WQD7011 NUMERICAL OPTIMIZATION

Portfolio Optimisation using Quadratic Programming and Monte Carlo Methods

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Common Factors

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Chapter 1 Modern Portfolio Theory

1.1 Overview of Modern Portfolio Theory

Modern Portfolio Theory (MPT), developed by Harry Markowitz in the early 1950s, is a fundamental framework in modern finance used to construct portfolios that optimize the balance between risk and return. A lot of contemporary methods for capital budgeting, risk management, asset allocation, investment analysis, and decision-making under uncertainty still revolve around the principles presented by this model (Zopounidis et al., 2014). The theory provides a systematic method for investors to diversify their investments across various assets, thereby minimizing risk while achieving desired returns. This approach contrasts with the traditional practice of selecting individual securities independently. At the heart of portfolio theory lies the principle of diversification, which suggested that investing in a variety of assets can reduce the overall risk of a portfolio. Different assets often result in different responses to market conditions.

By carefully selecting a mix of assets with varying risk and return characteristics, investors can construct a portfolio that offers a more favourable risk-return trade-off. This is known as the *Mean-Variance* Portfolio Theory (MVPT) and whilst it has its flaws, it provides a good starting point in portfolio theory. In addition, the MVPT can explain the various reported responses of assets to news or financial events; some are found to respond both short- and long-term, while others only reflect long-term reactions (Fahmy, 2020).

1.2 Basic Terminologies and Definitions

Here are some key concepts in portfolio theory:

1.2.1 Portfolio Weights and Constraints

Definition 1. **(Portfolio Weight w_i)** *Portfolio weight w_i is the percentage/proportion of an investment portfolio that a particular asset A_i comprises.*

Generally, when building a portfolio, we ensure that a fixed sum of money is divided by a total of n assets $A_1, \cdots, A_n$. We are interested in what fraction/proportion of money is being put into each asset so that we can take the sum of money to be 1. Hence, we impose the following constraint:

$$\sum_{i=1}^{n} w_i = 1.$$

Other constraints can be put on w_i, for example, the exclusion of asset *short-selling*, which is to sell stock asset which one does not own at the time, in the hope of buying at a lower price before the delivery time. As such, we further define n constraints such that:

$$w_i \geq 0 \quad \forall i \in \{1, \cdots, n\}.$$

1.2.2 Measures of Return

Definition 2. **(Return on a Portfolio)** *The **return** on a portfolio is the percentage change in its value taking into account all cash in-flows and out-flows.*

In financial mathematics, we are generally interested in the future rather than in the past, so the return will normally be uncertain. It is therefore the *expected* return being more important than the actual *return*, with the *expected* return typically measured as the mean of the probability distribution of possible returns.

1.2.3 Measures of Risk

In MPT, there are various measures of risk, such as the Value at Risk (VaR), Sharpe Ratio and etc. In MVPT, the risk measures used are variance and covariance.

Definition 3. (Variance and Standard Deviation) *The variance measures the dispersion of returns around the expected return, while standard deviation is the square root of variance. It provides insights into the uncertainty or risk associated with an investment.*

Definition 4. (Covariance and Correlation) *Covariance is a statistical measure of the directional relationship between two assets. Covariance indicates the direction of the relationship, while correlation standardizes this relationship between -1 and 1.*

1.2.4 Defining Efficiency

The principal assumptions of MVPT are as follows:

1. Investors only care about mean and variances of return
2. Investors prefer higher means to lower means.
3. Investors prefer lower variances to higher variances.
4. The means, variances and covariances of the assets are known.

It simply means that investors want more money and less risk (risk-averse).

Definition 5. (Opportunity Set) *The set of all possible pairs of standard deviations and returns attainable from investing in a collection of assets is called the* ***opportunity set****.*

Definition 6. (Efficient Portfolio) *A portfolio is said to be* ***efficient*** *provided the following holds:*

- *No other portfolios in that opportunity set have at least as much expected return and lower standard deviation.*
- *No other portfolio in that opportunity set has higher return and standard deviation which is smaller or equal.*

In other words, no other portfolios performs at least as well as an efficient portfolio in terms of risk and return, and it means that an efficient portfolio lies on the edge of an opportunity set.

Definition 7. (Efficient Frontier) *The subset of the opportunity set which is efficient is called the efficient frontier.*

Put simply, the efficient frontier in MVPT represents the set of optimal portfolios that offer the highest expected return for a given level of risk or the lowest risk for a given level of expected return. The efficient frontier is typically depicted graphically by plotting the expected return on the y-axis and the standard deviation of the portfolio on the x-axis. Figure 1.1 illustrates this concept along with the Capital Market Line (CML), showing the risk-return trade-off of portfolios that include a risk-free asset.

Remark 1. *Note that if we replace standard deviation with variance, we will still get the same set of optimal portfolios as the mapping between variance and standard deviation is strictly increasing.*

Remark 2. *The efficiency is defined relative to a set of investment opportunities. If an asset has been added or removed, portfolios that were previously efficient will no longer be efficient.*

Remark 3. *For simplicity, we assume that the market has no risk-free asset, where we focus on the construction on the efficient frontier.*

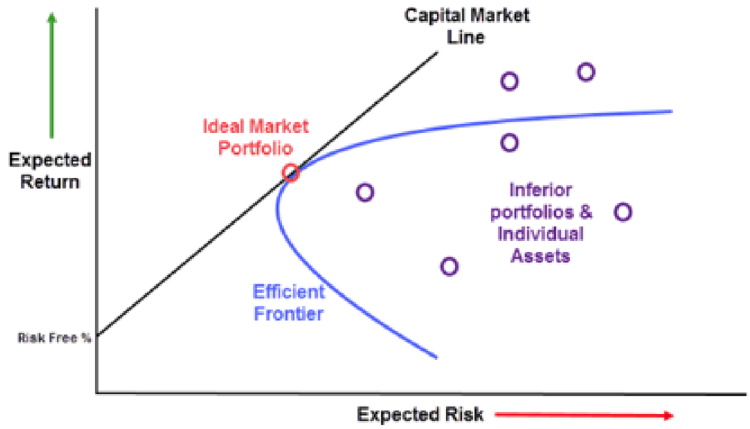


Figure 1.1: Efficient Frontier with CML

1.3 Optimisation Problems

Throughout this research, we aim to solve the problem of optimizing investment portfolios to achieve the best possible balance between risk and return. Specifically, we address three key optimization challenges:

1. **Mean Variance Portfolio Optimization:** We seek to find the optimal portfolio weights that minimize the overall risk (variance) for a given portfolio return. This involves solving a quadratic programming problem.

2. **Exploring Alternatives:** We explore Monte Carlo methods to determine an empirical efficient frontier and a comparison will be made with the theoretical efficient frontier. Challenges encountered will be discussed.

3. **Realistic Constraints:** To ensure practical applicability, we introduce realistic constraints, such as limiting the number of assets in the portfolio.

By addressing these problems, our objective of this report is to develop a comprehensive and realistic framework for portfolio optimization that balances risk and return while incorporating practical investment constraints. The dataset we choose will be implemented in **Quadratic Programming** and **Monte Carlo Simulation**. Following in the next chapter, we will discuss the formulation of our optimization problems in detail.

1.4 Dataset Description

Dataset: https://people.brunel.ac.uk/~mastjjb/jeb/orlib/portinfo.html

An important aspect of evaluating computational results for our portfolio optimization project is the availability of benchmark data sets. These data sets are essential for comparing the efficiency of our algorithms and the quality of the solutions obtained. For our study, we use publicly available data sets that are widely used in portfolio optimization research. Specifically, we utilize data derived from the asset price data of the **Hang Seng index** available from Beasley's OR-Library, labelled `port1.txt` and `portef1.txt`. The first dataset includes covariance matrices and expected return vectors for a range of asset sizes, and the latter contains the theoretical efficient frontier. To ensure practical applicability and the relevance of our research, we focus exclusively on the Hang Seng index.

Chapter 2 Mathematical Formulation

Suppose there are n assets $A_1, \cdots, A_n$ in the market. For $i, j \in \{1, \cdots, n\}$, define the random variable R_i as the return of asset A_i and consider the following notations:

- Let $\mu_i = \mathbb{E}(R_i)$ denote the expected return of asset i.
- Let $\rho_{ij} = Corr(R_i, R_j)$ denote the correlation between the return of asset A_i and A_j
- Let $\sigma_i = \sqrt{Var(R_i)}$ denote the volatility/standard deviation of return of asset A_i.
- Let $\sigma_{ij} = Cov(R_i, R_j) = \rho_{ij}\sigma_i\sigma_j$ denote the covariance between asset i and j.
- Let w_i denote the weightage of wealth invested in asset A_i, and let $\mathbf{w} \in \mathbb{R}^n$ denote the vector of portfolio weights, given by:

$$\mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

- Let $\mathbf{\Sigma} \in \mathbb{R}^{n \times n}$ be the covariance matrix for the returns on the assets in the portfolio with entries:
  - $\mathbf{\Sigma_{ij}} = \sigma_{ij}$ for $i \neq j$
  - $\mathbf{\Sigma_{ii}} = \sigma_{ii} = \sigma_i^2$ along the diagonals.
- Let $\mathbf{R} \in \mathbb{R}^n$ be the a vector of expected returns, given by:

$$\mathbf{R} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}$$

It follows that:

$\mathbf{w}^T\mathbf{\Sigma}\mathbf{w} \in \mathbb{R}$ is the variance of portfolio return.

$\mathbf{R}^T\mathbf{w} \in \mathbb{R}$ is the expected return on the portfolio.

Key assumptions:

1. Investors are risk averse.
2. No short-selling is allowed.
3. There is no risk-free asset in the market.

2.1 Standard MVPT

Let $\lambda \geq 0$ denote the risk aversion level, which is the tendency of people to prefer outcomes with low uncertainty to those outcomes with high uncertainty, even if the average outcome of the latter is equal to or higher in monetary value than the more certain outcome.

To find the efficient frontier, for given risk aversion level λ, we solve the following constrained optimisation problem:

$$
\begin{aligned}
\max_{\mathbf{w}\in\mathbb{R}^n} \quad & \mathbf{R}^T\mathbf{w} - \lambda \mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w} \\
\text{subject to} \quad & \sum_{i=1}^{n} w_i = 1 \\
& w_i \geq 0 \quad \forall i \in \{1, \cdots, n\}
\end{aligned}
\tag{2.1}
$$

An alternative approach to specifying the efficient frontier is to do so parametrically on the expected portfolio return $\mathbf{R}^T\mathbf{w}$, we have that for fixed return parameter $\tilde{\mu}$, we minimize the portfolio variance: given by:

$$
\begin{aligned}
\min_{\mathbf{w}\in\mathbb{R}^n} \quad & \mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w} \\
\text{subject to} \quad & \sum_{i=1}^{n} w_i = 1, \\
& \mathbf{R}^T\mathbf{w} = \tilde{\mu}, \\
& w_i \geq 0 \quad \forall i \in \{1, \cdots, n\}.
\end{aligned}
\tag{2.2}
$$

In a more compact manner, Problem 2.2 can be written as:

$$
\begin{aligned}
\min_{\mathbf{w}\in\mathbb{R}^n} \quad & \mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w} \\
\text{subject to} \quad & \mathbf{1}^T\mathbf{w} = 1, \\
& \mathbf{R}^T\mathbf{w} = \tilde{\mu}, \\
& \mathbf{w} \geq \mathbf{0},
\end{aligned}
$$

where $\mathbf{1}$ is a $n \times 1$ vector of ones, and $\mathbf{0}$ is a $n \times 1$ vector of zeroes.

2.2 Extension: A Cardinality Constraint

To make our project more comprehensive and realistic, we can impose additional realistic constraints, such as a cardinality constraint such that no more than K assets should be held in the portfolio, which is:

$$|supp(\mathbf{w})| \leq K, \quad \text{where } supp(\mathbf{w}) = \{i \mid w_i > 0\}$$

Chapter 3 Optimisation Techniques

3.1 Quadratic Programming

3.1.1 The Standard MVPT

$$
\begin{aligned}
\min_{\mathbf{w}\in\mathbb{R}^n} \quad & \mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w} \\
\text{subject to} \quad & \mathbf{1}^T\mathbf{w} = 1, \\
& \mathbf{R}^T\mathbf{w} = \tilde{\mu}, \\
& \mathbf{w} \geq \mathbf{0}.
\end{aligned}
$$

The objective function $\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}$ is a *multivariate* quadratic function. The covariance matrix $\boldsymbol{\Sigma}$ is shown to be positive definite as all of its eigenvalues are positive. Hence, the optimal solution computed will be unique, and globally minimum.

Using `quadprog(H,f,A,b,Aeq,beq,lb,ub,x0,options)` from `Optimization Toolbox` in Matlab, it aims to solve the Quadratic Programming problem with the following specifications, where $x = \mathbf{w}$:

$$\min_x \quad \frac{1}{2}x^T Hx + f^T x \quad \text{such that} \quad \begin{cases} A \cdot x \leq b, \\ Aeq \cdot x = beq, \\ lb \leq x \leq ub. \end{cases}$$

1. The constant $\frac{1}{2}$ in the objective function does not affect the final solution x, but only the value of the objective function. Since we are interested in portfolio weights, we can proceed without further manipulations.
2. We set $H = \boldsymbol{\Sigma}$, f to be empty.
3. $A \cdot x \leq b$ are inequality constraints. For convenience, we set these to be empty, and consider the inequality $\mathbf{w} \geq \mathbf{0}$ for ub.
4. $Aeq \cdot x = beq$ are equality constraints. For return parameter $\tilde{\mu}$, set:

$$Aeq = \begin{bmatrix} \mathbf{1}^T \\ \mathbf{R}^T \end{bmatrix} \quad \text{and} \quad Beq = \begin{bmatrix} 1 \\ \tilde{\mu} \end{bmatrix}.$$

5. Finally, set $lb = \mathbf{0}$, a $n \times 1$ vector of zeroes. ub need not be set as the first equality and the inequality ensures each weight w_i is bounded above (inclusive) by 1.
6. We do not provide an initial point x_0, and we set `options = optimoptions('quadprog', 'Display', 'off')`.

As such, we solve the QP problem for $\tilde{\mu} \in [\min \mathbb{E}(R_i), \max \mathbb{E}(R_i)]$ to obtain the efficient frontier.

3.1.2 MVPT with Cardinality Constraint

With the additional cardinality constraint:

$$|supp(\mathbf{w})| \leq K$$

where $supp(\mathbf{w}) = \{i \mid w_i > 0\}$, Problem 2.1 becomes a *Mixed Integer Quadratic Programming* (MIQP) problem as there is an additional discrete constraint.

MIQP is solvable using `intlinprog` Mixed-Integer Linear Programming (MILP) solver. The idea is to iteratively solve a sequence of MILP problems that locally approximate the MIQP problem

However, we do not proceed with this approach as it is computationally intensive. The computation for a single λ takes approximately 20 minutes, where we need several thousand λ's to trace the new efficient frontier.

3.2 Monte Carlo

Monte Carlo methods are a broad class of computational algorithms that rely on repeated random sampling to obtain numerical results. The fundamental concept is to use randomness to solve problems that might be deterministic in principle, such as the problem in Section3.1.2. These methods are widely used in various fields such as physics, finance, engineering, and statistics.

The core idea of Monte Carlo methods are as follows:

1. **Repeated Sampling:** The core idea behind Monte Carlo methods is to use random samples to estimate the properties of a system. This involves generating random variables and using these variables to simulate the behavior of the system.

2. **Law of Large Numbers:** Monte Carlo methods rely on the law of large numbers, which states that the average of the results from a large number of trials should be close to the expected value. This implies that as the number of simulations increases, the Monte Carlo estimate becomes more accurate.

Due to the computationally intensive nature of Monte Carlo methods, the Matlab `Parallel Computing Toolbox` will be used to perform parallel computations. For random number generation using distributions, the Matlab `Statistics and Machine Learning Toolbox` will also be used.

3.2.1 Monte Carlo Procedure: Standard MVPT

1. Randomly generate a large number of portfolio with weights that satisfies the constraints in Problem (2.2).
2. For each portfolio, compute the expected return and standard deviation.
3. For any target return in $[\min \mathbb{E}(R_i), \max \mathbb{E}(R_i)]$, find portfolios that are greater or equal to the target return, if they exists.
4. For the selected portfolios, find those with the lowest standard deviation to be the efficient portfolio.
5. Ensure uniqueness of portfolios.
6. For the efficient portfolios, plot them on $(Risk, Return)$ space and empirically connect the points to form the efficient frontier. Also determine the minimum variance portfolio.

3.2.2 Monte Carlo Procedure: Limited MVPT

The procedure is similar to Section 3.2.1, with a few differences.

1. Simulate the number of assets $\tilde{n}$ in the portfolio, ranging from $1, \cdots, K$.
2. Randomly select $\tilde{n}$ assets out of the n assets to form a portfolio.
3. Randomly generate $\tilde{n}$ asset weights for the portfolio which sums to 1 and are greater than 0.
4. Repeat Steps 1-3 for a large number of times.
5. For each portfolio, compute the expected return and standard deviation.
6. For any target return in $[\min \mathbb{E}(R_i), \max \mathbb{E}(R_i)]$, find portfolios that are greater or equal to the target return, if they exists.
7. For the selected portfolios, find those with the lowest standard deviation to be the efficient portfolio.
8. Ensure uniqueness of portfolios.
9. For the efficient portfolios, plot them on $(Risk, Return)$ space and empirically connect the points to form the limited efficient frontier. Also determine the minimum variance portfolio.

Chapter 4 Results and Analysis

4.1 Standard MVPT

4.1.1 Quadratic Programming

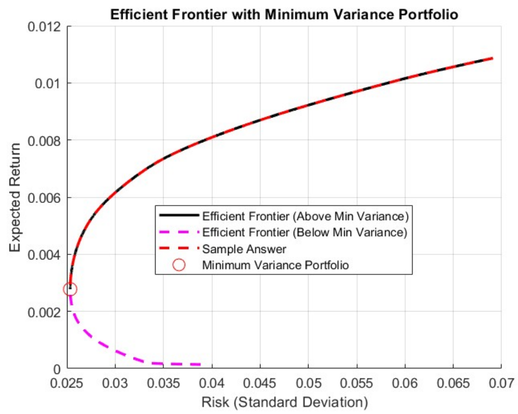
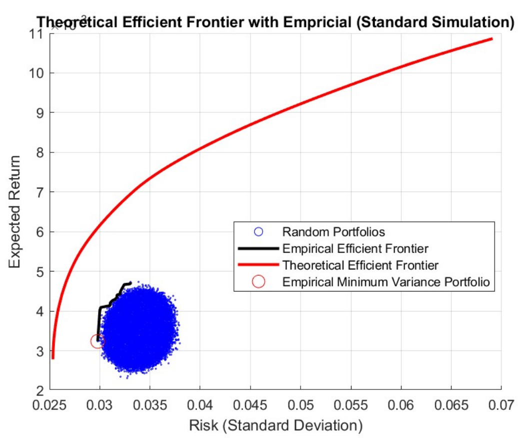


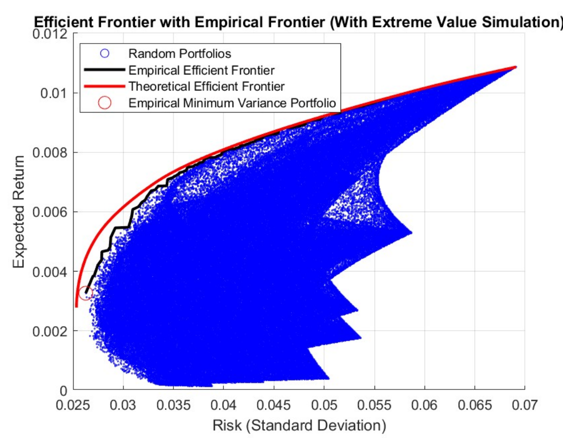
Figure 4.1: Efficient Frontier from solving the Quadratic Programming Problem

The results from the quadratic programming method demonstrate a clear and theoretically sound efficient frontier for the Standard Mean-Variance Portfolio Theory. Figure 4.1 highlights the efficient frontier, showing the optimal balance between risk and return, and aligns with the theoretical frontier from `portef1.txt`. The smooth curve indicates that as the risk (standard deviation) increases, the expected return also increases, which aligns with the principles of Modern Portfolio Theory (Markowitz, 1952). The identification of the minimum variance portfolio on this frontier further proved the method's capability to provide an optimal low-risk solution, making it a reliable approach for portfolio optimization (Elton & Gruber, 1997).

4.1.2 Monte Carlo



(a) Efficient Frontier using Monte Carlo with Uniformly Distributed Weights



(b) Efficient Frontier using Monte Carlo with Extreme Weights

Figure 4.2: Comparison of Efficient Frontiers

In contrast, the Monte Carlo simulation method offers a different perspective on the optimization problem. As shown in Figure 4.2a, Monte Carlo simulations with uniformly distributed weights yield an empirical efficient frontier that deviates far away from the theoretical efficient frontier. This deviation is due to the fact that there are 31 assets, and it is unlikely to get individual assets with weights that are close to 1, which corresponds to the portfolios near the efficient frontier with a higher expected return.

To overcome this, we included the sampling of weights using a non-uniform distribution, such as the Dirichlet distribution, to effectively sample those portfolio near the boundaries. As Figure 4.2b demonstrates, using extreme weights in Monte Carlo simulations leads to an empirical efficient frontier that is much closer to the theoretical efficient frontier. This adjustment indicates the importance of selecting an appropriate weight distribution method to improve the accuracy and reliability of Monte Carlo simulations in portfolio optimization, particularly when dealing with a larger number of assets (Glasserman, 2004).

4.2 Limited MVPT with Monte Carlo

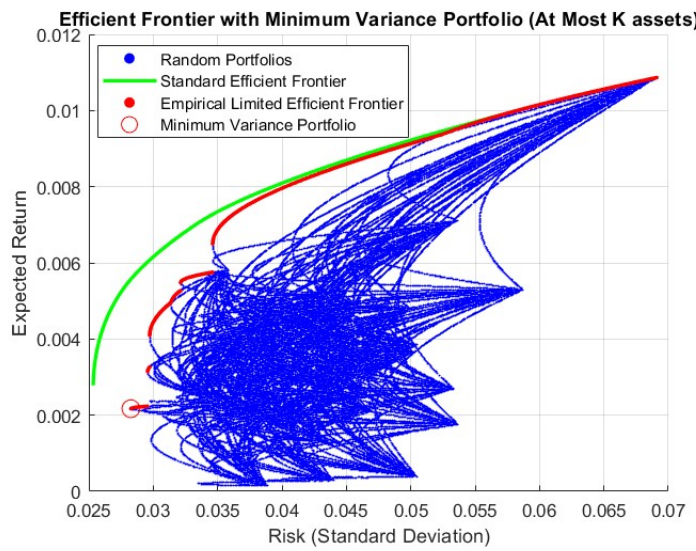


Figure 4.3: Limited Efficient Frontier using Monte Carlo

1. **Empirical Limited Efficient Frontier (Red Line):** This represents the efficient frontier under the constraint of having at most K assets. It is more curved, and discontinuous and lies below the standard efficient frontier, indicating a less optimal trade-off between risk and return due to the imposed constraint.
2. **Minimum Variance Portfolio (Red Circle):** This is the portfolio with the lowest risk among the constrained set, highlighting the best achievable outcome under the given limitations.

The limited MVPT using Monte Carlo simulations, shown in Figure 4.3, suggested that the practical implications of imposing realistic constraints, such as limiting the number of assets in the portfolio. The empirical limited efficient frontier is lower and more curved compared to the standard efficient frontier, indicating a less optimal risk-return trade-off, as suggested by Chang et al., 2000. This result highlights the impact of practical constraints on portfolio optimization, demonstrating that while theoretical models provide a useful foundation, real-world applications require adaptations that account for these limitations, ultimately affecting the efficiency of the optimized portfolios (Kellerer et al., 2000).

Chapter 5 Conclusion and Future Work

In conclusion, this study explores various methods for portfolio optimization within the framework of Modern Portfolio Theory (MPT), emphasizing the pursuit of an optimal balance between risk and return. Quadratic Programming (QP) is employed to derive a theoretically robust efficient frontier, effectively illustrating the trade-off between risk and return and identifying the minimum variance portfolio. Monte Carlo Simulations complement this by empirically validating the QP-derived efficient frontier through randomized portfolio generation, demonstrating practical insights and enhancing accuracy with adjusted sampling methods. The inclusion of realistic constraints, such as cardinality constraints in Limited Mean-Variance Portfolio Theory (MVPT), reveals their impact on the efficient frontier, illustrating how practical limitations can affect risk-return trade-offs compared to unconstrained models.

Moving forward, future research can enhance the computational efficiency of Monte Carlo simulations, integrate dynamic asset allocation strategies, explore alternative risk measures, conduct robustness analyses, and validate methodologies through real-world applications, thereby advancing portfolio optimization in financial mathematics and investment management.

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