SOMAIYA VIDYAVIHAR UNIVERSITY

K J Somaiya College of Engineering

3D Transformations…Numerical Practice

By

Prof. Vaibhav P. Vasani

Assistant Professor

Department of Computer Engineering

K. J. Somaiya College of Engineering

Somaiya Vidyavihar University

Somaiya TRUST

6.1 Define *tilting* as a rotation about the x axis followed by a rotation about the y axis: (a) find the tilting matrix; (b) does the order of performing the rotation matter?

(a) We can find the required transformation T by composing (concatenating) two rotation matrices:

$$T = R_{\theta_x,\mathbf{I}} \cdot R_{\theta_y,\mathbf{J}}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x & 0 \\ 0 & -\sin\theta_x & \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta_y & 0 & \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y & 0 \\ \sin\theta_y \sin\theta_x & \cos\theta_x & \cos\theta_y \sin\theta_x & 0 \\ \sin\theta_y \cos\theta_x & -\sin\theta_x & \cos\theta_y \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

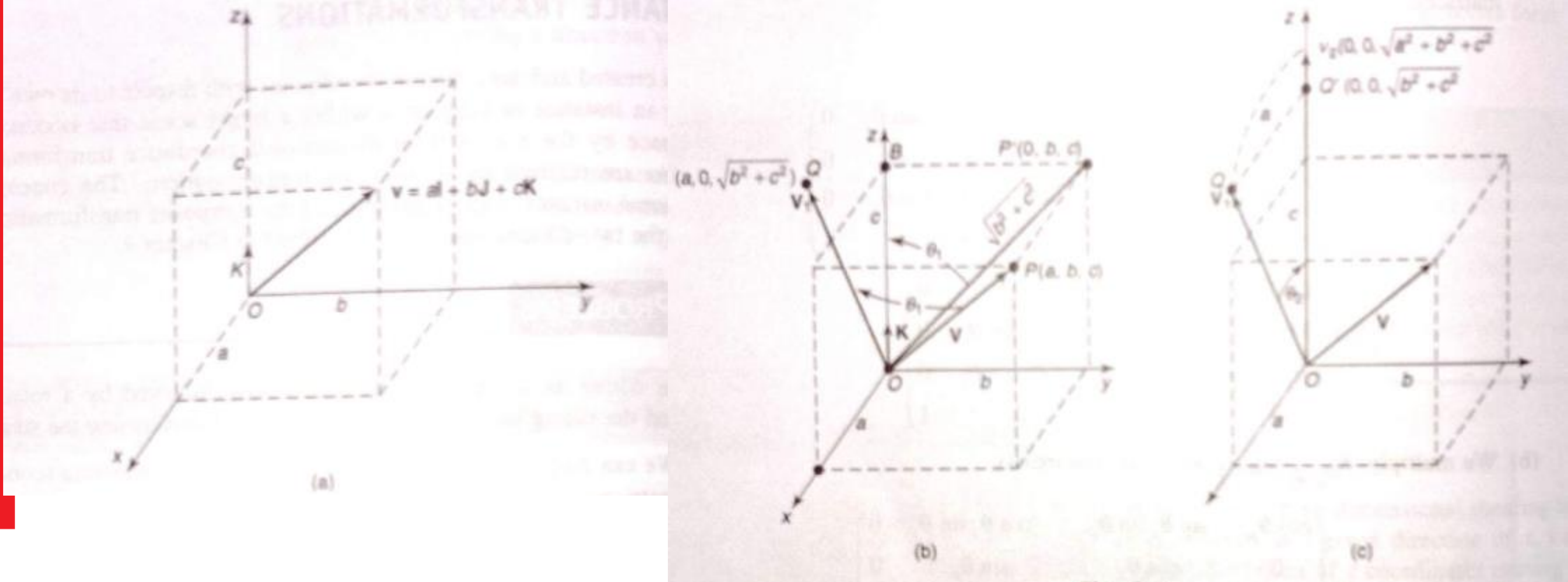
(b) We multiply $R_{\theta_x,\mathbf{J}} \cdot R_{\theta_y,\mathbf{I}}$ to obtain the matrix

$$= \begin{pmatrix} \cos\theta_y & \sin\theta_x \sin\theta_y & -\cos\theta_x \sin\theta_y & 0 \\ 0 & \cos\theta_x & \sin\theta_x & 0 \\ \sin\theta_y & -\sin\theta_x \cos\theta_y & \cos\theta_x \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is not the same matrix as in part (a); thus the order of rotation matters.

Find a transformation $A_{\mathbf{V}}$ which aligns a given vector $\mathbf{V}$ with the vector $\mathbf{K}$ along the positive z axis.

See Fig. 6.4(a). Let $\mathbf{V} = a\mathbf{I} + b\mathbf{J} + c\mathbf{K}$. We perform the alignment through the following sequence of transformations [Figs. 6.4(b) and 6.4(c)]:



(a) (b) (c)

Fig. 6.4

1. Rotate about the x axis by an angle θ_1 so that $\mathbf{V}$ rotates into the upper half of the xz plane (as the vector $\mathbf{V}_1$).
2. Rotate the vector $\mathbf{V}_1$ about the y axis by an angle $-\theta_2$ so that $\mathbf{V}_1$ rotates to the positive z axis (as the vector $\mathbf{V}_2$).

Implementing step 1 from Fig. 6.4(b), we observe that the required angle of rotation θ_1 can be found by looking at the projection of $\mathbf{V}$ onto the yz plane. (We assume that b and c are not both zero.) From triangle $OP'B$:

$$\sin\theta_1 = \frac{b}{\sqrt{b^2+c^2}} \qquad \cos\theta_1 = \frac{c}{\sqrt{b^2+c^2}}$$

The required rotation is

$$R_{\theta_1,\mathbf{I}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{c}{\sqrt{b^2+c^2}} & \dfrac{b}{\sqrt{b^2+c^2}} & 0 \\ 0 & -\dfrac{b}{\sqrt{b^2+c^2}} & \dfrac{c}{\sqrt{b^2+c^2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Applying this rotation to the vector **V** produces the vector $\mathbf{V}_1$ with the components $(a, 0, \sqrt{b^2 + c^2})$.

Implementing step 2 from Fig. 6.4(c), we see that a rotation of θ_2 degrees is required, and so from triangle OQQ':

$$\sin(-\theta_2) = -\sin\ \theta_2 = -\frac{a}{\sqrt{a^2 + b^2 + c^2}} \quad \text{and} \quad \cos(-\theta_2) = \cos\ \theta_2 = \frac{\sqrt{b^2 + c^2}}{\sqrt{a^2 + b^2 + c^2}}$$

Then

$$R_{-\theta_2,\mathbf{J}} = \begin{pmatrix} \dfrac{\sqrt{b^2+c^2}}{\sqrt{a^2+b^2+c^2}} & 0 & \dfrac{a}{\sqrt{a^2+b^2+c^2}} & 0 \\ 0 & 1 & 0 & 0 \\ -\dfrac{a}{\sqrt{a^2+b^2+c^2}} & 0 & \dfrac{\sqrt{b^2+c^2}}{\sqrt{a^2+b^2+c^2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since $|\mathbf{V}| = \sqrt{a^2+b^2+c^2}$, and introducing the notation $\lambda = \sqrt{b^2+c^2}$, we find

$$A_{\mathbf{V}} = R_{\theta_1,\mathbf{I}} \cdot R_{-\theta_2,\mathbf{J}}$$

$$= \begin{pmatrix} \dfrac{\lambda}{|\mathbf{V}|} & 0 & \dfrac{a}{|\mathbf{V}|} & 0 \\ \dfrac{-ab}{\lambda|\mathbf{V}|} & \dfrac{c}{\lambda} & \dfrac{b}{|\mathbf{V}|} & 0 \\ \dfrac{-ac}{\lambda|\mathbf{V}|} & \dfrac{-b}{\lambda} & \dfrac{c}{|\mathbf{V}|} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

If both b and c are zero, then $\mathbf{V} = a\mathbf{I}$, and so $\lambda = 0$. In this case, only a $\pm 90°$ rotation about the y axis is required. So if $\lambda = 0$, it follows that

$$A_{\mathbf{V}} = R_{-\theta_2,\mathbf{J}} = \begin{pmatrix} 0 & 0 & \frac{a}{|a|} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-a}{|a|} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In the same manner we calculate the inverse transformation that aligns the vector $\mathbf{K}$ with the vector $\mathbf{V}$:

$$A_{\mathbf{V}}^{-1} = (R_{\theta_1,\mathbf{I}} \cdot R_{-\theta_2,\mathbf{J}})^{-1} = R^{-1}_{-\theta_2,\mathbf{j}} \cdot R^{-1}_{\theta_1,\mathbf{I}} = R_{\theta_2,\mathbf{j}} \cdot R_{-\theta_1,\mathbf{I}}$$

$$= \begin{pmatrix} \frac{\lambda}{|\mathbf{V}|} & \frac{-ab}{\lambda|\mathbf{V}|} & \frac{-ac}{\lambda|\mathbf{V}|} & 0 \\ 0 & \frac{c}{\lambda} & \frac{-b}{|\mathbf{V}|} & 0 \\ \frac{a}{|\mathbf{V}|} & \frac{b}{|\mathbf{V}|} & \frac{c}{|\mathbf{V}|} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Let an axis of rotation L be specified by a direction vector $\mathbf{V}$ and a location point P. Find the transformation for a rotation of $\theta°$ about L. Refer to Fig. 6.5.

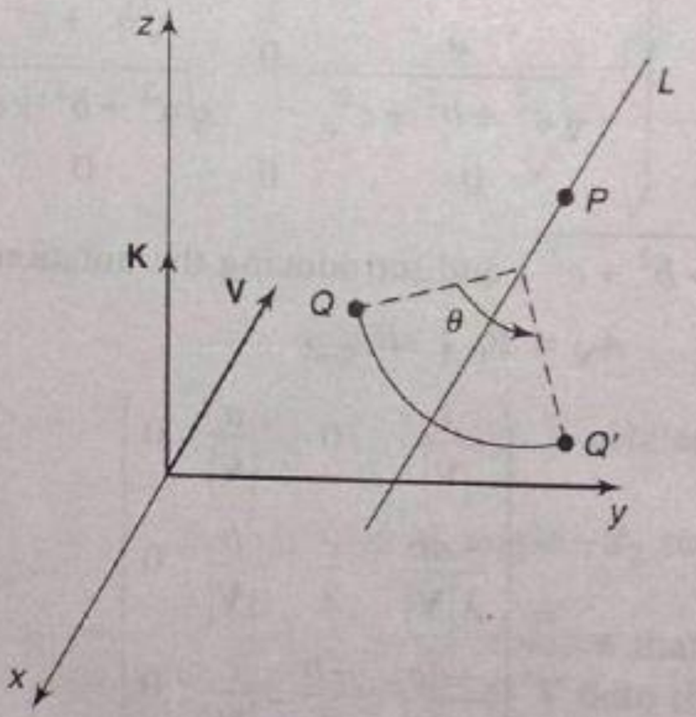


Fig. 6.5

We can find the required transformation by the following steps:

1. Translate P to the origin.
2. Align **V** with the vector **K**.
3. Rotate by $\theta°$ about **K**.
4. Reverse steps 2 and 1.

So

$$R_{\theta,L} = T_{-P} \cdot A_{\mathbf{V}} \cdot R_{\theta,\mathbf{K}} \cdot A_{\mathbf{V}}^{-1} \cdot T_{-P}^{-1}$$

Here, $A_{\mathbf{V}}$ is the transformation described in Problem 6.2.

6.4 The pyramid defined by the coordinates $A(0, 0, 0)$, $B(1, 0, 0)$, $C(0, 1, 0)$, and $D(0, 0, 1)$ is rotated 45° about the line L that has the direction $\mathbf{V} = \mathbf{J} + \mathbf{K}$ and passing through point $C(0, 1, 0)$ (Fig. 6.6). Find the coordinates of the rotated figure.

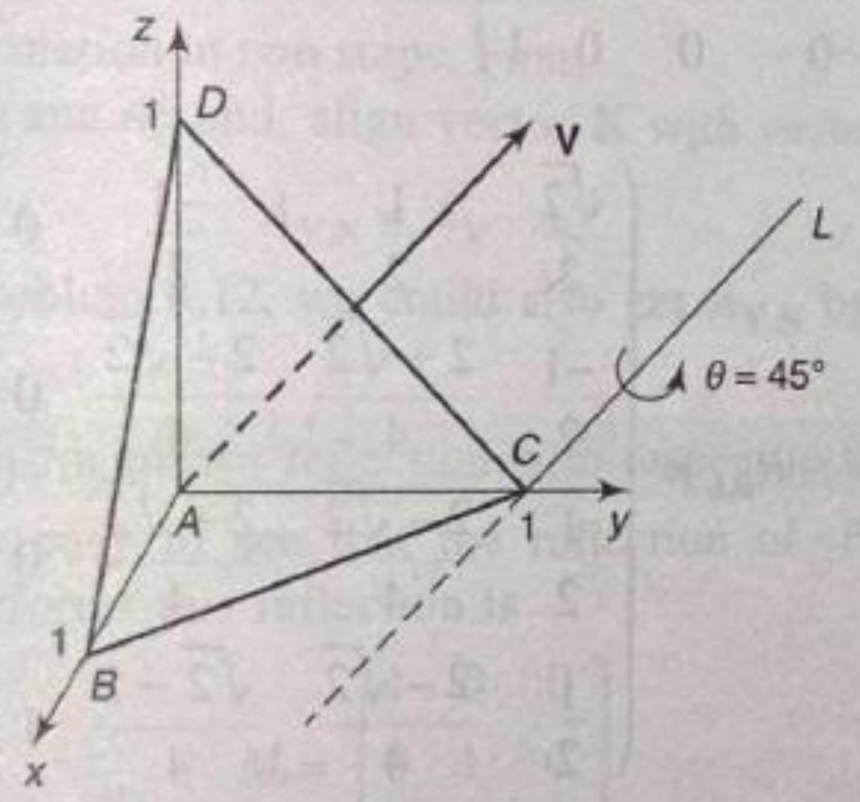


Fig. 6.6

From Solved Problem 6.3, the rotation matrix $R_{\theta,L}$ can be found by concatenating the matrices

$$R_{\theta,L} = T_{-P} \cdot A_{\mathbf{V}} \cdot R_{\theta,\mathbf{K}} \cdot A_{\mathbf{V}}^{-1} \cdot T_{-P}^{-1}$$

With $P = (0, 1, 0)$, then

$$T_{-P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Now $\mathbf{V} = \mathbf{J} + \mathbf{K}$. So from Problem 6.2, with $a = 0$, $b = 1$, $c = 1$, we find $\lambda = \sqrt{2}$, $|\mathbf{V}| = \sqrt{2}$, and

Also $$A_{\mathbf{V}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{\mathbf{V}}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_{45^\circ,\mathbf{K}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad T_{-P}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Then
$$R_{\theta,L} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{-1}{2} & 0 \\ \frac{-1}{2} & \frac{2+\sqrt{2}}{4} & \frac{2-\sqrt{2}}{4} & 0 \\ \frac{1}{2} & \frac{2-\sqrt{2}}{4} & \frac{2+\sqrt{2}}{4} & 0 \\ \frac{1}{2} & \frac{2-\sqrt{2}}{4} & \frac{\sqrt{2}-2}{4} & 1 \end{pmatrix}$$

To find the coordinates of the rotated figure, we apply the rotation matrix $R_{\theta,L}$ to the matrix of homogeneous coordinates of the vertices A, B, C, and D:

$$C = \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

So

$$C \cdot R_{\theta,L} = \begin{pmatrix} \frac{1}{2} & \frac{2-\sqrt{2}}{4} & \frac{\sqrt{2}-2}{4} & 1 \\ \frac{1+\sqrt{2}}{2} & \frac{4-\sqrt{2}}{4} & \frac{\sqrt{2}-4}{4} & 1 \\ 0 & 1 & 0 & 1 \\ 1 & \frac{2-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 1 \end{pmatrix}$$

The rotated coordinates are (Fig. 6.7).

$$A' = \left(\frac{1}{2}, \frac{2-\sqrt{2}}{4}, \frac{\sqrt{2}-2}{4}\right)$$

$$B' = \left(\frac{1+\sqrt{2}}{2}, \frac{4-\sqrt{2}}{4}, \frac{\sqrt{2}-4}{4}\right)$$

$$C' = (0, 0, 1)$$

$$D' = \left(1, \frac{2-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

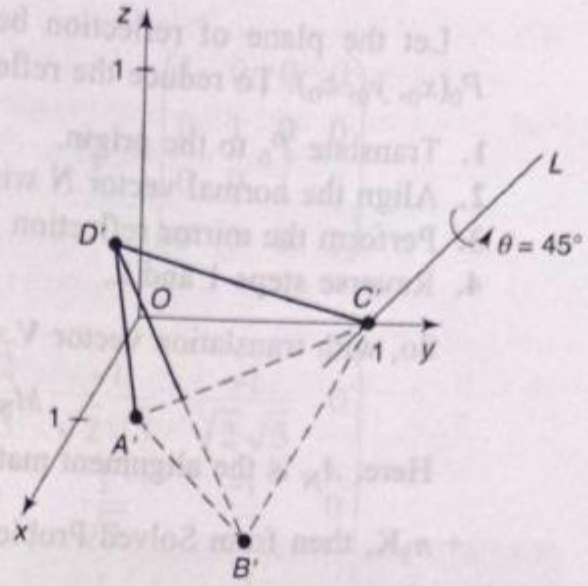


Fig. 6.7

6.5 Find a transformation $A_{V,N}$ which aligns a vector **V** with a vector **N**.

We form the transformation in two steps. First, align **V** with vector **K**, and second, align vector **K** with vector **N**. So from Problem 6.2,

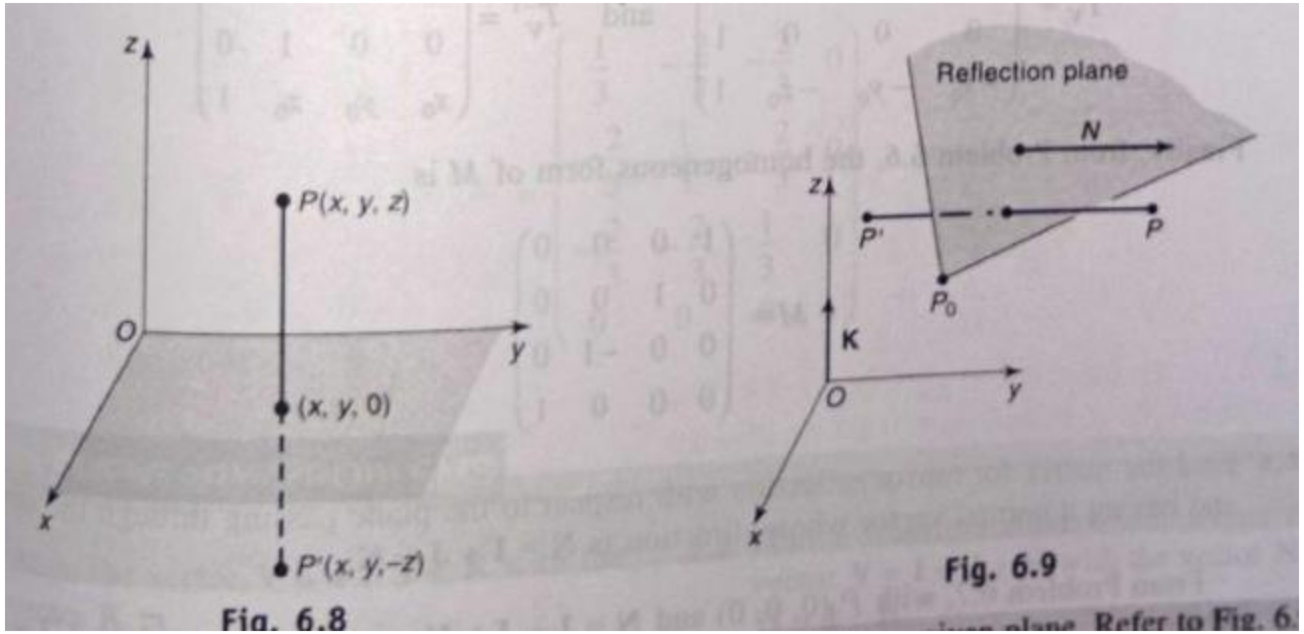
$$A_{V,N} = A_V \cdot A_N^{-1}$$

Referring to Solved Problem 6.12, we could also get $A_{V,N}$ by rotating **V** towards **N** about the axis $\mathbf{V} \times \mathbf{N}$.

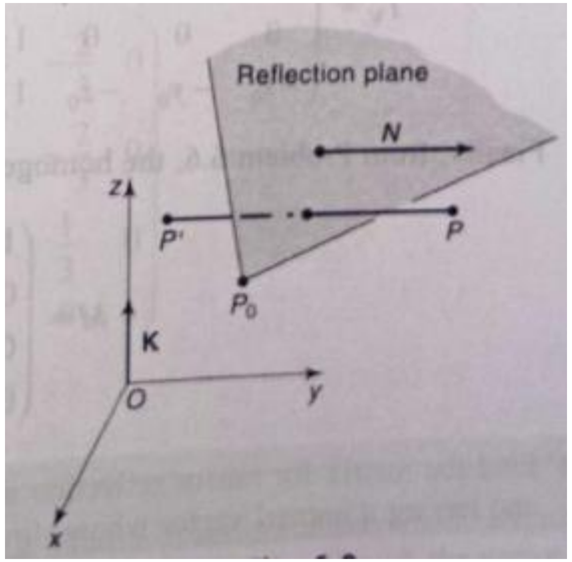
6.6 Find the transformation for mirror reflection with respect to the xy plane.

From Fig. 6.8, it is easy to see that the reflection of $P(x, y, z)$ is $P'(x, y, z)$. The transformation that performs this reflection is

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$


z
P(x, y, z)
O
y
(x, y, 0)
x
P'(x, y, -z)
Fig. 6.8
Reflection plane
N
z
P'
P
P₀
K
O
y
x
Fig. 6.9

Find the transformation for mirror reflection with respect to given plane



Let the plane of reflection be specified by a normal vector **N** and a reference point $P_0(x_0, y_0, z_0)$. To reduce the reflection to a mirror reflection with respect to the xy plane:

1. Translate P_0 to the origin.
2. Align the normal vector **N** with the vector **K** normal to the xy plane.
3. Perform the mirror reflection in the xy plane (Problem 6.6).
4. Reverse steps 1 and 2.

So, with translation vector $\mathbf{V} = -x_0\mathbf{I} - y_0\mathbf{J} - z_0\mathbf{K}$

$$M_{\mathbf{N},P_0} = T_{\mathbf{V}} \cdot A_{\mathbf{N}} \cdot M \cdot A_{\mathbf{N}}^{-1} \cdot T_{\mathbf{V}}^{-1}$$

Here, $A_{\mathbf{N}}$ is the alignment matrix defined in Problem 6.2. So if the vector $\mathbf{N} = n_1\mathbf{I} + n_2\mathbf{J} + n_3\mathbf{K}$, then from Solved Problem 6.2, with $|\mathbf{N}| = \sqrt{n_1^2 + n_2^2 + n_3^2}$ and $\lambda = \sqrt{n_2^2 + n_3^2}$, we find

$$A_{\mathbf{N}} = \begin{pmatrix} \frac{\lambda}{|\mathbf{N}|} & 0 & \frac{n_1}{|\mathbf{N}|} & 0 \\ \frac{-n_1 n_2}{\lambda|\mathbf{N}|} & \frac{n_3}{\lambda} & \frac{n_2}{|\mathbf{N}|} & 0 \\ \frac{-n_1 n_3}{\lambda|\mathbf{N}|} & \frac{-n_2}{\lambda} & \frac{n_3}{|\mathbf{N}|} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_{\mathbf{N}}^{-1} = \begin{pmatrix} \frac{\lambda}{|\mathbf{N}|} & \frac{-n_1 n_2}{\lambda|\mathbf{N}|} & \frac{-n_1 n_3}{\lambda|\mathbf{N}|} & 0 \\ 0 & \frac{n_3}{\lambda} & \frac{-n_2}{\lambda} & 0 \\ \frac{n_1}{|\mathbf{N}|} & \frac{n_2}{|\mathbf{N}|} & \frac{n_3}{|\mathbf{N}|} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In addition

$$T_{\mathbf{V}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -x_0 & -y_0 & -z_0 & 1 \end{pmatrix} \quad \text{and} \quad T_{\mathbf{V}}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x_0 & y_0 & z_0 & 1 \end{pmatrix}$$

Finally, from Problem 6.6, the homogeneous form of M is

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

6.8 Find the matrix for mirror reflection with respect to the plane passing through the origin and having a normal vector whose direction is $\mathbf{N} = \mathbf{I} + \mathbf{J} + \mathbf{K}$.

From Problem 6.7, with $P_0(0, 0, 0)$ and $\mathbf{N} = \mathbf{I} + \mathbf{J} + \mathbf{K}$, we find $|\mathbf{N}| = \sqrt{3}$ and $\lambda = \sqrt{2}$. Then

$$T_{\mathbf{V}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (\mathbf{V} = 0\mathbf{I} + 0\mathbf{J} + 0\mathbf{K}) \qquad T_{\mathbf{V}}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_{\mathbf{N}} = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 \\ \frac{-1}{\sqrt{2}\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{-1}{\sqrt{2}\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{\mathbf{N}}^{-1} = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & \frac{-1}{\sqrt{2}\sqrt{3}} & \frac{-1}{\sqrt{2}\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The reflection matrix is

$$M_{N,O} = T_V \cdot A_N \cdot M \cdot A_N^{-1} \cdot T_V^{-1}$$

$$= \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & 0 \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} & 0 \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

6.1 Align the vector $\mathbf{V} = \mathbf{I} + \mathbf{J} + \mathbf{K}$ with the vector $\mathbf{K}$.

6.2 Find a transformation which aligns the vector $\mathbf{V} = \mathbf{I} + \mathbf{J} + \mathbf{K}$ with the vector $\mathbf{N} = 2\mathbf{I} - \mathbf{J} - \mathbf{K}$.

Thank you