第二章：有限维向量空间

记号：$F \to R \vee C, V \to F$上的向量空间

1 张成空间与线性无关

惯例 1.1. 向量构成的组（向量组）不用（）表示。(1,2,3),(3,4,5)表示一个R^3中长度为2的向量组

定义 1.2. *线性组合（Linear combination）*

V中一组向量$x_1, \ldots, x_n$的线性组合：$v = a_1x_1 + \cdots + a_nx_n, a_i \in F$

例 1.3. $(17, -4, 2)$是$(2, 1, -3), (1, -2, 4)$的线性组合$(17, -4, 2) = 6(2, 1, -3) + 5(1, -2, 4)$

定义 1.4. *张成空间(span)(线性张成空间)*

一组向量$x_1, x_2, \ldots, x_n$定义的集合$\{v \colon v = a_1x_1 + a_2x_2 + \cdots + a_nx_n, a_i \in F\}$称为$x_1, \ldots, x_n$的张成空间，记作：

$$\operatorname{span}(x_1, \ldots, x_n) = \{v \colon v = a_1x_1 + \cdots + a_nx_n, a_n \in F\}$$

特殊的：$\operatorname{span}(\varnothing) = \{0\}$

定理 1.5. *线性张成空间是包含这组向量的最小子空间*

证明.

$$\begin{aligned}
v_1, \ldots, v_n \in V \to \operatorname{span}(v_1, \ldots, v_n) &\subset \mathbb{V} \\
\mathbf{0} = 0v_1 + \cdots + 0v_n &\in \operatorname{span}(v) \\
\forall x, y \in \operatorname{span}(v), x + y = a_1v_1 + \ldots a_nv_n &+ b_1v_1 + \cdots + b_nv_n \\
=(a_1 + b_1)v_1 + \cdots + (a_n + b_n)v_n &\in \operatorname{span}(v) \\
\forall x \in \operatorname{span}(v), \lambda x = \lambda a_1x_1 + \cdots + \lambda a_nx_n &\in \operatorname{span}(v) \\
\to \operatorname{span}(v) &\in \mathbb{V}
\end{aligned}$$

$$\begin{aligned}
\forall \mathbb{U} \subset \mathbb{V} \wedge v \in \mathbb{U} &\to \operatorname{span}(v) \subset \mathbb{U} \\
&\to \operatorname{span}(v)\text{是最小的子空间}
\end{aligned}$$

□

定义 1.6. *张成(spans)：$\operatorname{span}(v) = V$,称$v_1, \ldots, v_n$张成$V$*

eg:$(1, 0, 0), (0, 1, 0), (0, 0, 1)$张成$F^3$

定义 1.7. *有限维向量空间(finite-dimensional vector space)*

一个向量空间可以由该空间中的某个向量组张成，称此向量空间是有限维的。

$\forall n \in N^+, F^n$是有限维向量空间。

定义 1.8. *多项式(polynomial)*

函数$p \colon F \to F$,若存在$a_0, \ldots a_n \in F \to \forall z \in F, p(z) = a_0 + a_1z + \cdots + a_nz^n$称p为系数属于F的多项式。

$\mathcal{P}(F)$是系数属于F的全体多项式构成的集合。

在通常的加法和标量乘法下，$\mathcal{P}(F)$是F上的向量空间。$\mathcal{P}(F) \subset F^F$

定义 1.9. *多项式的次数(degree of polynomial)*

$\forall p \in \mathcal{P}(F)$，*若* $\exists a_0, \ldots, a_{n-1}, a_n \neq 0 \in F \to \forall z \in F, p(z) = a_0 + \ldots a_n z^n$ *则说*p*的次数为*n，$\deg(p) = n$

规定： $\deg(p \equiv 0) = -\infty$

定义 1.10. $\mathcal{P}_n(F)$:

对于$n \in N^+, \mathcal{P}_n(F)$*表示系数在*$F$*中且* $\deg(p) \leqslant n$*的所有多项式构成的集合。*

$\mathcal{P}_n(F) = \operatorname{span}(1, z, \ldots, z^n)$

例 1.11. $\forall n \in N \to \mathcal{P}_n(F)$是有限维向量空间。

定义 1.12. *无限维向量空间(infinite-dimensional vector space)*

不属于有限维向量空间的向量空间称为无限维向量空间。

例 1.13. 证明：$\mathcal{P}(F)$是无限维向量空间

证明.

$$\begin{gathered}\forall x_1, \ldots, x_n \in \mathcal{P}(F), \deg(\operatorname{span}(x_1, \ldots, x_n)) = m \\ \exists z^{m+1} \in \mathcal{P}(F) \to z^{m+1} \notin \operatorname{span}(x_1, \ldots, x_n)\end{gathered}$$

□

定义 1.14. *线性无关(linearly independent)*

$x_1, \ldots, x_n$*线性无关：* $a_i \in F, a_1 x_1 + \cdots + a_n x_n = 0 \to a_i = 0$

规定： ()*线性无关*

等价于$\forall v \in \operatorname{span}(x), v$*的表示唯一（证明类似之前）*

例 1.15.

1. $\forall v_0 \in V, (v_0)$线性无关$\Leftrightarrow v_0 \neq 0$
2. $\forall x, y \in V, x, y$线性无关$\Leftrightarrow \forall \lambda \to x \neq \lambda y$
3. F^4中的$(1,0,0,0), (0,1,0,0), (0,0,1,0)$线性无关
4. $\forall n \in N, \mathcal{P}(F)$中的$1, z, z^2, \ldots, z^n$线性无关
5. 一个线性无关向量组$x_1, \ldots, x_n$中不重复使用的元素构成的任意子组也线性无关

定义 1.16. *线性相关(linearly dependent)*

$x_1, x_2, \ldots, x_n \in V$,*若它们不线性相关，则称为线性相关*

例 1.17. 线性相关组

1. F^3中的$(2,3,1), (1,-1,2), (7,3,8)$线性相关：$2(2,3,1) + 3(1,-1,2) + (-1)(7,3,8) = (0,0,0)$
2. 验证：F^3中的向量组$(2,3,1), (1,-1,2), (7,3,c)$线性相关$\Leftrightarrow c = 8$
3. 若V中的向量组中的一个向量可以被其余向量用线性组合表示，则向量组线性相关
4. 包含0向量的向量组线性相关。

引理 1.18. *线性相关*

设$v_1,...v_n$是V中的一个线性相关的向量组。则

$$\begin{gathered}\forall i\in 1...n\rightarrow v_i\in\operatorname{span}(v_1,\ldots,v_{i-1})\\ \forall i\in 1...n\rightarrow\operatorname{span}(v_1,\ldots,v_{j-1},v_{j+1},\ldots,v_n)=\operatorname{span}(v_1,\ldots,v_n)\end{gathered}$$

证明.

$$\begin{gathered}v_1,\ldots,v_n\text{线性相关}\rightarrow a_1v_1+\cdots+a_nv_n=0\\ v_n=-\tfrac{a_1v_1+\cdots+a_{n-1}v_{n-1}}{a_n}\\ v_n\in\operatorname{span}(v_1,\ldots,v_{n-1})\end{gathered}$$

$$\begin{gathered}\forall v\in\operatorname{span}(v_1,\ldots,v_n),\exists a_1,\ldots,a_n\in F\rightarrow v=a_1v_1+\cdots+a_nv_n\\ \text{由第一条},v_n\in\operatorname{span}(v_1,\ldots,v_{n-1})\rightarrow v=a_1v_1+\cdots+a_{n-1}v_{n-1}+a_n(b_1v_1+\cdots+b_{n-1}v_{n-1})\\ \rightarrow\operatorname{span}(v_1,\ldots v_{n-1})=\operatorname{span}(v_1,\ldots,v_n)\end{gathered}$$

$$\begin{gathered}\text{但特殊的}i=1\rightarrow v_1=0\\ v_1\in\operatorname{span}(\varnothing)\\ \operatorname{span}(\varnothing)=\operatorname{span}(v_1=0)\end{gathered}$$

□

定理 1.19. *有限维线性空间中：线性无关组的长度$\leqslant$张成组的长度*

证明. *设$x_1,\ldots,x_m$线性无关*，$\operatorname{span}(y_1,\ldots,y_n)=V$

$$\begin{gathered}\operatorname{span}(y_1,\ldots,y_n)=V\rightarrow\operatorname{span}(y_1,\ldots,y_n)=\operatorname{span}(y_1,\ldots,y_n,x_1)\\ \text{引理1.18}\rightarrow\operatorname{span}(y_2,\ldots y_n,x_1)=V\\ \text{重复可得}\rightarrow\operatorname{span}(y_m,\ldots y_n,x_1,\ldots x_j)=V\\ \text{设}m>n\rightarrow\operatorname{span}(x_1,\ldots x_{j<m})=V\rightarrow x_{j+1}=a_1x_1+\cdots+a_nx_n\\ \text{这与}x_1,\ldots,x_m\text{线性无关矛盾}\\ \rightarrow m\leqslant n\end{gathered}$$

□

例 1.20.

1. 组$(1,2,3),(4,5,8),(9,6,7),(-3,2,8)\in R^3$线性相关
2. $\forall x,y,z\in R^4,\operatorname{span}(x,y,z,w)\neq R^4$

定理 1.21. *有限维线性空间的子空间都是有限维的*

证明.

$$\begin{gathered}\mathbb{U}\subset\mathbb{V}\rightarrow\mathbb{U}\text{有限维}\\ \mathbb{U}=\{0\}\rightarrow\mathbb{U}\text{有限}\\ \mathbb{U}\neq\{0\},\forall v_0\in\mathbb{U}\wedge v_0\neq 0\\ \mathbb{U}=\operatorname{span}(v_0)\rightarrow\mathbb{U}\text{有限}\\ \mathbb{U}\neq\operatorname{span}(v_0)\rightarrow\exists v_1\in\mathbb{U}\wedge v_1\notin\operatorname{span}(v_0)\\ \mathbb{U}=\operatorname{span}(v_0,v_1)\\ \ldots\\ \forall v_i\in\mathbb{U}\subset\mathbb{V}\rightarrow\text{最多取}\dim(\mathbb{V})\text{个}v\\ \rightarrow\dim(\mathbb{U})\leqslant\dim(\mathbb{V})\end{gathered}$$

□

习题2.A

1. 证明：$\operatorname{span}(v_1,v_2,v_3,v_4)=V\rightarrow \operatorname{span}(v_1-v_2,v_2-v_3,v_3-v_4,v_4)=V$

   $$\begin{aligned}&v_1=v_1-v_2+v_2-v_3+v_3-v_4+v_4\\&v_2=v_2-v_3+v_3-v_4+v_4\\&v_3=v_3-v_4+v_4\\&v_4=v_4\end{aligned}$$

2. 证明：例

3. 计算：$t\rightarrow(3,1,4),(2,-3,5),(5,9,t)$在$R^3$中线性相关

   $$\begin{gathered}a(3,1,4)+b(2,-3,5)=(5,9,t)\\3a+2b=5\\a-3b=9\\\rightarrow a+\frac{2}{3}b=\frac{5}{3}\\\rightarrow\frac{11}{3}b=\frac{5-27}{3}\rightarrow 11b=-22\rightarrow b=-2\\a+6=9\rightarrow a=3\\3\times4-2\times5=2\\\rightarrow t=2\end{gathered}$$

4. 证明：

5. 证明：

   a. $\mathbb{C}$视为$\mathbb{R}$上的向量空间，组$1+i,1-i$线性无关

   $$\begin{gathered}\forall a,b\in R,a(1+i)+b(1-i)=0\rightarrow\\a+b+(a-b)\,i=0\rightarrow a+b=0,a-b=0\rightarrow a=0,b=0\end{gathered}$$

   b. $\mathbb{C}$视为$\mathbb{C}$上的向量空间，组$1+i,1-i$线性相关

   $$1+i=b(1-i)\rightarrow b=\frac{1+i}{1-i}=\frac{(1+i)^2}{2}=\frac{2i}{2}=i$$

6. Proof:v_1,v_2,v_3,v_4线性无关$\rightarrow v_1-v_2,v_2-v_3,v_3-v_4,v_4$线性无关

   设线性相关：$v_4=a(v_1-v_2)+b(v_2-v_3)+c(v_3-v_4)$

   $$\begin{gathered}=av_1-av_2+bv_2-bv_3+cv_3-cv_4\\=av_1+(b-a)v_2+(c-b)v_3-cv_4\\v_4=\frac{av_1+(b-a)v_2+(c-b)v_3}{1+c}\end{gathered}$$

   $c\neq-1\rightarrow v_4$用v_1,v_2,v_3线性表示。矛盾

   $c=-1\rightarrow v_1,v_2,v_3$线性相关。矛盾

   $\rightarrow$它们线性无关

7. Proof or Counterexample:若$v_1,v_2,\ldots,v_m$在$\mathbb{V}$中线性无关$\rightarrow 5v_1-4v_2,v_2,v_3,\ldots v_m$线性无关

   $v_1,\ldots,v_m$线性无关$\rightarrow v_2,\ldots,v_m$线性无关

   若$5v_1-4v_2,\ldots,v_m$线性相关$\rightarrow 5v_1-4v_2=a_2v_2+\cdots+a_mv_m$

   $5v_1=(a_2+4)v_2+\cdots+a_mv_m$与$v_1,\ldots v_m$线性无关矛盾

   $\rightarrow 5v_1-4v_2,\ldots,v_m$线性无关

8. Proof or Counterexample:若$v_1,v_2,\ldots v_n$线性无关，$\forall\lambda\in F\wedge\lambda\neq0\rightarrow\lambda v_1,\ldots,\lambda v_n$线性无关

$$
\begin{aligned}
0 &= a_1v_1+\cdots+a_nv_n\\
\rightarrow \lambda 0 &= a_1\lambda v_1+\cdots+a_n\lambda v_n\\
\rightarrow 0 &= \lambda(a_1v_1+\cdots+a_nv_n)\\
&\rightarrow \text{线性无关}
\end{aligned}
$$

9. Proof or Counterexample:若$v_1,\dots,v_n$，$w_1,\dots,w_n$都线性无关$\rightarrow v_1+w_1,\dots,v_n+w_n$线性无关

   $v_1=(1,0), v_2=(0,1), w_1=(-1,0), w_2=(0,-1)\rightarrow v_1+w_1=(0,0)\rightarrow v_1+w_1,\dots,v_n+w_n$相关

10. Proof: $v_1,\dots v_n$线性无关，$w\in V, v_1+w, v_2+w,\dots,v_n+w$线性相关$\rightarrow w\in\operatorname{span}(v_1,\dots,v_n)$

$$
\begin{gathered}
v_1+w,\dots,v_n+w\text{线性相关}\\
0=a_1(v_1+w)+\ldots+a_n(v_n+w), a_1,\dots,a_n\neq 0\\
-(a_1+\ldots+a_n)w=a_1v_1+\cdots+a_nv_n\\
\text{若}\sum a_i\neq 0\rightarrow w\in\operatorname{span}(v_1,\dots v_n)\\
\text{若}\sum a_i=0\rightarrow a_1v_1+\ldots a_nv_n=0\rightarrow a_1,\dots,a_n=0\text{矛盾}\\
\rightarrow w\in\operatorname{span}(v_1,\dots,a_n)
\end{gathered}
$$

11. Proof:$v_1,\dots,v_n$线性无关，$w\in V, v_1+w,\dots,v_n+w$线性无关$\Leftrightarrow w\notin\operatorname{span}(v_1,\dots v_n)$

$$
\begin{gathered}
w\notin\operatorname{span}(v_1,\dots,v_n)\rightarrow v_1+w,\dots v_n+w\text{线性无关}\\
v_i+w\in\operatorname{span}(v_1,\dots,v_n,w)\\
0=\sum a_i(v_i+w)\rightarrow\sum a_iv_i+\sum a_iw=0\\
\sum a_iv_i=-\sum a_iw\rightarrow a_i=0\\
\rightarrow v_i+w\text{线性无关}
\end{gathered}
$$

$$
\begin{gathered}
v_i+w\text{线性无关}\rightarrow w\notin\operatorname{span}(v)\\
v_i+w\text{线性无关}\rightarrow 0=\sum a_i(v_i+w)\rightarrow a_i=0\\
0=\sum a_iv_i+\sum a_iw\rightarrow a_i=0\\
\forall a_i\neq 0\rightarrow -\sum a_iv_i\neq\sum a_iw\\
\rightarrow w\notin\operatorname{span}(v)
\end{gathered}
$$

12. Proof:$\forall v_1,\dots,v_6\in\mathcal{P}_4(F), v_1,\dots,v_6$线性相关

$$
\begin{gathered}
\dim(\mathcal{P}_4(F))=4\rightarrow\forall p\in\mathcal{P}_4(F)\rightarrow v=a_0+a_1x+\cdots+a_4x^4\\
\mathcal{P}_4(F)=\operatorname{span}(1,x,x^2,x^3,x^4)
\end{gathered}
$$

$$
\begin{gathered}
\text{设}v_1,\dots,v_6\text{线性无关}\rightarrow\operatorname{span}(v_1,\dots,v_5)=\mathcal{P}_4(F), v_6\in\mathcal{P}_4(F)\\
v_6=\sum a_iv_i\text{矛盾}\\
\rightarrow v\text{线性相关}
\end{gathered}
$$

13. Proof:$\forall v_1,\dots,v_4\in\mathcal{P}_4(F)\rightarrow\operatorname{span}(v_1,\dots,v_4)\neq\mathcal{P}_4(F)$

   $\forall v_1,\dots,v_4\in\mathcal{P}_4(F),\dim(\operatorname{span}(v_1,\dots,v_4))<\dim(\operatorname{span}(v_1,\dots,v_5))=\mathcal{P}_4(F)$

14. Proof:$\dim(V)=\infty\Leftrightarrow\exists v_1,\dots\in V,\forall m\in N^+\rightarrow v_1,\dots,v_m$线性无关

$$
\begin{gathered}
\dim(V)=\infty\rightarrow\exists v_1,\dots\in V,\forall m\in N^+\rightarrow v_1,\dots,v_m\text{线性无关}\\
\dim(V)=\infty\rightarrow\forall v\in S,\operatorname{span}(S)\neq V\\
\forall m\in N^+, v_1,\dots,v_m\in S\rightarrow v_1,\dots,v_m\text{线性无关}\\
\forall m\in N^+\rightarrow v_1,\dots,v_m\text{线性无关}\rightarrow\dim(V)=\infty\\
\text{反证：}\dim(V)\neq\infty\rightarrow\dim(V)=d\\
d=\operatorname{span}(v_1,\dots,v_d)\rightarrow m=d+1,\operatorname{span}(v_1,\dots,v_{m+1})\leqslant V\\
\text{故}v_1,\dots,v_{m+1}\text{线性相关}
\end{gathered}
$$

15. Proof:F^∞是无限维的

$$
\begin{gathered}
\text{设}F^\infty\text{有限维}\rightarrow\forall x\in S,\operatorname{card}(S)\in N^+\rightarrow F^\infty=\operatorname{span}(S)\\
\forall v\in F^\infty, v=a_0v_0+\ldots+a_nv_n\\
w=v_{n+1}\in F^\infty, w\notin\operatorname{span}(S)!\\
\rightarrow F^\infty\text{不是有限维}
\end{gathered}
$$

16. Proof:$C^{[0,1]}$是无限维的

$$\begin{gathered}\forall x\in S,\operatorname{card}(S)\in N^+,\operatorname{span}(S)=C^{[0,1]}\\ \forall f\in\operatorname{span}(S)f=a_0f_0+\cdots+a_nf_n\\ g=e^f\in C^{[0,1]},e^f\notin\operatorname{span}(S)\text{超越函数}\\ ???\end{gathered}$$

17. Proof:$p_1,\ldots,p_m\in\mathcal{P}_m(F)\wedge p_1(2)=\ldots=p_m(2)=0\to p_0,\ldots,p_m$线性相关

$$\begin{gathered}p_1(2)=\cdots=p_m(2)=0\to(x-2)q_1=\ldots(x-2)q_m,q_m\in\mathcal{P}_{m-1}\\ q_1,\ldots q_n\in\mathcal{P}_m\text{线性相关}\to\lambda q_i\text{线性相关}\\ \to p_1,\ldots,p_m\text{线性相关}\end{gathered}$$

2 基(basis)

定义 2.1. *基(basis)：若V中的一个向量组v线性无关且*$\operatorname{span}(v)=V$*,称v为V的基*

定理 2.2. *基的判定准则:*$\boldsymbol{v}$*是V的基*$\Leftrightarrow\forall v\in V\to v=a_0v_0+\cdots+a_nv_n,a_i\in F,a_i$唯一

证明.

$$\begin{gathered}\boldsymbol{v}\text{是}V\text{的基}\to\forall v\in V\to v=a_0v_0+\cdots+a_nv_n,a_i\in F,a_i\text{唯一}\\ \operatorname{span}(\boldsymbol{v})=V\to\forall v\in V\to v=b_0v_0+\cdots+b_nv_n\\ v=b_0v_0+\cdots+b_nv_n=c_0v_0+\cdots+c_nv_n\\ 0=(v-v)=\sum(b_i-c_i)v_i\wedge v_i\text{线性无关}\\ \to b_i=c_i\end{gathered}$$

□

$$\begin{gathered}\forall v\in V,v=a_0v_0+\cdots+a_nv_n,a_i\in F,a_i\text{唯一}\to\boldsymbol{v}\text{是}V\text{的基}\\ \forall v\in V,v=a_0v_0+\cdots+a_nv_n\to v\in\operatorname{span}(\boldsymbol{v})\\ \to\operatorname{span}(\boldsymbol{v})=V\\ \text{设}v=a_0v_0+\cdots+a_nv_n=b_nv_0+\cdots+b_nv_n\to a_i=b_i\\ \text{故0的表示唯一}\to\boldsymbol{v}\text{线性无关}\end{gathered}$$

例 2.3. 一些基

1. 组$(1,0,\ldots,0),(0,1,\ldots,0),\ldots,(0,0,\ldots,1)$是$F^n$的基，称为$F^n$的标准基
2. 组$(1,2),(3,5)$是F^2的基
3. 组$(1,2,-4),(7,-5,6)$在F^3中线性无关，但不是F^3的基
4. 组$(1,2),(3,5),(4,13)$张成V^2但不是F^2的基
5. 组$(1,1,0),(0,0,1)$是$\{(x,x,y)\colon x,y\in F\}$的基
6. 组$(1,-1,0),(1,0,-1)$是$\{(x,y,z)\in F^3\colon x+y+z=0\}$的基
7. 组$1,z,z^2,\ldots,z^n$是$\mathcal{P}_m(F)$的基

定理 2.4. $\operatorname{span}(\boldsymbol{v})=V\to\exists\boldsymbol{b}\subset\boldsymbol{v}$

证明.

$$\begin{gathered}
\operatorname{span}(\boldsymbol{v}) = V \rightarrow \forall v \in V \rightarrow v = a_0 v_0 + \cdots + a_n v_n \\
\text{若线性无关，则称为基over} \\
\text{若线性相关, 由定理1.18}\exists (v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n) \\
\rightarrow \operatorname{span}(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots v_j) = \operatorname{span}(v_0, \ldots, v_n) \\
\text{反复重复上述步骤直到得到一个子集}\boldsymbol{b} \subset \boldsymbol{v} \wedge b_1, \ldots, b_n\text{线性无关} \\
\boldsymbol{b}\text{为}V\text{的基}
\end{gathered}$$

□

推论 2.5. *每个有限维向量空间都有基*

定理 2.6. *有限维向量空间中，无关组$\boldsymbol{v}$可以扩充为基$\boldsymbol{b}$*

证明.

$$\begin{gathered}
\boldsymbol{v} \in V \wedge v_0, \ldots, v_m\text{线性无关} \\
b_0, \ldots, b_n\text{为基} \rightarrow \operatorname{span}(b_0, \ldots, b_n) = \operatorname{span}(b_0, \ldots b_n, v_0, \ldots, v_m) = V \\
v_0, \ldots, v_n\text{线性无关} \rightarrow \operatorname{span}(v_0, \ldots, v_m, b_0, \ldots, b_{j-1}, b_{j+1}, \ldots, b_n) = \operatorname{span}(v_0, \ldots b_n)\ 1.18 \\
\text{重复以上得: } \operatorname{span}(v_0, \ldots, v_m, b_{t0}, b_{t1}, \ldots b_{tk}) = V
\end{gathered}$$

□

定理 2.7. *有限维空间V的每个子空间都是V的直和的项*

有限维空间V,$\mathbb{U} \subset V, \exists \mathbb{W} \rightarrow \mathbb{U} \oplus \mathbb{W} = V$

证明.

$$\begin{gathered}
\mathbb{U} \subset V \rightarrow \text{有限维 定理1.21} \\
\exists \boldsymbol{b}_u \in \mathbb{U} \rightarrow \forall u \in \mathbb{U}, u = a_0 b_{u0} + \cdots + a_n b_{un} \\
V\text{的基}\boldsymbol{b}_V = \operatorname{span}(\boldsymbol{b}_u, \boldsymbol{b}_w) \\
\mathbb{U} \oplus \mathbb{W} = V \Leftrightarrow \mathbb{U} + \mathbb{W} = V \wedge \mathbb{U} \cap \mathbb{W} = \{0\} \\
\forall v \in V \rightarrow v = \sum a_i b_{ui} + \sum t_j b_{wj} \\
\rightarrow V = \mathbb{U} + \mathbb{W} \\
\\
\forall x \in \mathbb{U} \cap \mathbb{W} \rightarrow x = \sum a_i b_{ui} = \sum t_j b_{wj} \\
0 = \sum a_i b_{ui} - \sum t_j b_{wj}, \boldsymbol{b}\text{线性无关} \\
\rightarrow a_i = t_j = 0 \\
\rightarrow x = 0 \\
\\
\rightarrow V = \mathbb{U} \oplus \mathbb{W}
\end{gathered}$$

□

习题2.B

1. example:一个基的所有向量空间

   $$\begin{gathered}
   0 \in V, b \in V \wedge b \neq 0, \lambda b \in V \\
   V = \{0, \lambda b, \lambda \in F\}
   \end{gathered}$$

2. 证明2.3

3. Compute:

   a. $\mathbb{U} = \{(x_1, \ldots, x_5) \in R^5 \colon x_1 = 3x_2, x_3 = 7x_4\}$求$\boldsymbol{b}_U$

   $(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1)$

b. 扩张$\boldsymbol{b}_U \to \boldsymbol{b}_{R^5}$

$$(3,1,0,0,0),(0,0,7,1,0),(0,0,0,0,1),(1,0,0,0,0),(0,0,1,0,0)$$

c. $\mathbb{W}, \mathbb{W} \to \mathbb{U} \oplus \mathbb{W} = R^5$

$$\mathbb{W} = \operatorname{span}((1,0,0,0,0),(0,0,1,0,0))$$

4. Compute:???这里做的都在R中，而没考察C

a. $\mathbb{U} = \{(z_1,\ldots,z_5) \in \mathbb{C}^5, 6z_1 = z_2, z_3 + 2z_4 + 3z_5 = 0\}$,求$\boldsymbol{b}_U$

$$\begin{aligned} z_3 + 2z_4 + 3z_5 &= 0 \\ \to z_5 = 2, z_4 = 0 &\to z_3 = 6 \\ z_5 = 2, z_3 = 0 &\to z_2 = 3 \\ (1,6,0,0,0),(0,0,6,0,2),&(0,0,0,3,2) \end{aligned}$$

b. 扩张$\boldsymbol{b}_U \to \boldsymbol{b}_{\mathbb{C}^5}$

$$\begin{gathered} (1,6,0,0,0),(0,0,6,0,2),(0,0,0,3,2) \\ (0,1,0,0,0),(0,0,0,0,1) \end{gathered}$$

c. $\mathbb{W}, \mathbb{W} \to \mathbb{U} \oplus \mathbb{W} = \mathbb{C}^5$

$$\mathbb{W} = \operatorname{span}((0,1,0,0,0),(0,0,0,0,1))$$

5. Proof or Disproof:$\exists \boldsymbol{b} \in \mathcal{P}_3(F) \to \deg(b_0) \neq 2, \ldots, \deg(b_3) \neq 2$

$$\begin{aligned} \forall p, q \in \mathcal{P}_3(F) \wedge \deg(p) &= 2 \\ \deg(ap + bq) &= \max(\deg(ap) + \deg(bq)) \\ &= \max(\deg(p), \deg(q)) = 2 \\ &\to \text{不成立} \end{aligned}$$

6. Proof:$\boldsymbol{b} \in V \to b_1 + b_2, b_2 + b_3, b_3 + b_4, b_4$也是基

$$\begin{aligned} \forall v \in V, x = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 \\ = a_1(b_1 + b_2 - b_2 - b_3 + b_3 + b_4 - b_4) \\ + a_2(b_2 + b_3 - b_3 - b_4 + b_4) \\ + a_3(b_3 + b_4 - b_4) \\ + a_4b_4 \\ = a_1(u_1 - u_2 + u_3 - u_4) \\ + a_2(u_2 - u_3 + u_4) \\ + a_3(u_3 - u_4) \\ + a_4u_4 \\ = a_1u_1 - a_1u_2 + a_1u_3 - a_1u_4 + a_2u_2 - a_2u_3 + a_2u_4 + a_3u_3 - a_3u_4 + a_4u_4 \\ = a_1u_1 + (-a_1 + a_2)u_2 + (a_1 - a_2 + a_3)u_3 + (-a_1 + a_2 - a_3 + a_4)u_4 \\ \to \operatorname{span}(u_1, u_2, u_3, u_4) = V \end{aligned}$$

$$\begin{gathered} \text{线性无关: 带入0易证} \\ \to \boldsymbol{u}\text{也是}V\text{的基} \end{gathered}$$

7. Proof or CounterExample:$v_1,\ldots,v_4$是V的基，$\mathbb{U} \subset V \wedge v_1, v_2 \in \mathbb{U}, v_3, v_4 \notin \mathbb{U} \to v_1, v_2$是$\mathbb{U}$的基

$$\begin{aligned} \forall u \in \mathbb{U}, x = a_1v_1 + a_2v_2 + a_3v_4 + a_4v_4 \\ \to a_3, a_4 = 0 \text{ 否则} 0,0,1,0 \to a_3 \in \mathbb{U} \\ \to x = a_1v_1 + a_2v_2 \\ \to \mathbb{U} = \operatorname{span}(v_1, v_2) \end{aligned}$$

8. Proof: $\mathbb{U},\mathbb{W}\subset V\wedge\mathbb{U}\oplus\mathbb{W}=V$, $u_1,\ldots,u_n$是$\mathbb{U}$的基，$w_1,\ldots,w_m$是$\mathbb{W}$的基$\rightarrow u_1,\ldots,u_n,w_1,\ldots,w_m$是$\mathbb{V}$的基

$$\begin{aligned}&\forall x\in V, x=u+w=a_1u_1+\ldots+a_nu_n+b_1w_1+\cdots+b_mw_m\\&\qquad\rightarrow \operatorname{span}(v_1,\ldots v_n,w_1,\ldots,w_m)=V\\&\mathbb{U}\cap\mathbb{W}=\{0\}\rightarrow a_1u_1+\cdots+a_nu_n+b_1w_1+\cdots+b_mw_m=0\\&\qquad\rightarrow a_1,\ldots,a_n,b_1,\ldots,b_m=0\\&\qquad\rightarrow u_1,\ldots,u_n,w_1,\ldots,w_m\text{线性无关}\end{aligned}$$

3 维数(dimension)

定理 3.1. *基的长度不依赖与基的选取*

有限维向量空间的任意两个基的长度相同

证明. $\boldsymbol{b}_1$线性无关$\wedge\operatorname{span}(\boldsymbol{b}_2)=V\rightarrow\operatorname{length}(\boldsymbol{b}_1)\leqslant\operatorname{length}(\boldsymbol{b}_2)$ □

定义 3.2. *维数(dimension)*

有限维向量空间的基的长度称为这个向量空间的维数:$\dim(V)=\operatorname{length}(\boldsymbol{b})$

若V是有限维的，记为dim V

定理 3.3. *子空间的维数*

$\mathbb{U}\subset V\rightarrow\dim\mathbb{U}<\dim\mathbb{V}$

证明. $\mathbb{U}\subset V\rightarrow\mathbb{U}=a_1v_1+\cdots+a_nv_n\rightarrow\dim\mathbb{U}\leqslant\dim\mathbb{V}$ □

定理 3.4. *V中每个长度为dim V的线性无关组都是基*

证明.

$$\begin{aligned}&\text{线性无关组}\boldsymbol{v},\operatorname{span}(\boldsymbol{v})\leqslant\operatorname{span}(\boldsymbol{v},\boldsymbol{b})\\&\operatorname{length}(\boldsymbol{v})=\mathrm{dimV}\rightarrow\operatorname{span}(\boldsymbol{v})=V\end{aligned}$$ □

定理 3.5. *具有适当长度的张成组是基*

若V是有限维的，则V的每个长度为dim V的张成向量组都是V的基

证明.

$$\begin{aligned}&\mathrm{dimV}=n,\operatorname{span}(v_1,\ldots,v_n)=V\\&\qquad\rightarrow\operatorname{span}(v_{j1},\ldots,v_{\mathrm{jn}})=V\\&\qquad\operatorname{length}(\boldsymbol{b})=\mathrm{dimV}\\&\rightarrow v_{j1},\ldots,v_{jn}=v_1,\ldots,v_n\\&\qquad\rightarrow v_1,\ldots,v_n\text{是基}\end{aligned}$$ □

定理 3.6. $\mathbb{U}_1,\mathbb{U}_2\subset V\rightarrow\dim(\mathbb{U}_1+\mathbb{U}_2)=\dim\mathbb{U}_1+\dim\mathbb{U}_2-\dim(\mathbb{U}_1\cap\mathbb{U}_2)$

证明.

$$设\mathbb{U}_1\cap\mathbb{U}_2=\{0\}\to\dim(\mathbb{U}_1+\mathbb{U}_2)=\dim\mathbb{U}_1+\dim\mathbb{U}_2$$
$$设\mathbb{U}_1\cap\mathbb{U}_2\neq\{0\}\to\boldsymbol{b}_{12}=b_1,\ldots,b_n是\mathbb{U}_1\cap\mathbb{U}_2的基\ ???这块需要补充证明\mathbb{U}_1\cap\mathbb{U}_2是个线性空间$$
$$\mathbb{U}_1=\operatorname{span}(\boldsymbol{b}_{12},\boldsymbol{b}_{-1}),\mathbb{U}_2=\operatorname{span}(\boldsymbol{b}_{12},\boldsymbol{b}_{-2})$$
$$\to\dim\mathbb{U}_1=n+\operatorname{length}(\boldsymbol{b}_{-1}),\dim\mathbb{U}_2=n+\operatorname{length}(\boldsymbol{b}_{-2})$$
$$0=a_1v_1+\ldots+a_nv_n+b_1u_{11}+\cdots+b_mu_{1m}+c_1u_{21}+\cdots+c_lu_{2l}$$
$$\to\sum_{i=1}^m b_iu_{1i}=-(\sum_{i=1}^n a_iv_i+\sum_{i=1}^l c_iu_{2l})$$
$$\sum b_iu_{1i}\in\mathbb{U}_1,\sum a_iv_i+\sum c_iu_{2l}\in\mathbb{U}_2 \qquad \square$$
$$又由于\boldsymbol{b}_{-1}\notin\mathbb{U}_2$$
$$\to(\sum b_iu_{1i})\cap(\sum a_iv_i+\sum c_iu_{2l})=\{0\}$$
$$\to b_i=0\to a_i,c_i=0$$
$$\to上式中系数全为0$$
$$\to\dim(\mathbb{U}_1+\mathbb{U}_2)=\dim\mathbb{U}_1+\dim\mathbb{U}_2-\dim(\mathbb{U}_1\cap\mathbb{U}_2)$$
$$???细节有点不太完整，主要在\{\sum b_iu_{1i}\}\cap\mathbb{U}_2=\{0\}这里。书上的没看懂$$

习题2.C

1. Proof:设V是有限维的，$\mathbb{U}\subset V\wedge\dim\mathbb{U}=\dim V\to\mathbb{U}=V$

$$\dim\mathbb{U}=\dim V=n$$
$$\to\operatorname{span}(\boldsymbol{b}_U)=\mathbb{U}$$
$$\mathbb{U}\subset V\to\operatorname{span}(\boldsymbol{b}_U)\subset\operatorname{span}(\boldsymbol{b}_U,\boldsymbol{b}_V)=V$$
$$\boldsymbol{b}_U线性无关\to\operatorname{length}(\boldsymbol{b}_U)=\operatorname{length}(\boldsymbol{b}_V)$$
$$\to\operatorname{span}(\boldsymbol{b}_U)=V$$
$$\to\mathbb{U}=V$$

2. Proof:R^2的子空间为：$\{0\},R^2,R^2$中过原点的所有直线

$$\{0\},R^2\subset R^2$$
$$\forall\mathbb{U}\in R^2\wedge\mathbb{U}\neq\{0\}$$
$$0\in\mathbb{U}$$
$$\to\mathbb{U}=\{(x,y)\in R^2:\forall u,v\in\mathbb{U},u+v\in\mathbb{U},\forall\lambda\in F,\lambda u\in\mathbb{U}\}$$
$$\mathbb{U}=\{(x,y)\in R^2\},\forall\lambda\in F,\forall x\in\mathbb{U},\lambda x\in\mathbb{U}$$
$$\to\mathbb{U}至少包含一条直线$$
$$\forall y\notin\lambda x\wedge y\in\mathbb{U}\to x,y线性无关$$
$$\operatorname{span}(x,y)=R^2矛盾$$
$$\to\mathbb{U}为所有过原点的直线$$

3. Proof:R^3的子空间为：$\{0\},R^3,R^3$中过原点的所有直线,R^3中过原点的所有平面

$$\{0\},R^3\subset R^3$$
$$\mathbb{U}^1=\{(x,y,z)\in R^3\},\forall\lambda\in F,\forall v\neq0\in\mathbb{U}^1\to\lambda v\in\mathbb{U}^1$$
$$\to\mathbb{U}^1至少包含一条直线$$
$$\forall u\notin\{\lambda v\}\wedge u\in\mathbb{U}^2\to u,v线性无关$$
$$\dim(\operatorname{span}(u,v))=2\neq\dim R^3$$
$$\forall x,y\in\mathbb{U}^2,x+y=(a_1,a_2,a_3)+(b_1,b_2,b_3)$$
$$Ax+By+Cz=0\to$$
$$A(a_1+b_1)+B(a_2+b_2)+C(a_3+b_3)=Aa_1+Ba_2+Ca_3+Ab_1+Bb_2+Cb_3=0$$
$$\to x+y\in\mathbb{U}$$
$$\forall\lambda\in F,v\in\mathbb{U},\lambda v=(\lambda a_1,\lambda a_2,\lambda a_3)$$
$$A\lambda a_1+B\lambda a_2+C\lambda a_3=\lambda(Aa_1+Ba_2+Ca_3)=\lambda0=0$$
$$\to\lambda v\in\mathbb{U}^2$$
$$\to\mathbb{U}^2\subset V$$

$$\forall z\neq0\in\mathbb{U}^3\wedge z\notin ax+by$$
$$\operatorname{span}(x,y,z)=R^3矛盾$$
$$\to R^3的子空间\{0\},R^3中过原点的直线，R^3中过原点的平面$$

4.

a. 设$\mathbb{U}=\{p\in\mathcal{P}_4(F)\colon p(6)=0\}$，求$\mathbb{U}$的一个基

$$
\begin{gathered}
p(6)=0\rightarrow p=q(x-6)\\
\mathbb{U}=\{p\in\mathcal{P}_4(F)\colon p=q(x-6)\}\\
q\in\mathcal{P}_3(F)\\
(x-6),(x-6)x,(x-6)x^2,(x-6)x^3
\end{gathered}
$$

b. 扩充上题结果到$\mathcal{P}_4(F)$的基

$$
\begin{gathered}
\mathcal{P}_4(F)\text{有一个基}1,z,z^2,z^3,z^4\\
1,(x-6),(x-6)x,(x-6)x^2,(x-6)x^3
\end{gathered}
$$

c. 求$\mathcal{P}_4(F)$的一个子空间$\mathbb{W}\rightarrow\mathcal{P}_4(F)=\mathbb{W}\oplus\mathbb{U}$

$\mathbb{W}=\mathrm{span}(1)=\{f\colon f(x)=C\}$

5.

a. $\mathbb{U}=\{p\in\mathcal{P}_4(F)\colon p''(6)=0\}$,求$\mathbb{U}$的一个基

$$
\begin{gathered}
p\in\mathcal{P}_4\rightarrow p''\in\mathcal{P}_2\\
(fg)'=f'g+fg'\\
p''(6)=0\rightarrow p''=a(x+b)(x-6)\\
\forall f\in\mathcal{P}_n(F),\,f'\in\mathcal{P}_{n-1}(F),\,f''\in\mathcal{P}_{n-2}(F)\\
(fg)''=(f'g+fg')'=f''g+f'g'+f'g'+fg''\\
\\
p''(6)=(x-6)q,\,q\in\mathcal{P}_1,\forall f\in q,\,f=a\,(x+b)\\
\int(x-6)(ax+b)\mathrm{d}x=Ax^3+Bx^2+Cx+D\\
ax^2+(b-6a)x-6\\
\iint(x-6)(ax+b)\mathrm{d}x=Ax^4+Bx^3+Cx^2+Dx+E\\
\\
\mathrm{span}(1,x,x^2,\textstyle\iint(x-6)(ax+b),(x-6)x^3)
\end{gathered}
$$

???没找到微分的对应的基???等Riesz表示定理学了再回来整活

b. 扩充上题结果到$\mathcal{P}_4(F)$的基

c. 求$\mathcal{P}_4$的子空间$\mathbb{W}\rightarrow\mathcal{P}_4(F)=\mathbb{W}\oplus\mathbb{U}$

6.

a. $\mathbb{U}=\{p=\mathcal{P}_4(F)\colon p(2)=p(5)\}$,求一个基

$$
\begin{gathered}
p(2)=p(5)=c\\
p(2)-c=p(5)-c=0\\
\rightarrow p-c=q_1\in G(x-2)\\
p-c=q_2\in G(x-5)\\
\rightarrow q\in\mathrm{span}((x-2)(x-5),(x-2)(x-5)x,(x-2)(x-5)x^2)\\
\rightarrow p(2)+p(2)\in\mathbb{U}\rightarrow p(2)=p(5)=0\rightarrow c=0\\
p=\mathrm{span}((x-2)(x-5),(x-2)(x-5)x,(x-2)(x-5)x^2)
\end{gathered}
$$

b. 扩充上题结果到$\mathcal{P}_4(F)$的基

$\mathrm{span}(1,x,(x-2)(x-5),(x-2)(x-5)x,(x-2)(x-5)x^2)$

c. 求$\mathcal{P}_4(F)$的子空间$\mathbb{W}\rightarrow\mathcal{P}_4(F)=\mathbb{W}\oplus\mathbb{U}$

$\mathbb{W}=\mathrm{span}(1,x)$

7.

a. $\mathbb{U}=\{p\in\mathcal{P}_4(F)\colon p(2)=p(5)=p(6)\}$，求一个基

$$(p+p)(2)=(p+p)(5)=(p+p)(6)$$
$$p(2)=p(5)=p(6)=0$$
$$\mathbb{U}=\operatorname{span}((x-2)(x-5)(x-6),(x-2)(x-6)(x-5)x)$$

b. 扩充上题结果到$\mathcal{P}_4(F)$的基

$\operatorname{span}(1,x,x^2,(x-2)(x-5)(x-6),(x-2)(x-5)(x-6)x)$

c. 求$\mathcal{P}_4(F)$的子空间$\mathbb{W}\rightarrow\mathcal{P}_4(F)=\mathbb{W}\oplus\mathbb{U}$

$\mathbb{W}=\operatorname{span}(1,x,x^2)$

8.

a. $\mathbb{U}=\left\{p\in\mathcal{P}_4(F)\colon\int_{-1}^{1}p=0\right\}$，求一个基

等Riesz表示定理学了再找这个积分对应的线性元素...
太菜了。。。

b. 扩充上题结果到$\mathcal{P}_4(F)$的基

c. 求$\mathcal{P}_4(F)$的子空间$\mathbb{W}\rightarrow\mathcal{P}_4(F)=\mathbb{W}\oplus\mathbb{U}$

9. Proof:$v_1,\ldots,v_n$在V中线性无关，$\forall w\in V\rightarrow\dim(\operatorname{span}(v_1+w,\ldots,v_n+w))\geqslant n-1$

$$\operatorname{span}(v_1+w,\ldots,v_n+w)\leqslant\operatorname{span}(v_1,\ldots,v_n)$$
$$w\in\operatorname{span}(v_1,\ldots,v_n)$$
$$w=a_1x_1+\cdots+a_nx_n$$
$$0=\sum a_i(x_i+w)=\sum a_ix_i+\sum a_iw$$
$$-\sum a_iw=\sum a_ix_i$$
$$(-\sum a_i)(t_1x_0+\cdots+t_nx_n)=\sum a_ix_i$$
$$\rightarrow(-\sum a_i)t_i=a_i$$
$$-\sum a_i=0\rightarrow a_i=0$$

10. Proof:$p_0,\ldots,p_n\in\mathcal{P}(F)\wedge\deg(p_j)=j\rightarrow p_0,\ldots,p_n$是$\mathcal{P}_n(F)$的基

$$\forall p_i\in\mathcal{P}_n(F)$$
$$\forall a\neq0\in F\rightarrow a\,p_i=a\,(a_0+a_1x+\cdots+a_ix^i)$$
$$=a\,a_0+\cdots+a\,a_nx^n\in\mathcal{P}_i(F)$$
$$\rightarrow\forall p_i,p_j\in p_n\text{线性无关}$$
$$\dim(\mathcal{P}_n(F))=n\rightarrow\operatorname{length}(\boldsymbol{b})=n$$
$$n=\operatorname{length}(p)$$
$$\rightarrow p\in\boldsymbol{B}$$

11. Proof:$\mathbb{U},\mathbb{W}\subset R^8,\dim\mathbb{U}=3,\dim\mathbb{W}=5,\mathbb{U}+\mathbb{W}=R^8\rightarrow R^8=\mathbb{U}\oplus\mathbb{W}$

$$\dim\mathbb{W}=5\rightarrow\mathbb{W}=\operatorname{span}(w_1,\ldots,w_5)$$
$$\mathbb{U}=\operatorname{span}(u_1,u_2,u_3)$$
$$\mathbb{U}+\mathbb{W}=\{v\in R^8\colon v=\sum a_iu_i+\sum b_iw_i\}$$
$$\forall v\in R^8\rightarrow v=\sum a_iv_i$$
$$R^8=\operatorname{span}(v_i)$$
$$\dim R^8=8\rightarrow\operatorname{length}\boldsymbol{b}=8$$
$$\operatorname{length}v_i=8$$
$$\rightarrow v_i\text{是基}\rightarrow v_i\text{线性无关}\rightarrow\mathbb{U}\cap\mathbb{W}=\{0\}$$

12. Proof:$\mathbb{U}, \mathbb{W} \subset R^9, \dim\mathbb{U} = \dim\mathbb{W} = 5 \rightarrow \mathbb{U} \cap \mathbb{W} \neq \{0\}$

$$\begin{gathered}\dim\mathbb{U} = \dim\mathbb{W} = 5 \rightarrow \mathbb{U} = \operatorname{span}(u_i), \mathbb{W} = \operatorname{span}(w_i)\\ \mathbb{U} + \mathbb{W} \subset R^9 \rightarrow \dim(\mathbb{U} + \mathbb{W}) \leqslant 9\\ \mathbb{U} + \mathbb{W} = \operatorname{span}(\boldsymbol{v}), \text{length}\, \boldsymbol{v} \leqslant 9\\ \forall a \in \mathbb{U} + \mathbb{W} = \sum a_i u_i + \sum b_i w_i\\ \rightarrow \mathbb{U} + \mathbb{W} = \operatorname{span}(\boldsymbol{u}, \boldsymbol{w})\\ \text{length}\mathbb{U} + \text{length}\mathbb{W} = 10 > 9\\ \rightarrow \boldsymbol{u}, \boldsymbol{w}\text{线性相关}\\ \rightarrow u_0 = \sum t_i w_i\\ \mathbb{U} \cap \mathbb{W} = \operatorname{span}(u_0) \neq \varnothing\end{gathered}$$

13. Proof:$\mathbb{U}, \mathbb{W} \subset \mathbb{C}^6, \dim\mathbb{U} = \dim\mathbb{W} = 4 \rightarrow \exists x, y \in \mathbb{U} \cap \mathbb{W} \wedge x \neq y \rightarrow \forall \lambda \in F, \lambda x \neq y$

$$\begin{gathered}\dim\mathbb{U} = \dim\mathbb{W} = 4 \rightarrow \mathbb{U} = \operatorname{span}(\boldsymbol{u}), \mathbb{W} = \operatorname{span}(\boldsymbol{w})\\ \mathbb{U} + \mathbb{W} \subset \mathbb{C}^6 \rightarrow \dim(\mathbb{U} + \mathbb{W}) \leqslant 6\\ u_1, u_2 \in \boldsymbol{u}\text{: } u_1 = \sum p_i w_i, u_2 = \sum q_i w_i\\ \text{只有这三种情况:}\\ \mathbb{U} \cap \mathbb{W} = \operatorname{span}(u_1, u_2, u_3, u_4) \rightarrow\\ \mathbb{U} \cap \mathbb{W} = \operatorname{span}(u_1, u_2, u_3) \rightarrow\\ \mathbb{U} \cap \mathbb{W} = \operatorname{span}(u_1, u_2) \rightarrow\end{gathered}$$

$$\begin{gathered}\text{设不存在这两个向量} \rightarrow \forall x, y \in \mathbb{U} \cap \mathbb{V} \wedge x \neq y \rightarrow \exists \lambda \neq 0 \in F, \lambda x = y\\ \rightarrow \dim(\mathbb{U} \cap \mathbb{V}) = 1\\ \text{与}\dim(\mathbb{U} \cap \mathbb{V}) \geqslant 2\text{矛盾}\\ \rightarrow \mathbb{U} \cap \mathbb{W}\text{至少有两个线性无关的向量组}\end{gathered}$$

14. Proof:$\mathbb{U}_1, \ldots, \mathbb{U}_n \subset V$均为有限维向量空间$\rightarrow \mathbb{U}_1 + \cdots + \mathbb{U}_n$有限维$\wedge \dim(\mathbb{U}_1 + \ldots + \mathbb{U}_n) \leqslant \dim\mathbb{U}_1 + \cdots + \dim\mathbb{U}_n$

$$\begin{gathered}\mathbb{U}_i\text{有限维} \rightarrow \mathbb{U}_i = \operatorname{span}(u_0, \ldots, u_n)\\ \mathbb{U}_1 + \cdots + \mathbb{U}_n = \operatorname{span}(u_{11}, \ldots u_{1n}, u_{21} \ldots, \ldots, u_{nn}). \operatorname{card}(\bigcup \boldsymbol{u}_i) = \sum \operatorname{card}(\boldsymbol{u}_i) \in N^+\\ \rightarrow \mathbb{U}_1 + \ldots + \mathbb{U}_n\text{有限维}\\ \mathbb{U}_1 + \cdots + \mathbb{U}_n = \operatorname{span}(\bigcup \boldsymbol{u}_i) \leqslant \sum \operatorname{span}(\boldsymbol{u}_i)\text{成立}\end{gathered}$$

15. Proof:V是有限维的$\wedge \dim\text{V} = n \geqslant 1 \rightarrow \exists \mathbb{U}_1, \ldots, \mathbb{U}_n \rightarrow V = \mathbb{U}_1 \oplus \cdots \oplus \mathbb{U}_n$

16. Proof:$\mathbb{U}_1, \ldots, \mathbb{U}_n$为$V$的一维子空间$\wedge \mathbb{U}_1 \oplus \cdots \oplus \mathbb{U}_n \rightarrow \dim(\mathbb{U}_1 \oplus \cdots \oplus \mathbb{U}_n) = \dim\mathbb{U}_1 + \cdots + \dim\mathbb{U}_n < \infty$

$$\begin{gathered}\mathbb{U}_i = \operatorname{span}(u_i) \rightarrow \mathbb{U}_1 + \ldots + \mathbb{U}_n = \operatorname{span}(\boldsymbol{u})\\ \mathbb{U}_1 \oplus \cdots \oplus \mathbb{U}_n \rightarrow \boldsymbol{u}\text{线性无关}\\ \dim(\mathbb{U}_1 \oplus \ldots \oplus \mathbb{U}_n) = \dim(\operatorname{span}(\boldsymbol{u})) = \text{length}\boldsymbol{u} = n\end{gathered}$$

17. Proof or Counter:$\dim(\mathbb{U}_1 + \mathbb{U}_2 + \mathbb{U}_3) = \dim\mathbb{U}_1 + \dim\mathbb{U}_2 + \dim\mathbb{U}_3 - \dim(\mathbb{U}_1 \cap \mathbb{U}_2) - \dim(\mathbb{U}_1 \cap \mathbb{U}_3) - \dim(\mathbb{U}_2 \cap \mathbb{U}_3) + \dim(\mathbb{U}_1 \cap \mathbb{U}_2 \cap \mathbb{U}_3)$

$$\begin{gathered}\dim(\mathbb{U}_1 + \mathbb{U}_2 + \mathbb{U}_3) = \dim((\mathbb{U}_1 + \mathbb{U}_2) + \mathbb{U}_3)\\ = \dim(\mathbb{U}_1 + \mathbb{U}_2) + \dim\mathbb{U}_3 - \dim((\mathbb{U}_1 + \mathbb{U}_2) \cap \mathbb{U}_3)\\ = \dim(\mathbb{U}_1 + \mathbb{U}_2) + \dim\mathbb{U}_3 - \dim(\mathbb{U}_1 \cap \mathbb{U}_3 + \mathbb{U}_2 \cap \mathbb{U}_3)\\ = \dim\mathbb{U}_1 + \dim\mathbb{U}_2 - \dim(\mathbb{U}_1 \cap \mathbb{U}_2) + \dim\mathbb{U}_3\\ - \dim(\mathbb{U}_1 \cap \mathbb{U}_3) - \dim(\mathbb{U}_2 \cap \mathbb{U}_3) + \dim(\mathbb{U}_1 \cap \mathbb{U}_3 \cap \mathbb{U}_2 \cap \mathbb{U}_3)\\ = \dim\mathbb{U}_1 + \dim\mathbb{U}_2 + \dim\mathbb{U}_3 - \dim(\mathbb{U}_1 \cap \mathbb{U}_2) - \dim(\mathbb{U}_1 \cap \mathbb{U}_3) - \dim(\mathbb{U}_2 \cap \mathbb{U}_3)\\ + \dim(\mathbb{U}_1 \cap \mathbb{U}_2 \cap \mathbb{U}_3)\end{gathered}$$

$$\begin{gathered}\text{这块需要补充证明}(\mathbb{U}_1 + \mathbb{U}_2) \cap \mathbb{U}_3 = \mathbb{U}_1 \cap \mathbb{U}_3 + \mathbb{U}_2 \cap \mathbb{U}_3\\ ???\end{gathered}$$