Real Analysis I

Lecture Notes

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Last updated: June 22, 2024

Abstract: These notes were prepared for a summer "bootcamp" to prepare graduate students in mathematics for the real analysis prelim exam. The sections, theorems, and definitions marked with $*$ below will stand for more advanced topics which are less likely to appear on exams.

Contents

1 Measure Theory

Definition 1 (Outer Measure)**.** *Let X be a set and $\mu : 2^X \to [0, \infty]$. We say μ is an outer measure if it is countably sub-additive and $\mu(\emptyset) = 0$.*

Definition 2 (Lebesgue Measure)**.** *The Lebesgue measure $\lambda_d : 2^{\mathbb{R}^d} \to [0, \infty]$ is defined by*

$$\lambda_d(E) := \inf \left\{ \sum_{k=1}^{\infty} vol(Q_k) : \quad E \subset \bigcup_{k=1}^{\infty} Q_k, \quad Q_k \ cubes \right\} \tag{1}$$

where $vol(Q)$ is the volume of the cube Q. It is an outer measure.

Definition 3 (Measure)**.** *Let X be a set and $\mathcal{F} \subset 2^X$ be a σ-algebra. A function $\mu : \mathcal{F} \to [0, \infty]$ is a measure if μ is countably additive. The sets $E \in \mathcal{F}$ are the measurable sets.*

Theorem 1 (Carethéodory's Theorem). *Let X be a set and μ be an outer measure. Then the set*

$$\mathcal{F} := \{E \subset X : \forall A \subset X, \quad \mu(A) = \mu(A \cap E) + \mu(A \cap E^c)\} \tag{2}$$

is a σ-algebra, and $\mu : \mathcal{F} \to [0, \infty]$ is a measure. We therefore say $E \in \mathcal{F}$ is measurable with respect to the outer measure μ.

Theorem 2 (Lebesgue Measurability). *For each $E \subset \mathbb{R}^d$, the following are equivalent*

- *For all $A \subset \mathbb{R}^d$, $\lambda_d(A) = \lambda_d(A \cap E) + \lambda_d(A \cap E^c)$.*
- *For each $\epsilon > 0$, there exists an open set $O \supset A$ such that $\lambda_d(O \setminus A) < \epsilon$.*

Theorem 3 (Measuring Tools). *Let μ be a measure on X. Let A_k be measurable for $k \in \mathbb{N}$.*

- *If $A_1 \subset A_2 \subset \cdots$ and $A^* = \bigcup_{k=1}^{\infty} A_k$, then $\mu(A^*) = \lim_{k \to \infty} \mu(A_k)$.*
- *If $A_1 \supset A_2 \supset \cdots$, $A_* = \bigcap_{k=1}^{\infty} A_k$, and $\mu(A_1) < \infty$ then $\mu(A_*) = \lim_{k \to \infty} \mu(A_k)$.*
- *Let us define*
  $$\limsup_{k \to \infty} A_k := \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} A_j.$$
  If $\sum_{k=1}^{\infty} \mu(A_k) < \infty$, then $\mu(\limsup_{k \to \infty} A_k) = 0$. This implication is known as the Borel-Cantelli Lemma.

Definition 4 (Singular and Absolutely Continuous). *Let μ, ν be two measures on X.*

- *We say μ is singular with respect to ν (written $\mu \perp \nu$) if there exists a measurable set $Z \subset X$ such that*
  $$\mu(Z) = \nu(X \setminus Z) = 0.$$
- *We say μ is absolutely continuous with respect to ν (written $\mu \ll \nu$) if for all measurable $A \subset X$,*
  $$\nu(A) = 0 \quad \implies \quad \mu(A) = 0.$$

Theorem 4 (Jordan Decompsition*). *Let $\mu : 2^X \to [-\infty, \infty]$ be a signed measure. Then there exists unique positive measures μ^+, μ^- such that $\mu = \mu^+ - \mu^-$.*

Theorem 5 (Hahn Decomposition*). *Let $\mu : 2^X \to [-\infty, \infty]$ be a signed measure. Then there exists measurable sets $P, N \subset X$ such that $P \cap N = \emptyset$ and $P \cup N = X$ with the property that*

$$E \subset P \implies \mu(E) \geq 0$$
$$E \subset N \implies \mu(E) \leq 0.$$

We call P the positive part and N the negative part of μ. They satisfy

$$\mu^+(E) = \mu(E \cap P), \qquad \mu^-(E) = -\mu(E \cap N).$$

Theorem 6 (Lebesgue Decomposition*). *Let μ, ν be σ-finite signed measures. Then there exists unique signed measures $\mu_\perp$ and $\mu_\ll$ on X such that $\mu_\perp \perp \nu$, $\mu_\ll \ll \nu$, and*

$$\mu = \mu_\perp + \mu_\ll.$$

Theorem 7 (Radon-Nikodym*). *Let μ, ν be two signed σ-finite measures on X. If $\mu \ll \nu$, then there exists a unique measurable function $f \in L^1(X, d\nu)$ such that*

$$\mu(E) = \int_E f d\nu.$$

We call f the Radon-Nikodym derivative $\frac{d\mu}{d\nu}$.

Definition 5 (Weak Limits). *Let $\{\mu_n\}_{n\in\mathbb{N}}, \mu$ be finite measures on a metric space X. We say μ_n weakly converges to μ (written $\mu_n \rightharpoonup \mu$) if for every bounded continuous function $\phi : X \to \mathbb{R}$ we have*

$$\int_X \phi d\mu_n \to \int_X \phi d\mu.$$