Linear Algebra

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Eigenvectors and Eigenvalues

- **Definition**: An **eigenvector** of a **square** matrix $A \in \mathbb{R}^{n \times n}$ is a **nonzero** vector $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ
In this case, λ is called an **eigenvalue** of A, and
such an $\mathbf{x}$ is called an ***eigenvector corresponding to $\boldsymbol{\lambda}$***.

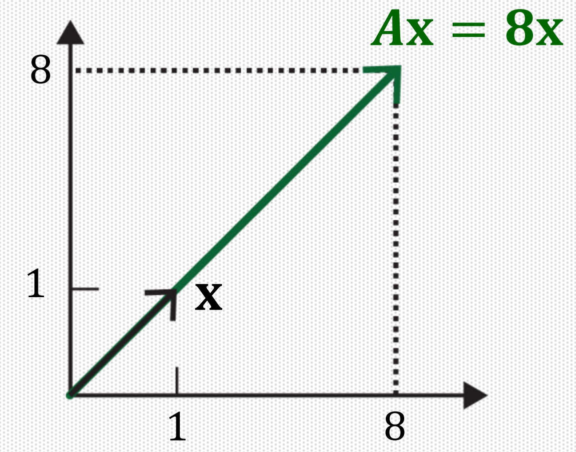
Transformation Perspective

- Consider a linear transformation $T(\mathbf{x}) = A\mathbf{x}$.
- If $\mathbf{x}$ is an eigenvector, then $T(\mathbf{x}) = A\mathbf{x} = \lambda\mathbf{x}$, which means the output vector has **the same direction** as $\mathbf{x}$, but the length is scaled by a factor of λ.

- **Example:** For $A = \begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix}$, an eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ since

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \end{bmatrix} = 8\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A \quad \mathbf{x} \quad = \quad 8 \; \mathbf{x}$$



Computational Advantage

- Which computation is faster between $\begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $8 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

Eigenvectors and Eigenvalues

- The equation $A\mathbf{x} = \lambda\mathbf{x}$ can be re-written as

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

- λ is an eigenvalue of an $n \times n$ matrix A if and only if this equation has a **nontrivial** solution (since $\mathbf{x}$ should be a nonzero vector).

Eigenvectors and Eigenvalues

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

- The set of *all* solutions of the above equation is the **null space** of the matrix $(A - \lambda I)$, which we call the **eigenspace** of A **corresponding to λ**.
- The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ, satisfying the above equation.

Null Space

- **Definition**: The **null space** of a matrix $A \in \mathbb{R}^{m \times n}$ is the set of all solutions of a homogeneous linear system, $A\mathbf{x} = \mathbf{0}$.

  We denote the null space of A as Nul A.

- For $A = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}$, $\mathbf{x}$ should satisfy the following:

$$\mathbf{a}_1^T\mathbf{x} = 0, \mathbf{a}_2^T\mathbf{x} = 0, \dots, \mathbf{a}_m^T\mathbf{x} = 0$$

- That is, $\mathbf{x}$ should be orthogonal to every row vector in A.

Null Space is a Subspace

- **Theorem**: The null space of a matrix $A \in \mathbb{R}^{m \times n}$, denoted as Nul A is a subspace of $\mathbb{R}^n$. In other words, the set of all the solutions of a system $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{R}^n$.

- **Note**: An eigenspace thus have a set of basis vectors with a particular dimension.

Orthogonal Complement

- If a vector $\mathbf{z}$ is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, then $\mathbf{z}$ is said to be **orthogonal to** W.
- The set of all vectors $\mathbf{z}$ that are orthogonal to W is called the **orthogonal complement** of W and is denoted by $W^\perp$ (and read as "W perpendicular" or simply "W perp").
- A vector $\mathbf{x} \in \mathbb{R}^n$ is in $W^\perp$ if and only if $\mathbf{x}$ is orthogonal to every vector in a set that spans W.
- $W^\perp$ is a subspace of $\mathbb{R}^n$.
- Nul $A = (\text{Row } A)^\perp$.
- Likewise, Nul $A^T = (\text{Col } A)^\perp$.

Fundamental Subspaces Given by A

- Nul $A = (\text{Row } A)^{\perp}$.
- Nul $A^T = (\text{Col } A)^{\perp}$.

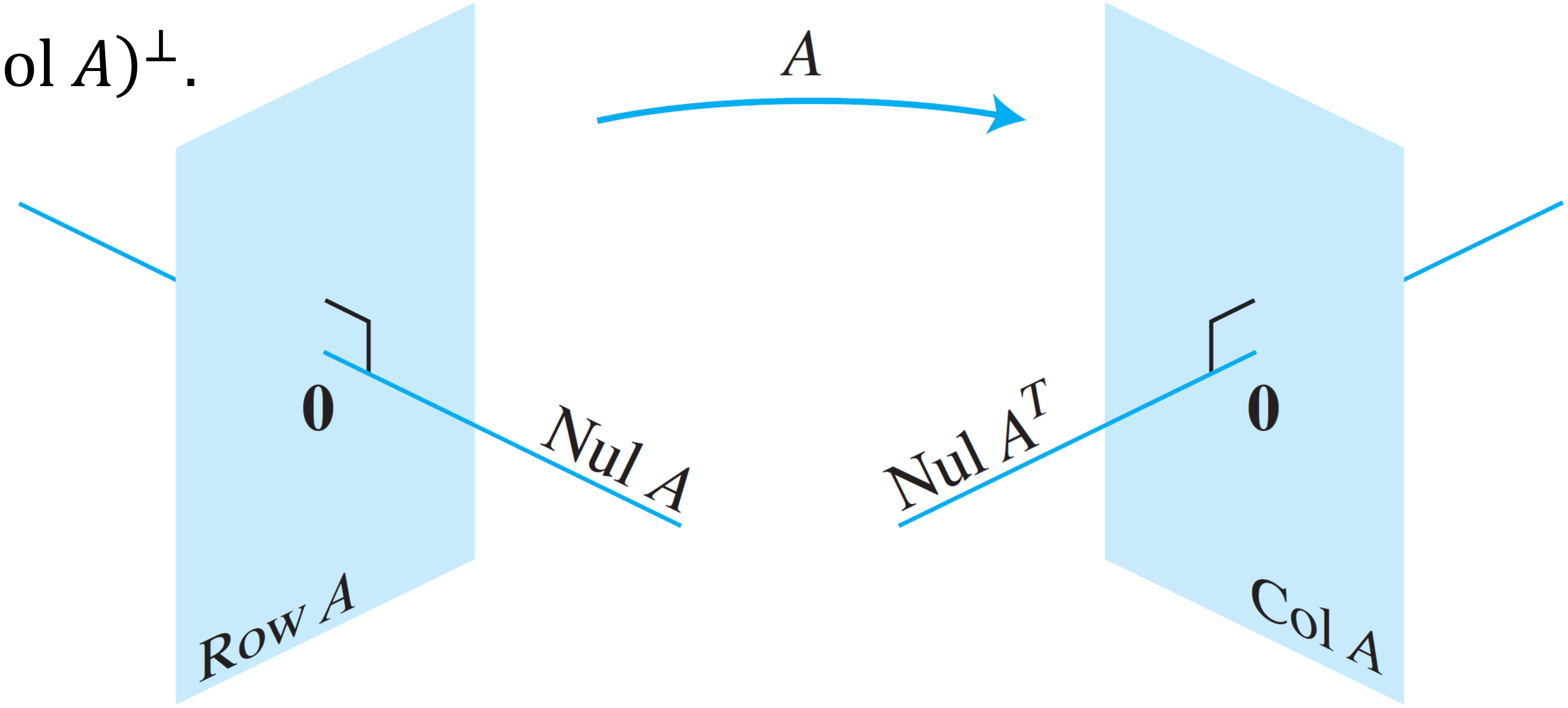


FIGURE 8 The fundamental subspaces determined by an $m \times n$ matrix A.

Example: Eigenvalues and Eigenvectors

- **Example**: Show that 8 is an eigenvalue of a matrix

  $A = \begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix}$ and find the corresponding eigenvectors.

- **Solution**: The scalar 8 is an eigenvalue of A if and only if the equation $(A - 8I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution:

$$(A - 8I)\mathbf{x} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

- The solution is $\mathbf{x} = c\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for any nonzero scalar c, which is Span $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

Example: Eigenvalues and Eigenvectors

- In the previous example, -3 is also an eigenvalue:

$$(A+3I)\mathbf{x} = \begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

- The solution is $\mathbf{x} = c\begin{bmatrix} 1 \\ -5/6 \end{bmatrix}$ for any nonzero scalar c, which is Span $\left\{\begin{bmatrix} 1 \\ -5/6 \end{bmatrix}\right\}$.

Characteristic Equation

- How can we find the eigenvalues such as 8 and -3?
- If $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then the columns of $(A - \lambda I)$ should be noninvertible.
- If it is invertible, $\mathbf{x}$ cannot be a nonzero vector since
$$(A - \lambda I)^{-1}(A - \lambda I)\mathbf{x} = (A - \lambda I)^{-1}\mathbf{0} \Longrightarrow \mathbf{x} = \mathbf{0}$$
- Thus, we can obtain eigenvalues by solving
$$\det(A - \lambda I) = 0$$
called a **characteristic equation**.
- Also, the solution is not unique, and thus $A - \lambda I$ has linearly dependent columns.

Example: Characteristic Equation

- In the previous example, $A = \begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix}$ is originally invertible since

$$\det(A) = \det \begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix} = 6 - 30 = -24 \neq 0.$$

- By solving the characteristic equation, we want to find λ that makes $A - \lambda I$ non-invertible:

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 6 \\ 5 & 3-\lambda \end{bmatrix}$$

$$= (2-\lambda)(3-\lambda) - 30$$

$$= -\lambda^2 - 5\lambda - 25 = (8-\lambda)(-3-\lambda) = 0$$

$\lambda = -3$ or 8

Example: Characteristic Equation

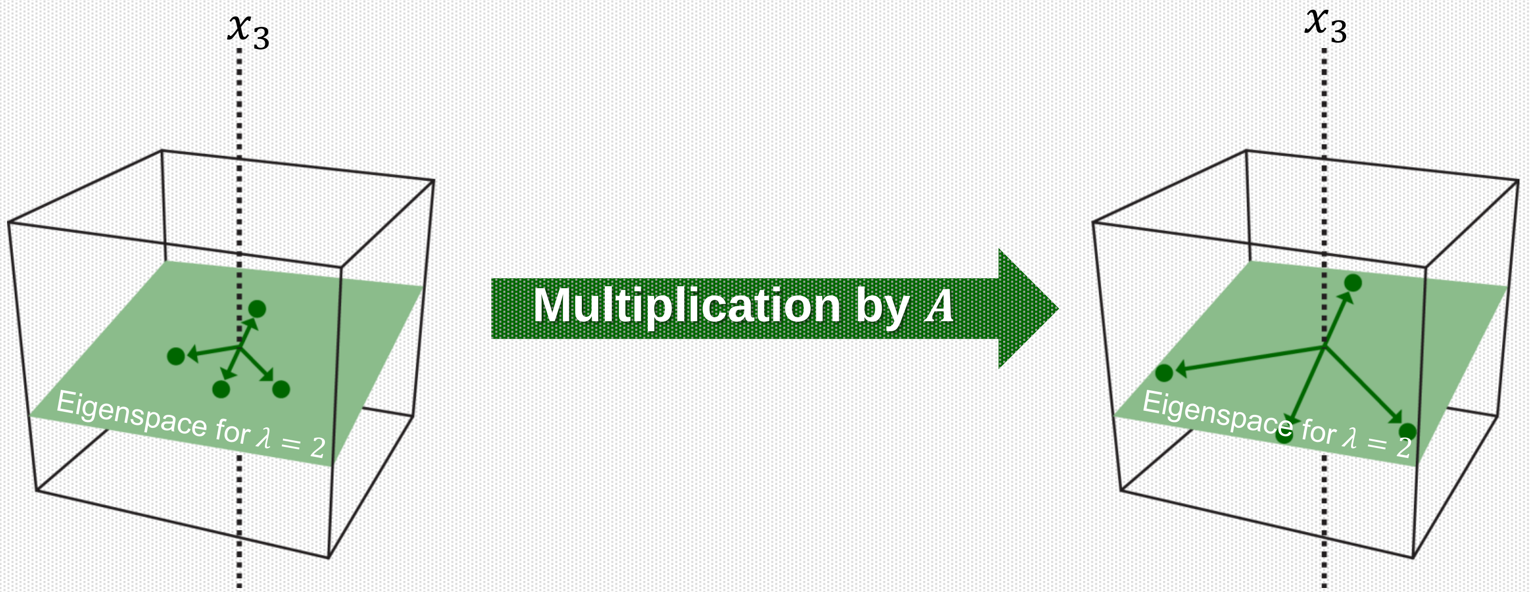
- Once obtaining eigenvalues, we compute the eigenvectors for each λ by solving

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

Eigenspace

- Note that the dimension of the eigenspace (corresponding to a particular λ) can be more than one. In this case, any vector in the eigenspace satisfies

$$T(\mathbf{x}) = A\mathbf{x} = \lambda\mathbf{x}$$



A acts as a dilation on the eigenspace.

Finding all eigenvalues and eigenvectors

- In summary, we can find all the possible eigenvalues and eigenvectors, as follows.
- First, find all the eigenvalue by solving the **characteristic equation**:

$$\det(A - \lambda I) = 0$$

- Second, for each eigenvalue λ, solve for $(A - \lambda I)\mathbf{x} = \mathbf{0}$ and obtain the set of basis vectors of the corresponding eigenspace.