Linear Algebra

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Lecture Overview

- Elements in linear algebra
- Linear system
- Linear combination, vector equation,
  Four views of matrix multiplication
- Linear independence, span and subspace
- Linear transformation
- Least squares
- Eigendecomposition
- <u>Singular value decomposition</u>

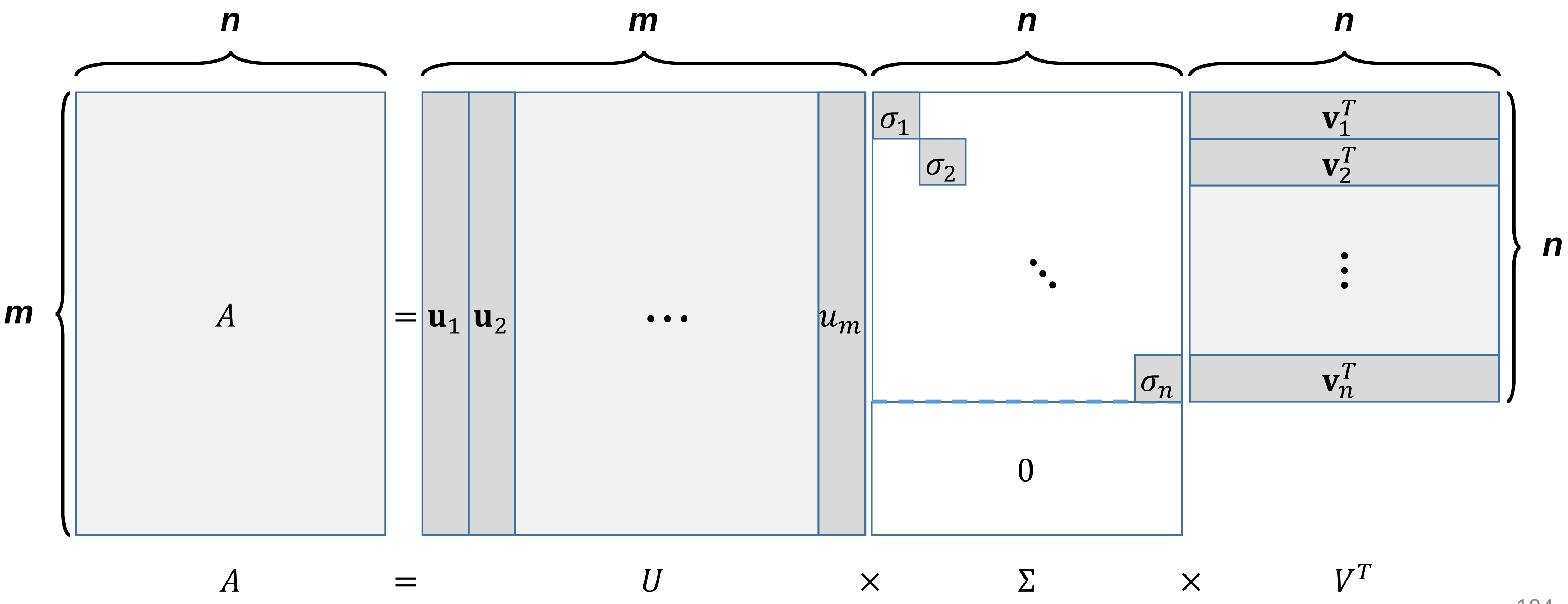
Singular Value Decomposition (SVD)

- Given a rectangular matrix $A \in \mathbb{R}^{m \times n}$,
  its singular value decomposition is written as
  $$A = U\Sigma V^T$$
  where
- $U \in \mathbb{R}^{m \times m}$, $V \in \mathbb{R}^{n \times n}$: matrices with orthonormal columns, providing an orthonormal basis of Col A and Row A, respectively
- $\Sigma \in \mathbb{R}^{m \times n}$: a diagonal matrix whose entries are in a decreasing order, i.e., $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min(m,n)}$

Basic Form of SVD

- Given a matrix $A \in \mathbb{R}^{m \times n}$ where $m > n$, SVD gives

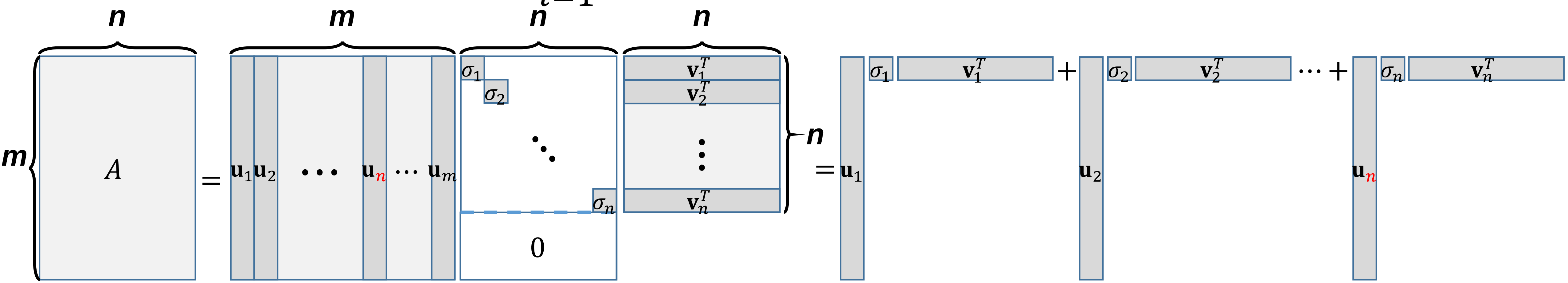
$$A = U\Sigma V^T$$



SVD as Sum of Outer Products

- A can also be represented as the sum of outer products

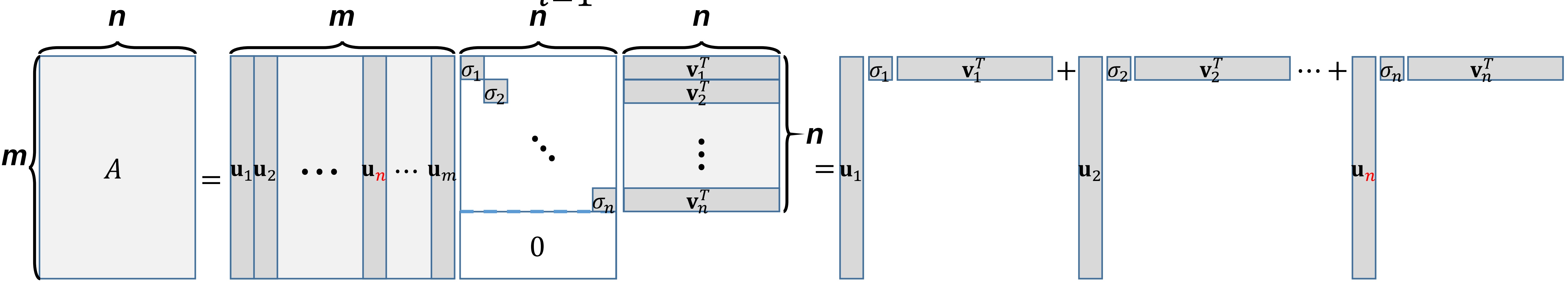
$$A = U\Sigma V^T = \sum_{i=1}^{n} \sigma_i \mathbf{u}_i \mathbf{v}_i^T, \qquad \text{where } \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$$

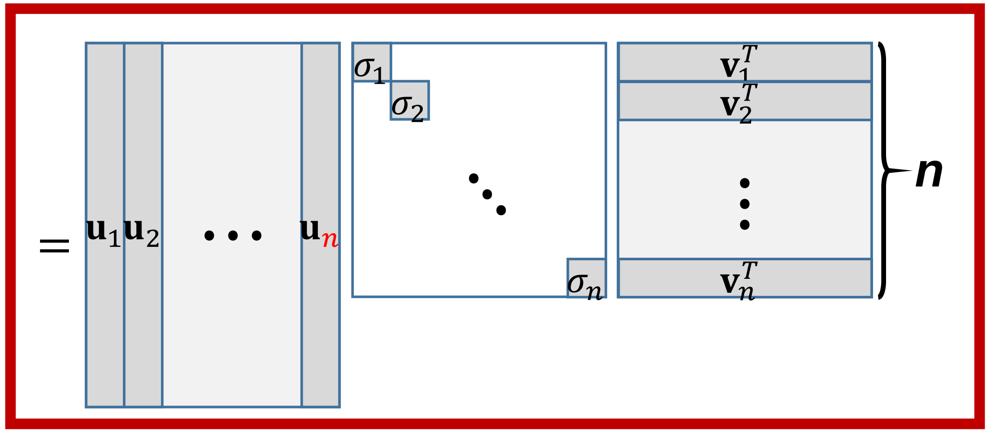


Reduced Form of SVD

- A can also be represented as the sum of outer products

$$A = U\Sigma V^T = \sum_{i=1}^{n} \sigma_i \mathbf{u}_i \mathbf{v}_i^T , \qquad \text{where } \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$$





Another Perspective of SVD

- We can easily find two orthonormal basis sets, $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ for Col A and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for Row A, by using, say, Gram–Schmidt orthogonalization.
- Are these unique orthonormal basis sets?

- No. Then, can we jointly find them such that

$$A\mathbf{v}_i = \sigma_i \mathbf{u}_i, \qquad \forall\, i = 1, \ldots, n$$

n

Another Perspective of SVD

- Let us denote $U = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n] \in \mathbb{R}^{m\times n}$, $V = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n] \in \mathbb{R}^{n\times n}$, and $\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_n \end{bmatrix} \in \mathbb{R}^{n\times n}$
- Consider $AV = A[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n] = [A\mathbf{v}_1 \quad A\mathbf{v}_2 \quad \cdots \quad A\mathbf{v}_n]$ and

$$U\Sigma = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n]\begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_n \end{bmatrix}$$
$$= [\sigma_1\mathbf{u}_1 \quad \sigma_2\mathbf{u}_2 \quad \cdots \quad \sigma_n\mathbf{u}_n]$$

- $AV = U\Sigma \Leftrightarrow [A\mathbf{v}_1 \quad A\mathbf{v}_2 \quad \cdots \quad A\mathbf{v}_n] = [\sigma_1\mathbf{u}_1 \quad \sigma_2\mathbf{u}_2 \quad \cdots \quad \sigma_n\mathbf{u}_n]$
- $V^{-1} = V^T$ since $V \in \mathbb{R}^{n\times n}$ has orthonormal columns.
- Thus $AV = U\Sigma \Leftrightarrow A = U\Sigma V^T$