Functions and Pigeon Hole Principle

Syllabus

5.1 Definition and types of functions: Injective, Surjective and Bijective

5.2 Composition, Identity and Inverse

5.3 Pigeon-hole principle , Extended Pigeon-hole principle

DEFINITION OF FUNCTIONS

Let A and B be non–empty sets. A **function** f from A to B, denoted as **$f : A \rightarrow B$**, is a relation from A to B such that for every $a \in A$, there exists a **unique** $b \in B$ such that $(a, b) \in f$.

Normally if $(a, b) \in f$, we write $f(a) = b$

An important point to be re-emphasised is that f is a relation with the following special property :

If $f(a) = b$ and $f(a) = c$

then $b = c$

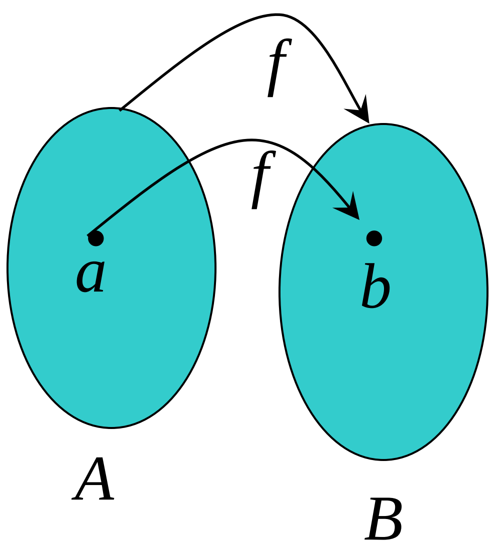
This condition implies that to each element $a \in A$, a unique element $b \in B$ should be assigned by the relation f.

Generic Functions

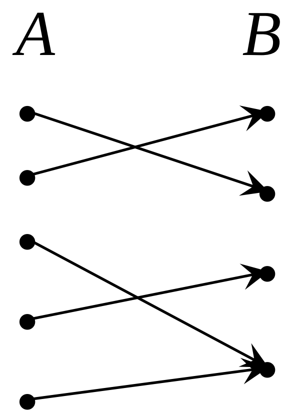
- A function f: X → Y is a relationship between elements of X to elements of Y, when each element from X is related to a unique element from Y
- X is called domain of f, range of f is a subset of Y so that for each element y of this subset there exists an element x from X such that y = f(x)
- Sample functions:
  - $f : R \rightarrow R$, $f(x) = x^2$
  - $f : Z \rightarrow Z$, $f(x) = x + 1$
  - $f : Q \rightarrow Z$, $f(x) = 2$

Graphical Representations

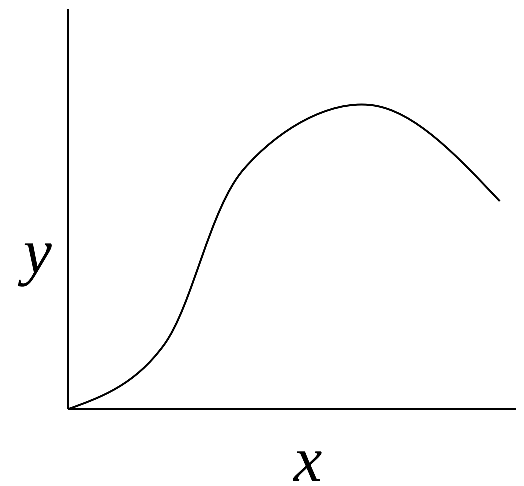
- Functions can be represented graphically in several ways:



Like Venn diagrams



Graph



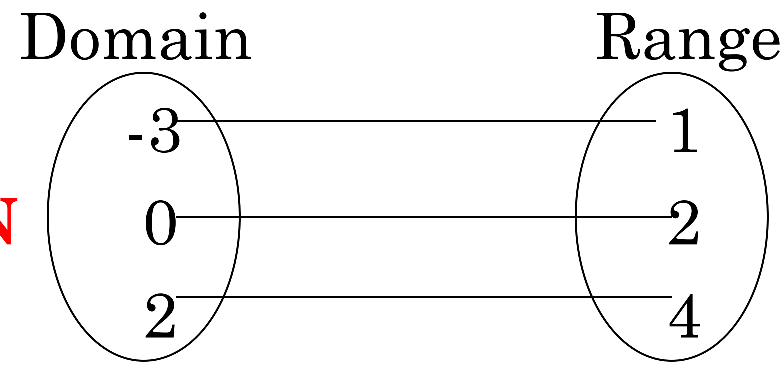
Plot

Some Function Terminology

- If $f:A\rightarrow B$, and $f(a)=b$ (where $a\in A$ & $b\in B$), then:
  - A is the *domain* of f.
  - B is the *codomain* of f.
  - b is the *image* of a under f.
  - a is a *pre-image* of b under f.
    - In general, b may have more than one pre-image.
  - The *range* $R\subseteq B$ of f is $\{b \mid \exists a\, f(a)=b\}$.

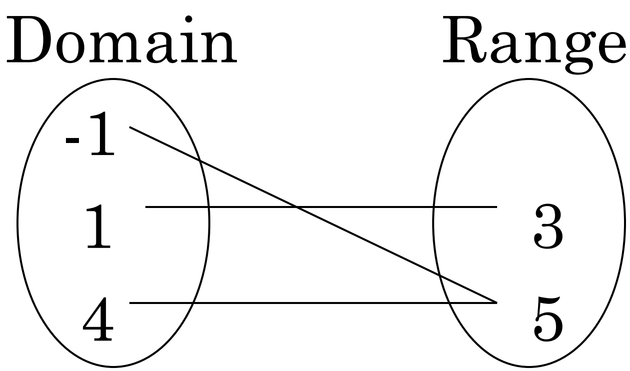
Example 1 {(-3,1),(0,2),(2,4)}

FUNCTION



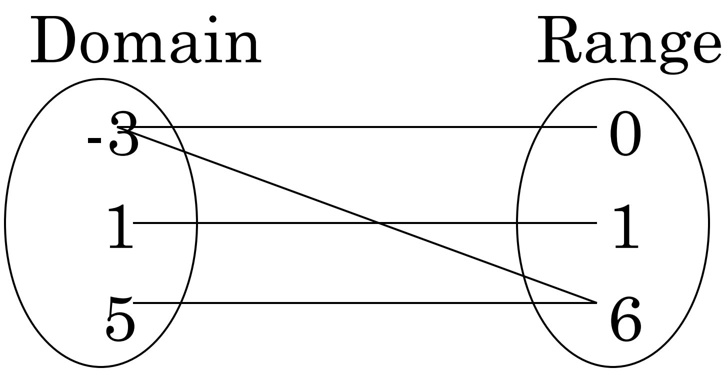
1 to 1

{(-1,5),(1,3),(4,5)}



FUNCTION

{(5,6),(-3,0),(1,1),(-3,6)}



Not a Function

Example 2: If f is the mod - 12 function, compute each of the following :

(i) $f(1259 + 743)$ (ii) $f(1259) + f(743)$

(iii) $f(2.319)$ (iv) $2 \cdot f(319)$

Solution:

(i) $f(1259 + 743) = f(2002)$

$= 2002 \% 12 = 10$

(ii) $f(1259) + f(743) = 1259\% 12 + 743\%12$

$= 11 + 11$

$= 22$

(iii) $f(2.319) = 2.319 \% 12$

$= 2.319$

(iv) $2 \cdot f(319) = 2 \times 319 \% 12 = 2 \times 7$

$= 14$

Example 3

Given $f(x) = 3x - 5$ and $g(x) = x^2 + 2$, find:

(a) $f(-3)$

$$f(x) = 3x - 5$$

$$f(-3) = 3(-3) - 5$$

$$= -9 - 5 = -14$$

(b) $g(2z)$

$$g(x) = x^2 + 2$$

$$g(2z) = (2z)^2 + 2$$

$$= (2)^2(z)^2 + 2$$

$$= 4z^2 + 2$$

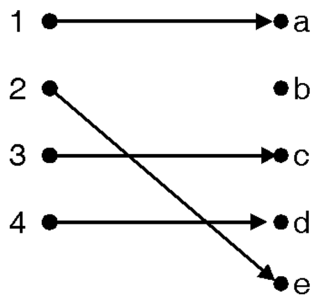
TYPES OF FUNCTIONS

- One to One (Injective)
- Onto (Surjective)
- One to One Onto Function (Bijective)

One-to-One Functions

- Function f : X → Y is called one-to-one (injective) when for all elements x1 and x2 from X if f(x1) = f(x2), then x1 = x2
- Iff every element of its range has only one pre-image. Only one element of the domain is mapped to any given one element of the range.

Example : Let A = { 1, 2, 3, 4 }, and B = { a, b , c, d, e }.



and f = {(1, a), (2, e), (3, c), (4, d)}.

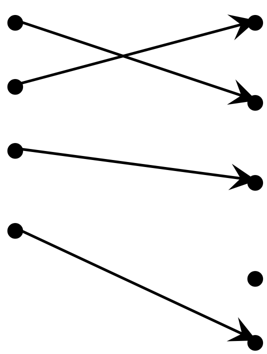
f (1) = a f (2) = e

f (3) = c f (4) = d

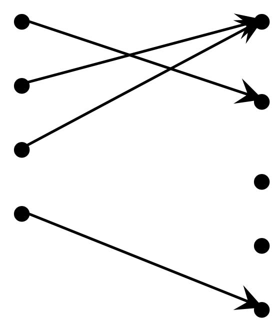
Given function *f* is one to one or injective function.

One-to-One Illustration

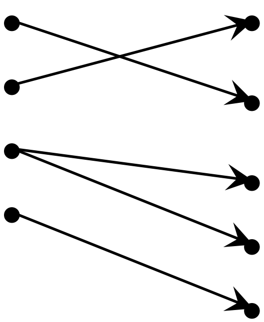
- Graph representations of functions that are (or not) one-to-one:



One-to-one



Not one-to-one



Not even a function!

- Let A = {1, 2, 3} and B = {a, b, c, d}. Which of the following is a one-to-one function?

Choices:

A. {(1,a), (2,c), (3,a)}
B. {(1,b), (2,d), (3,a)}
C. {(1,a), (2,a), (3,a)}
D. {(1,c), (2,b) (1,a), (3,d)}

Correct Answer: B

Onto Functions

Function f : X → Y is called onto (surjective) when given any element y from Y, there exists x in X so that f(x) = y

Determine whether the following functions is onto:

Example : Let A = { 1, 2, 3, 4 } and B = { a, b, c, d}.

and f = {(1, a), (2, d), (4, c), (3, b)}.

f (1) = a, f (2) = d, f (3) = b,

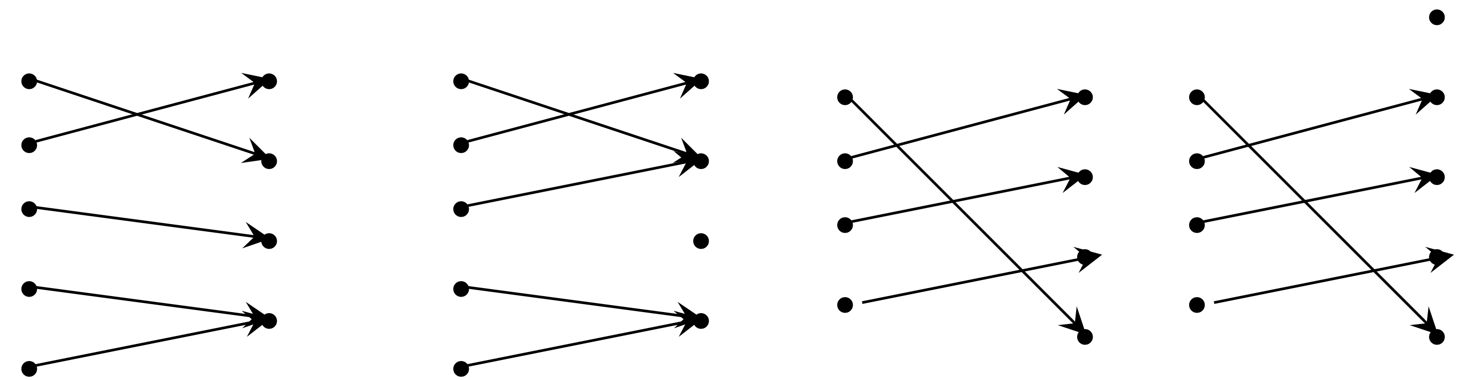
f (4) = c,

Ran (*f*) = { a, b, c, d } = B.

So, this function is onto or surjective function.

ILLUSTRATION OF ONTO

- Some functions that are or are not *onto* their codomains:



Onto (but not 1-1)

Not Onto (or 1-1)

Both 1-1 and onto

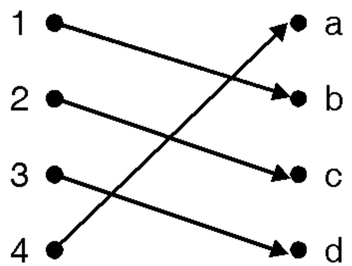
1-1 but not onto

One to One Onto(Bijective) Functions

- Function from A to B is said to be one to one onto (bijective) if it is both one to one and onto function
- Determine whether the following functions is bijective:

Example : Let A = { 1, 2, 3, 4 }, B = { a, b, c, d }.

and f = {(1, b), (2, c), (3, d), (4, a)}.



f(1) = b, f(2) = c,

f(3) = d, f(4) = a.

Given function *f* is one to one onto or bijective function.

Everywhere defined function

A function f is said to be everywhere defined if Dom(f) = A

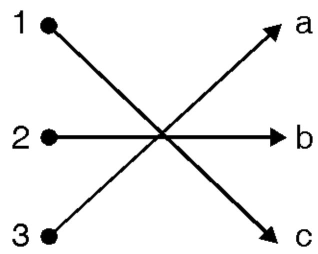
Example : Let A = {1, 2, 3 }, B = { a, b, c }.

and f = {(1, c), (2, b), (3, a)}.

f (1) = c, f (2) = b,

f (3) = a.

Dom (f) = { 1, 2, 3 }. Thus given function f is everywhere defined function.



PROBLEMS

Q1 : Is the following function one to one ?

$f : Z \to Z$, where $f(x) = 2x - 1$.

Soln. :

Let $a, b, \in z$ and $f(a) = f(b)$ then

$2a - 1 = 2b - 1.$

$a = b.$

$\therefore$ f is one to one function

Q 2 : Is the following function one to one ?

$g : Z \to Z$ where $g(x) = x^2$.

Soln. : $g(-2) = g(2) = 4$.

Two numbers are giving same output so above function is not one to one.

Q 3 : Let $A = \{ 0, -1, 1 \}$ and $B = \{ 0, 1\}$ Let $f : A \to B$ where $f(a) = |a|$. Is f onto?

Soln. :

The elements of B are 0 and 1.

Hence f is onto if we can find $x, y \in A$ such that $f(x) = 0$ and $f(y) = 1$.

Now, $f(0) = |0| = 0$ and

$f(-1) = |-1| = 1$.

Hence to each element b of B there is an element $a \in A$.

$\therefore f(a) = b$. Hence f is onto

Q 5.: Let A = B = R, the set of real numbers. Let $f : A \to B$ be given by the formula $f(x) = 2x^3 - 1$, and let $g : B \to A$ be given by

$$g(y) = \sqrt[3]{\frac{1}{2}Y + \frac{1}{2}}$$

Show that f is bijection between. A and B and g is a bijection between B and A.

Soln. :

A function from A to B is Bijection if it is one to one and onto

$\therefore$ for $f(x) = 2x^3 - 1$ to be one to one and onto.

If a, b, $\in$ A

Such that $f(a) = f(b)$

$\Rightarrow 2a^3 - 1 = 2b^3 - 1$

$a = b$

$\therefore$ f is one to one

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Now for $y = 2x^3 - 1$

$$1 + y = 2x^3$$

$$x^3 = \frac{1}{2} + \frac{y}{2}$$

$$\therefore \quad x = \sqrt[3]{\frac{1}{2} + \frac{y}{2}}$$

$\therefore$ for each $y \in B$. There is a unique x in A such that $f(x) = y$.

$\therefore$ f is onto.

$\therefore$ f is bijective function between A and B.

Similarly for $g : B \to A$ to be one to one and onto

$$g(a) = g(b) = \sqrt[3]{\frac{1}{2} + \frac{a}{2}} = \sqrt[3]{\frac{1}{2} + \frac{b}{2}}$$

$$\therefore \quad \frac{1}{2} + \frac{a}{2} = \frac{1}{2} + \frac{b}{a}$$

$$\Rightarrow \quad a = b$$

$\therefore$ g is one to one.

Also for $x = \sqrt[3]{\frac{1}{2} + \frac{y}{2}}$

$$x^3 = \frac{1}{2} + \frac{y}{2}$$

$$2x^3 = 1 + y$$

$$y = 2x^3 - 1$$

for each x in A. There is a corresponding y in B.
Such that $g(y) = x$

$\therefore$ g is onto function

So g is bijective function between B and A.

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COMPOSITION OF FUNCTIONS

- Let f be a function from A to B (i.e. $f : A \rightarrow B$) and g be a function from B to C (i.e. $g : B \rightarrow C$). Then the composition of f and g denoted as "g o f " is a relation from A to C, where $g \circ f(a) = g(f(a))$.
- $g \circ f : A \rightarrow C$ is also a function.
- Composition of two **one-to-one** functions is **one-to-one**
- Composition of two **onto** functions is **onto**

PROBLEMS

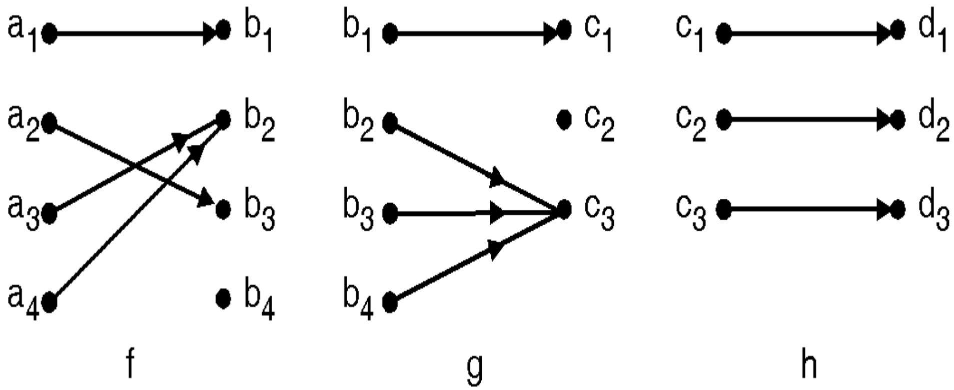
Q.1 : Let A = { 1, 2, 3 }, B = { a, b } and C = { 5, 6, 7 }.
Let f : A $\rightarrow$ B be
defined by f(1) = a. f(2) = a. f(3) = b.
i.e. f = {(1, a), (2, a), (3, b)}.
Let g : B $\rightarrow$ C be defined by
g(a) = 5 g(b) = 7 i.e. g = {(a, 5), (b, 7)}.
Find composition of f and g i.e. (g o f)

Soln. :

If f(1) =a and g(a) = 5 then g o f (1) = 5
If f(2) =a and g(a) = 5 then g o f (2) = 5
If f(3) =b and g(b) = 7 then g o f (3) = 7
i.e. (g o f) (1) =5
(g o f) (2) =5
(g o f) (3) =7
g o f = {(1, 5), (2, 5), (3, 7)}.

PROBLEMS

Q. 2 : $A = \{ a_1, a_2, a_3, a_4 \}$, $B = \{ b_1, b_2, b_3, b_4 \}$,
$C = \{c_1, c_2, c_3\}$, $D = \{ d_1, d_2, d_3 \}$.
(i) For the function f and g, determine g o f.
(ii) For the function f, g and h, determine
h o (g o f) and (h o g) o f



(i) $f : A \to B$ i.e. and $g : B \to C$ i.e.

$f(a_1) = b_1,$ $g(b_1) = c_1,$

$f(a_2) = b_3,$ $g(b_2) = c_3,$

$f(a_3) = b_2,$ $g(b_3) = c_3,$

$f(a_4) = b_2.$ $g(b_4) = c_3$

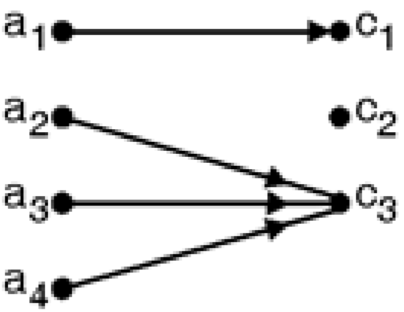
$$\begin{aligned} g \circ f &= g(f(a_1)) = g(b_1) = c_1 \\ &= g(f(a_2)) = g(b_3) = c_3 \\ &= g(f(a_3)) = g(b_2) = c_3 \\ &= g(f(a_4)) = g(b_2) = c_3 \end{aligned}$$

$\therefore$ $g \circ f(a_1) = c_1$

$g \circ f(a_2) = c_3$

$g \circ f(a_3) = c_3$

$g \circ f(a_4) = c_3$



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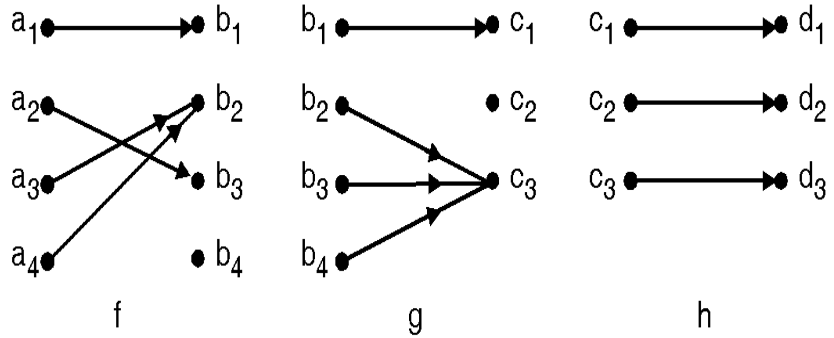


$\therefore$

(ii) h o (g o f) $= h(g \circ f(a_1))$
$= h(c_1)$
$= d_1$.
$= h(g \circ f(a_2))$
$= h(c_3)$
$= d_3$.
$= h(g \circ f(a_3))$
$= h(c_3)$
$= d_3$.
$= h(g \circ f(a_4))$
$= h(c_3)$
$= d_3$.

For (h o g) o f

h o g $= h(g(b_1))$
$=$ **h (c_1)=d_1**
$= h(g(b_2))$
$=$ **h (c_3)=d_3**
$= h(g(b_3))$
$=$ **h (c_3)=d_3**
$= h(g(b_4))$
$=$ **h (c_3)=d_3**

h o g (b_1) $= d_1$
h o g (b_2) $= d_3$
h o g (b_3) $= d_3$
h o g (b_4) $= d_3$

(h o g) o f $= (h \circ g)(f(a_1))$
$= hog(b_1)$
$= d_1$
$= (h \circ g)(f(a_2))$
$= hog(b_3)$
$= d_3$
$= (h \circ g)(f(a_3))$
$= hog(b_2)$
$= d_3$
$= (h \circ g)(f(a_4))$
$= hog(b_2)$
$= d_3$

Q. 3 : Consider the above function $f(x) = 2x - 3$. Find a formula for the composition functions (i) $f^2 = f \circ f$ and (ii) $f^3 = f \circ f \circ f$.

Soln. :

(i) $(f \circ f)(x) = f(f(x))$

$= f(2x - 3)$

$= 2(2x - 3) - 3$

$= 4x - 6 - 3$

$= 4x - 9$

(ii) $(f \circ f \circ f)(x) = f(f(f(x)))$

$= f(f(2x - 3)) = f(4x - 9)$

$= 2(4x - 9) - 3$

$= 8x - 18 - 3$

$= 8x - 21$

IDENTITY FUNCTION

Let A be a non-empty set. Then we can always define a function

$f : A \to A$ (i. e. $B = A$) as $f(a) = a$ for all $a \in A$

f is called the Identify function on A and is denoted by I_A.

$I_A \quad = \quad \{(a,a) \mid a \in A\}$

Example :

Let $A = \{1, 2, 3\}$ and $f : A \to A$ is identify function since

$f(1) = 1$

$f(2) = 2$

$f(3) = 3$

Inverse Functions

Let $f : A \to B$, be function, then

$f^{-1} : B \to A$ is called the **inverse** mapping of f

f^{-1} is the set defined as

$$f^{-1} \quad = \quad \{(b, a) \mid (a, b) \in f\}.$$

A function f for which f^{-1} exists is called invertible.

Let $A = \{ 1, 2, 3 \}$ and f be the function defined on A such that $f(1) = 2$, $f(2) = 3$, $f(3) = 1$. Then $f^{-1} : A \to A$ is defined by

$$f^{-1}(1)=\{ 3 \}.$$

$$f^{-1}(3)=\{ 2 \}.$$

$$f^{-1}(2)=\{ 1 \}.$$

$$f^{-1}=\{(1, 3)\ (3, 2), (2, 1)\}.$$

Ex. : Let f be a function, from $A = \{1, 2, 3, 4\}$ to $B = \{a, b, c, d\}$. Determine whether f^{-1} is a function.

(i) $f = \{(1, a), (2, a), (3, c), (4, d)\}$.

(ii) $f = \{(1, a), (2, c), (3, b), (4, d)\}$.

Soln. : $f(1) = \{a\}$ $\quad$ $f(2) = \{a\}$.

$f(3) = \{c\}$ $\quad$ $f(4) = \{d\}$.

$f^{-1}(a) = \{1, 2\}$ $\quad$ $f^{-1}(c) = \{3\}$.

$f^{-1}(d) = \{4\}$.

f^{-1} is not a function, since $f^{-1}(a) = \{1, 2\}$. Hence f is not invertable.

(ii) $f(1) = \{a\}$ $\quad$ $f(2) = \{c\}$.

$f(3) = \{b\}$ $\quad$ $f(4) = \{d\}$.

$f^{-1}(a) = \{1\}$ $\quad$ $f^{-1}(c) = \{2\}$.

$f^{-1}(b) = \{3\}$ $\quad$ $f^{-1}(d) = \{4\}$.

f^{-1} is a function. Hence f is invertible.

Ex. : Let $f : Z \to Z$ where $f(x) = x^2 - 1$. Is f Invertible?

Soln. :

We have $f(x) = x^2 - 1$;

For $x = 1$ and -1

$f(1) = 0$ and $f(-1) = 0$

$\therefore \quad f^{-1}(0) = \{1, -1\}$.

$f^{-1}(n)$ is not a single value function. Hence f is not invertible.

Ex. : Function $f(x) = (4x + 3) / (5x - 2)$. Find f^{-1}.

Soln.: To find f^{-1}

Set $y = f(x)$

and then interchange x and y as follow.

$$y = (4x + 3) / (5x - 2)$$

One interchanging x and y.

$$x = (4y + 3) / (5y - 2)$$

$$5xy - 2x = 4y + 3$$

$$5xy - 4y = 2x + 3$$

$$y(5x - 4) = (2x + 3)$$

$$y = \frac{(2x + 3)}{(5x - 4)}$$

$$f^{-1}(x) = \frac{(2x + 3)}{(5x - 4)}$$

- <u>Steps</u>

Set y = f (x)

Interchange x and y

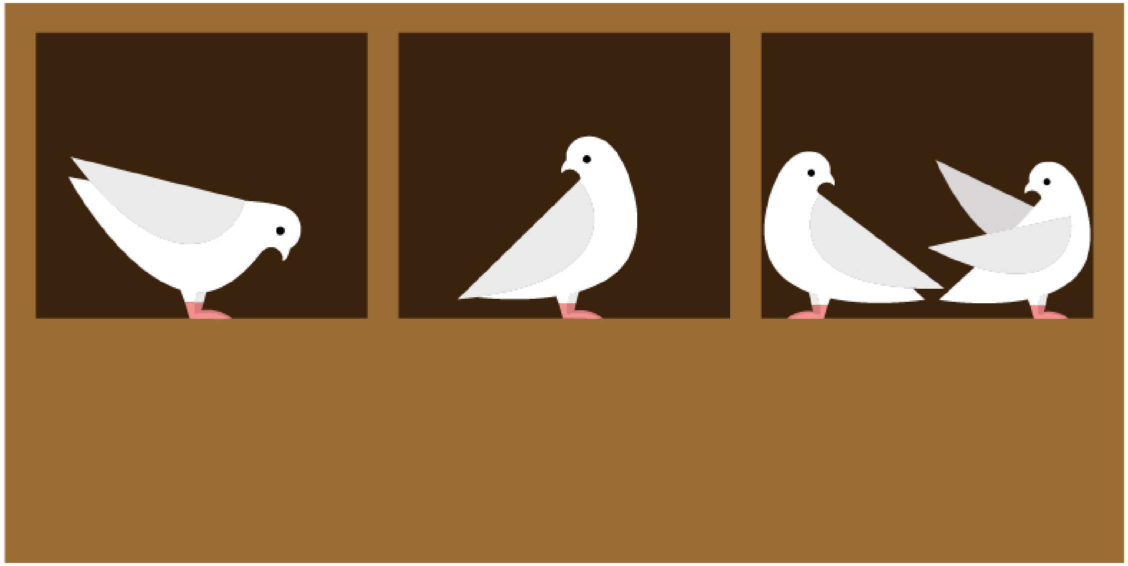
Solve for y which is $f^{-1}(x)$

Problems: Find $f^{-1}(x)$

1) $f(x) = 1 / (x - 2)$
2) $f(x) = (x + 1) / x$
3) $f(x) = x^3 + 2$
4) $f(x) = 5x - 7$
5) $f(x) = 8 / (9 - 3x)$
6) $f(x) = (4x + 3) / (5x - 2)$
7) $f(x) = (7 + 4x) / (6 - 5x)$

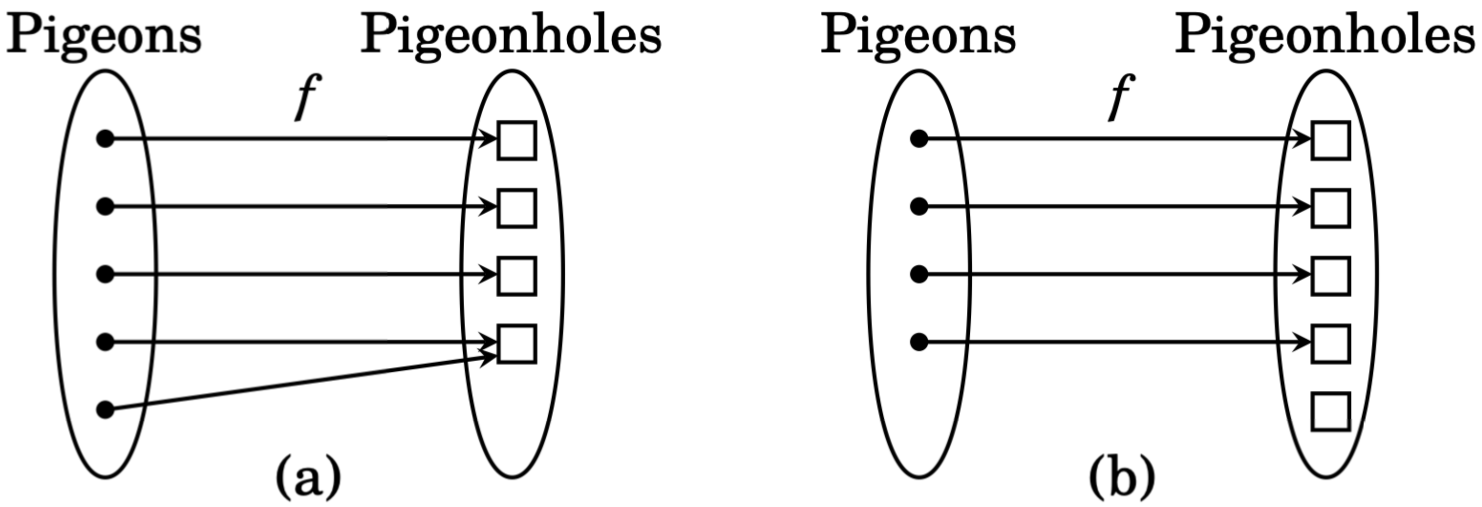
Pigeonhole Principle

If n pigeons are assigned to m pigeonholes, and m < n, then at least one pigeonhole contains two or more pigeons.



The Pigeonhole Principle (function version)

Suppose A and B are finite sets and $f:A\rightarrow B$ is any function.
1. If $|A|>|B|$, then f is not injective.
2. If $|A|<|B|$, then f is not surjective.



THE EXTENDED PIGEONHOLE PRINCIPLE

If there are m pigeonholes and more than 2m pigeons, then three or more pigeons will have to be assigned to at least one of the pigeonholes.

If n and m are positive integers, then $\lfloor n / m \rfloor$ stands for the largest integer less than or equal to the rational number n/m. Thus $\lfloor 3/2 \rfloor$ is 1, $\lfloor 9/4 \rfloor$ is 2 and $\lfloor 6/3 \rfloor$ is 2.

Theorem : (The extended pigeonhole principle)

If n pigeons are assigned to m, pigeonholes, then one of the pigeonholes must obtain at least $\lfloor (n-1) / m \rfloor + 1$ pigeons.

Pigeonhole Principle

Ex. 1 : If eight people are chosen in anyway from some group, at least two of them will have been born on the same day of the week.

Soln. : Here each person (pigeon) is assigned to the day of the week (pigeonhole) on which he or she was born. Since there are eight people and only seven days of the week, the pigeon hole principle tells us least two people must be assigned to the same day of the week.

Ex. 2 : Show that if any five numbers from 1 to 8 are chosen, then two of them will add upto 9.

Soln. : Construct four different sets, each containing two numbers that add up to 9 as follows : A1={ 1, 8 }, A2={ 2, 7 }, A3={ 3, 6 }, A4={ 4, 5 }

Each of the five numbers chosen must belong to one of these sets. Since there are only four sets, the pigeonhole principle tells us that two of the chosen numbers belong to the same set. These numbers add upto 9.

Ex. 3 : Show that it any 30 people are selected, then we may choose a subset at 5 so that all 5 were born on the same day of the week.

Soln. : Assign each person to the day of the week on which she or he was born. Then 30 pigeons are being assigned to 7 pigeonholes. By the extended pigeonhole principle with n = 30 and m = 7, at least $\lfloor (30 - 1) / 7 \rfloor + 1$ or 5 of the people must have been born on the same day of the week.

Ex. 4 : Show that if 30 dictionaries in a library contain a total of 61,327 pages, then one of the dictionaries must have at least 2045 pages.

Soln. : Let the pages be the pigeons and the dictionaries the pigeonholes. Assign each page to the dictionary in which it appears. Then by the extended pigeonhole principle, one dictionary must contain at least

$\lfloor 61, 326 / 30 \rfloor + 1$ or 2045 pages.

Ex. 5 : Six friends discover that they have total of 2161 Rs. with them on a trip to the movies. Show that one or more of them must have at least 361 Rs.

Soln. : Let the rupees be the pigeons and the number of friends is number of pigeonholes. Then by the extended pigeonhole principle one friend must have at least

$\lfloor 2160 / 6 \rfloor + 1$ or 361 rupees.

Ex. 6 : Show that if seven numbers from 1 to 12 are chosen, then 2 of them will add upto 13.

Soln. : Construct 6 different sets each containing two numbers that add upto 13 as follows.

A1	=	{ 1, 12 },	A2	=	{ 8, 5 },
A3	=	{ 7, 6 },	A4	=	{ 11, 2 },
A5	=	{ 9, 4 },	A6	=	{ 10, 3 }.

Each of the seven numbers chosen must belong to one of the sets. Since there are only 6 sets, the pigeonhole principle tells us that two of the chosen numbers belong to the same set. These numbers add upto 13.

Ex. 7 : Show that 7 colors are used to paint 50 bicycles, at least 8 bicycles will be of same color.

Soln. : If n pigeons are assigned to m pigeonholes, and m < n, then at least one pigeonhole contains two or more pigeons.

By the extended pigeonhole principle at least.

$\lfloor (50 - 1) / 7 \rfloor + 1 = 8$ will be of the same color.

Ex. 8 : What is the minimum number of students required in a discrete structures class to be sure that at least six will receive the same grade, if there are five possible grades A, B, C, D, E

Soln. : By extended pigeon hole principle

$$\left\lfloor \frac{(n-1)}{5} \right\rfloor + 1 = 6$$

$$\therefore \quad \frac{n-1}{5} = 6-1$$

$$\therefore \quad \frac{n-1}{5} = 5$$

$$\therefore \quad n-1 = 25$$

$$\therefore \quad n = 26$$

26 Students are required in a discrete structures class.