
Linear Algebra

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Diagonalization

- We want to change a given square matrix $A \in \mathbb{R}^{n \times n}$ into a diagonal matrix via the following form:

$$D = V^{-1}AV$$

where $V \in \mathbb{R}^{n \times n}$ is an **invertible** matrix and $D \in \mathbb{R}^{n \times n}$ is a diagonal matrix. This is called a **diagonalization** of A .

- It is not always possible to diagonalize A . For A to be diagonalizable, an **invertible V should exist** such that $V^{-1}AV$ becomes a diagonal matrix.



Finding V and D

- How can we find an invertible P and the resulting diagonal matrix $D = V^{-1}AV$?
- $D = V^{-1}AV \Rightarrow VD = AV$
- Let us represent the following:
- $V = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n]$ where \mathbf{v}_i 's are column vectors of V
- $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$



Finding V and D

- $AV = A[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n] = [A\mathbf{v}_1 \quad A\mathbf{v}_2 \quad \cdots \quad A\mathbf{v}_n]$

- $VD = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$
 $= [\lambda_1 \mathbf{v}_1 \quad \lambda_2 \mathbf{v}_2 \quad \cdots \quad \lambda_n \mathbf{v}_n]$

- $AV = VD \Leftrightarrow [A\mathbf{v}_1 \quad A\mathbf{v}_2 \quad \cdots \quad A\mathbf{v}_n] = [\lambda_1 \mathbf{v}_1 \quad \lambda_2 \mathbf{v}_2 \quad \cdots \quad \lambda_n \mathbf{v}_n]$



Finding V and D

- Equating columns, we obtain

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1, A\mathbf{v}_2 = \lambda_2\mathbf{v}_2, \dots, A\mathbf{v}_n = \lambda_n\mathbf{v}_n$$

- Thus, $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ should be **eigenvectors** and $\lambda_1, \lambda_2, \dots, \lambda_n$ should be **eigenvalues**.
- Then, For $VD = AV \Rightarrow D = V^{-1}AV$ to be true, V should be **invertible**.
- In this case, the resulting diagonal matrix D has eigenvalues as diagonal entries.



Diagonalizable Matrix

- For V to be invertible,
 V should be a **square** matrix in $\mathbb{R}^{n \times n}$, and
 V should have **n linearly independent columns**.
- Recall columns of V are eigenvectors.
Hence, A should have n linearly independent eigenvectors.
- It is not always the case, but if it is, A is **diagonalizable**.



Eigendecomposition

- If A is diagonalizable, we can write $D = V^{-1}AV$.
- We can also write $A = VDV^{-1}$.
which we call **eigendecomposition** of A .
- A being diagonalizable is equivalent to A having **eigendecomposition**.



Linear Transformation via Eigendecomposition

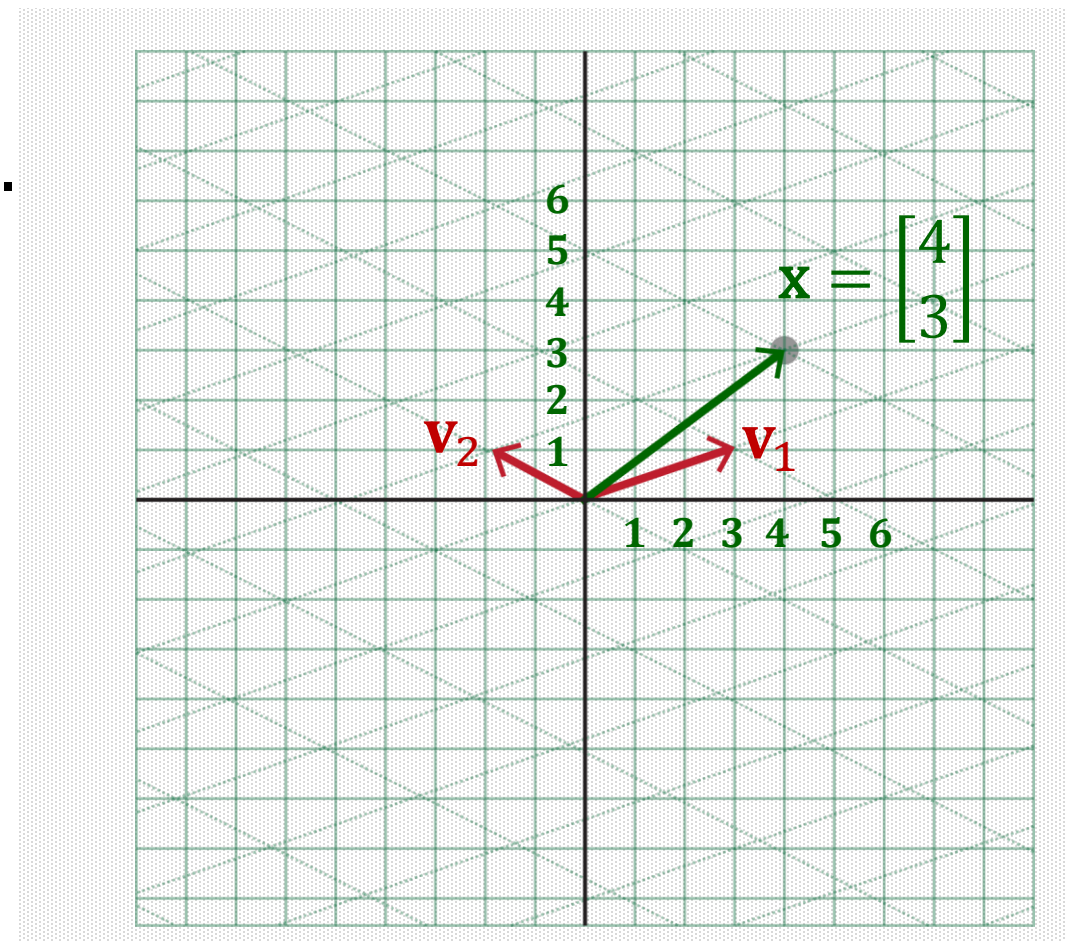
- Suppose A is diagonalizable, thus having eigendecomposition

$$A = VDV^{-1}$$

- Consider the linear transformation $T(\mathbf{x}) = A\mathbf{x}$.
- $T(\mathbf{x}) = A\mathbf{x} = VDV^{-1}\mathbf{x} = V(D(V^{-1}\mathbf{x}))$.

Change of Basis

- Suppose $A\mathbf{v}_1 = -1\mathbf{v}_1$ and $A\mathbf{v}_2 = 2\mathbf{v}_2$.
- $T(\mathbf{x}) = A\mathbf{x} = VDV^{-1}\mathbf{x} = V(D(V^{-1}\mathbf{x}))$
- Let $\mathbf{y} = V^{-1}\mathbf{x}$. Then,
$$V\mathbf{y} = \mathbf{x}$$
- \mathbf{y} is a new coordinate of \mathbf{x} with respect to a new basis of eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2\}$.



$$\mathbf{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = P\mathbf{y} = [\mathbf{v}_1 \quad \mathbf{v}_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 2\mathbf{v}_1 + 1\mathbf{v}_2 \Rightarrow \mathbf{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

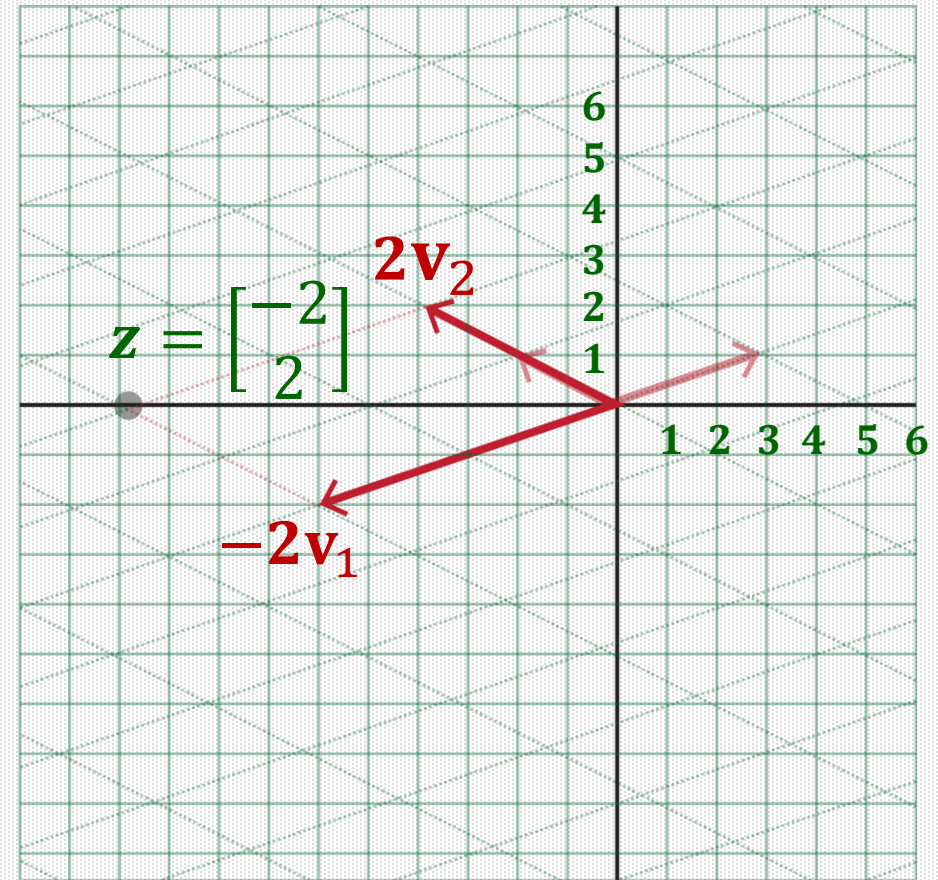
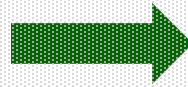
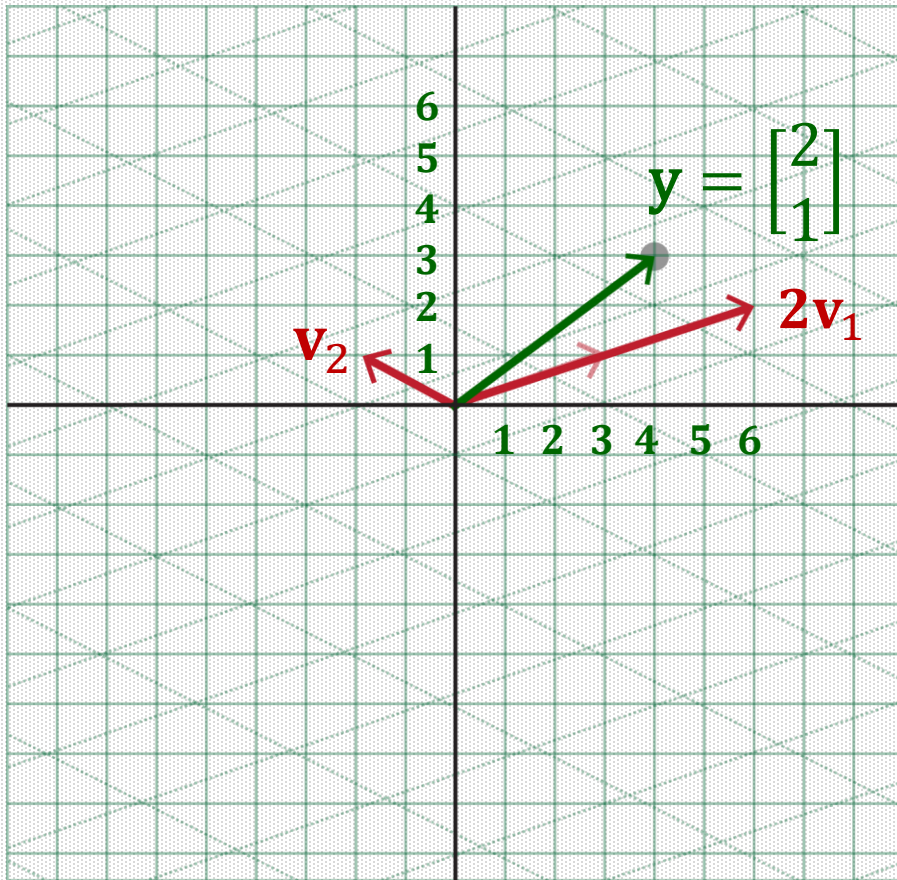


Element-wise Scaling

- $T(\mathbf{x}) = V(D(V^{-1}\mathbf{x})) = V(D\mathbf{y})$
- Let $\mathbf{z} = D\mathbf{y}$. This computation is a simple **Element-wise scaling** of \mathbf{y} .
- **Example:** Suppose $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$. Then

$$\mathbf{z} = D\mathbf{y} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} (-1) \times 2 \\ 2 \times 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

Dimension-wise Scaling





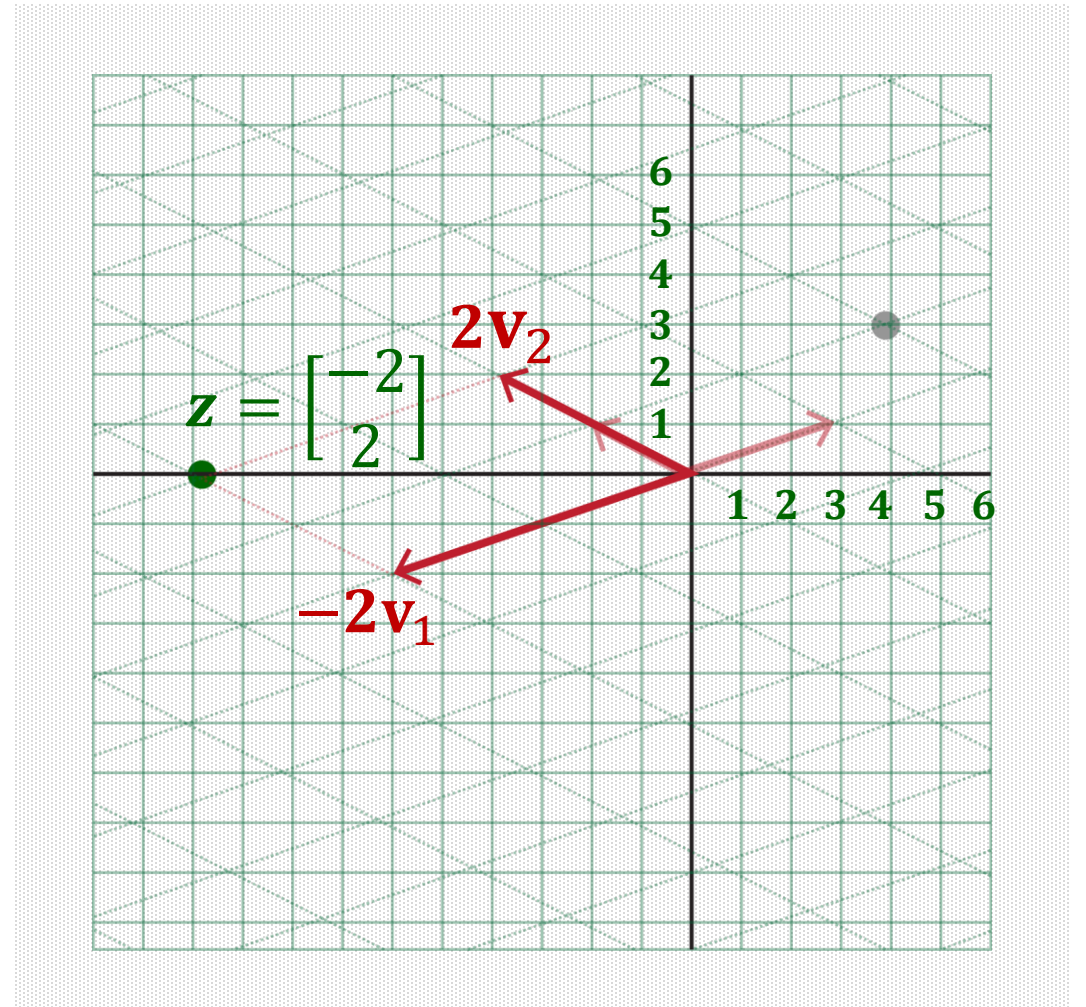
Back to Original Basis

- $T(\mathbf{x}) = V(D\mathbf{y}) = V\mathbf{z}$
- \mathbf{z} is still a coordinate based on the new basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.
- $V\mathbf{z}$ converts \mathbf{z} to another coordinates based on the original standard basis.
- That is, $V\mathbf{z}$ is a linear combination of \mathbf{v}_1 and \mathbf{v}_2 using the coefficient vector \mathbf{z} .
- That is,

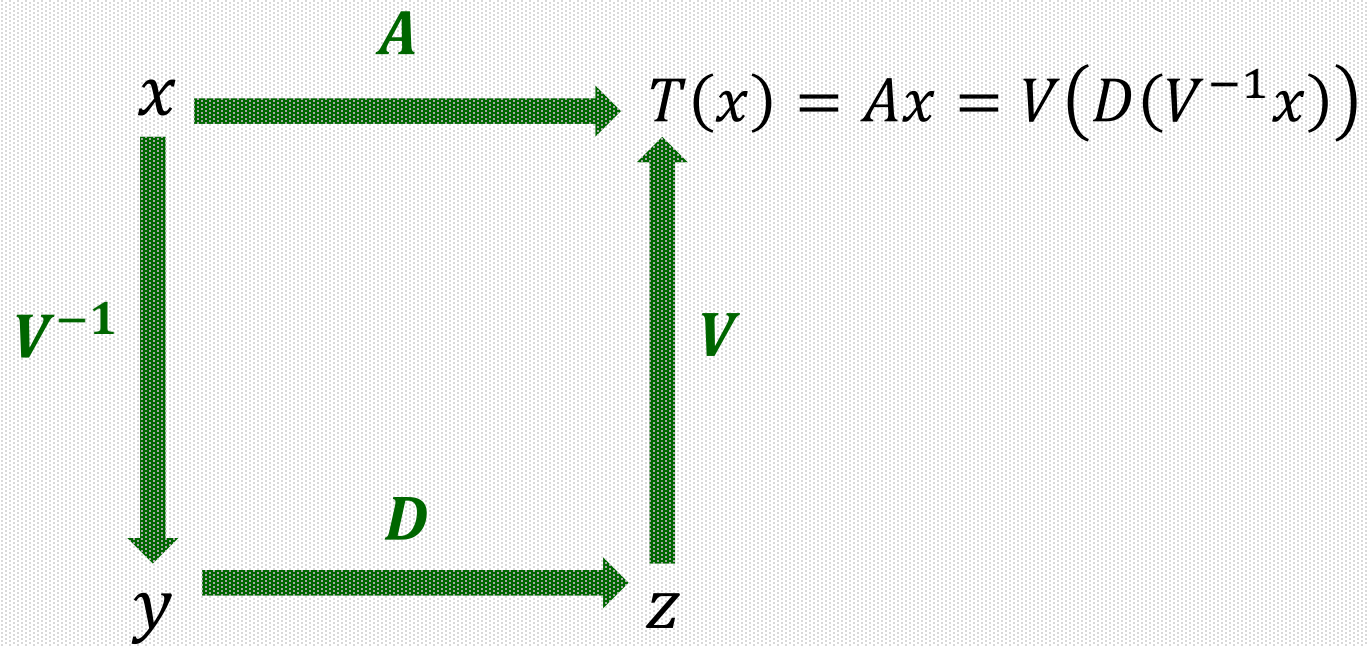
$$V\mathbf{z} = [\mathbf{v}_1 \quad \mathbf{v}_2] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \mathbf{v}_1 z_1 + \mathbf{v}_2 z_2$$

Back to Original Basis

- $$\begin{aligned} T(\mathbf{x}) &= V\mathbf{z} = [\mathbf{v}_1 \quad \mathbf{v}_2] \begin{bmatrix} -2 \\ 2 \end{bmatrix} \\ &= -2\mathbf{v}_1 + 2\mathbf{v}_2 \\ &= -2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -10 \\ 0 \end{bmatrix} \end{aligned}$$



Overview of Transformation using Eigendecomposition





Linear Transformation via A^k

- Now, consider recursive transformation $A \times A \times \cdots \times A \mathbf{x} = A^k \mathbf{x}$.
- If A is diagonalizable, A has eigendecomposition

$$A = VDV^{-1}$$

- $A^k = (VDV^{-1})(VDV^{-1}) \cdots (VDV^{-1}) = VD^kV^{-1}$
- D^k is simply computed as

$$D^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n^k \end{bmatrix}$$



Linear Transformation via A^k

- $A^k \mathbf{x} = VD^kV^{-1}\mathbf{x}$ can be computed in the similar manner to the previous example.
- It is much faster to compute $V \left(D^k (V^{-1}\mathbf{x}) \right)$ than to compute $A^k \mathbf{x}$.



Further Study

- Determining whether a matrix $A \in \mathbb{R}^{n \times n}$ diagonalizable.
 - Geometric multiplicity should be equal to algebraic multiplicity.
 - As a special case, A has n distinct eigenvalues, A is diagonalizable.
 - Lay Ch5.3
- Solving $(A - \lambda I)\mathbf{x} = \mathbf{0}$ for a given eigenvalue λ
 - Lay Ch1.5