1.2:2

Suppose S has the LUBP and GLBP and $X, Y \subseteq S$ and $X, Y = \varnothing$
If $x < y$ $\forall x \in X$ and $\forall y \in Y$, Then is $\sup X < \inf Y$?

Proof. This is not true, look at $N = \{1/n \mid n \in \mathbb{N}\}$, then $M = \{-n \mid n \in N\}$, $m < n$ for each $m \in M$ and $n \in N$, but $\sup M = \inf N = 0$ □

1.2:3

Let S be an ordered set with LUBP, and let $A_i (i \in I)$ be a family of nonempty subsets of S. Suppose each A_i is bounded above, let $\alpha_i = \sup A_i$, and suppose that the set $\{\alpha_i \mid i \in I\}$ is bounded above. Show that $\cup_{i \in I} A_i$ is bounded above and that $\sup \cup_{i \in I} A_i = \sup\{\alpha_i \mid i \in I\}$

Proof. Since $\{\alpha_i \mid i \in I\}$ is bounded above it has a supremum α since it is a subset of S. Let $a \in A = \cup_{i \in I} A_i$, then there is some j st $a \in A_j$, then $a \leq \alpha_j \leq \alpha$. Since this is true for all $a \in A$, then the set A is bounded above with upper bound α. Since $A \subseteq S$, and it is bounded above it has a supremum, call it α', we will now show that $\alpha = \alpha'$, since α is an upper bound of A, then $\alpha' \leq \alpha$. If we can show that $\alpha \leq \alpha'$, then we are done. α' is an upper bound of A, so then it is an upper bound of A_i for all $i \in I$. So then $\alpha_i \leq \alpha'$ for each i so α' is an upper bound of the set of supremums. Since α is the supremum of that set, then $\alpha \leq \alpha'$. So then $\alpha = \alpha'$. □

- Suppose not all sets are bounded above. Show then that A is unbounded

  Proof. There is some A_i that is not bounded above, then for each $x \in S$ there is some $x' \in A_i$ such that $x < x'$. Then since $A \subseteq S$, for each $a \in S$, there is some $x \in A_i \subseteq A$ such that $a < x$ so then A is not bounded above. □

- Suppose that each A_i is bounded above but the set of their supremums is not bounded above.

  Proof. Call the set of supremums I. So since I is not bounded above, for each $x \in S$, there is some $\alpha_j \in I$ such that $x < \alpha_j$. Then since since α_j is the supremum of A_j and $x < \alpha_j$, then x is not an upper bound of A_j so there is some $a \in A_j \subseteq A$ such that $x < a$. Since we can do this for each $x \in S$, A is not bounded above. □

When each A_i is bounded above, show that $A = \cap_{i \in I} A_i$ is bounded above.

Proof. Let $a \in A \subseteq A_j$, since A_j is bounded above, there is some $\beta \in S$ such that $a \leq \beta$. So then A is bounded above. □

Must A be nonempty?

Proof. This statment is vacuously true, since in symbolic form, the statment would be

$$(\exists x \in S)\,(\forall a \in A)\,(x < a) \quad \iff \quad \exists x \forall a\,(x \in S \wedge (a \in A \implies x < a))$$

This then becomes

$$\exists x \forall a\,(x \in S \wedge (\mathbf{F} \implies x < a)) \quad \implies \quad \exists x \forall a\,(x \in S \wedge \mathbf{T})$$

Then we pick any $x \in S$ and this statment is true as long as S is nonempty. □

Suppose $A \neq \varnothing$. What is the relationship between $\sup A$ and the numbers α_i?

Proof. I conjecture that $\sup A = \inf I$. To show this, we will do the following. Note that every element of I is an upper bound of A so A has a sup. Let $x \in A$, since $x \leq \alpha_i \quad \forall \alpha_i \in I$ since $A \subseteq A_i$ for all A_i, then I is bounded below and nonempty, so it has an inf. Call $\sup A = a$ and $\inf I = b$. Since a is an upper bound of A, then $a \in I$ so $a \geq b$. Now we will show $a \leq b$. In order to show this, suppose $a > b$, then since b is an inf, then a is not a lower bound of I, so there is some $\alpha \in I$ such that $a > \alpha$. But α is an upper bound of A and a is its supremum, to this is a contradiction, so then $a = b$. □

1.2:4

Let S be a nonempty ordered set such that every nonempty subset $E \subseteq S$ has both a least upper bound and greatest lower bound. Suppose that $f : S \to S$ is a monatonically increasing function, $(\forall x, y \in S, x \leq y \implies f(x) \leq f(y))$.

Show there is an $x \in S$ such that $x = f(x)$

Proof. Suppose not for contradiction, then $\forall x \in S$, either $x < f(x)$ or $x > f(x)$. Construct the sets $A := \{x \in S \mid x < f(x)\}$, and $B := \{x \in S \mid x > f(x)\}$. Since these are subsets of S, if they are nonempty then they have supremums and infimums. At least one would be nonempty otherwise $S = \varnothing$. Suppose $B = \varnothing$, then $x < f(x)$ for all $x \in S$. Then A has an inf and sup, call them a and b respectively. Note that $b = \sup S$ since $S = A$. Since $b \in S = A$, then $b < f(b)$, but $f(b) \in S$ which which contradicts the fact that b is an upper bound of S. Similar for $A = \varnothing$. Now without loss of generality, look only at A and consider $\alpha = \sup A$. We will consider $\alpha \in A$ and $\alpha \in B$. Suppose $\alpha \in A$, then $\alpha < f(\alpha)$. Since $f(\alpha), \alpha \in S$ and $\alpha \leq f(\alpha)$, then $f(\alpha) \leq f(f(\alpha))$. Now since $\alpha = \sup A$ and $\alpha < f(\alpha)$, then $f(\alpha)$, is not in A. Since $S \setminus A = B$, then $f(\alpha) \in B$, so then $f(\alpha) > f(f(\alpha))$ contradicting what we showed above. Now suppose $\alpha \in B$, then $\alpha > f(\alpha)$. Now let $x \in A$, then since $\alpha = \sup A$, then $x \leq \alpha$ since α is an upper bound of A. Therfore $f(x) \leq f(\alpha)$, and since $x \in A$, then $x < f(x)$. So $x < f(x) \leq f(\alpha) < \alpha$. So $x < f(\alpha)$ for all $x \in A$. This means that $f(\alpha)$ is an upper bound of A, but $f(\alpha) < \alpha$, contradicting that α is the supremum of A. Similar for B with analyzing its infimum. Therfore we have a contradiction, so there must be some $x \in S$ such that $x = f(x)$

□

1.2:5

- Let S be an ordered set such that for any 2 elements $p < r$ in S, there is an element $q \in S$ with $p < q < r$. Suppose that α and β are elements of S such that for every $x \in S$ with $x > \alpha$ one has $x \geq \beta$. Show that $\beta \leq \alpha$.

  Proof. Suppose not, then $\beta > \alpha$. Since they are in S, there is some $\gamma \in S$ such that $\beta > \gamma > \alpha$. Note that $\gamma > \alpha$, but $\gamma < \beta$ contradicting the hypothesis. So then $\beta \leq \alpha$ □

- Show by example that this is not true if density of S is not required.

  Proof. Let $S = \{1, 2, 3\}$, then let $\alpha = 1$ and $\beta = 2$. then the hypothesis are satisfied since for each element larger than α, it is larger than or equal to β. □

1.2:6

- Find subset $E \subseteq S_1 \subseteq S_2 \subseteq S_3 \subseteq \mathbb{Q}$ such that E has a least upper bound in S_1 but not in S_2 but does in S_3.

  Proof. Let $E = [0, 1) \cap \mathbb{Q}$. Then let $S_1 = E \cup \{2\}$. Note that E has no supremum in itself, in fact it has no upper bounds in E. It does in S_1. It is 2. Now let

  $$S_2 = S_1 \cup ((1, 2] \cap \mathbb{Q})$$

  Note that every element of $S_2 \setminus S_1$ is an upper bound of E. But it has no smallest element, so E has no least upper bound. Then let $S_3 = [0, 2]$. the supremum of E in this case is then 1 $\square$

- Prove for any example with the properties dscribed in (a) (not just the example given), the least upper bound of E in S_1 must be different than the one in S_3.

  Proof. Suppose they are the same, then call the supremum α. Note that α is not a supremum in S_2, so then either α is no longer an upper bound of E or there is some $\gamma \in S_2$ such that $\alpha > \gamma$ and γ is an upper bound of E. Note that since α is an upper bound of E as a subset of S_1, then $(\forall x \in E)\, \alpha \geq x$ and $\alpha \in S_1$. But $S_1 \subseteq S_2$, so then $\alpha \in S_2$ and $(\forall x \in E)\, \alpha \geq x$. So then α is still an upper bound of E in S_2. Now E has a supremum in S_3, call is α_3 and $\gamma \in S_3$. Since γ is an upper bound of E, then $\gamma \geq \alpha_3 = \alpha$. So then $\alpha > \gamma \geq \alpha$ which is a contradiction. $\square$

- Can there exist an example with the properties of (a) such that $E = S_1$?

  Proof. Suppose that there is. Then E has a supremum in S_1 call it α. Then $\alpha \in E$. Now since E does not have a supremum in S_2, there is some $\gamma \in S_2$ such that $\gamma < \alpha$ and $\gamma \geq x$ for all $x \in E$. So then $\gamma < \alpha$ and $\gamma \geq \alpha$ since $\alpha \in E$. Which is a contradiction. $\square$

1.2:7

Let S be an ordered set and $E \subseteq S$ and $x \in S$. If one translates the statment "x is the supremum of E" directly into symbols, one gets

$$((\forall y \in E)\, x \geq y) \wedge ((\forall z \in S)((\forall y \in E)\, z \geq y) \implies z \geq x)$$

This leads one to wonder if there are any simpler ways to express this property. Prove that in fact x is the supremum of E if and only iff

$$(\forall y \in S)(y < x \iff ((\exists z \in E)(z > y)))$$

1.2:8

Prove that $\inf \{x + y + z \mid x, y, z \in \mathbb{R},\, 0 < x < y < z\} = 0$