第九章 多元函数

1 线性变换

有限维欧氏空间R^n内的向量的一类特殊集合

定义 1.1. *向量空间*

$$\begin{gathered}\text{非空集} X\in R^n\\ 1\quad \forall \boldsymbol{x},\boldsymbol{y}\in X\to \boldsymbol{x}+\boldsymbol{y}\in X\\ 2\quad \forall \boldsymbol{x}\in X,\forall\lambda\in R\to\lambda\boldsymbol{x}\in X\\ X\text{称为向量空间}\end{gathered}$$

线性组合

$$\begin{array}{cc} & \boldsymbol{x}_1,\ldots,\boldsymbol{x}_k\in R^n, c_1,\ldots,c_k\in R\\ \text{线性组合} & c_1\boldsymbol{x}_1+\cdots+c_k\boldsymbol{x}_k\\ \text{生成, 张成} & S\subset R^n, E\text{是}S\text{内的元素的所有线性组合构成的集}\end{array}$$

线性相关、线性无关

$$\begin{array}{cc} & k\text{个向量的集}\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k\\ \text{无关的} & c_1\boldsymbol{x}_1+\cdots+c_k\boldsymbol{x}_k=\boldsymbol{0}\to c_1=\cdots=c_k=0\\ \text{相关的} & \exists c_i\neq 0\to c_1\boldsymbol{x}_1+\cdots+c_k\boldsymbol{x}_k=\boldsymbol{0}\end{array}$$

Remark: *无关集必然没有零向量*

向量空间的维数

$$\begin{gathered}\text{向量空间}X\text{有}r\text{个向量的线性无关集}\wedge\text{没有}r+1\text{个向量的线性无关集}\\ \dim X=r\end{gathered}$$

Remark: $\dim\{0\}=0$

基

$$\text{如果一个}X\text{的无关集能够生成}X\text{，称这个无关子集为}X\text{的基}$$

坐标

$$\begin{gathered}\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\text{是}X\text{的基}\to\forall\boldsymbol{x}\in X,\boldsymbol{x}=c_1\boldsymbol{x}_1+\cdots+c_n\boldsymbol{x}_n\\ c_1,\ldots,c_n\text{称为}\boldsymbol{x}\text{在基}\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\text{下的坐标}\end{gathered}$$

标准基

$$\begin{gathered}e_i=(0,\ldots,1_i,\ldots,0)\\ \text{集}\{e_1,\ldots,e_n\}\text{称为}R^n\text{的标准基}\end{gathered}$$

定理 1.2. $\forall r\in N^+$.若X能由r个向量的集生成$\to\dim X\leqslant r$

证明.

$$\begin{gathered}\text{反证：若此定理不成立.向量空间}X\\ \text{无关集}Q=\{\boldsymbol{y}_1,\ldots,\boldsymbol{y}_{r+1}\},\text{但}X\text{能由}r\text{个向量的集合}S_0\text{生成}\\ \text{设}0\leqslant i<r.\text{生成}X\text{的的集}S_i=\{\boldsymbol{y}_1,\ldots\boldsymbol{y}_i\}\cup\{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{r-i}\},\boldsymbol{x}_i\in S_0\\ \to\textstyle\sum_{j=1}^{i+1}a_j\boldsymbol{y}_j+\sum_{k=1}^{r-i}b_k\boldsymbol{x}_k=\boldsymbol{0}\\ \text{若}b_k\text{都为}0,Q\text{的无关性使得}a_j\text{也全为}0.\text{矛盾}\\ \to\exists\boldsymbol{x}_k\in S_i,\boldsymbol{x}_k\in\operatorname{span}(T_i)=\operatorname{span}(S_i\cup\{\boldsymbol{y}_{i+1}\})\\ \text{去掉这个}\boldsymbol{x}_i,\text{剩下的集称为}S_{i+1}\\ \to\operatorname{span}S_{i+1}=\operatorname{span}T_i=X\end{gathered}$$

$$\begin{gathered}\text{使用此方法替换}S_0\to S_1,S_2,\ldots,S_r\\ S_r=\{\boldsymbol{y}_1,\ldots,\boldsymbol{y}_r\},\operatorname{span}S_r=X\\ \boldsymbol{y}_{r+1}\text{和}S_r\text{线性无关}\to\boldsymbol{y}_{r+1}\text{不在}\operatorname{span}S_r\text{中}\\ \text{矛盾}\\ \to\operatorname{span}E=X,\dim X\leqslant\operatorname{length}E\end{gathered}$$

推论 1.3. $\dim R^n = n$. *因为*$\boldsymbol{e}_n$*是*R^n*的基*

定理 1.4. *n维向量空间的性质*

$$\begin{array}{lc} 1 & X\text{中}n\text{个向量的集}E\text{能生成}X \Leftrightarrow E\text{是无关的} \\ 2 & X\text{必有基，}\ \operatorname{length} X = \dim X = n \\ 3 & 1 \leqslant r \leqslant n.\ \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_r\}\text{是}X\text{中的一个无关集} \to X\text{必有包含}\{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_r\}\text{的基} \end{array}$$

证明.

1
$$\begin{gathered} E = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}.\ \dim X = n \\ \forall \boldsymbol{y} \in X.\ \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n, \boldsymbol{y}\}\text{是相关的} \\ \text{若}E\text{是无关的} \to y \in \operatorname{span} E \to \operatorname{span} E = X \\ \text{若}E\text{相关} \to \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{i-1}, \boldsymbol{x}_{i+1}, \ldots, \boldsymbol{x}_n\} \subset E\text{是线性无关的} \\ \dim \operatorname{span} E < \dim X = n \to E\text{必然不能生成}X \\ \to n\text{个向量的}E\text{能生成}X \end{gathered}$$
$\square$

2 应该用一下选择公理???若能遍历整个向量空间必可构造

3 继续用选择公理遍历向量空间 $\to \{y_1, \ldots, y_r, x_1 \ldots x_{n-r}\}$必能选出来

定义 1.5. *线性变换*

$$\begin{gathered} \text{向量空间}X, Y.\text{映射}f\colon X \to Y \\ 1 \quad \forall x, y \in X \to f(x+y) = f(x) + f(y) \\ 2 \quad \forall x \in X, \forall \lambda \in F \to f(\lambda x) = \lambda f(x) \\ \text{称}f\text{为线性变换} \end{gathered}$$

$f \in \mathcal{L}(X, Y) \to f(0_X) = 0_Y.\ f(x + -x) = f(0_X) = f(x) + f(-x) = f(x) - f(x) = 0_Y$

线性变换可以使用 变换对两个空间中的基的作用 进行表示(矩阵)

$$\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}\text{是}X\text{的基} \to \forall \boldsymbol{x} \in X, \exists\text{唯一}\boldsymbol{c}_i, \boldsymbol{x} = \sum c_i \boldsymbol{x}_i$$

因此线性变换可以表示为

$$A\boldsymbol{x} = \sum_{i=1}^{n} c_i A \boldsymbol{x}_i$$

这表示 任意向量的线性变换 可以 用基的线性变换 的线性组合 表示。

定义 1.6. *线性算子*

$$X\text{是向量空间}.\mathcal{L}(X, X)\text{中的元素称为}X\text{上的线性算子}.\text{记为}\mathcal{L}(X)$$

定义 1.7. *线性算子的可逆*

$$\text{若线性算子}A\text{是单射且是满射} \to A\text{可逆},\text{记为}A^{-1}$$

$$A \in \mathcal{L}(X). \forall x \in X \to A A^{-1}(x) = A^{-1} A(x) = I_X(x) = x.\ A^{-1} \in \mathcal{L}(X)$$

定理 1.8. *有限维向量空间X上的算子. 单性和满性等价*

证明.

$$\begin{aligned}
&X\text{的基}\boldsymbol{x}.\\
A\text{单}\rightarrow Ax&=A(\textstyle\sum c_i x_i)\\
&=\textstyle\sum c_i A x_i\\
x_i\neq x_j&\rightarrow Ax_i\neq Ax_j\\
&\rightarrow \operatorname{range}A=\operatorname{span}(A\boldsymbol{x})\\
\text{设}A\boldsymbol{x}\text{是线性相关的}&\rightarrow 0=\textstyle\sum c_i A x_i, \exists c_i\neq 0\\
&\rightarrow A(\textstyle\sum c_i x_i)=0\\
\text{但}\forall x\neq 0, A(x)&\neq A(0)=0\\
&\rightarrow c_i=0\\
&\rightarrow A\boldsymbol{x}\text{线性无关}\\
&\rightarrow A\boldsymbol{x}\text{是}X\text{的基}\\
&\rightarrow \operatorname{range}A=X\\
&\rightarrow A\text{满}
\end{aligned}$$

□

$$\begin{aligned}
A\text{满}&\rightarrow \exists(Ax_1,\ldots,Ax_n)\text{是}X\text{的基}\\
&\rightarrow \forall x\in X, x=c_1Ax_1+\cdots+c_nAx_n\\
x&=A(\textstyle\sum c_i x_i)\\
\forall x\neq y\in X.\, Ax&=Ay\\
c_1Ax_1+\cdots+c_nAx_n&=d_1Ax_1+\cdots+d_nAx_n\\
x-y&=\textstyle\sum (c_i-d_i)Ax_i\\
&\rightarrow \exists c_i\neq d_i\\
\text{但}A\boldsymbol{x}\text{是线性无关的}&\rightarrow \forall i, c_i=d_i\\
&\rightarrow A\text{是单的}
\end{aligned}$$

$$\begin{aligned}
A\text{单}&\rightarrow A\text{满}\rightarrow A\text{可逆}\\
A\text{满}&\rightarrow A\text{单}\rightarrow A\text{可逆}
\end{aligned}$$

定理 1.9. *线性变换的性质*

$$\begin{array}{rr}
 & X,Y,Z\text{是向量空间}\\
1 & f,g\in\mathcal{L}(X,Y)\rightarrow c_1f+c_2g\in\mathcal{L}(X,Y)\\
2 & f\in\mathcal{L}(Y,Z), g\in\mathcal{L}(X,Y)\rightarrow fg\in\mathcal{L}(X,Z)
\end{array}$$

证明.

$$\begin{aligned}
1\quad \forall x,y\in X.\,(c_1f+c_2g)(x+y)&=c_1fx+c_2\,gx+c_1fy+c_2gy\\
&=(c_1f+c_2g)x+(c_1f+c_2g)y\\
\forall x\in X,\forall\lambda\in F.\,(c_1f+c_2g)(\lambda x)&=c_1f(\lambda x)+c_2g(\lambda x)\\
&=\lambda c_1f(x)+\lambda c_2g(x)\\
&=\lambda(c_1f+c_2g)\,x\\
&\rightarrow c_1f+c_2g\in\mathcal{L}(X,Y)
\end{aligned}$$

□

$$\begin{aligned}
2\quad \forall x,y\in X.\, fg(x+y)&=f(g(x)+g(y))=f(g(x))+f(g(y))\\
&=fg(x)+fg(y)\\
\forall x\in X,\forall\lambda\in F.\, fg(\lambda x)&=f(g(\lambda x))=f(\lambda g(x))=\lambda f(g(x))\\
&=\lambda fg\,(x)\\
&\rightarrow fg\in\mathcal{L}(X,Z)
\end{aligned}$$

定义 1.10. *线性映射的范数. 有界的最大值范数(因为这是线性映射，所以这个范数能用吧 阿巴阿巴)*

$$\begin{aligned}
&A\in\mathcal{L}(R^n,R^m)\\
A\text{的范数}\quad \|A\|&=\sup_{x\in R^n\wedge|x|\leqslant 1}|Ax|
\end{aligned}$$

Remark: $|Ax| \leqslant \|A\| \, |x|$

定理 1.11. *线性映射的一些性质*

$$\begin{array}{ccc} 1 & A \in \mathcal{L}(R^n, R^m) & \|A\| < \infty \wedge A\text{为}R^n\text{到}R^m\text{的一致连续映射} \\ 2 & A, B \in \mathcal{L}(R^n, R^m), c \in R & \|A + B\| \leqslant \|A\| + \|B\|; \|cA\| = |c| \, \|A\| \\ & \text{度量(距离)} & d(A, B) = \|A - B\|. \mathcal{L}(R^n, R^m)\text{是度量空间} \\ 3 & A \in \mathcal{L}(R^n, R^m), B \in \mathcal{L}(R^m, R^n) & \|BA\| \leqslant \|B\| \, \|A\| \end{array}$$

证明.

$$\begin{array}{ll} 1 & \boldsymbol{e}\text{是}R^n\text{中的标准基} \\ & \boldsymbol{x} = \sum c_i e_i. \, |\boldsymbol{x}| \leqslant 1 \rightarrow |c_i| \leqslant 1 \\ & |Ax| = |\sum c_i e_i| \leqslant \sum |c_i| \, |Ae_i| \leqslant \sum \, |Ae_i| \\ & \|A\| \leqslant \sum_{i=1}^n |Ae_i| < \infty \\ & \\ & x, y \in R^n. \, |Ax - Ay| \leqslant \|A\| \, |x - y| \\ & \rightarrow A\text{是一致连续的} \end{array}$$

$$\begin{array}{lll} 2 & & |(A + B)x| \leqslant |Ax + Bx| \leqslant |Ax| + |Bx| \leqslant (\|A\| + \|B\|)|x| \\ d & \text{正定性} & (A - B)0_X = 0_Y \rightarrow d(A + B) = \|A - B\| = \sup_{|x| \leqslant 1} \geqslant 0 \\ & \text{三角不等式} & \|A - C\| = \|A - B + B - C\| \leqslant \|A - B\| + \|B - C\| \end{array}$$

$$\begin{array}{ll} 3 & |(BA)x| = |B(A(x))| \leqslant \|B\| \, |Ax| \leqslant \|B\| \, \|A\| \, |x| \\ & \rightarrow \|BA\| \leqslant \|B\| \, \|A\| \end{array}$$

□

Remark: $\mathcal{L}(R^n, R^m)$空间现在有了度量，开集，连续性

定理 1.12. $\Omega := R^n$*上所有可逆线性算子的集*

$$\begin{array}{lll} 1 & A \in \Omega, B \in \mathcal{L}(R^n) \wedge \|B - A\| \, \|A^{-1}\| < 1 & \rightarrow \; B \in \Omega \wedge \Omega\text{开} \\ 2 & \Omega\text{是}\mathcal{L}(R^n)\text{的开子集}, \, f: A \rightarrow A^{-1}\text{在}\Omega\text{上是连续且可逆的} & \end{array}$$

证明.

$$\begin{array}{ll} 1 & \|A^{-1}\| = \frac{1}{\alpha}. \, \|B - A\| = \beta \rightarrow \beta < \alpha. \forall \boldsymbol{x} \in R^n \\ & \alpha |\boldsymbol{x}| = a |A^{-1}Ax| \leqslant \alpha \|A^{-1}\| \cdot |Ax| \\ & = |Ax| \leqslant |(A - B)x| + |Bx| \leqslant \beta |x| + |Bx| \\ & (\alpha - \beta) |\boldsymbol{x}| \leqslant |B\boldsymbol{x}|. \, (\boldsymbol{x} \in R^n) \\ & \alpha - \beta \neq 0 \rightarrow B\boldsymbol{x} \neq 0 \rightarrow B\text{是可逆的} \\ & \rightarrow B \in \Omega. \end{array}$$

$$\begin{array}{ll} 2 & \Omega\text{是开集} \\ & \forall A \in \Omega, \forall B \wedge d(A, B) = \|B - A\| < \alpha \rightarrow B \in \Omega \\ & \rightarrow A\text{是}\Omega\text{的内点} \rightarrow \Omega\text{是开集} \\ & \text{let: } B^{-1}\boldsymbol{y} = \boldsymbol{x} \rightarrow (\alpha - \beta) |B^{-1}\boldsymbol{y}| \leqslant |BB^{-1}\boldsymbol{y}| = |\boldsymbol{y}|. \, \boldsymbol{y} \in R^n \\ & |B^{-1}\boldsymbol{y}| \leqslant (\alpha - \beta)^{-1} \rightarrow \|B^{-1}\| \leqslant (\alpha - \beta)^{-1} \\ & B^{-1} - A^{-1} = B^{-1}(A - B)A^{-1} \\ & \rightarrow \|B^{-1} - A^{-1}\| \leqslant \|B^{-1}\| \, \|A - B\| \, \|A^{-1}\| \leqslant \frac{1}{\alpha - \beta} \cdot \beta \cdot \frac{1}{\alpha} = \frac{\beta}{\alpha(\alpha - \beta)} \\ & \lim_{B \rightarrow A} \beta = \lim_{B \rightarrow A} \|B - A\| = 0 \\ & \rightarrow \lim_{B \rightarrow A} \|B^{-1} - A^{-1}\| = 0 \\ & \rightarrow f: A = A^{-1}\text{是连续的} \end{array}$$

□

定义 1.13. *矩阵*

$$\{x_1,\ldots,x_n\}\text{和}\{y_1,\ldots,y_m\}\text{是}X\text{和}Y\text{的基}$$

$\rightarrow$

$$\forall A\in\mathcal{L}(X,Y), Ax_j=\sum_{i=1}^{m} a_{i,j}\cdot y_i.\ (1\leqslant j\leqslant n)$$

线性映射的矩阵

$$\mathcal{M}(A)=\begin{pmatrix} a_{1,1} & \cdots & a_{1,n}\\ \vdots & \ddots & \vdots\\ a_{m,1} & \cdots & a_{m,n}\end{pmatrix}_{m\times n}$$

$$Ax_j\text{对应于基}\boldsymbol{y}\text{的坐标出现在}\mathcal{M}(A)\text{的第}j\text{列}.\text{因此称}Ax_j\text{为列向量}$$

$$\operatorname{range}A=\operatorname{span}(\mathcal{M}(A)x_1,\ldots,\mathcal{M}(A)x_n)$$

$$\boldsymbol{x}=\sum c_jx_j\rightarrow A\boldsymbol{x}=\sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{i,j}\cdot c_j\right)y_i$$

$$\rightarrow A\boldsymbol{x}\text{的坐标为}\sum_j a_{i,j}c_j.$$

Remark

$$Ax_j=\sum_i a_{i,j}y_i;\ A\boldsymbol{x}=\sum_j a_{i,j}c_j$$

矩阵乘法

$$Z\text{是向量空间},\text{具有基}\{z_1,\ldots,z_p\}$$

$$By_i=\sum_k b_{k,i}\cdot z_k$$

$$\begin{aligned}(BA)x_j&=B\left(\sum_i a_{i,j}\cdot y_i\right)\\&=\sum_i a_{i,j}By_i\\&=\sum a_{i,j}\left(\sum_k b_{k,i}\cdot z_k\right)\\&=\sum_k\left(\sum_i b_{k,i}\cdot a_{i,j}\right)z_k\end{aligned}$$

$$\boldsymbol{z}\text{线性无关}\rightarrow c_{k,j}=\sum_i b_{k,i}\cdot a_{i,j}$$

$$\rightarrow\mathcal{M}(BA)=\mathcal{M}(B)\cdot\mathcal{M}(A)$$

$$\rightarrow\mathcal{M}(BA)_{i,j}=\sum_k b_{i,k}\cdot a_{k,j}$$

矩阵元素与连续性

$$\boldsymbol{x},\boldsymbol{y}\text{是}R^n,R^m\text{的标准基}$$

勾股定理?

$$|A\boldsymbol{x}|^2=\sum_i\left(\sum_j a_{i,j}\cdot c_j\right)^2$$

Schwarz不等式

$$\leqslant\sum_i\left(\sum_j a_{i,j}^2\cdot\sum_j c_j^2\right)$$

$$=\sum_{i,j} a_{i,j}^2|\boldsymbol{x}|^2$$

$$\rightarrow\|A\|\leqslant\left(\sum_{i,j} a_{i,j}^2\right)^{\frac{1}{2}}$$

$$\rightarrow\forall A,B\in\mathcal{L}(R^n,R^m)\rightarrow\|B-A\|\leqslant\left(\sum_{i,j}(b_{i,j}-a_{i,j})^2\right)^2$$

$$\rightarrow a_{i,j}\text{连续}\rightarrow A\text{连续}$$

$$\text{矩阵中的元素都是连续函数}\rightarrow\text{矩阵映射也是连续}$$

$$S\text{是度量空间},a_{1,1},\ldots,a_{m,n}\text{是}S\text{上的实连续函数}$$

$$\forall p\in S, A_p\in\mathcal{L}(R^n,R^m).\mathcal{M}(A_p)=(a_{i,j}(p))$$

???讨论元素到线性映射的映射是要干什么

$$\rightarrow p\rightarrow A_p\text{是}S\rightarrow\mathcal{L}(R^n,R^m)\text{的连续映射}$$

2 微分法

导数定义的扩充

引理 2.1. $f\colon R\to R$的导数定义的推广

$$f\colon(a,b)\to R. x\in(a,b)$$
$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\in R$$
$$\to f(x+h)-f(x)=f'(x)h+r(h).\ \lim_{h\to 0}\frac{r(h)}{h}=0$$
$$\text{此时可以有记法:}\frac{\mathrm{d}}{\mathrm{d}x}\colon(D(a,b),h)\to R^R\cdot h,\frac{\mathrm{d}}{\mathrm{d}x}(f,h)=f'(x)h$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(f+g)=\frac{\mathrm{d}}{\mathrm{d}x}f+\frac{\mathrm{d}}{\mathrm{d}x}g;\frac{\mathrm{d}}{\mathrm{d}x}(\lambda f)=\lambda\frac{\mathrm{d}}{\mathrm{d}x}f$$
$$\to\frac{\mathrm{d}}{\mathrm{d}x}\text{是}R\to R\text{的可微函数到实函数空间上的线性算子}$$

考虑$f\colon(a,b)\to R^m$的函数

$$\boldsymbol{f}'(x)=\lim_{h\to 0}\frac{\boldsymbol{f}(x+h)-\boldsymbol{f}(x)}{h}=\boldsymbol{y}\in R^m$$
$$\to\boldsymbol{f}(x+h)-\boldsymbol{f}(x)=h\boldsymbol{y}+\boldsymbol{r}(h).\ \lim_{h\to 0}\frac{\boldsymbol{r}(h)}{h}=\boldsymbol{0}$$
$$\text{而}h\boldsymbol{y}\text{是}h\text{的线性映射:}\boldsymbol{y}(x+y)=\boldsymbol{y}x+\boldsymbol{y}y;\boldsymbol{y}(\lambda x)=\lambda\boldsymbol{y}x$$
$$R^m\cong\mathcal{L}(R,R^m)\to\boldsymbol{f}'(x)\text{看作是在}x\text{点处的线性映射}\boldsymbol{f}'\in\mathcal{L}(R,R^m)$$
$$\boldsymbol{f}\text{可微}:=\lim_{h\to 0}\frac{\boldsymbol{f}(x+h)-\boldsymbol{f}(x)-\boldsymbol{f}'(x)h}{h}=0$$
$$\Leftrightarrow\lim_{h\to 0}\frac{|\boldsymbol{f}(x+h)-\boldsymbol{f}(x)-\boldsymbol{f}'(x)h|}{|h|}=0$$

定义 2.2. *多元向量函数的可微性*

$$E\text{是}R^n\text{中的开集，}\boldsymbol{f}\colon E\to R^m.\boldsymbol{x}\in E.$$
$$\text{若}\exists A\in\mathcal{L}(R^n,R^m)\to\lim_{\boldsymbol{h}\to\boldsymbol{0}}\frac{|\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h})-\boldsymbol{f}(\boldsymbol{x})-A\boldsymbol{h}|}{|\boldsymbol{h}|}=0$$
$$\text{称}\boldsymbol{f}\text{在}\boldsymbol{x}\text{处可微, 记作: }\boldsymbol{f}'(\boldsymbol{x})=A$$
$$\forall\boldsymbol{x}\in E,\boldsymbol{f}\text{可微}\to\boldsymbol{f}\text{在}E\text{上可微}$$

Remark:

$\boldsymbol{h}\in R^n.\boldsymbol{h}$趋近0 $\to|\boldsymbol{h}|$趋近0
E开$\to\boldsymbol{x}+\boldsymbol{h}\in E\to\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h})$有定义
$\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h})-\boldsymbol{f}(\boldsymbol{x})-A\boldsymbol{h}\in R^m$
分子上的范数是R^m上的范数
分母上的范数是R^n上的范数

定理 2.3. *多元向量值函数导数的唯一性*

$$E\text{是开集},\boldsymbol{x}\in E.\boldsymbol{f}\colon E\to R^m.$$
$$A_1,A_2\in\mathcal{L}(R^n,R^m)\text{都满足在}\boldsymbol{x}\text{处的导数定义}$$
$$\to A_1=A_2$$

证明.

$$B=A_1-A_2$$
$$|B\boldsymbol{h}|\leqslant|\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h})-\boldsymbol{f}(\boldsymbol{x})-A_1\boldsymbol{h}|+|\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h})-\boldsymbol{f}(\boldsymbol{x})-A_2\boldsymbol{h}|$$
$$\to\lim_{\boldsymbol{h}\to\boldsymbol{0}}\frac{|B\boldsymbol{h}|}{|\boldsymbol{h}|}=0.\text{(这里其实可以看出来}B\text{变换后的}\boldsymbol{h}\text{为0了，因为}B\text{可以被行列式度量)}$$
$$\to\forall\boldsymbol{h}\in R^n,B\boldsymbol{h}=0\to B=\boldsymbol{0}_{n\to m}$$

□

定义 2.4. *多元函数的一些定义*

$$
\begin{array}{ll}
1 & \boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h})-\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{f}'(\boldsymbol{x})\boldsymbol{h}+\boldsymbol{r}(\boldsymbol{h}) \\
 & \boldsymbol{r}\colon R^n\rightarrow R^m, \lim_{\boldsymbol{h}\rightarrow\boldsymbol{0}}\frac{|\boldsymbol{r}(\boldsymbol{h})|}{|\boldsymbol{h}|}=0 \\
2 & \boldsymbol{f}\text{在}E\text{内可微}, \boldsymbol{f}'(\boldsymbol{x})\in(R^m)^{(E)}.\boldsymbol{f}'(\boldsymbol{x})\text{是在}\boldsymbol{x}\text{点的线性函数}A\colon\in\mathcal{L}(R^n,R^m) \\
3 & \text{在}\boldsymbol{x}\text{点上}, \lim_{\boldsymbol{h}\rightarrow\boldsymbol{0}}\frac{|\boldsymbol{r}(\boldsymbol{h})|}{|\boldsymbol{h}|}=0.\text{即}\boldsymbol{f}\text{在}\boldsymbol{x}\text{点连续} \\
4 & \boldsymbol{f}'\text{称为}\boldsymbol{f}\text{的微分或称为}\boldsymbol{f}\text{的全导数}
\end{array}
$$

例 2.5. 线性映射的导数

$$
\begin{gathered}
A\in\mathcal{L}(R^n,R^m).\boldsymbol{x}\in R^n \\
A(\boldsymbol{x}+\boldsymbol{h})=A(\boldsymbol{x})+A(\boldsymbol{h}) \\
A(\boldsymbol{x}+\boldsymbol{h})-A(\boldsymbol{x})=A(\boldsymbol{h}) \\
\lim_{\boldsymbol{h}\rightarrow\boldsymbol{0}}\left(\frac{A(\boldsymbol{x}+\boldsymbol{h})-A(\boldsymbol{x})}{\boldsymbol{h}}-B\boldsymbol{h}\right)=\left(\lim_{\boldsymbol{h}\rightarrow\boldsymbol{0}}\frac{A(\boldsymbol{h})}{\boldsymbol{h}}-B\boldsymbol{h}\right)=0 \\
\rightarrow A=B \\
\rightarrow A'=A
\end{gathered}
$$

定理 2.6. *复合函数求导法则。链式法则*

$$
\rightarrow \quad
\begin{gathered}
E\text{是}R^n\text{的开集}, \boldsymbol{f}\colon E\rightarrow R^m.\ \boldsymbol{f}\text{在}\boldsymbol{x}_0\text{可微} \\
F\text{是包含}\boldsymbol{f}(E)\text{的开集}; \boldsymbol{g}\colon F\rightarrow R^k.\ \boldsymbol{g}\text{在}\boldsymbol{f}(\boldsymbol{x}_0)\text{是可微的} \\
\boldsymbol{F}(x)=\boldsymbol{g}\circ\boldsymbol{f}\colon R^n\rightarrow R^k \\
\boldsymbol{F}\text{在}\boldsymbol{x}_0\text{是可微的且} \\
\boldsymbol{F}'(\boldsymbol{x}_0)=\boldsymbol{g}'(\boldsymbol{f}(\boldsymbol{x}_0))\boldsymbol{f}'(\boldsymbol{x}_0)
\end{gathered}
$$

证明.

$$
\begin{gathered}
\boldsymbol{y}_0=\boldsymbol{f}(\boldsymbol{x}_0), A=\boldsymbol{f}'(\boldsymbol{x}_0), B=\boldsymbol{g}'(\boldsymbol{y}_0).\text{在}U_{\boldsymbol{x}_0}(r)\text{和}U_{\boldsymbol{y}_0}(r)\text{的开集内取}\boldsymbol{h}\text{和}\boldsymbol{k} \\
\boldsymbol{u}(\boldsymbol{h})=\boldsymbol{f}(\boldsymbol{x}_0+\boldsymbol{h})-\boldsymbol{f}(x_0)-A\boldsymbol{h} \\
\boldsymbol{v}(\boldsymbol{k})=\boldsymbol{g}(\boldsymbol{y}_0+\boldsymbol{k})-\boldsymbol{g}(\boldsymbol{y}_0)-B\boldsymbol{h} \\
\rightarrow|\boldsymbol{u}(\boldsymbol{h})|=\varepsilon(\boldsymbol{h})\,|\boldsymbol{h}|;\ |\boldsymbol{v}(\boldsymbol{k})|=\eta(\boldsymbol{h})\,|\boldsymbol{k}|. \\
\lim_{\boldsymbol{h}\rightarrow\boldsymbol{0}}\boldsymbol{u}(\boldsymbol{h})=0;\ \lim_{\boldsymbol{k}\rightarrow\boldsymbol{0}}\boldsymbol{v}(\boldsymbol{k})=0 \\
\forall\boldsymbol{h}\in U_{\boldsymbol{x}_0}(r).\boldsymbol{k}=\boldsymbol{f}(\boldsymbol{x}_0+\boldsymbol{h})-\boldsymbol{f}(\boldsymbol{x}_0) \\
|\boldsymbol{k}|=|A\boldsymbol{h}+\boldsymbol{u}(\boldsymbol{h})|\leqslant(\|A\|+\varepsilon(\boldsymbol{h}))|\boldsymbol{h}| \\
\boldsymbol{F}(\boldsymbol{x}_0+\boldsymbol{h})-\boldsymbol{F}(\boldsymbol{x}_0)-BA\boldsymbol{h}=\boldsymbol{g}(\boldsymbol{y}_0+\boldsymbol{k})-\boldsymbol{g}(\boldsymbol{y}_0)-BA\boldsymbol{h} \\
=B(\boldsymbol{k}-A\boldsymbol{h})+\boldsymbol{v}(\boldsymbol{k}) \\
=B\boldsymbol{u}(\boldsymbol{h})+\boldsymbol{v}(\boldsymbol{k}) \\
\\
\boldsymbol{h}\neq\boldsymbol{0}\rightarrow\frac{|\boldsymbol{F}(\boldsymbol{x}_0+\boldsymbol{h})-\boldsymbol{F}(\boldsymbol{x}_0)-BA\boldsymbol{h}|}{|\boldsymbol{h}|}\leqslant\|B\|\varepsilon(\boldsymbol{h})+(\|A\|+\varepsilon(\boldsymbol{h}))\eta(\boldsymbol{k}) \\
\lim_{\boldsymbol{h}\rightarrow\boldsymbol{0}}\varepsilon(\boldsymbol{h})=0; \lim_{\boldsymbol{h}\rightarrow\boldsymbol{0}}\boldsymbol{k}=(\|A\|+\varepsilon(\boldsymbol{h}))|\boldsymbol{h}|=0 \\
\lim_{h\rightarrow 0}\eta(\boldsymbol{k})=\frac{|\boldsymbol{k}|}{|\boldsymbol{v}(\boldsymbol{k})|}=0 \\
\rightarrow\lim_{\boldsymbol{h}\rightarrow\boldsymbol{0}}\boldsymbol{F}'(\boldsymbol{x}_0)=BA
\end{gathered}
$$

□

定义 2.7. *偏导数*

$$
\begin{array}{ll}
 & \text{开集}E\in R^n.\ \boldsymbol{f}\colon E\rightarrow R^m.\boldsymbol{e},\boldsymbol{u}\text{是}R^n,R^m\text{的标准基} \\
 & \boldsymbol{f}(\boldsymbol{x})=\sum_{i=1}^m f_i(\boldsymbol{x})\boldsymbol{u}_i.\ (\boldsymbol{x}\in E) \\
f\text{关于第}j\text{个变量的偏导数} & (D_jf_i)(\boldsymbol{x})=\lim_{t\rightarrow 0}\frac{f_i(\boldsymbol{x}+te_j)-f_i(\boldsymbol{x})}{t}\in R \\
 & \frac{\partial f_i}{\partial x_j}\text{记为}f_i\text{关于}x_j\text{的偏导数}
\end{array}
$$

Remark: 对于单变量函数，仅有一个偏导数.可微即偏导数存在。

对于多变量函数。各个偏导数都连续，或至少有界 才能推出可微性

但是反向的，可微则各个偏导数都存在并且它们唯一确定了线性变换$\boldsymbol{f}'$

定理 2.8. *多元函数可微则各个偏导数存在*

$$\rightarrow \begin{array}{c}\text{开集} E\subset R^n.\ \boldsymbol{f}\colon E\rightarrow R^m.\ \boldsymbol{f}\text{在}\boldsymbol{x}\in E\text{可微}\\ \text{偏导数} D_j f_i(\boldsymbol{x})\text{存在}\\ \boldsymbol{f}'(\boldsymbol{x})\cdot e_j=\sum_{i=1}^m (D_j f_i)(\boldsymbol{x})u_i.\ (1\leqslant j\leqslant n)\end{array}$$

证明.

$$\begin{array}{c}\text{对于确定的分量} j.\\ \boldsymbol{f}\text{在}\boldsymbol{x}\text{可微}\rightarrow \boldsymbol{f}(\boldsymbol{x}+t\boldsymbol{e}_j)-\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{f}'(\boldsymbol{x})(t\boldsymbol{e}_j)+\boldsymbol{r}(t\boldsymbol{e}_j)\quad |t|<\delta\text{在}U_{\boldsymbol{x}}(r)\text{中}\\ \lim_{t\rightarrow 0}\sum_{i=1}^m \frac{f_i(\boldsymbol{x}+t\boldsymbol{e}_j)-f_i(\boldsymbol{x})}{t}u_i=\boldsymbol{f}'(\boldsymbol{x})\boldsymbol{e}_j\\ \rightarrow\lim_{t\rightarrow 0}\frac{f_i(\boldsymbol{x}+t\boldsymbol{e}_j)-f_i(\boldsymbol{x})}{t}u_i\text{都有限}\qquad\qquad \text{定理4.10}\\ \rightarrow D_j f_i\text{存在}\end{array}$$

□

推论 2.9. *多元向量函数的导数可以被偏导数矩阵唯一表示*

$$\boldsymbol{f}\in R^n\rightarrow R^m;\ \boldsymbol{f}'(\boldsymbol{x})=\begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n}\end{pmatrix}_{m,n}$$

$$\boldsymbol{f}'(\boldsymbol{x})\boldsymbol{h}=\sum_{i=1}^m\left(\sum_{j=1}^n (D_j f_i)(\boldsymbol{x})h_j\right)u_i$$

定义 2.10. *一些向量值多元函数的定义*

$$1\qquad\begin{array}{c}\text{参变量函数}\\ \boldsymbol{f}(\boldsymbol{x})=\boldsymbol{f}(\boldsymbol{x}(t))\\ \text{使用链式法则}\rightarrow \boldsymbol{f}'(\boldsymbol{x})=\boldsymbol{f}'(\boldsymbol{x}(t))\boldsymbol{x}'(t)\\ =\sum_{i=1}^n\frac{\partial f_i}{\partial \boldsymbol{x}}(\boldsymbol{x}(t))\cdot\boldsymbol{x}'(t)\end{array}$$

向量值多元函数的梯度

$$2\qquad\begin{array}{c}\text{梯度}\\ (\nabla f)(\boldsymbol{x})=\sum_{i=1}^n (D_i f)(\boldsymbol{x})\cdot\boldsymbol{e}_i\\ =\left(\frac{\partial f}{\partial x_1}(\boldsymbol{x}),\ldots,\frac{\partial f}{\partial x_n}(\boldsymbol{x})\right)\\ f'(t)=\langle(\nabla f(\boldsymbol{x}(t))),\boldsymbol{x}'(t)\rangle\end{array}$$

$$
\begin{array}{rr}
3 & \text{向量值函数的方向导数} \\
& \boldsymbol{x}\in E,\boldsymbol{u}\in R^n\wedge|\boldsymbol{u}|=1 \\
& \gamma(t)=\boldsymbol{x}+t\boldsymbol{u}.t\in R \\
& \rightarrow\gamma'(t)=\boldsymbol{u} \\
& \boldsymbol{f}'(0)=\langle(\nabla f)(\boldsymbol{x}),\boldsymbol{u}\rangle \\
& \boldsymbol{f}'(t)=\boldsymbol{f}(0)=\boldsymbol{f}(x+t\boldsymbol{u})-\boldsymbol{f}(\boldsymbol{x}) \\
\text{称为}\boldsymbol{f}\text{沿着}\boldsymbol{u}\text{方向} & \rightarrow\lim_{t\to 0}\dfrac{\boldsymbol{f}(\boldsymbol{x}+t\boldsymbol{u})-\boldsymbol{f}(\boldsymbol{x})}{t}=\langle(\nabla\boldsymbol{f})(\boldsymbol{x}),\boldsymbol{u}\rangle \\
\text{的方向导数} & D_{\boldsymbol{u}}f=\lim_{t\to 0}\dfrac{\boldsymbol{f}(\boldsymbol{x}+t\boldsymbol{u})-\boldsymbol{f}(\boldsymbol{x})}{t}=\langle(\nabla\boldsymbol{f}),\boldsymbol{u}\rangle
\end{array}
$$

Remark: 当$\boldsymbol{u}$沿着$\nabla\boldsymbol{f}$方向时,$D_{\boldsymbol{u}}\boldsymbol{f}$达到最大值

$$
\begin{aligned}
\boldsymbol{u}&=\sum_n u_i\boldsymbol{e}_i \\
(D_{\boldsymbol{u}}f)(\boldsymbol{x})&=\sum_{i=1}(D_if)(\boldsymbol{x})u_i
\end{aligned}
$$

Remark. 可微性是非常强的性质，可微意味着梯度存在，方向导数必然也存在

梯度表示在$\boldsymbol{x}$点上各个方向的偏导数表示的向量。

对于任意坐标的线性变换，方向导数最大值的方向是协变的。$T\boldsymbol{f}=T\boldsymbol{u}$

实际上对坐标的非线性变换应该也一样的。只要是1-1的。。。

定理 2.11.

$$
\begin{array}{rc}
& \text{凸开集}E\subset R^n.\ \boldsymbol{f}\colon E\to R^m.\boldsymbol{f}\text{在}E\text{内可微} \\
& \forall\boldsymbol{x}\in E,\|\boldsymbol{f}'(\boldsymbol{x})\|\leqslant M\in R \\
\rightarrow & \forall\boldsymbol{a},\boldsymbol{b}\in E,|\boldsymbol{f}(\boldsymbol{a})-\boldsymbol{f}(\boldsymbol{b})|\leqslant M\ |\boldsymbol{b}-\boldsymbol{a}|
\end{array}
$$

证明.

$$
\begin{array}{c}
\boldsymbol{a},\boldsymbol{b}\in E.\forall t\in R \\
\gamma(t)=(1-t)\boldsymbol{a}+t\boldsymbol{b}\in E \\
\boldsymbol{g}(t)=\boldsymbol{f}(\gamma(t)) \\
\rightarrow\boldsymbol{g}'(t)=\boldsymbol{f}'(\gamma(t))\cdot\gamma'(t)=\boldsymbol{f}'(\gamma(t))(\boldsymbol{b}-\boldsymbol{a}) \\
\rightarrow\forall t\in[0,1],|g'(t)|\leqslant\|\boldsymbol{f}'(t)\|\ |\boldsymbol{b}-\boldsymbol{a}|\leqslant M\ |\boldsymbol{b}-\boldsymbol{a}| \\
\rightarrow|\boldsymbol{g}(1)-\boldsymbol{g}(0)|=|\boldsymbol{f}(\boldsymbol{a})-\boldsymbol{f}(\boldsymbol{b})|\leqslant M\ |\boldsymbol{b}-\boldsymbol{a}|
\end{array}
$$

□

推论 2.12. *若导数恒为零向量，则函数是常函数*

$$
\forall\boldsymbol{x}\in E,|\boldsymbol{f}(\boldsymbol{a})-\boldsymbol{f}(\boldsymbol{b})|=\|\boldsymbol{f}'(\boldsymbol{x})\||\boldsymbol{b}-\boldsymbol{a}|=0\cdot|\boldsymbol{b}-\boldsymbol{a}|=0
$$

定义 2.13. *连续可微*

$$
\begin{array}{rc}
& \text{开集}E\subset R^n.\ \boldsymbol{f}\colon E\to R^m. \\
\text{连续可微} & \boldsymbol{f}'\colon E\to\mathcal{L}(R^n,R^m)\text{连续}
\end{array}
$$

进一步

$$
\begin{array}{c}
\forall\boldsymbol{x}\in E,\forall\varepsilon>0,\exists\delta>0,\boldsymbol{y}\in E\wedge|\boldsymbol{x}-\boldsymbol{y}|<\delta \\
\|\boldsymbol{f}'(\boldsymbol{y})-\boldsymbol{f}'(\boldsymbol{x})\|<\varepsilon \\
\text{记为}\boldsymbol{f}\in\mathscr{C}'(E)
\end{array}
$$

定理 2.14. *连续可微 当且仅当 所有偏导数都连续*

$$\text{开集}E\subset R^n.\boldsymbol{f}\colon E\to R^m.$$
$$\boldsymbol{f}\in\mathscr{C}'(E)\Leftrightarrow\forall i,j,D_jf_i=\frac{\partial f_j}{\partial x_i}\text{在}E\text{上连续}$$

证明.

$$\begin{gathered}
\boldsymbol{f}\in\mathscr{C}'(E)\to D_jf_i\text{在}E\text{上连续}\\
(D_jf_i)(\boldsymbol{x})=\langle(\boldsymbol{f}'(\boldsymbol{x})\boldsymbol{e}_j),\boldsymbol{u}_i\rangle\\
(D_jf_i)(\boldsymbol{y})-(D_jf_i)(\boldsymbol{x})=((\boldsymbol{f}'(\boldsymbol{y})-\boldsymbol{f}'(\boldsymbol{x}))\boldsymbol{e}_j)\cdot\boldsymbol{u}_i\\
|\boldsymbol{u}_i|=|\boldsymbol{e}_j|=1\\
|(D_jf_i)(\boldsymbol{y})-(D_jf_i)(\boldsymbol{x})|\leqslant|(\boldsymbol{f}'(\boldsymbol{y})-(\boldsymbol{f}'(\boldsymbol{x}))\cdot\boldsymbol{e}_j|\\
\leqslant\|\boldsymbol{f}'(\boldsymbol{y})-\boldsymbol{f}'(\boldsymbol{x})\|\\
\to D_jf_i\text{连续}
\end{gathered}$$

$$\begin{gathered}
D_jf_i\text{在}E\text{上连续}\to\boldsymbol{f}\in\mathscr{C}'(E)\\
\text{因为}\boldsymbol{x}\text{连续}\Leftrightarrow x_i\text{都连续}\\
\text{考虑}E\to R\text{上的函数, 即多变量函数}\\
\forall\boldsymbol{x}\in E,E\text{开}\to\exists U_{\boldsymbol{x}}(r)\subset E\\
D_jf\text{连续}\to|(D_jf)(\boldsymbol{y})-(D_jf)(\boldsymbol{x})|<\tfrac{\varepsilon}{n}\\
\boldsymbol{h}=\sum h_j\boldsymbol{e}_j\wedge|\boldsymbol{h}|<r\\
\boldsymbol{v}_0=\boldsymbol{0}.\boldsymbol{v}_k=\textstyle\sum_1^k h_i\boldsymbol{e}_i\\
f(\boldsymbol{x}+\boldsymbol{h})-f(\boldsymbol{x})=\textstyle\sum_{j=1}^n(f(\boldsymbol{x}+\boldsymbol{v}_j)-f(\boldsymbol{x}+\boldsymbol{v}_{j-1}))\\
|\boldsymbol{v}_k|<r\wedge U_{\boldsymbol{x}}(r)\text{是凸的}\\
\to\forall\boldsymbol{y}\in\lambda(\boldsymbol{x}+\boldsymbol{v}_j)+(1-\lambda)(\boldsymbol{x}+\boldsymbol{v}_{j-1})\wedge\lambda\in[0,1]\to\boldsymbol{y}\in U_{\boldsymbol{x}}(r)\\
\boldsymbol{v}_j=\boldsymbol{v}_{j-1}+h_j\boldsymbol{e}_j\\
\to f(\boldsymbol{x}+\boldsymbol{v}_j)-f(\boldsymbol{x}+\boldsymbol{v}_{j-1})=h_j(D_jf)(\boldsymbol{x}+\boldsymbol{v}_{j-1}+\theta_jh_j\boldsymbol{e}_j).\theta\in(0,1)\quad\text{中值定理}\\
\to|h_j(D_jf(\boldsymbol{x}+\boldsymbol{v}_{j-1}+\theta_jh_j\boldsymbol{e}_j))-h_j(D_jf)(\boldsymbol{x})|<\tfrac{\varepsilon}{n}\\
\to\forall|\boldsymbol{h}|<r,|f(\boldsymbol{x}+\boldsymbol{h})-\boldsymbol{f}(\boldsymbol{x})-\textstyle\sum_{j=1}^n h_j(D_jf)(\boldsymbol{x})|\\
\leqslant\tfrac{1}{n}\textstyle\sum_{j=1}^n|h_j|\varepsilon\leqslant|\boldsymbol{h}|\varepsilon\\
\to f\text{在}\boldsymbol{x}\text{处可微}
\end{gathered}$$

Remark:这里表示$\boldsymbol{f}$*在*$\boldsymbol{x}$*可微，则导数*$\boldsymbol{f}'(\boldsymbol{x})$*是一个线性函数。对于任何向量确定一个数*$\sum h_j(D_jf)(\boldsymbol{x})$.

线性映射$\boldsymbol{f}'(\boldsymbol{x})=\begin{pmatrix}D_1f_1&\cdots&D_nf_1\\\vdots&\ddots&\vdots\\D_1f_m&\cdots&D_nf_m\end{pmatrix}$. 偏导数$D_jf_i$都连续$\to\boldsymbol{f}'(\boldsymbol{x})$连续$\to\boldsymbol{f}'\in\mathscr{C}'(E)$ □

3 凝缩原理

任何完备度量空间都有有效的不动点定理

定义 3.1. *凝缩函数*

$$X\text{是度量空间，}\varphi\colon X\to X$$
$$\text{凝缩函数}\quad\forall x,y\in X,\exists c<1\to d(\varphi(x),\varphi(y))\leqslant cd(x,y)$$

定理 3.2. *凝缩原理, 压缩映象原理*

$$\begin{array}{ll}&X\text{是完备度量空间},\varphi\text{是}X\to X\text{上的凝缩函数}\\ \to&\exists\text{唯一}x\in X\to\varphi(x)=x\qquad\varphi\text{在}X\text{有唯一不动点}\end{array}$$

证明.

$$
\begin{gathered}
\text{存在性}\\
\forall x_0\in X,x_{n+1}=\varphi(x_n).\,n\in N\\
d(x_{n+1},x_n)=d(\varphi(x_n),\varphi(x_{n-1}))\leqslant c\,d(x_n,x_{n-1})\\
\to d(x_{n+1},x_n)\leqslant c^n d(x_1,x_0).\,n\in N\\
n<m\to d(x_n,x_m)\leqslant \textstyle\sum_{i=n+1}^{m} d(x_i,x_{i-1})\\
\leqslant(c^n+c^{n-1}+\cdots+c^{m-1})d(x_1,x_0)\\
\leqslant((1-c)^{-1}d(x_1,x_0))c^n\\
\to x_n\text{是Cauchy序列}\\
X\text{完备}\to x_n\text{收敛于一个极限点}x\\
\varphi(x)=\lim_{n\to\infty}\varphi(x_n)=\lim_{n\to\infty}x_{n+1}=x\\
\to x\text{是不动点}
\end{gathered}
$$

Remark: φ在X上连续$\wedge$一致连续 □

4 反函数定理

连续可微的映射$\boldsymbol{f}$，在线性变换$\boldsymbol{f}'(\boldsymbol{x})$可逆的点$\boldsymbol{x}$的领域内是可逆的

定理 4.1. *反函数定理*

$$
\begin{gathered}
\text{开集}E\subset R^n,\,\boldsymbol{f}\colon E\to R^m.\,\boldsymbol{f}\in\ell'(E)\\
\boldsymbol{a}\in E,\,\boldsymbol{f}'(\boldsymbol{a})\text{可逆},\boldsymbol{b}=\boldsymbol{f}(\boldsymbol{a})
\end{gathered}
$$

$$\to$$

$$
\begin{array}{ll}
1 & \text{开集}U,V\subset R^n.\,\boldsymbol{a}\in U,\boldsymbol{b}\in V.\,\boldsymbol{f}\text{在}U\text{上是}1-1\text{的}\wedge\boldsymbol{f}(U)=V\\
2 & \boldsymbol{g}\colon V\to U,\,\boldsymbol{g}=\boldsymbol{f}^{-1}.\,\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}))=\boldsymbol{x}\to\boldsymbol{g}\in\ell'(V)
\end{array}
$$

Remark: *可逆线性映射在某个点$\boldsymbol{x}$可微。则必在$U_{\boldsymbol{x}}(r)$内有反函数且反函数连续可微*

证明.

1

$$
\begin{gathered}
\boldsymbol{f}'(\boldsymbol{a})=A.\,A\text{可逆}\to\|A^{-1}\|=\frac{1}{\alpha}\\
\exists\lambda,2\lambda\|A^{-1}\|=1\\
\boldsymbol{f}'\text{在}a\text{连续},\exists U_a(r),\forall\boldsymbol{x}\in U_a(r)\subset E\to\|f'(\boldsymbol{x})-A\|<\lambda\\
\forall\boldsymbol{y}\in R^n=\boldsymbol{x}+A^{-1}(\boldsymbol{y}-\boldsymbol{f}(\boldsymbol{x})).\,\boldsymbol{x}\in E\\
\boldsymbol{\varphi}'(\boldsymbol{x})=I-A^{-1}\boldsymbol{f}'(\boldsymbol{x})=A^{-1}(A-\boldsymbol{f}'(\boldsymbol{x}))\\
\to\|\boldsymbol{\varphi}'(\boldsymbol{x})\|\leqslant\|A^{-1}\|\,\|A-\boldsymbol{f}'(\boldsymbol{x})\|=\frac{1}{a}\cdot\lambda<\frac{1}{2}\\
\to\boldsymbol{x}_1,\boldsymbol{x}_2\in U\to|\boldsymbol{\varphi}(\boldsymbol{x}_1)-\boldsymbol{\varphi}(\boldsymbol{x}_2)|\leqslant\|\boldsymbol{\varphi}'(\boldsymbol{x})\|\,|\boldsymbol{x}_1-\boldsymbol{x}_2|\\
\to\boldsymbol{\varphi}\text{是凝缩映射}\\
\to\boldsymbol{\varphi}\text{在}U\text{内最多有一个不动点}\boldsymbol{x}\in U,\,\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{y}\\
\to\boldsymbol{f}\text{在}U\text{内是}1-1\text{的}\\
V=\boldsymbol{f}(U),\,\boldsymbol{y}_0\in V.\\
\boldsymbol{x}_0\in U,\,\boldsymbol{f}(\boldsymbol{x}_0)=\boldsymbol{y}_0\\
\text{设}B=U_{\boldsymbol{x}_0}(r),\bar{B}\subset U\\
\forall\boldsymbol{y}\in V,|\boldsymbol{y}-\boldsymbol{y}_0|<\lambda r\\
|\boldsymbol{\varphi}(\boldsymbol{x}_0)-\boldsymbol{x}_0|=|A^{-1}(\boldsymbol{y}-\boldsymbol{y}_0)|<\|A^{-1}\|\lambda r=\frac{r}{2}\\
\forall\boldsymbol{x}\in\bar{B},|\boldsymbol{\varphi}(\boldsymbol{x})-\boldsymbol{x}_0|\leqslant|\boldsymbol{\varphi}(\boldsymbol{x})-\boldsymbol{\varphi}(\boldsymbol{x}_0)|+|\boldsymbol{\varphi}(\boldsymbol{x}_0)-\boldsymbol{x}_0|\\
<\frac{1}{2}|\boldsymbol{x}-\boldsymbol{x}_0|+\frac{r}{2}\leqslant r\\
\to\boldsymbol{\varphi}(\boldsymbol{x})\in B.\,\boldsymbol{x}_1,\boldsymbol{x}_2\in\bar{B}\to|\boldsymbol{\varphi}(\boldsymbol{x}_1)-\boldsymbol{\varphi}(\boldsymbol{x}_2)|\leqslant\frac{1}{2}|\boldsymbol{x}_1-\boldsymbol{x}_2|\\
\bar{B}\text{是}R^n\text{有界闭集}\to\bar{B}\text{是完备集}\\
\to\boldsymbol{\varphi}\text{是}\bar{B}\text{内的凝缩函数}\\
\to\exists\boldsymbol{x}\in\bar{B},\,\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{y}\\
\to\boldsymbol{y}\in\boldsymbol{f}(\bar{B})\subset\boldsymbol{f}(U)=V\\
\text{开集内可以构造闭集完全描述开集???}
\end{gathered}
$$

□

2

$$
\begin{gathered}
\boldsymbol{y}\in V\,,\boldsymbol{y}+\boldsymbol{k}\in V\\
\to\exists\boldsymbol{x}\in U\,,\boldsymbol{x}+\boldsymbol{h}\in U\to\boldsymbol{y}=\boldsymbol{f}(\boldsymbol{x});\boldsymbol{y}+\boldsymbol{k}=\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h})\\
\varphi(\boldsymbol{x}+\boldsymbol{h})-\varphi(\boldsymbol{x})=\boldsymbol{h}+A^{-1}(\boldsymbol{f}(\boldsymbol{x})-\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h}))=\boldsymbol{h}-A^{-1}\boldsymbol{k}\\
|\boldsymbol{h}-A^{-1}\boldsymbol{k}|\leqslant\tfrac{1}{2}|\boldsymbol{h}|\to|A^{-1}\boldsymbol{k}|\geqslant\tfrac{1}{2}|\boldsymbol{h}|\\
|\boldsymbol{h}|\leqslant2\|A^{-1}\|\;|\boldsymbol{k}|=\tfrac{1}{\lambda}|\boldsymbol{k}|\\
\to\boldsymbol{f}'(\boldsymbol{x})\text{可逆},T=(\boldsymbol{f}'(\boldsymbol{x}))^{-1} \qquad 1.12\,1\\
\boldsymbol{g}(\boldsymbol{y}+\boldsymbol{k})-\boldsymbol{g}(\boldsymbol{y})-T\boldsymbol{k}=\boldsymbol{h}-T\boldsymbol{k}\\
=-T(\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h})-\boldsymbol{f}(\boldsymbol{x})-\boldsymbol{f}'(\boldsymbol{x})\boldsymbol{h})\\
\frac{|\boldsymbol{g}(\boldsymbol{y}+\boldsymbol{k})-\boldsymbol{g}(\boldsymbol{y})-T\boldsymbol{k}|}{|\boldsymbol{k}|}\leqslant\frac{\|T\|}{\lambda}\cdot\frac{|\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h})-\boldsymbol{f}(\boldsymbol{x})-\boldsymbol{f}'(\boldsymbol{x})\boldsymbol{h}|}{|\boldsymbol{h}|}\\
\lim_{\boldsymbol{k}\to\boldsymbol{0}}|\boldsymbol{h}|\leqslant\tfrac{1}{\lambda}|\boldsymbol{k}|\to\lim_{\boldsymbol{k}\to0}\boldsymbol{h}=\boldsymbol{0}\\
\to\lim_{\boldsymbol{k}\to\boldsymbol{0}}\frac{|\boldsymbol{g}(\boldsymbol{y}+\boldsymbol{k})-\boldsymbol{g}(\boldsymbol{y})-T\boldsymbol{y}|}{|\boldsymbol{k}|}=0\to\boldsymbol{g}'(\boldsymbol{y})=T=(\boldsymbol{f}'(\boldsymbol{x}))^{-1}\\
\to\boldsymbol{g}'(\boldsymbol{y})=(\boldsymbol{f}'(\boldsymbol{g}(\boldsymbol{y})))^{-1}.\ \boldsymbol{y}\in V\\
\boldsymbol{g}\colon V\to U\,,\boldsymbol{g}\text{是满且可微}\\
\boldsymbol{f}'\colon U\to\mathcal{L}(R^n)\text{的所有可逆的}\Omega\text{中的连续映射}\to\boldsymbol{g}\text{是}\Omega\to\Omega\text{的连续映射} \qquad 1.12\,2\\
\to\boldsymbol{g}\in\ell'(V)
\end{gathered}
$$

Remark: 只有最后使用1.122的时候没有用到$\boldsymbol{f}$的连续可微性。连续可微条件太强。

定理 4.2.

$$
\begin{gathered}
\text{开集}E\in R^n,\ \boldsymbol{f}\colon E\to R^n,\ \boldsymbol{f}\in\ell'(E).\\
\boldsymbol{f}'(\boldsymbol{x})\text{在}E\text{上都可逆}\\
\to\quad\forall\text{开集}W\subset E,\ \boldsymbol{f}(W)\text{是}R^n\text{的开子集}
\end{gathered}
$$

Remark: 这定理中的假定对E中的元素都存在领域，在领域内可逆。可以说$\boldsymbol{f}$在每个局部都是可逆的，但不一定是在E内可逆。

5 隐函数定理

根据反函数定理，若函数在每个局部上都可逆，考虑R^n上的度量拓扑(积拓扑)。这样就可以允许一些不可逆的点也具有某种极限意义下的反函数。并且可以考虑反函数的导数。

f是平面上的连续可微函数，f在(a,b)满足$f(a,b)=0\wedge\frac{\partial f}{\partial y}\neq0$.在$(a,b)$内有一个开集内$f$可逆。

如果(a,b)内,$\frac{\partial f}{\partial x}\neq0$,就在$(a,b)$上$f$有反函数使得$x=f^{-1}(y)$

但是因为反函数定理中各种连续性，这表明可以在一些点上是可以不需要偏导数不等于0的存在的

定义 5.1. *记号*

$$
\begin{gathered}
\boldsymbol{x}=(x_1,\ldots,x_n)\in R^n\\
\boldsymbol{y}=(y_1,\ldots,y_m)\in R^m\\
(\boldsymbol{x},\boldsymbol{y})=(x_1,\ldots,x_n,y_1,\ldots,y_m)\in R^{n+m}
\end{gathered}
$$

$$
\begin{gathered}
A\in\mathcal{L}(R^{n+m},R^n)\text{可以分为}A_x,A_y\\
A_x\boldsymbol{h}=A(\boldsymbol{h},\boldsymbol{0});\ A_y\boldsymbol{y}=A(0,\boldsymbol{k})
\end{gathered}
$$

定理 5.2. *线性映射是可以作用在向量的不同区块上的*

$$
\begin{gathered}
A\in\mathcal{L}(R^{n+m},R^n),A_x\text{可逆}\\
\forall\boldsymbol{k}\in R^m,\exists\text{唯一}\boldsymbol{h}\in R^n\to A(\boldsymbol{h},\boldsymbol{k})=\boldsymbol{0}\\
\boldsymbol{h}=-(A_x)^{-1}A_y\boldsymbol{k}
\end{gathered}
$$

证明.

$$
\begin{aligned}
&A(\boldsymbol{h}+\boldsymbol{k})=\boldsymbol{0}\Leftrightarrow A_x\boldsymbol{h}+A_y\boldsymbol{k}=\boldsymbol{0}\\
&\quad A_x\text{可逆}\to\boldsymbol{h}=-A_x^{-1}A_y\boldsymbol{k}
\end{aligned}
$$

Remark: A_x可逆$\Leftrightarrow\operatorname{range}A_x=R^n$.这表示随便$\operatorname{range}A_y$都可以被$A_x$表示掉，加起来等于0就行 □

定理 5.3. *隐函数定理*

$$
\begin{gathered}
\text{开集}E\in R^{n+m}.\ \boldsymbol{f}\colon E\to R^n,\ \boldsymbol{f}\in\mathscr{C}'(E)\\
\exists(\boldsymbol{a},\boldsymbol{b})\in E,\ \boldsymbol{f}(\boldsymbol{a},\boldsymbol{b})=\boldsymbol{0}\\
A=\boldsymbol{f}'(\boldsymbol{a},\boldsymbol{b}),\text{假定}A_x\text{可逆}\\
\to\ 1\quad\exists\text{开集}U\subset R^{n+m},\exists\text{开集}W\subset R^m.\ (\boldsymbol{a},\boldsymbol{b})\in U,\boldsymbol{b}\in W\\
\forall\boldsymbol{y}\in W,\exists\text{唯一}\boldsymbol{x},(\boldsymbol{x},\boldsymbol{y})\in U,\ \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})=\boldsymbol{0}\\
x=\boldsymbol{g}(y)\\
\to\ 2\qquad\boldsymbol{g}\colon W\to R^n,\boldsymbol{g}\in\mathscr{C}'(W).\ \boldsymbol{g}(\boldsymbol{b})=\boldsymbol{a}\\
\boldsymbol{f}(\boldsymbol{g}(\boldsymbol{y}),\boldsymbol{y})=\boldsymbol{0},\ \boldsymbol{y}\in W\\
\boldsymbol{g}'(\boldsymbol{b})=-(A_x)^{-1}A_y
\end{gathered}
$$

Remark:

$$
\boldsymbol{f}=\begin{cases}f_1(x_1,\ldots,x_n,y_1,\ldots,y_m)=\boldsymbol{0}\\ \cdots\\ f_n(x_1,\ldots,x_n,y_1,\ldots,y_m)=\boldsymbol{0}\end{cases}
$$

$$
A_x\text{可逆}\to\begin{pmatrix}D_1f_1&\cdots&D_nf_1\\ \vdots&\ddots&\vdots\\ D_1f_m&\cdots&D_nf_m\end{pmatrix}_{n\times m}=\begin{pmatrix}\frac{\partial f_1}{\partial x_1}&\cdots&\frac{\partial f_1}{\partial x_n}\\ \vdots&&\\ \frac{\partial f_m}{\partial x_1}&\cdots&\frac{\partial f_m}{\partial x_n}\end{pmatrix}_{n\times m}\qquad(m=n)
$$

这表明: 多元函数如果存在一组变量使得关于这组变量可逆，则必然使得整个函数可逆

证明.

$$
\begin{gathered}
1\qquad F(\boldsymbol{x},\boldsymbol{y})=(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}),\boldsymbol{y}).\ (\boldsymbol{x},\boldsymbol{y})\in E\\
\to F\in\mathscr{C}'(E).\ F'(\boldsymbol{a},\boldsymbol{b})\text{是}L(R^{n+m})\text{的可逆元}\\
\boldsymbol{f}(\boldsymbol{a},\boldsymbol{b})=\boldsymbol{0}\to\boldsymbol{f}(\boldsymbol{a}+\boldsymbol{h},\boldsymbol{b}+\boldsymbol{h})=A(\boldsymbol{h},\boldsymbol{k})+\boldsymbol{r}(\boldsymbol{h},\boldsymbol{k}).\lim_{\boldsymbol{h}\to\boldsymbol{0}}\tfrac{|\boldsymbol{r}(\boldsymbol{h},\boldsymbol{k})|}{|(\boldsymbol{h},\boldsymbol{k})|}=0\\
\boldsymbol{F}(\boldsymbol{a}+\boldsymbol{h},\boldsymbol{b}+\boldsymbol{k})-\boldsymbol{F}(\boldsymbol{a},\boldsymbol{b})=(\boldsymbol{f}(\boldsymbol{a}+\boldsymbol{h},\boldsymbol{b}+\boldsymbol{k}),\boldsymbol{k})\\
=(A(\boldsymbol{h},\boldsymbol{k}),\boldsymbol{k})+(\boldsymbol{r}(\boldsymbol{h},\boldsymbol{k}),\boldsymbol{0})\\
\to\boldsymbol{F}'(\boldsymbol{a},\boldsymbol{b})\in\mathcal{L}(R^{n+m});\ \boldsymbol{F}'(\boldsymbol{h},\boldsymbol{k})=(A(\boldsymbol{h},\boldsymbol{k}),\boldsymbol{k})\\
\boldsymbol{0}=\boldsymbol{F}'(\boldsymbol{h},\boldsymbol{k})=(A(\boldsymbol{h},\boldsymbol{k}),\boldsymbol{k})\to A(\boldsymbol{h},\boldsymbol{0})=\boldsymbol{0}.\ A\text{可逆}\to\boldsymbol{h}=\boldsymbol{0}\\
\to\boldsymbol{F}'(\boldsymbol{a},\boldsymbol{b})\text{是}1-1\text{的}\to\boldsymbol{F}'\text{可逆}
\end{gathered}
$$

$$
\begin{gathered}
\exists\text{开集}U,V\in R^{n+m},(\boldsymbol{a},\boldsymbol{b})\in U,(\boldsymbol{0},\boldsymbol{b})\in V.\ \boldsymbol{F}\colon U\to V\wedge\boldsymbol{F}\text{可逆}\\
\text{let:}\ W=\{(\boldsymbol{0},\boldsymbol{y})\in R^{n+m}\colon\}\cap V\to\boldsymbol{b}\in W\\
V\text{开}\to W\text{开}\\
\boldsymbol{y}\in W,\exists(\boldsymbol{x},\boldsymbol{y})\in U\to(\boldsymbol{0},\boldsymbol{y})=\boldsymbol{F}(\boldsymbol{x},\boldsymbol{y})\to\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})=\boldsymbol{0}\\
\text{设}\boldsymbol{x}'\text{也能使得上式成立}\\
\boldsymbol{F}(\boldsymbol{x}',\boldsymbol{y})=(\boldsymbol{f}(\boldsymbol{x}',\boldsymbol{y}),\boldsymbol{y})=(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}),\boldsymbol{y})=\boldsymbol{F}(\boldsymbol{x},\boldsymbol{y})\\
\text{因为}\boldsymbol{F}\text{在}U\text{中是}1-1\text{的}\to\boldsymbol{x}'=\boldsymbol{x}
\end{gathered}
$$

2 $$\begin{gathered}\boldsymbol{y}\in W.\,\boldsymbol{g}(\boldsymbol{y})\to(\boldsymbol{g}(\boldsymbol{y}),\boldsymbol{y})\in U\wedge\boldsymbol{f}(\boldsymbol{g}(\boldsymbol{y}),\boldsymbol{y})=\mathbf{0}\\ \to\boldsymbol{F}(\boldsymbol{g}(\boldsymbol{y}),\boldsymbol{y})=(\mathbf{0},\boldsymbol{y})\\ \boldsymbol{G}\colon V\to U.\text{若}\boldsymbol{G}\text{满}\wedge\boldsymbol{G}=(\boldsymbol{F})^{-1}\\ \to\boldsymbol{G}\in\ell'(V)\wedge(\boldsymbol{g}(\boldsymbol{y}),\boldsymbol{y})=\boldsymbol{G}(\mathbf{0},\boldsymbol{y})\\ \boldsymbol{G}\in\ell'(V)\to\boldsymbol{g}\in\ell'(V)\\ \text{let:}\,\Phi(\boldsymbol{y})=(\boldsymbol{g}(\boldsymbol{y}),\boldsymbol{y})\\ \Phi'(\boldsymbol{y})\boldsymbol{k}=(\boldsymbol{g}'(\boldsymbol{y})\boldsymbol{k},I\boldsymbol{k}).\,\boldsymbol{y}\in W,\boldsymbol{k}\in R^m\\ \boldsymbol{f}(\Phi(\boldsymbol{y}))=\mathbf{0}\\ \to\boldsymbol{f}'(\Phi(\boldsymbol{y}))\Phi'(\boldsymbol{y})=\mathbf{0}\\ \boldsymbol{y}=\boldsymbol{b}\text{时},\Phi(\boldsymbol{y})=(\boldsymbol{a},\boldsymbol{b}),\,\boldsymbol{f}'(\Phi(\boldsymbol{y}))=A\\ \to A\Phi'(\boldsymbol{b})=0\\ \forall\boldsymbol{k}\in R^m.\,A_x\boldsymbol{g}'(\boldsymbol{b})\boldsymbol{k}+A_y\boldsymbol{k}=A(\boldsymbol{g}'(\boldsymbol{b})\boldsymbol{k},\boldsymbol{k})=A\Phi'(\boldsymbol{b})\boldsymbol{k}=\mathbf{0}\\ \to A_x\boldsymbol{g}'(\boldsymbol{b})+A_y=\mathbf{0}\\ \to\boldsymbol{g}'(\boldsymbol{b})=-(A_x)^{-1}A_y\end{gathered}$$

□

Remark: 将$A_x\boldsymbol{g}'(\boldsymbol{b})+A_y=\mathbf{0}$写成分量形式得到

$$\begin{gathered}\sum_{j=1}^{n}(D_jf_i)(\boldsymbol{a},\boldsymbol{b})D_kg_j(\boldsymbol{b})=-(D_{n+k}f_i)(\boldsymbol{a},\boldsymbol{b})\\ \Leftrightarrow\qquad\sum_{j=1}^{n}\left(\frac{\partial f_i}{\partial x_j}\right)\left(\frac{\partial g_i}{\partial y_k}\right)=-\left(\frac{\partial f_i}{\partial y_k}\right).\,1\leqslant i\leqslant n;1\leqslant k\leqslant m\\ \text{对于每个}k\text{来说，这是以}\frac{\partial g_i}{\partial y_k}\text{为未知量，}n\text{个线性方程的方程组}\end{gathered}$$

例 5.4. $n=2;m=3$

$$\boldsymbol{f}\colon R^5\to R^2$$

$$\begin{cases}f_1(x_1,x_2,y_1,y_2,y_3)=2e^{x_1}+x_2y_1-4y_2+3\\ f_2(x_1,x_2,y_1,y_2,y_3)=x_2\cos x_1-6x_1+2y_1-y_3\end{cases}$$

$$\begin{gathered}a=(0,1);b=(3,2,7)\to\boldsymbol{f}(\boldsymbol{a},\boldsymbol{b})=\mathbf{0}\\ A=\boldsymbol{f}'(\boldsymbol{a},\boldsymbol{b})=\begin{pmatrix}\frac{\partial f_1}{\partial x_1}&\frac{\partial f_1}{\partial x_2}&\frac{\partial f_1}{\partial y_1}&\frac{\partial f_1}{\partial y_2}&\frac{\partial f_1}{\partial y_3}\\ \frac{\partial f_2}{\partial x_1}&\frac{\partial f_2}{\partial x_2}&\frac{\partial f_2}{\partial y_1}&\frac{\partial f_2}{\partial y_2}&\frac{\partial f_2}{\partial y_3}\end{pmatrix}=\begin{pmatrix}2e^{x_1}&y_1&x_2&-4&0\\ -x_2\sin x_1-6&\cos x_1&2&0&-1\end{pmatrix}\\ \boldsymbol{f}'(0,1,3,2,7)=\begin{pmatrix}2&3&1&-4&0\\ -6&1&2&0&-1\end{pmatrix}\\ A_x=\begin{pmatrix}2&3\\ -6&1\end{pmatrix},A_y=\begin{pmatrix}1&-4&0\\ 2&0&-1\end{pmatrix}\\ \to\exists\boldsymbol{g}\colon R^3\to R^2,\boldsymbol{g}(\boldsymbol{b})=\boldsymbol{a};\qquad\text{隐函数定理}\\ \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{b}),\boldsymbol{b})=\mathbf{0}\\ \boldsymbol{g}'(\boldsymbol{b})=A_x^{-1}A_y=\begin{pmatrix}2&3\\ -6&1\end{pmatrix}^{-1}\begin{pmatrix}1&-4&0\\ 2&0&-1\end{pmatrix}\\ =\frac{1}{20}\begin{pmatrix}1&-3\\ 6&2\end{pmatrix}\begin{pmatrix}1&-4&0\\ 2&0&-1\end{pmatrix}=\begin{pmatrix}\frac{1}{4}&\frac{1}{5}&-\frac{3}{20}\\ -\frac{1}{2}&\frac{6}{5}&\frac{1}{10}\end{pmatrix}=\begin{pmatrix}\frac{\partial g}{\partial y_1}&\frac{\partial g}{\partial y_2}&\frac{\partial g}{\partial y_3}\\ \frac{\partial g}{\partial y_1}&\frac{\partial g}{\partial y_2}&\frac{\partial g}{\partial y_3}\end{pmatrix}\end{gathered}$$

6 秩定理

反函数定理和隐函数定理表示连续可微的映射在一点的局部性质， 这利用了线性映射$F'(x)$在x的矩阵性质。

秩定理也表明这样的函数具有某种性质

定理 6.1. *有限维线性映射的值域和核都是线性空间*

$$
\begin{gathered}
X,Y\text{是向量空间}, A\in\mathcal{L}(X,Y)\\
\ker A=\operatorname{null}A=\{x\in X\colon Ax=\mathbf{0}\}\\
\operatorname{range}A=\{Ax\colon x\in X\}\\
\operatorname{null}A\text{是线性空间}\\
\operatorname{range}A\text{是线性空间}
\end{gathered}
$$

证明.

$$
\begin{gathered}
A\mathbf{0}_X=\mathbf{0}_Y\rightarrow\mathbf{0}_X\in\ker A=\operatorname{null}A\\
x,y\in\operatorname{null}A. A(x+y)=A(x)+A(y)=\mathbf{0}_Y+\mathbf{0}_Y=\mathbf{0}_Y\\
\rightarrow x+y\in\operatorname{null}A\\
x\in\operatorname{null}A, A(\lambda x)=\lambda A(x)=\lambda\mathbf{0}_Y=\mathbf{0}_Y\\
\rightarrow\lambda x\in\operatorname{null}A\\
\rightarrow\operatorname{null}A\text{是线性空间}
\end{gathered}
$$

$$
\begin{gathered}
\mathbf{0}_X\in X, A\mathbf{0}_X=\mathbf{0}_Y\rightarrow\mathbf{0}_X\in\operatorname{range}A\\
\forall x,y\in\operatorname{range}A\rightarrow A(m)=x; A(n)=y\\
\rightarrow A(m)+A(n)=A(m+n)=x+y\\
\rightarrow x+y\in\operatorname{range}A\\
\forall x\in\operatorname{range}A. \lambda x=\lambda Am=A(\lambda m)\\
\rightarrow\lambda x\in\operatorname{range}A\\
\rightarrow\operatorname{range}A\text{是子空间}
\end{gathered}
$$

□

定义 6.2. *射影*

$$X\text{是向量空间}. P\text{是}X\text{的射影}:=P\in\mathcal{L}(X), P^2=P.$$

定理 6.3. *射影的初等性质*

$$
\begin{array}{ll}
1 & P\text{是}X\text{中的射影}\rightarrow\forall x\in X, x=x_1+x_2, x_1\in\operatorname{null}P, x_2\in\operatorname{range}P\\
2 & X\text{是有限维向量空间}, X_1\text{是}X\text{的一个向量空间}\rightarrow\exists P\in\mathcal{L}(X), \operatorname{range}P=X_1
\end{array}
$$

证明.

$$
\begin{array}{ll}
1\ \text{存在} & \forall x\in X. x_1=Px. x_2=x-x_1\\
 & Px_2=P(x-x_1)=P(x)-P(x_1)=P(x)-P^2(x)=0\\
 & \rightarrow x_2\in\operatorname{null}P; x_1=\operatorname{range}P\\
\ \ \text{唯一} & Px=P(x_1+x_2)=P(x_1)+P(x_2)=P(x_1)+\mathbf{0}\rightarrow x_1=Px
\end{array}
$$

$$
\begin{array}{ll}
2 & X_1=\{0\}. P=\mathbf{0}. P^2=\mathbf{0}\\
 & X_1=\operatorname{span}(\boldsymbol{u})\rightarrow X=\operatorname{span}(\boldsymbol{u},\boldsymbol{v})\\
 & \rightarrow\forall x\in X\rightarrow x=\boldsymbol{au}+\boldsymbol{bv}\\
 & \text{let}: Px=P(\boldsymbol{au}+\boldsymbol{bv})=P(\boldsymbol{au})=\boldsymbol{au}\\
 & P^2x=P^2(\boldsymbol{au}+\boldsymbol{bv})+P(\boldsymbol{au})=P(\boldsymbol{au}+\mathbf{0}\boldsymbol{v})=\boldsymbol{au}=Px\\
 & \operatorname{range}P=\operatorname{range}\boldsymbol{u}=X_1
\end{array}
$$

□

定理 6.4. *秩定理*

前提 $$m,n,r\in N.m\geqslant r,n\geqslant r.$$
$$开集E\in R^n.F\colon E\to R^m,F是\mathscr{C}'的,\forall \boldsymbol{x}\in E,\operatorname{rank}F'(x)=r$$
$$\forall \boldsymbol{a}\in E,A=F'(\boldsymbol{a}),\operatorname{range}A=Y_1,P是R^m中的射影，\operatorname{range}P=Y_1.Y_2=\operatorname{null}P$$

结论 $$\exists开集U,开集V\subset R^n,\boldsymbol{a}\in U,U\subset E.\exists V\to U上的1-1连续可微映射H$$
$$F(H(\boldsymbol{x}))=Ax+\varphi(Ax)\ (\boldsymbol{x}\in V)$$
$$\varphi是把开集A(V)\subset Y_1映入Y_2内的连续可微映射$$

证明.

$$r=0\to F(x)在U_x(r)内是常量。V=U.H(x)=I_X.\varphi(0)=F(a)显然成立$$
$$r>0$$
$$\dim Y_1=r\to Y_1有基\boldsymbol{y}_r.选z_i\in R^n\to A\boldsymbol{z}=\boldsymbol{y}$$
$$S\colon Y_1\to R^n,S(c_1y_1+\cdots+c_ny_n)=c_1z_1+\cdots+c_nz_n$$
$$\to AS\boldsymbol{y}=A\boldsymbol{z}=\boldsymbol{y}$$
$$G\colon E\to R^n,G(x)=x+SP(F(x)-Ax)$$
$$G'(x)=I_{R^n}+SP(A-A)=I_{R^n}$$
$$反函数定理\to R^n中有开集U和V,\boldsymbol{a}\in U,G\colon U\to V上的1-1映射$$
$$G的逆H也是连续可微的$$
$$PA=A,ASPA=ASA=IA=A\to ASPA=A$$
$$AG(x)=PF(x)$$
$$P(F(H(x)))=Ax$$
$$\Psi(x)=F(H(x))-Ax$$
$$PA=A\to\forall\boldsymbol{x}\in V,P\Psi(x)=\boldsymbol{0}.$$
$$\to\Psi是V到Y_2内的连续可微映射$$
$$V是开集\to A(V)是开集$$
$$x_1,x_2\in V,Ax_1=Ax_2\to\Psi(x_1)=\Psi(x_2)$$
$$???$$

□

Remark: 几何意义

$$\boldsymbol{y}\in F(U).\exists\boldsymbol{x}\in V\to\boldsymbol{y}=F(H(\boldsymbol{x}))$$
$$\to P\boldsymbol{y}=A\boldsymbol{x}$$
$$\to\boldsymbol{y}=P\boldsymbol{y}+\varphi(P(\boldsymbol{y}))$$
$$这表明\boldsymbol{y}被射影P\boldsymbol{y}所确定$$
$$P限制在F(U)内，P是F(U)\to A(V)的1-1映射$$
$$F(U)是“r维曲面”,在A(V)的每个点上恰有一个F(U)的一个点$$

7 行列式

行列式是方阵的实函数

定义 7.1. *行列式*

逆序数 $$\mathrm{s}(j_1,\ldots,j_n)=\prod_{p<q}\operatorname{sgn}(j_q-j_p)$$
行列式 $$\det A=\sum_{(j_1,\ldots,j_n)\in\operatorname{prem}(1,\ldots,n)}s(j_1,\ldots,j_n)a_{1,j_1}a_{2,j_2}\ldots a_{n,j_n}$$
列向量的行列式 $$\det A=\det(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)$$

定理 7.2. *行列式的性质*

$$\begin{array}{rc} 1 & \det I = 1 \\ 2 & \det(x_1,\dots,x_n)\text{是每个}x_n\text{的线性函数} \\ 3 & \det(x_1,\dots,x_i,\dots,x_j,\dots,x_n) = -\det(x_1,\dots,x_j,\dots,x_i,\dots,x_n) \\ 4 & x_i = x_j \to \det(x_1,\dots,x_i,\dots,x_j,\dots,x_n) = 0 \end{array}$$

证明.

1 $$\det I = \det s(1,\dots,n) a_{1,1} \cdots a_{n,n} = 1$$

2
$$\begin{aligned} \det(a_1,\dots,a_n) &= \sum s(j_1,\dots,j_n) a_{1,j_1} \cdots a_{n,j_n} \\ \det(a_i + a_j) &= \sum s(j_1,\dots,j_n) a_{1,j_1} \cdots (a_{i_1,j_i} + a_{i_2,j_i}) \cdots a_{n,j_n} \\ &= \sum s(j_1,\dots,j_n) a_{1,j_1} \cdots a_{i,j_i} \cdots a_{n,j_n} + \sum s(j_1,\dots,j_n) a_{1,j_1} \cdots a_{i_2,j_n} \\ &= \det(A_1) + \det(A_2) \\ \det(x_1,\dots,x_n) &= \sum s(j_1,\dots,j_n) a_{1,j_1} \cdots \lambda a_{i,j_i} \cdots a_{n,j_n} \\ &= \lambda \sum s(j_1 \cdots j_n) a_{1,j_1} \cdots a_{n,j_n} \\ &= \lambda \det(x_1,\dots,x_n) \\ &\to \det(x_1,\dots,x_n) \in \mathcal{L}(x_i) \end{aligned}$$

□

3
$$\begin{aligned} s(j_1,\cdots,j_p,\cdots,j_q \cdots j_n) &= -s(j_1,\cdots,j_q,\cdots,j_p,\cdots j_n) \\ \to \det(x_1,\dots,x_i,\dots,x_j,\dots,x_n) &= -\det(x_1,\dots,x_j,\dots,x_i,\dots,x_n) \end{aligned}$$

4
$$\begin{aligned} \det(x_1,\dots,x_i,\dots,x_j,\dots,x_n) &= \det(x_1,\dots,x_j,\dots,x_i,\dots,x_n) \\ &\to \det(x_1,\dots,x_n) = 0 \end{aligned}$$

定理 7.3. *行列式是可乘的*

$$A, B \in \mathcal{L}(F^n). \det(AB) = (\det A) \times \det(B)$$

证明.

$$\begin{aligned} \Delta_B(x_1,\dots,x_n) &= \Delta_B A = \det(BA) \\ \det(BA) &= \det(BA_{\cdot,1},\dots,BA_{\cdot,n}) \\ \to \Delta_B A = \Delta_B\left(\sum a_{i,1}\boldsymbol{e}_i, x_2,\dots,x_n\right) &= \sum a_{i,1} \Delta_B(e_i, x_2,\dots,x_n) \\ \to \Delta_B A &= \sum a_{i_1,1} a_{i_2,2} \dots a_{i_n,n} \\ \Delta_B(e_1,\dots,e_n) &= t(i_1,\dots,i_n) \Delta_B(e_1,\dots,e_n) \\ \Delta_B(e_1,\dots,e_n) &= \det B \\ \to \det(BA) &= \left\{\sum a_{i_1,1} \cdots a_{i_n,n} t(i_1,\dots,i_n)\right\} \det B \\ &= \det AB \end{aligned}$$

□

定理 7.4. *线性算子可逆的充要条件是行列式不为0*

证明.

$$\begin{aligned} & A\text{可逆} \to \text{行列式不为0} \\ \det A \det A^{-1} = \det I = 1 &\to \det A, \det A^{-1} \neq 0 \\ & \text{行列式不为0} \to A\text{可逆} \\ \det A = 0 &\to \text{存在}a\text{的特征值为0} \\ \to Av = 0v \to Av &= 0 \to \operatorname{null} A \neq \{0\} \\ & \to A\text{不可逆} \end{aligned}$$

□

定理 7.5. *算子的行列式与基的选择无关*

证明.

$$
\begin{aligned}
&\boldsymbol{e},\boldsymbol{u}\text{是}V\text{的两个基}\\
\mathcal{M}(T,\boldsymbol{e}) &= \mathcal{M}(T,\boldsymbol{e},\boldsymbol{u})^{-1}\mathcal{M}(T,\boldsymbol{u})\mathcal{M}(T,\boldsymbol{e},\boldsymbol{u}) \qquad \text{基变换公式}\\
\to\det(\mathcal{M}(T,\boldsymbol{e})) &= \det(\mathcal{M}(T,\boldsymbol{e},\boldsymbol{u})^{-1}\mathcal{M}(T,\boldsymbol{u})\mathcal{M}(T,\boldsymbol{e},\boldsymbol{u}))\\
&= \det\mathcal{M}(T,\boldsymbol{e},\boldsymbol{u})^{-1}\times\det\mathcal{M}(T,\boldsymbol{u})\times\det\mathcal{M}(T,\boldsymbol{e},\boldsymbol{u})\\
&= \det\mathcal{M}(T,\boldsymbol{u})\times\det(\mathcal{M}(T,\boldsymbol{e},\boldsymbol{u})^{-1}\mathcal{M}(T,\boldsymbol{e},\boldsymbol{u}))\\
&= \det\mathcal{M}(T,\boldsymbol{u})
\end{aligned}
$$

□

定义 7.6. *函数行列式*

$$
\begin{aligned}
&\text{开集}E\subset R^n.\ \boldsymbol{f}\colon E\to R^n,\ \boldsymbol{f}\text{在}\boldsymbol{x}\in E\text{可微}\\
\text{行列式}\quad & J_{\boldsymbol{f}(\boldsymbol{x})} = \det(\boldsymbol{f}'(\boldsymbol{x}))\text{称为}\boldsymbol{f}\text{在}\boldsymbol{x}\text{的函数行列式}\\
\text{使用记号}\quad & J_{\boldsymbol{f}(\boldsymbol{x})} = \frac{\partial(y_1,\ldots,y_n)}{\partial(x_1,\ldots,x_n)}
\end{aligned}
$$

8 高阶导数

定义 8.1. *高阶偏导数，高阶可微函类*

$$
\begin{aligned}
&\text{开集}E\subset R^n.\ f\colon E\to R^m\\
&f\text{的偏导数}D_1 f,\ldots,D_n f\\
\text{二阶偏导数}\quad & D_{i,j}f = D_i(D_j(f)) = \frac{\partial^2 f}{\partial x_i\partial x_j}
\end{aligned}
$$

$$
\begin{aligned}
\text{偏导数与可微性}\quad & \text{和一阶偏导数一样，即使两个偏导数都存在也不能推得高阶可微}\\
& \text{若}n\text{阶偏导数在}x\text{点连续}\to\text{在}x\text{点}n\text{阶可微}
\end{aligned}
$$

定理 8.2. *逐变量中值定理(名字是我自己瞎起的)*

$$
\begin{aligned}
&\text{开集}E\subset R^2.\ f\colon E\to E.\ D_1 f\text{和}D_{2,1}f\text{在}E\text{的每个点都存在}\\
&Q\subset E\text{是闭矩形，}Q\text{的边和变量轴平行.}\ (a,b),(a+h,b+k)\text{是矩形的对顶点}\\
&\Delta(f,Q) = f(a+h,b+k)-f(a+h,b)-f(a,b+k)+f(a,b)\\
\to\quad &\exists(x,y)\in Q\to\Delta(f,Q) = hk(D_{2,1}f)(x,y)
\end{aligned}
$$

Remark: D_1和$D_{2,1}$在E上都存在表明必可以先对D_1方向取一个增量，再在$D_{2,1}$上进一步取增量

证明.

$$
\begin{aligned}
u(t) &= f(t,b+k)-f(t,b)\\
\to\Delta(f,Q) &= u(a+h)-u(a)\\
&= hu'(x)\\
&= h[(D_1 f)(x,b+k)-(D_1 f)(x,b)]\\
&= hk(D_{2,1}f)(x,y)
\end{aligned}
$$

□

定理 8.3. *一阶偏导数和一个二阶偏导数存在。二阶偏导数连续则对应另一个二阶偏导数必存在且相等*

$$
\begin{aligned}
&D_1 f, D_2 f, D_{1,2}\text{在}E\text{上都存在}\\
&D_{1,2}\text{在}(a,b)\text{上连续}\to D_{2,1}f(a,b)\text{存在}\wedge(D_{2,1}f)(a,b) = (D_{1,2}f)(a,b)
\end{aligned}
$$

Remark: $f\in\mathscr{C}''(R)\to D_{1,2}f = D_{2,1}f$

证明.

$$
\begin{aligned}
A=&(D_{2,1}f)(a,b),\forall\varepsilon>0,\forall(x,y)\in Q\\
&|A-(D_{2,1}f)(x,y)|<\varepsilon\\
&\to\left|\tfrac{\Delta(f,Q)}{hk}-A\right|<\varepsilon\\
\to D_2\text{存在}\to&\left|\tfrac{(D_2f)(a+h,b)-(D_2f)(a,b)}{h}-A\right|<\varepsilon\\
&\to(D_{1,2}f)(a,b)=A
\end{aligned}
$$

□

9 积分的微分法

这里主要研究微分和积分的可交换性

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b\varphi(x,t)\mathrm{d}x=\int_a^b\frac{\partial\varphi}{\partial t}(x,t)\mathrm{d}x$$

定理 9.1. *积分和微分可交换的充分条件*

$$
\begin{array}{cc}
1 & a\leqslant x\leqslant b,c\leqslant t\leqslant d,\varphi(x,t)\text{有定义}\\
2 & \alpha\text{是}[a,b]\text{上的增函数}\\
3 & \forall t\in[c,d],\varphi(x,t)\text{可积}\\
4 & c<s<d.\forall\varepsilon>0,\exists\delta>0,\forall x\in[a,b],t\in(s-\delta,s+\delta)\\
 & \to|(D_2\varphi)(x,t)-(D_2\varphi)(x,s)|<\varepsilon\\
 & f(t)=\displaystyle\int_a^b\varphi(x,t)\mathrm{d}\alpha(x)\ \ (c\leqslant t\leqslant d)\\
\to & (D_2\varphi)(x,s)\text{对}x\text{可积},f'(s)\text{存在}\\
 & f'(s)=\displaystyle\int_a^b(D_2\varphi)(x,s)\mathrm{d}\alpha(x)\\
\text{Remark} & D_1\varphi=\dfrac{\partial\varphi}{\partial x};D_2\varphi=\dfrac{\partial\varphi}{\partial t}
\end{array}
$$

证明.

$$
\begin{aligned}
&\Psi(x,t)=\tfrac{\varphi(x,t)-\varphi(x,s)}{t-s}.0<|t-s|<\delta\\
&\to\forall u\in(s,t)\to\Psi(x,t)=(D_2\varphi)(x,u)\\
\to&\text{条件4表明}:|\Psi(x,t)-(D_2\varphi)(x,s)|<\varepsilon.\ (x\in[a,b],0<|t-s|<\delta)\\
&\tfrac{f(t)-f(s)}{t-s}=\textstyle\int_a^b\Psi(x,t)\mathrm{d}\alpha(x)\\
&\text{由于}t\in[c,d]\text{是有界闭区间}\to\forall t,\Psi(x,t)\text{一致收敛于}D_2(x,s)\\
&\forall t\in[c,d],\Psi(x,t)\in\mathfrak{R}(\alpha)\\
\to&\text{对}s\text{取一个序列}s_n\to t,\text{得到}f'(t)=\lim_{s_n\to t}\textstyle\int_a^b\Psi(x,s_n)\mathrm{d}\alpha(x)\\
&=\textstyle\int_a^b(D_2\varphi)\mathrm{d}\alpha(x)
\end{aligned}
$$

□

例 9.2. 定理9.1中的$[a,b]$可以扩张到$(-\infty,+\infty)$.

$$
\begin{aligned}
f(t)&=\int_{-\infty}^{\infty}e^{-x^2}\cos(xt)\mathrm{d}x\\
g(t)&=\int_{-\infty}^{\infty}xe^{-x^2}\sin(xt)\mathrm{d}x
\end{aligned}
$$

这两个积分都存在且绝对收敛。$e^{-x^2}\cos(xt)\leqslant e^{-x^2}.xe^{-x^2}\sin(xt)\leqslant|x|e^{-x^2}$

注意到

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{-x^2}\cos(xt))=-xe^{-x^2}\sin(xt)$$

因此可以断定$f'(t)=g(t).t\in R$

证明.

$$\beta>0.\frac{\cos(\alpha+\beta)-\cos\alpha}{\beta}+\sin\alpha=\frac{1}{\beta}\int_{\alpha}^{\alpha+\beta}(\sin\alpha-\sin t)\mathrm{d}t$$
$$|\sin\alpha-\sin t|\leqslant|t-\alpha|\to\frac{1}{\beta}\int_{\alpha}^{\alpha+\beta}(\sin\alpha-\sin t)\mathrm{d}t\leqslant\frac{1}{\beta}\int_{\alpha}^{\alpha+\beta}|t-\alpha|\,\mathrm{d}t$$
$$\leqslant\frac{1}{\beta}\frac{1}{2}|(\alpha+\beta-\alpha)|^2=\frac{\beta}{2}$$
$$\text{对}\beta<0\text{也有类似结果}$$
$$\to\left|\frac{\cos(\alpha+\beta)-\cos\alpha}{\beta}+\sin\alpha\right|\leqslant|\beta|$$
$$h\neq0.\alpha=xt;\beta=xh$$
$$\to\left|\frac{f(t+h)-f(t)}{h}-g(t)\right|\leqslant|h|\int_{-\infty}^{\infty}x^2e^{-x^2}\mathrm{d}x$$
$$\to f'(t)=g(t)$$
$$f(t)=2\int_{-\infty}^{\infty}xe^{-x^2}\frac{\sin(xt)}{t}\mathrm{d}x$$
$$\to tf(t)=-2g(t)$$
$$\to2f'(t)+tf(t)=0$$
$$f(0)=\sqrt{\pi}\to f(t)=\sqrt{\pi}e^{-\frac{t^2}{4}}$$

□

习题