

Fourier Series

$$f(x) = (a)_0 \text{ term } + (a)_1 \text{ term } + (f)_1 x + (b)_1 \sin x + \dots$$

- Dirichlet's Conditions \rightarrow $f(x)$ is a function defined in the interval $(a, a+2L)$

Consider a single-valued function

$$(1) f(x) \text{ is defined in the interval } (a, a+2L) \text{ and } f(x) = f(x+2L)$$

which satisfies following conditions known as
Dirichlet's Conditions.

- ① $f(x)$ is defined in the interval $(a, a+2L)$
and $f(x) = f(x+2L)$
- ② $f(x)$ is a continuous function or has finite number
of discontinuities in the interval $(a, a+2L)$
- ③ $f(x)$ has no maxima or minima or has finite number
of maxima or minima

If these 3 conditions are satisfied, $f(x)$ can be written as :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)] \quad \begin{matrix} \text{Fourier series of} \\ f(x) \text{ in interval} \\ (a, a+2L) \end{matrix}$$

$$\text{where } a_0 = \frac{1}{L} \int_a^{a+2L} f(x) dx,$$

$$a_n = \frac{1}{L} \int_a^{a+2L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L} \int_a^{a+2L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Fourier
Coefficients

$$\text{Ans. } \int_{-\pi}^{\pi} f(x) dx = \int_{-\pi}^{\pi} (\pi - x)^2 dx = \frac{1}{3} \left[(\pi - x)^3 \right]_{-\pi}^{\pi} = \frac{1}{3} \cdot 4\pi^3 = \frac{4\pi^3}{3}$$

\rightarrow Fourier series in interval $(0, 2\pi)$ is $\sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$

Q. Obtain fourier series for $f(x) = (\pi - x)^2$, $0 \leq x \leq 2\pi$

$$(x_0)_{\text{ans}} + (x_0)_{\text{ans}} \cdot \frac{f(\pi)}{2} = \text{ans}$$

$$\text{and } f(x+2\pi) = f(x).$$

$$\text{Deduce: (i) } \frac{\pi^2}{6} = -1 + \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right) - \frac{f(\pi)}{2} = \text{ans}$$

$$\text{(ii) } \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

$$\text{(iii) } \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{f(\pi)}{2} = \text{ans}$$

$$\text{(iv) } \frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{f(\pi)}{4} = \text{ans}$$

$$\text{Ans. } a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{4\pi} \int_0^{2\pi} (\pi - x)^2 dx =$$

$$= \frac{1}{4\pi} \left[\frac{(\pi - x)^3}{3} \right]_0^{2\pi} = \frac{2\pi}{12\pi} = \left(\frac{1}{6}\pi^3 - \frac{1}{3}\pi^3 \right) =$$

$$\therefore a_0 = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot \cos(nx) dx = \frac{1}{4\pi} \int_0^{2\pi} (\pi - x)^2 \cdot \cos(nx) dx$$

Remember!! Ans. (ii)

when period = $(0, 2\pi)$ or $(-\pi, \pi)$

$$\cos\left(\frac{n\pi x}{l}\right) \rightarrow \cos(nx)$$

$$\sin\left(\frac{n\pi x}{l}\right) \rightarrow \sin(nx)$$

$$= \frac{1}{4\pi} \int_0^{2\pi} (\pi - x)^2 \cdot \cos(nx) dx =$$

$$= \frac{1}{4\pi} \left[(\pi - x)^2 \cdot \frac{\sin(nx)}{n} \right]_0^{2\pi} - 2(\pi - x)(-1) \cdot \left(-\frac{\cos(nx)}{n^2} \right) + 2 \cdot \left(-\frac{\sin(nx)}{n^3} \right) \Big|_0^{2\pi}$$

$$= -\frac{1}{2n^2\pi} (-\pi - \pi) = \frac{-1}{2n^2\pi} (-2\pi) = \frac{1}{n^2}$$

$$\therefore a_n = \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} \frac{(\pi-x)^2}{4} \cdot \sin(nx) dx$$

$$\therefore b_n = \frac{1}{4\pi} \left[(\pi-x)^2 \left(-\frac{\cos(nx)}{n} \right) - (-2)(\pi-x) \left(-\frac{\sin(nx)}{n^2} \right) + 2 \left(\frac{\cos(nx)}{n^3} \right) \right]_0^{2\pi}$$

$$\therefore b_n = \frac{1}{4\pi n} \left[-(\pi-x)^2 \cdot \cos(nx) + 2\cos(nx) \right]_0^{2\pi} = \frac{1}{4\pi n} \left[-\pi^2 \cdot 1 + 2 \cdot \frac{(-\pi^2) \cdot 1 - 2}{n^2} \right] = 0$$

$$\therefore \left(\frac{\pi}{2}\right)^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cdot \cos(nx)$$

(i) Putting $n=0$,

$$\therefore \frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{12} + \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

$$\therefore \frac{\pi^2 - \pi^2}{4} = \frac{\pi^2}{12} = \frac{1 + 9(4+1) + 1 + \dots}{12 \cdot 2^2 \cdot 3^2 \cdot 4^2} \quad \text{--- (1)}$$

Hence Proved.

(ii) Putting $x=\pi$, $a_n = \frac{1}{\pi} \int_0^{2\pi} \sum_{n=1}^{\infty} (-1)^n \cos(nx) dx = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \sin(nx) \Big|_0^{2\pi} = 0$

$$\therefore \frac{\pi^2 - \pi^2}{4} = \frac{\pi^2}{12} = \frac{1 + 9(4+1) + 1 + \dots}{12 \cdot 2^2 \cdot 3^2 \cdot 4^2} \quad \text{--- (2)}$$

$$\therefore \frac{\pi^2 - \pi^2}{4} = \frac{1}{12} + \frac{1}{12} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \quad \text{--- (2)} \quad \text{Hence Proved.}$$

(iii) Adding (1) and (2) deductions,

$$\frac{\pi^2 + \pi^2}{6} = \frac{1}{12} + \frac{1}{12} + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{4^2} + \dots$$

$$\frac{18\pi^2}{72} = \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \dots \quad \therefore \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Hence Proved.

~~(iii)~~) Parseval's Identity \rightarrow

$$\frac{1}{L} \int_a^{a+2L} f^2(x) dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \quad \text{[use whenever you need square of summation series]}$$

(iv) $a_0 = \frac{\pi^2}{6}$, $a_n = \frac{1}{h^2}$, $b_n = 0$

(ii) choose (ii)

\therefore Using Parseval's Identity,

$$\frac{1}{16} \times \frac{1}{\pi} \int_0^{2\pi} (\pi - x)^5 dx = \frac{\pi^4}{72} + \sum_{n=1}^{\infty} \frac{1}{h^4} =$$

$$\therefore \frac{1}{16\pi} \left[(\pi - x)^5 \right]_0^{2\pi} = \frac{\pi^4}{72} + \sum_{n=1}^{\infty} \frac{1}{h^4}$$

$$\therefore \frac{\pi^4}{40} - \frac{\pi^4}{72} \left[\sum_{n=1}^{\infty} \frac{1}{h^4} \right] =$$

$$\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \dots$$

Hence Proved,

$$0 \left[((x_0)_{\text{adj}} - (x_0)_{\text{m123}}) \cdot x_{-9} \right] \frac{1}{\pi} =$$

$$((a-1) \cdot 1 - (a-1) \cdot x_{-9}) \frac{1}{(1+5)} =$$

$$\left(\frac{a-1}{\pi} \right) \frac{1}{1+5} = ad$$

\leftarrow (be unitary)

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Q. If $f(x) = e^{-x}$ in $(0, 2\pi)$.

Find value of $\int_{-\infty}^{\infty} \frac{(-1)^n}{n^2+1} dx$

(ii) $\operatorname{cosech}(\pi)$

$$\text{Ans. } a_0 = \frac{1}{\pi} \int_0^{2\pi} e^{-x} dx$$

$$= \frac{-1}{\pi} \left[e^{-x} \right]_0^{2\pi} = \frac{1 - e^{-2\pi}}{\pi}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} e^{-x} \cos(nx) dx$$

$$= \frac{1}{\pi} \left[\frac{e^{-x}}{n^2+1} (-\cos(nx) + n \sin(nx)) \right]_0^{2\pi}$$

$$= \frac{1}{(n^2+1)\pi} \left(e^{-2\pi} \cdot (-1) - 1 \cdot (-1) \right) = \frac{1}{n^2+1} \cdot \left(\frac{1 - e^{-2\pi}}{\pi} \right)$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} e^{-x} \sin(nx) dx$$

$$= \frac{1}{\pi} \left[\frac{e^{-x}}{n^2+1} (-n \sin(nx) - n \cos(nx)) \right]_0^{2\pi}$$

$$= \frac{1}{\pi(n^2+1)} \left(e^{-2\pi} \cdot (-n) - 1 \cdot (-n) \right)$$

$$b_n = \frac{n}{n^2+1} \left(\frac{1 - e^{-2\pi}}{\pi} \right)$$

(continued) →

$$\therefore f(x) = \left(\frac{1-e^{-2\pi}}{2\pi} \right) + \frac{1-e^{-2\pi}}{\pi} \sum_{n=1}^{\infty} \frac{(\cos(nx) + 2n \cdot \sin(nx))}{n^2+1}$$

(i) Putting $x=\pi$, $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1} = \text{trig}(\pi) \text{cosech}(\pi)$

$$\therefore e^{-\pi} = \left(\frac{1-e^{-2\pi}}{2\pi} \right) + \left(\frac{1-e^{-2\pi}}{\pi} \right) \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} \quad \text{LHS} = \text{RHS}$$

$$\therefore e^{-\pi} = \left(\frac{1-e^{-2\pi}}{2\pi} \right) + \left(\frac{1-e^{-2\pi}}{2\pi} \right) + \left(\frac{1-e^{-2\pi}}{\pi} \right) \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$$

$$\therefore \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{\pi \cdot e^{-\pi}}{1-e^{-2\pi}} = \frac{\pi (\text{trig})}{e^{\pi}-e^{-\pi}} \quad \text{LHS} = \text{RHS}$$

$$(ii) \therefore \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{\pi}{2(e^{\pi}-e^{-\pi})} = \frac{\pi \cdot \text{cosech}(\pi)}{2} \quad \text{LHS} = \text{RHS}$$

$$\therefore \text{cosech}(\pi) = \frac{(2 \cdot \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}) \text{trig}(\pi)}{\pi} \quad \text{LHS} = \text{RHS}$$

$$\left[\frac{(\text{trig}(\pi-2)) \text{trig}}{\pi-2} + \frac{(\text{trig}(\pi+2)) \text{trig}}{\pi+2} \right] \quad \text{LHS} = \text{RHS}$$

$$\left[\frac{(\text{trig}(\pi-2)) \text{trig}}{\pi-2} + \frac{(\text{trig}(\pi+2)) \text{trig}}{\pi+2} \right] \quad \text{LHS} = \text{RHS}$$

$$[(\text{trig}) \text{trig}] = (\text{trig}) \text{trig} \cdot (\text{trig}) \text{trig} = \text{trig}(\pi) \text{trig} \cdot \text{trig}^2$$

$$\left[\frac{1}{\pi-2} + \frac{1}{\pi+2} \right] (\text{trig}) \text{trig} = \text{trig} \quad \text{LHS} = \text{RHS}$$

$$\left[\frac{1}{\pi-2} + \frac{1}{\pi+2} \right] (\text{trig}) \text{trig} = \text{trig} \quad \text{LHS} = \text{RHS}$$

Every, but trigon used

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~~Q.~~ $f(x) = \cos(px)$ in $(0, 2\pi)$ where p is not an integer.

Deduce that (i) $\pi \cot(p\pi) = \frac{1}{p} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{pn} + \frac{1}{p-n} \right]$

$$(ii) \pi \cot(2p\pi) = \frac{1}{2p} + p \sum_{n=1}^{\infty} \frac{1}{p^2 - n^2}$$

Ans. $a_0 = \frac{1}{\pi} \int_0^{2\pi} \cos(px) dx$

$$= \frac{1}{\pi} \left[\frac{\sin(px)}{p} \right]_0^{2\pi} = \frac{1}{p\pi} (\sin(2p\pi) - \sin(0))$$

$$a_0 = \frac{\sin(2p\pi)}{2p\pi}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} \cos(px) \cdot \cos(n\pi x) dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} \cos(px) \cdot \cos(n\pi x) dx$$

$$\therefore a_n = \frac{1}{2\pi} \int_0^{2\pi} (\cos((p+n)x) + \cos((p-n)x)) dx$$

$$= \frac{1}{2\pi} \left[\frac{\sin((p+n)x)}{p+n} + \frac{\sin((p-n)x)}{p-n} \right]_0^{2\pi}$$

$$= \frac{1}{2\pi} \left(\frac{\sin((p+n)2\pi)}{p+n} + \frac{\sin((p-n)2\pi)}{p-n} \right)$$

But, $\sin(p+n)2\pi = \sin(2p\pi) \cdot \cos(2n\pi) = \sin(2p\pi)$

$$\therefore a_n = \frac{\sin(2p\pi)}{2\pi} \left[\frac{1}{p+n} + \frac{1}{p-n} \right]$$

$$a_n = \frac{\sin(2p\pi)}{2\pi} \left(\frac{2p}{p^2 - n^2} \right)$$

(continued) →

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(px) \cdot \sin(nx) dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (\sin((p+n)x) - \sin((p-n)x)) dx = 0$$

$$= \frac{1}{2\pi} \left[\frac{\cos((p+n)x)}{p+n} - \frac{\cos((p-n)x)}{p-n} \right] \quad (i) \text{ : result}$$

~~*~~ But $\cos(p \pm n)2\pi = \cos(2p\pi)$ $\Rightarrow (i) \text{ (ii)}$

$$\therefore b_n = \frac{1}{2\pi} \left[\frac{\cos(2p\pi) - 1}{p-n} - \frac{\cos(2p\pi) - 1}{p+n} \right] \quad (n \neq 0)$$

$$b_n = \frac{\cos(2p\pi) - 1}{2\pi} \left(\frac{2n}{p^2 - n^2} \right) = \frac{n(\cos(2p\pi) - 1)}{p^2 - n^2} \quad //$$

$$\therefore f(x) = \cos(px) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

$$= \frac{\sin(2p\pi)}{2p\pi} + \frac{p \sin(2p\pi)}{\pi} \sum_{n=1}^{\infty} \frac{\cos(nx)}{p^2 - n^2} + \frac{(\cos(2p\pi) - 1)}{\pi} \sum_{n=1}^{\infty} \frac{n \sin(nx)}{p^2 - n^2}$$

(i) Putting $x = \pi$,

$$\cos(p\pi) = \frac{\sin(2p\pi)}{2p\pi} + p \sin(2p\pi) \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{p+n} + \frac{1}{p-n} \right) = 0$$

$$\therefore \cos(p\pi) = 2 \sin(p\pi) \cos(p\pi) + p^2 \sin(p\pi) \cos(p\pi) \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{p+n} + \frac{1}{p-n} \right]$$

$$\therefore \pi \cot(p\pi) = \frac{1}{p} \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{p+n} + \frac{1}{p-n} \right] \quad \text{Hence Proved.} //$$

(ii) Putting $x = 2\pi$,

$$\cos(2p\pi) = \frac{\sin(2p\pi)}{2p\pi} + p \sin(2p\pi) \sum_{n=1}^{\infty} \frac{1}{p^2 - n^2}$$

$$\therefore \pi \cot(2p\pi) = \frac{1}{2p} + p \sum_{n=1}^{\infty} \frac{1}{p^2 - n^2} \quad (1 - \frac{1}{p^2}) = np$$

Q.E.D. \blacksquare

Q. Find the Fourier series in the interval $(-\pi, \pi)$ where,

$$f(x) = \begin{cases} 0, & -\pi \leq x < 0 \\ \sin(x), & 0 \leq x \leq \pi \end{cases}$$

$$\text{Deduce: (i)} \frac{1}{1 \cdot 3} \frac{x(4+1) \cdot 100}{100} + \frac{1}{5 \cdot 7} \frac{x(4+9) \cdot 100}{100} = 11$$

$$(ii) (\pi - 2) = \frac{1 \cdot 3}{1 \cdot 3} \frac{x(4+5) \cdot 200}{200} + \frac{1 \cdot 3}{5 \cdot 7} \frac{x(4+9) \cdot 200}{200} = 7.68$$

$$\text{Ans. } f(-x) = \begin{cases} 0, & 0 < x \leq \pi \\ \sin(-x), & -\pi \leq x \leq 0 \end{cases}$$

$\therefore f(x)$ is neither even nor odd.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(x) dx = -1 \left[\cos(x) \right]_{-\pi}^{\pi} = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(x) \cos(nx) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\sin((n+1)x) - \sin((n-1)x)) dx$$

$$\therefore a_n = \frac{1}{2\pi} \left[\frac{\cos((n+1)\pi)}{n+1} - \frac{\cos((n-1)\pi)}{n-1} \right] = (-1)^{n+1}$$

$$[\text{But } \cos((n \pm 1)\pi) = \cos(n\pi) \cdot \cos(\pi) \mp \sin(n\pi) \cdot \sin(\pi)]$$

$$a_n = \frac{1}{2\pi} \left[\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n+1}}{n+1} + \frac{1}{n+1} - \frac{1}{n-1} \right]$$

$$= \frac{1}{2\pi} = \frac{(-1)^{n+1}}{2\pi} \left[\frac{1}{n+1} - \frac{1}{n-1} \right]$$

$$a_n = \frac{(-1)^{n+1} - 1}{\pi(n^2 - 1)}$$

(continued) \rightarrow

$$\exists \quad \underline{a_n} = \frac{1}{\pi} \int_{-\pi}^{\pi} -2 \cdot \begin{cases} 1 & \text{if } n=2k \\ 0 & \text{if } n=2k+1 \end{cases}, \quad \text{if } n=2k \\ \underline{a_n} = \frac{1}{\pi} \int_{-\pi}^{\pi} -2 \cdot \begin{cases} 1 & \text{if } n=2k \\ 0 & \text{if } n=2k+1 \end{cases}, \quad \text{if } n=2k+1$$

$$\therefore a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} -2 \cdot \begin{cases} \frac{1}{n(n+1)} \sin(2nx), & \text{if } n=2k \\ 0, & \text{if } n=2k+1 \end{cases}$$

(since at $k=1, n=3$
so a_3 term \downarrow)

$$\left. \begin{aligned} a_1 &= \frac{1}{\pi} \int_0^\pi \sin(x) \cdot \cos(x) dx = 1 + 0 = 0 = (i) \\ &= \frac{1}{2\pi} \int_0^\pi \sin(2x) dx = \frac{1}{2\pi} \left[-\frac{\cos(2x)}{2} \right]_0^\pi \\ a_1 &= 0, \end{aligned} \right]$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_0^\pi \sin(n) \sin(nx) dx \\ &= \frac{1}{2\pi} \int_0^\pi (\cos((n-1)x) - \cos((n+1)x)) dx \\ &= \frac{1}{2\pi} \left[\frac{\sin((n-1)x)}{n-1} - \frac{\sin((n+1)x)}{n+1} \right]_0^\pi \\ &= 0 \quad \forall (n \neq 1) \end{aligned}$$

$$\left. \begin{aligned} b_1 &= \frac{1}{\pi} \int_0^\pi \sin(x) \cdot \sin(x) dx \\ &= \frac{1}{2\pi} \int_0^\pi [1 - \cos(2x)] dx \\ &\text{forward solving} \\ &= \frac{1}{2\pi} \left[x - \frac{\sin(2x)}{2} \right]_0^\pi = \frac{1}{2\pi} \times \pi \end{aligned} \right]$$

$$b_1 = \frac{1}{2\pi}$$

(continued) →

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$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

$$= \frac{1}{\pi} - \frac{2}{\pi} \sum_{k=1}^{\infty} \left(\frac{1}{4k^2-1} \cos(2kx) \right) + \frac{\sin(x)}{2}$$

(i) Putting $x=0$,

$$f(0) = 0 = \frac{1}{\pi} - \frac{2b}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)(2k+1)} + 0$$

$$\therefore 1 = 2 \sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)}$$

$$\therefore \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \dots = \frac{1}{2}$$

Hence Proved.

(ii) Putting $x=\pi/2$,

$$f(\pi/2) = 1 = \frac{1}{\pi} \left[\sum_{k=1}^{\infty} \left(\frac{\cos(k\pi)}{4k^2-1} \right) + 2 \right]$$

$$\therefore \left(1 - \frac{1}{\pi} - \frac{1}{2} \right) \times \frac{\pi}{2} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{4k^2-1}$$

$$\therefore \frac{\pi-2}{4} = \frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \frac{1}{7 \cdot 9} + \dots$$

Hence Proved.

Q. $f(x) = |\cos x|$ in $(-\pi, \pi)$

hence, find Fourier series for $|\sin(x)|$

Ans. $f(-x) = f(x)$

$\Rightarrow f(x)$ is even function ($\Rightarrow b_n = 0$)

$\left[\begin{array}{l} \text{If } f(x) \text{ is even, } b_n = 0 \\ \text{if } f(x) \text{ is odd, } b_n = T.F. \end{array} \right]$

$$\therefore a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |\cos x| dx$$

$$= \frac{2}{\pi} \int_0^{\pi} |\cos x| dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} \cos x dx - \int_{\pi/2}^{\pi} \cos x dx \right]$$

$$= \frac{2}{\pi} \left([\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi} \right)$$

$$\therefore a_0 = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} |\cos x| \cos nx dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} \cos x \cos nx dx - \int_{\pi/2}^{\pi} \cos x \cos nx dx \right]$$

$$a_n = \frac{1}{\pi} \left(\int_0^{\pi/2} (\cos(n+1)x + \cos(n-1)x) dx - \int_{\pi/2}^{\pi} (\cos(n+1)x + \cos(n-1)x) dx \right)$$

$$= \frac{1}{\pi} \left(\left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right]_0^{\pi/2} - \left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right]_{\pi/2}^{\pi} \right)$$

$$[\sin(n+1)\pi/2 = \pm \cos(n\pi/2)]$$

$$\therefore a_n = \frac{-4}{\pi(n^2-1)} \cdot \cos\left(\frac{n\pi}{2}\right)$$

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28 P.T.S.

Finding, a_1 ,

$$\begin{aligned}
 a_1 &= \frac{2}{\pi} \left(\int_0^{\pi/2} \cos^2(x) dx - \int_{\pi/2}^{\pi} \cos^2(x) dx \right), \quad \text{and} \\
 &= \frac{1}{\pi} \left(\int_0^{\pi/2} (\cos(2x) + 1) dx - \int_{\pi/2}^{\pi} (\cos(2x) + 1) dx \right) \\
 &= \frac{1}{\pi} \left([\sin(2x) + x]_0^{\pi/2} - [\sin(2x) + x]_{\pi/2}^{\pi} \right)
 \end{aligned}$$

$$a_1 = 0$$

$$\text{But, } \cos(n\pi/2) = \begin{cases} (-1)^k, & \text{if } n=2k \\ 0, & \text{if } n=2k+1 \end{cases}$$

$$\therefore a_n = \begin{cases} \frac{-4(-1)^k}{\pi(n^2-1)}, & \text{if } n=2k \\ 0, & \text{if } n=2k+1 \end{cases}$$

$$\therefore f(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{4k^2-1} \cos(2kx)$$

Replacing x by $x+\pi/2$,

$$|\cos(\pi/2+x)| = |-8\sin x| = |\sin x|$$

$$|\cos(x+\pi/2)| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{4k^2-1} \cos(2k(x+\pi/2))$$

$$|\cos(x+\pi/2)| = \frac{2}{\pi} \cdot \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k (-1)^k}{4k^2-1} \cdot \cos(2kx)$$

$$\therefore |\sin x| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2kx)}{4k^2-1}$$

Q. $f(x) = \begin{cases} x + \frac{\pi}{2}, & -\pi < x < 0 \\ \frac{\pi}{2} - x, & 0 < x < \pi \end{cases}$

Deduce: (i) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$
(ii) $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots$

Ans. $f(-x) = \begin{cases} \frac{\pi}{2} - x, & 0 < x < \pi \\ x + \frac{\pi}{2}, & -\pi < x < 0 \end{cases}$

 $= f(x)$
 $\therefore f(x)$ is even, $b_n = 0$

$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi} f(x) dx \quad (\text{since even})$

$\therefore \frac{2}{\pi} \int_0^{\pi} (\frac{\pi}{2} - x) dx = \frac{2}{\pi} \left[\frac{\pi x}{2} - \frac{x^2}{2} \right]_0^{\pi} = \pi^2$

$a_0 = 0$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = \frac{2}{\pi} \int_0^{\pi} (\frac{\pi}{2} - x) \cos(nx) dx$
 $a_n = \frac{2}{\pi} \left[\left(\frac{\pi}{2} \right) \left(\frac{\sin(nx)}{n} \right) \right]_0^{\pi} - \left[\left(-1 \right)^n \left(\frac{\cos(nx)}{n} \right) \right]_0^{\pi}$
 $= -\frac{2}{\pi n^2} \left(\cos(n\pi) - \cos(0) \right)$

$a_n = \frac{-2}{\pi n^2} \left((-1)^n - 1 \right)$

$\therefore a_n = \begin{cases} \frac{4}{\pi n^2}, & \text{if } n = 2k-1 \\ 0, & \text{if } n = 2k \end{cases}$

$$\therefore f(x) = \frac{q_0}{2} + \sum_{n=1}^{\infty} q_n \cos(nx) \quad \forall x > 0$$

$$= \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos((2k-1)x)}{(2k-1)^2}$$

$$(i) f(0) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

Since $f(x)$ is discontinuous at 0,

$$f(0) = \frac{1}{2} \left(\lim_{x \rightarrow 0^-} f(x) + \lim_{x \rightarrow 0^+} f(x) \right)$$

$$= \frac{1}{2} \left[\lim_{x \rightarrow 0^-} (x + \frac{\pi}{2}) + \lim_{x \rightarrow 0^+} (\frac{\pi}{2} - x) \right]$$

$$\text{Now } x = \frac{\pi}{2} \Rightarrow \frac{1}{2} (\frac{\pi}{2} + \frac{\pi}{2}) = \frac{\pi}{2}$$

$$\frac{\pi}{2} = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

$$\therefore \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \quad \text{Hence Proved.}$$

(iii) Using Parseval's Identity, $\int_0^{\pi} f^2(x) dx = q_0^2 + \sum_{n=1}^{\infty} q_n^2$

$$\frac{2}{\pi} \int_0^{\pi} f^2(x) dx = q_0^2 + \sum_{n=1}^{\infty} q_n^2 \quad (\text{for even})$$

$$\therefore \frac{2}{\pi} \int_0^{\pi} (\frac{\pi}{2} - x)^2 dx = 0 + \sum_{k=1}^{\infty} \frac{16}{(2k-1)^4}$$

\downarrow (Solving integral)

$$\therefore \frac{\pi^2}{6} = \frac{16}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4}$$

$$\therefore \frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots$$

\downarrow Hence Proved.

Q. Interval $(0, 2L)$, $f(x) = 2x - x^2$, $(0 \leq x \leq 3)$, [Period = 3]

Ans. $2L = 3$

$$\therefore L = \frac{3}{2}$$

$$q_0 = \frac{1}{L} \int_0^{2L} f(x) dx = \frac{2}{3} \int_0^3 (2x - x^2) dx$$

$$= \frac{2}{3} \left[x^2 - \frac{x^3}{3} \right]_0^3 = \frac{2}{3} [9 - 8\frac{1}{3}]$$

$$q_0 = 0$$

$$a_n = \frac{1}{L} \int_0^{2L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{3} \int_0^3 (2x - x^2) \cos\left(\frac{2n\pi x}{3}\right) dx$$

$$\therefore q_n = \frac{2}{3} \left[(2x - x^2) \cdot \left(\sin\left(\frac{2n\pi x}{3}\right) \right) - (2-2x) \left(-\cos\left(\frac{2n\pi x}{3}\right) \right) \right]_0^3$$

$$= \left[+(-2) \left(-\sin\left(\frac{2n\pi x}{3}\right) \right) \right]_0^3$$

$$= \frac{-2}{3} \int_0^3 = \frac{3}{2n^2\pi^2} \left[(2-2x) \left(\cos\left(\frac{2n\pi x}{3}\right) \right) \right]_0^3$$

$$= \frac{3}{2n^2\pi^2} (-6)$$

$$\therefore a_n = \frac{-9}{n^2\pi^2}$$

$$b_n = \frac{1}{L} \int_0^{2L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{3} \int_0^3 (2x - x^2) \sin\left(\frac{2n\pi x}{3}\right) dx$$

$$S = b_0 + \sum b_n = \frac{2}{3} \left[(2x - x^2) \left(\frac{-\cos(\frac{2n\pi x}{3})}{\frac{2n\pi}{3}} \right) - (2-2x) \left(\frac{+\sin(\frac{2n\pi x}{3})}{4n^2\pi^2/9} \right) \right. \\ \left. + (-2) \left(\frac{\cos(\frac{2n\pi x}{3})}{8n^3\pi^3/27} \right) \right]_0^3$$

$$\therefore b_n = - \left[(2x - x^2) \left(\frac{\cos(\frac{2n\pi x}{3})}{\frac{2n\pi}{3}} \right) + 2 \left(\frac{\cos(\frac{2n\pi x}{3})}{\frac{4n^2\pi^2}{9}} \right) \right]_0^3 \times \frac{1}{n\pi} \\ = -\frac{1}{n\pi} \left[(-3)(1) + (2) \left(\frac{9}{4n^2\pi^2} \right) - 0 - (2) \left(\frac{9}{4n^2\pi^2} \right) \right]$$

$$b_n = \frac{3}{n\pi}$$

$$\therefore f(x) = \frac{2x - x^2}{2} = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\pi x) + b_n \sin(n\pi x)]$$

$$\therefore 2x - x^2 = -\frac{9}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{\cos(2n\pi x/3)}{n^2} \right) + \frac{3}{\pi} \sum_{n=1}^{\infty} \left(\frac{\sin(2n\pi x/3)}{n} \right)$$

$$\int ((\text{cos } n\pi x)^2) dx = \int 1 dx = \frac{x}{1}$$

$$(2\pi) \cdot \frac{1}{2} = \frac{\pi}{2}$$

$$\int (\text{cos } n\pi x) \text{cos } (m\pi x) dx = \frac{1}{2} \int (\text{cos } (n-m\pi x) + \text{cos } (n+m\pi x)) dx$$

$$\int (\text{cos } n\pi x) \text{sin } (m\pi x) dx = \frac{1}{2} \int (\text{sin } (n-m\pi x) - \text{sin } (n+m\pi x)) dx$$

$$\int (\text{sin } n\pi x) \text{sin } (m\pi x) dx = \frac{1}{2} \int (\text{cos } (n-m\pi x) - \text{cos } (n+m\pi x)) dx$$

Q. Find Fourier Series of $f(x) = \begin{cases} 0, & -2 < x < -1 \\ 1+x, & -1 < x < 0 \\ 1-x, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$

$$f(x) = \begin{cases} 0, & -2 < x < -1 \\ 1+x, & -1 < x < 0 \\ 1-x, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$$

Ans. $\therefore f(-x) = \begin{cases} 0, & -2 < x < -1 \\ 1-x, & -1 < x < 0 \\ 0, & 1 < x < 2 \end{cases}$

$f(x) = f(-x)$ $\therefore f(x)$ is even, $b_n = 0$

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx = \frac{1}{2} \int_{-2}^2 f(x) dx$$

$$= \frac{1}{2} \int_0^2 (1-x) dx \quad (\text{since even})$$

$$a_0 = \frac{1}{2} \int_0^2 (1-x) dx = \left[\frac{x - \frac{x^2}{2}}{2} \right]_0^2 = \frac{1}{2} \left[2 - \frac{2}{2} \right] = \frac{1}{2}$$

$$\therefore a_0 = \frac{1}{2}$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2} \int_0^2 f(x) \cos\left(\frac{n\pi x}{2}\right) dx$$

$$\therefore a_n = \int_0^2 (1-x) \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \left[\frac{(1-x) \sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} - \frac{(+1)(+\cos\left(\frac{n\pi x}{2}\right))}{\left(\frac{n\pi}{2}\right)^2} \right]_0^2$$

$$\therefore a_n = \left[\frac{-4}{n^2\pi^2} \left[\cos\left(\frac{n\pi x}{2}\right) \right]_0^2 \right]$$

$$a_n = \frac{-4}{n^2\pi^2} \left(\cos\left(\frac{n\pi}{2}\right) - 1 \right)$$

P.D.

S.P. Pg

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$= \frac{1}{4} - \frac{4}{16\pi^2} \sum_{n=1}^{\infty} \left[\frac{\cos(n\pi x)}{n^2} \times (\cos\left(\frac{n\pi}{2}\right) - 1) \right]$$

→ Parseval's Identity for periodic functions ↗

For Even:

$$\boxed{\frac{2}{L} \int_0^L f^2(x) dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2)}$$

for odd:

$$\boxed{\frac{2}{L} \int_0^L f^2(x) dx = \sum_{n=1}^{\infty} (b_n^2)}$$

55 pages
Half-Range Series \rightarrow when entire Fourier Series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})]$$

\rightarrow Half-Range Sine Series: $\frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L})$

Half-Range Sine series for the function $f(x)$ in the interval $(0, L)$ is given by :

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\text{where, } b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$(1 - n(-1)^n) \cdot \frac{1}{n\pi} = \frac{(-1)^{n+1}}{n}$$

\rightarrow Half-Range Cosine Series:

Half-Range Cosine Series for the function $f(x)$ in the interval $(0, L)$ is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$\text{where, } a_0 = \frac{2}{L} \int_0^L f(x) dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{L} \int_0^L f(x) d\left(\frac{n\pi x}{L}\right) = \left[f(x) \frac{n\pi}{L} \right]_0^L$$

$$= \frac{1}{L} \int_0^L f(x) d\left(\frac{n\pi}{L} x\right) = \frac{1}{L} \int_0^L f(x) d\left(\frac{n\pi}{L} x\right)$$

$$= \frac{1}{L} \int_0^L f(x) d\left(\frac{n\pi}{L} x\right) = \frac{1}{L} \int_0^L f(x) d\left(\frac{n\pi}{L} x\right)$$

Q. Find half-range cosine series for $f(x) = x \sin x$ in $(0, \infty)$ and deduce that $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots = \frac{\pi^4}{96}$

Ans. $a_0 = \frac{2}{2} \int_0^2 x dx = \left[\frac{x^2}{2} \right]_0^2 = 2$

$\therefore a_0 = 2$

$$a_n = \frac{2}{2} \int_0^2 x \cos\left(\frac{n\pi x}{2}\right) dx = \left[x \sin\left(\frac{n\pi x}{2}\right) + \left(1\right)\left(+\cos\left(\frac{n\pi x}{2}\right)\right) \right]_0^2$$

$$\therefore a_n = \frac{4}{n^2\pi^2} ((-1)^n - 1)$$

$$\therefore a_n = \begin{cases} 0 & \text{if } n = 2k \text{ (even)} \\ \frac{-8}{n^2\pi^2} & \text{if } n = 2k-1 \end{cases}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) = \frac{8}{\pi^2} - \frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{\cos((2k-1)\pi x)}{(2k-1)^2}$$

By Parseval's Identity

$$\frac{2}{1} \int_0^1 f^2(x) dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\therefore \int_0^2 x^2 dx = 2 + \frac{64}{\pi^4} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4}$$

$$\therefore \frac{8}{3} = \frac{64}{\pi^4} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4}$$

$$\therefore \frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots$$

Hence Proved.

Q. Find half-range cosine series of period 2π to represent $f(x) = \sin x$ in the interval $(0, \pi)$. Deduce that:

$$(i) \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \dots = \frac{1}{2}$$

$$(ii) \frac{1}{2 \cdot 3^2} + \frac{1}{3^2 \cdot 5^2} + \frac{1}{5^2 \cdot 7^2} + \dots = \frac{\pi^2 - 8}{16}$$

$$(iii) \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Ans. $a_0 = \frac{2}{\pi} \int_0^\pi f(x) dx = \frac{2}{\pi} \int_0^\pi \sin x dx = \frac{2}{\pi} [-\cos x]_0^\pi = \frac{4}{\pi}$

$$= \frac{2}{\pi} \left[-\cos x \right]_0^\pi \quad \therefore a_0 = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi} \int_0^\pi \sin x \cdot \cos \left(\frac{n\pi x}{\pi} \right) dx$$

$$= \frac{-2}{\pi} \int_0^\pi (\sin(n+1)x + \sin(n-1)x) dx$$

$$= \frac{1}{\pi} \left[\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right]_0^\pi$$

$$\therefore a_n = \frac{-4}{\pi(n^2-1)}, \text{ for } n=2k \quad (n \neq 1)$$

Checking a_1 ↴
(check!)

$$a_1 = 0$$

$\text{trigonometric} \cong \frac{2\pi}{\pi} - 4 \sum_{k=1}^{\infty} \cos(2k\pi)$ and also separation by $\frac{1}{4}$ multiple of π will be $\pi/2 - (n\pi)$.

$$\frac{1}{2} + \dots + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2} \quad (\text{i})$$

(i) \therefore Putting $n=0$,

$$0 = \frac{2}{\pi} - 4 \sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)} \quad (\text{ii})$$

$$\therefore \frac{\pi^2}{8} = \frac{(1+(1\pi))^2 + 1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} \quad // \text{ Hence Proved.}$$

(ii) Using Parseval's Identity \downarrow (complete!)

(Use according to which half-range series!)

$$\left| \frac{nb(n(n+1)\pi/2 + n(n+1)\pi/2)}{n^2} \right|^2 \frac{1}{n^2} = nb^2$$

$$\left| \frac{nb(n(n+1)\pi/2 + n(n+1)\pi/2)}{n^2} \right|^2 \frac{1}{n^2} =$$

$$\left[\frac{n(n+1)\pi/2 + n(n+1)\pi/2}{n^2} \right]^2 =$$

$$(1+\frac{1}{n})^2 \pi^2 = n^2 \pi^2, \quad \frac{1}{n^2} = nb^2 \Rightarrow (1+\frac{1}{n})^2$$

It is the answer
(Required)

(iii) Putting $x = \frac{\pi}{2}$, $\Rightarrow (\pi, 0)$ lies on the unit circle.

$$1 = \frac{12}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2} \left[\frac{1}{2k-1} - \frac{1}{2k+1} \right] (-1)^{k+1}$$

$\therefore \left(\frac{-2}{\pi}\right) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2} \left(\frac{1}{2k-1} - \frac{1}{2k+1} \right) (-1)^{k+1}$ of form $\pi = \frac{4}{2} \sum_{k=1}^{\infty} \frac{1}{2k-1} - \frac{1}{2k+1}$

$$\therefore \frac{\pi}{4} = \frac{1}{2} \left[\left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{7} - \frac{1}{9} \right) + \dots \right]$$

$$\therefore \frac{\pi}{4} = \frac{1}{2} \left[\frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} \right) + \frac{1}{2} \left(\frac{1}{7} - \frac{1}{9} \right) + \dots \right]$$

$$\therefore \frac{\pi}{4} = \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \dots$$

Hence Proved.

$$\text{Now } (x_1 \cos \theta_1 + y_1 \sin \theta_1)^m = (x_1 \cos \theta_1)^m \cdot (y_1 \sin \theta_1)^m$$

$$= m(x_1 \cos \theta_1)^{m-1} \cdot x_1 \cos \theta_1 \cdot (y_1 \sin \theta_1)^m$$

$$= m(x_1 \cos \theta_1)^{m-1} \cdot x_1 \cos \theta_1 \cdot \frac{m}{m-1} \cdot \frac{(m-1)!}{(m-2)!} \cdot (y_1 \sin \theta_1)^m$$

similarly

Q. Prove that in the interval $(0, \pi)$, $e^{ax} - e^{-ax}$ is $\frac{2}{\pi} \left[\frac{\sin x}{a^2+1} - \frac{2\sin(2x)}{a^2+4} + \frac{3\sin(3x)}{a^2+9} - \dots \right]$

$$\frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}} = \frac{2}{\pi} \left[\frac{\sin x}{a^2+1} - \frac{2\sin(2x)}{a^2+4} + \frac{3\sin(3x)}{a^2+9} - \dots \right]$$

Ans. We need to find half-range sine series for the function

$$e^{ax} - e^{-ax} = f(x) \text{ in the interval } (0, \pi)$$

$$\therefore b_n = \frac{2}{\pi} \int_0^\pi (e^{ax} - e^{-ax}) \sin(nx) dx$$

$$= \frac{2}{\pi} \left[\int_0^\pi e^{ax} \sin(nx) dx - \int_0^\pi e^{-ax} \sin(nx) dx \right]$$

$$\left[\text{We know, } \int e^{ax} \sin(nx) dx = \frac{e^{ax}}{a^2+n^2} (a\sin(nx) - n\cos(nx)) \right]$$

$$\therefore b_n = \frac{2}{\pi} \left[\left[\frac{e^{ax}}{a^2+n^2} (a\sin(nx) - n\cos(nx)) \right]_0^\pi - \left[\frac{e^{-ax}}{a^2+n^2} (a\sin(nx) - n\cos(nx)) \right]_0^\pi \right]$$

$$b_n = \frac{2}{(a^2+n^2)\pi} \left[e^{a\pi}(-n)\cos(n\pi) + e^{-a\pi}n\cos(n\pi) + n - n \right]$$

$$b_n = \frac{-2n(-1)^n}{\pi(a^2+n^2)} (e^{a\pi} - e^{-a\pi}) = \frac{2n(-1)^{n+1}}{\pi(a^2+n^2)} (e^{a\pi} - e^{-a\pi})$$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) = \frac{2(e^{a\pi} - e^{-a\pi})}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot \sin(nx) \cdot n}{a^2+n^2}$$

$$\therefore \frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}} = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot \sin(nx) \cdot n}{a^2+n^2}$$

$$\therefore \frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}} = \frac{2}{\pi} \left[\frac{\sin(x)}{a^2+1} - \frac{2\sin(2x)}{a^2+4} + \frac{3\sin(3x)}{a^2+9} - \dots \right]$$

Hence Proved,

- Complex form of Fourier Series \rightarrow useful methods

Complex form of Fourier Series f_i for the function $f(x)$ in the interval $(a, a+2l)$ is given by :

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{inx}{l}}$$

$$\text{where } C_n = \frac{1}{2l} (a_n - ib_n)$$

$$= \frac{1}{2l} \int_a^{a+2l} f(x) \cdot e^{-\frac{inx}{l}} dx$$

$$\text{Also, } C_0 = \frac{a_0}{2}$$

$$C_n = \frac{1}{2} (a_n + ib_n)$$

Q. Obtain complex form of Fourier series for ~~$f(x)$~~ .

$$f(x) = \begin{cases} a, & \text{if } 0 < x < l \\ -a, & \text{if } l < x < 2l \end{cases}$$

a is a positive constant. Hence deduce corresponding trigonometric Fourier series.

Ans.

$$\begin{aligned} \therefore C_n &= \frac{1}{2l} \int_0^{2l} f(x) \cdot e^{-inx/l} dx \\ &= \frac{1}{2l} \left[a \int_0^l e^{-inx/l} dx - a \int_l^{2l} e^{-inx/l} dx \right] \\ &= \frac{a}{2l} \left[\left[\frac{e^{-inx/l}}{-in\pi/l} \right]_0^l - \left[\frac{e^{-inx/l}}{-in\pi/l} \right]_l^{2l} \right] \end{aligned}$$

$$C_n = \frac{a}{2l} \left(\frac{-l}{in\pi} \right) \left[(e^{-in\pi} - 1) - (e^{-i2n\pi} - e^{-in\pi}) \right]$$

[we know, $e^{i\theta} = \cos\theta + i\sin\theta$, $\bar{e}^{i\theta} = \cos\theta - i\sin\theta$]

$$\therefore C_n = \frac{ai}{2n\pi} \left[(-1)^n - 1 - 1 + (-1)^n \right] = \frac{ai}{n\pi} \left[(-1)^n - 1 \right] \quad (n \neq 0)$$

$$\begin{aligned} \therefore C_0 &= \frac{a_0}{2} = \frac{1}{2l} \int_0^{2l} f(x) dx \\ &= \frac{a}{2l} \left[\int_0^l dx - \int_l^{2l} dx \right] = 0 \end{aligned}$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} C_n e^{-inx/l}$$

$$f(x) = \frac{a}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n - 1}{n} \cdot e^{-inx/l} \quad (n \neq 0)$$

(continued) →

we know, $C_n = \frac{1}{2} (a_n - ib_n)$ for most values left. \therefore
 $(x^s - 1) \ln(x^s) dx + (xs) dx = (x^s + 1) \ln(x^s) dx$

$$C_{-n} = \frac{1}{2} (a_n + ib_n) + \left(\frac{x^s - 1}{s} \right) = (x^s + 1) \cdot 2ia$$

$$\therefore C_n + C_{-n} = a_n$$

$$\therefore a_n = \frac{ai}{n\pi} [(-1)^n - 1] - \frac{ai}{n\pi} [(-1)^{n+1} - 1] = n$$

$$\therefore a_n = 0$$

$$\text{Also, } C_n - C_{-n} = -ib_n$$

$$\therefore b_n = i(C_n - C_{-n})$$

$$= i \left[\frac{ai}{n\pi} [(-1)^n - 1] + \frac{ai}{n\pi} [(-1)^{n+1} - 1] \right]$$

$$b_n = -\frac{2a}{n\pi} [(-1)^n - 1] \quad ((n\pi - 0)) dx = 0$$

$$a_0 = 2C_0 = 0$$

$$\therefore f(n) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$= -\frac{2a}{\pi} \sum_{n=1}^{\infty} \underbrace{[(-1)^n - 1]}_{n\pi - 0} \cdot \sin(nx)$$

$$= \frac{4a}{\pi} \sum_{n=1}^{\infty} \underbrace{\sin((n\pi - 0)x)}_{\sin(n\pi - 0)} =$$

Q: Find complex form of Fourier Series

for $f(x) = \sinh(2x) + \cosh(2x)$ in $(-5, 5)$.

Ans.

$$\therefore f(x) = \left(\frac{e^{2x} - e^{-2x}}{2}\right) + \left(\frac{e^{2x} + e^{-2x}}{2}\right)$$

$$= e^{2x}$$

$$C_n = \frac{1}{2L} \int_a^{a+2L} f(x) e^{-inx} dx$$

$$= \frac{1}{10} \int_{-5}^5 e^{2x} \cdot e^{-inx} dx = \frac{1}{10} \int_{-5}^5 e^{x(2-\frac{in\pi}{5})} dx$$

$$= \frac{1}{10} \left[\frac{e^{(\frac{10-in\pi}{5})x}}{\frac{10-in\pi}{5}} \right]_{-5}^5$$

$$\therefore C_n = \frac{1}{10} \times \frac{5}{10-in\pi} \left[e^{(\frac{10-in\pi}{5})5} - e^{(\frac{10-in\pi}{5})(-5)} \right]$$

$$\therefore C_n = \underline{\sinh(10-in\pi)}$$

$10-in\pi$
and dividing

Multiplying λ with conjugate, $(10+in\pi)$

$$C_n = \frac{(10+in\pi) \times \sinh(10-in\pi)}{100+n^2\pi^2}$$

\therefore

$$\therefore f(n) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{intx}$$

$(\text{con}) \text{ and } \mathbb{R} = (\text{re}) + i\text{im}$

$$= \sum_{n=-\infty}^{\infty} \frac{(10+in\pi) \times \sinh(10-in\pi) \times e^{intx}}{100+n^2\pi^2}$$

\parallel

Q. Find complex form of Fourier series. for a periodic function

$$\text{for } f(x) = \begin{cases} 0, & 0 < x < l \\ a, & l < x < 2l \end{cases}$$

Ans. Does it make sense, if $l > 0$ and $a < 0$?

$$\text{Ans. } C_n = \frac{1}{2l} \int_0^{2l} f(x) e^{-inx/l} dx$$

$$= \frac{1}{2l} \left[\int_0^l 0 \cdot e^{-inx/l} dx + \int_l^{2l} a \cdot e^{-inx/l} dx \right]$$

$$= \frac{1}{2l} \left[a \cdot \frac{-e^{-inx/l}}{-in\pi/l} \right]_l^{2l}$$

$$= -\frac{a}{2l} \times \frac{l}{in\pi/l} \times [e^{-i2\pi n} - e^{-in\pi}]$$

$$= \frac{ai}{2n\pi} [\cos(2\pi n) - i\sin(2\pi n) - \cos(\pi n) + i\sin(\pi n)]$$

$$C_n = \frac{ai}{2n\pi} [i(1 - (-1)^n)]$$

(n ≠ 0, since n in denominator)

∴ finding C_0 , ∴ $C_0 = a_0/2$

$$\therefore C_0 = \frac{1}{2l} \int_0^{2l} f(x) dx$$

$$= \frac{1}{2l} \left[ax \right]_l^{2l} = \frac{a}{2} (2l - l) \text{ as } C_0 = a/2$$

$$\text{Total form: } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{ai}{2n\pi} [1 - (-1)^n] \cdot e^{inx/l}$$

∴ for a periodic and continuous

function, if $f(x)$ is even, then $a_n = 0$ for all n

if $f(x)$ is odd, then $b_n = 0$ for all n

if $f(x)$ is periodic, then $a_0 = 0$ for all n