Tutorial - 10 Z - transform

Q1. Find Z-transform of $(3^k + 5^k)$, $k \geq 0$

A.

$$Z[3^k + 5^k] = Z[3^k] + Z[5^k] \quad \ldots\ldots \text{ Linearity prop.}$$

$$= \sum_{k=0}^{\infty} 3^k z^{-k} + \sum_{k=0}^{\infty} 5^k z^{-k}$$

$$= \left[1 + \frac{3}{z} + \left(\frac{3}{z}\right)^2 + \ldots\right] + \left[1 + \left(\frac{5}{z}\right) + \left(\frac{5}{z}\right)^2 + \ldots\right]$$

For $1 + \frac{3}{z} + \left(\frac{3}{z}\right)^2 + \ldots$, $a = 1$, $r = 3/z$

$$\therefore 1 + \frac{3}{z} + \left(\frac{3}{z}\right)^2 + \ldots = \frac{a}{1-r} = \frac{1}{1 - 3/z} = \frac{z}{z-3}$$

$$|z| > |3|$$

$$\text{|||}^{ly} \quad 1 + \frac{5}{z} + \left(\frac{5}{z}\right)^2 + \ldots = \frac{a}{1-r} = \frac{1}{1 - 5/z} = \frac{z}{z-5}$$

$$|z| > 5$$

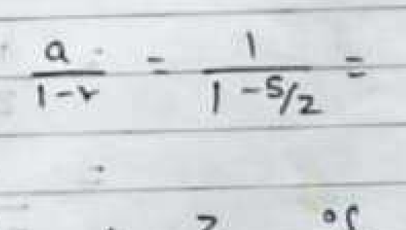
$$\therefore Z[3^k + 5^k] = \frac{z}{z-3} + \frac{z}{z-5} \quad \text{if } |z| > 5 \text{ \& } |z| > 3$$

$$= \frac{z^2 - 5z + z^2 - 3z}{(z-3)(z-5)} \quad \text{if } |z| > 5$$

$$= \frac{2z^2 - 8z}{(z-3)(z-5)}$$

$$\therefore \boxed{Z[3^k + 5^k] = \frac{2z(z-4)}{(z-3)(z-5)} \quad \text{if } |z| > 5}$$

ROC :-



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Q2. Find z-transform of $[e^{-3k} \sin 2k]$.

A. $$Z[\sin 2k] = Z\left[\frac{e^{+i2k} - e^{-i2k}}{2i}\right]$$

$$= \frac{1}{2i}\left[Z[e^{2ik}] - Z[e^{-2ik}]\right]$$

$$= \frac{1}{2i}\left[\sum_{k=0}^{\infty} e^{2ik} z^{-k} - \sum_{k=0}^{\infty} e^{-2ik} z^{-k}\right]$$

$$= \frac{1}{2i}\left[\sum_{k=0}^{\infty} \left(\frac{e^{2i}}{z}\right)^k - \sum_{k=0}^{\infty}\left(\frac{e^{-2i}}{z}\right)^k\right]$$

$$= \frac{1}{2i}\left[\frac{1}{1-\frac{e^{2i}}{z}} - \frac{1}{1-\frac{e^{-2i}}{z}}\right] \text{ for } \left|\frac{e^{2i}}{z}\right| < 1 \text{ \& } \left|\frac{e^{-2i}}{z}\right| < 1$$

$$= \frac{1}{2i}\left[\frac{z}{z-e^{2i}} - \frac{z}{z-e^{-2i}}\right] \text{ for } |z| > 1$$

$$= \frac{1}{2i}\left(\frac{z(e^{2i} - e^{-2i})}{z^2 - 2z(e^{2i} + e^{-2i}) + 1}\right) \text{ for } |z| > 1$$

$$= \frac{z \sin 2}{z^2 - 2z\cos 2 + 1} = F(z) \text{ for } |z| > 1$$

$$Z[e^{-3k} \sin 2k] = F(e^3 z) \quad \ldots\ldots \text{ Change of scale prop}$$

$$= \frac{(e^3 z)\sin 2}{(e^3 z)^2 - 2(e^3 z)\cos 2 + 1}$$

$$\boxed{Z[e^{-3k} \sin 2k] = \frac{e^3 z \sin 2}{e^6 z^2 - 2e^3 z \cos 2 + 1}}$$

Q3. Find z-transform of $[k^2 a^{k-1}]$, $k \geq 0$

A. $$Z[a^k] = \sum_{k=0}^{\infty} a^k z^{-k}$$
$f(k) = a^k$

$$= \sum_{k=0}^{\infty}\left(\frac{a}{z}\right)^k = 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \ldots\ldots$$

$$= \frac{1}{1-\frac{a}{z}} \qquad \left|\frac{a}{z}\right| < 1$$

$$Z[a^k] = \frac{z}{z-a} \text{ for } |z| > a$$

$$= F(z)$$

$$f(k-1)$$

$$Z[\cancel{a^{k-1}}] = Z[a^{k-1}] = z^{-1} F(z) \quad \text{.... Shifting prop}$$

$$= \frac{1}{\cancel{z}} \times \frac{\cancel{z}}{z-a}$$

$$= \frac{1}{z-a} = F_1(z) \text{ for } |z| > a$$

$$Z[k^2 a^{k-1}] = \left(-z\frac{d}{dz}\right)\left(-z\frac{d}{dz}\right) F_1(z)$$

$$= \left(-z\frac{d}{dz}\right)\left(-z\frac{d}{dz}\right) \frac{1}{z-a}$$

$$= \left(-z\frac{d}{dz}\right) \frac{z}{(z-a)^2}$$

$$= \frac{-z\left[(z-a)^2 - 2z(z-a)\right]}{(z-a)^4}$$

$$= \frac{z\cancel{(z-a)}(z+a)}{(z-a)^{\cancel{4}\,3}}$$

$$\boxed{Z[k^2 a^{k-1}] = \frac{z(z+a)}{(z-a)^3}} \quad \text{for } |z| > a$$

Q4. Find inverse Z-transform of $F(z) = \frac{1}{z-2}$

when i) $|z| < 2$ ii) $|z| > 2$

A. i) if $|z| < 2$ i.e. $1 < \left|\frac{2}{z}\right|$

$$F(z) = \frac{1}{\frac{\cancel{z+a}}{z-2}} = \frac{1}{2\cancel{a}\left(\frac{z}{\cancel{a}\,2} - 1\right)} = -\frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1}$$

$$= \frac{-1}{2}\left[1+\frac{z}{a}+\left(\frac{z}{a}\right)^2+\dots+\left(\frac{z}{a}\right)^{k-1}+\dots\right] \quad \dots \frac{1}{1-x} \quad |x|<1$$

$$= \cancel{-\frac{1}{2}\sum_{k=0}^{\infty} 2^{k} z} \; - \left[\frac{1}{2}+\frac{z}{2^2}+\left(\frac{z^2}{2^3}\right)+\dots\frac{z^{k-1}}{2^{k}}\right]$$

$$= \cancel{-\left[\frac{1}{2}+\frac{z}{2^2}+\frac{z^2}{2^3}+\dots\right]} \quad \cancel{-\frac{1}{2}-\frac{z}{2}-z}$$

$$= \cancel{-\sum_{k=0}^{-\infty} 2^{k-1} z^{k}} = \cancel{\sum_{k=0}^{-\infty}\left(-2^{k-1}\right) z^{-k}}$$

$\cancel{\therefore z^{-1}[F(z)] =}$ ~~Coefficient of~~ $\cancel{z^k} =$

$$= \cancel{-\frac{1}{2}} - 2^{-1} - z\cdot 2^{-2} - z^2 2^{-3} + \dots z^k 2^{-k-1}$$

Coefficient of $z^k = -2^{-k-1}, \quad k \geq 0$

Coefficient of $z^{-k} = -2^{k-1}, \quad k \leq 0$

$$\therefore \boxed{z^{-1}[F(z)] = -2^{k-1}}, \quad k \leq 0$$

~~Q5. Find inverse z-transform of~~ $\cancel{F(z) = \frac{z+2}{z^2-2z+1}}$

ii) if $|z| > 2$ i.e. $1 > \left|\frac{2}{z}\right|$

$$F(z) = \frac{1}{z-a} = \frac{1}{z\left(1-\frac{a}{z}\right)} = \frac{1}{z}\left(1-\frac{a}{z}\right)^{-1}$$

$$(1-x)^{-1} = 1+x+x^2+x^3+\dots$$

$$F(z) = \frac{1}{z}\left[1+\frac{a}{z}+\left(\frac{a}{z}\right)^2+\left(\frac{a}{z}\right)^3+\dots\frac{a^{k-1}}{z^{k-1}}\right]$$

$$= \frac{1}{z}+\frac{a}{z^2}+\frac{a^2}{z^3}+\dots+\frac{a^{k-1}}{z^{k}}$$

$$= \cancel{\sum_{k=0}^{\infty} a^{k-1} z^{-k}} \; \sum_{k=1}^{\infty} a^{k-1} z^{-k}$$

$\therefore$ ~~Coefficient of~~ $\cancel{z^{-k}}$

Compare with $Z[f(k)] = \sum_{k=0}^{\infty} f(k) z^{-k}$ ~~$k \le 0$~~

$\therefore f(k) = a^{k-1} 2^{k-1} \ldots$ ~~$k \le 0$~~ $k \ge 1$

$\therefore \boxed{Z^{-1}[F(z)] = 2^{k-1}} \ldots \quad k \ge 1$

Q5. Find inverse Z-transform of $F(z) = \dfrac{z+2}{z^2-2z+1}$, $|z|>1$

A. if $|z|>1$ i.e. $1 > \left|\dfrac{1}{z}\right|$

$$F(z) = \frac{z+2}{z^2-2z+1} = \frac{z+2}{(z-1)^2}$$

$$\frac{z+2}{(z-1)^2} = \frac{A}{(z-1)} + \frac{B}{(z-1)^2}$$

$$z+2 = A(z-1) + B$$

$$z+2 = zA + \cancel{(A-B)}\ (B-A)$$

comparing RHS and LHS

$A = 1$ ~~$A-B=2$~~ $B-A=2$

$\therefore B = 3$

$$\therefore F(z) = \frac{1}{z-1} + \frac{3}{(z-1)^2}$$

$$= \frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{3}{z^2\left(1-\frac{1}{z}\right)^2}$$

$$= \frac{1}{z}\left[1-\frac{1}{z}\right]^{-1} + \frac{3}{z^2}\left[1-\frac{1}{z}\right]^{-2}$$

$$= \frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\ldots\right] + \frac{3}{z^2}\left[1+\frac{2}{z}+\frac{3}{z^2}+\ldots\right]$$

$$= \left[\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots\right] + 3\left[\frac{1}{z^2}+\frac{2}{z^3}+\frac{3}{z^4}+\ldots\right]$$

$$= \left[z^{-1}+z^{-2}+z^{-3}+\ldots\right] + 3\left[z^{-2}+2z^{-3}+3z^{-4}+\ldots\right]$$

$$= \left[z^{-1} + z^{-2} + z^{-3} + \dots + z^{-k} + \dots \right] + 3\left[z^{-2} + 2z^{-3} + 3z^{-4} + \dots + (k-1)z^{-k} \right]$$

$$z^{-1}[F(z)] = 1 + 3(k-1)$$

$$\boxed{z^{-1}[F(z)] = 3k-2}$$

Q6. Find inverse Z- transform of $F(z) = \frac{z^2}{(z-3)(z-2)}$ using convolution theorem.

A. $F(z) = \frac{z^2}{(z-3)(z-2)} = \frac{z}{z-3} \times \frac{z}{z-2}$

$F_1(z) = \frac{z}{z-3}$ & $F_2(z) = \frac{z}{z-2}$

$= \frac{\cancel{z}}{\cancel{z}(1-\frac{3}{z})}$ $\qquad = \frac{\cancel{z}}{\cancel{z}(1-\frac{2}{z})}$

$= \left[1-\frac{3}{z}\right]^{-1}$ $\qquad = \left[1-\frac{2}{z}\right]^{-1}$

Using $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

$= 1 + \left(\frac{3}{z}\right) + \left(\frac{3}{z}\right)^2 + \dots$ $\qquad = 1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \dots$

$= 1 + 3z^{-1} + 3^2z^{-2} + \dots + 3^n z^{-n} + \dots$ $\qquad = 1 + 2z^{-1} + 2^2z^{-2} + \dots + 2^n z^{-n} \dots$

Coefficient of $z^{-n} = 3^n$ $\qquad$ Coefficient of $z^{-n} = 2^n$

$\therefore z^{-1}[F_1(z)] = 3^n = f_1(n)$ $\qquad z^{-1}[F_2(z)] = 2^n = f_2(n)$

Using Convolution th^m,

$$z^{-1}[F(z)] = z^{-1}[F_1(z)F_2(z)] = \sum_{m=0}^{n} f_2(m)\, f_1(n-m)$$

$$z^{-1}[F(z)] = \cancel{\sum_{m=0}^{n} 3^m 2^{n-m}} \; \sum_{m=0}^{n} 2^m 3^{n-m}$$

$$= 3^n \sum_{m=0}^{n} \left(\frac{2}{3}\right)^m$$

$$= 3^n \left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \dots + \left(\frac{2}{3}\right)^n\right]$$

$$\cancel{= 3^n \times \frac{1}{1-2/3}}$$

$$\cancel{= 3^n \times \frac{3}{1}}$$

$$\boxed{\cancel{z^{-1}[F(z)] = 3^{n+1}}}$$

$$= 3^n \times \frac{1-(2/3)^{n+1}}{1-2/3}$$

$$= \cancel{3^n} \times \frac{3^{n+1}-2^{n+1}}{3-2} \times \frac{\cancel{3}}{\cancel{3^{n+1}}}$$

$$\boxed{z^{-1}[F(z)] = 3^{n+1} - 2^{n+1}}$$