Linear Algebra

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Orthogonal Projection Perspective

- Back to the case of invertible $C = A^T A$, consider the orthogonal projection of $\mathbf{b}$ onto Col A as

$$\hat{\mathbf{b}} = f(\mathbf{b}) = A\hat{\mathbf{x}} = A(A^T A)^{-1} A^T \mathbf{b}$$

Orthogonal and Orthonormal Sets

- **Definition**: A set of vectors $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is an **orthogonal set** if each pair of distinct vectors from the set is orthogonal
  That is, if $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$.

- **Definition**: A set of vectors $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is an **orthonormal set** if it is an orthogonal set of unit vectors.

- Is an orthogonal (or orthonormal) set also a linearly independent set? What about its converse?

Orthogonal and Orthonormal Basis

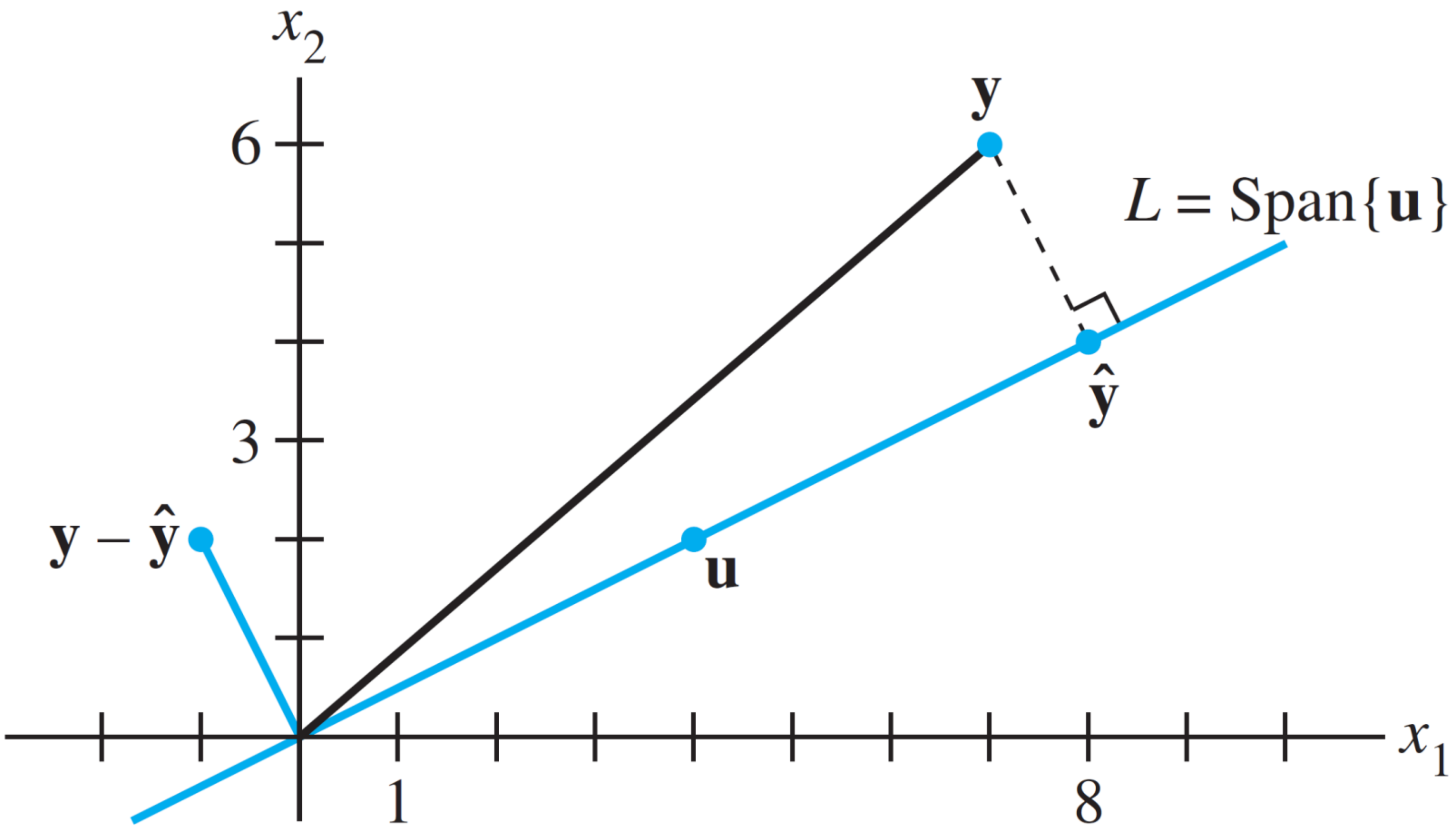
- Consider basis $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of a p-dimensional subspace W in $\mathbb{R}^n$.
- Can we make it as an orthogonal (or orthonormal) basis?
  - Yes, it can be done by Gram–Schmidt process. → QR factorization.
- Given the orthogonal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ of W,
  let's compute the orthogonal projection of $\mathbf{y} \in \mathbb{R}^n$ onto W.

Orthogonal Projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto Line

- Consider the orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto one-dimensional subspace L.

- $\hat{\mathbf{y}} = \text{proj}_L \mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u}$

- If $\mathbf{u}$ is a unit vector, $\hat{\mathbf{y}} = \text{proj}_L \mathbf{y} = (\mathbf{y} \cdot \mathbf{u})\mathbf{u}$



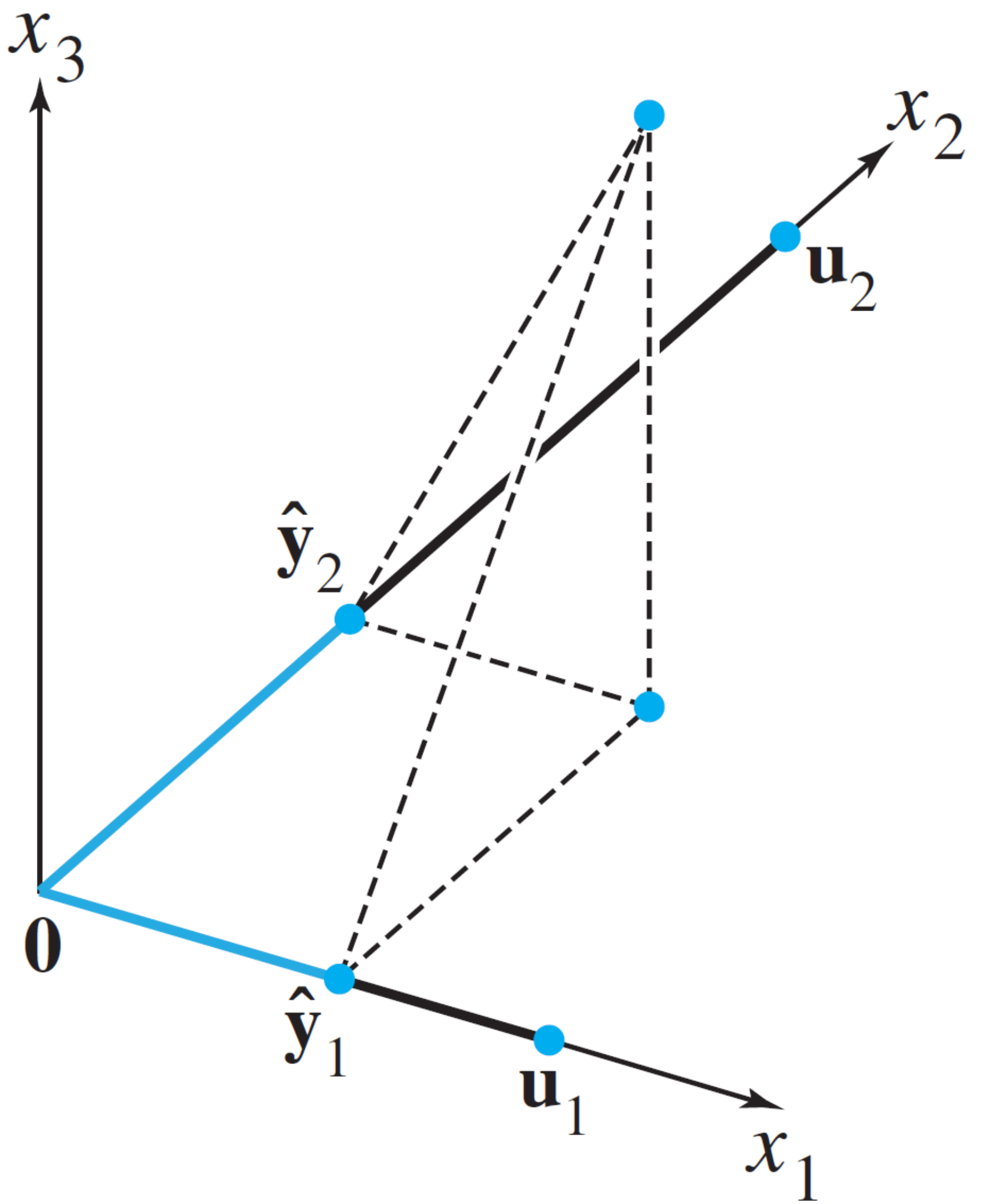
Orthogonal Projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto Plane

- Consider the orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto two-dimensional subspace W

- $\hat{\mathbf{y}} = \text{proj}_L \mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2$

- If $\mathbf{u}_1$ and $\mathbf{u}_2$ are unit vectors,
  $\hat{\mathbf{y}} = \text{proj}_L \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2)\mathbf{u}_2$

- Projection is done independently on each orthogonal basis vector.



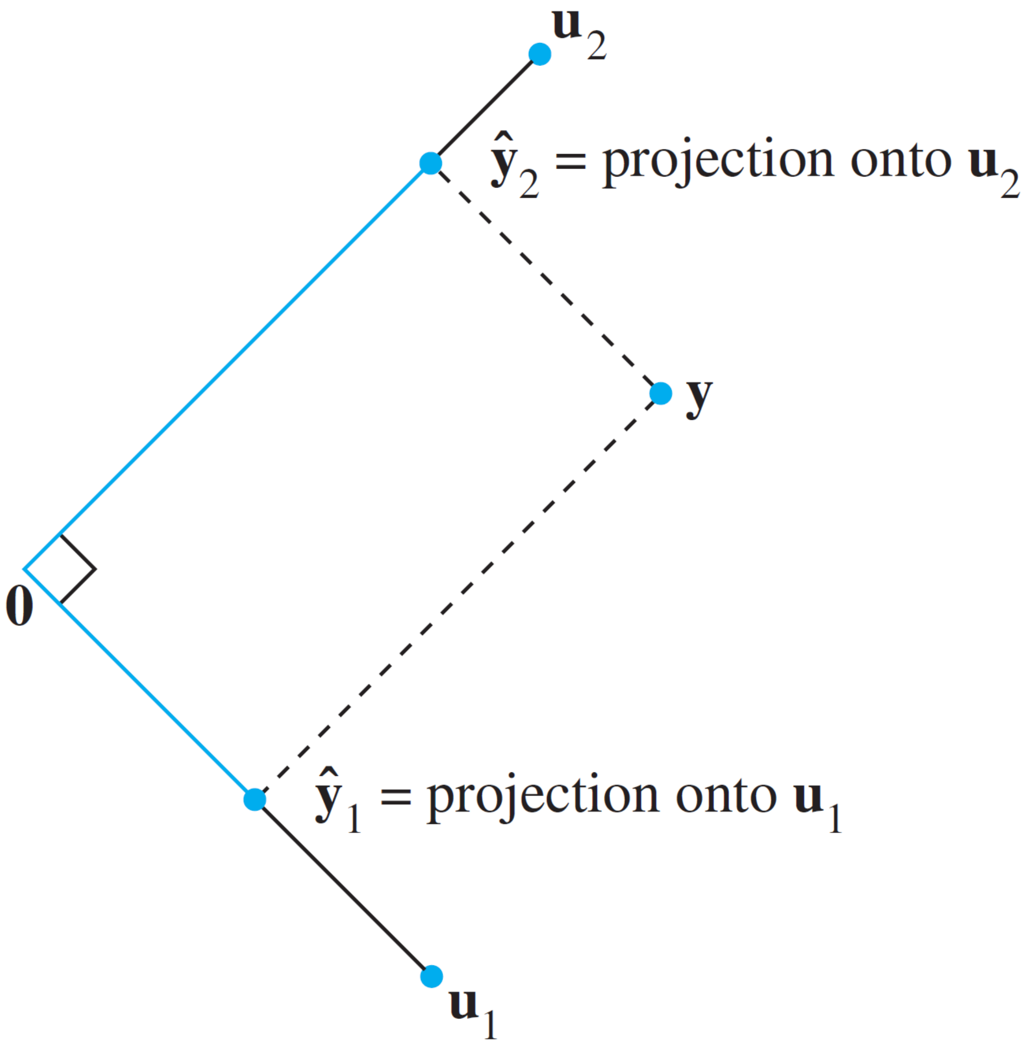
Orthogonal Projection when $\mathbf{y} \in W$

- Consider the orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto two-dimensional subspace W, where $\mathbf{y} \in W$

- $\hat{\mathbf{y}} = \mathrm{proj}_L \mathbf{y} = \mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2$

- If $\mathbf{u}_1$ and $\mathbf{u}_2$ are unit vectors,
  $\hat{\mathbf{y}} = \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2)\mathbf{u}_2$

- The solution is the same as before. Why?



Transformation: Orthogonal Projection

- Consider a transformation of orthogonal projection $\hat{\mathbf{b}}$ of $\mathbf{b}$, given orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ of a subspace W:

$$\begin{aligned}
\hat{\mathbf{b}} = f(\mathbf{b}) &= (\mathbf{b} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{b} \cdot \mathbf{u}_2)\mathbf{u}_2 \\
&= (\mathbf{u}_1^T\mathbf{b})\mathbf{u}_1 + (\mathbf{u}_2^T\mathbf{b})\mathbf{u}_2 \\
&= \mathbf{u}_1(\mathbf{u}_1^T\mathbf{b}) + \mathbf{u}_2(\mathbf{u}_2^T\mathbf{b}) \\
&= (\mathbf{u}_1\mathbf{u}_1^T)\mathbf{b} + (\mathbf{u}_2\mathbf{u}_2^T)\mathbf{b} \\
&= (\mathbf{u}_1\mathbf{u}_1^T + \mathbf{u}_2\mathbf{u}_2^T)\mathbf{b} \\
&= [\mathbf{u}_1 \quad \mathbf{u}_2]\begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \end{bmatrix}\mathbf{b} = UU^T\mathbf{b} \quad \Rightarrow \text{linear transformation!}
\end{aligned}$$

Orthogonal Projection Perspective

- Let's verify the following, when $A = U = [\mathbf{u}_1 \quad \mathbf{u}_2]$ has orthonormal columns:
  Back to the case of invertible $C = A^T A$, consider the orthogonal projection of $\mathbf{b}$ onto Col A as
  $$\hat{\mathbf{b}} = A\hat{\mathbf{x}} = A(A^T A)^{-1} A^T \mathbf{b} = f(\mathbf{b})$$

- $C = A^T A = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \end{bmatrix} [\mathbf{u}_1 \quad \mathbf{u}_2] = I$. Thus,
  $$\hat{\mathbf{b}} = A\hat{\mathbf{x}} = A(A^T A)^{-1} A^T \mathbf{b} = A(I)^{-1} A^T \mathbf{b} = AA^T \mathbf{b} = UU^T \mathbf{b}$$