Module 5:- Vector Integration

(i) Line Integral

v.imp sure coming (ii) → Green's theorem → (Sure coming) (1 sure Q 6 marks)

(iii) Stoke's theorem (1 question 5 marks)

(iv) Gauss Divergence theorem

Q) $\int_C \bar{F} \cdot d\bar{r}$

$\bar{F} = f_1 i + f_2 j + f_3 k$

$d\bar{r} = dx i + dy j + dz k$

$\bar{F} \cdot d\bar{r} = f_1 dx + f_2 dy + f_3 dz$

$\int_{A(0,0,0)}^{B(1,1,1)} f_1 dx + f_2 dy + f_3 dz$

$A \xrightarrow[C]{} B$

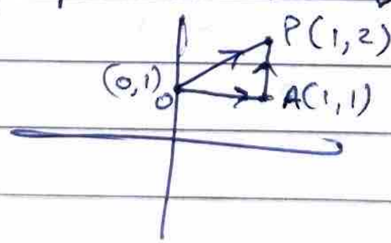
Q) Evaluate $\int_C \bar{F} \cdot d\bar{r}$

where $\bar{F} = (x^2 - y) i + (y^2 + x) j$

(i) ~~from~~ From $O(0,1)$ to $P(1,2)$ along the line OP.

(ii) Along the st. line from $(0,1)$ to $(1,1)$ and then along the line $(1,1)$ ~~and then~~ to ~~and~~ $(1,2)$

(iii) along the parabola $x = t$; $y = t^2 + 1$



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Ans:- $\int_C \bar{F}\cdot d\bar{r} = \int_C (x^2-y)\,dx + (y^2+x)\,dy$

(i) Along the line joining $O(0,1)$ to $P(1,2)$

C_1:- eqn of OP

$$\frac{y-y_1}{x-x_1} = \frac{y_1-y_2}{x_1-x_2}$$

$$\frac{y-1}{x-0} = \frac{1-2}{0-1} = 1$$

$$y-1 = x$$

$$y = x+1$$

$$\boxed{dy = dx}$$

$$\int_{C_1} \bar{F}\cdot d\bar{r} = \int_0^1 \left(x^2-(x+1)\right)dx + \left((x+1)^2+x\right)dx$$

$$= \int_0^1 (x^2-x-1)\,dx + (x^2+1+3x)\,dx$$

$$= \int_0^1 (2x^2+2x)\,dx = \left(\frac{2x^3}{3}+x^2\right)_0^1 = \frac{2}{3}+1$$

$$= \boxed{\frac{5}{3}}$$

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(ii) C_2 :- O to A + A to P

eqⁿ of OA :- $y = 1$; $dy = 0$

eqⁿ of AP :- $x = 1$; $dx = 0$

$$\int_{C_2} \bar{F} \cdot d\bar{r} = \int_{OA} \bar{F} \cdot d\bar{r} + \int_{P} \bar{F} \cdot d\bar{r}$$

$$= \int_0^1 (x^2 - 1)\, dx + \int_1^2 (y^2 + 1)\, dy$$

$$= \left[\frac{x^3}{3} - x\right]_0^1 + \left[\frac{y^3}{3} + y\right]_1^2$$

$$= \left(\frac{1}{3} - 1\right) + \left(\frac{8}{3} + 2 - \frac{1}{3} - 1\right)$$

$$= 3 - \cancel{2} + \cancel{2} - \frac{1}{3}$$

$$= \boxed{\frac{8}{3}}$$

(iii) along the parabola :-

C_3 :- $x = t$ $\quad dx = dt$

$t \to 0$ to 1 since $\nearrow$ $y = t^2 + 1$ $\quad dy = 2t\,dt$

$t \to 0$ to 1

$$\int_{C_3} \bar{F} \cdot d\bar{r} = \int_0^1 \left(t^2 - (t^2 + 1)\right) dt + \left((t^2+1)^2 + t\right) 2t\,dt$$

$$= \int_0^1 (-dt) + (t^4 + 2t^2 + 1 + t)\, 2t\,dt$$

$$= \int_0^1 -1 + 2t^5 + 4t^3 + 2t + 2t^2)\, dt$$

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$$= \left[-t + \frac{2t^6}{6} + \frac{4t^4}{4} + \frac{2t^2}{2} + \frac{2t^3}{3} \right]_0^1$$

$$= \boxed{2}$$

Note:- Here we can conclude that vector integration path dependent.

But If $\bar{F}$ is a conservative field then the value of integral does not depend on the path.

Also if we integrate a conservative field over a closed curve then the value of integral will be zero.

$\bar{F}$ is conservative if curl $= 0$

If field is conservative work done is zero.

Work done means to find integral.

Q) Evaluate :- $\int_C \bar{F} \cdot d\bar{r}$

where $\bar{F} = yz\,i + zx\,j + xy\,k$

and c is the ~~pro~~ portion of the curve.

$\bar{r} = (a\cos t)\,i + (b\sin t)\,j + ct\,k$

from $t = 0$ to $t = \frac{\pi}{4}$

$x = a\cos t \qquad y = b\sin t \qquad z = ct$

$dx = -a\sin t\,dt \qquad dy = b\cos t\,dt \qquad dz = c\,dt$

$\int \bar{F} \cdot d\bar{r} = \int yz\,dx + zx\,dy + xy\,dz$

$\cancel{\bar{F} = (b\sin t)(ct)\,i + (c \cdot a\cos t)\,j + (a\cos t)(b\sin t)}$

$\cancel{\int \bar{F} \cdot d\bar{r} = \int [(bct\sin t)\,i + (ca\cos t)\,j + (ab\sin t\cos t)\,k]\,dt}$

$=$

$$\int \bar{F}\cdot d\bar{r} = \int (bct\sin t)\,dx + (ca\cos t)\,dy + (ab\sin t\cos t)\,dz$$

$$= \int_0^{\pi/4} [(bct\sin t)(-a\sin t)\,dt + (ca\cos t)(b\cos t)\,dt + (ab\sin t\cos t)(c)\,dt]$$

$$= \int_0^{\pi/4} -(abct\sin^2 t) + (abct\cos^2 t) + (abc\sin t\cos t)$$

$$= abc \int_0^{\pi/4} (t\cos^2 t + \sin t\cos t - t\sin^2 t)\,dt$$

$$= abc \int_0^{\pi/4} \left(t\cos^2 t + \frac{\sin 2t}{2} - t\sin^2 t\right) dt$$

$$= abc$$

$$\int \bar{F}\cdot d\bar{r} = \int_0^{\pi/4} -abct\sin^2 t\,dt + abct\cos^2 t\,dt + abc\sin t\cos t\,dt$$

$$= abc \int_0^{\pi/4} (t(\cos^2 t - \sin^2 t) + \sin t\cos t)\,dt$$

$$= abc \int_0^{\pi/4} t\cos 2t + \frac{\sin 2t}{2}\,dt$$

$$= abc \left[t\left(\frac{\sin 2t}{2}\right) + (1)\left(\frac{-\cos 2t}{4}\right) - \frac{\cos 2t}{4} \right]_0^{\pi/4}$$

$$= abc\left[\frac{\pi}{4}\cdot\frac{\sin \pi/2}{2}\right]$$

$$= \frac{\pi abc}{8}$$