Last Time: Field axioms.

Bubble lemma:

If $x \in \mathbb{R}$ is nonzero, then $\exists N \in \mathbb{N}$ s.t. for any representative (x_k) of x, there is a $K \in \mathbb{N}$ so that whenever $k \geq K$, $|x_k| \geq \frac{1}{N}$.

Side note, say (x_k), a Cauchy sequence in $\mathbb{Q}$, represents

Order ~~of~~ on $\mathbb{R}$

Basic idea: Extend positivity & ">" $\mathbb{Q}$ to $\mathbb{R}$.

What does it mean for x to be positive: if $x = [(x_k)]$ then x_k are eventually positive.

Problem: $[(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots)] = 0$.

The fix: we say $x \in \mathbb{R}$ is positive if $\exists N \in$ s.t. for any representative (Cauchy sequence in $\mathbb{Q}$ (x_k)) of x, $\exists K$ s.t. $k \geq K \rightarrow x_k \geq \frac{1}{N}$

Thm: Fix $x \in \mathbb{R}$. Then exactly one of the following is true:

① x is positive ② $-x$ is positive ③ $x = 0$

Pf: Assume $x \neq 0$. By the bubble lemma, there is some $N \in \mathbb{N}$ so that for any representative (x_k) of x, $\exists k \geq K$ s.t. $|x_k| \geq \frac{1}{N}$. This means $\forall k \geq K$, either $x_k \geq \frac{1}{N}$ or $-x_k \geq \frac{1}{N}$ ($x_k \leq -\frac{1}{N}$)

Claim that $\exists K' \in \mathbb{N}$ s.t. $x_k \geq \frac{1}{N}$ for all $k \geq K'$ or $x_k \leq -\frac{1}{N}$ for all $k \geq K'$. If not, then $\forall K' \in \mathbb{N}$ we can find $k \geq K'$ s.t. $x_k \geq \frac{1}{N}$ & $l \geq K'$ s.t. $x_l \leq -\frac{1}{N}$.

i.e. $\exists \infty$-many k, l, s.t. $x_k \geq \frac{1}{N}$ & $x_l \leq -\frac{1}{N}$. This means $|x_k - x_l| = x_k - x_l \geq \frac{2}{N}$, contradicting (x_k) is Cauchy.

Thus K' exists. If $x_k \geq \frac{1}{N}$ for all $k \geq K'$, x is positive & if $x_k \leq -\frac{1}{N}$ for all $k \geq K'$, $-x$ is ~~negative~~ positive.

(note if $(x_k') \sim (x_k)$ & $\frac{1}{N}$ is as in bubble lemma, then $\nexists K'$ s.t. $x_k' \leq -\frac{1}{N}$ for all $k \geq K'$ since this means $|x_k - x_k'| > \frac{2}{N}$ contradicting $(x_k) \sim (x_k')$)

Def. We say x is negative if $-x$ is positive. If $x, y \in \mathbb{R}$, we say $x > y$ if $x-y$ is positive. $x \geq y$ if $x-y$ is positive or zero. ($x > 0$ means x is positive). Furthermore, define $|x|$ to be:
$|x| = x$ if $x \geq 0$ & $|x| = -x$ if $x < 0$.
(Hw: show if $x = [(x_k)] \Rightarrow |x| = [(|x_k|)]$)

Corollary of the above theorem: Let $x, y \in \mathbb{R}$. Exactly one of the following is true
① $x > y$ ② $x = y$ ③ $x < y$.

Prop: If $x = [(x_k)]$ & $y = [(y_k)]$ ((x_k), (y_k) Cauchy sequences in $\mathbb{Q}$.) & $\exists K \in \mathbb{N}$ s.t $k \geq K \rightarrow x_k \geq y_k$, then $x \geq y$.
Pf. Assume $\neg(x \geq y)$. Then $x < y$. This means $y - x > 0$.
This means $\exists N \in \mathbb{N}$, $\exists K \in \mathbb{N}$ s.t $k \geq K \rightarrow y_k - x_k \geq \frac{1}{N}$.
i.e. $\forall k \geq K$, $y_k \geq x_k + \frac{1}{N} \Rightarrow y_k > x_k$, a contrary to assumpsion.

Q: $x_k > y_k$ for all k, is $x > y$?
No. Consider $x_k = \frac{1}{k}$ $y_k = 0$. Both represent 0.

Prop: $\forall x, y \in \mathbb{R}$, $|x+y| \leq |x| + |y|$
Pf: Let (x_k) represent x & (y_k) represent y for Cauchy sequences (x_k) & (y_k) in $\mathbb{Q}$. $|x+y|$ is represented by $(|x_k + y_k|)$. $|x| + |y|$ is represented by $(|x_k| + |y_k|)$
Note by triangle inequality in $\mathbb{Q}$, $|x_k + y_k| \leq |x_k| + |y_k|$ for all k.
By previous prop. $|x+y| \leq |x| + |y|$.

Thm: (Archemedian Property): If $x \in \mathbb{R}$ & $x > 0$, then $\exists N \in \mathbb{N}$ s.t $x \geq \frac{1}{N}$.
Pf. Let $x > 0$, $\exists N$ s.t. for any rep. (x_k) of x, $\exists K$ s.t $k \geq K$ implies $x_k \geq \frac{1}{N}$.
Observe: $x = [(x_k)]$ & $\frac{1}{N} = [(\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \ldots)]$. Since $k \geq K \rightarrow x_k \geq \frac{1}{N}$.
It follows that $x \geq \frac{1}{N}$ by prop above.

Thm: Density of $\mathbb{Q}$ in $\mathbb{R}$: $\forall n \in \mathbb{N}$ & $x \in \mathbb{R}$, $\exists q_n \in \mathbb{Q}$ s.t. $|x - q_n| \leq \frac{1}{n}$.

Pf. Let (x_k) be a Cauchy sequence in $\mathbb{Q}$ representing x. Thus, given $n \in \mathbb{N}$, $\exists K$ s.t. $k, l \geq K \to |x_k - x_l| < \frac{1}{n}$.

Fix $l \geq K$. Claim $|x - x_l| < \frac{1}{n}$. Why?

$|x - x_l| = [(|x_1 - x_l|, |x_2 - x_l|, |x_3 - x_l|, \cdots, |x_k - x_l|, \cdots)]$. If $k \geq K$, then $|x_k - x_l| \leq \frac{1}{n} \Rightarrow |x - x_l| \leq \frac{1}{n}$. So let $q_n = x_l$.