Linear Algebra

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Computing SVD

- First, we form AA^T and A^TA and compute eigendecomposition of each:

$$AA^T = U\Sigma V^T V\Sigma^T U^T = U\Sigma\Sigma^T U^T = U\Sigma^2 U^T$$
$$A^TA = V\Sigma^T U^T U\Sigma V^T = V\Sigma^T\Sigma U^T = V\Sigma^2 V^T$$

- Can we find the following?
  1. Orthogonal eigenvector matrices U and V
  2. Eigenvalues in Σ^2 that are all positive
  3. Eigenvalues in Σ^2 that are shared by AA^T and A^TA
- Yes, since AA^T and A^TA are symmetric positive (semi-)definite.
  - More details in the next slides.

Diagonalization of Symmetric Matrices

- In general, $A \in \mathbb{R}^{n \times n}$ is diagonalizable if and only if n linearly independent eigenvectors exist.

- How about a symmetric matrix $S \in \mathbb{R}^{n \times n}$, where $S^T = S$?

- S is always diagonalizable.
- Furthermore, S is orthogonally diagonalizable, meaning that their eigenvectors are not only linearly independent, but also orthogonal to each other.

Spectral Theorem of Symmetric Matrices

Consider a symmetric matrix $S \in \mathbb{R}^{n\times n}$, where $S^T = S$.

- A has n real eigenvalues, counting multiplicities.
- The dimension of the eigenspace for each eigenvalue equals the multiplicity of λ as a root of the characteristic equation.
- The eigenspaces are mutually orthogonal. That is, eigenvectors corresponding to different eigenvalues are orthogonal.
- To sum up, A is orthogonally diagonalizable.

- Proofs in Lay Ch7.1

Spectral Decomposition

Eigendecomposition of a symmetric matrix,
also known as spectral decomposition, is represented as

- $$S = UDU^{-1} = UDU^T = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 \mathbf{u}_1 & \lambda_2 \mathbf{u}_2 & \cdots & \lambda_n \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix}$$

$$= \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T$$

- Each term, $\lambda_i \mathbf{u}_j \mathbf{u}_j^T$ can be viewed as a projection matrix onto the subspace spanned by $\mathbf{u}_j$, scaled by its eigenvalue λ_i.

Positive Definite Matrices

- **Definition**: $A \in \mathbb{R}^{n\times n}$ is positive definite if $\mathbf{x}^T A\mathbf{x} > 0, \quad \forall \mathbf{x} \neq \mathbf{0}$.
- **Definition**: $A \in \mathbb{R}^{n\times n}$ is positive semi-definite
  if $\mathbf{x}^T A\mathbf{x} \geq 0, \quad \forall \mathbf{x} \neq \mathbf{0}$.

- **Theorem**: $A \in \mathbb{R}^{n\times n}$ is positive definite
  if and only if the eigenvalues of A are all positive.
- Proofs in Lay Ch7.2

Symmetric Positive Definite Matrices

- If $S \in \mathbb{R}^{n \times n}$ is symmetric and positive-definite, then the spectral decomposition will have all positive eigvenvalues:
- $S = UDU^T = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix}$

  $= \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T$

  where $\lambda_j > 0, \forall j = 1, \cdots, n$

Back to Computing SVD

- In the following,

$$AA^T = U\Sigma V^T \mathrm{V}\Sigma^T U^T = U\Sigma\Sigma^T U^T = U\Sigma^2 U^T$$
$$A^T A = V\Sigma^T U^T U\Sigma V^T = V\Sigma^T \Sigma U^T = V\Sigma^2 V^T$$

- Can we prove that both AA^T and $A^T A$ are symmetric positive-(semi-)definite?
- Symmetric: $(AA^T)^T = AA^T$ and $(A^T A)^T = A^T A$
- Positive-(semi-)definite
  - $\mathbf{x}^T AA^T \mathbf{x} = (A^T \mathbf{x})^T (A^T \mathbf{x}) = \|A^T \mathbf{x}\|^2 \geq 0$
  - $\mathbf{x}^T A^T A\mathbf{x} = (A\mathbf{x})^T (A\mathbf{x}) = \|A\mathbf{x}\|^2 \geq 0$
- Thus, we can find
  1. Orthogonal eigenvector matrices U and V
  2. Eigenvalues in Σ^2 that are all positive

Things to Note

- Given any rectangular matrix $A \in \mathbb{R}^{m \times n}$, its SVD always exists.

- Given a square matrix $A \in \mathbb{R}^{n \times n}$, its eigendecomposition does not always exist, but its SVD always exists.

- Given a square, symmetric positive (semi-)definite matrix $S \in \mathbb{R}^{n \times n}$, its eigendecomposition always exists, and it is actually the same as its SVD.