Math 453

Goal of course: Gain a true understanding of the structure of $\mathbb{R}$ & see how it makes calculus work.

Sets, Functions & Cardinalities

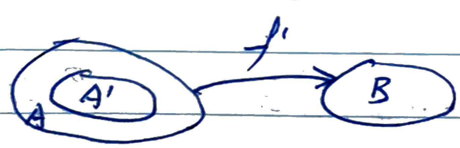
Notation: "$x \in A$" means "x is an element of the set A"
"$A \subseteq B$" means "A is a subset of B" (proper subset: $A \subsetneq B$)

Observe $x \in A \Leftrightarrow \{x\} \subseteq A$

Def: Let A & B be sets. A function $f: A \to B$ is a rule assigning to each $x \in A$ exactly one $f(x) \in B$.
A: domain B: codomain.
Warning: f must be defined everywhere. i.e. $f: \mathbb{N} \to \mathbb{N}$ $f(n) = \frac{1}{n}$

but $f: \mathbb{N} \to \mathbb{Q}$ (rational #'s) given by $f(n) = \frac{1}{n}$ is a function

Images & Inverse Images: Let $f: A \to B$ be a function



If $A' \subseteq A$ we define the ~~inverse~~ image of A' to be:
$$f(A') = \{ b \in B \mid b = f(a) \text{ for some } a \in A' \}$$
$$= \{ f(a) \mid a \in A' \}$$

If $B' \subseteq B$, the inverse image of B' is:
$$f^{-1}(B') = \{ a \in A \mid f(a) \in B' \}$$ (always exists even if f^{-1} does not)

Ex: $f: \mathbb{Q} \to \mathbb{Q}$ given by $f(x) = x^2$ $f^{-1}(\{1\}) = \{-1, 1\}$ $f^{-1}(\{2\}) = \emptyset$
$f^{-1}(4)$ makes no sense

Prop: $f(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f(A_i)$

Pf: "$\subseteq$" Let $b \in f(\bigcup_{i\in I} A_i)$. This means $\exists x \in \bigcup_{i\in I} A_i$ s.t. $f(x) = b$.

Since $x \in \bigcup_{i\in I} A_i$, $\exists i \in I$ so that $x \in A_i$

Since $x \in A_i$, this means $f(x) \in f(A_i)$. This means $f(x) \in \bigcup_{i\in I} f(A_i)$ i.e. $b \in \bigcup_{i\in I} A_i$

Hence $f(\bigcup_{i\in I} A_i) \subseteq \bigcup_{i\in I} f(A_i)$. "$\supseteq$" is HW.

Injectivity, surjectivity, bijectivity: Let $f: A \to B$ be a function.

- We say f is injective (1-1) if $\forall a, a' \in A$, $f(a) = f(a') \to a = a'$

  This is equivalent to $f^{-1}(\{b\})$ having at most one element for all $b \in B$.

- We say f is surjective (onto) if for every $b \in B$, there exists $a \in A$ so that $b = f(a)$

  This equivalent to $f^{-1}(\{b\})$ having at least one element for all $b \in B$.

- f is bijective if both injective & surjective.

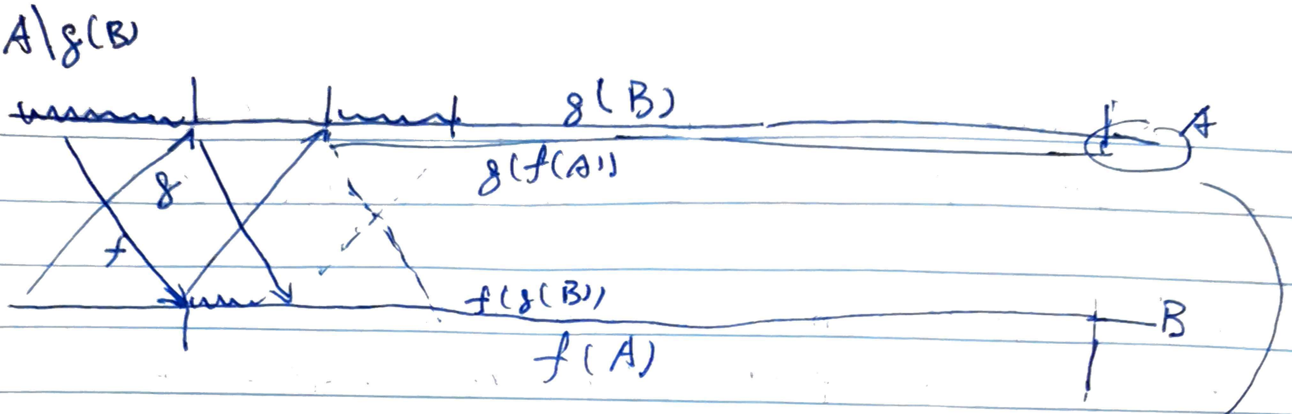
Fact $f: A \to B$ is bijective iff it has an inverse function $g: B \to A$ s.t. $f \circ g$ & $g \circ f$ are the identity function

Def: We say A & B have the same cardinality if $\exists h: A \to B$ bijective.

Useful Theorem for cardinality: (Schröder-Bernstein Theorem)

Suppose A & B are sets & $\exists f: A \to B$ injective & $\exists g: B \to A$ injective. Then A and B have the same cardinality.

(Sketch of proof): Goal: Construct $h: A \to B$ bijection using f & g.



left over piece
$\bigcap_{n=1}^{\infty} \cancel{f}\, g(f(g(\dots(f(A))\dots))$

Define h as follows:

$$h(a) = \begin{cases} f(a) & \text{if } a \in [A \setminus g(B)] \cup [g(f(A)) \setminus g(f(g(B)))] \cup \dots \cup (g\circ f)^n(A) \setminus (g\circ f)^n[g(B)] \\ f(a) & \text{if } a \in \bigcap_{n=1}^{\infty} [g\circ f]^n(A) \\ \text{The unique element of } g^{-1}(\{a\}) & \text{if } a \in g(B)\setminus g(f(A)) \dots \underbrace{\cup\, (f\circ g)^n(B) \setminus (g\circ f)^{n+1}(A) \cup \dots}_{\text{everything is in } g(B)} \end{cases}$$

Observation: If A is infinite, then $\exists$ an injective $f: \mathbb{N} \to A$

Why: Pick $a_1 \in A$, then pick $a_2 \in A \setminus \{a_1\}$. Inductively pick $a_n \in A \setminus \{a_1 \dots a_{n-1}\}$
Now define $f: \mathbb{N} \to A$ by $f(n) = a_n$. f is injective

Thm: Let A be an infinite set, TFAE:
① A is countable
② $\exists\ g: \mathbb{N} \to A$ surjective.
③ $\exists\ f: A \to \mathbb{N}$ injective.

Pf: ① → ②: We know $\exists h: \mathbb{N} \to A$ bijective. In particular, h is surjective.
② → ③: Assume $\exists g: \mathbb{N} \to A$ surjective. We want to build $f: A \to \mathbb{N}$ injective
Observe $\forall a \in A$, $g^{-1}(\{a\}) \neq \emptyset$ (Note: $g^{-1}(\{a\}) \subseteq \mathbb{N}$) Note if $a \neq a'$, $g^{-1}(\{a\}) \cap g^{-1}(\{a'\}) = \emptyset$. Otherwise if both contain m, then
$(m) = a$ & $(m = a'$ the as is a

Let L_a be the least element in $g^{-1}(\{a\})$. Let $f: A \to \mathbb{N}$ be defined by $f(a) = L_a$. Note $a \neq a' \Rightarrow L_a \neq L_{a'}$ as $g^{-1}(\{a\}) \cap g^{-1}(\{a'\}) = \emptyset \to$ f is injective.

③ → ① : If A is infinite, $\exists h: \mathbb{N} \to A$ injective. Since we also know $\exists f: A \to \mathbb{N}$ injective, this implication follows from Schröder-Bernstein.

Ex: $\mathbb{N} \times \mathbb{N}$ is countable.

Why? $\mathbb{N} \times \mathbb{N}$ is infinite, so we just need injective $h: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Define h by: $h(m,n) = 2^m 3^n$. By fundamental theorem of arithmetic,

$$h(m_1, n_1) = h(m_2, n_2) \Leftrightarrow 2^{m_1} 3^{n_1} = 2^{m_2} 3^{n_2} \Leftrightarrow m_1 = m_2,\ n_1 = n_2.$$

Thus h is injective.

Ex. If $A_1, A_2, \ldots, A_n$ are countable, then so is $\bigcup_{i=1}^{\infty} A_i$.

Why: $\forall i, \exists f: \mathbb{N} \to A_i$ injective. Write $f_i(j) = a_{ij} \in A_i$;

i.e $A_i = \{a_{i1}, a_{i2}, a_{i3}, \ldots\}$

Define $F: \mathbb{N} \times \mathbb{N} \to \bigcup_{i=1}^{\infty} A_i$ by $F(i,j) = a_{ij}$. Note f is surjective.

Note if $h: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is a bijection, then $F \circ h: \mathbb{N} \to \bigcup_{i=1}^{\infty}$ is onto.

Corollary: $\mathbb{Z}$ is countable: $\mathbb{Z} = \{1, 2, 3, \ldots\} \cup \{0, -1, -2, -3\}$.

$\mathbb{Q}$ is countable $\mathbb{Q} = \bigcup_{i=1}^{\infty} \{\frac{n}{i} \mid n \in \mathbb{Z}\}$.

← elements are functions

Def: If A & B are sets, set $A^B = \{f \mid f: B \to A\}$.

Ex: $A^{\mathbb{N}} = \{f \mid f: \mathbb{N} \to A\}$. In this case, let $a \in A^{\mathbb{N}}$ i.e $a: \mathbb{N} \to A$

$a(1) \in A,\ a(2) \in A, \cdots a(n) \in A$.

Really, a is a sequence in A: $a(1) = a_1, \cdots a(n) = a_n$

$a = (a_1, a_2, a_3, a_4, \ldots) \in A^{\mathbb{N}}$

Side note. Can visualize $A^{\mathbb{N} \times \mathbb{N}} \to$ an array $(a_{ij})_{i,j \in \mathbb{N}}$ of elements of A.