Chapter 11

Nonplanar Graphs

We saw in Chapter 10 that if G is a maximal planar graph containing two nonadjacent vertices u and v, then the graph $G + uv$ obtained by adding the edge uv is nonplanar. Although nonplanar, the graph $G + uv$ is very close to being planar. We now look at various ways of measuring how close nonplanar graphs are to being planar.

11.1 The Crossing Number of a Graph

Nonplanar graphs cannot, of course, be embedded in the plane. Hence, whenever one attempts to draw a nonplanar graph in the plane, some of its edges must cross. This rather simple observation leads to a concept.

The **crossing number** $\text{cr}(G)$ of a graph G is the minimum number of crossings (of its edges) among the drawings of G in the plane. Before proceeding further, we comment on assumptions we are making regarding all drawings under consideration. In particular, we assume that

- adjacent edges never cross
- two nonadjacent edges cross at most once
- no edge crosses itself
- no more than two edges cross at a point of the plane
- the (open) arc in the plane corresponding to an edge of the graph contains no vertex of the graph.

A few observations will prove useful. If $G \subseteq H$, then $\text{cr}(G) \leq \text{cr}(H)$; while if H is a subdivision of G, then $\text{cr}(G) = \text{cr}(H)$. A graph G is planar if and only if $\text{cr}(G) = 0$. In particular, if G is a maximal planar graph of order $n \geq 3$ and size m, then $m = 3n - 6$ and $\text{cr}(G) = 0 = m - 3n + 6$. If $m > 3n - 6$ and so $m - 3n + 6 > 0$, then G is nonplanar and so $\text{cr}(G) \geq 1$. In fact, the number

$m - 3n + 6$ provides a lower bound for the crossing number of a graph of order $n \geq 3$ and size m.

Theorem 11.1 *If G is a graph of order $n \geq 3$ and size m, then*

$$\operatorname{cr}(G) \geq m - 3n + 6.$$

Proof. Let there be given a drawing of G in the plane with $\operatorname{cr}(G) = c$ crossings. At each crossing, a new vertex is introduced, producing a plane graph H of order $n + c$ and size $m + 2c$. Since H is planar, it follows by Theorem 10.3 that

$$m + 2c \leq 3(n + c) - 6.$$

Thus, $\operatorname{cr}(G) = c \geq m - 3n + 6$. ■

While the lower bound for the crossing number of a graph can be useful in determining $\operatorname{cr}(G)$ for certain graphs G, this bound can differ significantly from $\operatorname{cr}(G)$. For example, for large integers s and t, let $H = P_s \,\square\, P_t$. Then this planar graph H can be embedded in the plane where one region has a $(2s + 2t - 4)$-cycle for its boundary, while the boundary of all other regions are 4-cycles. Edges can be added to H to produce a maximal planar graph G. By appropriately selecting nonadjacent vertices x and y of G, it follows that $\operatorname{cr}(G + xy) \geq 1$ by Theorem 11.1 but $G + xy$ can have a large crossing number.

Crossing Numbers of Complete Graphs

One class of graphs whose crossing number has been a subject of study is complete graphs. By Theorem 11.1, it follows for $n \geq 3$ that

$$\operatorname{cr}(K_n) \geq \binom{n}{2} - 3n + 6 = \frac{(n-3)(n-4)}{2}. \tag{11.1}$$

Richard K. Guy [112] discovered an even better lower bound for $\operatorname{cr}(K_n)$.

Theorem 11.2 *For $n \geq 5$, $\operatorname{cr}(K_n) \geq \frac{1}{5}\binom{n}{4}$.*

Proof. We proceed by induction on n. For $n = 5$, we have $\frac{1}{5}\binom{n}{4} = \frac{1}{5}\binom{5}{4} = 1$. Since K_5 is nonplanar, $\operatorname{cr}(K_5) \geq 1$. In fact, the drawing of K_5 in Figure 11.1 with one crossing shows that $\operatorname{cr}(K_5) = 1$.

Assume that $\operatorname{cr}(K_{n-1}) \geq \frac{1}{5}\binom{n-1}{4}$ for an integer $n \geq 6$. Let there be a drawing of K_n in the plane with $\operatorname{cr}(K_n)$ crossings. When a vertex of K_n is deleted, a drawing of K_{n-1} is obtained, where the number of crossings in this copy of K_{n-1} is at least $\operatorname{cr}(K_{n-1})$. Hence, the n vertex-deleted subgraphs of K_n produce n graphs K_{n-1} having a total of at least $n \operatorname{cr}(K_{n-1})$ crossings.

A crossing in K_n involves two nonadjacent edges, say uv and xy. This crossing occurs in every vertex-deleted subgraph K_{n-1} of K_n except when u, v, x

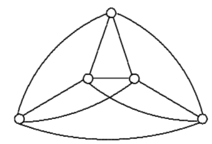

Figure 11.1: A drawing of K_5 with one crossing

or y is deleted; that is, this crossing occurs in $n-4$ subgraphs K_{n-1} of K_n. Thus, the total number of crossings in these n drawings of vertex-deleted subgraphs K_{n-1} of K_n is $(n-4)\,\mathrm{cr}(K_n)$. Hence,

$$(n-4)\,\mathrm{cr}(K_n) \geq n\,\mathrm{cr}(K_{n-1}) \geq \frac{n}{5}\binom{n-1}{4}.$$

Therefore,

$$\mathrm{cr}(K_n) \geq \frac{n}{5(n-4)}\binom{n-1}{4} = \frac{1}{5}\binom{n}{4},$$

as desired. ■

The lower bounds for $\mathrm{cr}(K_n)$ given in (11.1) and in Theorem 11.2 are the same when $n = 5$ and $n = 6$, while

$$\frac{1}{5}\binom{n}{4} > \frac{(n-3)(n-4)}{2}$$

when $n \geq 7$. Thus the lower bound $\mathrm{cr}(K_n) \geq \frac{1}{5}\binom{n}{4}$ is an improvement over that given in (11.1). When $n = 6$, these bounds state that $\mathrm{cr}(K_6) \geq 3$. The drawing of K_6 with three crossings in Figure 11.2 shows that $\mathrm{cr}(K_6) = 3$.

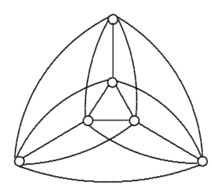

Figure 11.2: A drawing of K_6 with three crossings

It has been shown by Jaroslav Blažek and Milan Koman [30] and Richard K. Guy [111], among others, that for complete graphs,

$$\mathrm{cr}(K_n) \leq \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor. \qquad (11.2)$$

Guy conjectured, in fact, that the upper bound in (11.2) is, in fact, the crossing number of K_n for all positive integers n. As far as known results are concerned, the best obtained is the following.

Theorem 11.3 *For* $1 \le n \le 12$,

$$\mathrm{cr}(K_n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor. \tag{11.3}$$

Guy [112] established Theorem 11.3 for $1 \le n \le 10$, while Shengjun Pan and R. Bruce Richter [182] verified Theorem 11.3 for $n = 11$ and $n = 12$.

Turán's Brick-Factory Problem

We now turn to the crossing number of complete bipartite graphs. The problem of determining $\mathrm{cr}(K_{s,t})$ has a memorable history. This problem is sometimes referred to as **Turán's Brick-Factory Problem**, named for the Hungarian mathematician Paul Turán (1910–1976).

Born in Budapest, Hungary, Paul Turán displayed remarkable mathematical ability at a very early age. Turán was one of many young students in Budapest who studied graph theory under Dénes König. Turán met Paul Erdős in September 1930 and became and remained friends with him. Turán received a Ph.D. in 1935 from the University of Budapest, with a dissertation in number theory. By the end of 1935, Turán had seven papers in print. Despite his outstanding record at such an early age, Turán had great difficulty securing a faculty position because of his Jewish heritage. He could only make a living as a private mathematics tutor although he continued his research.

While Turán finally secured a position in 1938 (as a school teacher), his personal situation had grown worse. After the German invasion of Poland which began World War II, Hungary was not involved in the war at first but was nevertheless greatly influenced by Nazi policies. In 1940 Turán was sent to a labor camp. Indeed, Turán was in and out of several labor camps during the war. Turán himself [238] wrote:

> *We worked near Budapest, in a brick factory. There were some kilns where the bricks were made and some open storage yards where the bricks were stored. All the kilns were connected by rail with all the storage yards. The bricks were carried on small wheeled trucks to the storage yards. All we had to do was to put the bricks on the trucks at the kilns, push the trucks to the storage yards, and unload them there. We had a reasonable piece rate for the trucks, and the work itself was not difficult; the trouble was only at the crossings. The trucks generally jumped the rails there, and the bricks fell out of them; in short this caused a lot of trouble and loss of time which was precious to all of us. We were all sweating and cursing at such*

> *occasions, I too; but nolens volens the idea occurred to me that this loss of time could have been minimized if the number of crossings of the rails had been minimized. But what is the minimum number of crossings? I realized after several days that the actual situation could have been improved, but the exact solution of the general problem with s kilns and t storage yards seemed to be very difficult ... the problem occurred to me again ... at my first visit to Poland where I met Zarankiewicz. I mentioned to him my 'brick-factory'–problem ... and Zarankiewicz thought to have solved (it). But Ringel found a gap in his published proof, which nobody has been able to fill so far – in spite of much effort. This problem has also become a notoriously difficult unsolved problem ....*

Crossing Numbers of Complete Bipartite Graphs

Kazimierz Zarankiewicz [262] thought he had proved that

$$\operatorname{cr}(K_{s,t}) = \left\lfloor \frac{s}{2} \right\rfloor \left\lfloor \frac{s-1}{2} \right\rfloor \left\lfloor \frac{t}{2} \right\rfloor \left\lfloor \frac{t-1}{2} \right\rfloor \tag{11.4}$$

but, in actuality, he had only verified that the right-hand expression of (11.4) is an upper bound for $\operatorname{cr}(K_{s,t})$. As it turned out, both Paul C. Kainen and Gerhard Ringel found flaws in Zarankiewicz's argument. Hence, (11.4) remains only a conjecture. The best general result on crossing numbers of complete bipartite graphs is the following, due to the combined work of Daniel Kleitman [144] and Douglas Woodall [260].

Theorem 11.4 *If s and t are positive integers with $s \le t$ and either* (1) $s \le 6$ *or* (2) $s = 7$ *and $t \le 10$, then*

$$\operatorname{cr}(K_{s,t}) = \left\lfloor \frac{s}{2} \right\rfloor \left\lfloor \frac{s-1}{2} \right\rfloor \left\lfloor \frac{t}{2} \right\rfloor \left\lfloor \frac{t-1}{2} \right\rfloor.$$

It follows, therefore, from Theorem 11.4 that

$$\operatorname{cr}(K_{3,t}) = \left\lfloor \frac{t}{2} \right\rfloor \left\lfloor \frac{t-1}{2} \right\rfloor, \qquad \operatorname{cr}(K_{4,t}) = 2 \left\lfloor \tfrac{t}{2} \right\rfloor \left\lfloor \tfrac{t-1}{2} \right\rfloor,$$

$$\operatorname{cr}(K_{5,t}) = 4 \left\lfloor \frac{t}{2} \right\rfloor \left\lfloor \frac{t-1}{2} \right\rfloor \quad \text{and} \quad \operatorname{cr}(K_{6,t}) = 6 \left\lfloor \tfrac{t}{2} \right\rfloor \left\lfloor \tfrac{t-1}{2} \right\rfloor$$

for all t. For example, $\operatorname{cr}(K_{3,3}) = 1$, $\operatorname{cr}(K_{4,4}) = 4$, $\operatorname{cr}(K_{5,5}) = 16$, $\operatorname{cr}(K_{6,6}) = 36$ and $\operatorname{cr}(K_{7,7}) = 81$. A drawing of $K_{4,4}$ with four crossings is shown in Figure 11.3.

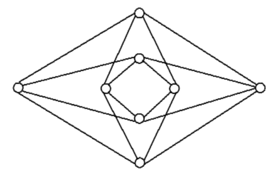

Figure 11.3: A drawing of $K_{4,4}$ with four crossings

As would be expected, the situation regarding crossing numbers of complete k-partite graphs, $k \geq 3$, is even more complicated. For the most part, only bounds and highly specific results have been obtained in these cases. On the other hand, some of the proof techniques employed have been enlightening. As an example, the following result of Arthur T. White [254, p. 67] establishes the crossing number of $K_{2,2,3}$.

Theorem 11.5 *The crossing number of* $K_{2,2,3}$ *is* 2.

Proof. The graph $K_{2,2,3}$ has order 7 and size 16. Suppose that $\mathrm{cr}(K_{2,2,3}) = c$. Since $K_{3,3}$ is nonplanar and $K_{3,3} \subseteq K_{2,2,3}$, it follows that $K_{2,2,3}$ is nonplanar and so $c \geq 1$. Let there be given a drawing of $K_{2,2,3}$ in the plane with c crossings. At each crossing we introduce a new vertex, producing a connected plane graph G of order $n = 7 + c$ and size $m = 16 + 2c$. By Theorem 10.3, $m \leq 3n - 6$.

Let u_1u_2 and v_1v_2 be two (nonadjacent) edges of $K_{2,2,3}$ that cross in the given drawing, giving rise to a new vertex. If G is a maximal planar graph, then $C = (u_1, v_1, u_2, v_2, u_1)$ is a cycle of G, implying that the subgraph induced by $\{u_1, u_2, v_1, v_2\}$ in $K_{2,2,3}$ is K_4. However, $K_{2,2,3}$ contains no such subgraph; thus, G is not a maximal planar graph and so $m < 3n - 6$. Therefore,

$$16 + 2c < 3(7 + c) - 6,$$

from which it follows that $c \geq 2$. The inequality $c \leq 2$ follows from the fact that there exists a drawing of $K_{2,2,3}$ with two crossings (see Figure 11.4). ■

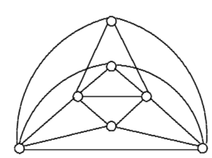

Figure 11.4: A drawing of $K_{2,2,3}$ with two crossings

Crossing Numbers of Cartesian Products of Graphs

Other graphs whose crossing numbers have been investigated with little success are the n-cubes Q_n. Since Q_n is planar for $n = 1, 2, 3$, $\text{cr}(Q_n) = 0$ for each such n. Roger Eggleton and Richard Guy [76] have shown that $\text{cr}(Q_4) = 8$ but $\text{cr}(Q_n)$ is unknown for $n \geq 5$. One might observe that

$$Q_4 = K_2 \,\square\, K_2 \,\square\, K_2 \,\square\, K_2 = C_4 \,\square\, C_4$$

so that $\text{cr}(C_4 \,\square\, C_4) = 8$. This raises the question of determining $\text{cr}(C_s \,\square\, C_t)$ for $s, t \geq 3$. For the case $s = t = 3$, Frank Harary, Paul Kainen and Allen Schwenk [120] showed that $\text{cr}(C_3 \,\square\, C_3) = 3$. Their proof consisted of the following three steps:

Step 1. Exhibiting a drawing of $C_3 \,\square\, C_3$ with three crossings so that $\text{cr}(C_3 \,\square\, C_3) \leq 3$.

Step 2. Showing that $C_3 \,\square\, C_3 - e$ is nonplanar for every edge e of $C_3 \,\square\, C_3$ so that $\text{cr}(C_3 \,\square\, C_3) \geq 2$.

Step 3. Showing, by case exhaustion, that it is impossible to have a drawing of $C_3 \,\square\, C_3$ with exactly two crossings so that $\text{cr}(C_3 \,\square\, C_3) \geq 3$ (see Exercise 6).

Richard D. Ringeisen and Lowell W. Beineke [200] then significantly extended this result by determining $\text{cr}(C_3 \,\square\, C_t)$ for all integers $t \geq 3$.

Theorem 11.6 *For each integer $t \geq 3$,*

$$\text{cr}(C_3 \,\square\, C_t) = t.$$

Proof. We label the vertices of $C_3 \,\square\, C_t$ by the $3t$ ordered pairs $(0, j), (1, j)$ and $(2, j)$, where $j = 0, 1, \ldots, t-1$, and, for convenience, we let

$$u_j = (0, j),\ v_j = (1, j) \text{ and } w_j = (2, j).$$

First, we note that $\text{cr}(C_3 \,\square\, C_t) \leq t$. This observation follows from the fact that there exists a drawing of $C_3 \,\square\, C_t$ with t crossings. A drawing of $C_3 \,\square\, C_4$ with four crossings is shown in Figure 11.5. Drawings of $C_3 \,\square\, C_t$ with t crossings for other values of t can be given similarly.

To complete the proof, we show that $\text{cr}(C_3 \,\square\, C_t) \geq t$. We verify this by induction on $t \geq 3$. For $t = 3$, we recall the previously mentioned result $\text{cr}(C_3 \,\square\, C_3) = 3$.

Assume that $\text{cr}(C_3 \,\square\, C_k) \geq k$ for some integer $k \geq 3$, and consider the graph $C_3 \,\square\, C_{k+1}$. We show that $\text{cr}(C_3 \,\square\, C_{k+1}) \geq k + 1$. Let there be given a drawing of $C_3 \,\square\, C_{k+1}$ with $\text{cr}(C_3 \,\square\, C_{k+1})$ crossings. We consider two cases.

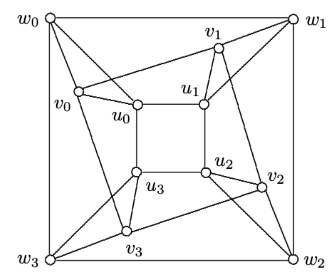

Figure 11.5: A drawing of $C_3 \square C_4$ with four crossings

Case 1. *No edge of any triangle* $T_j = G[\{u_j, v_j, w_j\}]$, $j = 0, 1, \ldots, k$, *is crossed.* For $j = 0, 1, \ldots, k$, define

$$H_j = G[\{u_j, v_j, w_j, u_{j+1}, v_{j+1}, w_{j+1}\}],$$

where the subscripts are expressed modulo $k+1$. We show that for each $j = 0, 1, \ldots, k$, the number of times edges of H_j are crossed totals at least 2. Since, by assumption, no triangle T_j has an edge crossed and since every edge not in any triangle T_j belongs to exactly one subgraph H_j, it will follow that there are at least $k+1$ crossings in the drawing because then every crossing of an edge in H_j involves either two edges of H_j or an edge of H_j and an edge of H_i for some $i \neq j$.

If two of the edges $u_j u_{j+1}$, $v_j v_{j+1}$ and $w_j w_{j+1}$ cross each other, then two edges of H_j are crossed. Assume then that no two edges of H_j cross each other. Thus, H_j is a plane subgraph in the drawing of $C_3 \square C_{k+1}$ (see Figure 11.6). The triangle T_{j+2} must lie within some region of H_j. If T_{j+2} lies in a region of H_j bounded by a triangle, say T_j, then at least one edge of the cycle $(u_0, u_1, \ldots, u_k, u_0)$, for example, must cross an edge of T_j, contradicting our assumption. Thus, T_{j+2} must lie in a region of H_j bounded by a 4-cycle, say $(u_j, u_{j+1}, w_{j+1}, w_j, u_j)$. However then, edges of the cycle $(v_0, v_1, \ldots, v_k, v_0)$ must cross edges of the cycle $(u_j, u_{j+1}, w_{j+1}, w_j, u_j)$ at least twice and, hence, edges of H_j at least twice, as asserted.

Case 2. *Some triangle, say* T_0, *has at least one of its edges crossed.* Suppose that $\operatorname{cr}(C_3 \square C_{k+1}) < k+1$. Then the graph $C_3 \square C_{k+1} - E(T_0)$, which is a subdivision of $C_3 \square C_k$, is drawn with fewer than k crossings, contradicting the inductive hypothesis. ∎

The only other result giving the crossing number of graphs $C_s \square C_t$ is the following theorem of Beineke and Ringeisen [19].

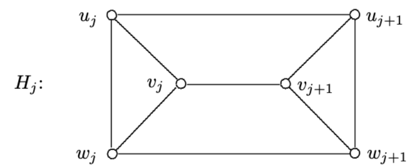

Figure 11.6: The subgraph H_j in the proof of Theorem 11.6

Theorem 11.7 *For all* $t \geq 4$,

$$\mathrm{cr}(C_4 \,\Box\, C_t) = 2t.$$

Beineke and Ringeisen [19] also determined $\mathrm{cr}(K_4 \,\Box\, C_t)$ for $t \geq 4$.

Theorem 11.8 *For all* $t \geq 4$,

$$\mathrm{cr}(K_4 \,\Box\, C_t) = 3t.$$

Fáry's Theorem

In a planar embedding of a graph G, an edge of G can be any curve, including a straight-line segment. The **rectilinear crossing number** $\overline{\mathrm{cr}}(G)$ of a graph G is the minimum number of crossings among all those drawings of G in the plane in which each edge is a straight-line segment. Since the crossing number $\mathrm{cr}(G)$ considers all drawings of G in the plane (not just those for which edges are straight-line segments), we have the obvious inequality

$$\mathrm{cr}(G) \leq \overline{\mathrm{cr}}(G). \tag{11.5}$$

Clearly, $\overline{\mathrm{cr}}(G) \geq 0$ for every planar graph G. However, an interesting feature of planar graphs is that they can be embedded in the plane so that every edge is a straight-line segment. Such an embedding is referred to as a **straight-line embedding**. This result is known as Fáry's theorem but was proved independently not only by István Fáry [89] but by Sherman K. Stein [225] and Klaus Wagner [250] as well.

Theorem 11.9 (**Fáry's Theorem**) *If G is a planar graph, then*

$$\overline{\mathrm{cr}}(G) = 0.$$

Proof. If the rectilinear crossing number of every maximal planar graph is 0, then the rectilinear crossing number of every planar graph is 0. Hence, it suffices to prove the theorem for maximal planar graphs. This result is obvious for K_1 and K_2.

We prove by induction on $n \geq 3$ that for every maximal plane graph G of order n, the boundary of whose exterior region contains the vertices u, v and w, there exists a straight-line embedding of G, each region of which has the same boundary as the given planar embedding of G, and whose exterior region has boundary vertices u, v and w. The result is certainly true for $n = 3$ and $n = 4$. Assume that the statement is true for all maximal plane graphs of order k for some integer $k \geq 4$. Let G be a maximal plane graph of order $k + 1$ whose exterior region has boundary vertices u, v and w.

By Corollary 10.5, G contains a vertex $x \notin \{u, v, w\}$ such that $\deg x = r$ and $3 \leq r \leq 5$. Let $N_G(x) = \{x_1, x_2, \ldots, x_r\}$. Remove x from G and let R be the region of $G - x$ whose boundary vertices are $N_G(x)$. Add $r - 3$ edges to the region R of $G - x$ so that a maximal plane graph G' results. Since G' is a maximal plane graph of order k, it follows by the induction hypothesis that there is a straight-line embedding of G' resulting in a graph G'' each region of which has the same boundary and whose exterior region has boundary vertices u, v and w.

Now remove the $r-3$ edges that were added to $G-x$ to produce a straight-line embedding G^* of $G - x$ such that the boundary of the region R^* with boundary vertices $N_G(x)$ is an r-gon, where $3 \leq r \leq 5$. If the r-gon is convex, then the vertex x can be added anywhere in R^* and joined to the vertices of $N_G(x)$ by straight-line segments, producing a straight-line embedding of G.

Suppose that the r-gon P is not convex and so $r = 4$ or $r = 5$. We then triangulate P by adding $r-3$ straight-line segments. If $r = 4$, then one straight-line segment is added. Place a new vertex x on this segment and remove this segment. Straight-line segments can then be drawn from x to each of the vertices x_i $(1 \leq i \leq 4)$, producing a straight-line embedding of G. See Figure 11.7.

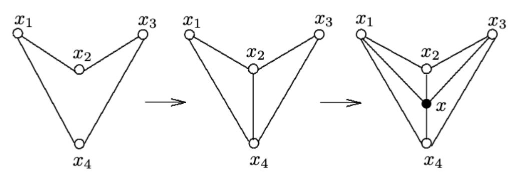

Figure 11.7: A step $(r = 4)$ in the proof of Theorem 11.9

If P is a pentagon, then two straight-line segments can be added to triangulate P, producing three triangles. One of the five vertices of P, say x_1, lies on each of these three triangles. One of these triangles, say T, is a neighbor of the other two triangles. Since a triangle is convex, a straight-line segment can be drawn in T and so in P to each of x_2, x_3, x_4 and x_5 (see Figure 11.8), again producing a straight-line embedding of G. ■

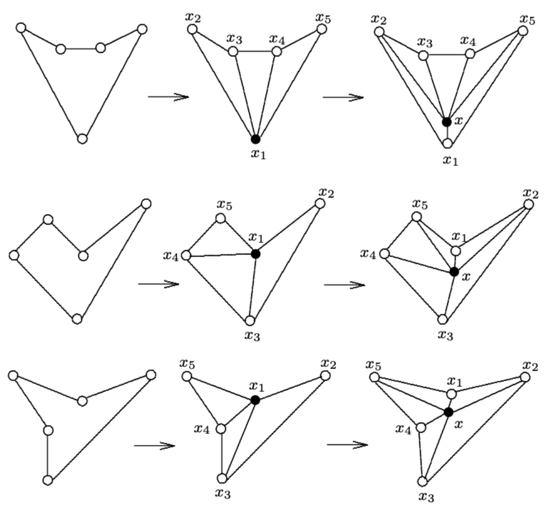

Figure 11.8: A step ($r = 5$) in the proof of Theorem 11.9

The Art Gallery Problem

By Fáry's Theorem, there is a straight-line embedding of every planar graph. In the proof of this theorem, we used the fact that within the interior of each triangle, quadrilateral and pentagon, a point can be placed that can be joined by straight-line segments to the vertices of the polygon P so that they lie within the interior of P. Actually this is a special case of a result from geometry:

The Art Gallery Problem

> *Suppose that a certain art gallery consists of a single large room with n walls on which paintings are hung. What is the minimum number of security guards that must be hired and stationed in the gallery to guarantee that for every painting hung on a wall there is a guard who has straight line vision of the artwork?*

This problem was posed in 1973 by the geometer Victor Klee after a discussion with Vašek Chvátal. It was shown by Chvátal [56], with a simpler proof by Steve Fisk [91], that no more than $\lfloor n/3 \rfloor$ guards are needed and that examples exist where $\lfloor n/3 \rfloor$ guards are required. In the case of an art gallery with three, four or five walls, only one guard need be hired, which is the fact used in the proof of Theorem 11.9.

It has been conjectured that $\mathrm{cr}(K_{s,t}) = \overline{\mathrm{cr}}(K_{s,t})$ for all positive integers s and t. In the case of complete graphs, not only is $\overline{\mathrm{cr}}(K_n) = \mathrm{cr}(K_n)$ for $1 \le n \le 4$ (when K_n is planar), it is also known that $\overline{\mathrm{cr}}(K_n) = \mathrm{cr}(K_n)$ for $5 \le n \le 7$ and $n = 9$. However, $\mathrm{cr}(K_8) = 18$ and $\overline{\mathrm{cr}}(K_8) = 19$ (see Guy [112]), so strict inequality in (11.5) is indeed a possibility. Furthermore, Alex Brodsky, Stephane Durocher and Ellen Gethner [37] showed that $\overline{\mathrm{cr}}(K_{10}) = 62$, while $\mathrm{cr}(K_{10}) = 60$. The members Oswin Aichholzer, Franz Aurenhammer and Hannes Krasser of the *Rectilinear Crossing Number Project* determined $\overline{\mathrm{cr}}(K_n)$ for all n with $11 \le n \le 21$ except $n = 20$. In particular

- $\mathrm{cr}(K_{11}) = 100$ and $\overline{\mathrm{cr}}(K_{11}) = 102$ and
- $\mathrm{cr}(K_{12}) = 150$ and $\overline{\mathrm{cr}}(K_{12}) = 153$.

In a paper by Bernardo M. Ábrego, Silvia Fernández-Merchant and Gelasio Salazar [1], the values of $\overline{\mathrm{cr}}(K_n)$ are given for $5 \le n \le 30$, including $\overline{\mathrm{cr}}(K_{30}) = 9726$.

11.2 The Genus of a Graph

While only planar graphs can be embedded in the plane, there is a host of common and increasingly complex surfaces on which a nonplanar graph G might possibly be embedded. Determining the simplest of these surfaces on which G can be embedded gives an indication of how close G is to being planar. We will see that the embeddings of interest are those called 2-cell embeddings and discuss the possible surfaces on which a given graph has such an embedding.

We have seen that a graph G is planar if G can be drawn in the plane in such a way that no two edges cross and that such a drawing is called an embedding of G in the plane or a planar embedding. Furthermore, a graph G can be embedded in the plane if and only if G can be embedded on (the surface of) a sphere.

Of course, not all graphs are planar. Indeed, Kuratowski's theorem (Theorem 10.18) and Wagner's theorem (Theorem 10.21) describe conditions (involving the two nonplanar graphs K_5 and $K_{3,3}$) under which G can be embedded in the plane. Graphs that are not embeddable in the plane (or on a sphere) may be embeddable on other surfaces, however.

Embeddings on a Torus

A common surface on which a graph may be embedded is the **torus**, a doughnut-shaped surface (see Figure 11.9(a)). Two different embeddings of the (planar) graph K_4 on a torus are shown in Figures 11.9(b) and 11.9(c).

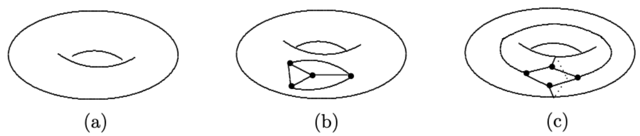

Figure 11.9: Embedding K_4 on a torus

While it is easy to see that every planar graph can be embedded on a torus, some nonplanar graphs can be embedded on a torus as well. For example, embeddings of K_5 and $K_{3,3}$ on a torus are shown in Figures 11.10(a) and 11.10(b).

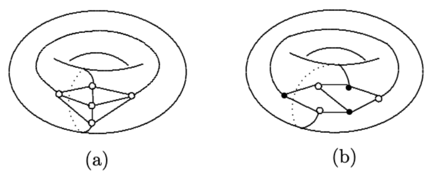

Figure 11.10: Embedding K_5 and $K_{3,3}$ on a torus

Another way to represent a torus and to visualize an embedding of a graph on a torus is to begin with a rectangular piece of (flexible) material as in Figure 11.11 and first make a cylinder from it by identifying sides a and c, which are the same after the identification occurs. Sides b and d are then circles. These circles are then identified to produce a torus.

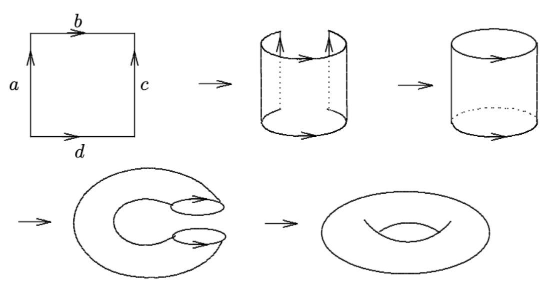

Figure 11.11: Constructing a torus

After seeing how a torus can be constructed from a rectangle, it follows that the points labeled A in the rectangle in Figure 11.12(a) represent the same point on the torus. This is also true of the points labeled B and the points labeled C. Figures 11.12(b) and 11.12(c) show embeddings of K_5 and $K_{3,3}$ on the torus. There are five regions in the embedding of K_5 on the torus shown in Figure 11.12(b) and R is a single region in this embedding. Moreover, there are three regions in the embedding of $K_{3,3}$ on the torus shown in Figure 11.12(c) and R' is a single region.

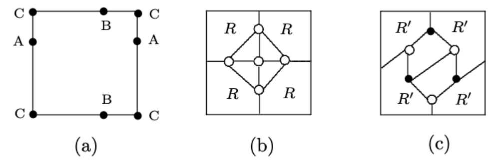

Figure 11.12: Embedding K_5 and $K_{3,3}$ on a torus

Another way to represent a torus and an embedding of a graph on a torus is to begin with a sphere, insert two holes in its surface (as in Figure 11.13(a)) and attach a handle on the sphere, where the ends of the handle are placed over the two holes (as in Figure 11.13(b)). An embedding of K_5 on the torus constructed in this manner is shown in Figure 11.13(c).

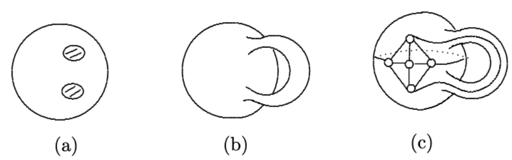

Figure 11.13: Embedding K_5 on a torus

While a torus is a sphere with one handle, a sphere with k handles, $k \geq 0$, is called a **surface of genus** k and is denoted by S_k. Thus, S_0 is a sphere and S_1 is a torus. The surfaces S_k are the **orientable surfaces**.

Let G be a nonplanar graph. When drawing G on a sphere, some edges of G will cross. The graph G can always be drawn so that only two edges cross at any point of intersection. At each such point of intersection, a handle can be suitably placed on the sphere so that one of these two edges passes over the handle and the intersection of the two edges has been avoided. Consequently, every graph can be embedded on some orientable surface. The smallest nonnegative integer k such that a graph G can be embedded on S_k is called the **genus** of G and is denoted by $\gamma(G)$. Therefore, $\gamma(G) = 0$ if and only if G is planar; while

$\gamma(G) = 1$ if and only if G is nonplanar but G can be embedded on the torus. An embedding of a graph G on the torus is called a **toroidal embedding** of G. In particular,

$$\gamma(K_5) = 1 \text{ and } \gamma(K_{3,3}) = 1.$$

Figure 11.14(a) shows an embedding of a disconnected graph H on a sphere. In this case, $n = 8$, $m = 9$ and $r = 4$. Thus $n - m + r = 8 - 9 + 4 = 3$. That $n-m+r \neq 2$ is not particularly surprising as the Euler Identity (Theorem 10.1) requires that H be a *connected* plane graph. Although this is a major reason why we will restrict our attention to connected graphs here, it is not the only reason. There is a desirable property possessed by every embedding of a connected planar graph on a sphere that is possessed by no embedding of a disconnected planar graph on a sphere.

2-Cell Embeddings of Graphs

Suppose that G is a graph embedded on a surface S_k, $k \geq 0$. A region of this embedding is a **2-cell** if every closed curve in that region can be continuously deformed in that region to a single point. (Topologically, a region is a 2-cell if it is homeomorphic to a disk.) While the closed curve C in R in the embedding of the graph on a sphere shown in Figure 11.14(b) can in fact be continuously deformed in R to a single point, the curve C' cannot. Hence R is not a 2-cell in this embedding.

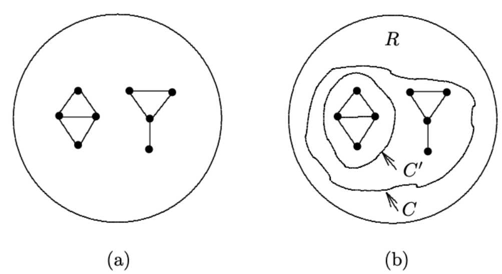

Figure 11.14: An embedding on a sphere that is not a 2-cell embedding

An embedding of a graph G on some surface is a **2-cell embedding** if every region in the embedding is a 2-cell. Consequently, the embedding of the graph shown in Figure 11.14(a) is not a 2-cell embedding. It turns out, however, that every embedding of a connected graph on a sphere is necessarily a 2-cell embedding. Of course, such a graph is necessarily planar. If a connected graph is embedded on a surface S_k where $k > 0$, then the embedding may or may not be a 2-cell embedding, however. For example, the embedding of K_4 in Figure 11.9(b) is not a 2-cell embedding. The curves C and C' shown in Figures 11.15(a) and 11.15(b) cannot be continuously deformed to a single point in the region in which these curves are drawn. On the other hand, the embedding of K_4 shown in Figure 11.9(c) and shown again in Figure 11.15(c) is a 2-cell embedding.

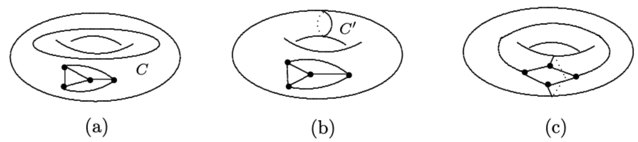

Figure 11.15: Non-2-cell and 2-cell embeddings of K_4 on the torus

The Generalized Euler Identity

The embeddings of K_4, K_5 and $K_{3,3}$ on a torus given in Figures 11.9(c), 11.10(a) and 11.10(b), respectively, are all 2-cell embeddings. Furthermore, in each case, $n - m + r = 0$. As it turns out, if G is a connected graph of order n and size m that is 2-cell embedded on a torus resulting in r regions, then it is always the case that $n - m + r = 0$. This fact together with the Euler Identity (Theorem 10.1) are special cases of a more general result. The mathematician Simon Antoine Jean Lhuilier spent much of his life working on problems related to the Euler Identity. Lhuilier, like Euler, was from Switzerland and was taught mathematics by one of Euler's former students (Louis Bertrand). Lhuilier saw that the Euler Identity did *not* hold for graphs embedded on spheres containing handles. In fact, he proved a more general form of this identity [156].

Theorem 11.10 (**The Generalized Euler Identity**) *If G is a connected graph of order n and size m that is 2-cell embedded on a surface of genus $k \geq 0$, resulting in r regions, then*

$$n - m + r = 2 - 2k.$$

Proof. We proceed by induction on k. If G is a connected graph of order n and size m that is 2-cell embedded on a surface of genus 0, then G is a plane graph. By the Euler Identity, $n - m + r = 2 = 2 - 2 \cdot 0$. Thus the basis step of the induction holds.

Assume, for every connected graph G' of order n' and size m' that is 2-cell embedded on a surface S_k for some nonnegative integer k, resulting in r' regions, that

$$n' - m' + r' = 2 - 2k.$$

Let G be a connected graph of order n and size m that is 2-cell embedded on S_{k+1}, resulting in r regions. We may assume, without loss of generality, that no vertex of G lies on any handle of S_{k+1} and that the edges of G are drawn on the handles so that a closed curve can be drawn around each handle that intersects no edge of G more than once.

Let H be one of the $k+1$ handles of S_{k+1}. There are necessarily edges of G on H; for otherwise, the handle belongs to a region R in which case any closed curve around H cannot be continuously deformed in R to a single point, contradicting the assumption that R is a 2-cell. We now draw a closed curve C around H, which intersects some edges of G on H but intersects no edge more than once. Suppose that there are $t \geq 1$ points of intersection of C and the edges on H. At each point of intersection, a new vertex is introduced. Each of the t edges then becomes two edges. Also, the segments of C between vertices become edges. We add two vertices of degree 2 along C to produce two additional edges. (This guarantees that the resulting structure will be a graph, not a multigraph.)

Let G_1 be the graph just constructed, where G_1 has order n_1, size m_1 and r_1 regions. Then

$$n_1 = n + t + 2 \text{ and } m_1 = m + 2t + 2.$$

Since each portion of C that became an edge of G_1 is in a region of G, the addition of such an edge divides that region into two regions, each of which is a 2-cell. Since there are t such edges,

$$r_1 = r + t.$$

We now cut the handle H along C and "patch" the two resulting holes, producing two duplicate copies of the vertices and edges along C (see Figure 11.16). Denote the resulting graph by G_2, which is now 2-cell embedded on a surface S_k.

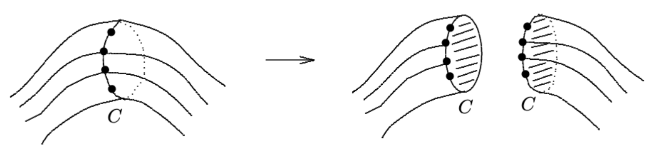

Figure 11.16: Converting a 2-cell embedding of G_1 on S_{k+1} into a 2-cell embedding of G_2 on S_k

Let G_2 have order n_2, size m_2 and r_2 regions, all of which are 2-cells. Then

$$n_2 = n_1 + t + 2,\ m_2 = m_1 + t + 2 \text{ and } r_2 = r_1 + 2.$$

Furthermore, $n_2 = n + 2t + 4$, $m_2 = m + 3t + 4$ and $r_2 = r + t + 2$. By the induction hypothesis, $n_2 - m_2 + r_2 = 2 - 2k$. Therefore,

$$\begin{aligned} n_2 - m_2 + r_2 &= (n + 2t + 4) - (m + 3t + 4) + (r + t + 2) \\ &= n - m + r + 2 = 2 - 2k. \end{aligned}$$

Therefore, $n - m + r = 2 - 2(k + 1)$. ■

We noted that every embedding of a connected planar graph G in the plane is always a 2-cell embedding of G. This fact is a special case of a useful result obtained by J. W. T. (Ted) Youngs [261].

Theorem 11.11 *Every embedding of a connected graph G of genus k on S_k, where k is a nonnegative integer, is a 2-cell embedding of G on S_k.*

Lower Bounds for the Genus of a Graph

With the aid of Theorems 11.10 and 11.11, we have the following.

Corollary 11.12 *If G is a connected graph of order n and size m that is embedded on a surface of genus $\gamma(G)$, resulting in r regions, then*

$$n - m + r = 2 - 2\gamma(G).$$

The following result is a consequence of Corollary 11.12.

Theorem 11.13 *If G is a connected graph of order $n \geq 3$ and size m, then*

$$\gamma(G) \geq \frac{m}{6} - \frac{n}{2} + 1.$$

Proof. Since the result is obviously true for $n = 3$, we may assume that $n \geq 4$. Suppose that G is embedded on a surface of genus $\gamma(G)$, resulting in r regions. By Corollary 11.12, $n - m + r = 2 - 2\gamma(G)$. Let $R_1, R_2, \ldots, R_r$ be the regions of G and let m_i be the number of edges on the boundary of R_i $(1 \leq i \leq r)$. Thus, $m_i \geq 3$ for $1 \leq i \leq r$. Since every edge is on the boundary of either one or two regions, it follows that

$$3r \leq \sum_{i=1}^{r} m_i \leq 2m$$

and so $3r \leq 2m$. Therefore,

$$6 - 6\gamma(G) = 3n - 3m + 3r \leq 3n - 3m + 2m = 3n - m. \tag{11.6}$$

Solving (11.6) for $\gamma(G)$, we have $\gamma(G) \geq \frac{m}{6} - \frac{n}{2} + 1$. ■

Theorem 11.13 is a generalization of Theorem 10.3, for when G is planar (and so $\gamma(G) = 0$) Theorem 11.13 becomes Theorem 10.3. The lower bound for $\gamma(G)$ presented in Theorem 11.13 can be improved when more information on cycle lengths in G is available (see Exercise 13).

Theorem 11.14 *If G is a connected graph of order n, size m and girth k, then*

$$\gamma(G) \geq \frac{m}{2}\left(1 - \frac{2}{k}\right) - \frac{n}{2} + 1.$$

The following result is a consequence of Theorem 11.14 that includes bipartite graphs as a special case. Recall that a graph is called triangle-free if it contains no triangles.

Corollary 11.15 *If G is a connected, triangle-free graph of order $n \geq 3$ and size m, then*

$$\gamma(G) \geq \frac{m}{4} - \frac{n}{2} + 1.$$

While there is no general formula for the genus of an arbitrary graph, the following result by Joseph Battle, Frank Harary and Yukihiro Kodama [13] implies that, as far as genus formulas are concerned, only 2-connected graphs need be investigated.

Theorem 11.16 *If G is a graph having blocks $B_1, B_2, \ldots, B_k$, then*

$$\gamma(G) = \sum_{i=1}^{k} \gamma(B_i).$$

The following corollary is a consequence of the preceding result (see Exercise 14).

Corollary 11.17 *If G is a graph with components $G_1, G_2, \ldots, G_k$, then*

$$\gamma(G) = \sum_{i=1}^{k} \gamma(G_i).$$

The Genus of Some Well-Known Graphs

As is often the case when no general formula exists for the value of a parameter for an arbitrary graph, formulas (or partial formulas) are established for certain families of graphs. Ordinarily the first classes to be considered are the complete graphs, the complete bipartite graphs and the n-cubes. The genus offers no exception to this statement.

According to Theorem 11.13,

$$\gamma(K_5) \geq \tfrac{1}{6},\ \gamma(K_6) \geq \tfrac{1}{2} \ \text{ and } \ \gamma(K_7) \geq 1.$$

This says that all three graphs K_5, K_6 and K_7 are nonplanar. Of course, we already knew this by Corollary 10.8. Because K_5 is nonplanar, so too are K_6 and K_7. We have seen that $\gamma(K_5) = 1$. Actually, $\gamma(K_7) = 1$ as well.

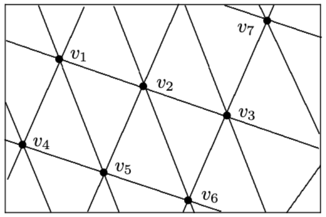

Figure 11.17: An embedding of K_7 on the torus

Figure 11.17 shows an embedding of K_7 on the torus where the vertex set of K_7 is $\{v_1, v_2, \ldots, v_7\}$. Because K_6 is nonplanar and K_6 is a subgraph of a graph that can be embedded on a torus, $\gamma(K_6) = 1$.

Applying Theorem 11.13 to the complete graph K_n, $n \geq 3$, we have

$$\gamma(K_n) \geq \frac{\binom{n}{2}}{6} - \frac{n}{2} + 1 = \frac{(n-3)(n-4)}{12}.$$

Since $\gamma(K_n)$ is an integer,

$$\gamma(K_n) \geq \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil. \tag{11.7}$$

Born in India, Ted Youngs (1910–1970) received his Ph.D. in 1934. In 1964 he was appointed one of the first faculty members of the University of California Santa Cruz. Born in Austria, Gerhard Ringel (1919–2008) was one of the pioneers of graph theory. He received his Ph.D. from the University of Bonn in Germany in 1951. In addition to his mathematical skills, Ringel was well known for his interest in collecting and breeding butterflies. Ringel left Germany in 1970 to join Youngs at the University of California Santa Cruz in order to complete a proof of a famous theorem in graph theory, which will be discussed in Chapter 16. However, the truth of this theorem required showing that the lower bound for $\gamma(K_n)$ in (11.7) is, in fact, the value of $\gamma(K_n)$, which Ringel and Youngs were successful in accomplishing [204].

Theorem 11.18 *For every integer $n \geq 3$,*

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil.$$

Ringel [203] was also successful in obtaining a formula for the genus of every complete bipartite graph.

Theorem 11.19 *For every two integers $s, t \geq 2$,*

$$\gamma(K_{s,t}) = \left\lceil \frac{(s-2)(t-2)}{4} \right\rceil.$$

In particular, Theorem 11.19 implies that a complete bipartite graph G can be embedded on a torus if and only if G is planar or is a subgraph of $K_{4,4}$ or $K_{3,6}$.

A formula for the genus of the n-cube was found by Ringel [201] and by Beineke and Harary [18]. We prove this result to illustrate some of the techniques involved.

Theorem 11.20 *For $n \geq 2$, the genus of the n-cube is given by*

$$\gamma(Q_n) = (n-4) \cdot 2^{n-3} + 1.$$

Proof. Since the n-cube is a triangle-free graph of order 2^n and size $n \cdot 2^{n-1}$, it follows by Corollary 11.15 that

$$\gamma(Q_n) \geq (n-4) \cdot 2^{n-3} + 1.$$

To verify the reverse inequality, we employ induction on n. In fact, we show for every integer $n \geq 2$, that there is an embedding of Q_n on the surface of genus $(n-4) \cdot 2^{n-3} + 1$ such that the boundary of every region is a 4-cycle and such that there exist 2^{n-2} regions with pairwise disjoint boundaries. Since Q_2 and Q_3 are planar and $(n-4) \cdot 2^{n-3} + 1 = 0$ for $n = 2$ and $n = 3$, this is certainly true for $n = 2$ and $n = 3$.

Assume for an integer $k \geq 4$ that there is an embedding of Q_{k-1} on the surface S of genus $(k-5) \cdot 2^{k-4} + 1$ such that the boundary of every region is a 4-cycle and such that there exist 2^{k-3} regions with pairwise disjoint boundaries. Since the order of Q_{k-1} is 2^{k-1}, each vertex of Q_{k-1} belongs to the boundary of precisely one of the aforementioned 2^{k-3} regions. Furthermore, let Q_{k-1} be embedded on another copy S' of the surface of genus $(k-5) \cdot 2^{k-4} + 1$ such that the embedding of Q_{k-1} on S' is a "mirror image" of the embedding of Q_{k-1} on S (that is, if v_1, v_2, v_3, v_4 are the vertices of the boundary of a region of Q_{k-1} on S, where the vertices are listed clockwise about the 4-cycle, then there is a region on S' with the vertices v_1, v_2, v_3, v_4 on its boundary listed counterclockwise).

We now consider the 2^{k-3} distinguished regions of S together with the corresponding regions of S' and join each pair of associated regions by a handle. The addition of the first handle produces the surface of genus $2[(k-5) \cdot 2^{k-4} + 1]$ while the addition of each of the other $2^{k-3} - 1$ handles results in an increase of 1 to the genus. Thus, the surface just constructed has genus $(k-4) \cdot 2^{k-3} + 1$. Now each set of four vertices on the boundary of a distinguished region can be joined to the corresponding four vertices on the boundary of the associated region so that the four edges are embedded on the handle joining the regions. It is now immediate that the resulting graph is isomorphic to Q_k and that every region is bounded by a 4-cycle. Furthermore, each added handle gives rise to four regions, "opposite" ones of which have disjoint boundaries, so there exist 2^{k-2} regions of Q_k that are pairwise disjoint. ■

The Möbius Strip and Projective Plane

There are other surfaces on which graphs can be embedded. The **Möbius strip** (or **Möbius band**) is a one-sided surface that can be constructed from a rectangular piece of material by giving one side of the rectangle a half-twist (or a rotation through 180^o) and then identifying opposite sides of the rectangle (see Figure 11.18). Thus, A represents the same point on the Möbius strip. The Möbius strip is named for the German mathematician August Ferdinand Möbius who discovered it in 1858 (even though the mathematician Johann Benedict Listing discovered it shortly before Möbius).

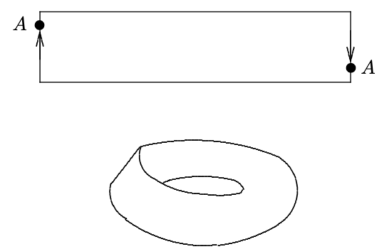

Figure 11.18: The Möbius strip

Certainly, every planar graph can be embedded on the Möbius strip. Figure 11.19 shows that $K_{3,3}$ can also be embedded on the Möbius strip.

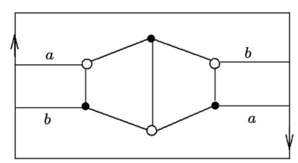

Figure 11.19: An embedding of $K_{3,3}$ on the Möbius strip

Of more interest are the nonorientable surfaces (or the nonorientable 2-dimensional manifolds), the simplest example of which is the projective plane. The **projective plane** can be represented by identifying opposite sides of a rectangle in the manner shown in Figure 11.20(a). Note that A represents the same point in the projective plane, as does B. Figure 11.20(b) shows an embedding of K_5 on the projective plane.

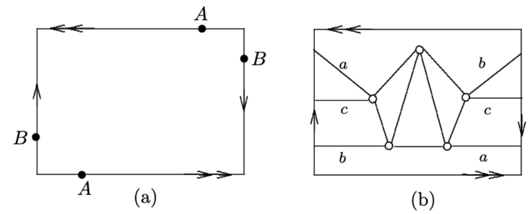

Figure 11.20: An embedding of K_5 on the projective plane

The projective plane can also be represented by a circle where antipodal pairs of points on the circumference are the same point. Using this representation, we can give an embedding of K_6 on the projective plane shown in Figure 11.21.

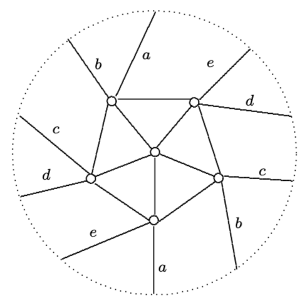

Figure 11.21: An embedding of K_6 on the projective plane

For the embedding of K_5 on the projective plane shown in Figure 11.20(b), $n = 5$, $m = 10$ and $r = 6$; while for the embedding of K_6 shown in Figure 11.21, $n = 6$, $m = 15$ and $r = 10$. In both cases, $n - m + r = 1$. In fact, for any connected graph of order n and size m that is 2-cell embedded on the projective plane, resulting in r regions,

$$n - m + r = 1.$$

Recall that S_k denotes the surface of genus k. Thus, S_0 represents the sphere (or plane), S_1 represents the torus and S_2 represents the **double torus** (or sphere with two handles). We have already mentioned that the torus can be represented as a square with opposite sides identified. More generally, the surface S_k $(k > 0)$ can be represented as a regular $4k$-gon whose $4k$ sides can be listed in clockwise order as

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \ldots a_k b_k a_k^{-1} b_k^{-1}, \tag{11.8}$$

where, for example, a_1 is a side directed clockwise and a_1^{-1} is a side also labeled a_1 but directed counterclockwise. These two sides are then identified in a manner consistent with their directions. Thus, the double torus can be represented by a regular octagon, as shown in Figure 11.22. The "two" points labeled X are actually the same point on S_2 while the "eight" points labeled Y are, in fact, a single point.

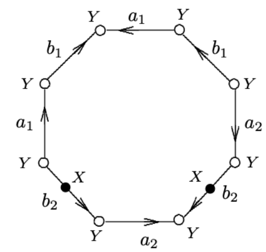

Figure 11.22: A representation of the double torus

Although it is probably obvious that there exist numerous graphs that can be embedded on the surface S_k of a given nonnegative integer k, it may not be entirely obvious that there always exist graphs for which a 2-cell embedding on S_k exists.

Theorem 11.21 *For every nonnegative integer k, there exists a connected graph that has a 2-cell embedding on S_k.*

Proof. For $k = 0$, every connected planar graph has the desired property; thus, we assume that $k > 0$.

We represent S_k as a regular $4k$-gon whose $4k$ sides are described and identified as in (11.8). First, we define a pseudograph H as follows. At each vertex of the $4k$-gon, let there be a vertex of H. Actually, the identification process associated with the $4k$-gon implies that there is only one vertex of H. Let each side of the $4k$-gon represent an edge of H. The identification produces $2k$ distinct edges, each of which is a loop. This completes the construction of H. Hence, the pseudograph H has order 1 and size $2k$. Furthermore, there is only one region, namely the interior of the $4k$-gon; this region is clearly a 2-cell. Therefore, there exists a 2-cell embedding of H on S_k.

To convert the pseudograph H into a graph, we subdivide each loop twice, producing a graph G having order $4k + 1$, size $6k$ and again a single 2-cell region. ■

Figure 11.23 illustrates the construction given in the proof of Theorem 11.21 in the case of the torus S_1. The graph G so constructed is shown in Figure 11.23(a). In Figure 11.23(b)-(e), we see a variety of ways of visualizing the embedding. In Figure 11.23(b), a 3-dimensional embedding is described. In Figures 11.23(c) and (d), the torus is represented as a rectangle with opposite sides identified. (Figure 11.23(c) is the actual drawing described in the proof of the theorem.) In Figure 11.23(e), a portion of G is drawn in the plane. Then two circular holes are made in the plane and a tube (or handle) is placed over the plane joining the two holes. The edge uv is then drawn over the handle, completing the 2-cell embedding.

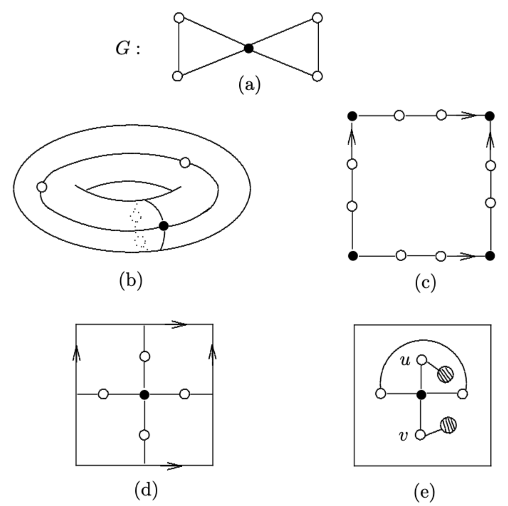

Figure 11.23: A graph 2-cell embedded on the torus

The graphs G constructed in the proof of Theorem 11.21 are planar. Hence, for every nonnegative integer k, there exist planar graphs that can be 2-cell embedded on S_k. It is also true that for every planar graph G and every positive integer k, there exists an embedding of G on S_k that is *not* a 2-cell embedding. In general, for a given graph G and *positive* integer k with $k > \gamma(G)$, there always exists an embedding of G on S_k that is not a 2-cell embedding, which can be obtained from an embedding of G on $S_{\gamma(G)}$ by adding $k - \gamma(G)$ handles to the interior of some region of G. If $k = \gamma(G)$ and G is connected, then by Theorem 11.11, every embedding of G on S_k is a 2-cell embedding. Of course, if $k < \gamma(G)$, there is no embedding whatsoever of G on S_k.

11.3 The Graph Minor Theorem

We have seen by Wagner's theorem (Theorem 10.21) that a graph G is planar if and only if neither K_5 nor $K_{3,3}$ is a minor of G. That is, Wagner's theorem is a **forbidden minor** characterization of planar graphs – in this case, two forbidden minors: K_5 and $K_{3,3}$. A natural question to ask is whether a forbidden minor characterization may exist for graphs that can be embedded on other surfaces. It was shown by Daniel Archdeacon and Philip Huneke [10] that there are exactly 35 forbidden minors for graphs that can be embedded on the projective plane. More general results involving minors have been obtained. Klaus Wagner conjectured that in every infinite collection of graphs, there are always two graphs where one is isomorphic to a minor of the other. In what must be considered one of the major theorems of graph theory, Neil Robertson and Paul Seymour [206] verified this conjecture. Its lengthy proof is a consequence of a sequence of several papers that required years to complete.

Theorem 11.22 (**The Graph Minor Theorem**) *In every infinite set of graphs, there are two graphs where one is* (*isomorphic to*) *a minor of the other.*

One consequence of Theorem 11.22 is another major theorem in graph theory, also due to Robertson and Seymour [206]. A set S of graphs is said to be **minor-closed** if for every graph G in S, every minor of G also belongs to S. For example, the set S of planar graphs is minor-closed because every minor of a planar graph is planar; that is, if $G \in S$ and H is a minor of G, then $H \in S$ as well.

Theorem 11.23 *Let S be a minor-closed set of graphs. Then there exists a finite set M of graphs such that $G \in S$ if and only if no graph in M is a minor of G.*

Proof. Let $\overline{S}$ be the set of graphs not belonging to S and let M be the set of all graphs F in $\overline{S}$ such that every proper minor of F belongs to S. We claim that this set M has the required properties. First, we show that $G \in S$ if and only if no graph in M is a minor of G.

First, suppose that there is some graph $G \in S$ such that there is a graph F that is a minor of G and for which $F \in M$. Since $G \in S$ and S is minor-closed, it follows that $F \in S$. However, since $F \in M$, we have $F \in \overline{S}$, which is a contradiction.

For the converse, assume to the contrary that there is a graph $G \in \overline{S}$ such that no graph in M is a minor of G. We consider two cases.

Case 1. *All of the proper minors of G are in S.* Then by the defining property of M, it follows that $G \in M$. Since $G \in M$ and G is a minor of itself, this contradicts our assumption that no graph in M is a minor of G.

Case 2. *Some proper minor of G, say G', is not in S.* Thus, $G' \in \overline{S}$. Then G' either satisfies the condition of Case 1 or of Case 2. We proceed in this manner as long as we remain in Case 2, producing a sequence $G = G^{(0)}, G^{(1)}, G^{(2)}, \ldots$ of proper minors.

If this process terminates, we have a finite sequence

$$G = G^{(0)}, G' = G^{(1)}, \ldots, G^{(p)},$$

where each graph in the sequence is a proper minor of all those graphs that precede it. Then $G^{(p)} \in M$, which returns us to Case 1. Otherwise, the sequence $G = G^{(0)}, G' = G^{(1)}, G^{(2)}, \ldots$ is infinite and where each graph $G^{(i)}$, $i \geq 1$, is a proper minor of all those graphs that precede it. This, however, is impossible since for each $i \geq 0$, either the order of $G^{(i+1)}$ is less than that of $G^{(i)}$ or the orders are the same and the size of $G^{(i+1)}$ is less than that of $G^{(i)}$.

It therefore remains only to show that M is finite. Assume, to the contrary, that M is infinite. By the Graph Minor Theorem, M contains two graphs, H_1 and H_2 say, such that one is a minor of the other. Suppose that H_2 is a minor (necessarily a proper minor) of H_1. Since every proper minor of H_1 belongs to S, it follows that $H_2 \in S$. However, since $H_2 \in M$, it follows that $H_2 \in \overline{S}$, producing a contradiction. ■

We now return to the question concerning the existence of a forbidden minor characterization for graphs embeddable on a surface S_k of genus $k \geq 0$. Certainly, if G is a sufficiently small graph (in terms of its order and/or size), then G can be embedded on S_k. Hence, if we begin with a graph F that cannot be embedded on S_k and perform successive edge contractions, edge deletions and vertex deletions, then eventually we arrive at a graph F' that also cannot be embedded on S_k but such that any additional edge contraction, edge deletion or vertex deletion of F' produces a graph that *can* be embedded on S_k. Such a graph F' is said to be **minimally nonembeddable on** S_k. Consequently, a graph F' is minimally nonembeddable on S_k if F' cannot be embedded on S_k but every proper minor F' can be embedded on S_k. Thus, the set of graphs embeddable on S_k is minor-closed. As a consequence of the Graph Minor Theorem, we have the following.

Theorem 11.24 *For each integer $k \geq 0$, the set of minimally nonembeddable graphs on S_k is finite.*

Consequently, for each nonnegative integer k, there is a finite set $\mathcal{M}_k$ of graphs such that a graph G is embeddable on S_k if and only if no graph in $\mathcal{M}_k$ is a minor of G. Of course, the set of minimally nonembeddable graphs on the sphere is $\mathcal{M}_0 = \{K_5, K_{3,3}\}$. Although the number of minimally nonembeddable graphs on the torus is finite, the actual value of this number is not known. However, it is known that this number exceeds 800 and so $|\mathcal{M}_1| > 800$.

in this manner as long as we remain in Case 2, producing a sequence $G = G^{(0)}, G^{(1)}, G^{(2)}, \ldots$ of proper minors.

If this process terminates, we have a finite sequence

$$G = G^{(0)}, G' = G^{(1)}, \ldots, G^{(p)},$$

where each graph in the sequence is a proper minor of all those graphs that precede it. Then $G^{(p)} \in M$, which returns us to Case 1. Otherwise, the sequence $G = G^{(0)}, G' = G^{(1)}, G^{(2)}, \ldots$ is infinite and where each graph $G^{(i)}$, $i \geq 1$, is a proper minor of all those graphs that precede it. This, however, is impossible since for each $i \geq 0$, either the order of $G^{(i+1)}$ is less than that of $G^{(i)}$ or the orders are the same and the size of $G^{(i+1)}$ is less than that of $G^{(i)}$.

It therefore remains only to show that M is finite. Assume, to the contrary, that M is infinite. By the Graph Minor Theorem, M contains two graphs, H_1 and H_2 say, such that one is a minor of the other. Suppose that H_2 is a minor (necessarily a proper minor) of H_1. Since every proper minor of H_1 belongs to S, it follows that $H_2 \in S$. However, since $H_2 \in M$, it follows that $H_2 \in \overline{S}$, producing a contradiction. ■

We now return to the question concerning the existence of a forbidden minor characterization for graphs embeddable on a surface S_k of genus $k \geq 0$. Certainly, if G is a sufficiently small graph (in terms of its order and/or size), then G can be embedded on S_k. Hence, if we begin with a graph F that cannot be embedded on S_k and perform successive edge contractions, edge deletions and vertex deletions, then eventually we arrive at a graph F' that also cannot be embedded on S_k but such that any additional edge contraction, edge deletion or vertex deletion of F' produces a graph that *can* be embedded on S_k. Such a graph F' is said to be **minimally nonembeddable on** S_k. Consequently, a graph F' is minimally nonembeddable on S_k if F' cannot be embedded on S_k but every proper minor F' can be embedded on S_k. Thus, the set of graphs embeddable on S_k is minor-closed. As a consequence of the Graph Minor Theorem, we have the following.

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Exercises for Chapter 11

Section 11.1: The Crossing Number of a Graph

1. Draw K_7 in the plane with nine crossings.

2. Determine $\text{cr}(K_{3,3})$ without using Theorem 11.4.

3. Show that $\text{cr}(K_{5,5}) \leq 16$.

4. Determine $\text{cr}(K_{2,2,2})$.

5. Determine $\text{cr}(K_{1,2,3})$.

6. Show that $2 \leq \text{cr}(C_3 \,\Box\, C_3) \leq 3$ without using Theorem 11.6.

7. (a) It is known that $\text{cr}(W_4 \,\Box\, K_2) = 2$, where W_4 is the wheel $C_4 \vee K_1$ of order 5. Draw $W_4 \,\Box\, K_2$ in the plane with two crossings.

   (b) Prove or disprove: If G is a nonplanar graph containing an edge e such that $G - e$ is planar, then $\text{cr}(G) = 1$.

8. Prove that $\overline{\text{cr}}(C_3 \,\Box\, C_t) = t$ for $t \geq 3$.

9. Give an example of a straight-line embedding of a maximal planar graph of order exceeding 4 containing exactly four vertices of degree 5 or less.

10. Let G be a connected planar graph. Prove or disprove: If $\text{cr}(G \,\Box\, K_2) = 0$, then G is outerplanar.

Section 11.2: The Genus of a Graph

11. Determine $\gamma = \gamma(K_{4,4})$ without using Theorem 11.19 and label the regions in a 2-cell embedding of $K_{4,4}$ on the surface of genus γ.

12. (a) Show that $\gamma(G) \leq \text{cr}(G)$ for every graph G.

    (b) Prove that for every positive integer k, there exists a graph G such that $\gamma(G) = 1$ and $\text{cr}(G) = k$.

13. Prove Theorem 11.14: *If G is a connected graph of order n and size m whose smallest cycle has length k, then $\gamma(G) \geq \frac{m}{2}\left(1 - \frac{2}{k}\right) - \frac{n}{2} + 1$.*

14. Use Theorem 11.16 to prove Corollary 11.17: *If G is a graph with components $G_1, G_2, \ldots, G_k$, then $\gamma(G) = \sum_{i=1}^{k} \gamma(G_i)$.*

15. Show for every two integers $s, t \geq 2$ that
$$\gamma(K_{s,t}) \geq \left\lceil \frac{(s-2)(t-2)}{4} \right\rceil.$$

16. (a) Find a lower bound for $\gamma(K_{3,3} \vee \overline{K}_n)$.
    (b) Determine $\gamma(K_{3,3} \vee \overline{K}_n)$ exactly for $n = 1, 2, 3$.

17. Determine $\gamma(K_2 \,\square\, C_4 \,\square\, C_6)$.

18. Prove, for every positive integer γ, that there exists a connected graph G of genus γ.

19. Prove, for each positive integer k, that there exists a connected planar graph G such that $\gamma(G \,\square\, K_2) \geq k$.

20. Show, in a manner similar to the embedding of $K_{3,3}$ shown in Figure 11.19, that K_5 can be embedded on the Möbius strip.

21. By Theorem 11.18, $\gamma(K_7) = 1$. Let there be an embedding of K_7 on the torus and let R_1 and R_2 be two neighboring regions. Let G be the graph obtained by adding a new vertex v in R_1 and joining v to the vertices on the boundaries of both R_1 and R_2. What is $\gamma(G)$?

22. The graph H is a certain 6-regular graph of order 12. It is known that $G = H \,\square\, K_2$ can be embedded on S_3. What is $\gamma(G)$?

23. Does there exist a graph G containing two nonadjacent vertices u and v such that $\gamma(G) = \gamma(G + uv)$ but $\operatorname{cr}(G) \neq \operatorname{cr}(G + uv)$?

24. For an r-regular graph H of order n where $r \equiv 5 \pmod 6$, the graph $G = H \,\square\, K_2$ can be embedded on the surface S_k where $k = \frac{(r-5)n}{6} + 1$. Show that $\gamma(G) = k$.

Section 11.3: The Graph Minor Theorem

25. (a) Show that the set $\mathcal{F}$ of forests is a minor-closed family of graphs.
    (b) What are the forbidden minors of $\mathcal{F}$?

26. Is the set B of bipartite graphs a minor-closed family of graphs? If so, what are the forbidden minors of B?

27. We have seen that the set of planar graphs is a minor-closed family of graphs. Is the set O of outerplanar graphs a minor-closed family of graphs? If so, what are the forbidden minors of O?

28. Use the Graph Minor Theorem (Theorem 11.22) to show that for any infinite set $S = \{G_1, G_2, G_3, \ldots\}$ of graphs, there exist infinitely many pairwise disjoint 2-element sets $\{i, j\}$ of integers such that one of G_i and G_j is a minor of the other.