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Travail de Maturité

Stochastic Modelling of Snow Depth in Mountains

Mountain Top, Bernese Oberland, Switzerland – Source: Personal Collection

Number of Words: 6174

Under the Supervision of Dr. Paul Maley

Research Question:

How can stochastic modelling help us make accurate representations of snow depth on mountains?

Contents

1. Introduction

Since the rise of winter sports in post-WW1 Europe, the world has developed a multi-million-dollar industry that has constantly been expanding to welcome more tourists every single year. From the summits of the Tierra del Fuego in Argentina to the Japanese Alps in Hokkaido, passing by the Bavarian mounts, the winter sports industry is responsible for tens of thousands of jobs and is often of major economic importance.

Figure 1 - Davos, Circa 1930 - Source: J. Gaberell[1]

The purpose of this research paper is to create a stochastic model that can help us better understand and predict snow depth in mountains. We will try in this introduction to demonstrate the economic implications of snow depth.

Switzerland, for example has an economy mostly based on its leading services and pharmaceutical and high-precision industries; however, winter tourism is an important source of income for this small alpine country. It is estimated that ski resorts are responsible for over

[1] vintageski (n.d.). *Vintage Ski*. [online] Vintage Ski. Available at: https://vintageski.tumblr.com/post/32739061217/the-slopes-are-a-bit-loose-above-the-parsenn-h%C3%BCtte [Accessed 28 Oct. 2021].

1% of the country's GDP but this figure increases to 10% in some of the country's regions that rely heavily on the sector such as Valais and Grisons.[2]

In the United States, where ski resorts have managed to transform a niche into a mass-market tourism industry, it is estimated by the National Ski Areas Association (NSAA) that over 59 million people visited ski resorts nation-wide in 2018.[3] This represents almost four times the number of tourists that visited the city of Dubai that same year (16 million).[4] This impressive figure represents the extent of the winter industry and its importance for the tourism sector.

Skiing remains the most popular attraction for tourists visiting these regions, followed by snowboarding. Except for artificial snow surfaces which have undergone major progress in recent years, snow remains essential for both activities. A minimum amount of snow depth is required for slopes to be ploughed; this depth often depends on the ski slope. This snow depth doesn't only depend on snowfall, which can be irregular during the winter season. Generally, it also depends on climatic factors such as temperature, wind speed, rain, or anthropogenic factors such as the number of skiers passing. This sometimes leads to periods with insufficient snow for ski infrastructures to operate.

With the rising uncertainties brought by climate change, ski resorts located at lower altitudes have been trying to diversify their sources of income by giving tourists the opportunity to try new activities such as mountain-biking or cross-country skiing, which require little to no snow. As illustrated in the study of climate written by the Swiss Federal Research Institute,[5] winters are not only getting shorter on the long-term, but the snow depth is also highly volatile and does not follow the same schemes as it did 50 years ago.

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[2] RTS (2020). *Tourisme, sports de neige, l'"or blanc" pèse lourd dans l'économie alpine*. [online] RTS Info. Available at: https://www.swissinfo.ch/fre/tourisme--sports-de-neige--l--or-blanc--pèse-lourd-dans-l-économie-alpine/46225450 [Accessed 23 Sep. 2021].

[3] NSAA (2020). *OTTKE END OF SEASON AND DEMOGRAPHIC REPORT 2019/20*. [online] NSAA. Available at: https://nsaa.org/webdocs/Media_Public/IndustryStats/Historical_Skier_Days_1979_1920.pdf [Accessed 23 Sep. 2021].

[4] Dubai Online. (n.d.). *Dubai Tourism Statistics - Visitor Numbers, Number Of Hotels And Rooms*. [online] Available at: https://www.dubai-online.com/essential/tourism-statistics/[Accessed 23 Sep. 2021].

[5] Peter Muster (n.d.). *Snow and climate change - SLF*. [online] www.slf.ch. Available at: https://www.slf.ch/en/snow/snow-and-climate-change.html. [Accessed 23 Sep. 2021].

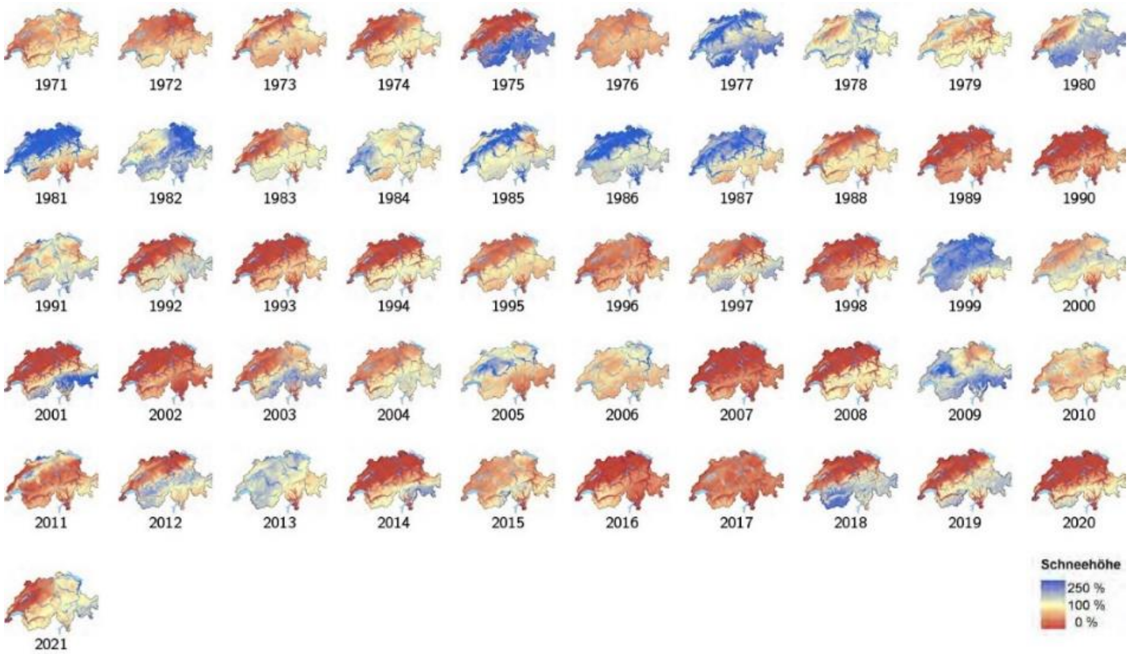

Figure 2 - Snow Depth from 1971 to 2021 in Switzerland - Source: Swiss Federal Research Institute WSL (ETH Domain)[6]

Figure 2 illustrates the state of snow depth in Switzerland over the years. The blue colour represents higher than average snow depth whereas red represents under the average. As we go further in time towards today, we notice that the regions affected as well as the damage created is more and more uneven. This is an excellent illustration of the growing uncertainties ski resorts must face every season. Although action to counter climate change is being taken, it is undeniable that uncertainties in snow depth will rise, as proven by the RCP pathways which illustrate the different possible paths climate change might take.[7]

[6] Peter Muster (n.d.). *Snow and climate change - SLF*. [online] www.slf.ch. Available at: https://www.slf.ch/en/snow/snow-and-climate-change.html. [Accessed 23 Sep. 2021].

[7] What are the RCPs? (n.d.). [online] Available at: https://coastadapt.com.au/sites/default/files/infographics/15-117-NCCARFINFOGRAPHICS-01-UPLOADED-WEB%2827Feb%29.pdf. [Accessed 23 Sep. 2021].

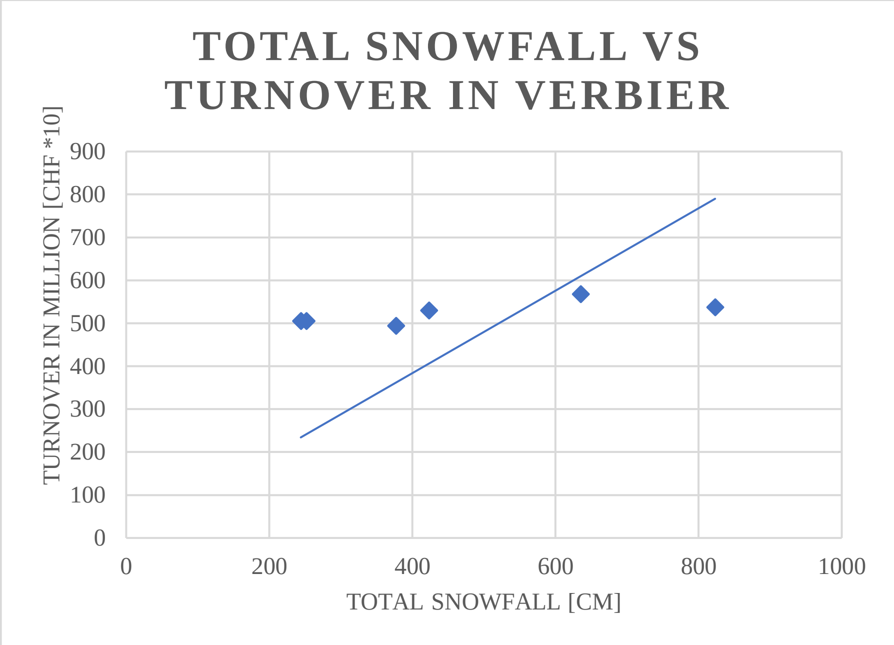

Figure 3 - Data Source: Téléverbier SA & onthenow.co.uk[8]

The example of Verbier's Infrastructure Operator TéléVerbier acutely demonstrates this deeply rooted link between the profitability of infrastructure operators and snowfall amounts. We have disregarded the 2019-2020 as the turnaround was heavily impacted by the Covid pandemic. We can observe in figure 3 that turnover increases alongside snowfall. Indeed, high snowfall leads to an increase in tourist attractivity yielding a higher turnover overall.

Supporting this is the Pearson Correlation Ratio for this dataset that can be estimated at $r = 0.7$ indicating a correlation. Even though correlation does not imply causation, we may still assume that a low level of snow can cause a decrease in the earnings of ski resorts. Indeed, several other examples of the economic impact of low snowfall seem to follow this tendency; a season with poor or little snow can have devastating consequences.

In 2018, the US state of New Mexico suffered from low snowfall; the ski resorts located in the region only saw about half of the usual snow. The attendance at ski resorts plummeted from 750'000 to 534'000 visitors. This represents an estimated 100 million U.S. Dollars in loss of

[8] verbier4vallees.ch. (n.d.). *A propos de Téléverbier - Verbier 4Vallées*. [online] Available at: https://www.televerbier.ch/fr/televerbier/informations-financieres/televerbier-communiques-financiers.html [Accessed 23 Sep. 2021]. & www.onthesnow.co.uk. (n.d.). *Verbier Snowfall Statistics | Historical Snow | OnTheSnow*. [online] Available at: https://www.onthesnow.co.uk/valais/verbier/historical-snowfall.html?&y=2018 [Accessed 23 Sep. 2021].

income for the winter tourism industry of the state.[9] In a 2020 documentary, the AFP (Agence France-Presse) tackles the subject of low snow levels in the Pyrenees range.[10] From the interviews, we learn that the lack of snow that year is not only linked to the higher-than-average temperatures but also to a storm in December of that same year. This example illustrates the fact that snow depth does not only depend on climatic factors but can also depend on a certain group of random events occurring during the winter season.

As insecurities linked to snowfall rise, the ability to model and predict snowfall is becoming a necessity for numerous players in the industry. We will therefore be asking ourselves how stochastic modelling can help us make accurate representations of snow depth on mountains. We will first try to understand the stakes of the mathematical modelling of snowfall, through the discovery of different applications. Then, we will attempt to better understand snow depth on mountains and to recognize patterns from historical data. After having better understood how snow depth works, we will be modelling the phenomenon in the most accurate way possible. The last part of this research paper will be to build a computer program that will allow us to apply this mathematical model, in order to evaluate its accuracy and make predictions. It is important to note that our model will only focus on the short term within a season and not on the long term.

[9] Outside Online. (2018). *What the Worst Winter in 60 Years Did to Ski Resorts*. [online] Available at: https://www.outsideonline.com/outdoor-adventure/snow-sports/winter-really-was-bad-everyone-thought/ [Accessed 23 Sep. 2021].

[10] www.youtube.com. (n.d.). *Manque de neige: comment les Pyrénées s'adaptent au réchauffement climatique | AFP Reportage*. [online] Available at: https://www.youtube.com/watch?v=O4EkKs-TrnU. [Accessed 23 Sep. 2021].

2. Applications of Mathematical Modelling of Snow Depth

Several stakeholders require or request mathematical models of climatic weather phenomena. The underlying predictions resulting from the models can help with organization, planification, anticipation, and risk management, allowing more efficient preventive measures.

A simple application of snow depth predictions would be to allow ski resorts to better inform the public about the state of ski slopes. Indeed, skiers are always curious, when planning their trip, as to how much snow there will be on the mountain tops, independently from the weather of the day. This could enhance the number of visitors.

The most important application of these snow depth predictions is for ski resorts themselves. Thanks to the advances of technologies in ski slope preparation and snow ploughing during the past 40 years, ski resort operators are now able to prepare the ski slopes to accommodate skiers on a specific type of snow. For example, ploughing with a very low snow depth needs to be done using special machines to make sure dirt isn't raked out. These ploughing techniques allow ski resorts to extend the ski season by preserving the snow depth. The ability to predict future snow depth allows ploughs to be done in an organised and optimised way to better tackle the snow problem and extend the ski season to maximise profit.

Figure 4 - Ski Slope Being Ploughed - Source: PistenBully[11]

[11] PistenBully | Pistenraupen & Pistenfahrzeuge. (n.d.). *PistenBully 600*. [online] Available at: https://www.pistenbully.com/home/fahrzeuge/pistenbully-600.html [Accessed 28 Oct. 2021].

To lengthen the ski season, some ski resorts have even invested in artificial snow-making technologies such as snow guns or snow cannons. These complex systems can only be used when numerous criteria are satisfied (i.e. low enough temperature, sun exposure, and restricted operation timeframes), but most importantly they are heavily reliant on important quantities of water. As most ski resorts are not able to find water in sufficient amounts, and are therefore limited in the quantity of snow they are able to produce throughout the season, short-term predictions would allow ski resorts to better manage their water reserves, lengthen the ski season, and preserve the environment.

Insurances also show great interest in accurate snowfall predictions. Indeed, to minimize losses in case of insufficient snowfall, most ski resorts nowadays sign insurance contracts. Every year, ski resorts will pay a fee to the insurer and in case of low snowfall, the insurer will organize a pay-out to all affected ski resorts if they satisfy the criteria defined in the contract. However, to ensure good cash management, insurers invest in a large array of securities which have different durations. Therefore, the ability for insurers to predict future snowfall is essential to efficiently organize the flow of redemptions versus the pay-outs needed.

Shot-term snowfall predictions can also be used as financial instruments by investors through the use of derivatives. A snow derivative is a financial product traded over the counter that allows parties to hedge against the risk of snow related loss.[12] The issuer manages their risk more efficiently that way by transferring part of the inherent risk to the investor who will try to trade these contracts. These derivatives can also be treated as futures or options on futures, both products allowing different investment strategies. The ability to make short-term predictions of snow depth is key for traders to anticipate the future price of these contracts as they are based on it. Most of them trade on niche markets such as the CME (Chicago Mercantile Exchange). However, due to low trading volumes, the CME has discontinued all snow-based derivatives.[13] These products might make a comeback in the future if the market can find a way to better understand them.

[12] Investopedia. (2019). *Weather Derivative*. [online] Available at: https://www.investopedia.com/terms/w/weatherderivative.asp. [Accessed 23 Sep. 2021].
[13] Fortune. (n.d.). *Wall Street Still Can't Figure Out How to Make Money off Snow*. [online] Available at: https://fortune.com/2017/02/09/wall-street-snow-derivatives/ [Accessed 23 Sep. 2021].

The most fundamental use of snow depth predictions is in the domain of hydrology. Having the ability to understand and predict snow depth is essential to better understand snow melt. As snow melts and water in rivers rises, knowing the future flow will help plan for example damns, hydroelectric plants and much more. Not only this, but it could also save lives. In the summer of 2021, Germany saw its deadliest floods ever, maybe a precise prediction of water flows could have helped to prevent casualties.[14]

We have looked at a few applications of snow depth models, however, there are countless applications one might use to resolve precise problems.

[14] Welle (www.dw.com), D. (n.d.). *Floods in Germany | DW | 09.10.2021*. [online] DW.COM. Available at: https://www.dw.com/en/floods-in-germany/t-58300604. [Accessed 23 Sep. 2021].

3. Statistical Study of Snow Falls and Snow Depth

Our snow depth predictions will be done taking a statistical approach, modelled independently from meteorological weather forecast. Instead, it will be done looking at historical snow depth data. By analyzing many datasets located at different altitudes, on different continents, we will try to better understand the phenomenon and find trends, which will help us model it.

The first study we will be looking at was set up by the Center for Snow & Avalanche Studies in Southern Colorado. Located in an untouched area, it provides a perfect location for the study of snow depth on mountains and ski resorts.

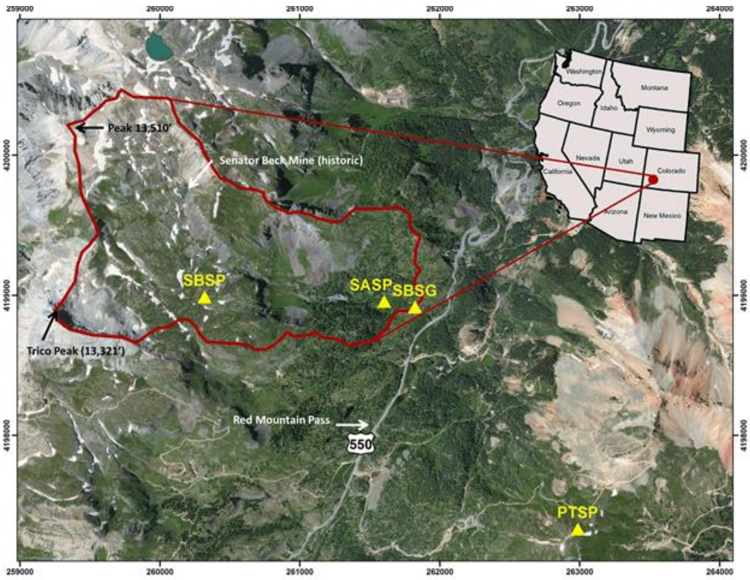

Figure 5 - Satellite Imagery of the Senator Beck Basin Study Area - Source: Center for Snow and Avalanche Studies[15]

Senator Beck Study Plot (SBSP), at an altitude of 3692 meters, is one of the locations where measurements were made with several instruments such as a laser snow gauge or a thermometer. For the purpose of this research, we will be analyzing snow depth data which is more relevant than snow depth. For example, in case of a very large snowfall and then high temperatures, studying snowfall to find snow depth would be an unnecessary complication.

[15] Anon, (n.d.). *SENATOR BECK BASIN STUDY AREA: A MOUNTAIN SYSTEM OBSERVATORY | Center for Snow and Avalanche Studies*. [online] Available at: https://snowstudies.org/senator-beck-study-area-overview/ [Accessed 23 Sep. 2021].

Figure 6 - Laser Snow Gauge at SBSP - Center for Snow and Avalanche Studies[16]

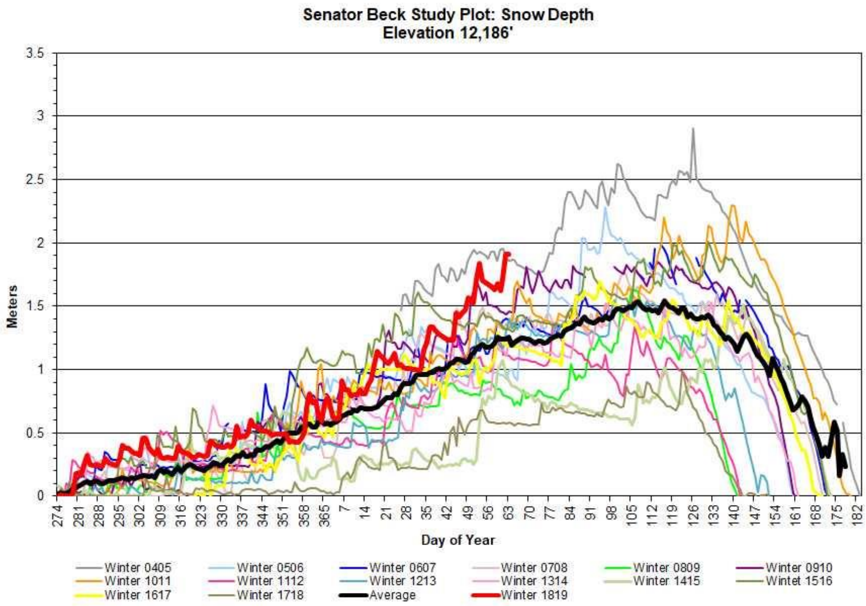

Figure 7 - Graph Illustrating Different Winters' Snow Depth as a Function of Time in Senator Beck, USA - Source: Center for Snow and Avalanche Studies[17]

Figure 7 illustrates numerous years of snow depth over time. The black line represents the average which allows us to identify a "typical" snow depth over time graph.

[16] *SENATOR BECK BASIN STUDY AREA: A MOUNTAIN SYSTEM OBSERVATORY | Center for Snow and Avalanche Studies*.
[17] *SENATOR BECK BASIN STUDY AREA: A MOUNTAIN SYSTEM OBSERVATORY | Center for Snow and Avalanche Studies*.

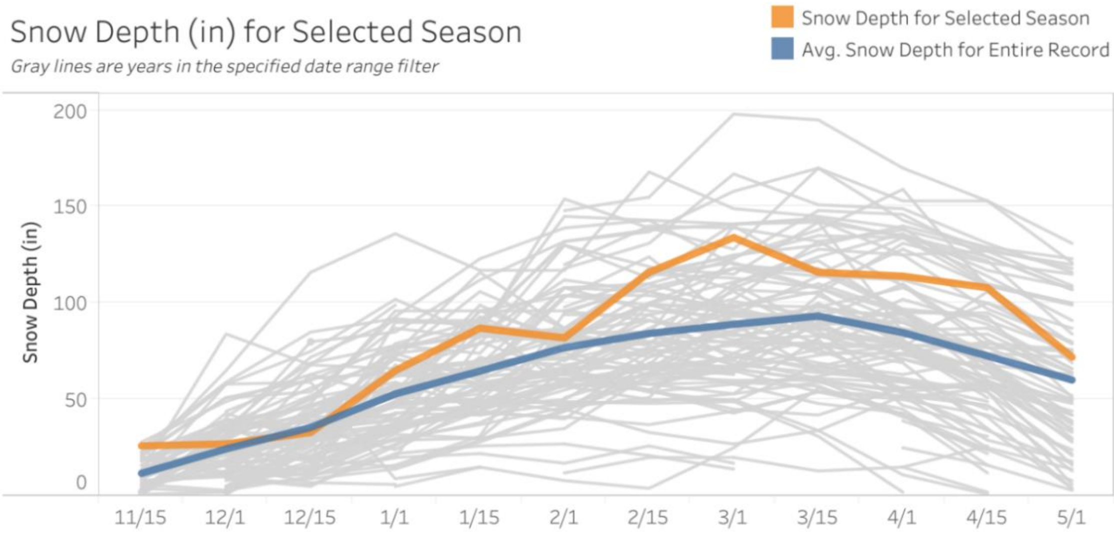

Figure 8 - Graph Illustrating Different Winters' Snow Depth as a Function of Time in Crystal Mountain, USA - Source: University of Washington[18]

Figure 8, based on a data sample taken in the State of Washington on the United States' Eastern Coast also endorses the average snow depth pattern studied in the previous example and therefore confirms that this pattern can be observed at other locations.

After the study of many other datasets, one can conclude that there is a trend on most mountains to have similar snow depth repartitions over time. The continuous initial climb is because there is a snow accumulation from the beginning to the season until spring. After this period, the snow starts melting at a rate higher than the initial accumulation. Even though some locations might behave slightly differently, this general trend is universal.

[18] Washington.edu. (2021). *Mountain Snow Depth | Office of the Washington State Climatologist*. [online] Available at: https://climate.washington.edu/climate-data/snowdepth/ [Accessed 28 Oct. 2021].

4. Construction of a Stochastic Model

We will now tackle the central part of this research paper where we will find a model that reliably represents snow depth. We will first start by finding a function which we will call "base function", that follows the trend we have identified in section 3, which will give us a reliable representation of snow depth over time and then applying a Wiener process to it, transforming the model into a stochastic one, therefore, illustrating random and extreme occurrences.

4.1. Base Function

Taking account of the previously studied cases; a good mathematical approximation of the average snowfall would be a Beta distribution.

The Beta probability distribution function can be defined as:

$\beta(x, \alpha, \beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$; with α and β our shape parameters.

And $B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$;

$B(\alpha, \beta)$ is the Beta function, also called the Euler integral.

Usually, the Beta distribution is used in probabilities to assess random variables like for example normal or Gaussian distributions. However, it can also be used as a "common" function which we will define in our model between [0,1].

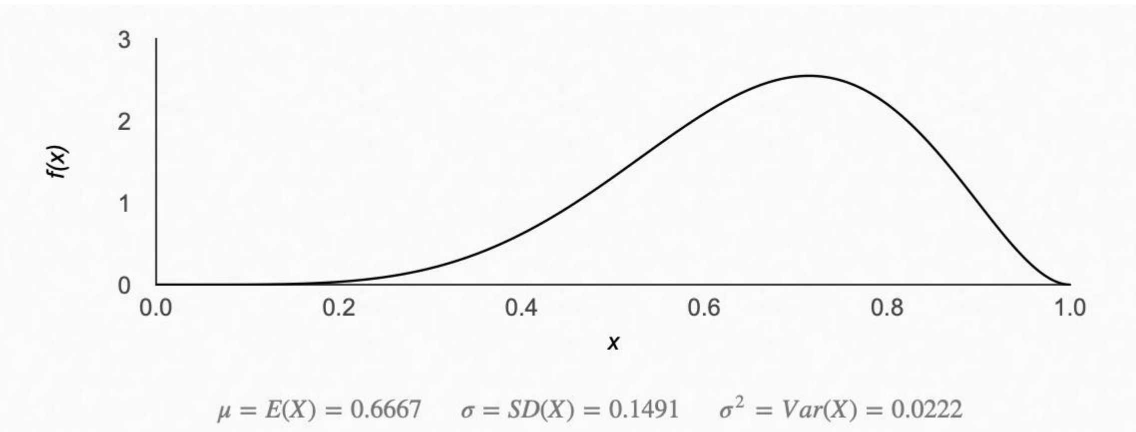

Figure 9 - Beta Distribution with the Two Defined Shape Parameters

We will be using the two shape factors, $\alpha = 6\,;\beta = 3$, which best fits the trend we found in part 3 of this paper. Figure 9, 10 and 11 are generated thanks to the University of Iowa's (Matt Bognar) Beta Distribution Tool.[19]

We will be basing our stochastic model on this Beta distribution function as it allows a good foundation for our next section, stochastic modelling.

4.2. Wiener Process

In this sub-section, we will be introducing the basics of stochastic modelling, even though we will be taking a numerical path, it will allow a better understanding of models and will facilitate the construction of a computer program to resolve our model.

In comparison to deterministic models, stochastic models always include at least one random variable and will thus never give the same result when simulated. These models for example allow us to better simulate rare occurrence such as extreme weather events by considering both probabilistic and random factors into the equation. For example, when stochastically modelling a wildlife population, the simulation could contain a rapid decreases in specimen numbers in a simulation; this could represent the rise of a predator in the region. Although inaccurate most times, modelling allows the party to accurately estimate the risk of an occurrence.

As previously mentioned, by definition, a stochastic model will produce different outcomes each time as at least one random variable is contained in the model. Therefore, by generating thousands if not millions of simulations, we will be able to see numerous possible outcomes a ski season might have, basing ourselves on our Beta function. We will be using the simulated outcomes to recognize patterns and the more outcomes we find, the more patterns we can recognize. Therefore, by comparing our model's outcomes to an ongoing ski season, one might find similarities to one generated stochastic model and predict the upcoming snow depth based off that correlated model.

[19] homepage.divms.uiowa.edu. (n.d.). *Beta Distribution Applet/Calculator*. [online] Available at: https://homepage.divms.uiowa.edu/~mbognar/applets/beta.html [Accessed 23 Sep. 2021].

According to the law of large numbers, if we were to perform this experiment a very large number of times, we would get closer to the theoretical value which in our case is the Beta function.[20] This shows that even though our stochastic models might heavily differ, if we average them down, we will find our base function.

Some occurrences are impossible in real-life, for example, as illustrated in figure 11, point 3, there cannot be a flat line in our snow depth for such a period of time as snow evolves and cannot stay a constant. This is also impossible in our stochastic model as the probability of having a same random variable in our Wiener process is close to 0, our stochastic model will therefore take into account this possibility. Furthermore, illustrated at points 1 and 2 of figure 10, are rare occurrences, these events have an extremely low probability of occurring, however, they are still possible for example in case of an avalanche, an earthquake, or an extreme snow blizzard. Their rare occurrence is also considered in our stochastic model as we will be using a deviation in our model that will be low therefore making such a decrease or increase extremely rare. This shows how stochastic models adapt extremely well to the modelling of snow depth.

Two Graphs Illustrating the Advantages of Using Stochastic Modelling:

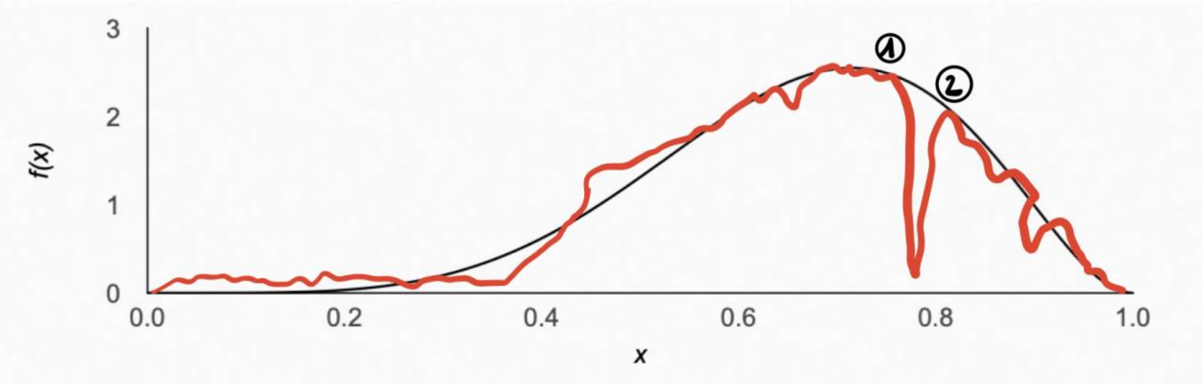

Figure 10 - Beta Function with Possible Stochastic Simulation

[20] Dekking, F.M. and Al, E. (2010). *A modern introduction to probability and statistics : understanding why and how*. London: Springer. [Accessed 23 Sep. 2021].

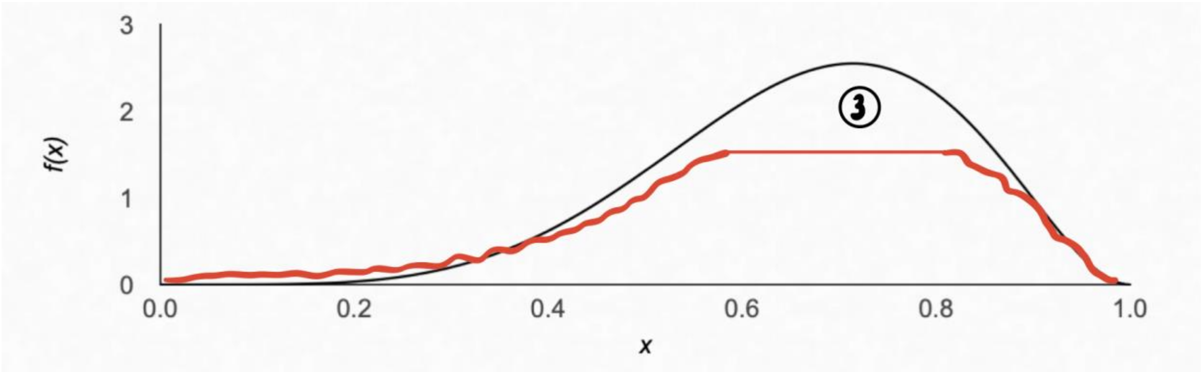

Figure 11 - Beta Distribution with Impossible Stochastic Simulation

In stochastic modelling, one may principally use two methods:

- Continuous stochastic processes which can be described as $\{X_t\}_{t \geq 0}$
- Discrete stochastic processes which can be described as $X_0, X_1, X_2, X_3, \ldots$

Discrete-time stochastic processes are most often preferred as they are simpler to apply than continuous-time processes. Nonetheless, in our case we will be using a continuous-time stochastic process and will be working at small intervals within it.

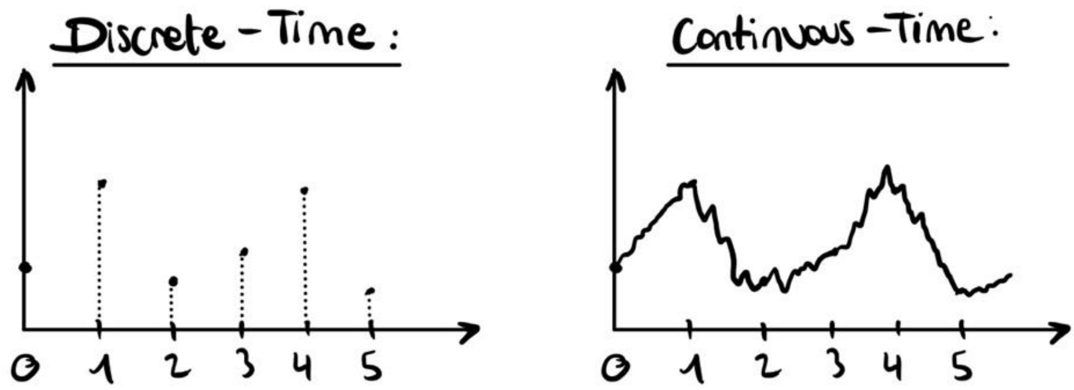

Figure 12 - Comparison Between Discrete and Continuous Stochastic Processes - Source: Personal Collection

As mentioned earlier, our model will be based on the Beta repartition function. As by itself it isn't a stochastic process as there is no random variable, we need to add a stochastic component. The simplest way to apply this idea is to use a single-dimension Wiener process, a type of stochastic process which we will be introducing in the next paragraph.

A Wiener process is a continuous-time stochastic process that was named after Norbert Wiener, a mathematician who famously investigated the works of Robert Brown. Scottish botanist

Robert Brown first described the phenomena in 1827 after studying the movement of pollen in water, which he then described as the sum of the current position and a random increment, meaning the movement is independent from its previous position.[21]

A Wiener process is a stochastic process $\{W_t\}_{t\geq 0}$ indexed by nonnegative real numbers t with the following properties:[22]

1) $W_0 = 0$;
2) In the interval [0,1], the function t $\rightarrow$ W_t is continuous in t;
3) The process $\{W_t\}_{t\geq 0}$ has stationary and independent increments;
4) The integral $W_{t+s} - W_s$ has normal distribution (0,t)

With (0,t) the interval on which we study the process.

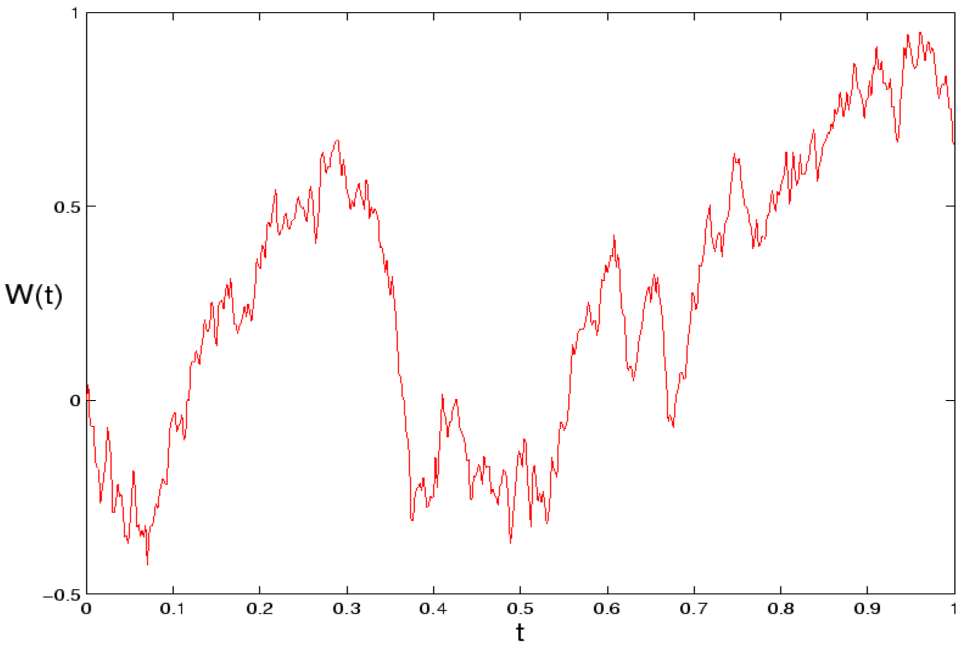

Figure 13 - Random Walk of a Winer Process – Source: University of California, Santa Barbara[23]

[21] The Editors of Encyclopedia Britannica (2017). Brownian motion | physics. In: *Encyclopædia Britannica*. [online] Available at: https://www.britannica.com/science/Brownian-motion. [Accessed 23 Sep. 2021].
[22] BROWNIAN MOTION. (n.d.). [online] Available at: https://galton.uchicago.edu/~lalley/Courses/313/BrownianMotionCurrent.pdf. [Accessed 23 Sep. 2021].
[23] sites.me.ucsb.edu. (n.d.). *A Standard Wiener Process*. [online] Available at: https://sites.me.ucsb.edu/~moehlis/APC591/tutorials/tutorial7/node2.html. [Accessed 23 Sep. 2021].

A realization of a standard Wiener process can be seen in figure 13; as this is a stochastical model, this is only one of the infinite numbers of possible realizations.

The Wiener process we will be using will be:

$$d(t) \ = \ \beta(t) + \ \sigma W;$$

However, because of the difficulties we might encounter when trying to solve these complex differential equations, we will be evaluating them "numerically", at small increments:

$$\Delta d(t) = \Delta\beta(t) + \Delta\sigma W;$$

Where dt is our snow depth function, $\beta(t)$ our Beta probability distribution function, σ the coefficient of our deviation and W our deviation.

This form allows us to bypass complicated differential equations and worked with small time differences offering a more complete numerical solution.

As a reminder, the Beta probability distribution is defined as:

$$\beta(t, \alpha, \beta) = \frac{1}{B(\alpha, \beta)} t^{\alpha-1}(1-t)^{\beta-1}$$

Using the shape parameters mentioned in the previous section ($\alpha = 6\,; \beta = 3$), we can calculate that $B(6{,}3) = {}^{1}\!/_{168}$, as obtained by using the following integral found in our definition in sub-section 4.1:

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx$$

$$= {}^{1}\!/_{168}$$

This will simplify our Beta probability function to:

$$\beta(t, \alpha, \beta) = 168t^5(1-t)^2;$$

5. Creation of a Numerical Solution

The manual calculation of stochastic processes is extremely complicated. Having a numerical solution will not only simplify the process, it will also significantly improve its efficiency. We will be coding this solution using the Python programming language. This language is highly polyvalent thanks to the many libraries made for it which can be used to expand our programming horizons. Furthermore, it is free and fully open source. In comparison, other mathematics-oriented programming languages such as MATLAB might be easier to use and more efficient but cost thousands of dollars in base software and extensions every single year.

5.1. Main Frame

In the first part of this section, we will be creating the main body of our program which we will thereafter use to first evaluate the accuracy of our model and then make predictions of future snow depth.

5.1.1. Initial Settings

```
import pandas as pd
import matplotlib.pyplot as pl

n = 500 #number of models generated
T = 275 #time intervals
total = 1 #total function interval
cf = []
cc = []

dg = pd.DataFrame() #all generated models dataframe
df = pd.read_excel('sheet.xlsx') #template sheet
dr = pd.read_excel('result.xlsx') #template sheet (results)
```

To start our program, we must set our initial settings. We import two libraries, *pandas* which allows us to treat data with a higher efficiency thanks to the use of dataframes and *matploylib.pyplot* which will allow us to plot large arrays with higher efficiency than for instance Excel. We also set the *n* parameter which indicates how many models our program will generate as well as the *T* parameter is the number of intervals. We usually choose the number of days in our data sample. The *total* parameter is the total function interval, usually set to 1 as the Beta function is defined on [0,1]. We also create two empty lists which we will be using later. The last step is to create an empty dataframe *dg*, and two template Excel sheets, *df* and *dr*.

5.1.2. Function Calculations

```
for k in range(n):
```

Even though it might seem like an obvious step, it is important to notice that we will be iterating the function creation *n* times.

```
    dt = total/(T+1) #time interval
    t = dt #current time
    for i in range(T):
        b = 168 *(t**5)*((1-t)**2)
        t += dt
        df.loc[i,'Beta(t)'] = b
```

We first calculate our *dt* parameter which is the time interval of our function, as mentioned earlier we take a numerical approach which involves the choice of a “Δ”. We then set the initial time and then iterate *T* times the calculation of our Beta function, update the current time and add the result to our *df* dataframe under the Beta(t) column using a localiser.

```
    from scipy.stats import norm
    x = 0
    dev = 0.25
    w0 = 0
    for i in range(T+1):
        w = (x+norm.rvs(scale=dev**2))
        w0 += w
        df.loc[i,'W(t)'] = w0
```

The second aspect of our program consists in creating the stochastic part of our model. This will be done by finding a numerical solution to a Wiener process. We start by importing a function from the *scipy.stats* library called *norm*, standing for normal distribution which is used in the choice of random variable. We must also define two initial parameters, *x* and *dev*. *x* indicates the initial state of the Brownian motion and *dev* indicated the deviation the model will feature, as mentioned, this is a mere direction, as because of stochasticity, the function could largely exceed this indicator. We then iterate over the number of intervals previously chosen the creation of Wiener Δs. The *norm.rvs* allows us to generate random continuous variables following our dev^2 as a *scale* or the standard deviation of the function. We then cumulate *w* (Wiener) into *w0* (Cumulation) and add this generated variable to our data frame under the ‘W(t)’ column.

5.1.3. Depth Model

```
    for i in range(T+1):
        if df.iloc[i,0] + df.iloc[i,1] > 0:
            df.loc[i,'D(t)'] = df.iloc[i,0] + df.iloc[i,1]
        else:
            df.loc[i,'D(t)'] = 0

    for i in range(T):
        dg.loc[i,k] = df.iloc[i,2]
```

The third part of the program is to add these two parts of our function to get the final result of our function d(t). We must first adjust some values, as at the initial stages of the Beta function, the values are usually close to 0 and the Brownian process might issue negative values therefore outputting negative depth values. To avoid this, we first verify if the value of Beta(t) + W(t) is larger than 0, if not we assign D(t) the value of 0 and add it to the data frame in the 'D(t)' column. In the opposite case, we simply add the two values and copy them to the 'D(t)' column. The last part is to add this season to our data frame of all generated seasons. To achieve this, we iterate in *T* and copy all the values into our *dg* data frame in the *k* column and the *i* row.

To illustrate our model, we will be generating 10 simulations and plotting them with the *matplotlib.pyplot* library, with different *dev* to illustrate our ability to guide deviation.

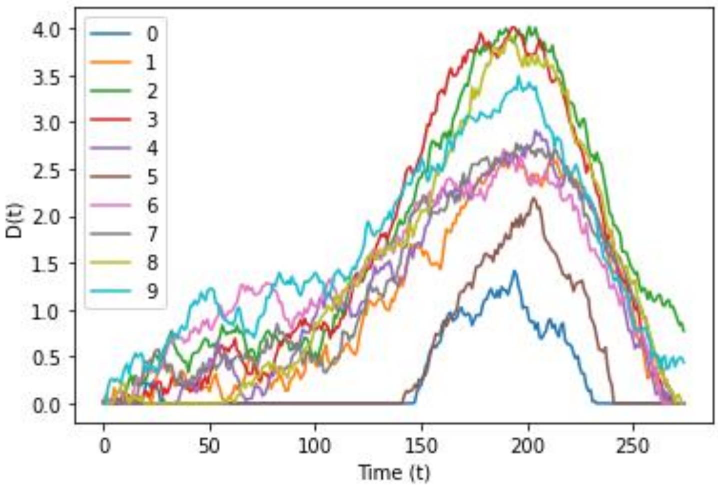

Figure 14 – 10 Depth Simulations with dev=0.25

This illustration shows us the stochasticity of our model with paths taking a very different and unique shape depending on the computer generation.

5.2. Correlation to a Real Season

In this part, we will be concentrating on evaluating the accuracy of our model by writing a program that evaluates correlations between our model and real snow depth data.

```
ds = pd.read_csv('arbaz.txt',sep = ';') #import of current season as CSV
```

We first import our season data, in this case the Arbaz data was provided by MeteoSwiss as a CSV file and convert it to a dataframe *ds*.

```
for i in range(T):
    cc.append(ds.iloc[i,2])
```

We then copy the information from the second columns of our *ds* dataframe which contains the depth data into a list, this will allow us to run the *scipy.stats* correlation function.

```
from scipy.stats import pearsonr
for i in range(n):
    for j in range(T):
        cf.append(dg.iloc[j,i])
    z = pearsonr(cc,cf)
    dr.loc[i,'Correlation'] = z[0]
    dr.loc[i,'Model Number'] = i
    cf.clear()
dr.sort_values(by=['Correlation'],inplace=True, ascending=False)
```

To find correlations between our models and the current season, we import the *scipy.stats pearsonr* function which enables us to run a Pearson correlation index between two lists. As we already have the current season stored in the *cc* list, we now add the values from each model into a temporary list *cf*. We therefore append each row (j) for each column (i) and add the correlation result to our *df* dataframe, also adding the model of this number. Before moving onto the next simulation, we erase the values of *cf*. After these steps, we arrange the correlations by descending order.

```
print('General Accuracy: (full season only)',dr['Correlation'].mean())
```

Finally, we print the mean value of all the correlations calculated by our model.

As we now have the ability to compare our generated models to real data, we must first modify parameters to better approximate real winter conditions. We will therefore evaluate different *dev* coefficients in our model and see which deviation is more adapted to our examples.

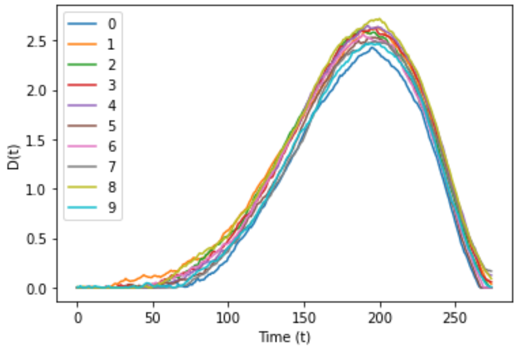

Figure 15 - Depth Simulations with dev = 0.1

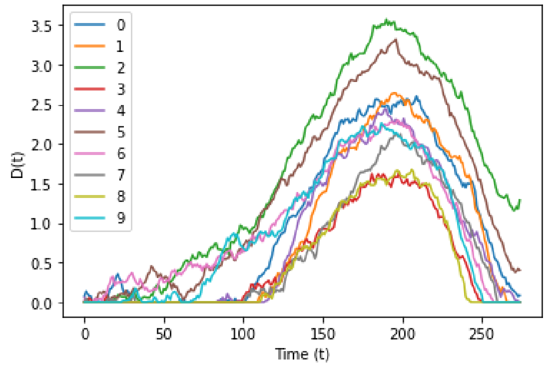

Figure 16 - Depth Simulations with dev = 0.2

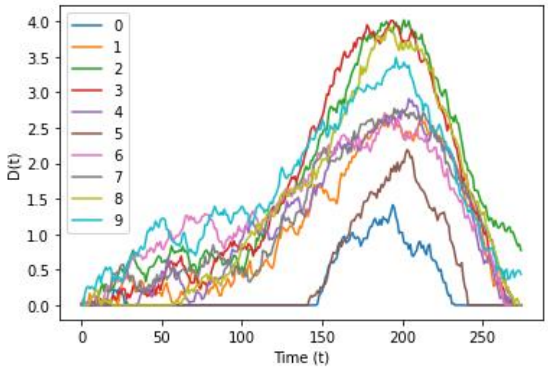

Figure 17 - Depth Simulations with dev = 0.25

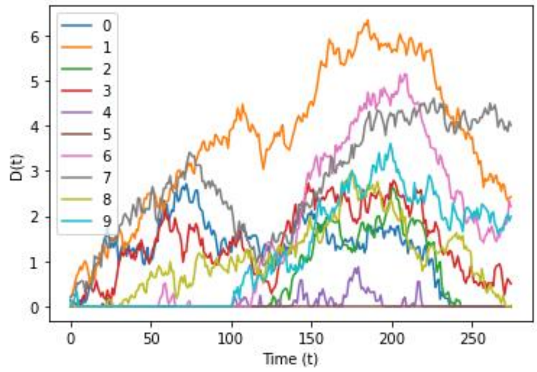

Figure 18 - Depth Simulations with dev = 0.4

These graphics in the table above illustrate how the *dev* variable can influence results, at *dev*=0.1 is similar to our Beta function whereas at 0.4, the function doesn't look like it at all and almost looks like a random walk.

Even though we could have chosen a *dev* simply by looking at the previous graphs, we will analyse the correlation to a real example and see how they correlate.

Dev:	Average Correlation to Arbaz (VS) 2019-2020
0.1	0.83
0.2	0.8
0.25	0.75
0.4	0.55

It seems that as the deviation increases, correlation decreases. At first, one might be tempted to choose a *dev* of 0.1 as it has a high correlation, however, it might not be the right choice as the stochasticity is low and we do not have the illustration of rare occurrence events, simulations have a too big overlap. On the other hand, choosing a *dev* of 0.4 makes little sense as it is much less correlated, and it also simulates too many rare occurrences. A *dev* of 0.25 seems to be the right compromise between correlation and stochasticity, which is why we will be using it in our future analysis. It is important to note that different *dev* might be more effective when applied to different locations and seasons, this example here only gives us an idea of which *dev* might be accurate in general.

5.3. Predictions for a Real Season

The second part of our program focuses on analysing data from an ongoing season and making predictions. We will then be looking at different years, locations and time intervals to evaluate the accuracy of our model.

As for the "Correlation to a Real Season" part, we will be using the "Main Frame" and adding on it to write this program.

```
ds = pd.read_csv('arbaz.txt',sep = ';') #import of current season as CSV
```

We first start by importing a current season as a CSV file, using the *pd.read_csv* function.

```
N = int(0.8*T)
```

A new parameter must be defined, *N* will allow us to evaluate only a part of the season. As we will be using historical data, we must only extract the first part of the season to predict the second part. For example, by choosing 0.8, one chooses to evaluate the initial 80% of a season.

```
for i in range(N):
    cc.append(ds.iloc[i,2])
```

We then iterate *N* times the copying of snow depth data into a list *cc*. This will later allow us to use the *pearsonr* function which is incompatible with dataframe columns.

```
from scipy.stats import pearsonr #itterating the correlations
for i in range(n):
    for j in range(N):
        cf.append(dg.iloc[j,i])
    z = pearsonr(cc,cf)
    dr.loc[i,'Correlation'] = z[0]
    dr.loc[i,'Model Number'] = i
    cf.clear()
dr.sort_values(by=['Correlation'],inplace=True, ascending=False)
```

The next step is to run correlation between the current season *cc* and the *n* number of generated models to see which is more accurate.

After testing many ways to calculate Pearson correlations, I determined that using the *scipy.stats pearsonr* library was the best for our program. We therefore import the function.

We start by having a main iteration *n* times. Inside of this iteration we iterate in *N* the copying of a generated model into a list, as mentioned earlier in order to work with the pearsonr function. We then run the correlation and add it to the *dr* dataframe as well as adding the model number to allow us to find it. We then clear the *cf* dataframe to be able to re-iterate.

The final step is to arrange the models by most accurate to least accurate using the *sort_values* function.

```
ooo = dr.iloc[0,0]
oooo = dr.iloc[1,0]
zero = (dg.iloc[N,ooo]+dg.iloc[N,oooo])/2
zeror = ds.iloc[N,2]
```

The final goal of this program is to evaluate the precision of the chosen model. Therefore, we will first find the two models with the highest correlation, looking up the model number in the *dr* dataframe. The *ooo* variable is highest correlated model and the *oooo* is the second highest correlated. We then search for the last data point of the model *zero*. To remove impurities in the data we make an average of the two highest correlated models. We also make a *zeror* variable which takes the last value of the current season data.

```
def percentage(x,y):
    return ((100*x)/y)-100
```

In the next step, we will be calculating percentages, so we need to define a function which does that.

```
pred = []
for i in range(50):
    temp = ((dg.iloc[N+i,ooo]+dg.iloc[N+i,oooo])/2)
    pred.append(percentage(temp,zero))
```

We then iterate for 50 days the percentage difference between the average of the two ‘best’ zeros and their predicted values. We then add this percentage to a list *pred*.

```
real = []
for i in range(50):
    real.append(percentage(ds.iloc[N+i,2],zeror))
```

The last step is to also calculate the percentage difference between the real zero and future predicted values. We then store them in a real list.

```
pl.plot(pred)
pl.plot(real)
```

To illustrate our results, we plot the real and predicted values to evaluate the precision of our model when doing predictions of future snow depth.

We will now be testing this program on a different dataset than Arbaz, where we trained our model.

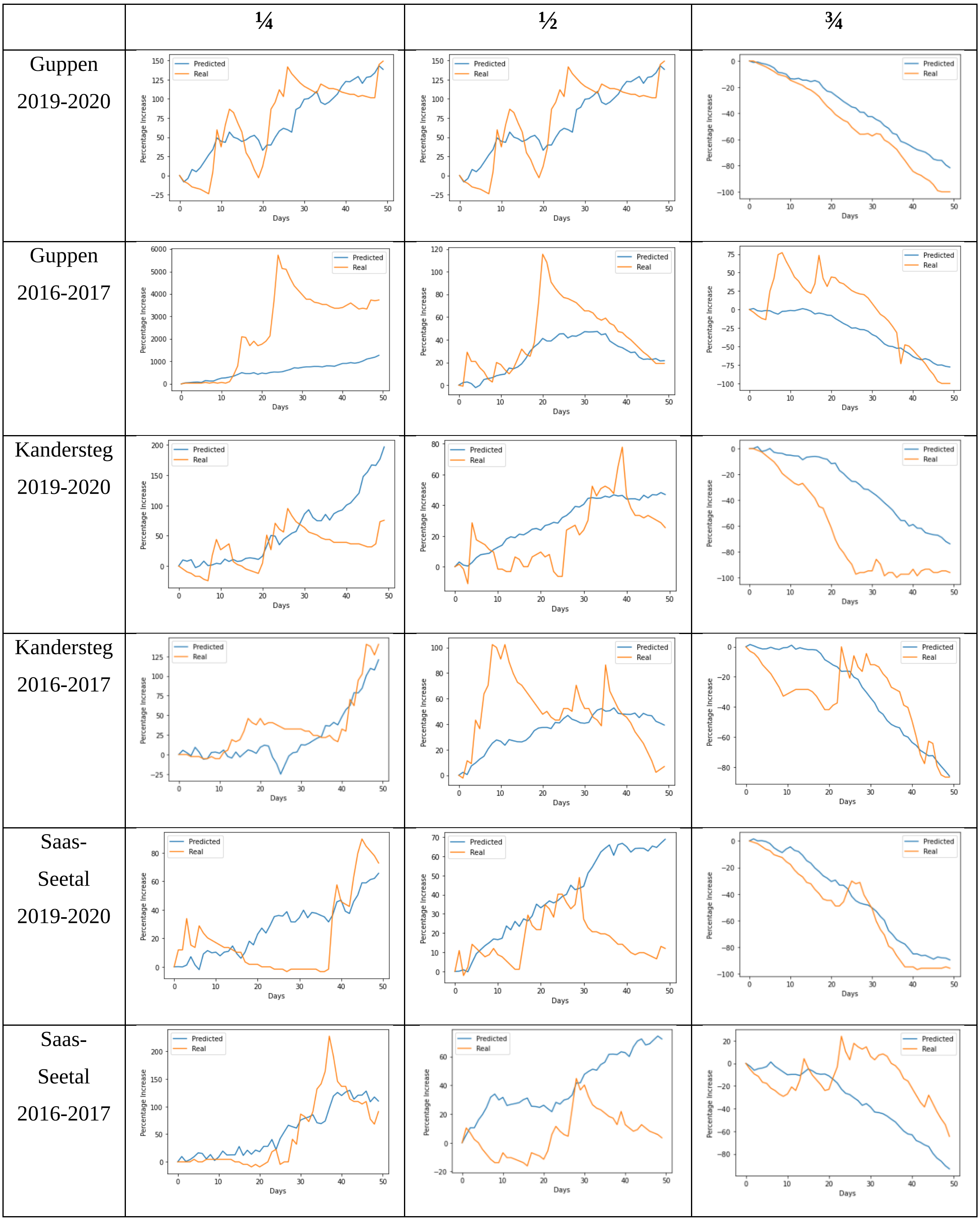

Figure 19 - Result Table

The above table compares the accuracy of prediction as a function of time in different locations at ¼, ½ and ¾ of a season. Each graph illustrates the relationship between our predictions and the real snow depth. We have chosen to make predictions for different years and locations as well as distinct times during the season to identify the highlights and defects of our model. We will be evaluating the accuracy of these simulations in our conclusion.

6. Conclusion

With the growing uncertainties linked to climate change, the growing interest in financial products linked to snow, the control over waterways, and much more, the ability to better understand and predict snow depth is becoming a necessity. We can therefore imply that the domain of mathematical modelling of snow depth will be exponentially growing in the coming years.

After analysing in chapter 3 two specific cases, we were able to recognize a trend in snow depth looking at multi-year averages. We then concluded that this pattern is universal to all mountains because of the earth’s climate.

We then established that an accurate way to mathematically model this pattern was to use the Beta function which we then called as our base function. After defining stochasticity and Wiener processes, we added one to our model therefore making it stochastic.

After creating a base program, we were able to test our model and fine tune parameters in order to enhance our model. This was done by calculating Pearson correlation ratios and plotting graphs to choose which parameters better fit mountains.

The second part of our program helped us to make predictions at different times of a season, year and locations. By simple observation of the results in the result table (figure 19), we noticed that extremely volatile datasets won’t be predicted as well as the others. However, we observed that in these cases, our model still gives an approximation similar to a trendline. The model seems relatively accurate at the beginning of seasons. However, for example in Gruppen 2016-2017, the predictions weren’t correct because of extremely large and sudden snowfalls. The phase at which our model is the least accurate remains the middle of seasons. This happens because the middle of seasons usually have very volatile snowfalls, something our model struggles to predict. Our model clearly exceeds expectations on the last part of the season. It is fair to assume that because our program has had more datapoints to train itself on, that it is able to better predict snow depth. Overall, the results are very satisfying. Even though there can be large discrepancies when snowfalls are too volatile, it is most of the time relatively accurate.

Our model still contains some flaws, for instance, all the parameters chosen arbitrarily such as the shape factors or *dev*, it might not be the most accurate. The Beta function could probably be replaced by another base function more accurate at representing the season averages. Another flaw lies within the testing of our model: even though we proved that the pattern of snow depth over time is universal, it slightly differs from one location to another. However, we only tested it on locations within Switzerland. This flaw is linked to the pure difficulty of accessing snow depth data outside of Switzerland for academic purposes.

There are numerous ways in which one could improve the model we have built. To better analyse the data samples and better identify patterns, one might want to implement a machine learning feature, correcting the choice of parameters made arbitrarily. The program, when fed with as many datapoints as possible will identify trends by itself and therefore enhance our ability to model the phenomenon. Even without the use of machine learning, a better base function can be found, not only by improving the parameters we chose but also by finding other functions that give a better approximation. The Wiener process could also be improved, with a more complex application of it. Major improvement could also come from our program. In its current state, the program isn't highly optimized and requires a longer run time. One could optimize by for example using Taylor series and simplify function calculation, gaining significant performance, which could allow for more simulations to be created, which might help getting better results.

Finally, as proven by our results, stochastic modelling can help to model snow depth as a function of time and therefore allow us to make reliable predictions and give us a better understanding of the phenomenon.

7. Appendix

7.1. Personal Review

Writing this research paper was extremely interesting. I was not only given the opportunity to research stochastic modelling, a subject I had been eying for a while, but was also able to learn more about something I am passionate about, snow.

I faced many challenges while writing this paper. First, starting with the mathematical and digital tools used to obtain results which are difficult to use. The most difficult aspect of all was having the creativity to find a base function. As no academic work on the stochastic modelling of snow depth had ever been done, I had to find all the models by myself. This major challenge helped me to develop my creativity and curiosity.

Overall, I was very happy to work on this research paper and was very satisfied with the results.

7.2. Definitions

- AFP: Agence France-Presse, French international news agency.
- CME: Chicago Mercantile Exchange.
- CSV File: Coma-Separated Values File.
- Data frame: Matrix-like list of variables whose columns can contain different types (string, integer, …).[24]
- Matplotlib.pylpot: Python library used to plot different types of graphs.[25]
- MeteoSwiss: Federal Office of Meteorology and Climatology.
- Scypy.stats: Python library that includes numerous functions useful in statistics.[26]

[24] Geneva, E.H., U. of (n.d.). *Data frames*. [online] www.unige.ch. Available at: http://www.unige.ch/ses/sococ/cl/r/concepts/dataframe.e.html [Accessed 28 Oct. 2021].
[25] matplotlib.org. (n.d.). *History — Matplotlib 3.4.3 documentation*. [online] Available at: https://matplotlib.org/stable/users/history.html [Accessed 28 Oct. 2021].
[26] Scipy.org. (2019). *Statistical functions (scipy.stats) — SciPy v1.3.3 Reference Guide*. [online] Available at: https://docs.scipy.org/doc/scipy/reference/stats.html. [Accessed 23 Sep. 2021].

- Pandas: Open-source Python library launched in 2008 that allows programmers to manipulate dataframes within Python.[27]
- Pearson Correlation Ratio: Measures the strength of a linear association between two variables, included within [-1,1]. -1 representing an inverse correlation and 1 representing a perfect correlation.[28]
- RCP Pathways: Representative Concentrative Pathway, describe four 21st century different pathways of atmospheric concentrations of CO_2.[29]

[27] pandas.pydata.org. (n.d.). *pandas - Python Data Analysis Library*. [online] Available at: https://pandas.pydata.org/about/ [Accessed 23 Sep. 2021].

[28] Laerd Statistics (2018). *Pearson Product-Moment Correlation - When you should run this test, the range of values the coefficient can take and how to measure strength of association.* [online] Laerd.com. Available at: https://statistics.laerd.com/statistical-guides/pearson-correlation-coefficient-statistical-guide.php. [Accessed 23 Sep. 2021].

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All datasets used in section 5.3 were provided by MeteoSwiss via their IDAWEB platform under an academic license:

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7.4. Acknowledgments

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