MOD 3

DFT & IDFT

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nK}{N}}$$

$$x(n) = \sum_{K=0}^{N-1} x(K)\, e^{j\frac{2\pi nK}{N}} \times \frac{1}{N}$$

Property

1) Linearity

id $X_1(K) = DFT[x_1(n)]$ & $X_2(K) = DFT[x_2(n)]$

then $DFT[a_1 x_1(n) + a_2 x_2(n)] = a_1 X_1(K) + a_2 X_2(K)$

2) Periodicity

id $x(K)$ is the N-point od DFT od $x(n)$, then

$x(K+N) = X(K)$

3) Convolution

id $x_1(K) = DFT[x_1(n)]$ & $x_2(K) = DFT[x_2(n)]$

then $DFT[x_1(n) \circledast x_2(n)] = X(K) \cdot X_2(K)$

4) Multiplication

id $x_1(K) = DFT[x_1(n)]$ & $x_2(K) = DFT[x_2(n)]$, then

$DFT[x_1(n)\, x_2(n)] = \frac{1}{N}[X_1(K) \circledast x_2(K)]$

5) Time Reversal

$DFT[x(N-n)] = X(N-K)$

6) Time Shift

$DFT[x((n-m))_N] = X(K)\, e^{-j\frac{2\pi Km}{N}}$

eg) $x(n) = \{1, 2, 3, 4\}$

$N = 4$

$$x(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n k}{N}} = \sum_{0}^{3} x(n)\, e^{-j\frac{\pi}{2}nk}$$

<u>For K=0</u>

$$x(0) = \sum_{0}^{3} x(n)\, e^{-j\frac{\pi}{2}n\times 0} = [1+2+3+4] = 10$$

<u>For K=1</u>

$$x(1) = \sum_{0}^{3} x(n)\, e^{-j\frac{\pi}{2}n\times 1} = [1\times e^{0} + 2e^{-j\frac{\pi}{2}} + 3e^{-j\pi} + 4e^{-j3\pi/2}]$$

$$= 1 + 2(-j) + 3(-1) + 4(j)$$

$$= -2 + 2j$$

<u>For K=2</u>

$$x(2) = \sum_{0}^{3} x(n)\, e^{-j\frac{\pi}{2}n\times 2} = [1\times e^{0} + 2e^{-j\pi} + 3e^{-j2\pi} + 4e^{-j3\pi}]$$

$$= 1 + 2(-1) + 3(1) + 4(-1)$$

$$= -2$$

<u>For K=3</u>

$$x(3) = \sum_{0}^{3} x(n)\, e^{-j\frac{\pi}{2}n\times 3} = [1\times e^{0} + 2e^{-j\frac{3\pi}{2}} + 3e^{-j3\pi} + 4e^{-j\frac{9\pi}{2}}]$$

$$= 1 + 2(j) + 3(-1) + 4(-j)$$

$$= -2 - 2j$$

$\therefore x(K) = \{10, -2+2j, -2, -2-2j\}$

Apply IDFT

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} x(k)\, e^{j\frac{2\pi n k}{N}} = \frac{1}{4}\sum_{0}^{3} x(k)\, e^{j\frac{\pi}{2}nk}$$

<u>For n=0</u>

$$x(0) = \frac{1}{4}\sum_{0}^{3} x(k)\, e^{j\frac{\pi}{2}\times 0\times k} = \frac{1}{4}[10 - 2 + \cancel{2j} - 2 - 2\cancel{-2j}]$$

$$= 1$$

October

W	M	T	W	T	F	S	S
40				1	2	3	4
41	[illegible]	6	7	8	9	10	11
42	12	13	14	15	16	17	18
43	19	20	21	22	23	24	25
44	26	27	28	29	30	31	

For n= 1

$$x(1) = \frac{1}{4}\sum_{0}^{3} x(k) e^{j\frac{2\pi}{2}k} = \frac{1}{4}[10e^0 + (-2+2j)\cancel{(j)} e^{j\pi/2} + (-2)e^{j\pi}$$
$$+ (-2-2j)e^{j3\pi/2}]$$

$$= \frac{1}{4}[10 - \cancel{2j} - 2 \cancel{+2} \cancel{+2j} \cancel{+2}]$$
$$= 2$$

For n= 2

$$x(2) = \frac{1}{4}\sum_{0}^{3} x(k) e^{j\pi k} = \frac{1}{4}[10e^0 + (-2+2j)e^{j\pi} + (-2)e^{j2\pi}$$
$$+ (-2-2j)e^{j3\pi}]$$

$$= \frac{1}{4}[10 + \cancel{2} \cancel{-2j} \cancel{-2} + 2 \cancel{+2j}]$$

$$= 3$$

For n=3

$$x(3) = \frac{1}{4}\sum_{0}^{3} x(k) e^{j\frac{\pi}{2}\times 3k} = \frac{1}{4}[10e^0 + (-2+2j)e^{j3\pi/2} + (-2)e^{j3\pi}$$
$$+ (-2-2j)e^{j9\pi/2}]$$

$$= \frac{1}{4}[10 + \cancel{2j} + 2 + 2 \cancel{-2j} + 2]$$

$$= 4$$

$$x(n) = \{1, 2, 3, 4\}$$

Twiddle Factor

$$W_N^{nk} = e^{-j\frac{2\pi kn}{N}}$$

$$\begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \end{array} \leftarrow \text{Kernel}$$

$\boxed{\text{2D} \rightarrow \text{DFT \& IDFT}}$

DFT $\Rightarrow$ Kernel $\times f(x,y) \times$ Kernel^T

IDFT $\rightarrow$ (Kernel $\times f(u,v) \times$ Kernel^T) $\times \frac{1}{N^2}$

Determine convolution using DFT

$x_1(n) = \{2, 1, 2, 1\}$

$x_2(n) = \{1, 2, 3, 4\}$

$x_3(n) = ?$

$$x_1(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$x_2(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$$x_3(k) = x_1(k) \cdot x_2(k) = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix} = \begin{bmatrix} 60 \\ 0 \\ -4 \\ 0 \end{bmatrix}$$

October

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$$x_3(n) = \frac{1}{4}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -j & + & -1\\ 1 & j & -1 & -j\end{bmatrix}\begin{bmatrix}60\\ 0\\ -4\\ j\end{bmatrix} = \frac{1}{4}\begin{bmatrix}56\\ 64\\ 56\\ 64\end{bmatrix} = \begin{bmatrix}14\\ 16\\ 14\\ 16\end{bmatrix}$$

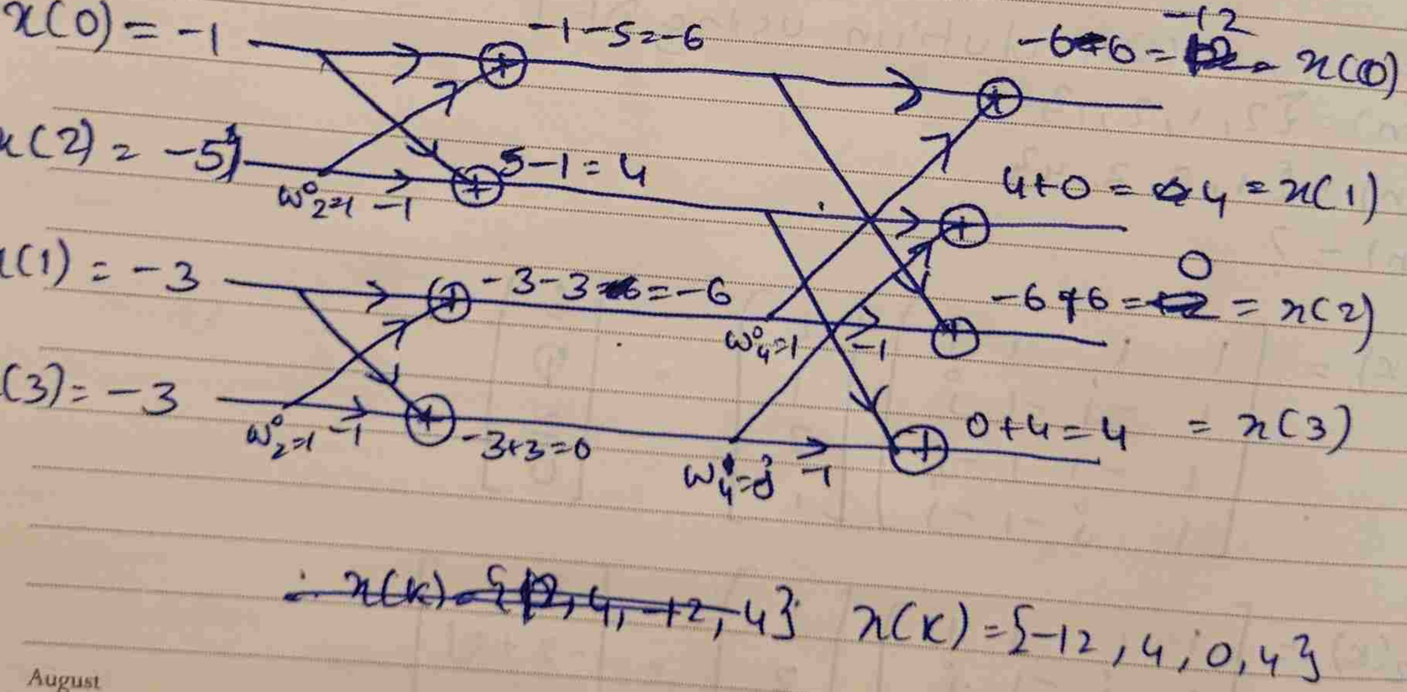
FFT

→ Method of computing DFT with reduced no° of calc.

DIT-FFT

$x(n) = \{-1, -3, -5, -3\}$

0 ⇒ 00	⟹	00 → 0
1 ⇒ 01	⟹	10 → 2
2 ⇒ 10	⟹	01 → 1
3 ⇒ 11	⟹	11 → 3



∴ ~~x(k) = {12, 4, -12, 4}~~ $x(k) = \{-12, 4, 0, 4\}$

Image Transformation

→ Useful for fast computation od convolution & corrlation

→ They do not change the info content present in the signal

→ Need

- Mathematical ~~convenience~~ convenience
- To extract more Information

1) Orthogonal sinusoidal basis Function

- One od the most powerful transform ~~ation~~ is the fourier transform.
- Widely used in image compression is DCT

2) Non sinusodial orthogonal basis Function

- One od the important advantages od wavelet transform is that signals can be represented in different resolutions.

3) Basis function depending on statistics od input signal

4) Directional transform

Unitary Transform

$AXA^H = I$ | Preserves the signal energy

$A^H = A^{*T}$

Sunday 20

Day (263 - 102)

Orthogonal

if A is unitary & has only real elements then it is Orthogonal

$A \cdot A' = I$

DCT

$$X[K] = \alpha(K) \sum_{n=0}^{N-1} x(n) \cos\left(\frac{(2n+1)\pi K}{2N}\right)$$

~~IDCT~~

$X(n)$ ~~$= \alpha(k) \sum_{k=0}^{N-1} x(k) \cos\left[\frac{(2n+1)}{}\right]$~~

$$\alpha(K) = \sqrt{\frac{1}{N}} \quad K=0$$

$$= \sqrt{\frac{2}{N}} \quad K \neq 0$$

$$\text{Kernel} = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.6532 & 0.2706 & -0.2706 & -0.6532 \\ 0.5 & -0.5 & -0.5 & 0.5 \\ 0.2706 & -0.6532 & 0.6532 & -0.2706 \end{bmatrix}$$

October

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IDCT

$$x(n) = \sum_{k=0}^{N-1} \alpha(k)\, x(k) \cos\left[\frac{(2n+1)k\pi}{2N}\right]$$

Hadamard

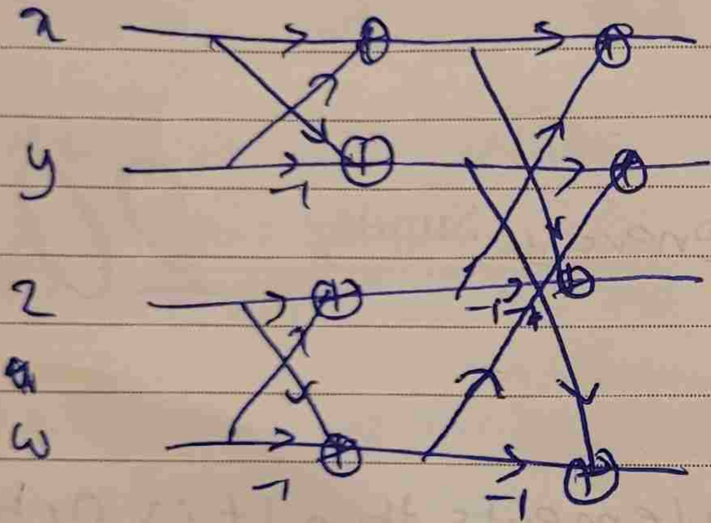
$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H_{2N} = \begin{bmatrix} H_N & H_N \\ H_N & H_{-N} \end{bmatrix}$$

2D $\Rightarrow T f T$

inverse $= \cancel{\frac{1}{N^2}} \cdot \frac{1}{N^2}\, T f T$

Butterfly



Walsh

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{array}{l} \Rightarrow 0 \\ \Rightarrow 3 \\ \Rightarrow 1 \\ \Rightarrow 2 \end{array} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

Same as Hadamard

Haar

2D $\Rightarrow H$ & H'

$$Hr_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad Hr_4 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

$$Hr_8 = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$

Property

1) Real & orthogonal
2) very fast transformation
3) Poor energy compaction for image

October

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Slant transform

→ Designed for image coding

→ $S_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$S_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ a+b & a-b & -a+b & -a-b \\ 1 & -1 & -1 & 1 \\ a-b & -a-b & a+b & -a+b \end{bmatrix} \qquad b = \frac{1}{\sqrt{5}}, \; a = \frac{2}{\sqrt{5}}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3/\sqrt{5} & 1/\sqrt{5} & -1/\sqrt{5} & -3/\sqrt{5} \\ 1 & -1 & -1 & 1 \\ 1/\sqrt{5} & -3/\sqrt{5} & 3/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$$

KL transform

→ It decorrelates the data which facilitates high degree of compression

→ Also known as eigen vector transform

→ Steps

i) Find the mean vector & the covariance matrix of x

ii) Find eigenvalues & then the eigenvectors of the covariance matrix

iii) Create transformation matrix T such as row of T (basis functions) are the eigenvectors

iv) $X = T[C_x - m]$

August

W	M	T	W	T	F	S	S
31						1	2
32	3	4	5	6	7	8	9
33	10	11	12	13	14	15	16
34	17	18	19	20	21	22	23
35	24	25	26	27	28	29	30
36	31						

eg $X = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$

Step 1) $x_0 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ $x_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

Step 2) Covariance

$$\bar{x} = \frac{1}{M} \sum_{k=0}^{M-1} x_k$$

$$\bar{x} = \frac{1}{2} \{ x_0 + x_1 \} = \frac{1}{2} \begin{bmatrix} 4 \\ -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \mathrm{cov}(x) = E[x x^T] - \bar{x}\bar{x}^T$$

$$\bar{x}\bar{x}^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$E[x x^T] = \frac{1}{M} \sum_{k=0}^{M-1} x_k x_k^T$$

$$= \frac{1}{2} \left[\begin{bmatrix} 4 \\ -1 \end{bmatrix} \begin{bmatrix} 4 & -1 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} \begin{bmatrix} -2 & 3 \end{bmatrix} \right]$$

$$= \frac{1}{2} \left[\begin{bmatrix} 16 & -4 \\ -4 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -6 \\ -6 & 9 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 10 & -5 \\ -5 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathrm{cov}(x) = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 0 \end{bmatrix}$$

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Step 3) Eigen value

$$|cov(x) - \lambda I| = 0$$

$$\left| \begin{bmatrix} 1 & -2 \\ -2 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} 1-\lambda & -2 \\ -2 & -\lambda \end{bmatrix} \right| = 0$$

$$(1-\lambda)(-\lambda) \cancel{-4} - (-2)(-2) = 0$$

$$\lambda^2 - \lambda - 4 = 0$$

$$\lambda = \frac{1 \pm 4.1231}{2}$$

$$\lambda_0 = 2.5615$$

$$\lambda_1 = -1.5615$$

Step 4) Eigen vector od cov(x)

$$(cov(x) - \lambda_0 I)\,\phi_0 = 0$$

$$\begin{bmatrix} -1.5615 & -2 \\ -2 & -2.5615 \end{bmatrix} \begin{bmatrix} \phi_{00} \\ \phi_{01} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $\phi_{01} = 1$

$$\therefore \phi_{00} = \frac{2}{-1.5615} = -1.2809$$

$$\phi_0 = \begin{bmatrix} -1.2808 \\ 1 \end{bmatrix}$$

August

W	M	T	W	T	F	S	S
31						1	2
32	3	4	5	6	7	8	9
33	10	11	12	13	14	15	16
34	17	18	19	20	21	22	23
35	24	25	26	27	28	29	30
36	31						

do the same for $\phi_1 = \begin{bmatrix} 0.7808 \\ 1 \end{bmatrix}$

Step 5) Normalization

$$\frac{\phi_0}{\|\phi_0\|} = \frac{1}{\sqrt{(-1.2808)^2 + 1}} \begin{bmatrix} -1.2808 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.7882 \\ 0.6154 \end{bmatrix}$$

$$\frac{\phi_1}{\|\phi_1\|} = \begin{bmatrix} 0.6154 \\ 0.7882 \end{bmatrix}$$

$$\therefore T = \begin{bmatrix} -0.7882 & 0.6154 \\ 0.6154 & 0.7882 \end{bmatrix}$$

Check $\Rightarrow TT^T = I$

Step 6) transform Matrix of the input matrix

$$Y_0 = T[x_0] = \begin{bmatrix} -3.7682 \\ 1.6734 \end{bmatrix}$$

$$Y_1 = T[x_1] = \begin{bmatrix} 3.4226 \\ 1.1338 \end{bmatrix}$$

$$Y = \begin{bmatrix} -3.7682 & 3.4226 \\ 1.6734 & 1.1338 \end{bmatrix}$$ //

Image Enchancement

i) Low pass filtering

→ $H(u,v) = \begin{cases} 1 & u^2+v^2 \leq D_o^2 \\ 0 & \text{o.w.} \end{cases}$ ← cutoff frequency which determines the amt of frequency components pass

→ D_0 controls the amount of blurring

→ Ringing Effect ⇒ Sharp cutoff frequencies produce an overshoot of image features whose frequency is close to the cutoff

→ Types

a) Ideal

b) butterworth

$$H(K,L) = \frac{1}{1+\left[\frac{\sqrt{K^2+L^2}}{D_0}\right]^{2n}}$$

→ Helps in lowering the ring effect.

c) Gaussian

$$H(u,v) = e^{-(u^2+v^2)/2\sigma^2} \Rightarrow e^{-(u^2+v^2)/2D_0^2}$$

2) Highpass filtering

→ Obtain from LP ⇒ $H_{Hp}(u,v) = 1 - H_{Lp}(u,v)$

→ Preserves high frequency

→ Enchances edges & fine details

→ ~~Type~~ $H(u,v) = \begin{cases} 1 & u \geq D_0 \\ 0 & \text{o.w} \end{cases}$, $H(u,v) = \begin{cases} 1 & u^2+v^2 \geq D_0^2 \\ 0 & \text{o.w} \end{cases}$

→ Types

a) Ideal

b) Butterworth →

$$H(u,v) = \frac{1}{1+[D_0/\sqrt{u^2+v^2}]^{2n}}$$

c) Gaussian

$$H(u,v) = 1 - e^{-(u^2+v^2)/2D_0^2}$$

3) Homomorphic Filtering

→ Enchances contrast

→ Reduce illumination artifacts

→ $f(x,y) = i(x,y)\, r(x,y)$

Illumination — Reflection

- Varies slowly — - Varies faster
- affects low freq — - affects high freq

→ Steps

i) Take $\ln(f(x,y)) = \ln(i(x,y)) + \ln(r(x,y))$

ii) Apply FT : $F(\ln(f(x,y)))$

iii) Apply $H(u,v) \Rightarrow Z(u,v)H(u,v) = Illum(u,v)H(u,v) + Refl(u,v)H(u,v)$

iv) Take inverse FT : $F^{-1}[Z(u,v)H(u,v)]$ or $s(x,y)$

v) Take exp $\Rightarrow e^{s(x,y)} = e^{i'(x,y)} e^{r'(x,y)}$

or $g(x,y) = i_0(x,y)\, r_0(x,y)$