Real Analysis I

Problems

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Abstract: Here's a list of practice problems organized by topic for the UT Austin preliminary examination on Real Analysis (i.e. the Analysis I exam). They have mostly come from the old prelim exam page, with some of my own exercises dispersed throughout.

Contents

1 Measure Theory

(1) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz transformation. Show that if A is a set of Lebesgue measure zero, then $T(A)$ also has Lebesgue measure zero.

(2) For any $r \geq 0$ and any $x \in \mathbb{R}^2$, define the closed unit ball

$$B_r(x) := \{y \in \mathbb{R}^2 : |y - x| \leq r\}.$$

Let $0 < c < 1$. Let E be a measurable subset of the unit square $Q = [0,1]^2 \subset \mathbb{R}^2$ with the property that for every $x \in Q$ and every $r > 0$ there exists a $y \in B_r(x)$ such that $B_{c|x-y|}(y) \subset E$. Prove that $Q \setminus E$ has Lebesgue measure zero.

(3) Let (X, d) be a compact metric space. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of positive Borel measures on X that converge in the weak* topology to a finite positive Borel measure μ. Show that for every compact $K \subset X$,

$$\mu(K) \geq \limsup_{n\to\infty} \mu_n(K).$$

(4) Let Z be a subset of R with measure zero. Show that the set $A = \{x^2 : x \in Z\}$ also has measure zero.

(5) Let $E \subset \mathbb{R}$ be a measurable set such that $0 < |E| < \infty$. Prove that for every $\alpha \in (0,1)$ there is an open interval I such that

$$|E \cap I| \geq \alpha I.$$

(6) Assume that μ is a finite Borel measure on $\mathbb{R}^n$, and that there exists a constant $0 < R < \infty$ such that the k-th moments of μ satisfy the bound

$$\int |x|^k d\mu < R^{k^r} \qquad \forall k \in \mathbb{N},$$

for some $0 < r \leq 1$. Prove that μ has bounded support contained in $\{x \in \mathbb{R}^n : |x| \leq R\}$ if $r = 1$ and in $\{x \in \mathbb{R}^n : |x| \leq 1\}$ if $0 < r < 1$.

7. Let μ be a measure in the plane for which all open squares are measurable, with the property that there exists $\alpha \geq 1$ such that if two open squares Q, Q' are translates of each other and their closures $\overline{Q}, \overline{Q'}$ have a non-empty intersection, then

$$\mu(\overline{Q}) \leq \alpha\mu(\overline{Q'}) < \infty.$$

(For Lebesgue measure, $\alpha = 1$. In general $\alpha \geq 1$.) Show that horizontal lines have zero μ-measure.

8. Let μ be a Borel measure on $[0, 1]$. Assume that

   (a) μ and Lebesgue measure are mutually singular.

   (b) $\mu([0, t])$ depends continuously on t.

   (c) For any function $f : [0, 1] \to \mathbb{R}$, if $f \in L^1(\text{Lebesgue})$ then $f \in L^1(\mu)$. (Note that f is finite valued)

   Show that $\mu \equiv 0$.

9. Show that the following notions of measurability are equivalent. Here, we let $\lambda : 2^{\mathbb{R}} \to [0, \infty]$ be the Lebesgue outer measure.

   (a) $E \subset \mathbb{R}$ is measurable iff for every $\epsilon > 0$ there exists an open set $O \supset E$ such that $\lambda(O \setminus E) < \epsilon$.

   (b) $E \subset \mathbb{R}$ is measurable iff for every set $A \subset \mathbb{R}$ (measurable or not) we have

$$\lambda(A \cap E) + \lambda(A \cap E^c) = \lambda(E).$$

2 Density and Limits in L^p spaces

1. Show that $\mathcal{C}_c(\mathbb{R}^n) := \{f \in \mathcal{C}(\mathbb{R}^n) : f \text{ has compact support}\}$ is dense in $L^1(\mathbb{R}^n)$.

2. Find an uncountable family of measurable functions $\mathcal{F} \subset \{f : \mathbb{R} \to \mathbb{R} \text{ measurable }\}$ that satisfies the following two conditions:

   (a) For all $f \in \mathcal{F}$, $\|f\|_{L^\infty(\mathbb{R})} = 1$

(b) For all $f, g \in \mathcal{F}$, we have $\|f - g\|_{L^\infty(\mathbb{R})} = 1$.

(Bonus: Show that this implies L^∞ is not separable.)

3 Let $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Show that if $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ then $f * g$ is bounded and continuous on $\mathbb{R}^n$.

4 Let B be the closed unit ball in $\mathbb{R}^n$, and let $\{f_k\}_{k\in\mathbb{N}}$ be nonnegative integrable functions on B. Assume that

(a) $f_k \to f$ almost everywhere.

(b) For every $\epsilon > 0$ there exists an $M > 0$ such that

$$\int_{\{x\in B: f_k(x) > M\}} f_k(x)dx < \epsilon \qquad \forall k \in \mathbb{N}.$$

Show that $f_k \to f$ in $L^1(B)$.

5 Let $\{f_k\}_{k\in\mathbb{N}} \subset L^p$ with $1 \leq p < \infty$. If $f_k \to f$ pointwise a.e. and $\|f_k\|_p \to \|f\|_p$, show that $\|f - f_k\|_p \to 0$.

6 Let $f \in L^1(\mathbb{R})$ and φ_ϵ be a mollifier. This measn that $\varphi_\epsilon(x) = \epsilon^{-1}\varphi(x/\epsilon)$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is a function that satisfies: $\varphi \geq 0$, φ is compactly supported, and $\int \varphi = 1$. Let $f_\epsilon := f * \varphi_\epsilon$. Show that

$$\int_{\mathbb{R}} \liminf_{\epsilon\to 0} |f_\epsilon| \leq \int_{\mathbb{R}} |f|.$$

7 Let $f \in L^1(X, \mu)$. Prove that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left| \int_A f d\mu \right| < \epsilon$$

for all measurable $A \subset X$ such that $\mu(A) < \delta$.

8. Let $p \in [1, \infty)$ and suppose $\{f_n\}_{n=1}^{\infty} \subset L^p(\mathbb{R})$ is a sequence that converges to 0 in $L^p(\mathbb{R})$. Prove that one can find a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to 0$ almost everywhere.

9. Show that, if $f \in L^4(\mathbb{R})$, then

$$\int |f(\lambda x) - f(x)|^4 dx \to 0$$

as $\lambda \to 1$.

10. Let f, g be bounded measurable functions on $\mathbb{R}^n$. Assume that g is integrable and satisfies $\int g = 0$. Define $g_k(x) = k^n g(kx)$ for $k \in \mathbb{N}$. Show that $f * g_k \to 0$ pointwise a.e. as $k \to \infty$.

11. Suppose that $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of integrable functions on $[0, 1]$ such that $\|f_n\|_{L^1([0,1])} \leq n^{-2}$ for all $n \in \mathbb{N}$. Show that $f_n \to 0$ pointwise a.e.

12. Let $f \in L^\infty(\mu)$ be a nonnegative bounded μ-measurable function. Consider the set R_f consisting of all positive real numbers w such that $\mu(\{x : |f(x) - w| \leq \epsilon\}) > 0$ for every $\epsilon > 0$.

   (a) Prove that R_f is compact.

   (b) Prove that $\|f\|_{L^\infty} = \sup R_f$.

13. Let $\{f_k\}_{k \in \mathbb{N}}$ be a sequence of $L^1([0, 1])$ functions such that $f_k \to f$ pointwise a.e. Show that $\|f_k - f\|_{L^1([0,1])} \to 0$ if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \int_A f_k \right| < \epsilon$$

for all k and all measurable sets $A \subset [0, 1]$ with measure $|A| < \delta$.

14. Let $f_k \to f$ a.e. on $\mathbb{R}$. Show that given $\epsilon > 0$, there exists E with $|E| < \epsilon$ so that $f_k \to f$ uniformly on $I \setminus E$ for any finite interval I.

15. Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) < \infty$. Show that a measurable function $f : X \to [0, \infty)$ is integrable if and only if $\sum_{n=0}^{\infty} \mu(\{x \in X : f(x) \geq n\})$ converges.

16. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $f \in L^1(\Omega)$. Prove that

$$\lim_{p \to 0} \left[\int_\Omega |f|^p d\mu \right]^{1/p} = \exp \left[\int_\Omega \log |f| d\mu \right]$$

where $\exp[-\infty] = 0$. To simplify the problem, you may assume that $\log |f| \in L^1(\Omega)$.

17. Let h be a bounded, measurable function such that for any interval I

$$\left| \int_I h \right| \leq |I|^{1/2}.$$

Let $h_\epsilon(x) = h(x/\epsilon)$. Show that for any A with $|A| < \infty$

$$\int_A h_\epsilon \to 0, \qquad \text{as } \epsilon \to 0$$

18. For $\frac{1}{p} + \frac{1}{q} = 1$, let

$$S = \{ f \in L^p(\mathbb{R}) : \ \text{support}(f) \subset [-1, 1], \text{ and } \|f\|_{L^p} \leq 1 \}$$

and let g be a fixed but arbitrary function in $L^q(\mathbb{R})$ with $\text{support}(g) \subset [-1, 1]$. Show that the image of S under the map $f \mapsto f * g$ is a compact set in $\mathcal{C}([-2, 2])$.

19. Let $\{f_k\}_{k \in \mathbb{N}}$ be a sequence on non-negative Lebesgue integrable functions on $\mathbb{R}^n$ such that

$$\sum_{k=1}^{\infty} \int (f_k - f_{k-1})_+ < \infty, \qquad \text{and} \qquad \lim_{k \to \infty} \int f_k = 0.$$

Show that $\limsup_{k \to \infty} f_k = 0$ a.e.

3 Convergence in Measure

1. Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of measurable functions on $\mathbb{R}^n$ that converge to zero in measure on $\{y \in \mathbb{R}^n : |y-x| \leq 1\}$ and are uniformly bounded in $L^2(\mathbb{R}^n)$. Show that f_k converges to zero in $L^1(\mathbb{R}^n)$.

2. Prove that, on a finite measure space, if $f_k \to f$ in measure and $g_k \to g$ in measure, then $f_k g_k \to fg$ in measure.

3. Suppose that $f_k \to f$ in measure on $\mathbb{R}$ with respect to Lebesgue measure and $|f_k| \leq g \in L^1(\mathbb{R})$ for all k. Prove that $f_k \to f$ in $L^1(\mathbb{R})$.

4. Let (X, Σ, μ) be a finite measure space and $1 \leq q < p < \infty$. Let $\{f_k\}_{k=1}^{\infty} \subset L^p(X,\mu)$ and $\|f_k\|_p \leq 1$ for all k. Assuming that $f_k \to f$ in measure, show that $f \in L^p(X,\mu)$ and that $\|f_k - f\|_q \to 0$.

4 Weak L^p and Fubini

1. Let H be a monotone function of f, a non-negative measurable function. Write

$$\int H(f(x))dx$$

in terms of $g(\lambda) := |\{f > \lambda\}|$.

2. Show that if $p > 1$ and $f \in L^p([0,\infty))$, then the *mean functional* of f, defined by

$$F(y) := \frac{1}{y}\int_0^y f(t)dt = \int_0^1 f(xy)dx$$

is also in $L^p([0,\infty))$ and moreover

$$\|F\|_p \leq \frac{p}{p-1}\|f\|_p.$$

(Hint: Use the generalized Minkowski inequality).

(3) Let f be a locally integrable function on $\mathbb{R}^2$. Assume that, for any given real numbers a and b outside of some set of measure zero, $f(x,a) = f(x,b)$ for almost every $x \in \mathbb{R}$ and $f(a,y) = f(b,y)$ for almost every $y \in \mathbb{R}$. Show that f is constant almost everywhere on $\mathbb{R}^2$.

(4) Let f, g be real valued measurable integrable functions on a measure space (X, μ) and let

$$F_t := \{x \in X : f(x) > t\}, \qquad G_t := \{x \in X : G(x) > t\}.$$

Prove that

$$\|f - g\|_1 = \int_{-\infty}^{\infty} \mu(F_t \Delta G_t) dt$$

where $F_t \Delta G_t = (F_t \setminus G_t) \cup (G_t \setminus F_t)$.

(5) Let $0 < q < p < \infty$. Let $E \subset \mathbb{R}^n$ be measurable with measure $|E| < \infty$. Let f be a measurable function on $\mathbb{R}^n$ such that

$$N := \sup_{\lambda > 0} \lambda^p |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|$$

is finite.

(a) Prove that $\int_E |f|^q$ is finite.

(b) Refine the argument of (a) to prove that

$$\int_E |f|^q \leq C N^{q/p} |E|^{1-q/p}$$

where C is a constant depending only on n, p, and q.

(6) Let $p > 0$. Give the definition of $L^p_{weak}(\mathbb{R})$ and the quasi-norm N_p on this space. Prove that simple functions are *not* dense in $L^p_{weak}(\mathbb{R})$. That is, there exists an $f \in L^p_{weak}(\mathbb{R})$ such that $N_p(f - h_k) \to 0$ fails to hold for every sequence of simple functions $\{h_k\}_{k=1}^{\infty}$.

(7) Let $1 < p < \infty$ and $f(x) = |x|^{-n/p}$ for $x \in \mathbb{R}^n$. Prove that $f \in L^p_{weak}(\mathbb{R}^n)$ and f is *not* the limit of a sequence of $\mathcal{C}_c^{\infty}(\mathbb{R}^n)$ functions.

5 Maximal Functions

1. For $f \in L^1(\mathbb{R})$ denote by Mf the restricted maximal function defined by

$$(Mf)(x) := \sup_{0<t<1} \frac{1}{2t} \int_{x-t}^{x+t} |f(z)|dz.$$

Show that $M(f * g) \leq (Mf) * (Mg)$ for all $f, g \in L^1(\mathbb{R})$.

2. For a function $f \in L^1(\mathbb{R}^2)$ let $\widetilde{M}f$ be the unrestricted maximal function

$$\widetilde{M}f(x_0, y_0) := \sup_Q \frac{1}{|Q|} \int_Q |f(x,y)|dxdy$$

where the supremum is over all closed cubes $Q \subset \mathbb{R}^2$ centered at (x_0, y_0).

(a) Show that $\widetilde{M}f(x_0, y_0) \leq M_1 M_2 f(x_0, y_0)$, where

$$M_1 f(x_0, y) := \sup_{k>0} \frac{1}{2k} \int_{x_0-k}^{x_0+k} |f(x,y)|dx, \qquad M_2 f(x, y_0) = \sup_{l>0} \frac{1}{2l} \int_{y_0-l}^{y_0+l} |f(x,y)|dy.$$

(b) Show that there exists a constant $C > 0$ such that if $f \in L^2(\mathbb{R}^2)$ then

$$\|\widetilde{M}f\|_{L^2(\mathbb{R}^2)} \leq C\|f\|_{L^2(\mathbb{R}^2)}.$$

3. Consider the Hardy-Littlewood maximal function (for balls)

$$Mf(x) := \sup_B \frac{1}{|B|} \int_B |f|, \qquad f(x) := \begin{cases} 1, & \text{if } |x| \leq 1 \\ 0, & \text{if } |x| > 1 \end{cases}$$

Prove that Mf belongs to $L^1_{weak}(\mathbb{R}^n)$.

6 Weak Derivatives and Absolute Continuity

1. Let $f, \{f_k\}_{k\in\mathbb{N}}$ be increasing functions on $[a,b]$. If $\sum_{k=1}^{\infty} f_k$ converges pointwise to f on $[a,b]$, show that $\sum_k f_k'$ converges to f' almost everywhere on $[a,b]$.

(2) Let $1 < p < \infty$. Assume $f \in L^p(\mathbb{R})$ satisfies

$$\sup_{0<|h|<1} \int \left| \frac{f(x+h) - f(x)}{h} \right|^p < \infty.$$

Show that f has a weak derivative $g \in L^p$, which by definition satisfies $\int \psi g = -\int \psi' f$ for every $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$.

(3) Assuming $f : [0,1] \to \mathbb{R}$ is absolutely continuous, prove that f is Lipschitz if and only if f' belongs to $L^\infty([0,1])$.

(4) Let f be a non-decreasing function on $[0,1]$. You may assume that f is differentiable a.e.

(a) Prove that

$$\int_0^1 f'(t)dt \leq f(1) - f(0).$$

(b) Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of non-decreasing functions on $[0,1]$ such that $F(x) = \sum_{n=1}^\infty f_n(x)$ converges for $x \in [a,b]$. Prove that $F'(x) = \sum_{n=1}^\infty f_n'(x)$.

(5) Is the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x\sin(1/x), & x > 0 \\ 0, & x = 0 \end{cases}$$

absolutely continuous on $[0,1]$? Explain fully.

(6) Let $f : \mathbb{R} \to \mathbb{R}$ be absolutely continuous with compact support, and let $g \in L^1(\mathbb{R})$. Prove that $f * g$ is absolutely continuous on $\mathbb{R}$.

7 Explicit Computations and Counterexamples

(1) Find a non-empty closed set in $L^2([0,1])$ which does not contain an element of minimal norm.

(2) Give an example of a sequence $\{f_n\}_{n\in\mathbb{N}} \subset L^1(\mathbb{R})$ such that $f_n \to 0$ a.e. on $\mathbb{R}$, but f_n does not converge to 0 in $L^1_{loc}(\mathbb{R})$.

(3) For any natural number n, construct a function $f \in L^1(\mathbb{R}^n)$ such that for any ball $B \subset \mathbb{R}^n$, f is not essentially bounded on B.

(4) Let $g \in L^1(\mathbb{R}^n)$ with $\|g\|_{L^1(\mathbb{R}^n)} < 1$. Prove that there is a unique $f \in L^1(\mathbb{R}^n)$ such that

$$f(x) + (f * g)(x) = e^{-|x|^2}, \qquad x \in \mathbb{R}^n \text{ a.e.}$$

(5) Provide an example of a sequence of measurable functions on $[0,1]$ which converges in L^1 to the zero function but does not converge pointwise a.e.

(6) Let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers in $[0,1]$ (possibly dense). Show that the series $\sum_k k^{-3/2}|x - x_k|^{-1/2}$ converges for almost every $x \in [0,1]$.

(7) Let f be a continuous function on $[0,1]$. Find

$$\lim_{n\to\infty} n \int_0^1 x^n f(x)dx.$$

Justify your answer.

(8) If $f(x,y) \in L^2(\mathbb{R}^2)$, show that $f(x+x^3, y+y^3) \in L^1(\mathbb{R}^2)$.