Linear Algebra

주재걸
고려대학교 컴퓨터학과

Lecture Overview

- Elements in linear algebra
- Linear system
- Linear combination, vector equation,
  Four views of matrix multiplication
- Linear independence, span, and subspace
- Linear transformation
- Least squares
- Eigendecomposition
- Singular value decomposition

Span and Subspace

- **Definition**: A **subspace** H is defined as a subset of $\mathbb{R}^n$ closed under linear combination:
  - For any two vectors, $\mathbf{u}_1, \mathbf{u}_2 \in H$, and any two scalars c and d, $c\mathbf{u}_1 + d\mathbf{u}_2 \in H$.
- Span $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is always a subspace. Why?
  - $\mathbf{u}_1 = a_1\mathbf{v}_1 + \cdots + a_p\mathbf{v}_p$, $\mathbf{u}_2 = b_1\mathbf{v}_1 + \cdots + b_p\mathbf{v}_p$
  - $$c\mathbf{u}_1 + d\mathbf{u}_2 = c\left(a_1\mathbf{v}_1 + \cdots + a_p\mathbf{v}_p\right) + d\left(b_1\mathbf{v}_1 + \cdots + b_p\mathbf{v}_p\right) = (ca_1 + db_1)\mathbf{v}_1 + \cdots + \left(ca_p + db_p\right)\mathbf{v}_p$$
- In fact, a subspace is always represented as Span $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$.

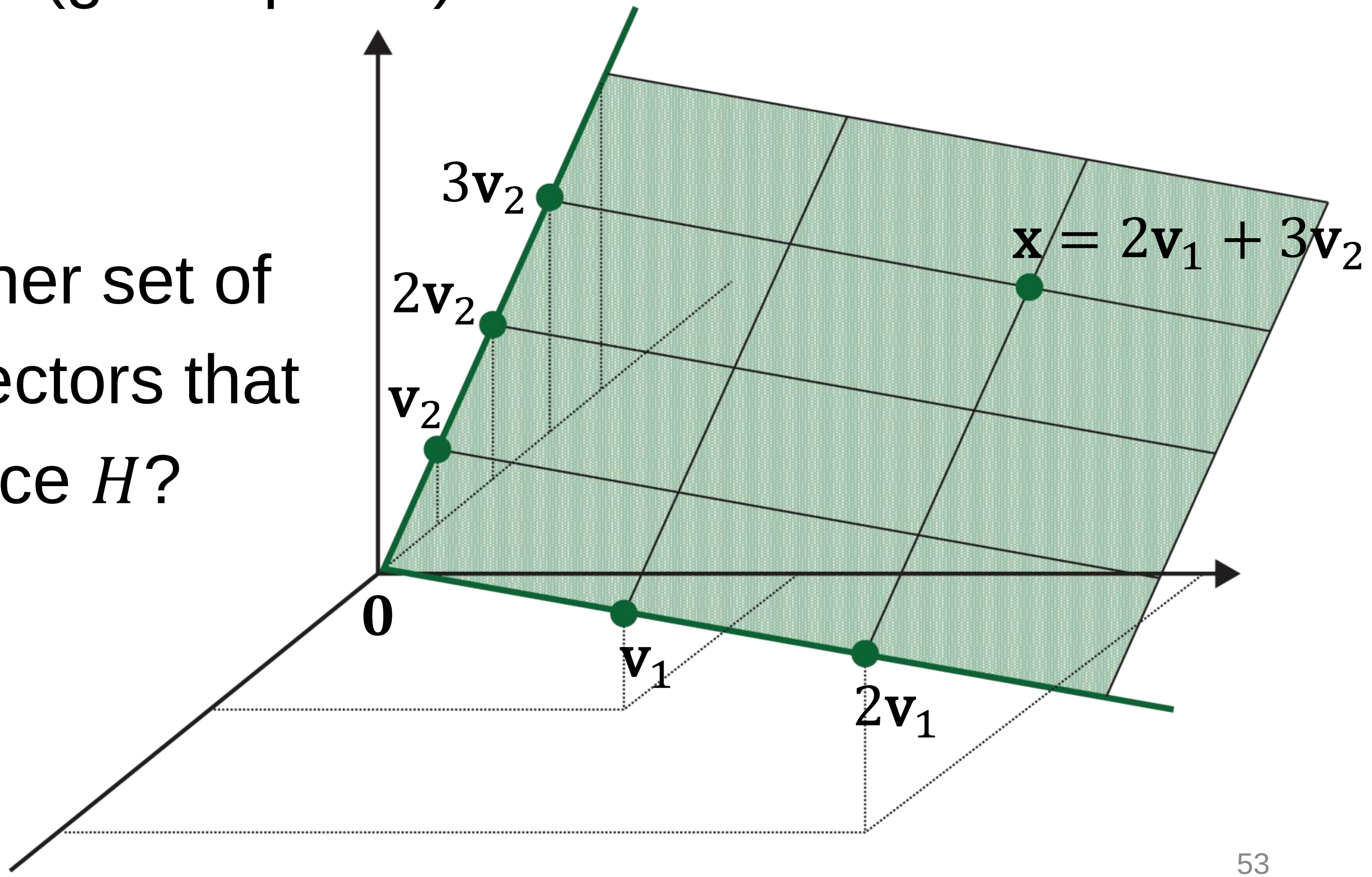
Basis of a Subspace

- **Definition**: A **basis** of a subspace H is a set of vectors that satisfies both of the following:
  - Fully spans the given subspace H
  - Linearly independent (i.e., no redundancy)

- In the previous example, where $H = \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ forms a plane, but $\mathbf{v}_3 = 2\mathbf{v}_1 + 3\mathbf{v}_2 \in \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of H, but not $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ nor $\{\mathbf{v}_1\}$ is a basis.

Non-Uniqueness of Basis

- Consider a subspace H (green plane).
- Is a basis unique?
- That is, is there any other set of linearly independent vectors that span the same subspace H?



Dimension of Subspace

- What is then unique, given a particular subspace H?
- Even though different bases exist for H, the number of vectors in any basis for H will be unique.
- We call this number as the dimension of H, denoted as dim H.
- In the previous example, the dimension of the plane is 2, meaning any basis for this subspace contains exactly two vectors.

Column Space of Matrix

- **Definition**: The **column space** of a matrix A is the subspace spanned by the columns of A. We call the column space of A as Col A.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \text{Col } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- What is dim Col A?

Matrix with Linearly Dependent Columns

- Given $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, note that $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$,

  i.e., the third column is a linear combination of the first two.

$$\text{Col } A = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right\} \Longrightarrow \text{Col } A = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

- What is dim Col A?

Rank of Matrix

- **Definition**: The **rank** of a matrix A, denoted by rank A, is the dimension of the column space of A:
  - rank A = dim Col A

Lecture Overview

- Elements in linear algebra
- Linear system
- Linear combination, vector equation,
  Four views of matrix multiplication
- Linear independence, span, and subspace
- Linear transformation
- Least squares
- Eigendecomposition
- Singular value decomposition

Summary So Far

- Scalars, vectors, matrices, and their operations such as addition, scalar multiple, matrix multiplication, transpose
- Linear system: solving using inverse matrix
- Matrix equation and vector equation
- Linear combination and Span
  - When does the solution of a linear system exist?
- Four views of matrix multiplication: inner product, column combination, row combination, sum of rank-1 outer products
- Linear independence
  - If the solution of a linear system exists, when is it unique or many?
- Subspace
  - Subset of vectors in $\mathbb{R}^n$ closed under linear combination
  - Basis and dimension
  - Column space and rank of a matrix