Linear Algebra

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Lecture Overview

- Elements in linear algebra
- Linear system
- Linear combination, vector equation,
  Four views of matrix multiplication
- Linear independence, span, and subspace
- Linear transformation
- Least squares
- Eigendecomposition
- Singular value decomposition

Recall: Linear System

- Recall the matrix equation of a linear system:

Person ID	Weight	Height	Is_smoking	Life-span
1	60kg	5.5ft	Yes (=1)	66
2	65kg	5.0ft	No (=0)	74
3	55kg	6.0ft	Yes (=1)	78

$$\Rightarrow \begin{bmatrix} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$$A \quad \mathbf{x} = \mathbf{b}$$

- Or, a vector equation is written as

$$\Rightarrow \begin{bmatrix} 60 \\ 65 \\ 55 \end{bmatrix} x_1 + \begin{bmatrix} 5.5 \\ 5.0 \\ 6.0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 = \mathbf{b}$$

Uniqueness of Solution for $A\mathbf{x} = \mathbf{b}$

- The solution exists only when $\mathbf{b} \in \mathrm{Span}\,\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

$$\begin{bmatrix} 60 \\ 65 \\ 55 \end{bmatrix} x_1 + \begin{bmatrix} 5.5 \\ 5.0 \\ 6.0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$$\mathbf{a}_1 x_1 + \quad \mathbf{a}_2 x_2 \quad + \quad \mathbf{a}_3 x_3 = \quad \mathbf{b}$$

- If the solution exists for $A\mathbf{x} = \mathbf{b}$, when is it unique?
- It is unique when $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are linearly independent.
- Infinitely many solutions exist when $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are linearly dependent.

Linear Independence

(Practical) Definition:

- Given a set of vectors $\mathbf{v}_1, \cdots, \mathbf{v}_p \in \mathbb{R}^n$, check if $\mathbf{v}_j$ can be represented as a linear combination of the previous vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{j-1}\}$ for $j = 1, \dots, p$, e.g.,

$$\mathbf{v}_j \in \text{Span}\left\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{j-1}\right\} \text{ for some } j = 1, \dots, p?$$

- If at least one such $\mathbf{v}_j$ is found, then $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is linearly dependent.
- If no such $\mathbf{v}_j$ is found, then $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is linearly independent.

Linear Independence

(Formal) Definition:

- Consider $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 \cdots + x_p\mathbf{v}_p = \mathbf{0}$.
- Obviously, one solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$,

  which we call a trivial solution.

- $\mathbf{v}_1, \cdots, \mathbf{v}_p$ are linearly independent if this is the only solution.
- $\mathbf{v}_1, \cdots, \mathbf{v}_p$ are linearly dependent if this system also has other nontrivial solutions, e.g., at least one x_i being nonzero.

Two Definitions are Equivalent

- If $\mathbf{v}_1, \cdots, \mathbf{v}_p$ are linearly dependent, consider a nontrivial solution.
- In the solution, let's denote j as the last index such that $x_j \neq 0$.
- Then, one can write $x_j \mathbf{v}_j = -x_1 \mathbf{v}_1 - \cdots - x_{j-1} \mathbf{v}_{j-1}$,
  and safely divide it by x_j, resulting in

$$\mathbf{v}_j = -\frac{x_1}{x_j}\mathbf{v}_1 - \cdots - \frac{x_{j-1}}{x_j}\mathbf{v}_{j-1} \in \text{Span}\left\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}\right\}$$

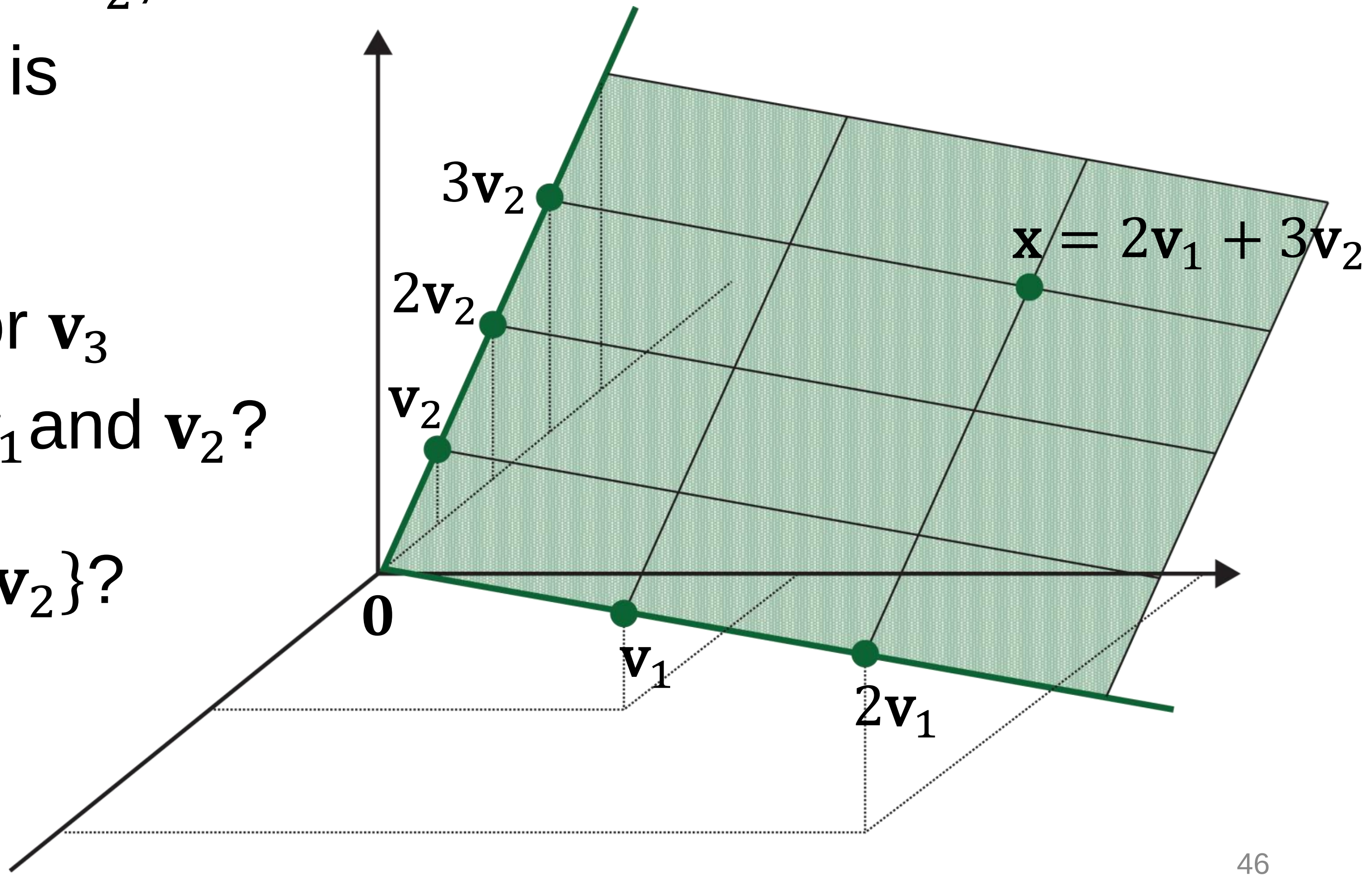
which means $\mathbf{v}_j$ can be represented as a linear combination of the previous vectors.

Geometric Understanding of Linear Dependence

- Given two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, Suppose Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ is the plane on the right.

- When is the third vector $\mathbf{v}_3$ linearly dependent of $\mathbf{v}_1$ and $\mathbf{v}_2$?

- That is, $\mathbf{v}_3 \in$ Span $\{\mathbf{v}_1, \mathbf{v}_2\}$?



Linear Dependence

- A linearly dependent vector does not increase Span!
- If $\mathbf{v}_3 \in$ Span $\{\mathbf{v}_1, \mathbf{v}_2\}$, then

$$\text{Span}\,\{\mathbf{v}_1,\,\mathbf{v}_2\} = \text{Span}\,\{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3\},$$

- Why?
- Suppose $\mathbf{v}_3 = d_1\mathbf{v}_1 + d_2\mathbf{v}_2$, then the linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ can be written as

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = (c_1 + d_1)\mathbf{v}_1 + (c_1 + d_1)\mathbf{v}_2$$

which is also a linear combination of $\mathbf{v}_1, \mathbf{v}_2$.

Linear Dependence and Linear System Solution

- Also, a linearly dependent set produces multiple possible linear combinations of a given vector.
- Given a vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$, suppose the solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, i.e., $3\mathbf{v}_1 + 2\mathbf{v}_2 + 1\mathbf{v}_3 = \mathbf{b}$.
- Suppose also $\mathbf{v}_3 = 2\mathbf{v}_1 + 3\mathbf{v}_2$, a linearly dependent case.
- Then, $3\mathbf{v}_1 + 2\mathbf{v}_2 + 1\mathbf{v}_3 = 3\mathbf{v}_1 + 2\mathbf{v}_2 + (2\mathbf{v}_1 + 3\mathbf{v}_2) = 5\mathbf{v}_1 + 5\mathbf{v}_2$, so $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$ is another solution. Many more solutions exist.

Linear Dependence and Linear System Solution

- Actually, many more solutions exist.
- e.g., $3\mathbf{v}_1 + 2\mathbf{v}_2 + 1\mathbf{v}_3 = 3\mathbf{v}_1 + 2\mathbf{v}_2 + (2\mathbf{v}_3 - 1\mathbf{v}_3)$
  $= 3\mathbf{v}_1 + 2\mathbf{v}_2 + 2(2\mathbf{v}_1 + 3\mathbf{v}_2) - 1\mathbf{v}_3 = 7\mathbf{v}_1 + 8\mathbf{v}_2 - 1\mathbf{v}_3$,

  thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ -1 \end{bmatrix}$ is another solution.

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