Chapter 18

BY 隐函数定理及应用

根据闭图像定理，函数的所有图像$\{(x,f(x))|x\in D\}$在二维空间是一维子空间

对于某些导数为无穷的点采用高维空间理论处理

弥补了所有存在切线的点(切线平行于y轴的点)的微分理论

1 隐函数

1. 二元函数的隐函数：

$$F(x,y)\colon R^2\to R$$
$$F(x,y)=0\text{方程的解集构成了}x,y\text{平面上的曲线}$$
$$\exists I,J\subset R,\exists x\in I,\exists\text{唯一}y\in J\to(x,y)\in E\wedge F(x,y)=0$$
$$\text{称方程确定了}I\text{上的，值域在}J\text{内的隐函数}$$
$$\text{Ex}\quad\text{这个定义可以定义出一大类隐函数}f(x)\begin{cases}\sqrt{1-x^2} & x\in Q\\ -\sqrt{1-x^2} & x\notin Q\end{cases}$$

2. 隐函数存在、唯一性定理：

$$\begin{cases}F\text{在}P_0\text{的某个邻域连续}\\ F(P_0)=0\\ F\text{在}U_{P_0}\text{内存在连续偏导数}F_y\\ F_y(P_0)\neq 0\end{cases}\Rightarrow\begin{cases}F\text{在}U_{P_0}\text{上唯一确定了}U_{x_0}\text{上的函数}y=f(x)\\ f(x)\text{在}U_{x_0}\text{上连续}\end{cases}$$

Pr

$$\text{存在性}$$
$$\text{由于}F_y(P_0)\neq 0\text{不妨设}F_y(P_0)>0$$
$$F_y>0\to\text{在}[x_0-\beta,x_0+\beta]\times[y_0-\beta,y_0+\beta]\text{上}F_y>0$$
$$\to\forall x\in[x_0-\beta,x_0+\beta],F(x,y_0-\beta)<0\wedge F(x,y_0+\beta)>0$$
$$F\text{连续}\to F(x,y_0-\beta),F(x,y_0+\beta)\text{连续}$$
$$\to\forall y\in[y_0-\beta,y_0+\beta],\exists\text{唯一}x\in[x_0-\beta,x_0+\beta]\wedge F(x,y)=0$$
$$\text{连续性}$$
$$F\text{在}[x_0-\beta,x_0+\beta]\times[y_0-\beta,y_0+\beta]\text{上一致连续}$$
$$\to\exists\varepsilon>0,d(P-P_0)<\delta\to|y-y_0|<\varepsilon$$
$$\to|x-x_0|<\delta\to|y-y_0|<\varepsilon$$

Re

$$\text{此定理只是充分条件、不是必要条件}$$
$$y^3-x^3=0,\text{在}(0,0)F_y=0;\text{但仍然确定唯一的函数}y=x$$

3. 隐函数可微性定理

$$\begin{cases}\text{隐函数存在、唯一定理}\\ F_x\text{ 在}P_0\text{的邻域内连续}\end{cases}\Rightarrow\begin{cases}\text{隐函数在}U_{x_0}\text{处可微}\\ f'(x)=-\frac{F_x(P_0)}{F_y(P_0)}\end{cases}$$

Pr

$$F(x,y)=0;F(x+\Delta x,y+\Delta y)=0$$
$$0=F(x+\Delta x,y+\Delta y)-F(x,y)$$
$$=F_x(x+\theta\Delta x,y+\theta\Delta y)\Delta x+F_y(x+\theta\Delta x,y+\theta\Delta y)\Delta y$$
$$\Rightarrow\frac{\Delta y}{\Delta x}=-\frac{F_x(x+\theta\Delta x,y+\theta\Delta y)}{F_y(x+\theta\Delta x,y+\theta\Delta y)}$$
$$\text{取极限，由于}F_x,F_y\text{的连续性得到}\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F_x(P_0)}{F_y(P_0)}$$

4. 隐函数的高阶导数

$$
\begin{aligned}
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{F_x}{F_y}\right)\\
&= -\frac{\left(F_{xx}+F_{xy}\frac{\mathrm{d}y}{\mathrm{d}x}\right)F_y - F_x\left(F_{yx}+F_{yy}\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{(F_y)^2}\\
&= -\frac{F_{xx}F_y + F_{xy}\left(-\frac{F_x}{F_y}\right)F_y - F_xF_{yx} - F_{yy}\left(-\frac{F_x}{F_y}\right)F^x}{(F_y)^2}\\
&= -\frac{F_{xx}F_y^2 - F_{xy}F_xF_y - F_xF_{yx}F_y + F_{yy}F_x^2}{(F_y)^3}\\
&= \frac{F_xF_yF_{xy} + F_xF_yF_{yx} - F_{xx}F_y^2 - F_{yy}F_x^2}{(F_y)^3}\\
F_{xy} = F_{yx} &\rightarrow \frac{2F_xF_yF_{xy} - F_{xx}F_y^2 - F_{yy}F_x^2}{F_y^3}
\end{aligned}
$$

5. 隐函数的极值问题:

若隐函数在开集D上可微，则在开集内的隐函数的极值方法使用函数的极值方法求得

设隐函数满足可微性条件

$$F(x,y)=0 \wedge F_y(x,y)\neq 0;$$

$$y_x(x,y)=0 \rightarrow -\frac{F_x}{F_y}=0 \rightarrow F_x(x,y)=0$$

先求y_x为0的点，解方程$\begin{cases} F(x,y)=0 \\ F_x(x,y)=0 \end{cases} \rightarrow P_0$

$$y_{xx}=\frac{2F_xF_yF_{xy}-F_x^2F_{yy}-F_y^2F_{xx}}{F_y^3}, F_x=0 \rightarrow y_{xx}=\frac{-F_y^2F_{xx}}{F_y^3}=-\frac{F_{xx}}{F_y}$$

极值第二条件: $\begin{cases} y_{xx}>0 & \text{在}P_0\text{取得极小值} \\ y_{xx}<0 & \text{在}P_0\text{取得最大值} \\ y_{xx}=0 & \text{需要更高阶导数判断} \end{cases}$

6. 多元函数决定的隐函数:

$$\begin{cases} F(x_1,x_2,\dots,x_n,y)\text{在}P_0\text{的领域内连续} \\ F(P_0)=0 \\ \text{各个偏导数}F_{x_1},F_{x_2},\dots,F_y\text{在}U_{P_0}\text{连续} \\ F_y(P_0)\neq 0 \end{cases} \Rightarrow \begin{cases} \exists U_{P_0}, F\text{唯一确定了函数}y=f(x_i) \\ \exists U_{P_0}, \text{隐函数}y=f(x_i)\text{在}U_{P_0}\text{上连续} \\ f_{x_i}=-\frac{F_{x_i}}{F_y} \end{cases}$$

1.1 隐函数组，(多元向量值函数)

1. 四元两个方程的隐函数组

$$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}, F,G\colon R^4 \rightarrow R$$

若开集$V\subset R^4$上函数，在R^2上开区域E上的点(x,y)

确定唯一的$(u,v)\in E \rightarrow (x,y,u,v)\in V$

$\wedge F(x,y,u,v)=G(x,y,u,v)=0$

称为隐函数组在D上确定的向量函数

2. 函数行列式、Jacobi雅各比行列式

$$n\text{个一元函数关于}n\text{组变量的偏导数构成的矩阵}$$

$$\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}\boldsymbol{x}}=\begin{pmatrix}\frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n}\\ \vdots & \ddots & \vdots\\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n}\end{pmatrix}$$

$$\det\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}=\begin{vmatrix}\frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n}\\ \vdots & & \vdots\\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n}\end{vmatrix}=\frac{\partial(F_1,\cdots,F_n)}{\partial(x_1,\cdots,x_n)}$$

3. 隐函数定理:

$$\begin{cases}F,G\text{在}U_{P_0}\text{上连续}\\ F(P_0)=G(P_0)=0\\ F,G\in\ell'(U_{P_0})\\ J=\frac{\partial(F,G)}{\partial(u,v)}(P_0)\neq 0\end{cases}\Rightarrow\begin{cases}\exists U_{P_0}\text{上唯一确定了两个函数}u=f(x,y);v=g(x,y)\\ u=f,v=g\text{在}U_{P_0}\text{上连续}\\ f,g\in\ell'(U_{P_0});\frac{\partial f}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)};\frac{\partial f}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)};\\ \frac{\partial g}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)};\frac{\partial g}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,y)}\end{cases}$$

4. 一般反函数定理:

$$\text{矩阵}\boldsymbol{F}'(\boldsymbol{a})=\begin{pmatrix}\frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n}\\ \vdots & & \\ \frac{\partial F_n}{\partial x_1} & & \frac{\partial F_n}{\partial x_n}\end{pmatrix}\text{在}P_0\text{可逆}\to\text{在}P_0\text{必存在反函数}$$

$$\begin{cases}F\colon R^n\to R^m,F\in\ell'(E)\\ \boldsymbol{a}\in E,F'(a)=A_F(\boldsymbol{x})\text{可逆},\boldsymbol{b}=\boldsymbol{f}(\boldsymbol{a})\end{cases}\Rightarrow\begin{cases}\exists U,V\subset R^n.F\text{在}U\text{上是}1-1\text{的}\wedge F(U)=V\\ G\colon V\to U,G=F^{-1}.G(F(\boldsymbol{x}))=\boldsymbol{x},G\in\ell'(E)\end{cases}$$

Re：这里的定义域的维数n和像空间的维数一般是$m>n$的否则降维后无法还原

5. 一般隐函数定理:

$$\begin{cases}\text{开集}E\subset R^{n+m}.\boldsymbol{F}\colon E\to R^n.\boldsymbol{F}\in\ell'(E)\\ \exists(\boldsymbol{a},\boldsymbol{b})\in E,\boldsymbol{F}(\boldsymbol{a},\boldsymbol{b})=\boldsymbol{0}\\ A=\boldsymbol{F}'(\boldsymbol{a},\boldsymbol{b})\\ \text{存在一组变量}x_i\text{使得}A_{\boldsymbol{x}}\text{可逆}.\text{假定是前}n\text{个}x_i\end{cases}$$

$$\Rightarrow\begin{cases}\begin{cases}\exists\text{开}U\subset R^{n+m},\exists\text{开}W\subset R^m,(\boldsymbol{a},\boldsymbol{b})\in U,\boldsymbol{b}\in W\\ \forall\boldsymbol{y}\in W,\exists\text{唯一}\boldsymbol{x},(\boldsymbol{x},\boldsymbol{y})\in U\wedge\boldsymbol{F}(\boldsymbol{x},\boldsymbol{y})=\boldsymbol{0},\text{确定函数}\boldsymbol{x}=\boldsymbol{G}(\boldsymbol{y})\end{cases}\\ \begin{cases}\boldsymbol{G}\colon W\to R^n.\boldsymbol{G}\in\ell'(W).\boldsymbol{G}(\boldsymbol{b})=\boldsymbol{a}\\ \forall\boldsymbol{y}\in W,\boldsymbol{F}(\boldsymbol{G}(\boldsymbol{y}),\boldsymbol{y})=\boldsymbol{0}\\ \boldsymbol{g}'(\boldsymbol{b})=-(A_x)^{-1}A_y\end{cases}\end{cases}$$

Re　这表明，只要函数的导数矩阵为满秩的
在该点，矩阵各个最大线性列无关组，可以线性表示关于其它变量的偏导数值

1.2 反函数组与坐标变换

关于多元函数反函数存在的充分条件

1. 函数组$\begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}: D\subset R^2\to R^2$是一个映射$T$

$$\begin{gathered}\text{若}T\text{在}D\text{上是}1-1\text{的，则存在}T^{-1}\text{: range}\,T\to D.\text{称为}T\text{的逆映射}\\ u\equiv u(x(u,v),y(u,v));v\equiv v(x(u,v),y(u,v))\text{称为}D\text{上的反函数组}\end{gathered}$$

2. 反函数组定理：

$$\begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}\text{满足}\begin{cases} u,v\in\mathscr{C}'(D) \\ P_0\text{是}D\text{的内点} \\ u_0=u(P_0),v_0=v(P_0) \\ \frac{\partial(u,v)}{\partial(x,y)}(P_0)=\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}\neq 0 \end{cases}$$

$$\begin{array}{ll}
\Rightarrow & \exists U_{(u(P_0),v(P_0))}\text{上存在唯一的反函数}\begin{cases} x=x(u,v) \\ y=y(u,v) \end{cases} \\
\wedge & \begin{cases} x_0=x(u_0,v_0) \\ y_0=y(u_0,v_0) \end{cases}\wedge\forall(u,v)\in U_{(u_0,v_0)},(x(u,v),v(u,v))\in U_{P_0} \\
\wedge & \text{反函数组在}U_{(u_0,v_0)}\text{上存在连续的一阶连续偏导数} \\
\wedge & \dfrac{\partial x}{\partial u}=\dfrac{1}{J_{(u,v)/(x,y)}}\dfrac{\partial v}{\partial y};\dfrac{\partial x}{\partial v}=-\dfrac{1}{J}\dfrac{\partial u}{\partial y} \\
 & \dfrac{\partial y}{\partial u}=-\dfrac{1}{J}\dfrac{\partial v}{\partial x};\dfrac{\partial y}{\partial v}=\dfrac{1}{J}\dfrac{\partial u}{\partial x} \\
\text{Re} & \text{函数与反函数的Jocabi行列式乘积为1} \\
 & \dfrac{\partial(u,v)}{\partial(x,y)}\cdot\dfrac{\partial(x,y)}{\partial(u,v)}=1 \\
 & \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}=\begin{pmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial v} \end{pmatrix}
\end{array}$$

1.3 几何应用

1. 平面曲线的切法线：

$$\begin{array}{ll}
 & F(x,y)=0\text{确定的曲线的切线和法线} \\
 & F(x,y)\text{在}P_0\text{的某领域上满足隐函数定理的条件} \\
\text{切线} & y-y_0=f'(x_0)(x-x_0) \\
 & F_x(P_0)(x-x_0)+F_y(P_0)(y-y_0)=0 \\
 & \\
\text{法线} & F_y(P_0)(x-x_0)-F_x(P_0)(y-y_0)=0
\end{array}$$

2. 空间曲线的切线和法平面:

$$\text{空间曲线}\begin{cases} x=x(t) \\ y=y(t) \\ z=z(t) \end{cases};t\in[\alpha,\beta]$$

$$\text{切线}\qquad x,y,z\text{在}t_0\text{处可微且}(x')^2+(y')^2+(z')^2\neq 0$$

$$\frac{x-x_0}{x'(t_0)}=\frac{y-y_0}{y'(t_0)}=\frac{z-z_0}{z'(t_0)}$$

$$\text{法平面}\qquad x'(t_0)(x-x_0)+y'(t_0)(y-y_0)+z'(t_0)(z-z_0)=0$$

$$\text{空间曲线}\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$$

$$\text{若它在}P_0\text{满足隐函数定理的条件,}\ \frac{\partial(F,G)}{\partial(x,y)}(P_0)\neq 0$$

$$\Rightarrow x=\varphi(z);y=\psi(z);$$

$$\frac{\mathrm{d}x}{\mathrm{d}z}=-\frac{\frac{\partial(F,G)}{\partial(z,y)}}{\frac{\partial(F,G)}{\partial(x,y)}};\frac{\mathrm{d}y}{\mathrm{d}z}=-\frac{\frac{\partial(F,G)}{\partial(x,z)}}{\frac{\partial(F,G)}{\partial(x,y)}}$$

$$\text{切线}\qquad \frac{x-x_0}{\frac{\mathrm{d}x}{\mathrm{d}z}|_{P_0}}=\frac{y-y_0}{\frac{\mathrm{d}y}{\mathrm{d}z}|_{P_0}}=\frac{z-z_0}{1}$$

$$\frac{x-x_0}{\frac{\partial(F,G)}{\partial(y,z)}|_{P_0}}=\frac{y-y_0}{\frac{\partial(F,G)}{\partial(z,x)}|_{P_0}}=\frac{z-z_0}{\frac{\partial(F,G)}{\partial(x,y)}|_{P_0}}$$

$$\text{法平面}\quad \frac{\partial(F,G)}{\partial(y,z)}|_{P_0}(x-x_0)+\frac{\partial(F,G)}{\partial(z,x)}|_{P_0}(y-y_0)+\frac{\partial(F,G)}{\partial(x,y)}|_{P_0}(z-z_0)=0$$

3. 曲面的切平面与法线

$$\text{空间曲面}F(x,y,z)=0;P_0=(x_0,y_0,z_0)$$

$$F\text{在点}P_0\text{处满足隐函数定理的条件}\wedge F_z(P_0)\neq 0$$

$$\text{切平面}\qquad z-z_0=-\frac{F_x(P_0)}{F_z(P_0)}(x-x_0)-\frac{F_y(P_0)}{F_z(P_0)}(y-y_0)$$

$$\text{一般式}\qquad F_x(P_0)(x-x_0)+F_y(P_0)(y-y_0)+F_z(P_0)(z-z_0)=0$$

$$\text{法线}\qquad \frac{x-x_0}{\left(-\frac{F_x(P_0)}{F_z(P_0)}\right)}=\frac{y-y_0}{\left(-\frac{F_y(P_0)}{F_z(P_0)}\right)}=\frac{z-z_0}{-1}$$

$$\text{一般式}\qquad \frac{x-x_0}{F_x(P_0)}=\frac{y-y_0}{F_y(P_0)}=\frac{z-z_0}{F_z(P_0)}$$

$$\text{Re}\quad \operatorname{grad}F(P)\text{是等值面}F(x,y,z)=c\text{在点}P\text{的法向量}\boldsymbol{n}=(F_x(P),F_y(P),F_z(P))$$

$$\text{Re}\quad \text{空间曲线可以看作两个空间曲面的交线}$$

$$\text{由于空间曲面的切线垂直于两个空间曲面的法线}$$

$$\rightarrow\text{空间曲线的切向量等于两个法向量的外积}$$

$$\boldsymbol{\tau}=\boldsymbol{n}_1\times\boldsymbol{n}_2=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ F_x(P_0) & F_y(P_0) & F_z(P_0) \\ G_x(P_0) & G_y(P_0) & G_z(P_0) \end{vmatrix}$$

$$=\frac{\partial(F,G)}{\partial(y,z)}|_{P_0}\boldsymbol{i}+\frac{\partial(F,G)}{\partial(z,x)}|_{P_0}\boldsymbol{j}+\frac{\partial(F,G)}{\partial(x,y)}|_{P_0}\boldsymbol{k}$$

1.4 条件极值

求函数极值问题已经解决;

求在某些函数确定的约束下求函数的极值，函数约束称为条件

1. $$\begin{gathered}F(\boldsymbol{x})\text{是目标函数;求此函数的极值}\\ \varphi(\boldsymbol{x})=0\text{是一组自变量之间的约束}\end{gathered}$$
   求满足条件V的前提下，F的最大值称为条件极值问题

2. 对于f,φ都是二元函数求f的极值；拉个朗日乘数法的引入：

$$\begin{gathered}
\text{曲线}\varphi=0\text{是曲线方程，在}P_0\text{满足隐函数定理的条件}\\
y=g(x)\\
\text{化为求}f(x,g(x))=h(x)\text{的极值}\\
f\text{在}P_0\text{可微}\Rightarrow h'(x_0)=f_x(P_0)+f_y(P_0)g'(x)=0\\
g'(x_0)=-\frac{\varphi_x(P_0)}{\varphi_y(P_0)}\\
f_x(P_0)\varphi_y(P_0)-f_y(P_0)\varphi_x(P_0)=0\\
\text{这是两条曲线}\begin{cases}f(x,y)=h\\ \varphi(x,y)=0\end{cases}\text{的交点}\\
\text{若在}P_0\text{点取得极值},\exists\lambda_0\begin{cases}f_x(P_0)+\lambda_0\varphi_x(P_0)=0\\ f_y(P_0)+\lambda_0\varphi_y(P_0)=0\\ \varphi(P_0)=0\end{cases}\\
\text{引入辅助变量}\lambda\text{和辅助函数}\\
L(x,y,\lambda)=f(x,y)+\lambda\varphi(x,y)\\
\text{从而将条件极值问题变成无条件极值问题，}\lambda\text{称为拉格朗日乘数}
\end{gathered}$$

3. 拉格朗日定理：

$$\begin{gathered}
\text{条件极值问题}\begin{cases}f(\boldsymbol{x})\\ \varphi_k(\boldsymbol{x})=0\end{cases}\\
f,\varphi_i\in\ell'(D)\\
\text{若}D\text{的内点}P_0\text{是上述问题的极值点}\\
\left.\begin{pmatrix}\frac{\partial\varphi_1}{\partial x_1}&\cdots&\frac{\partial\varphi_1}{\partial x_n}\\ \vdots&&\\ \frac{\partial\varphi_m}{\partial x_1}&\cdots&\frac{\partial\varphi_m}{\partial x_n}\end{pmatrix}_{m\times n}\right|\text{在}P_0\text{点的秩为}m(\text{约束个数})\\
\Rightarrow\ \exists m\text{个常数}\boldsymbol{\lambda}_m\text{使得}(\boldsymbol{x}_0,\boldsymbol{\lambda}_0)\text{为拉格朗日函数组解；也是拉个朗日函数的稳定点}\\
\begin{cases}L_{x_1}=\frac{\partial f}{\partial x_1}+\sum_{i=1}^k\lambda_i\frac{\partial\varphi_i}{\partial x_1}=0\\ \cdots\\ L_{x_n}=\frac{\partial f}{\partial x_n}+\sum_{i=1}^k\lambda_i\frac{\partial\varphi_i}{x_n}=0\\ L_{\lambda_1}=\varphi_1(\boldsymbol{x})=0\\ \cdots\\ L_{\lambda_m}=\varphi_m(\boldsymbol{x})=0\end{cases}
\end{gathered}$$

极值点必在稳定点内取得，因此代入这些稳定点进行运算即可求得条件极值

2 Trick

1. 把$F(x,y)$中的y看成x的函数; $F(x,f(x))=0\Rightarrow F_x+F_y y'=0$

2. 也可以直接在原式中直接求对某个自变量的微分得到其它变量关于此变量的微分表达式

$$\begin{gathered}
0=F(x,y)=F(x,f(y))\\
0=\mathrm{d}F=F_x+F_y y_x \qquad\qquad y_x=-\frac{F_x}{F_y}\\
0=\mathrm{d}^2F=F_{xx}+F_{xy}y_x+(F_{yx}+F_{yy}y_x)y_x+F_y y_{xx}\\
\to y_{xx}=\frac{-F_{xx}-F_{xy}y_x-F_{yx}y_x-F_{yy}(y_x)^2}{F_y}\\
\to y_{xx}=\frac{-F_{xx}-2F_{xy}\left(-\frac{F_x}{F_y}\right)-F_{yy}\left(\frac{F_x}{F_y^2}^2\right)}{F_y}\\
\to y_{xx}=\frac{2F_{xy}F_xF_y-F_{xx}F_y^2-F_{yy}F_x^2}{F_y^3}
\end{gathered}$$

3. 使用一般的隐函数定理的计算步骤

$$\begin{gathered}
F\colon R^{m+n}\to R^n;\\
\text{求}F\text{的导数矩阵}A_F(n\times(m+n));\\
\text{从}A_F\text{的}m+n\text{列中选择要求的}n\text{个变量的列}Y(n\times n)\\
\text{剩余的变量构成的矩阵为}X(n\times m)\\
\text{使用公式}Y_X=-Y^{-1}X;
\end{gathered}$$

eg

$$\begin{gathered}
F=\begin{cases} x^2+y^2-\frac{z^2}{2}\\ x+y+z-2\end{cases};F'=\begin{pmatrix} 2x & 2y & -z\\ 1 & 1 & 1\end{pmatrix}\\
\text{因此在}x\neq y\text{的点};\begin{pmatrix} 2x & 2y\\ 1 & 1\end{pmatrix}\text{可逆}\\
=\left(\frac{1}{2x-2y}\right)\begin{pmatrix} 1 & -2y\\ -1 & 2x\end{pmatrix}\\
G'(z)=\begin{pmatrix} x(z)\\ y(z)\end{pmatrix}'=\frac{-1}{2x-2y}\begin{pmatrix} 1 & -2y\\ -1 & 2x\end{pmatrix}\begin{pmatrix} -z\\ 1\end{pmatrix}\\
\frac{\partial y}{\partial z}=\sim\begin{pmatrix} -z-2y\\ z+2x\end{pmatrix}=\frac{z+2y}{2x-2y};\frac{\partial x}{\partial z}=\frac{z+2x}{2y-2x}
\end{gathered}$$

eg

$$\begin{gathered}
F=\begin{cases} x^2+y^2+z^2-a^2\\ x^2+y^2-ax\end{cases}\\
F'=\begin{pmatrix} 2x & 2y & 2z\\ 2x-a & 2y & 0\end{pmatrix}\\
y\neq z\wedge y\neq 0\text{时}\begin{pmatrix} 2y & 2z\\ 2y & 0\end{pmatrix}\text{可逆}\\
-\begin{pmatrix} 0 & \frac{1}{2y}\\ \frac{1}{2z} & \frac{-1}{2z}\end{pmatrix}\begin{pmatrix} 2x\\ 2x-a\end{pmatrix}=\begin{pmatrix} \frac{a-2x}{2y}\\ \frac{-a}{2z}\end{pmatrix}=\begin{pmatrix} \frac{\mathrm{d}y}{\mathrm{d}x}\\ \frac{\mathrm{d}z}{\mathrm{d}x}\end{pmatrix}
\end{gathered}$$

eg

$$F=\begin{pmatrix} x-u^2-yv \\ y-v^2-xu \end{pmatrix}$$
$$F'=\begin{pmatrix} 1 & -v & -2u & -y \\ -u & 1 & -x & -2v \end{pmatrix}$$
$$4uv-xy\neq 0\text{处可逆}$$
$$\text{求}H'(x,y)=-\begin{pmatrix} -2u & -y \\ -x & -2v \end{pmatrix}^{-1}\begin{pmatrix} 1 & -v \\ -u & 1 \end{pmatrix}$$
$$=\frac{-1}{4uv-xy}\begin{pmatrix} -2v & y \\ x & -2u \end{pmatrix}\begin{pmatrix} 1 & -v \\ -u & 1 \end{pmatrix}$$
$$=\begin{pmatrix} \frac{uy+2v}{4uv-xy} & -\frac{2v^2+y}{4uv-xy} \\ -\frac{2u^2+x}{4uv-xy} & \frac{2u+vx}{4uv-xy} \end{pmatrix}=\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

4. 反函数求导公式的推导

$$y\text{在}x_0\text{的某个邻域上有连续的导数}f'(x),f(x_0)=y_0$$
$$F(x,y)=y-f(x)=0$$
$$F(x_0,y_0)=0,F_y=1,F_x(x_0,y_0)=-f'(x_0)$$
$$\rightarrow\text{根据隐函数定理，有唯一的连续可微函数}x=g(y)$$
$$g'(y)=-\frac{F_y}{F_x}=-\frac{1}{-f'(x)}=\frac{1}{f'(x)}$$
$$\boldsymbol{g}'(\boldsymbol{y})=(\boldsymbol{f}'(\boldsymbol{x}))^{-1}$$

5. 拉格朗日乘数法的操作方法:

$$\text{目标函数}f(\boldsymbol{x}_n)\text{；约束}\boldsymbol{\varphi}_m(\boldsymbol{x})=0$$
$$\text{拉格朗日辅助函数}L(\boldsymbol{x}_n,\boldsymbol{\lambda}_m)=f(\boldsymbol{x}_n)+\sum_{i=1}^{m}\lambda_i\varphi_i(\boldsymbol{x}_n)$$
$$\text{求}L\text{的对于}(\boldsymbol{x}_n,\boldsymbol{\lambda}_m)\text{的各个偏导数}$$
$$\text{得到方程组}\begin{cases} L_{x_i}(\boldsymbol{x}_n,\boldsymbol{\lambda}_m)=0 \\ L_{\lambda_i}(\boldsymbol{x}_n,\boldsymbol{\lambda}_m)=0 \end{cases}$$
$$\text{解上述方程组得到}L\text{函数的各个稳定点}$$
$$\text{代入稳定点即得到函数的各个极值}$$