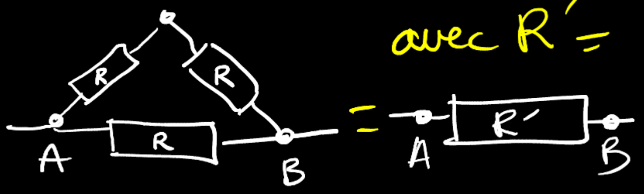
Le Théorème de Kirchhoff

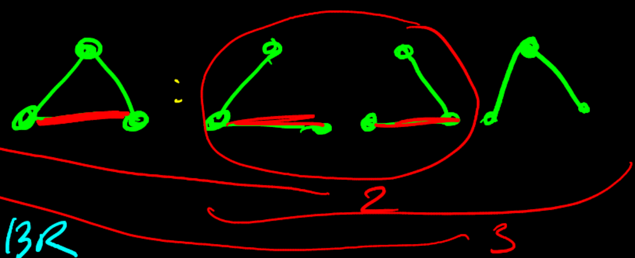
(permet de dénombrer les arbres couvrants)

Mise en jambes

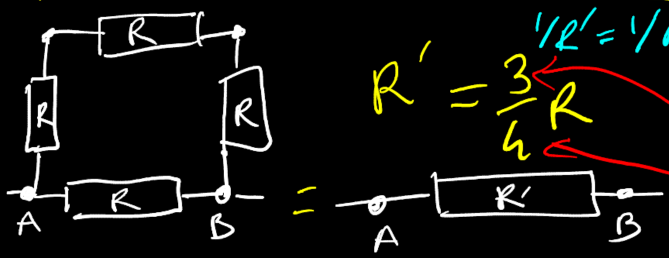
$$\frac{1}{R'} = \frac{1}{R} + \frac{1}{2R} = \frac{3}{2R}$$



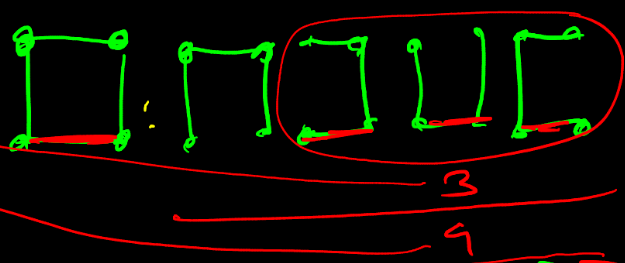
avec $R' = \frac{2}{3}R$

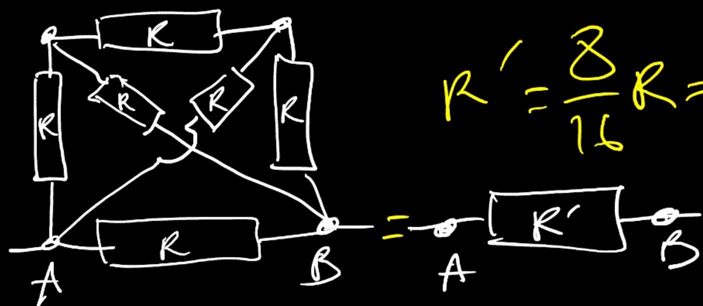


$1/R' = 1/R + 1/3R = 4/3R$

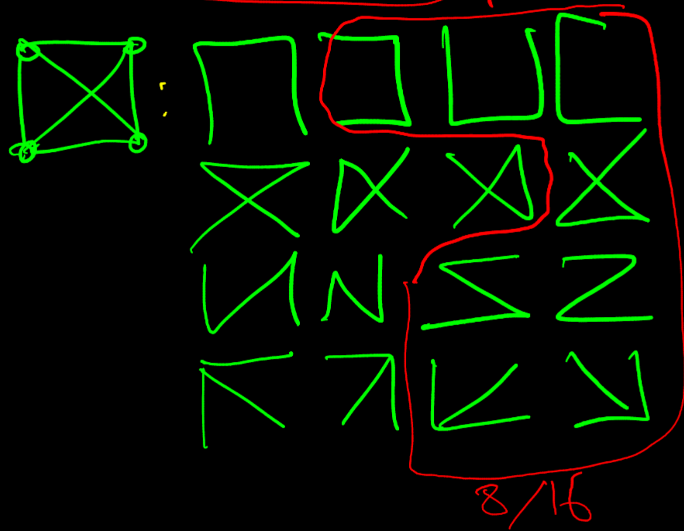


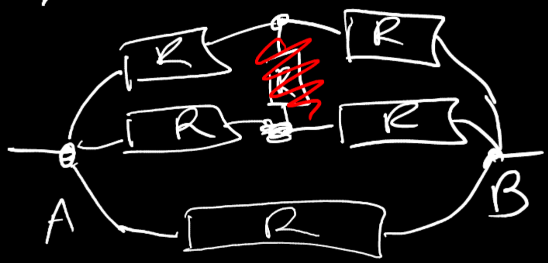
$R' = \frac{3}{4}R$





$R' = \frac{8}{16}R = \frac{R}{2}$

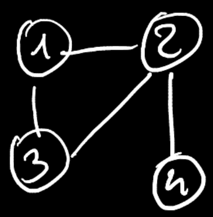




Théorème de Kirchhoff

Le nombre d'arbres couvrants d'un graphe G est égal à la valeur absolue de n'importe quel cofacteur de la matrice laplacienne de G. (génial !)

La matrice laplacienne, c'est la matrice des degrés moins la matrice d'adjacence

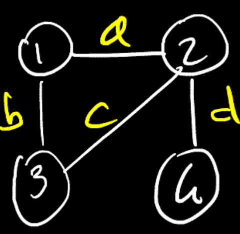


$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ 0 & -1 & 0 & 1 \end{vmatrix} = 2 \times 2 \times 1 - 1 \times (-1) \times (-1) = 3$$

Remarque 1

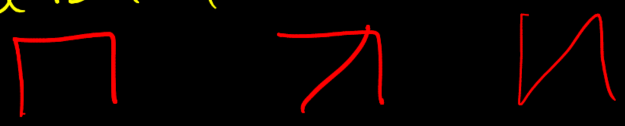
On peut faire une version "symbolique" de ce calcul :



$$\begin{vmatrix} a+b & -a & -b & 0 \\ \cancel{-a} & \cancel{a+c+d} & \cancel{-c} & \cancel{-d} \\ -b & -c & b+c & 0 \\ 0 & -d & 0 & d \end{vmatrix}$$

$$(a+b)(b+c)d - db^2 = abd + \cancel{b^2 d} + acd + bcd - \cancel{db^2}$$

$$abd + acd + bcd$$



Remarque 2

On peut redémontrer la formule de Cayley en appliquant Kirchhoff sur K_n.

$$\underbrace{\begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & n-1 \end{bmatrix}}_{(n \times n)} = L$$

calculons le cofacteur obtenu en virant ligne 1 et colonne 1

$$\underbrace{\begin{vmatrix} (n-1) & -1 & \cdots & -1 \\ -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & (n-1) \end{vmatrix}}_{(n-1)\times(n-1)} \underset{L_1 \leftarrow \sum_i L_i}{=} \begin{vmatrix} 1 & \cdots & \cdots & 1 \\ -1 & (n-1) & -1 \cdots & -1 \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & (n-1) \end{vmatrix} \underset{\forall i>1,\ L_i \leftarrow L_i + L_1}{=} \begin{vmatrix} 1 & \cdots & \cdots & 1 \\ 0 & n & 0 \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & n \end{vmatrix} = n^{n-2}$$