Chapter 10

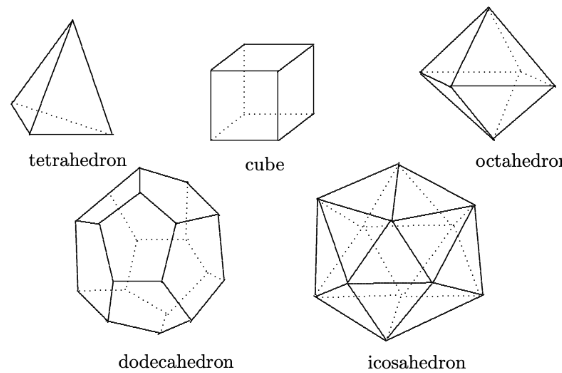
Planar Graphs

Of the methods we have described to represent a graph G, probably the most common and the most useful for determining whether G possesses particular properties of interest is that of presenting G by means of a drawing. There are occasions when the edges in a diagram may cross and other occasions when the edges in a diagram do not cross. If some pairs of edges in a diagram of a graph cross, then it may be that there are other drawings of this same graph when no edges cross – or perhaps this is not possible. It is the class of graphs that can be drawn in the plane without their edges crossing that will be of interest to us in this chapter. We will see that a result of Euler plays a central role in this study.

10.1 The Euler Identity

A **polyhedron** is a 3-dimensional object whose boundary consists of polygonal plane surfaces. These surfaces are typically called the **faces** of the polyhedron. The boundary of a face consists of the edges and vertices of the polygon. In this setting, the total number of faces in the polyhedron is commonly denoted by F, the total number of edges in the polyhedron by E and the total number of vertices by V. The best known polyhedra are the so-called **Platonic solids**: the **tetrahedron**, **cube** (**hexahedron**), **octahedron**, **dodecahedron** and **icosahedron**. These are shown in Figure 10.1, together with the values of V, E and F for these polyhedra.

During the 18th century, many letters (over 160) were exchanged between Leonhard Euler (who, as we saw in Chapter 5, essentially introduced graph theory to the world when he solved and then generalized the Königsberg Bridge Problem) and Christian Goldbach (well known for stating the conjecture that every even integer greater than 2 can be expressed as the sum of two primes). In a letter that Euler wrote to Goldbach on 14 November 1750, he stated a relationship that existed among the numbers V, E and F for a polyhedron and which would later become known as:



Platonic solid	V	E	F
tetrahedron	4	6	4
cube	8	12	6
octahedron	6	12	8
dodecahedron	20	30	12
icosahedron	12	30	20

Figure 10.1: The five Platonic solids

The Euler Polyhedral Formula *If a polyhedron has V vertices, E edges and F faces, then*

$$V - E + F = 2.$$

That Euler was evidently the first mathematician to observe this formula (which is actually an identity rather than a formula) may be somewhat surprising in light of the fact that Archimedes and René Descartes both studied polyhedra long before Euler. A possible explanation as to why others had overlooked this identity might be due to the fact that geometry had primarily been a study of distances.

The Euler Polyhedral Formula appeared in print two years later (in 1752) in two papers by Euler [87, 88]. In the first of these two papers, Euler stated that he had been unable to prove the formula. However, in the second paper, he presented a proof by dissecting polyhedra into tetrahedra. Although his proof was clever, he nonetheless made some missteps. The first generally accepted proof was obtained by the French mathematician Adrien-Marie Legendre [153].

Each polyhedron can be converted into a map and then into a graph by inserting a vertex at each meeting point of the map (which is actually a vertex of the polyhedron). This is illustrated in Figure 10.2 for the cube.

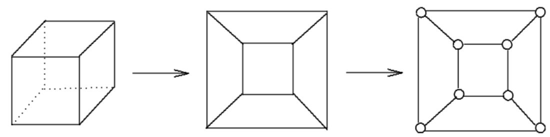


Figure 10.2: From a polyhedron to a map to a graph

The graphs obtained from the five Platonic solids are shown in Figure 10.3. These graphs have a property in which we will be especially interested: No two edges cross (intersect each other) in the graph.

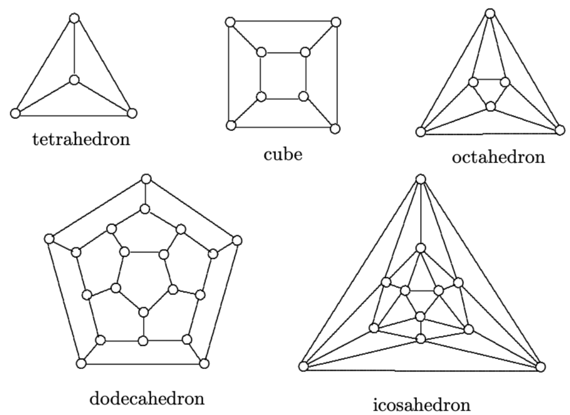


Figure 10.3: The graphs of the five Platonic solids

Planar Graphs

A graph G is called a **planar graph** if G can be drawn in the plane without any two of its edges crossing. Such a drawing is also called an **embedding of** G **in the plane**. In this case, the embedding is a **planar embedding**. A graph G that is already drawn in the plane in this manner is a **plane graph**. Certainly then, every plane graph is planar and every planar graph can be drawn as a plane graph. In particular, all five graphs of the Platonic solids are planar.

When those points in the plane that correspond to the vertices and edges of a plane graph G are removed from the plane, the resulting connected pieces of the plane are the **regions** of G. One of the regions is unbounded and is called the **exterior region** of G. For every planar embedding of a planar graph G and every region R in this planar embedding, there exists a planar embedding of G in which R is the exterior region. Consequently, for every edge e (or vertex v) of G, there is a planar embedding of G for which e (or v) lies on the boundary of the exterior region. When considering a plane graph G of a polyhedron, the faces of the polyhedron become the regions of G, one of which is the exterior region of G. On the other hand, a planar graph need not be the graph of any polyhedron. The plane graph H of Figure 10.4 is not the graph of any polyhedron. This graph has five regions, denoted by R_1, R_2, R_3, R_4 and R_5, where R_5 is the exterior region.

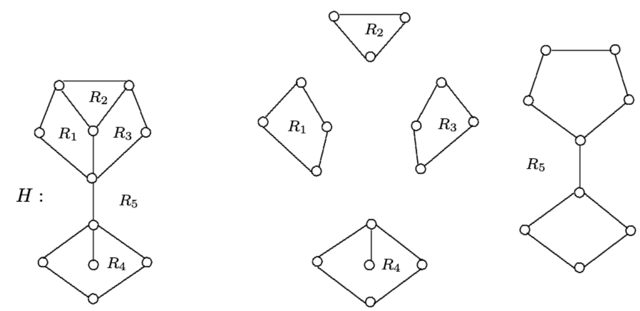


Figure 10.4: The boundaries of the regions of a plane graph

For a region R of a plane graph G, the vertices and edges incident with R form a subgraph of G called the **boundary** of R. Every edge of G that lies on a cycle belongs to the boundary of two regions of G, while every bridge of G belongs to the boundary of a single region. In Figure 10.4, the boundaries of the five regions of H are shown as well.

The five graphs G_1, G_2, G_3, G_4 and G_5 shown in Figure 10.5 are all planar, although G_1 and G_3 are not plane graphs. The graph G_1 can be drawn as G_2, while G_3 can be drawn as G_4. In fact, G_1 (and G_2) is the graph of the tetrahedron. For each graph, its order n, its size m and the number r of regions are shown as well.

Observe that $n - m + r = 2$ for each graph of Figure 10.5. Of course, this is not surprising for G_2 since this is the graph of a polyhedron (the tetrahedron) and $n = V$, $m = E$ and $r = F$. In fact, this identity holds for every connected plane graph.

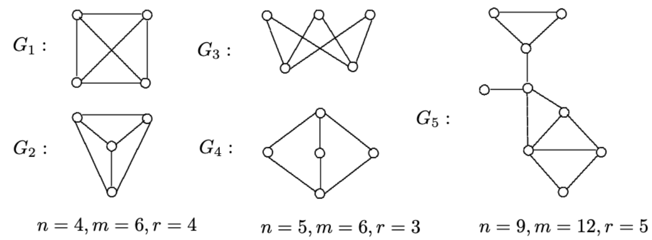


Figure 10.5: Planar graphs

Theorem 10.1 (**The Euler Identity**) *For every connected plane graph of order n, size m and having r regions,*

$$n - m + r = 2.$$

Proof. We proceed by induction on the size m of a connected plane graph. There is only one connected graph of size 0, namely K_1. In this case, $n = 1$, $m = 0$ and $r = 1$. Since $n - m + r = 2$, the base case of the induction holds.

Assume for a positive integer m that if H is a connected plane graph of order n' and size m', where $m' < m$ such that there are r' regions, then $n' - m' + r' = 2$. Let G be a connected plane graph of order n and size m with r regions. We consider two cases.

Case 1. *G is a tree.* In this case, $m = n - 1$ and $r = 1$. Thus $n - m + r = n - (n - 1) + 1 = 2$, producing the desired result.

Case 2. *G is not a tree.* Since G is connected and is not a tree, it follows by Theorem 3.10 that G contains an edge e that is not a bridge. In G, the edge e is on the boundaries of two regions. So in $G - e$ these two regions merge into a single region. Since $G - e$ has order n, size $m - 1$ and $r - 1$ regions and $m - 1 < m$, it follows by the induction hypothesis that $n - (m - 1) + (r - 1) = 2$ and so $n - m + r = 2$. ■

The Euler Polyhedron Formula is therefore a special case of Theorem 10.1. While Euler struggled with the verification of $V - E + F = 2$, he did not have the luxury of a developed graph theory at his disposal.

From Theorem 10.1, it follows that every two planar embeddings of a connected planar graph result in plane graphs having the same number of regions; thus one can speak of the number of regions of a connected planar graph. For planar graphs in general, we have the following result. (See Exercise 2.) Recall that $k(G)$ denotes the number of components of a graph G.

Corollary 10.2 *If G is a plane graph with n vertices, m edges and r regions, then*

$$n - m + r = 1 + k(G).$$

An Upper Bound for the Size of Planar Graphs

If G is a connected plane graph of order 4 or more, then the boundary of every region of G must contain at least three edges. This observation is helpful in showing that with respect to the order of a planar graph, its size cannot be too great.

Theorem 10.3 *If G is a planar graph of order $n \geq 3$ and size m, then*

$$m \leq 3n - 6.$$

Proof. Since the size of every graph of order 3 cannot exceed 3, the inequality holds for $n = 3$. So we may assume that $n \geq 4$. Futhermore, we may assume that the planar graphs under consideration are connected, for otherwise edges can be added to produce a connected graph. Suppose that G is a connected planar graph of order $n \geq 4$ and size m. Let there be given a planar embedding of G, resulting in r regions. By the Euler Identity, $n - m + r = 2$. Let $R_1, R_2, \ldots, R_r$ be the regions of G and suppose that we denote the number of edges on the boundary of R_i $(1 \leq i \leq r)$ by m_i. Then $m_i \geq 3$ for $1 \leq i \leq r$. Since each edge of G is on the boundary of at most two regions of G, it follows that

$$3r \leq \sum_{i=1}^{r} m_i \leq 2m.$$

Hence,

$$6 = 3n - 3m + 3r \leq 3n - 3m + 2m = 3n - m$$

and so $m \leq 3n - 6$. ■

By expressing Theorem 10.3 in its contrapositive form, we obtain the following reformulation of the theorem.

Theorem 10.4 *If G is a graph of order $n \geq 5$ and size m such that $m > 3n-6$, then G is nonplanar.*

There is an immediate consequence of this theorem.

Corollary 10.5 *Every planar graph contains a vertex of degree 5 or less.*

Proof. The result is obvious for planar graphs of order 6 or less. Let G be a graph of order n and size m all of whose vertices have degree 6 or more. Then $n \geq 7$ and

$$2m = \sum_{v \in V(G)} \deg v \geq 6n$$

and so $m \geq 3n$. By Theorem 10.4, G is nonplanar. ■

The Five Regular Polyhedra

We saw, by the Euler Polyhedron Formula, that if V, E and F are the number of vertices, edges and faces of a polyhedron, then

$$V - E + F = 2.$$

When dealing with a polyhedron P (as well as the graph of the polyhedron P), it is customary to represent the number of vertices of degree k by V_k and number of faces bounded by a k-cycle (k-sided faces) by F_k. It follows then that

$$2E = \sum_{k \geq 3} kV_k = \sum_{k \geq 3} kF_k. \tag{10.1}$$

By Corollary 10.5, every polyhedron has at least one vertex of degree 3, 4 or 5. As an analogue to this result, we have the following.

Theorem 10.6 *At least one face of every polyhedron is bounded by a k-cycle for some k where $k \in \{3, 4, 5\}$.*

Proof. Assume, to the contrary, that the statement is false. Then $F_3 = F_4 = F_5 = 0$. By equation (10.1),

$$2E = \sum_{k \geq 6} kF_k \geq \sum_{k \geq 6} 6F_k = 6 \sum_{k \geq 6} F_k = 6F.$$

Hence, $E \geq 3F$. Also,

$$2E = \sum_{k \geq 3} kV_k \geq \sum_{k \geq 3} 3V_k = 3 \sum_{k \geq 3} V_k = 3V.$$

By Theorem 10.1, $V - E + F = 2$ and so $3V - 3E + 3F = 6$. Hence, $6 = 3V - 3E + 3F \leq 2E - 3E + E = 0$, which is a contradiction. ■

Of the five Platonic solids shown in Figure 10.1, three are cubic polyhedra (the tetrahedron, cube and dodecahedron) as each vertex in these polyhedra has degree 3. The dodecahedron has twelve faces, all 5-sided. If the icosahedron (which has twelve vertices of degree 5 and twenty 3-sided faces) is “truncated”, replacing each vertex by a pentagon, another polyhedron results – namely a cubic polyhedron containing twelve 5-sided faces and twenty 6-sided faces. This is precisely what occurs with a soccer ball, which has 32 faces, 12 of which are pentagonal faces and 20 are hexagonal faces. Indeed, every cubic polyhedron containing only 5-sided and 6-sided faces must contain exactly twelve 5-sided faces. To see this, suppose that P is a cubic polyhedron with V vertices, E edges and F faces. Then $V = V_3$, $F = F_5 + F_6$ and, by the Euler Polyhedron Formula, $V - E + F = 2$. By (10.1), $2E = 3V_3 = 5V_5 + 6V_6$. Therefore,

$$\begin{aligned} 12 &= 6V - 6E + 6F = 6V_3 - 6E + 6(F_5 + F_6) \\ &= (10F_5 + 12F_6) - (15F_5 + 18F_6) + (6F_5 + 6F_6) = F_5. \end{aligned}$$

A **regular polyhedron** is a polyhedron whose faces are bounded by congruent regular polygons and whose polyhedral angles are congruent. In particular, for a regular polyhedron, $F = F_s$ for some s and $V = V_t$ for some t, where $s, t \in \{3, 4, 5\}$. For example, a cube is a regular polyhedron with $V = V_3$ and $F = F_4$. There are only four other regular polyhedra. These five regular polyhedra are the Platonic solids we saw in Figure 10.1. Over two thousand years ago, the Greeks were aware that there are only five such polyhedra.

Theorem 10.7 *There are exactly five regular polyhedra.*

Proof. Let P be a regular polyhedron and let G be an associated plane graph. Then $V - E + F = 2$, where V, E and F denote the number of vertices, edges and faces of P and the number of vertices, edges and regions of G. Therefore,

$$\begin{aligned} -8 &= 4E - 4V - 4F \\ &= 2E + 2E - 4V - 4F \\ &= \sum_{k\geq 3} kF_k + \sum_{k\geq 3} kV_k - 4\sum_{k\geq 3} V_k - 4\sum_{k\geq 3} F_k \\ &= \sum_{k\geq 3}(k-4)F_k + \sum_{k\geq 3}(k-4)V_k. \end{aligned} \tag{10.2}$$

Since G is regular, there exist integers s and t with $s, t \in \{3, 4, 5\}$ such that $F = F_s$ and $V = V_t$. Hence

$$-8 = (s-4)F_s + (t-4)V_t.$$

Moreover, $sF_s = 2E = tV_t$. If $s, t \geq 4$, then (10.2) yields $-8 = (s-4)F_s + (t-4)F_t \geq 0$, which is impossible. Hence either $s = 3$ or $t = 3$. This results in five possibilities for the pairs s, t.

Case 1. $s = 3$ *and* $t = 3$. Here we have

$$-8 = -F_3 - V_3 \text{ and } 3F_3 = 3V_3;$$

so $F_3 = V_3 = 4$. Thus P is the *tetrahedron*. (That the tetrahedron is the only regular polyhedron with $V_3 = F_3 = 4$ follows from geometric considerations.)

Case 2. $s = 3$ *and* $t = 4$. Therefore,

$$-8 = -F_3 \text{ and } 3F_3 = 4V_4.$$

Hence $F_3 = 8$ and $V_4 = 6$, implying that P is the *octahedron*.

Case 3. $s = 3$ *and* $t = 5$. In this case,

$$-8 = -F_3 + V_5 \text{ and } 3F_3 = 5V_5,$$

so $F_3 = 20$, $V_5 = 12$ and P is the *icosahedron*.

Case 4. $s = 4$ *and* $t = 3$. We find here that

$$-8 = -V_3 \text{ and } 4F_4 = 3V_3.$$

Thus $V_3 = 8$, $F_4 = 6$ and P is the *cube*.

Case 5. $s = 5$ *and* $t = 3$. For these values,

$$-8 = F_5 - V_3 \text{ and } 5F_5 = 3V_3.$$

Solving for F_5 and V_3, we find that $F_5 = 12$ and $V_3 = 20$, so P is the *dodecahedron*. ■

The graphs of the five regular polyhedra are shown in Figure 10.3.

The Graphs K_5 and $K_{3,3}$

Theorem 10.4 provides us with a large class of nonplanar graphs.

Corollary 10.8 *The graph K_5 is nonplanar.*

Proof. The graph K_5 has order $n = 5$ and size $m = 10$. Since $m = 10 > 9 = 3n - 6$, it follows by Theorem 10.4 that K_5 is nonplanar. ■

Since it is evident that any graph containing a nonplanar subgraph is itself nonplanar, it follows that once we know that K_5 is nonplanar, we can conclude that K_n is nonplanar for every integer $n \geq 5$. Of course, K_n is planar for $1 \leq n \leq 4$.

We will soon see that K_5 is an especially important nonplanar graph. Another important nonplanar graph is $K_{3,3}$. Since $K_{3,3}$ has order $n = 6$ and size $m = 9$ but $m < 3n - 6$, Theorem 10.4 cannot be used to establish the nonplanarity of $K_{3,3}$, however. On the other hand, we can use the fact that $K_{3,3}$ is bipartite to establish this property.

Corollary 10.9 *The graph $K_{3,3}$ is nonplanar.*

Proof. Suppose that $K_{3,3}$ is planar. Let there be given a planar embedding of $K_{3,3}$, resulting in r regions. Thus, by the Euler Identity, $n-m+r = 6-9+r = 2$ and so $r = 5$. Let $R_1, R_2, \ldots, R_5$ be the five regions and let m_i be the number of edges on the boundary of R_i $(1 \le i \le 5)$. Since $K_{3,3}$ is bipartite, $K_{3,3}$ contains no triangles and so $m_i \ge 4$ for $1 \le i \le 5$. Since every edge of $K_{3,3}$ lies on the boundary of a cycle, every edge of $K_{3,3}$ belongs to the boundary of two regions. Thus,

$$20 = 4r \le \sum_{i=1}^{5} m_i = 2m = 18,$$

which is impossible. ■

10.2 Maximal Planar Graphs

A planar graph G is **maximal planar** if the addition to G of any edge joining two nonadjacent vertices of G results in a nonplanar graph. Necessarily then, if a maximal planar graph G of order $n \ge 3$ and size m is embedded in the plane resulting in r regions, then the boundary of every region of G is a triangle and so $3r = 2m$. It then follows by the proof of Theorem 10.3 that $m = 3n - 6$. All of the graphs shown in Figure 10.6 are maximal planar.

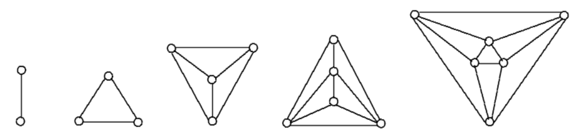


Figure 10.6: Maximal planar graphs

A graph G is **nearly maximal planar** if there exists a planar embedding of G such that the boundary of every region of G is a cycle, at most one of which is not a triangle. Thus, every maximal planar graph is nearly maximal planar (see Figure 10.7(a)). Also, the wheels $W_n = C_n \vee K_1$ $(n \ge 3)$ are nearly maximal planar (see Figure 10.7(b)). In addition, the graphs in Figures 10.7(c) and 10.7(d) (where the graph in Figure 10.7(d) is redrawn in Figure 10.7(e)) are nearly maximal planar.

We now derive some results concerning the degrees of the vertices of a maximal planar graph.

Theorem 10.10 *If G is a maximal planar graph of order 4 or more, then the degree of every vertex of G is at least 3.*

Proof. Let G be a maximal planar graph of order $n \ge 4$ and size m and let v be a vertex of G. Since $m = 3n - 6$, it follows that $G - v$ has order $n - 1$ and size $m - \deg v$. Since $G - v$ is planar and $n - 1 \ge 3$, it follows that

$$m - \deg v \le 3(n-1) - 6$$

and so $m - \deg v = 3n - 6 - \deg v \le 3n - 9$. Thus, $\deg v \ge 3$. ■

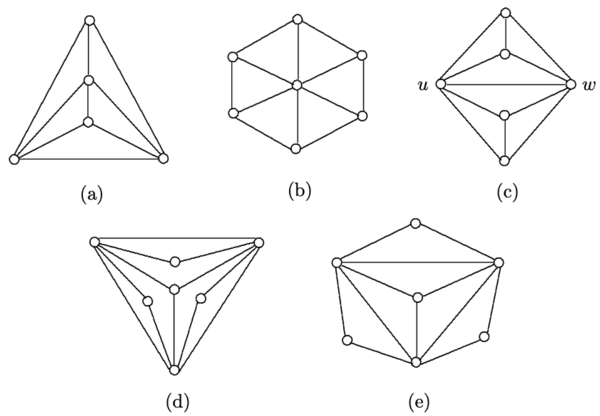


Figure 10.7: Nearly maximal planar graphs

Not only is the minimum degree of every maximal planar graph G of order 4 or more at least 3, the graph G is 3-connected (see Exercise 14).

In the early 20th century, Paul August Ludwig Wernicke studied under the supervision of Hermann Minkowski. (Dénes König, who, as we saw, wrote the first book on graph theory, also studied under Minkowski. Thus, Wernicke and König were "academic brothers".) By Corollary 10.5 and Theorem 10.10, every maximal planar graph of order 4 or more contains a vertex of degree 3, 4 or 5. In the very same year that he received his Ph.D., Wernicke [253] proved that every planar graph that doesn't have a vertex of a degree less than 5 must contain a vertex of degree 5 that is adjacent either to a vertex of degree 5 or to a vertex of degree 6. In the case of maximal planar graphs, Wernicke's result states the following.

Theorem 10.11 *If G is a maximal planar graph of order 4 or more, then G contains at least one of the following:* (1) *a vertex of degree* 3*,* (2) *a vertex of degree* 4*,* (3) *two adjacent vertices of degree* 5*,* (4) *two adjacent vertices, one of which has degree* 5 *and the other has degree* 6*.*

Proof. Assume, to the contrary, that there exists a maximal planar graph G of order $n \geq 4$ and size m containing none of (1)–(4). By Corollary 10.5, $\delta(G) = 5$. Let there be given a planar embedding of G, resulting in r regions. Then

$$n - m + r = 2.$$

Suppose that G has n_i vertices of degree i for $5 \le i \le \Delta(G) = \Delta$. Then

$$\sum_{i=5}^{\Delta} n_i = n \quad \text{and} \quad \sum_{i=5}^{\Delta} i n_i = 2m = 3r.$$

We now compute the number of regions whose boundary contains either a vertex of degree 5 or a vertex of degree 6. Since the boundary of every region is a triangle, it follows, by assumption, that no region has two vertices of degree 5 on its boundary or a vertex of degree 5 and a vertex of degree 6 on its boundary. On the other hand, the boundary of a region could contain two or perhaps three vertices of degree 6. Each vertex of degree 5 lies on the boundaries of five regions and every vertex of degree 6 lies on the boundaries of six regions. Furthermore, every region containing a vertex of degree 6 on its boundary can contain as many as three vertices of degree 6. Therefore, G has $5n_5$ regions whose boundary contains a vertex of degree 5 and at least $6n_6/3 = 2n_6$ regions whose boundary contains at least one vertex of degree 6. Thus,

$$\begin{aligned} r &\ge 5n_5 + 2n_6 \ge 5n_5 + 2n_6 - n_7 - 4n_8 - \cdots - (3\Delta - 20)n_\Delta \\ &= \sum_{i=5}^{\Delta}(20 - 3i)n_i = 20n - 3\sum_{i=5}^{\Delta} i n_i = 20(m - r + 2) - 3(2m) \\ &= (20m - 20r + 40) - 9r = (30r - 20r + 40) - 9r \\ &= r + 40, \end{aligned}$$

which is a contradiction. ■

The following result gives a relationship among the degrees of the vertices in a maximal planar graph of order at least 4.

Theorem 10.12 *Let G be a maximal planar graph of order $n \ge 4$ and size m containing n_i vertices of degree i for $3 \le i \le \Delta = \Delta(G)$. Then*

$$3n_3 + 2n_4 + n_5 = 12 + n_7 + 2n_8 + \cdots + (\Delta - 6)n_\Delta.$$

Proof. Since $m = 3n - 6$, it follows that $2m = 6n - 12$. Therefore,

$$\sum_{i=3}^{\Delta} i n_i = \sum_{i=3}^{\Delta} 6n_i - 12$$

and so

$$\sum_{i=3}^{\Delta}(6 - i)n_i = 12. \tag{10.3}$$

Hence, $3n_3 + 2n_4 + n_5 = 12 + n_7 + 2n_8 + \cdots + (\Delta - 6)n_\Delta$. ■

Discharging

Heinrich Heesch introduced the idea of assigning what is called a "charge" to each vertex of a planar graph as well as **discharging rules** which indicate how charges can be redistributed among the vertices. In a maximal planar graph G, every vertex v of G is assigned a **charge** of $6 - \deg v$. In particular, every vertex of degree 5 receives a charge of $+1$, every vertex of degree 6 receives a charge of 0, and every vertex of degree 7 or more receives a negative charge. By appropriately redistributing positive charges, some useful results can often be obtained. According to equation (10.3) in the proof of Theorem 10.12, the sum of the charges of the vertices of a maximal planar graph of order 4 or more is 12. This is restated below.

Theorem 10.13 *If G is a maximal planar graph of order $n \geq 4$, size m and maximum degree $\Delta(G) = \Delta$ such that G has n_i vertices of degree i for $3 \leq i \leq \Delta$, then*

$$\sum_{i=3}^{\Delta}(6-i)n_i = 12.$$

According to Theorem 10.10, every vertex of a maximal planar graph of order 4 or more has degree 3 or more. Consequently, no vertex can contribute more than 3 to the sum in Theorem 10.13. Because this sum is 12, we have the following corollary, which is an extension of Corollary 10.5.

Corollary 10.14 *If G is a maximal planar graph of order at least 4, then G contains at least four vertices whose degrees are at most 5.*

We now use the discharging method to give an alternative proof of Theorem 10.11.

Theorem 10.15 *If G is a maximal planar graph of order 4 or more, then G contains at least one of the following: (1) a vertex of degree 3, (2) a vertex of degree 4, (3) two adjacent vertices of degree 5, (4) two adjacent vertices, one of which has degree 5 and the other has degree 6.*

Proof. Assume, to the contrary, that there exists a maximal planar graph G of order $n \geq 4$, where there are n_i vertices of degree i for $3 \leq i \leq \Delta = \Delta(G)$ such that G contains none of (1)–(4). Thus, $\delta(G) = 5$. To each vertex v of G assign the charge $6 - \deg v$. Hence, each vertex of degree 5 receives a charge of $+1$, each vertex of degree 6 receives no charge, and each vertex of degree 7 or more receives a negative charge. By Theorem 10.13, the sum of the charges of the vertices of G is

$$\sum_{i=3}^{\Delta}(6-i)n_i = 12.$$

Let there be given a planar embedding of G. For each vertex v of degree 5 in G, redistribute its charge of $+1$ by moving a charge of $\frac{1}{5}$ to each of its five neighbors, resulting in v now having a charge of 0. Hence, the sum of the charges of the vertices of G remains 12. Since G contains neither (3) nor (4), no vertex of degree 5 or 6 will have its charges increased. Consider a vertex u with $\deg u = k \geq 7$. Thus u received an initial charge of $6 - k$. Because no consecutive neighbors of u in the embedding can have degree 5, the vertex u can receive an added charge of $+\frac{1}{5}$ from at most $k/2$ of its neighbors. After the redistribution of charges, the new charge of u is at most

$$6 - k + \frac{k}{2} \cdot \frac{1}{5} = 6 - \frac{9k}{10} < 0.$$

Hence, no vertex of G now has a positive charge. This is impossible, however, since the sum of the charges of the vertices of G is 12. ∎

Another result concerning maximal planar graphs that can be proved with the aid of the discharging method (see Exercise 19) is due to Philip Franklin [95].

Theorem 10.16 *If G is a maximal planar graph of order 4 or more, then G contains at least one of the following: (1) a vertex of degree 3, (2) a vertex of degree 4, (3) a vertex of degree 5 that is adjacent to two vertices, each of which has degree 5 or 6.*

10.3 Characterizations of Planar Graphs

In the preceding two sections, we discussed several characteristics of planar graphs. However, a fundamental question remains: For a given graph G, how does one determine whether G is planar or nonplanar? Of course, if G can be drawn in the plane without any of its edges crossing, then G is planar. On the other hand, if G cannot be drawn in the plane without edges crossing, then G is nonplanar. Nevertheless, it may very well be difficult to see how to draw a graph G in the plane without edges crossing or to know that such a drawing is impossible. We saw from Theorem 10.4 that if G has order $n \geq 3$ and size m where $m > 3n-6$, then G is nonplanar. Also, as a consequence of Theorem 10.4, we saw in Corollary 10.5 that if G contains no vertex of degree less than 6, then G is nonplanar. Of course, in that case, $m \geq 3n$.

Any graph that is a subgraph of a planar graph must surely be planar. Equivalently, every graph containing a nonplanar subgraph must itself be nonplanar. Thus, to show that a disconnected graph G is planar it suffices to show that each component of G is planar. Hence, when considering planarity, we may restrict our attention to connected graphs. Since a connected graph G is planar if and only if each block of G is planar (see Exercise 1), it is sufficient to concentrate on 2-connected graphs only.

According to Corollaries 10.8 and 10.9, the graphs K_5 and $K_{3,3}$ are nonplanar. Hence, if a graph G should contain K_5 or $K_{3,3}$ as a subgraph, then G

is nonplanar. For the maximal planar graph G of order 5 and size 9 shown in Figure 10.8 (that is, G is obtained by deleting one edge from K_5), we consider the graph $F = G \,\Box\, K_3$, shown in Figure 10.8. Thus, F consists of three copies of G, denoted by G_1, G_2 and G_3, where $u_1u_2 \notin E(G_1)$, $v_1v_2 \notin E(G_2)$ and $w_1w_2 \notin E(G_3)$. To make it easier to draw G, the nine edges of each graph G_i $(1 \le i \le 3)$ are not drawn. The graph F has order 15 and size $m = 42$. Since $m = 42 > 39 = 3n - 6$, it follows that F is nonplanar. Furthermore, it can be shown that no subgraph of F is isomorphic to either K_5 or $K_{3,3}$. Thus, despite the fact that F contains neither K_5 nor $K_{3,3}$ as a subgraph, the graph F is nonplanar. Consequently, there must exist some other explanation as to why this graph is nonplanar.

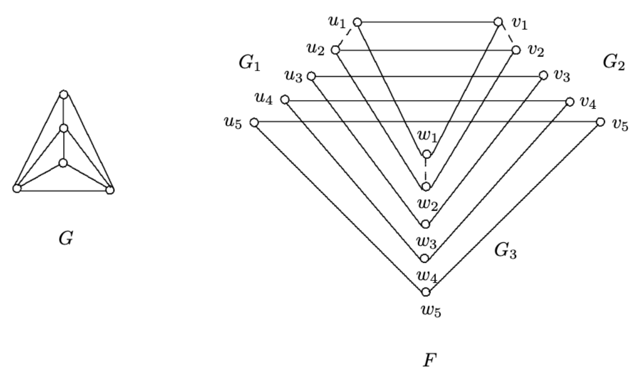


Figure 10.8: The graph $F = G \,\Box\, K_3$

Kuratowski's Theorem

A graph H is a **subdivision** of a graph G if either $H \cong G$ or H can be obtained from G by inserting vertices of degree 2 into some, all or none of the edges of G. Thus, for the graph G of Figure 10.9, all of the graphs H_1, H_2 and H_3 are subdivisions of G. Indeed, H_3 is also a subdivision of H_2.

Certainly, a subdivision H of a graph G is planar if and only if G is planar. Therefore, K_5 and $K_{3,3}$ are nonplanar as is any subdivision of K_5 or $K_{3,3}$. This provides a necessary condition for a graph to be planar.

Theorem 10.17 *A graph G is planar only if G contains no subgraph that is a subdivision of K_5 or $K_{3,3}$.*

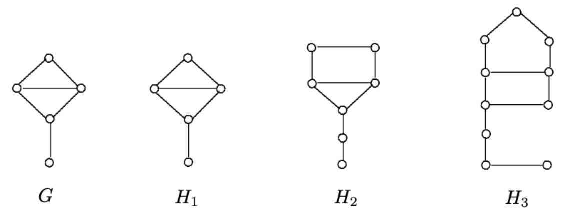


Figure 10.9: Subdivisions of a graph

The remarkable feature about this necessary condition for a graph to be planar is that the condition is also sufficient. The first published proof of this fact occurred in 1930. This theorem is due to the Polish topologist Kazimierz Kuratowski (1896–1980), who first announced this theorem in 1929. The title of Kuratowski's paper is "Sur le problème des courbes gauches en topologie" (On the problem of skew curves in topology), which suggests, and rightly so, that the setting of his theorem was in topology – not graph theory. Nonplanar graphs were sometimes called *skew graphs* during that period. The publication date of Kuratowski's paper was critical to having the theorem credited to him, for, as it turned out, later in 1930 the two American mathematicians Orrin Frink and Paul Althaus Smith submitted a paper containing a proof of this theorem as well but withdrew it after they became aware that Kuratowski's proof had preceded theirs, although just barely. They did publish a one-sentence announcement [96] of what they had accomplished in the *Bulletin of the American Mathematical Society* and, as the title "Irreducible non-planar graphs" of their note indicates, the setting for their proof was graph theoretical in nature.

It is believed by some that a proof of this theorem may have been discovered somewhat earlier by the Russian topologist Lev Semenovich Pontryagin, who was blind his entire adult life. Because the first proof of this theorem may have occurred in Pontryagin's unpublished notes, this result is sometimes referred to as the Pontryagin–Kuratowski theorem in Russia and elsewhere. However, since the possible proof of this theorem by Pontryagin did not satisfy the established practice of appearing in print in an accepted refereed journal, the theorem is generally credited to Kuratowski [151], and to Kuratowski alone.

Theorem 10.18 (**Kuratowski's Theorem**) *A graph G is planar if and only if G contains no subgraph that is a subdivision of K_5 or $K_{3,3}$.*

Proof. We have already noted the necessity of this condition for a graph to be planar. Hence, it remains to verify its sufficiency, namely that every graph containing no subgraph which is a subdivision of K_5 or $K_{3,3}$ is planar. Suppose that this statement is false. Then there is a nonplanar 2-connected graph G of minimum size containing no subgraph that is a subdivision of K_5 or $K_{3,3}$.

We claim, in fact, that G is 3-connected. Suppose that this is not the case. Then G contains a minimum vertex-cut consisting of two vertices x and y. Since G has no cut-vertices, it follows that each of x and y is adjacent to one or more vertices in each component of $G - \{x, y\}$. Let F_1 be one component of $G - \{x, y\}$ and let F_2 be the union of the remaining components of $G - \{x, y\}$. Furthermore, let

$$G_i = G[V(F_i) \cup \{x, y\}] \text{ for } i = 1, 2.$$

We consider two cases, depending on whether x and y are adjacent. Suppose first that x and y are adjacent. We claim that in this case at least one of G_1 and G_2 is nonplanar. If both G_1 and G_2 are planar, then there exist planar embeddings of these two graphs in which xy is on the boundary of the exterior region in each embedding. This, however, implies that G itself is planar, which is impossible. Thus G_1, say, is nonplanar. Since G_1 is a subgraph of G, it follows that G_1 contains no subgraph that is a subdivision of K_5 or $K_{3,3}$. However, the size of G_1 is less than the size of G, which contradicts the defining property of G. Hence, x and y must be nonadjacent.

Let f be the edge obtained by joining x and y, and let $H_i = G_i + f$ for $i = 1, 2$. If H_1 and H_2 are both planar, then, as above, there is a planar embedding of $G + f$ and of G as well. Since this is impossible, at least one of H_1 and H_2 is nonplanar, say H_1 is nonplanar. Because the size of H_1 is less than the size of G, the graph H_1 contains a subgraph F that is a subdivision of K_5 or $K_{3,3}$. Since G_1 contains no such subgraph, it follows that $f \in E(F)$. Let P be an $x - y$ path in G_2. By replacing f in F by P, we obtain a subgraph of G that is a subdivision of K_5 or $K_{3,3}$. This produces a contradiction. Hence, as claimed, G is 3-connected.

To summarize then, G is a nonplanar graph of minimum size containing no subgraph that is a subdivision of K_5 or $K_{3,3}$ and, as we just saw, G is 3-connected. Let $e = uv$ be an edge of G. Then $H = G - e$ is planar. Since G is 3-connected, H is 2-connected. By Theorem 3.4, there are cycles in H containing both u and v. Among all planar embeddings of H, choose one in which there is a cycle

$$C = (u = v_0, v_1, \ldots, v_\ell = v, \ldots, v_k = u)$$

containing u and v such that the number of regions interior to C is maximum.

It is convenient to define two subgraphs of H. By the *exterior subgraph* of H is meant the subgraph induced by those edges lying exterior to C and the *interior subgraph* of H is the subgraph induced by those edges lying interior to C. Both subgraphs exist, for otherwise the edge e could be added either to the exterior or interior subgraph of H so that the resulting graph (namely G) is planar.

No two distinct vertices of $\{v_0, v_1, \ldots, v_\ell\}$ or of $\{v_\ell, v_{\ell+1}, \ldots, v_k\}$ are connected by a path in the exterior subgraph of H, for otherwise there is a cycle in H containing u and v and having more regions interior to it than C has. Since G is nonplanar, there must be a $v_s - v_t$ path P in the exterior subgraph of H, where $0 < s < \ell < t < k$, such that v_s and v_t are the only vertices of P that belong to C. (See Figure 10.10.) In fact, the path P must be (v_s, v_t); for otherwise, if there is an interior vertex w on P, then, since G is 3-connected, there are three internally disjoint paths from w to C, creating a new cycle C' containing u and v such that C' has more regions interior to it than C has, which is a contradiction.

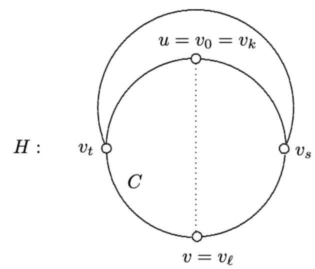


Figure 10.10: A step in the proof of Theorem 10.18

Let S be the set of vertices on C different from v_s and v_t, that is,

$$S = V(C) - \{v_s, v_t\},$$

and let H_1 be the component of $H - S$ that contains P. By the defining property of C, the subgraph H_1 cannot be moved to the interior of C in a plane manner. This fact together with the fact that $G = H + e$ is nonplanar implies that the interior subgraph of H must contain one of the following:

(1) A $v_a - v_b$ path with $0 < a < s$ and $\ell < b < t$ such that only v_a and v_b belong to C. (See Figure 10.11(a).)

(2) A vertex w not on C that is connected to C by three internally disjoint paths such that the terminal vertex of one such path P' is one of v_0, v_s, v_ℓ and v_t. If, for example, the terminal vertex of P' is v_0, then the terminal vertices of the other two paths are v_a and v_b, where $s \le a < \ell$ and $\ell < b \le t$ where not both $a = s$ and $b = t$ occur. (See Figure 10.11(b).) If the terminal vertex of P' is one of v_s, v_ℓ and v_t, then there are corresponding bounds for a and b for the terminal vertices of the other two paths.

(3) A vertex w not on C that is connected to C by three internally disjoint paths P_1, P_2 and P_3 such that the terminal vertices of these paths are three of the four vertices v_0, v_s, v_ℓ and v_t, say v_0, v_ℓ and v_s, respectively, together with a $v_c - v_t$ path P_4 ($v_c \ne v_0, v_\ell, w$), where v_c is on P_1 or P_2 and P_4 is disjoint from P_1, P_2 and C except for v_c and v_t. (See Figure 10.11(c).) The remaining choices for P_1, P_2 and P_3 produce three analogous cases.

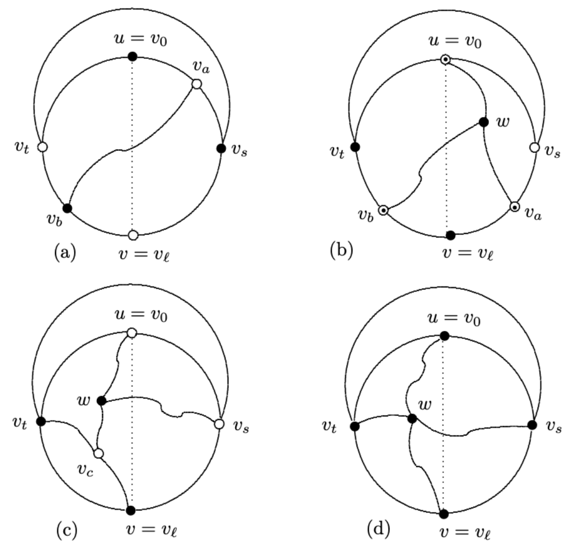


Figure 10.11: Situations (1)–(4) in the proof of Theorem 10.18

(4) A vertex w not on C that is connected to v_0, v_s, v_ℓ and v_t by four internally disjoint paths. (See Figure 10.11(d).)

In the first three cases, there is a subgraph of G that is a subdivision of $K_{3,3}$, while in the fourth case, there is a subgraph of G that is a subdivision of K_5. This is a contradiction. ■

While it is rarely easy to use Kuratowski's theorem to test a graph for planarity, a number of efficient algorithms have been developed that determine whether a graph is planar, including linear-time algorithms, the first of which was obtained by John Hopcroft and Robert Tarjan [131].

As a consequence of Kuratowski's theorem, the 4-regular graph G shown in Figure 10.12(a) is nonplanar since G contains the subgraph H in Figure 10.12(b) that is a subdivision of $K_{3,3}$.

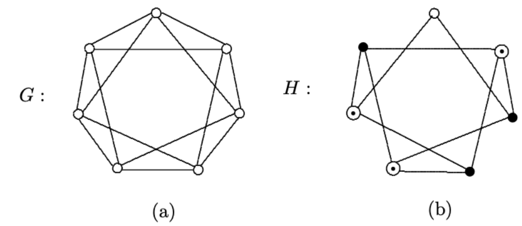


Figure 10.12: A nonplanar graph

Minors of Graphs

There is another characterization of planar graphs closely related to that given in Kuratowski's theorem. Before presenting this theorem, it is useful to introduce some additional terminology. If two adjacent vertices u and v in a graph G are identified, then we say that we have **contracted** the edge uv (denoting the resulting vertex by u or v). For the graph G of Figure 10.13, the graph G' is obtained by contracting the edge uv in G and where G'' is obtained by contracting the edge wy in G'.

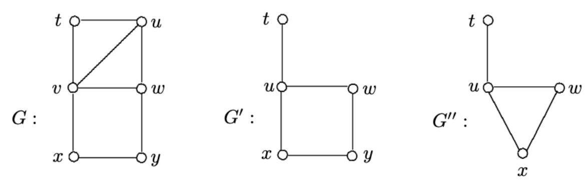


Figure 10.13: Contracting an edge

When dealing with edge contractions, it is often the case that we begin with a graph G, contract an edge in G to obtain a graph G', contract some edge in G' to obtain another graph G'', and so on, until finally arriving at a graph H. Any such graph H can be obtained in a different and perhaps simpler manner. In particular, H can be obtained from G by a succession of edge contractions if and only if the vertex set of H is the set of elements in a partition $\{V_1, V_2, \ldots, V_k\}$ of $V(G)$ where each induced subgraph $G[V_i]$ is connected and V_i is adjacent to V_j $(i \neq j)$ if some vertex in V_i is adjacent to some vertex in V_j in G. For example, in the graph G of Figure 10.13, if we were to let

$$V_1 = \{t\},\ V_2 = \{u, v\},\ V_3 = \{x\} \text{ and } V_4 = \{w, y\},$$

then the resulting graph H is shown in Figure 10.14. This is the graph G'' of Figure 10.13 obtained by successively contracting the edge uv in G and then the edge wy in G'.

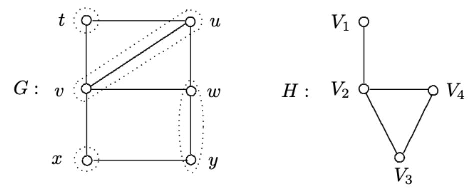


Figure 10.14: Edge contractions

A graph H is called a **minor** of a graph G if either $H \cong G$ or a graph isomorphic to H can be obtained from G by a succession of edge contractions, edge deletions and vertex deletions (in any order). Equivalently, H is a minor of G if $H \cong G$ or H can be obtained from a subgraph of G by a succession of edge contractions. Consequently, a graph G is a minor of itself. If H is a minor of G such that $H \ncong G$, then H is called a **proper minor** of G.

The planar graph H of Figure 10.14 is a minor of the planar graph G of that figure. In fact, every minor of a planar graph is planar.

Consider next the graph G_1 of Figure 10.15, where

$$V_1 = \{t_1, t_2\},\ V_2 = \{u_1, u_2, u_3, u_4\},$$
$$V_3 = \{v_1\},\ V_4 = \{w_1, w_2, w_3\},$$
$$V_5 = \{x_1, x_2\},\ V_6 = \{y_1\},\text{ and } V_7 = \{z_1\}.$$

Then the graph H_1 of Figure 10.15 can be obtained from G_1 by successive edge contractions. Thus, H_1 is a minor of G_1. By deleting the edge V_2V_6 and the vertices V_6 and V_7 from H_1 (or equivalently, deleting V_6 and V_7 from H_1), we see that K_5 is also a minor of G_1.

The example in Figure 10.15 serves to illustrate the following observation.

Theorem 10.19 *If a graph G is a subdivision of a graph H, then H is a minor of G.*

The converse of Theorem 10.19 is not true, however (see Exercise 30). The following is an immediate consequence of Theorem 10.19.

Theorem 10.20 *If G is a nonplanar graph, then K_5 or $K_{3,3}$ is a minor of G.*

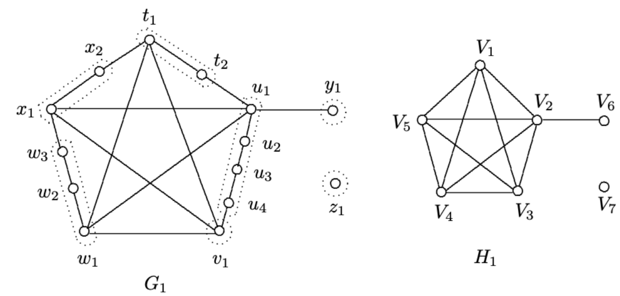


Figure 10.15: Minors of graphs

Wagner's Theorem

The German mathematician Klaus Wagner (1910–2000) showed that the converse of Theorem 10.20 is true [251] in 1937, only a year after obtaining his Ph.D. from Universität zu Köln (University of Cologne), thereby giving another characterization of planar graphs.

Theorem 10.21 (**Wagner's Theorem**) *A graph G is planar if and only if neither K_5 nor $K_{3,3}$ is a minor of G.*

Proof. We have already mentioned (in Theorem 10.20) that if a graph G is nonplanar, then either K_5 or $K_{3,3}$ is a minor of G. For the converse, suppose that G is a graph having K_5 or $K_{3,3}$ as a minor. If G were planar, then every minor of G is planar, contradicting the assumption that K_5 or $K_{3,3}$ is a minor of G. Thus, G is nonplanar. ■

More specifically, Theorem 10.21 can be proved with the aid of the following result.

Theorem 10.22 *Let G be a graph.*

(*a*) *If G has $K_{3,3}$ as a minor, then G contains a subdivision of $K_{3,3}$.*

(*b*) *If G has K_5 as a minor, then G contains either a subdivision of K_5 or a subdivision of $K_{3,3}$.*

Proof. Suppose first that $H = K_{3,3}$ is a minor of G. The graph H can be obtained by first deleting edges and vertices of G (if necessary), obtaining a connected graph G', and then by a succession of edge contractions in G'. We show, in this case, that G' contains a subgraph that is a subdivision of $K_{3,3}$ and so G', and G as well, is nonplanar.

Denote the vertices of H by U_i and W_i $(1 \leq i \leq 3)$, where $\{U_1, U_2, U_3\}$ and $\{W_1, W_2, W_3\}$ are the partite sets of H. Since H is obtained from G' by a succession of edge contractions, the subgraphs

$$F_i = G'[U_i] \text{ and } H_i = G'[W_i],\ 1 \leq i \leq 3,$$

are connected. Since $U_iW_j \in E(H)$ for $1 \leq i, j \leq 3$, there is a vertex $u_{i,j} \in U_i$ that is adjacent in H to a vertex $w_{i,j} \in W_j$. Among the vertices $u_{i,1}, u_{i,2}, u_{i,3}$ in U_i $(1 \leq i \leq 3)$, two or possibly all three may represent the same vertex. If $u_{i,1} = u_{i,2} = u_{i,3}$, then denote this vertex by u_i; if two of $u_{i,1}, u_{i,2}, u_{i,3}$ are the same, say $u_{i,1} = u_{i,2}$, then denote this vertex by u_i; if $u_{i,1}, u_{i,2}$ and $u_{i,3}$ are distinct, then let u_i denote a vertex in U_i that is connected to $u_{i,1}, u_{i,2}$ and $u_{i,3}$ by internally disjoint paths in F_i (possibly $u_i = u_{i,j}$ for some j). We proceed in the same manner to obtain vertices $w_i \in W_i$ for $1 \leq i \leq 3$. The subgraph of G induced by the previously described nine edges joining $U_1 \cup U_2 \cup U_3$ and $W_1 \cup W_2 \cup W_3$ together with the edge sets of all of the previously mentioned paths in F_i and H_j $(1 \leq i, j \leq 3)$ is a subdivision of $K_{3,3}$.

Next, suppose that $H = K_5$ is a minor of G. Then H can be obtained by first deleting edges and vertices of G (if necessary), obtaining a connected graph G', and then by a succession of edge contractions in G'. We show in this case that either G' contains a subgraph that is a subdivision of K_5 or G' contains a subgraph that is a subdivision of $K_{3,3}$.

We may denote the vertices of H by V_i $(1 \leq i \leq 5)$, where $G_i = G'[V_i]$ is a connected subgraph of G' and each subgraph G_i contains a vertex that is adjacent to G_j for each pair i, j of distinct integers where $1 \leq i, j \leq 5$. For $1 \leq i \leq 5$, let $v_{i,j}$ be a vertex of G_i that is adjacent to a vertex of G_j, where $1 \leq j \leq 5$ and $j \neq i$.

For a fixed integer i with $1 \leq i \leq 5$, if the vertices $v_{i,j}$ $(i \neq j)$ represent the same vertex, then denote this vertex by v_i. If three of the four vertices $v_{i,j}$ are the same, then we also denote this vertex by v_i. If two of the vertices $v_{i,j}$ are the same, the other two are distinct, and there exist internally disjoint paths from the coinciding vertices to the other two vertices, then we denote the two coinciding vertices by v_i. If the vertices $v_{i,j}$ are distinct and G_i contains a vertex from which there are four internally disjoint paths (one of which may be trivial) to the vertices $v_{i,j}$, then denote this vertex by v_i. Hence, there are several instances in which we have defined a vertex v_i. Should v_i be defined for all i $(1 \leq i \leq 5)$, then G' (and therefore G as well) contains a subgraph that is a subdivision of K_5.

We may assume then that for one or more integers i $(1 \leq i \leq 5)$, the vertex v_i has not been defined. For each such i, there exist distinct vertices u_i and w_i, each of which is connected to two of the vertices $v_{i,j}$ by internally disjoint (possibly trivial) paths, while u_i and w_i are connected by a path none of whose internal vertices are the vertices $v_{i,j}$ and where every two of the five paths have only u_i or w_i in common. If two of the vertices $v_{i,j}$ coincide, then we denote this vertex by u_i. If the remaining two vertices $v_{i,j}$ should also coincide, then we denote this vertex by w_i. We may assume that $i = 1$, that u_1 is connected to $v_{1,2}$ and $v_{1,3}$ and that w_1 is connected to $v_{1,4}$ and $v_{1,5}$, as described above. Denote the edge set of these paths by E_1.

We now consider G_2. If $v_{2,1} = v_{2,4} = v_{2,5}$, then let w_2 be this vertex and set $E_2 = \emptyset$; otherwise, there is a vertex w_2 of G_2 (which may coincide with $v_{2,1}$, $v_{2,4}$ or $v_{2,5}$) connected by internally disjoint (possibly trivial) paths to the distinct vertices in $\{v_{2,1}, v_{2,4}, v_{2,5}\}$. We then let E_2 denote the edge set of these paths. Similarly, the vertices w_3, u_2, and u_3 and the sets E_3, E_4, and E_5 are defined with the aid of the sets $\{v_{3,1}, v_{3,4}, v_{3,5}\}$, $\{v_{4,1}, v_{4,4}, v_{4,5}\}$ and $\{v_{5,1}, v_{5,2}, v_{5,3}\}$, respectively. The subgraph of G' induced by the union of the sets E_i and the edges $v_{i,j}v_{j,i}$ contains a subdivision of $K_{3,3}$ with partite sets $\{u_1, u_2, u_3\}$ and $\{w_1, w_2, w_3\}$. ■

In the proof of Theorem 10.22, it was shown that if K_5 is a minor of a graph G, then G contains a subdivision of K_5 or a subdivision of $K_{3,3}$. In other words, we were unable to show that G necessarily contains a subdivision of K_5. There is good reason for this, which is illustrated in the next example.

The Petersen graph P has order $n = 10$ and size $m = 15$. Since $m < 3n - 6$, no conclusion can be drawn from Theorem 10.3 regarding the planarity or nonplanarity of P. Nevertheless, the Petersen graph is, in fact, nonplanar. Theorems 10.18 and 10.21 give two ways to establish this fact. Figures 10.16(a) and 10.16(b) show P drawn in two ways. Since $P - x$ (shown in Figure 10.16(c)) is a subdivision of $K_{3,3}$, the Petersen graph is nonplanar. The partition $\{V_1, V_2, \ldots, V_5\}$ of $V(P)$ shown in Figure 10.16(d), where $V_i = \{u_i, v_i\}$, $1 \le i \le 5$, shows that K_5 in Figure 10.16(d) is a minor of P and is therefore nonplanar. Since P is a cubic graph, there is no subgraph of P that is a subdivision of K_5, however.

Outerplanar Graphs

We now turn our attention to a special class of planar graphs. A graph G is **outerplanar** if there exists a planar embedding of G so that every vertex of G lies on the boundary of the exterior region. Actually, if there is a planar embedding of G so that every vertex of G lies on the boundary of the same region of G, then G is outerplanar. The following two results provide characterizations of outerplanar graphs.

Theorem 10.23 *A graph G is outerplanar if and only if the join $G \vee K_1$ is planar.*

Proof. Let G be an outerplanar graph and suppose that G is embedded in the plane such that every vertex of G lies on the boundary of the exterior region. Then a vertex v can be placed in the exterior region of G and joined to all vertices of G in such a way that a planar embedding of $G \vee K_1$ results. Thus, $G \vee K_1$ is planar.

For the converse, assume that G is a graph such that $G \vee K_1$ is planar. Hence $G \vee K_1$ contains a vertex v that is adjacent to every vertex of G. Let

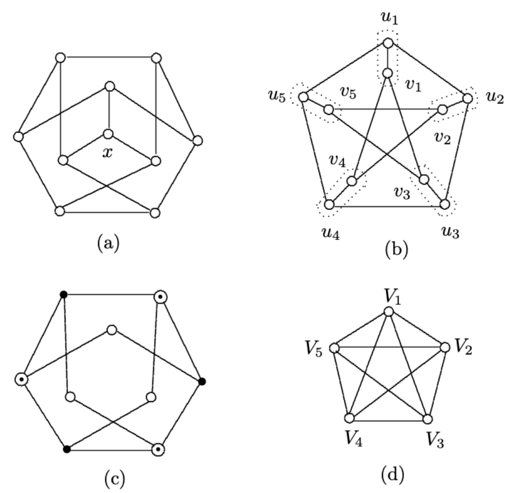


Figure 10.16: Showing that the Petersen graph is nonplanar

there be a planar embedding of $G \vee K_1$. Upon deleting the vertex v, we arrive at a planar embedding of G in which all vertices of G lie on the boundary of the same region. Thus, G is outerplanar. ■

The following characterization of outerplanar graphs is analogous to the characterization of planar graphs stated in Kuratowski's theorem.

Theorem 10.24 *A graph G is outerplanar if and only if G contains no subgraph that is a subdivision of K_4 or $K_{2,3}$.*

Proof. Suppose first that there exists some outerplanar graph G that contains a subgraph H that is a subdivision of K_4 or $K_{2,3}$. By Theorem 10.23, $G \vee K_1$ is planar. Since the subgraph $H \vee K_1$ of $G \vee K_1$ is a subdivision of K_5 or a subdivision of $K_{3,3}$, it follows that $G \vee K_1$ contains a subgraph that is a subdivision of K_5 or a subdivision of $K_{3,3}$ and so is nonplanar. This produces a contradiction.

For the converse, assume, to the contrary, that there exists a graph G that is not outerplanar but contains no subgraph that is a subdivision of K_4 or $K_{2,3}$. By Theorem 10.23, $G \vee K_1$ is not planar but contains no subgraph that is a subdivision of K_5 or $K_{3,3}$. This contradicts Theorem 10.18. ■

An outerplanar graph G is **maximal outerplanar** if the addition to G

of any edge joining two nonadjacent vertices of G results in a graph that is not outerplanar. Necessarily then, there is a planar embedding of a maximal outerplanar graph G of order at least 3, where the boundary of whose exterior region is a Hamiltonian cycle of G and the boundary of every other region is a triangle. Every maximal outerplanar graph of order 3 or more is therefore nearly maximal planar. We now describe some other facts about outerplanar graphs.

Theorem 10.25 *Every nontrivial outerplanar graph contains at least two vertices of degree* 2 *or less.*

Proof. Let G be a nontrivial outerplanar graph. The result is obvious if the order of G is 4 or less, so we may assume that the order of G is at least 5. Add edges to G, if necessary, to obtain a maximal outerplanar graph. Thus, the boundary of the exterior region of G is a Hamiltonian cycle of G. Among the chords of C, let uv be one such that uv and a $u-v$ path on C produce a cycle containing a minimum number of interior regions. Necessarily, this minimum is 1. Then the degree of the remaining vertex y on the boundary of this region is 2. There is such a chord wx of C on the other $u-v$ path of C, producing another vertex z of degree 2. In G, the degrees of y and z are therefore 2 or less. ∎

Theorem 10.26 *The size of every outerplanar graph of order $n \geq 2$ is at most $2n-3$.*

Proof. Let G be an outerplanar graph of order $n \geq 2$ and size m. By Theorem 10.23, $H = G \vee K_1$ is planar. Since H has order $n' = n+1$ and size $m' = m+n$, it follows that $m' \leq 3n' - 6$ and so $m + n \leq 3(n+1) - 6$. Thus, $m \leq 2n-3$. ∎

In view of Theorem 10.26, the size of a maximal outerplanar graph of order $n \geq 2$ is $2n-3$.

10.4 Hamiltonian Planar Graphs

In Sections 10.1 and 10.2, we saw necessary conditions for a connected graph to be planar and necessary conditions for a graph of order at least 4 to be maximal planar. In this section, we will be introduced to a single result, namely a necessary condition for a planar graph to be Hamiltonian.

Grinberg's Theorem

Let G be a Hamiltonian planar graph of order n with Hamiltonian cycle C and let there be given a planar embedding of G. An edge of G not lying on C

is a **chord** of G. Every chord and every region of G then lies interior to C or exterior to C. For $i = 3, 4, \ldots, n$, let r_i denote the number of regions interior to C whose boundary contains exactly i edges and let r'_i denote the number of regions exterior to C whose boundary contains exactly i edges.

The plane graph G of Figure 10.17 of order 12 is Hamiltonian. The edges of the Hamiltonian cycle $C = (v_1, v_2, \ldots, v_{12}, v_1)$ are drawn with bold lines. With respect to C, we have

$$r_3 = r'_3 = 1,\ r_4 = 3,\ r'_4 = 2,\ r_5 = r'_7 = 1,$$

while $r_i = 0$ for $6 \leq i \leq 12$ and $r'_i = 0$ for $i = 5, 6$ and $8 \leq i \leq 12$.

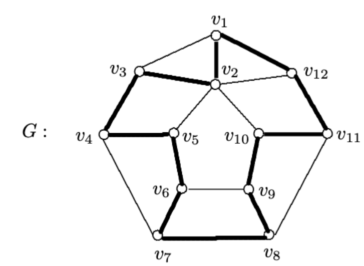


Figure 10.17: A Hamiltonian planar graph

In 1968 a necessary condition for a planar graph to be Hamiltonian was discovered by the Latvian mathematician Emanuels Ja. Grinberg [109].

Theorem 10.27 (**Grinberg's Theorem**) *For a plane graph G of order n with Hamiltonian cycle C,*

$$\sum_{i=3}^{n}(i-2)(r_i - r'_i) = 0.$$

Proof. Suppose that c chords of G lie interior to C. Then $c+1$ regions of G lie interior to C. Therefore,

$$\sum_{i=3}^{n} r_i = c + 1 \text{ and so } c = \sum_{i=3}^{n} r_i - 1.$$

Let N denote the result obtained by summing over all regions interior to C the number of edges on the boundary of each such region. Then each edge on C is counted once and each chord interior to C is counted twice, that is,

$$N = \sum_{i=3}^{n} ir_i = n + 2c.$$

Therefore,

$$\sum_{i=3}^{n} ir_i = n + 2c = n + 2\sum_{i=3}^{n} r_i - 2$$

and so

$$\sum_{i=3}^{n}(i-2)r_i = n - 2.$$

Similarly,

$$\sum_{i=3}^{n}(i-2)r_i' = n - 2.$$

Therefore, $\sum_{i=3}^{n}(i-2)(r_i - r_i') = 0$. ■

The following observations are quite useful in applying Theorem 10.27. Let G be a plane graph with Hamiltonian cycle C. Furthermore, suppose that an edge e of G is on the boundary of two regions R_1 and R_2 of G. If e is an edge of C, then one of R_1 and R_2 is in the interior of C and the other is in the exterior of C. If, on the other hand, e is not an edge of C, then R_1 and R_2 are either both in the interior of C or both in the exterior of C.

The Tutte Graph

In 1880, the British mathematician Peter Guthrie Tait conjectured that every 3-connected cubic planar graph is Hamiltonian. This conjecture was disproved in 1946 by William T. Tutte [239], who produced the graph G in Figure 10.18 as a counterexample. In addition to disproving Tait's conjecture, Tutte [243] proved that every 4-connected planar graph is Hamiltonian.

Since Theorem 10.27 gives a necessary condition for a planar graph to be Hamiltonian, this theorem also provides a sufficient condition for a planar graph to be non-Hamiltonian. We now see how Grinberg's theorem can be used to show that the plane graph of Figure 10.18 is not Hamiltonian. This graph is called the **Tutte graph** and has a great deal of historical interest. We will encounter this graph again in Chapter 17.

Assume, to the contrary, that the Tutte graph G is Hamiltonian. Then G has a Hamiltonian cycle C. Necessarily, C must contain exactly two of the three edges e, f_1 and f_2, say f_1 and either e or f_2. Similarly, C must contain exactly two edges of the three edges e', f_2 and f_3. Since we may assume that C contains f_2, we may further assume that e is not on C. Consequently, R_1 and R_2 lie interior to C.

Let G_1 denote the component of $G - \{e, f_1, f_2\}$ containing w. Thus, G_1 contains a Hamiltonian $v_1 - v_2$ path P'. Therefore, $G_2 = G_1 + v_1v_2$ is Hamiltonian and contains a Hamiltonian cycle C' consisting of P' and v_1v_2. Applying Grinberg's theorem to G_2 with respect to C', we obtain

$$1(r_3 - r_3') + 2(r_4 - r_4') + 3(r_5 - r_5') + 6(r_8 - r_8') = 0. \qquad (10.4)$$

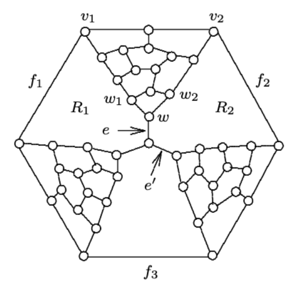


Figure 10.18: The Tutte graph

Since v_1v_2 is on C' and the exterior region of G_2 lies exterior to C', it follows that

$$r_3 - r_3' = 1 - 0 = 1 \quad \text{and} \quad r_8 - r_8' = 0 - 1 = -1.$$

Therefore, from (10.4), we have

$$2(r_4 - r_4') + 3(r_5 - r_5') = 5.$$

Necessarily, both ww_1 and ww_2 are edges of C' and so $r_4 \geq 1$, implying that either

$$r_4 - r_4' = 1 - 1 = 0 \text{ or } r_4 - r_4' = 2 - 0 = 2.$$

If $r_4 - r_4' = 0$, then $3(r_5 - r_5') = 5$, which is impossible. On the other hand, if $r_4 - r_4' = 2$, then $3(r_5 - r_5') = 1$, which is also impossible. Hence, G is not Hamiltonian.

For many years, Tutte's graph was the only known example of a 3-connected cubic planar graph that was not Hamiltonian. Much later, however, other such graphs have been found; for example, Grinberg himself found another counterexample to Tait's conjecture (see Exercise 39).

Exercises for Chapter 10

Section 10.1: The Euler Identity

1. Prove that a graph is planar if and only if each of its blocks is planar.

2. Prove Corollary 10.2: *If G is a plane graph with n vertices, m edges and r regions, then $n - m + r = 1 + k(G)$.*

3. Give an example of a graph G of order 8 such that G and $\overline{G}$ are planar.

4. Prove that if G is a planar graph of order 11, then $\overline{G}$ is nonplanar.

5. (a) Prove that the order of every 3-regular planar graph containing no triangle or 4-cycle is at least 20.
   (b) Show that the Petersen graph is nonplanar.

6. (a) Show that if G is a planar graph containing no vertex of degree less than 5, then G contains at least 12 vertices of degree 5.
   (b) Give an example of a planar graph that contains no vertex of degree less than 5.

7. Show that every graph G of order $n \geq 6$ that contains three spanning trees T_1, T_2 and T_3 such that every edge of G belongs to exactly one of these three trees is nonplanar.

8. If the boundary of every interior region of a plane graph G of order n and size m is a triangle and the boundary of the exterior region is a k-cycle ($k \geq 3$), express m in terms of n and k.

9. Determine all connected regular planar graphs G such that the number of regions in a planar embedding of G equals its order.

10. A cubic polyhedron P has only 5-sided, 6-sided and 7-sided faces. Determine a formula for the number of 5-sided faces in P.

Section 10.2: Maximal Planar Graphs

11. Give an example of two non-isomorphic maximal planar graphs of the same order.

12. Prove that there exists only one 4-regular maximal planar graph.

13. Prove that a planar graph of order $n \geq 3$ and size m is maximal planar if and only if $m = 3n - 6$.

14. Prove that every maximal planar graph of order 4 or more is 3-connected.

15. In a planar embedding of a maximal planar graph G of order 6, a vertex is placed in each interior region of G and joined to the vertices on its boundary, producing a graph H. Prove or disprove: H is Hamiltonian.

16. Determine all maximal planar graphs G of order 3 or more such that the number of regions in a planar embedding of G equals its order.

17. Determine whether the graph G shown in Figure 10.19 is nearly maximal planar.

G :

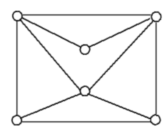


Figure 10.19: The graph G in Exercise 17

18. A nontrivial tree T of order n has the property that $\overline{T}$ is a maximal planar graph.

    (a) What is n?

    (b) Give an example of a tree T with this property.

19. Use a discharging method to prove Theorem 10.16: *If G is a maximal planar graph of order* 4 *or more, then G contains at least one of the following:* (1) *a vertex of degree* 3, (2) *a vertex of degree* 4, (3) *a vertex of degree* 5 *that is adjacent to two vertices, each of which has degree* 5 *or* 6.

20. If every vertex v of a nontrivial tree T is given a charge of $2 - \deg v$, then what is the sum of the charges of the vertices of T?

Section 10.3: Characterizations of Planar Graphs

21. Let U be a subset of the vertex set of a graph G, and suppose that $H = G[U]$ is an induced subgraph of G of order k. Let H' be the graph obtained by inserting a vertex of degree 2 into every edge of H. Let G' be the graph obtained by inserting a vertex of degree 2 into every edge of G.

    (a) Prove or disprove: If G is planar, then H' is planar.

    (b) What is $G'[U]$?

22. Let T be a tree of order at least 4, and let $e_1, e_2, e_3 \in E(\overline{T})$. Prove that $T + e_1 + e_2 + e_3$ is planar.

23. Let T be a tree of order at least 5, and let $e_1, e_2, \cdots, e_5 \in E(\overline{T})$. Let $G = T + \{e_1, e_2, \cdots, e_5\}$. Prove that if G does not contain a subdivision of $K_{3,3}$, then G is planar.

24. (a) Determine the order n, the size m and the number $3n - 6$ for the graph $K_4 \,\square\, K_2$.

    (b) What does the information in (a) say about the planarity of $K_4 \,\square\, K_2$?

    (c) Is $K_4 \,\square\, K_2$ planar or nonplanar?

25. Determine all graphs G of order $n \geq 5$ and size $m = 3n - 5$ such that for each edge e of G, the graph $G - e$ is planar.

26. Let $S_{a,b}$ denote the double star in which the degrees of the two vertices that are not end-vertices are a and b. Determine all pairs a, b of integers such that $\overline{S}_{a,b}$ is planar.

27. A nonplanar graph G of order 7 has the property that $G - v$ is planar for every vertex v of G.

    (a) Show that G does not contain $K_{3,3}$ as a subgraph.

    (b) Give an example of a graph with this property.

28. Determine all integers $n \geq 3$ such that $\overline{C}_n$ is planar.

29. Determine all integers $n \geq 3$ such that C_n^2 is nonplanar.

30. Show that the converse of Theorem 10.19 is not, in general, true.

31. It has been observed that if a graph H is a minor of a planar graph, then H is planar. Prove or disprove: If a minor H of a graph G is planar, then G is planar.

32. Let G be the graph shown in Figure 10.20.

    (a) Show that G contains $K_{3,3}$ as a subgraph.

    (b) Show that G does not contain a subdivision of K_5 as a subgraph.

    (c) Show that K_5 is a minor of G.

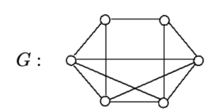


Figure 10.20: The graph G in Exercise 32

33. (a) Let G be a 4-regular graph of order 10 and size m. What can be deduced about the planarity of G by comparing the numbers m and $3n - 6$?

(b) Prove or disprove: There is a planar 2-connected 4-regular graph of order 10.

(c) Show that the 2-connected 4-regular graph H of order 10 shown in Figure 10.21 does not contain K_5 as a subgraph but does contain K_5 as a minor. What can you conclude from this?

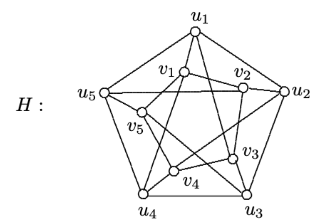


Figure 10.21: A 2-connected 4-regular graph of order 10

(d) Prove that if G is a graph of order $n \geq 5$ and size $m \geq 3n - 5$, then G need not contain K_5 as a subgraph but must contain a subgraph with minimum degree 4.

34. (a) What is the minimum possible order of a graph G containing only vertices of degree 3 and degree 4 and an equal number of each such that G contains a subdivision of K_5?

(b) Does the graph H of Figure 10.22 contain a subdivision of K_5 or a subdivision of $K_{3,3}$?

(c) Does the graph H of Figure 10.22 contain K_5 or $K_{3,3}$ as a minor?

(d) Is the graph H of Figure 10.22 planar or nonplanar?

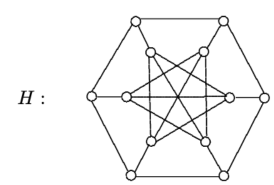


Figure 10.22: The graph H in Exercise 34

35. Determine all connected graphs G of order $n \geq 4$ such that $G \vee K_1$ is outerplanar.

36. For a positive integer k, a graph G of order $n > k$ and size m is said to have property π_k if (1) $m = kn - \binom{k+1}{2}$ and (2) for every induced subgraph H of order p and size q in G, where $k \leq p < n$, it follows that $q \leq kp - \binom{k+1}{2}$.

    (a) Show that $\delta(G) \geq k$.

    (b) Show that $\omega(G) \leq k + 1$.

    (c) What familiar class of graphs has property π_2? Show that there is a graph having property π_2 that does not belong to this class.

    (d) What familiar class of graphs has property π_3? Show that there is a graph having property π_3 that does not belong to this class.

Section 10.4: Hamiltonian Planar Graphs

37. Show, by applying Theorem 10.27, that $K_{2,3}$ is not Hamiltonian.

38. Show, by applying Theorem 10.27, that each of the graphs in Figure 10.23 is not Hamiltonian.

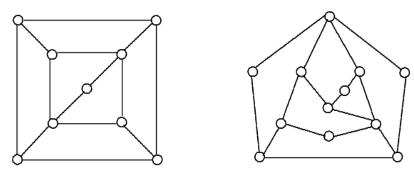


Figure 10.23: Graphs in Exercise 38

39. Show, by applying Theorem 10.27, that the **Grinberg graph** in Figure 10.24 is not Hamiltonian.

40. Show, by applying Theorem 10.27, that the **Herschel graph** in Figure 10.25 is not Hamiltonian.

41. Show, by applying Theorem 10.27, that no Hamiltonian cycle in the graph of Figure 10.26 contains both the edges e and f.

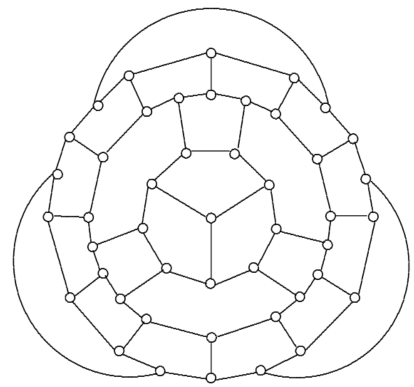


Figure 10.24: The Grinberg graph in Exercise 39

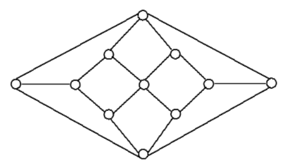


Figure 10.25: The Herschel graph in Exercise 40

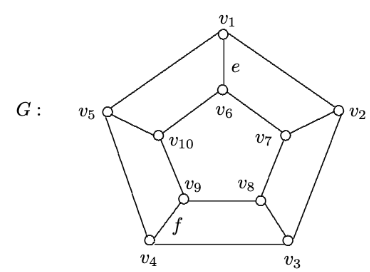


Figure 10.26: A Hamiltonian planar graph in Exercise 41