛半径

1. $\sum n^n z^n . R = 0$

2. $\sum \frac{z^n}{n!} . R = +\infty$

3. $\sum z^n . R = 1$. $|z| = 1$级数发散.$\lim |z^n| \neq 0$

4. $\sum \frac{z^n}{n} . R = 1 . z = 1$时级数发散.但在其它的$|z| = 1$上收敛

5. $\sum \frac{z^n}{n^2}. R=1. |z|=1$的所有点收敛. 比较验敛法$\sum \frac{z^n}{n^2} = \sum \frac{1}{n^2}$

11 分部求合法

处理积的级数。数列$\{a_n b_n\}$的级数

定理 11.1. *分部求和公式*

$$\begin{aligned} &\text{序列}\{a_n\}, \{b_n\}. A_n = \textstyle\sum_i^n a_i. A_{-1} = 0 \\ 0 \leqslant p \leqslant q \rightarrow &\textstyle\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \end{aligned}$$

$$\begin{aligned} &\textstyle\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n \\ &= \textstyle\sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1} \\ &= \textstyle\sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \end{aligned}$$

定理 11.2. $\sum a_n b_n$收敛的一个判断准则

$$\left.\begin{array}{ll} 1 & \sum a_n\text{的部分和}A_n\text{有界} \\ 2 & b_0 \geqslant b_1 \geqslant b_2 \geqslant \cdots . b_n\text{单调减} \\ 3 & \lim b_n = 0 \end{array}\right. \rightarrow \sum a_n b_n\text{收敛}$$

证明. *虽然分部求和公式无法应用在无穷求和上，但是可以用在Cauchy序列里.doge*

$$\begin{aligned} &A_n\text{有界} \rightarrow \exists M \in R^+. |A_n| < M. \\ &\lim b_n = 0. \forall \varepsilon > 0. \exists N, n > N \rightarrow b_n < \varepsilon \\ &\forall q \geqslant p \geqslant N \rightarrow \\ |\textstyle\sum_{n=p}^{q} a_n b_n| &= \left|\textstyle\sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right| \\ &\leqslant M \left|\textstyle\sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q + b_p\right| \qquad b_n - b_{n+1} \geqslant 0 \rightarrow \text{不改变方向} \\ &= 2M b_p \leqslant 2M\varepsilon \\ &\rightarrow \textstyle\sum a_n b_n\text{是Cauchy的} \\ &\rightarrow \textstyle\sum a_n b_n\text{收敛} \end{aligned}$$

□

定理 11.3. *交错级数的一个判别法. Leibnitz判别法*

$$\left.\begin{array}{ll} 1 & |c_1| \geqslant |c_2| \geqslant |c_3| \geqslant \cdots \\ 2 & c_{2m-1} \geqslant 0, c_{2m} \leqslant 0 \\ 3 & \lim c_n = 0 \end{array}\right. \rightarrow \sum c_n\text{收敛}$$

证明.

$$\begin{aligned} &a_n = (-1)^{n+1}, b_n = |c_n| \\ |a_n|\text{有界} \wedge b_n\text{单调减} \wedge \lim b_n = 0 \rightarrow &\textstyle\sum a_n b_n\text{收敛} \quad 11.2 \end{aligned}$$

□

定理 11.4. *幂级数收敛圆上的一个验敛法*

$$\left.\begin{array}{ll} 1 & \sum c_n z^n\text{的收敛半径} = 1 \\ 2 & c_0 \geqslant c_1 \geqslant c_2 \geqslant \cdots \\ 3 & \lim c_n = 0 \end{array}\right. \rightarrow \sum c_n z^n\text{在收敛圆上除了}z=1\text{之外都收敛}$$

证明.

$$a_n = z^n . b_n = c_n . \, |z| = 1 \wedge z \neq 1$$
$$\rightarrow |A_n| = |\textstyle\sum_{m=0}^n z^m| = \left|\frac{1-z^{n+1}}{1-z}\right| \leqslant \frac{2}{|1-z|}$$
$$\rightarrow |A_n| \text{有界}$$
□

12 绝对收敛

定义 12.1. *绝对收敛.* $\sum |a_n|$收敛

定理 12.2. *绝对收敛的级数一定收敛*

证明.

$$\sum |a_n| \text{收敛} \rightarrow \textstyle\sum_p^q |a_n| < \varepsilon$$
$$|\textstyle\sum_p^q a_n| \leqslant \sum_p^q |a_n| < \varepsilon$$
$$\rightarrow \textstyle\sum a_n \text{是柯西的}$$
$$\rightarrow \textstyle\sum a_n \text{收敛}$$

Remark: 正项级数的收敛就是绝对收敛

Remark: 幂级数在收敛圆内绝对收敛 □

注意 12.3. 比较验敛法、 根值验敛法、 比率验敛法都是绝对收敛的验敛法； 不能处理条件收敛的级数。分部求和法有时可以处理条件收敛的级数。

13 级数的加法和乘法

定义 13.1. *级数的加法*

$$\sum a_n = A, \sum b_n = B \rightarrow \sum a_n + b_n = A + B$$
$$\forall c \in F . \sum c a_n = cA$$

证明.

$$A_n = \textstyle\sum_{i=1}^n a_i; B_n = \sum_{i=1}^n b_i$$
$$A_n + B_n = \textstyle\sum_{i=1}^n (a_i + b_i)$$
$$\lim A_n = A \wedge \lim B_n = B \rightarrow \lim A_n + B_n = A + B \quad 1.3$$
□

$$\lim A_n = A, \lim cA_n = cA$$

定义 13.2. *级数的乘法 Cauchy积*

$$\sum a_n ; \sum b_n \qquad c_n = \sum_{k=0}^n a_k b_{n-k}$$
$$\text{definition} \quad \sum c_n = \sum a_n \sum b_n$$

来源

$$(a_1 + a_2 + \cdots)(b_1 + b_2 + \cdots +)$$
$$= a_1 b_1 + (a_1 b_2 + a_2 b_1) + (a_1 b_3 + a_2 b_2 + a_3 b_1) + \cdots$$

例 13.3. 收敛级数的积可以发散

$$A_n=\sum_{i=0}^n a_i;B_n=\sum_{i=0}^n b_i;C_n=\sum_{i=0}^n c_i$$
$$\lim A_n=A;\lim B_n=B$$
$$A_n=B_n=\sum\frac{(-1)^n}{\sqrt{n+1}}=1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}-\cdots$$
$$A_n,B_n\text{都收敛} \qquad 11.3$$
$$A_nB_n=1-\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{2}\sqrt{2}}+\frac{1}{\sqrt{3}}\right)-\left(\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{3}\sqrt{2}}+\frac{1}{\sqrt{2}\sqrt{3}}+\frac{1}{\sqrt{4}}\right)+\cdots$$
$$c_n=(-1)^n\sum_{i=0}^n\frac{1}{\sqrt{(n-k+1)(k+1)}}$$
$$(n-k+1)(k+1)=\left(\frac{n}{2}+1\right)^2-\left(\frac{n}{2}-1\right)^2\leqslant\left(\frac{n+2}{2}\right)^2$$
$$\frac{1}{\sqrt{(n-k+1)(k+1)}}\geqslant\frac{2}{n+2}$$
$$|c_n|\geqslant\sum_{i=0}^n\frac{2}{n+2}=\frac{2(n+1)}{n+2}$$
$$\lim|c_n|=2\geqslant 0$$
$$\rightarrow\sum c_n\text{发散}$$

定理 13.4. *Mertens. 两个收敛级数中至少有一个绝对收敛则乘积收敛且收敛到和的乘积*

$$\begin{array}{ll} 1 & \sum a_n\text{绝对收敛}.\sum a_n=A \\ 2 & \sum b_n=B \\ 3 & c_n=\sum_{k=0}^n a_kb_{n-k} \end{array}\quad\rightarrow\quad\sum_{n=0}^{\infty}c_n=AB$$

证明.

$$\text{let: }A_n=\sum_{i=0}^n a_i;B_n=\sum_{i=0}^n b_i;C_n=\sum_{i=0}^n c_i;\beta_n=B_n-B$$
$$C_n=a_0b_0+(a_0b_1+a_1b_0)+\cdots+(a_0b_n+a_1b_{n-1}+\cdots+a_nb_0)$$
$$=a_0B_n+a_1B_{n-1}+\cdots+a_nB_0$$
$$=a_0(B+\beta_n)+a_1(B+\beta_{n-1})+\cdots+a_n(B+\beta_0)$$
$$=A_nB+a_0\beta_n+a_1\beta_{n-1}+\cdots+a_n\beta_0$$
$$\text{let: }r_n=a_0\beta_n+a_1\beta_{n-1}+\cdots+a_n\beta_0$$
$$\lim C_n=AB\Leftrightarrow\lim r_n=0$$
$$\alpha=\sum|a_n|$$
$$\forall\varepsilon>0,n>N\rightarrow|\beta_n|<\varepsilon$$
$$|r_n|\leqslant|\beta_0a_n+\cdots\beta_Na_{n-N}|+|\beta_{N+1}a_{n-N-1}+\cdots+\beta_na_0|$$
$$\leqslant|\beta_0a_n+\cdots+\beta_Na_{n-N}|+\varepsilon\alpha$$
$$\lim a_n=0:n>M\rightarrow|a_n-a_{n-N}|<\varepsilon$$
$$\lim n\rightarrow\infty\ |\beta_0a_n+\cdots+\beta_Na_{n-N}|\leqslant\beta_nN\varepsilon$$
$$\rightarrow\lim|r_n|\leqslant(\beta_nN+\alpha)\varepsilon$$
$$\rightarrow\lim\sup|r_n|=0$$
$$\rightarrow\lim r_n=0$$

□

定理 13.5. *Abel.* $\sum a_n,\sum b_n,\sum c_n$分别收敛与$A,B,C.c_n=a_0b_n+a_1b_{n-1}+\cdots+a_nb_0\rightarrow\sum c_n=AB$

证明. 需要依赖函数连续性的证明。第八章 □

14 级数的重排

定义 14.1. *级数的重排*

$k_n = Z^+$, 且k_n是$1-1$的。

$$a'_n = a_{k_n}$$

$\sum a_{k_n}$是$\sum a_n$的重排

Remark: 级数重排的部分和序列可能是完全不同的数组成的。

例 14.2. 收敛级数的重排不收敛于原来的收敛值的例子

$$\text{级数}\sum \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

$$\text{重排}\sum \left(\frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n}\right) + = \frac{1}{1} + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$$

$$\text{let: } s = \sum \frac{(-1)^{n+1}}{n} \rightarrow s < 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

$$s'_n = \sum \left(\frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n}\right).\ n \geqslant 1 \rightarrow \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} > 0$$

$$\rightarrow s'_n\text{是单调增的}$$

$$s'_3 = \frac{1}{1} + \frac{1}{3} - \frac{1}{2} = \frac{5}{6}.s' > s'_3 \rightarrow s' \neq s$$

$$(s'_n\text{是收敛的})$$

定理 14.3. *Riemann. 任意广义实数a,b，条件收敛的实级数一定有重排使得上极限收敛到b，下极限收敛到a。*

$$\begin{array}{ccc} \text{实级数}\sum a_n\text{条件收敛} & & \exists\text{重排}\sum a_{n_k} \\ -\infty \leqslant \alpha \leqslant \beta \leqslant +\infty & \rightarrow & \rightarrow \liminf s'_n = \alpha, \limsup s'_n = \beta \end{array}$$

证明.

$$\text{let: } p_n = \frac{|a_n| + a_n}{2}, q_n = \frac{|a_n| - a_n}{2}$$

$$\rightarrow p_n - q_n = a_n; p_n + q_n = |a_n|$$

p_n, q_n都发散:

$$\sum p_n, \sum q_n\text{收敛} \rightarrow \sum p_n + q_n = \sum |a_n|\text{收敛.矛盾}$$

$$\text{Assume: } p_n\text{发散}, q_n\text{收敛} \rightarrow \sum a_n = \sum (p_n - q_n) = \sum p_n - \sum q_n$$

$$\rightarrow \sum a_n\text{发散.矛盾}$$

$$\rightarrow p_n, q_n\text{都发散}$$

选取正项和负项:

$$a_n \geqslant 0: P_i = a_n;$$

$$a_n < 0: Q_i = -a_n;$$

$$P_n\text{与}p_n\text{在}a_n < 0\text{的点上不同，在}a_n \geqslant 0\text{的点上相同} \rightarrow P_n\text{发散}$$

$$\rightarrow Q_n\text{发散}$$

□

$$\{m_n\}, \{k_n\} \rightarrow \text{级数}P_1 + \cdots + P_{m_1} - Q_1 - \ldots - Q_{k_1}$$

$$+P_{m_1+1} + \cdots + P_{m_2} - Q_{k_1+1} - \cdots - Q_{k_2} + \ldots$$

$\{m_n\}, \{k_n\}$存在:

$$\exists a_n, b_n \in R \wedge a_n \rightarrow a, b_n \rightarrow b.a_n < b_n.b_1 > 0$$

$$\exists m \rightarrow P_1 + \cdots P_m > b_1.m_1 = \min\{m\} \qquad P_n\text{发散}$$

$$\exists k \rightarrow P_1 + \cdots + P_m - Q_1 - \cdots - Q_k < a_1.k_1 = \min\{k\} \qquad Q_n\text{发散}$$

$$\text{在有限维空间中，正项序列发散} \rightarrow \text{不是Cauchy序列}$$

$\rightarrow$

$$\text{可以序列取到}\{m_n\}, \{k_n\}$$

$$\text{let: } \{x_n\} = P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + \cdots +$$

$$|x_n - b_n| \leqslant P_{m_n}; |x_n - a_n| \leqslant Q_{k_n}$$

$$\lim{}_{n\rightarrow\infty} P_n = \lim_{n\rightarrow\infty} Q_n = 0$$

$$\rightarrow \liminf x_n = a, \limsup x_n = b$$

定理 14.4. *绝对收敛的级数的任意重排收敛于原来值*

$$\sum |a_n|\text{收敛}\rightarrow\forall\text{重排}\sum a_m\text{收敛}$$

证明.

$$\begin{gathered}\sum |a_n|\text{收敛}\rightarrow\sum_{n=k}^{m}|a_n|\text{收敛} \qquad\qquad\qquad \text{Cauchy}\\ \text{重排}\left|\sum_{n=k}^{m}a_n'\right|\leqslant\sum_{n=k}^{m}|a_n'|\\ \text{部分和中包含的最大原项}a_n'=a_n\text{是确定的}\\ \text{let:}\max\{a_n'=n\}=N.\text{重排的前}n\text{项和中对应原数列中最大的}n\\ \sum_N^m|a_n'|\leqslant\sum_k^m|a_n|<\varepsilon\\ \rightarrow\sum a_n'\text{是柯西的}\\ \rightarrow\sum a_n'\text{收敛}\end{gathered}$$

□

这里没写清楚。重排的部分和必有原级数中的项。
选取$N=\max\{\text{Cauchy选取的}N,\text{重排的最大项}N\}$.
则其余的Cauchy和必然小于前面的这些项之和.

$$\begin{gathered}\exists n\in N^+\rightarrow|s_n-s_n'|\text{中的前}N\text{项相同的都被消掉}\rightarrow|s_n-s_n'|\leqslant|s_m-s_n|=\varepsilon\\ \rightarrow\lim s_n=\lim s_n'\end{gathered}$$

习题

1. Proof: $\{a_n\}$收敛$\rightarrow\{|a_n|\}$收敛；$\{|a_n|\}$收敛$\rightarrow\{a_n\}$不一定收敛

$$\begin{gathered}a_n\text{收敛}:\forall\varepsilon>0,\exists N\in N^+,n>N\rightarrow d(a_n,a)<\varepsilon\\ |a_n|\geqslant 0\rightarrow a\geqslant 0;\ |a_n|<0\rightarrow a\leqslant 0\\ ||a_n|-|a||\leqslant|a_n-a|<\varepsilon\\ \rightarrow|a_n|\text{收敛}\\ a_n=(-1)^n.a_n\text{不收敛.但}\lim|a_n|=1\end{gathered}$$

2. Compute: $\lim_{n\rightarrow\infty}\left(\sqrt{n^2+n}-n\right)$

$$\begin{gathered}\lim_{n\rightarrow\infty}\sqrt{n^2+n}-n\\ =\lim_{n\rightarrow\infty}\frac{n}{\sqrt{n^2+n}+n}\\ \leftarrow\frac{\sqrt{n^2+n}+n}{n}=\frac{\sqrt{n^2+n}}{n}+1\\ \sqrt{n^2}=n\leqslant\sqrt{n(n+1)}\leqslant n+\frac{1}{2}\\ \lim\frac{\sqrt{n^2+n}}{n}\leqslant\lim\frac{n+\frac{1}{2}}{n}=1\\ 1=\lim\frac{n}{n}\leqslant\lim\frac{\sqrt{n(n+1)}}{n}\\ \rightarrow 1\leqslant\lim\frac{\sqrt{n(n+1)}}{n}\leqslant 1\rightarrow\lim\frac{\sqrt{n(n+1)}}{n}=1\\ \rightarrow\lim\frac{\sqrt{n^2+n}+n}{n}=1+1=2\\ \rightarrow\frac{n}{\sqrt{n^2+n}+n}=\frac{1}{2}\end{gathered}$$

不能用连续性。构造夹逼准则，序列的夹逼准则可以绕过函数连续性

3. Proof: $s_1=\sqrt{2}, s_{n+1}=\sqrt{2+\sqrt{s_n}}$. Proof: $\{s_n\}$收敛$\wedge s_n<2$

$$
\begin{gathered}
s_1=\sqrt{2}<2. s_{n+1}=\sqrt{2+\sqrt{s_n}}<\sqrt{2+2}=2\\
\rightarrow\{s_n\}\text{有上界}2.\\
\frac{s_{n+1}}{s_n}=\frac{\sqrt{2+\sqrt{s_n}}}{s_n}=\sqrt{\frac{2+\sqrt{s_n}}{s_n^2}}\geqslant\sqrt{\frac{2+\sqrt{s_n}}{s_n}}=\sqrt{\frac{2}{s_n}+\frac{1}{\sqrt{s_n}}}\\
\geqslant\sqrt{1}=1\\
\rightarrow s_{n+1}>s_n\\
\rightarrow s_n\text{是单调数列}\\
\rightarrow s_n\text{收敛}
\end{gathered}
$$

4. Compute: 上下极限 $s_1=0; s_{2m}=\frac{s_{2m-1}}{2}; s_{2m+1}=\frac{1}{2}+s_{2m}$

$$
\begin{gathered}
s_1=0; s_2=\frac{0}{2}=0; s_3=\frac{1}{2}+0=\frac{1}{2}; s_4=\frac{1}{4}; s_5=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}; s_6=\frac{3}{8}; s_7=\frac{7}{8}\cdots\\
s_{2m+1}=\frac{1}{2}+s_{2m}\rightarrow s_{2m+1}>s_{2m}\\
s_{2m}=\frac{1}{2}s_{2m-1}\rightarrow s_{2m}\leqslant s_{2m-1}\\
s_{2(m+1)+1}=\frac{1}{2}+s_{2(m+1)}=\frac{1}{2}+\frac{1}{2}s_{2m+1}\\
s_{2(m+1)}=\frac{1}{2}s_{2m+1}=\frac{1}{2}\left(\frac{1}{2}+s_{2m}\right)
\end{gathered}
$$

$$
\begin{gathered}
s_1<1\\
\text{设}s_{2m-1}<1. s_{2m+1}=\frac{1}{2}+\frac{1}{2}s_{2m-1}<\frac{1}{2}+\frac{1}{2}=1\rightarrow s_{2m+1}<1\text{有界}\\
s_{2m+1}=\frac{1}{2}+\frac{1}{2}s_{2m-1}\geqslant\frac{1}{2}s_{2m-1}+\frac{1}{2}s_{2m-1}=s_{2m-1}\rightarrow s_{2m+1}\text{单调增}\\
\rightarrow s_{2m+1}\text{极限存在}\\
s_{2m+1}=\frac{1}{2}+s_{2m}>\frac{1}{2}\rightarrow s_{2m+1}\geqslant s_{2m}
\end{gathered}
$$

$$
\begin{gathered}
s_0<\frac{1}{2}\\
\text{设}s_{2m}<\frac{1}{2}. s_{2(m+1)}=\frac{1}{2}\left(\frac{1}{2}+s_{2m}\right)\leqslant\frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}\right)=\frac{1}{2}\rightarrow s_{2m}<\frac{1}{2}\text{有界}\\
s_{2(m+1)}=\frac{1}{2}\left(\frac{1}{2}+s_{2m}\right)\geqslant\frac{1}{2}(s_{2m}+s_{2m})=s_{2m}\rightarrow s_{2m}\text{单调增}\\
s_{2m}\text{极限存在}
\end{gathered}
$$

$$
\begin{gathered}
\limsup s_n=\lim s_{2m+1}=\frac{1}{2}+\frac{1}{2}(s_{2m-1})=\frac{1}{2}+\frac{1}{2}\left(\frac{1}{2}+s_{2m-3}\right)\\
=\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^4}+\cdots+\\
=1\\
\liminf s_n=\lim s_{2m}=\frac{1}{2}\left(\frac{1}{2}+s_{2(m-1)}\right)=\frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}\left(\frac{1}{2}+s_{2(m-2)}\right)\right)=\cdots\\
=\frac{1}{2^2}+\frac{1}{2^3}+\cdots\\
=\frac{1}{2}
\end{gathered}
$$

5. Proof: $\forall\{a_n\},\{b_n\}$收敛$\wedge a+b$不是$+\infty+-\infty$这种类型。Proof: $\limsup(a_n+b_n)\leqslant\limsup a_n+$

$\limsup b_n$

$$\limsup(a_n+b_n)=\sup\{a_{n_k}+b_{n_k}: n_k\text{是一个子序列}\}$$
$$\limsup a_n=\sup\{a_{n_i}\}.\limsup b_n=\sup\{b_{n_j}\}$$
$$\text{若}\limsup(a_n+b_n)=+\infty\to\forall M\in R^+,\exists N\in N^+,n>N\to a_n+b_n>M$$
$$a_n>M-b_n.\text{由于}M\text{的任意性},a_n>M.\text{同理}b_n>M$$
$$\to\limsup a_n=+\infty,\limsup b_n=+\infty\to\text{显然成立}$$
$$\limsup a_n=+\infty,\limsup b_n=b\in R\to\text{显然成立}$$
$$\limsup a_n=a\in R,\limsup b_n=b\in R$$
$$\limsup a_n=a\to\forall a_{n_i},a_{n_u}\geqslant a_{n_i}.\limsup b_n=b\to\forall b_{n_i},b_{n_v}\geqslant b_{n_i}$$
$$\limsup a_n+b_n=a_{n_i}+b_{n_i}\leqslant a_{n_u}+b_{n_v}=\limsup a_n+\limsup b_n$$

6. 验敛.

a. $a_n=\sqrt{n+1}-\sqrt{n}$

$$\sum(\sqrt{n+1}-\sqrt{n})=\sqrt{n+1}-1=\infty$$

b. $a_n=\frac{\sqrt{n+1}-\sqrt{n}}{n}$

$$\frac{\sqrt{n+1}-\sqrt{n}}{n}=\frac{1}{n(\sqrt{n+1}+\sqrt{n})}\leqslant\frac{1}{n(2\sqrt{n})}=\frac{1}{2}\frac{1}{n^{3/2}}.p\text{级数}\to\text{收敛}$$

c. $a_n=(\sqrt[n]{n}-1)^n$

$$\limsup\sqrt[n]{|(\sqrt[n]{n}-1)^n|}=\lim\sqrt[n]{n}-1=0\to\text{根值验敛法}\to\text{收敛}$$

d. $a_n=\frac{1}{1+z^n},z\in C$

$$\frac{a_{n+1}}{a_n}=\frac{1}{1+|z^{n+1}|}\cdot\frac{1+|z^n|}{1}=\frac{1+|z^n|}{1+|z^{n+1}|}<1$$
$$\lim\sqrt[n]{\frac{1}{1+|z^n|}}\leqslant\sqrt[n]{\frac{1}{|z^n|}}=\frac{1}{|z|}$$
$$\to|z|<1\text{时收敛},|z|>1\text{时发散}$$
$$|z|=1\wedge z\neq1\to z^n\text{不是确定的数}\to\text{发散}$$
$$z=1\to\frac{1}{2}>0\to\text{发散}$$

7. Proof: $a_n\geqslant 0.\sum a_n$收敛. Proof: $\sum\frac{\sqrt{a_n}}{n}$收敛

$$\forall\varepsilon>0,\sum a_n\text{收敛}\to s_n\text{单调有界}$$
$$a_n\geqslant0\to\frac{\sqrt{a_n}}{n}\geqslant0$$
$$\to\sum\frac{\sqrt{a_n}}{n}\text{单调}$$
$$\sqrt{x}\text{是凸函数, Jensen: }t\sqrt{x}+(1-t)\sqrt{y}\leqslant\sqrt{x+y}$$
$$tf(a)+(1-t)f(b)\leqslant f(ta+(1-t)b)$$
$$\frac{1}{m}\sqrt{a}+\frac{1}{2}\sqrt{b}\leqslant\sqrt{\frac{a+b}{2}}$$
$$\frac{\sqrt{a_n}}{n}+\frac{\sqrt{a_{n+1}}}{n+1}+\cdots+\frac{\sqrt{a_{n+m}}}{n+m}\leqslant\frac{\sqrt{a_n}+\sqrt{a_{n+1}}+\cdots+\sqrt{a_{n+m}}}{n}$$
$$\leqslant\frac{1}{n}\sqrt{\frac{a_n+a_{n+1}+\cdots+a_{n+m}}{m}}=\frac{1}{n}\sqrt{\frac{\varepsilon}{m}}<\frac{1}{nm}\sqrt{\varepsilon}$$
$$\to\frac{\sqrt{a_n}}{n}\text{收敛}$$

8. Proof: a_n收敛, b_n单调有界. Proof: $\sum a_n b_n$收敛

$$\begin{gathered}b_n\text{单调有界} \to b_n\text{收敛}\\ |b_n| < M\\ |\textstyle\sum_{n=i}^{j} a_n b_n| < M |\sum_{n=i}^{j} a_n|\\ = M\varepsilon\\ \to \textstyle\sum a_n b_n\text{收敛}\end{gathered}$$

9. Compute: 收敛半径

   a. $\sum n^3 z^n$

   $$\begin{gathered}\limsup \sqrt[n]{|n^3|} = \lim \sqrt[n]{n^3} = (\sqrt[n]{n})^3 = 1^3 = 1\\ \to R = \tfrac{1}{1} = 1\end{gathered}$$

   b. $\sum \frac{2^n}{n!} z^n$

   $$\limsup \sqrt[n]{\left|\tfrac{2^n}{n!}\right|} = \lim \sqrt[n]{\tfrac{2^n}{n!}} = \tfrac{2}{\sqrt[n]{n!}}$$
   Stirling: $n! \sim n^{n+1/2} \to R = +\infty$.

   c. $\sum \frac{2^n}{n^2} z^n$

   $$\limsup \sqrt[n]{\left|\tfrac{2^n}{n^2}\right|} = \lim \sqrt[n]{\tfrac{2^n}{n^2}} = \lim \tfrac{2}{\sqrt[n]{n^2}} = 2 \to R = \tfrac{1}{2}$$

   d. $\sum \frac{n^3}{3^n} z^n$

   $$\limsup \sqrt[n]{\left|\tfrac{n^3}{3^n}\right|} = \lim \sqrt[n]{\tfrac{n^3}{3^n}} = \tfrac{1}{3} \to R = 3$$

10. Proof: 幂级数$\sum a_n z^n, a_n \in Z$.无限个$a_n \neq 0$. Proof: 收敛半径$R \leqslant 1$

$$\begin{gathered}R > 1 \to \limsup \sqrt[n]{|a_n|} < 1\\ a_n \neq 0 \to |a_n| \neq 0\\ \forall x \in R^+. \lim \sqrt[n]{x} = 1 \to \exists n > N, \exists a_n \to \lim \sqrt[n]{a_n} = 1\\ \to \limsup \sqrt[n]{a_n} \geqslant 1\\ \to R \leqslant 1\end{gathered}$$

11. $a_n > 0, s_n = \sum_{i=1}^{n} a_i$. $\sum a_i$发散

   a. Proof: $\sum \frac{a_n}{1+a_n}$发散

   $$\begin{gathered}\forall n \in N^+, \exists i > j \to \textstyle\sum_{n=j}^{i} a_n > \varepsilon\\ \left(\frac{x}{1+x}\right)' = (x(1+x)^{-1})' = -x(1+x)^{-2} + (1+x)^{-1} < 0\\ \to \frac{1}{1+x} - \frac{x}{(1+x)^2} = \frac{1}{(1+x)^2} \geqslant 0\\ \to \frac{x}{1+x}\text{是凸函数.}\\ \text{Jense:}\, t f(a) + (1-t) f(b) \leqslant f(t a + (1-t) b)\\ \textstyle\sum_{n=j}^{i} \frac{a_i}{1+a_i} \leqslant \frac{a_j + a_{j+1} + \cdots + a_{j+m}}{1 + a_j + a_{j+1} + \cdots + a_{j+m}}\\ = \frac{\varepsilon}{1+\varepsilon}\\ ???\end{gathered}$$

   b. Proof: $\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geqslant 1 - \frac{s_N}{s_{N+k}}$; $\sum \frac{a_n}{s_n}$发散

   c. Proof: $\frac{a_n}{s_n^2} \leqslant \frac{1}{s_{n-1}} - \frac{1}{s_n}$; $\sum \frac{a_n}{s_n^2}$收敛

d. Proof or Disproof: $\sum\frac{a_n}{1+na_n}$收敛; $\sum\frac{a_n}{1+n^2a_n}$收敛

12. $a_n>0\wedge\sum a_n$收敛. $r_n=\sum_{m=n}^{\infty}a_m$

    a. Proof: $m<n\rightarrow\frac{a_m}{r_m}+\cdots+\frac{a_n}{r_n}>1-\frac{r_n}{r_m}$; $\sum\frac{a_n}{r_n}$发散

    b. Proof: $\frac{a_n}{\sqrt{r_n}}<2(\sqrt{r_n}-\sqrt{r_{n+1}})$; $\sum\frac{a_n}{\sqrt{r_n}}$收敛

13. Proof: 两个绝对收敛的级数的Cauchy积也绝对收敛

$$\begin{aligned}&\textstyle\sum|a_n|\text{收敛，}\sum|b_n|\text{收敛}\\&\rightarrow\textstyle\sum|a_n|\sum|b_n|\text{收敛.}\quad 13.4\\&\quad|a_n|>0,|b_n|\geqslant 0\\&\rightarrow|a_n|\cdot|b_{k-n}|\geqslant 0\\&\quad\rightarrow|c_n|\geqslant 0\\&\quad\rightarrow\text{绝对收敛}\end{aligned}$$

14. s_n为复数序列, $\sigma_n=\frac{s_0+s_1+\cdots+s_n}{n+1}$

    a. Proof: $\lim s_n=s$. Proof: $\lim\sigma_n=s$

$$\begin{gathered}\lim s_n=s, i,j>N\rightarrow d(s_i,s_j)<\varepsilon\\d(\sigma_i,\sigma_j)=\frac{s_0+\cdots+s_i}{i+1}-\frac{s_0+\cdots++s_j}{j+1}\\\text{let: }i>j.\,d(\sigma_i,\sigma_j)\leqslant\frac{s_0+\cdots+s_i-(s_0+\cdots+s_j)}{j+1}\\\leqslant\frac{1}{j+1}(s_{i+1}+\cdots+s_j)=\frac{1}{j+1}(j-i+1)\varepsilon\\=\frac{j-i+1}{j+1}<\varepsilon\\\rightarrow\lim\sigma_n\text{收敛}\end{gathered}$$

$$\begin{gathered}\lim\frac{1}{n+1}(s_0+\cdots+s_n)-s\\=\lim\frac{1}{n+1}(s_0+\cdots+s_N)+\frac{1}{n+1}((n-N)(s-\varepsilon))-s\\=0+\lim\frac{n-N}{n+1}(s-\varepsilon)-s\\=s-\varepsilon-s=-\varepsilon\\\rightarrow d\Big(\lim\frac{1}{n+1}(\textstyle\sum s_n),s\Big)<\varepsilon\\\rightarrow\lim\frac{1}{n+1}\textstyle\sum s_n=s\end{gathered}$$

    b. Example: s_n. $\lim\sigma_n=0\wedge s_n$不收敛

    $s_n=(-1)^n$.$\lim\sigma_n=\frac{(-1)^n}{n+1}$收敛

    c. Proof or Counterexample:$\exists s_n.\forall n\in N^+.\,s_n>0.\lim\sigma_n=0\wedge\limsup s_n=\infty$

$$\begin{gathered}\text{不可能.}\\\limsup s_n=\infty\rightarrow\forall M\in R^+.\exists n>N\rightarrow s_n>M\\\lim\sigma_n=0.\lim\frac{1}{n+1}\textstyle\sum s_n=0\\\rightarrow\sum_m^n\frac{1}{i+1}\textstyle\sum s_i<\varepsilon\\\rightarrow i\in n,m\rightarrow s_i>n+1\\\rightarrow\sum_m^n\frac{1}{i+1}\textstyle\sum s_i\geqslant\frac{1}{n+1}\sum_m^n\sum s_i\geqslant\frac{1}{n+1}(n+1)=1\\\rightarrow\sigma_n\text{不是Cauchy的}\rightarrow\sigma_n\text{不收敛}\end{gathered}$$

    d. Proof: $n\geqslant 1, a_n=s_n-s_{n-1}$. Proof: $s_n-\sigma_n=\frac{1}{n+1}\sum_{i=1}^n i a_i$; $\lim(na_n)=0\wedge\sigma_n$收敛$\rightarrow s_n$收敛.

$$s_n-\sigma_n=s_n-\frac{1}{n+1}\sum s_n=\frac{n}{1+n}s_n-\frac{1}{1+n}\sum_0^{n-1}s_n$$
$$a_n=s_n-s_{n-1}\rightarrow s_n=a_n+s_{n-1}$$
$$s_n=\sum a_n+s_0$$
$$\rightarrow\frac{n}{1+n}\left(\sum_1^n a_n+s_0\right)-\frac{1}{1+n}\sum_{i=0}^{n-1}\left(\sum_{j=1}^{i-1}(a_j+s_0)\right)$$
$$=\frac{n}{1+n}\left(ns_0+\sum_1^n a_n\right)-\frac{1}{1+n}\frac{(n-1)n}{2}s_0-\frac{1}{1+n}\sum_{i=1}^{n-1}(n-i)\,a_i$$
$$=\left(\frac{n^2}{1+n}-\frac{n^2-n}{(1+n)2}\right)s_0+\frac{n}{1+n}\sum_1^n a_i-\frac{1}{1+n}\sum_{i=1}^{n-1}(n-i)a_i$$
$$=\frac{n(n+1)}{2(1+n)}s_0+\frac{1}{1+n}\left(\sum_1^n na_i-\sum_1^{n-1}(n-i)a_i\right)$$
$$=\frac{n}{2}s_0+\frac{1}{1+n}\sum_1^n i\,a_i$$
???
$$\lim na_n=0\rightarrow\lim n\,(s_n-s_{n-1})=0$$
$$\lim s_n=\lim\left(s_0+\sum a_i\right)$$
$$=s_0+\lim\sum a_i$$
$$\leftarrow\lim\frac{1}{n+1}\sum ia_i\text{收敛}$$
$$\sum_n^m s_n=\sum_{i=n}^m\left(s_0+\sum_{j=1}^i a_j\right)$$
$$=(m-n+1)s_0+\sum_{i=1}^n\sum_{j=1}^n a_j+\sum_{i=n}^m\sum_{j=n}^i a_j$$
$$=(m-n+1)s_0+\sum_{i=1}^n na_i+\sum_{i=n}^m(i-n+1)a_i$$
$$\lim na_i=0$$
???

e. Proof: $M<\infty.\forall n\in N^+,\ |na_n|\leqslant M.\lim\sigma_n=\sigma$. Proof: $\lim s_n=\sigma$

15. 推广各个定理到R^k

16. $\alpha,x_1>\sqrt{\alpha}.x_{n+1}=\frac{1}{2}\left(x_n+\frac{\alpha}{x_n}\right)$.

a. Proof: x_n单调减. $\lim x_n=\sqrt{\alpha}$

$$\frac{x_{n+1}}{x_n}=\frac{\left(x_n+\frac{\alpha}{x_n}\right)}{2x_n}=\frac{1}{2}+\frac{\alpha}{2x_n^2}$$
$$x_1>\sqrt{\alpha}\rightarrow x_1^2>\alpha\rightarrow\frac{\alpha}{x_n^2}<1$$
$$\rightarrow\frac{x_{n+1}}{x_n}<1\rightarrow x_{n+1}<x_n$$
$$\rightarrow x_n\text{单调减}$$

$$x_{n+1}=\frac{1}{2}\left(x_n+\frac{\alpha}{x_n}\right)=\frac{1}{2}x_n+\frac{1}{2}\frac{\alpha}{x_n}$$
$$\geqslant\frac{1}{2}(x_n+\sqrt{\alpha})$$
$$x_nx_{n+1}=x_n\frac{1}{2}\left(x_n+\frac{\alpha}{x_n}\right)=\frac{1}{2}(x_n^2+\alpha)$$

$$x_n>\alpha\rightarrow x_{n+1}=\frac{1}{2}\left(x_n+\frac{\alpha}{x_n}\right)$$
$$\geqslant\frac{1}{2}(2\sqrt{\alpha})=\sqrt{\alpha}$$
$$\rightarrow x_n\geqslant\sqrt{\alpha}$$
$$\rightarrow x_n\text{有极限}$$

$$x_n\text{收敛}\rightarrow\lim\frac{x_{n+1}}{x_n}=\frac{\frac{1}{2}\left(x_n+\frac{\alpha}{x_n}\right)}{x_n}=\frac{x_n^2+\alpha}{2x_n^2}=1$$
$$\rightarrow\lim(x_n^2+\alpha)=\lim 2x_n^2$$
$$\rightarrow\lim x_n^2+\alpha=2\lim x_n^2$$
$$\rightarrow\alpha=\lim x_n^2$$
$$\rightarrow\lim x_n=\sqrt{\alpha}$$

b. Proof: $\varepsilon_n=x_n-\sqrt{\alpha}$. Proof: $\varepsilon_{n+1}=\frac{\varepsilon_n^2}{2x_n}<\frac{\varepsilon_n^2}{2\sqrt{\alpha}}.\beta=2\sqrt{\alpha}\rightarrow\varepsilon_{n+1}<\beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n}$

$$
\begin{aligned}
x_{n+1}-\sqrt{\alpha}&=\frac{1}{2}\Big(x_n+\frac{\alpha}{x_n}\Big)-\sqrt{\alpha}\\
\varepsilon_n&=x_n-\sqrt{\alpha}\\
\varepsilon_{n+1}&=\frac{1}{2}\Big(x_n+\frac{\alpha}{x_n}-2\sqrt{\alpha}\Big)\\
&=\frac{1}{2}\Big(\frac{x_n^2+\alpha-2x_n\sqrt{\alpha}}{x_n}\Big)\\
&=\frac{1}{2}\Big(\frac{(x_n-\sqrt{a})^2}{x_n}\Big)\\
&=\frac{\varepsilon_n^2}{2x_n}<\frac{\varepsilon_n^2}{2\sqrt{\alpha}} \qquad\qquad x_n>\sqrt{\alpha}
\end{aligned}
$$

$$
\begin{aligned}
&\varepsilon_{n+1}<\frac{\varepsilon_n^2}{2\sqrt{\alpha}}\to\varepsilon_{n+1}<\frac{\varepsilon_n^2}{2\sqrt{\alpha}}<\frac{\Big(\frac{\varepsilon_{n-1}^2}{2\sqrt{\alpha}}\Big)^2}{2\sqrt{\alpha}}\\
&<\frac{\varepsilon_1^{2^n}}{(2\sqrt{\alpha})^{2^0+2^1+\cdots+2^n}}=2\sqrt{\alpha}\Big(\frac{\varepsilon_1}{2\sqrt{\alpha}}\Big)^{2^n}
\end{aligned}
$$

c. Proof: $\alpha=3,x_1=2.$ Proof: $\frac{\varepsilon_1}{\beta}<\frac{1}{10}\to\varepsilon_5<4\cdot 10^{-16}.\varepsilon_6<4\cdot 10^{-32}$

$$
\begin{aligned}
&\alpha=3,x_1=2.\\
&\frac{\varepsilon_1}{\beta}=\frac{1}{2}\Big(2+\frac{3}{2}\Big)=\frac{7}{4}\\
&\frac{\varepsilon_1}{\beta}<\Big(\frac{2-\sqrt{3}}{2\sqrt{3}}\Big)=0.07735\ldots<\frac{1}{10}\\
&\frac{\varepsilon^5}{\beta}<\Big(\frac{\varepsilon_1}{\beta}\Big)^{2^4}=10^{-16}\\
&\varepsilon^5<\beta\cdot 10^{-16}.\beta=2\sqrt{3}<4\\
&\to\varepsilon^5<4\cdot 10^{-16}
\end{aligned}
$$

17. $\alpha>1,x_1>\sqrt{\alpha}.x_{n+1}=\frac{\alpha+x_n}{1+x_n}=x_n+\frac{\alpha-x_n^2}{1+x_n}$

a. Proof: $x_1>x_3>x_5>\cdots$

$$
\begin{aligned}
&x_{2n+1}-x_{2n-1}\\
&=\frac{\alpha+x_{2n}}{1+x_{2n}}-x_{2n-1}\\
&=\frac{\alpha+\frac{\alpha+x_{2n-1}}{1+x_{2n-1}}}{1+\frac{\alpha+x_{2n-1}}{1+x_{2n-1}}}-x_{2n-1}\\
&=\frac{\frac{\alpha(1+x_{2n-1})+\alpha+x_{2n-1}}{1+x_{2n-1}}}{\frac{1+x_{2n-1}+\alpha+x_{2n-1}}{1+x_{2n-1}}}-x_{2n-1}\\
&=\frac{2\alpha+(\alpha+1)x_{2n-1}}{1+\alpha+2x_{2n-1}}-x_{2n-1}\\
&x_1>\sqrt{\alpha}>1\to 2\alpha+\alpha+1x_{2n-1}>1+\alpha+2x_{2n-1}\\
&\to x_{2n+1}-x_{2n-1}>0\\
&\to x_{2n+1}>x_{2n-1}
\end{aligned}
$$

b. Proof: $x_2<x_4<x_6<\cdots$

$$
\begin{aligned}
&x_2=\frac{\alpha+x_n}{1+x_n}.\\
&\frac{x_{2(n+1)}}{x_{2n}}=\frac{x_{2n+1}}{}
\end{aligned}
$$

c. Proof: $\lim x_n=\sqrt{\alpha}$

d. Compute: 估计x_n的收敛速度

18. $\alpha>0,x_1>\sqrt{\alpha}.p>0.x_{n+1}=\frac{p-1}{p}x_n+\frac{\alpha}{p}x_n^{-p+1}$.计算是否收敛，估计收敛速度

19. $\forall a=\{a_n\colon a_n=0\vee 2\}.x(a)=\sum_{n=1}^{\infty}\frac{a_n}{3^n}$. Proof: $\{x(a)\}$是Cantor集

20. Proof: $\{x_n\}$是度量空间X中的Cauchy序列, $\exists\{x_{n_i}\}\to\lim x_{n_i}=x$.Proof: $\lim x_n=x$

$$\begin{gathered}\lim x_{n_i}=x\to\forall\varepsilon>0,\exists N_1\in N^+,n_i>N\to d(x_{n_i},x)<\varepsilon\\ \forall\varepsilon>0,\exists N_2\in N^+,i,j>N\to d(x_i,x_j)<\varepsilon\\ \text{let } N=\max(N_1,N_2).\, d(x_{n_i},x_j)<\varepsilon\\ j\text{的任意性}\to n>n_i\to d(x_n,x_j)<\varepsilon\\ d(x_n,x)\leqslant d(x_n,x_{n_i})+d(x_{n_i},x)=2\varepsilon\\ \to\lim x_n=x\end{gathered}$$

21. Proof: $\{E_n\colon E_n\in$度量空间X,E_n有界$\wedge$闭$\}.E_n\supset E_{n+1}\wedge\lim\operatorname{diam}E_n=0\to\operatorname{card}\bigcap_{n=1}^{\infty}E_n=1$

22. Proof: Baire.X是完备度量空间, $\{G_n\colon G_n$是X的稠密开子集$\}$. Proof: $\bigcap_{n=1}^{\infty}G_n\neq\varnothing$

23. Proof:$\{p_n\},\{q_n\}$是度量空间X中的Cauchy序列. Proof: $\{d_n=d(p_n,q_n)\}$收敛.

24. X是度量空间

a. Proof: Cauchy序列$\{p_n\},\{q_n\}$的关系$\sim$. $\lim d(p_n,q_n)=0\to p_n,q_n\in\sim$.Proof:$\sim$是等价关系

$$\begin{array}{ccc}\forall p_n,q_n.\lim d(p_n,q_n)=0 & & \\ \lim d(p_n,q_n)=\lim 0=0 & \to & (p_n,p_n)\in\sim\\ \lim d(p_n,q_n)=0\to\lim d(q_n,p_n)=0 & \to & (p_n,q_n)\in\sim\to(q_n,p_n)\in\sim\\ \lim d(x_n,y_n)=0,\lim d(y_n,z_n)=0 & & \\ \to d(x_n,z_n)\leqslant d(x_n,y_n)+d(y_n,z_n) & & \\ =0+0=0 & \to & (x_n,y_n)\in\sim\wedge(y_n,z_n)\in\sim\to(x_n,z_n)\in\sim\\ & & \\ \to & & \sim\text{是等价关系}\end{array}$$

b. Proof: X^*是上述等价类的集.$P\in X^*$, $Q\in X^*$, $\{p_n\}\in P$, $\{q_n\}\in Q$. $\Delta(P,Q)=\lim d(p_n,q_n)$.Proof: Δ是度量

$$\begin{gathered}\Delta(x,x)=\lim d(x_{1n},x_{2_n})=d(x,x)=0\\ \Delta(x,y)=\lim d(x_{1n},x_{2n})=d(x_1,x_2)=d(x_2,x_1)=\lim d(x_{2n},x_{1n})=\Delta(y,x)\\ \Delta(x,y)+\Delta(y,z)=\lim d(x_n,y_n)+\lim d(y_n,z_n)\\ =d(x,y)+d(y,z)\geqslant d(x,z)=\lim d(x_n,z_n)=\Delta(x,z)\\ \to\Delta\text{是度量}\end{gathered}$$

c. Proof: X^*是完备的

$$\begin{gathered}\forall x_n\in X^*.\forall\varepsilon>0,\exists N\in N^+,\forall i,j>N\to\Delta(x_i,x_j)<\varepsilon\\ x_n\colon\lim a_n=x_i,\lim b_n=x_j,\ldots\\ \Delta(x_i,x_j)<\varepsilon\to\lim d\,(a_n,b_n)<\varepsilon\\ \to\lim a_n=\lim b_n=x\\ \to\Delta(x_n,x)\leqslant\Delta(x_n,a_n)+\Delta(a_n,x)\\ =2\varepsilon\\ \to\lim x_n=x\in X^*\\ \to X^*\text{是完备的}\end{gathered}$$

d. Proof:$\forall x\in X.\exists\{p\}\in X;P_p\in X^*\wedge\{p\}\in P_p$. Proof: $\Delta(P_p,P_q)=d(p,q)$

$$\begin{gathered}\Delta(P_p,P_q)=\lim d(P_p,P_q)\\ =\lim d(P_p,P_q)\\ \leqslant\lim d(P_p,p)+\lim d(P_q,q)+\lim d(p,q)\\ p\text{是Cauchy的}\to p\text{是收敛的}\to d\,(P_p,p)<\varepsilon\\ =2\varepsilon+\lim d(p,q)=d(p,q)\\ \to\Delta(P_p,P_q)\leqslant\Delta(p,q)\end{gathered}$$

???需要反向证明$\Delta(p,q)\leqslant\Delta(P_p,P_q)$

e. Proof: $\varphi: X \to X^*. \varphi(p) = \{p\}. \varphi(X)$在$X^*$中稠密; X完备 $\to X = X^*$

25. Construct: X是度量空间.$X \subset Q. d(x,y) = |x-y|$.求$X$的完备化

$X^* = R.$
这是实数的Cauchy定义