C1

1 单元练习1.1

1.1.1

1. $f(x)=\frac{x}{\sqrt{1+x^2}}$.求$f_n(x)=f\circ f\circ\cdots\circ f(x)$

$$
\begin{aligned}
f_2(x) &= f\circ f(x)\\
&= f\left(\frac{x}{\sqrt{1+x^2}}\right)\\
&= \frac{\frac{x}{\sqrt{1+x^2}}}{\sqrt{1+\frac{x^2}{1+x^2}}}\\
&= \frac{\frac{x}{\sqrt{1+x^2}}}{\sqrt{\frac{1+2x^2}{1+x^2}}}\\
&= \frac{x}{\sqrt{1+2x^2}}\\
f_3(x) &= f(f_2(x))\\
&= f\left(\frac{x}{\sqrt{1+2x^2}}\right)\\
&= \frac{\frac{x}{\sqrt{1+2x^2}}}{\sqrt{1+\frac{x^2}{1+2x^2}}}\\
&= \frac{x}{\sqrt{1+3x^2}}\\
f_n(x) &= \frac{x}{\sqrt{1+nx^2}}\\
f_{n+1}(x) &= f(f_n(x))\\
&= f\left(\frac{x}{\sqrt{1+nx^2}}\right)\\
&= \frac{\frac{x}{\sqrt{1+nx^2}}}{\sqrt{1+\frac{x^2}{1+nx^2}}}\\
&= \frac{x}{\sqrt{1+(n+1)x^2}}\\
&\to\ f_n(x)=\frac{x}{\sqrt{1+nx^2}}
\end{aligned}
$$

2. $f(x)=\frac{x}{x-1}$. Pf: $f\circ\cdots\circ f(x)=f(x)$. 求$f\left(\frac{1}{f(x)}\right), (x\neq 0, x\neq 1)$

$$
\begin{gathered}
f\circ f=f\left(\frac{x}{x-1}\right)\\
=\frac{\frac{x}{x-1}}{\frac{x}{x-1}-1}=\frac{\frac{x}{x-1}}{\frac{x-x+1}{x-1}}\\
=\frac{x}{1}=x\\
f\circ f\circ f(x)=f(f\circ f(x))\\
=f(x)\\
f\left(\frac{1}{f(x)}\right)=f(g(f(x)))\\
=f\left(g\left(\frac{x}{x-1}\right)\right)\\
=f\left(\frac{x-1}{x}\right)\\
=\frac{\frac{x-1}{x}}{\frac{x-1}{x}-1}=\frac{x-1}{-1}=1-x
\end{gathered}
$$

1.1.2是否存在函数在[0,1]内都取有限值(并非有界)，在此区间的任意点的任意领域内无界

$$
\begin{aligned}
&\text{考虑} f(x) = \begin{cases} n & x = \frac{m}{n}. m/n\text{互质}, n > 0 \\ 0 & x \notin Q \end{cases} \\
&\forall x \in [0,1], f(x) = n \in R\text{有限} \\
&\forall x \in [0,1], \exists \frac{m}{n} \in U_x(r) \\
&\rightarrow \frac{2m+1}{2n} \in U_x(r) \\
&\rightarrow \frac{km+1}{kn} \in U_x(r) \\
&\rightarrow f\text{在}U_x(r)\text{内无界}
\end{aligned}
$$

1.1.3 说明有无穷多函数，$f \circ f = I_R$

$$
\begin{aligned}
&f \circ f = I_R\text{，}I_R\text{可逆} \rightarrow f\text{可逆} \\
&(f \circ f)^{-1} = f^{-1} \circ f^{-1} = I_R^{-1} = I_R \\
&g\colon (0,+\infty) \rightarrow (-\infty,0). g(x) = -x \\
&f = \begin{cases} g(x) & (0,+\infty) \\ 0 & 0 \\ g^{-1}(x) & (-\infty,0) \end{cases} \\
&f \circ f = f\left(\begin{cases} g(x) < 0 & (0,+\infty) \\ 0 & 0 \\ g^{-1}(x) > 0 & (-\infty,0) \end{cases} \right) \\
&= \begin{cases} f \circ g(x) \\ f(0) \\ f \circ g^{-1}(x) \end{cases} = \begin{cases} \begin{cases} g \circ g(x) & g(x) > 0 & \text{不存在的} \\ 0 & g(x) = 0 & \text{不存在的} \\ g^{-1} \circ g(x) & g(x) < 0 \end{cases} \\ 0 \\ \begin{cases} g \circ g(x) & g(x) > 0 & \text{不存在的} \\ 0 & g(x) = 0 & \text{不存在的} \\ g \circ g^{-1} & g(x) < 0 \end{cases} \end{cases} \\
&\rightarrow f \circ f = I_R
\end{aligned}
$$

1.14 f是R上的基函数，$f(1) = a; \forall x \in R$, $f(x+2) - f(x) = f(2)$

1. 用a表示$f(2)$和$f(5)$

$$
\begin{aligned}
&f(x+2) - f(x) = f(2) \\
&f(1+2) - f(2) = f(2) \\
&\rightarrow f(3) = 2f(2) \\
&f(-1+2) - f(-1) = f(2) \\
&f(1) - f(-1) = f(2) \\
&\rightarrow 2f(1) = f(2) \\
&f(2) = 2a \\
&f(5) = f(1+2) - f(1) = f(2) \\
&f(3) = f(2) + f(1) = 3a \\
&f(3+2) - f(3) = f(2) \\
&f(5) = f(2) + f(3) \\
&= 2a + 3a = 5a
\end{aligned}
$$

2. 求a的值使得2是f的周期函数

$$
\begin{aligned}
&f(x+2) = f(x) \\
&\rightarrow f(2) = 0 \\
&f(2) = 2a = 0 \rightarrow a = 0
\end{aligned}
$$

1.1.5 $f(x) = x - [x]. g(x) = \tan x.$说明$f + g$; $f - g$是不是周期函数

π是无理数，不能被整数给穷举到$k\pi$

$\to f+g; f-g$都不是周期函数

2 单元练习1.2

$$\lim_{n\to\infty} x_n = a. \text{Pf}: \sqrt[3]{x_n} = \sqrt[3]{a}$$

连续性 $\sqrt[3]{x}$是连续的 $\to \lim_{n\to\infty}\sqrt[3]{x_n} = \sqrt[3]{\lim_{n\to\infty} x_n} = \sqrt[3]{a}$

定义 $|\sqrt[3]{x_n} - \sqrt[3]{a}|$; $|x_n - a| = (\sqrt[3]{x_n})^3 - (\sqrt[3]{a})^3 = (\sqrt[3]{x_n} - \sqrt[3]{a})((\sqrt[3]{x_n})^2 + \sqrt[3]{x_n}\sqrt[3]{a} + (\sqrt[3]{a})^2)$

1.2.1

$$\to |\sqrt[3]{x_n} - \sqrt[3]{a}| = \frac{|x_n - a|}{(\sqrt[3]{x_n})^2 + \sqrt[3]{x_n}\sqrt[3]{a} + (\sqrt[3]{a})^2}$$
$$= \frac{|x_n - a|}{(\sqrt[3]{x_n} + \frac{1}{2}\sqrt[3]{a})^2 + \frac{3}{4}(\sqrt[3]{a})^2} \leqslant \frac{|x_n - a|}{\frac{3}{4}(\sqrt[3]{a})^2}$$
$$\text{let}: \varepsilon = \frac{3}{4}(\sqrt[3]{a})^{-2}\delta$$
$$\delta = d(\sqrt[3]{x_n}, \sqrt[3]{a}) \leqslant \frac{3}{4}(\sqrt[3]{a})^{-2}\delta = \varepsilon$$

1.2.2 用$\varepsilon - N$证明

$$\lim_{n\to\infty}\sqrt[n]{n} = 1$$

对数连续性

$$\ln(\sqrt[n]{n}) = \frac{1}{n}\ln n$$
$$\ln(\lim_{n\to\infty}\sqrt[n]{n}) = \lim_{n\to\infty}\ln\sqrt[n]{n}$$
$$= \lim_{n\to\infty}\frac{1}{n}\ln n$$
$$= \lim_{n\to\infty}\frac{(\ln n)'}{(n)'} = \lim_{n\to\infty}\frac{(\frac{1}{n})}{1} = \lim_{n\to\infty}\frac{1}{n} = 0 \qquad L'H$$
$$\to \ln(\lim_{n\to\infty}\sqrt[n]{n}) = 0 = \ln(1)$$
$$\to \lim_{n\to\infty}\sqrt[n]{n} = 1 \qquad \ln\text{可逆(单)}$$

1.

$\varepsilon - N$

$$|\sqrt[n]{n} - 1| = |\sqrt[n]{n} - \sqrt[n]{1}|$$
$$n - 1 = (\sqrt[n]{n})^n - (\sqrt[n]{1})^n$$
$$= (\sqrt[n]{n} - \sqrt[n]{1})((\sqrt[n]{n})^{n-1} + \cdots + \sqrt[n]{n})$$
$$\to d(\sqrt[n]{n}, 1) = \frac{n-1}{(\sqrt[n]{n})^{n-1} + \cdots + \sqrt[n]{n}}$$
$$\leqslant \frac{n-1}{(\sqrt[n]{n})^{n-1}} = \frac{n-1}{n^{1-\frac{1}{n}}}$$
$$\lim_{n\to\infty}\frac{n-1}{n^{\frac{n-1}{n}}} = \frac{\lim n - 1}{\text{pow}(\lim n, \lim\frac{n-1}{n})} = \lim\frac{n-1}{n^1} = 1$$
$$\to$$

2.

$$\lim_{n\to\infty}\frac{n^3}{q^n} = 0. \ |q| < 1$$

对数

$$\ln\left(\lim_{n\to\infty}\frac{n^3}{q^n}\right) = \lim_{n\to\infty}\ln\frac{n^3}{q^n}$$
$$= \lim_{n\to\infty}(\ln n^3 - \ln q^n)$$
$$\lim_{n\to\infty}(3\ln n - n\ln q)$$
$$\lim_{n\to\infty}\frac{3}{\ln q}\cdot\frac{\ln n}{n} = 0$$
$$\to \lim_{n\to\infty} 3\ln n - n\ln q = -\infty$$
$$\to \ln\left(\lim_{n\to\infty}\frac{n^3}{q^n}\right) = -\infty \to \lim_{n\to\infty} n^3 q^n = 0$$

3

$$\lim_{n\to\infty}\frac{\ln n}{n^2}=0$$
$$\lim_{n\to\infty}\frac{\ln n}{n}\cdot\frac{1}{n}\leqslant\frac{1}{n}$$
$$\varepsilon=\frac{1}{\delta}\to\lim_{n\to\infty}\frac{\ln n}{n^2}=0$$

1.2.3

$$设\lim_{n\to\infty}a_n=a.\ x_n=\frac{a_1+2a_2+\cdots+na_n}{1+2+\cdots+n}.\ \text{Pf: 用}\varepsilon-N\ \lim_{n\to\infty}x_n=a$$

$$\text{两段法:}$$
$$1\quad |x_n-a|=\left|\frac{a_1+2a_2+\cdots+na_n}{1+2+\cdots+n}-a\right|=\left|\frac{(a_1-a)+2(a_2-a)+\cdots+n(a_n-a)}{1+2+\cdots+n}\right|$$
$$\leqslant\frac{2}{n(n+1)}(|a_1-a|+2|a_2-a|+\cdots+n|a_n-a|)$$
$$\lim_{n\to\infty}a_n=a\to\forall\varepsilon>0,\exists N\in N^+,n>N_1\to|a_n-a|<\varepsilon$$
$$\to\frac{2}{n(n+1)}\left(\sum_{i=1}^{N}i(a_i-a)+\sum_{i=N+1}^{n}i\varepsilon\right)$$
$$=\frac{2}{n(n+1)}\sum_{i=1}^{N}i(a_i-a)+\frac{2}{n(n+1)}\sum_{i=N+1}^{n}i\varepsilon$$
$$\leqslant\frac{2}{n(n+1)}\sum_{i=1}^{N}i(a_i-a)+\frac{2}{n(n+1)}\sum_{i=1}^{n}i\varepsilon$$
$$=\frac{2}{n(n+1)}\sum_{i=1}^{N}i(a_i-a)+\frac{2}{n(n+1)}\cdot\frac{n(n+1)}{2}\cdot\varepsilon$$
$$\text{对于}\sum_{i=1}^{N}i(a_i-a)\text{是前}N\text{项差是有限的}\to\exists n>N_2\to\frac{2}{n(n+1)}\sum_{i=1}^{N}i(a_i-a)<\varepsilon$$
$$\to n=\max(N_1,N_2),\ |x_n-a|<2\varepsilon$$
$$\to\lim_{n\to\infty}x_n=a$$

1.2.4

$$x_n=\sum_{k=2}^{n}\frac{\cos k}{k(k-1)}.\ \text{Pf: }x_n\text{收敛}$$
$$|x_n-x_m|=\left|\sum_{k=2}^{n}\frac{\cos k}{k(k-1)}-\sum_{k=2}^{m}\frac{\cos k}{k(k-1)}\right|$$
$$=\left|\sum_{k=m+1}^{n}\frac{\cos k}{k(k-1)}\right|$$
$$\left|\frac{\cos k}{k(k-1)}\right|=\frac{|\cos k|}{k(k-1)}\leqslant\frac{1}{k(k-1)}=\frac{1}{k-1}-\frac{1}{k}$$
$$\sum_{k=m+1}^{n}\left(\frac{1}{k-1}-\frac{1}{k}\right)=\left(\frac{1}{m}-\frac{1}{m+1}+\frac{1}{m+1}-\frac{1}{m+2}+\cdots+\frac{1}{n-1}-\frac{1}{n}\right)$$
$$=\left(\frac{1}{m}-\frac{1}{n}\right)<\frac{1}{m}$$
$$\varepsilon=\frac{1}{N}.\ n>N\text{时}|x_n-x_m|<\varepsilon$$
$$\to x_n\text{是cauchy序列}\to x_n\text{收敛}$$

1.2.5

$$\text{数列}\{a_n\}\text{若}\lim_{n\to\infty}\frac{\sum a_i}{n}=a\in R.\,\text{Pf: }\lim_{n\to\infty}\frac{a_n}{n}=0$$

1
$$\lim_{n\to\infty}\frac{\sum a_i}{n}=a\to\lim_{n\to\infty}a_n=a$$
$$\to\lim_{n\to\infty}\frac{a_n}{n}=\frac{\lim_{n\to\infty}a_n}{\lim_{n\to\infty}n}=\frac{a}{\lim_{n\to\infty}n}=0$$

2
$$\text{Remark: }\frac{a_n}{n}=\frac{\sum_1^n a_i}{n}-\frac{\sum_1^{n-1}a_i}{n-1}\cdot\frac{n-1}{n}$$
$$\lim_{n\to\infty}\frac{a_n}{n}=\lim\frac{\sum_1^n a_i}{n}-\lim\frac{\sum_1^{n-1}a_i}{n-1}\cdot\lim\frac{n-1}{n}$$
$$=a-\frac{a}{n-1}\cdot 1=a$$