Construction of $\mathbb{R}$

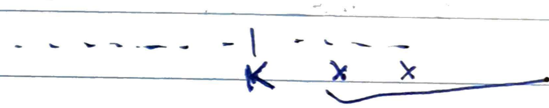
Basic question: which sequences in $\mathbb{Q}$ "should" converge?

Idea: Clearly need $\lim_{k\to\infty} |x_{k+1} - x_k| = 0$.

Not enough: $1, 2, 2\frac{1}{2}, 3, 3\frac{1}{3}, 3\frac{2}{3}, 4, 4\frac{1}{4}, \ldots$

Need <u>all</u> terms close ~~to~~ together: Arrived at Cauchy Sequence.

Def: We say a sequence (x_k) of rational numbers is Cauchy if for every $n \in \mathbb{N}$ there is a $K \in \mathbb{N}$ so that whenever $k \in \mathbb{N}$ & $l \in \mathbb{N}$, $k, l \geq K$, then $|x_k - x_l| < \frac{1}{n}$.



any two terms less than $\frac{1}{n}$ apart.

Formally: $\forall n \in \mathbb{N}, \exists K \in \mathbb{N}, \forall k \in \mathbb{N}, \forall l \in \mathbb{N}, (k \geq K) \wedge (l \geq K) \to |x_k - x_l| < \frac{1}{n}$

Ex: $1, 1.4, 1.414, 1.4142, \ldots$

Could define $\mathbb{R}$ as the space of Cauchy sequences. Problem $(0, 0, \ldots)$ & $(1, \frac{1}{2}, \frac{1}{3} \ldots)$ fix by identifying "equivalent" Cauchy sequences.

Def: Let (x_k) & (y_k) be Cauchy sequences of rationals. We say (x_k) is equivalent to (y_k) (i.e. $(x_k) \sim (y_k)$) whenever

~~$\forall n \in \mathbb{N}, \exists k \in \mathbb{N}, (k \geq K) \Rightarrow |x_k - y_k| < \frac{1}{n}$~~

$\forall n \in \mathbb{N}, \exists K \in \mathbb{N}, \forall k \in \mathbb{N}, (k \geq K) \to |x_k - y_k| < \frac{1}{n}$.

Proposition: "$\sim$" is an equivalence relation.

Pf: Must show

① $x_k \sim x_k$ ② $x_k \sim y_k \to y_k \sim x_k$ ③ $x_k \sim y_k \wedge y_k \sim z_k \to x_k \sim z_k$

Pf ①: Let $n \in \mathbb{N}$. Choose $K=1$. If $k \geq 1$, $|x_k - x_k| = 0 < \frac{1}{n}$. Thus $x_k \sim x_k$

Pf ②: L.T.R.

Pf ③: Use: Triangle inequality on $\mathbb{Q}$: $|x+y| \leq |x| + |y|$.

(use ↑ to prove $|x-y| = ||x| - |y||$ in hw

Scratch work: Bound $|x_k - z_k| = |(x_k - y_k) + (y_k - z_k)| \leq |x_k - y_k| + |y_k - z_k|$.

Fix $n \in \mathbb{N}$. Choose $K_1 \in \mathbb{N}$ s.t $k \geq K_1$ implies $|x_k - y_k| < \frac{1}{2n}$.

Choose $K_2 \in \mathbb{N}$ s.t $k \geq K_2$ implies $|y_k - z_k| < \frac{1}{2n}$

If $k \geq \max\{K_1, K_2\}$, then $|x_k - z_k| \leq |x_k - y_k| + |y_k - z_k| < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$.

Thus $x_k \sim z_k$.

Def: The set of real numbers, $\mathbb{R}$, is the set of equivalence classes of Cauchy sequences in $\mathbb{Q}$.

$[(x_k)] \to$ equivalence classes of $[(x_k)]$

We identify $\mathbb{Q} \subseteq \mathbb{R}$ by $q_k \mapsto [(q_k, q_k, q_k, \dots)]$

Arithmetic on $\mathbb{R}$

We want: $[(x_k)] + [(y_k)] = [(x_k + y_k)]$ & $[(x_k)][(y_k)] = [(x_k \cdot y_k)]$.

Must show definition is independent of representatives & $(x_k + y_k)$ and $(x_k \cdot y_k)$ are Cauchy.

Prop: let (x_k) & (y_k) be Cauchy then $(x_1 + x_2)$ & $(x_1 \cdot x_2)$ are cauchy.

Pf: Use $|(x_k + y_k) - (x_l + y_l)| = |(x_k - x_l) + (y_k - y_l)| \leq |x_k - x_l| + |y_k - y_l|$

Let $n \in \mathbb{N}$. Choose $K_1 \in \mathbb{N}$ s.t $k, l \geq K_1$ imply $|x_k - x_l| < \frac{1}{n}$

Choose $K_2 \in \mathbb{N}$ s.t $k, l \geq K_2$ imply $|y_k - y_l| < \frac{1}{n}$

Then if $k, l \geq \max\{K_1, K_2\}$, then $|(x_k + y_k) - (x_l + y_l)| \leq |x_k - x_l| + |y_k - y_l| < \frac{\overset{\text{fixed}}{2}}{\underset{\text{vary}}{n}}$

$\Rightarrow (x_k + y_k)$ is Cauchy

② (Note $|x_k y_k - x_\ell y_\ell| = |x_k y_k - x_\ell y_k + x_\ell y_k - x_\ell y_\ell|$

$$\leq |x_k y_k - x_\ell y_k| + |x_\ell y_k - x_\ell y_\ell|$$

$$= |y_k| \cdot |x_k - x_\ell| + |x_\ell| \, |y_k - y_\ell|.$$

Must control $|x_\ell|$ & $|y_k|$.

Lemma: If (x_k) is Cauchy, it is bounded i.e $\exists N \in \mathbb{N}$ s.t $|x_k| \leq N$ $\forall k$.

Pf. of lemma: Pick K s.t $k, l \geq K \rightarrow |x_k - x_\ell| \leq 1$. In particular, for all $k \geq K$, $|x_k - x_K| \leq 1$ since $|x_k| - |x_K| \leq |x_k - x_K| \leq 1$

$\forall k \geq K$, $|x_k| \leq |x_K| + 1$. Thus, choose $N \in \mathbb{N}$ s.t. $N \geq \max\{|x_1| \cdots |x_{K-1}|, |x_K|+1\}$

by construction. $|x_k| \leq N$ for all k.

Now show $(x_k y_k)$ Cauchy. Pick M s.t $|x_k| < M$ for all k & N s.t. $|y_k| < N$ for all ~~k~~ k.

Pick K_1 s.t $k \geq K_1 \Rightarrow |x_k - x_\ell| < \frac{1}{n}$ & K_2 s.t $k \geq K_2 \Rightarrow |y_k - y_\ell| < \frac{1}{n}$.

If $k \geq \max\{K_1, K_2\}$, then.

$|x_k y_k - x_\ell y_\ell| \leq |y_k| \cdot |x_k - x_\ell| + |x_\ell| \, |y_k - y_\ell| < N \frac{1}{n} + M \frac{1}{n} = \frac{N+M}{n}$. ← fixed ε can vary.

∴ $[(x_k y_k)]$ is Cauchy.

A sequence $(x_k)_{k=1}^{\infty}$ in $\mathbb{Q}$ is Cauchy whenever
$\forall n \in \mathbb{N}, \exists K \in \mathbb{N}, \forall k \in \mathbb{N}, \forall \ell \in \mathbb{N}, [(k \geq K) \wedge (\ell \geq K)] \to |x_k - x_\ell| < \frac{1}{n}$.

Two Cauchy sequences (x_k) & (y_k) in $\mathbb{Q}$ are equivalent if

$\forall n \in \mathbb{N}, \exists K \in \mathbb{N}, \forall k \in \mathbb{N}, (k \geq K) \to |x_k - y_k| < \frac{1}{n}$

Prop: If $(x_k), (x_k'), (y_k), (y_k')$ are Cauchy sequences in $\mathbb{Q}$. with $(x_k) \sim (x_k')$, $(y_k) \sim (y_k')$, then:

① $(x_k + y_k) \sim (x_k' + y_k')$
② $(x_k y_k) \sim (x_k' y_k')$

Pf: ① $|(x_k + y_k) - (x_k' + y_k')| \leq |x_k - x_k'| + |y_k - y_k'|$.
Let $n \in \mathbb{N}$. Choose $K_1 \in \mathbb{N}$ s.t $k > K_1 \to |x_k - x_k'| < \frac{1}{n}$.
Choose $K_2 \in \mathbb{N}$ st $k > K_2 \to |y_k - y_k'| < \frac{1}{n}$
If $k \geq \max\{K_1, K_2\}$, then $|(x_k + y_k) - (x_k' + y_k')| \leq |x_k - x_k'| + |y_k - y_k'| < \frac{2}{n}$.
This implies $(x_k + x_k') \sim (y_k + y_k')$

② L.T.R. use $|x_k y_k - x_k' y_k'| = |x_k y_k + x_k' y_k - x_k' y_k - x_k' y_k'|$.
$\leq |y_k||x_k - x_k'| + |x_k'||y_k - y_k'|$

Then argue as in final proof of last class

Def. Define addition & multiplication in $\mathbb{R}$ by $[(x_k)] + [(y_k)] = [(x_k + y_k)]$ & $[(x_k)] \cdot [(y_k)] = [(x_k y_k)]$

Theorem: $\mathbb{R}$ satisfies the field axioms:

① $\forall\, x, y \in \mathbb{R}$ s.t $x+y = y+x$; $x+(y+z) = (x+y)+z$

② $\exists\, 0 \in \mathbb{R}$ s.t $0 + x = 0$ for any $x \in \mathbb{R}$

③ $\forall\, x \in \mathbb{R}$, $\exists -x \in \mathbb{R}$ s.t $x + (-x) = 0$

④ $\forall x, y, z \in \mathbb{R}$, $x \cdot y = y \cdot x$; $x(yz) = (xy)z$

⑤ $\exists\, 1 \in \mathbb{R}$, $1 \neq 0$, s.t $1 \cdot x = x$ for all $x \in \mathbb{R}$.

⑥ $\forall\, x \in \mathbb{R}$, $x \neq 0 \rightarrow \exists x^{-1} \in \mathbb{R}$ s.t $xx^{-1} = 1$.

⑦ $\forall x, y, z \in \mathbb{R}$, $x(y+z) = xy + yz$

Pf: ① ④ ⑦ are clear as properties hold in $\mathbb{Q}$ (for instance: $x_k + y_k = y_k + x_k$ in $\mathbb{Q}$

for ② $0 = [(0_{\mathbb{Q}}, 0_{\mathbb{Q}}, \dots)]$

for ⑤ $1 = [(1_{\mathbb{Q}}, 1_{\mathbb{Q}}, \dots)]$

for ③, given $x = [(x_k)]$, for (x_k) Cauchy, then $-x = [(-x_k)]$ Clear that $(-x_k)$ is Cauchy.

for ⑥: If $x \neq 0$, $x = [(x_k)]$, we would like $x^{-1} = [(\frac{1}{x_k})]$

Note: $\left|\frac{1}{x_k} - \frac{1}{x_l}\right| = \left|\frac{x_l - x_k}{x_l x_k}\right| = \frac{1}{|x_k x_l|} \cdot |x_l - x_k|$

First need a lemma: (Bubble lemma): Suppose $x \in \mathbb{R}$ & $x \neq 0$. Then $\exists\, N \in \mathbb{N}$ so that for any Cauchy sequence $\{(x_k)\}$ representing x, $\exists\, K \in \mathbb{N}$ s.t $k \geq K$ implies $|x_k| \geq \frac{1}{N}$

(N is independent of representative x_k, but K depends on (x_k)).

Pf: Since $x \neq 0$, If (x_k) represents x, then $(x_k) \not\sim (0)$.

(Aside: $(x_k) \sim (0)$: $\forall n \in \mathbb{N}, \exists K \in \mathbb{N}, \forall k \in \mathbb{N}$ $(k \geq K) \Rightarrow |x_k - 0| = |x_k| < \frac{1}{n}$.
so $(x_k) \not\sim (0)$: $\exists M \in \mathbb{N}, \forall K \in \mathbb{N}, \exists k \in \mathbb{N}$, $(k \geq K) \wedge |x_k| \geq \frac{1}{M}$.)

This means there is some $M \in \mathbb{N}$ s.t for any $K \in \mathbb{N}$, one can find $k \in \mathbb{N}$ and $k \geq K$ s.t $|x_k| \geq \frac{1}{M}$.

Now, choose K' s.t $k, l \geq K' \rightarrow |x_k - x_l| < \frac{1}{2M}$

Fix $l \geq K'$ s.t. $|x_l| \geq \frac{1}{M}$. Then, if $k \geq K'$, we have

$|x_l| - |x_k| \leq |x_l - x_k| < \frac{1}{2M}$. This mean $|x_k| \geq |x_l| - \frac{1}{2M} \geq \frac{1}{M} - \frac{1}{2M} = \frac{1}{2M}$

Now, let (x_k') be any other Cauchy sequence representing x.
Choose L s.t. $k \geq L \Rightarrow |x_k' - x_k| < \frac{1}{4M}$.
Choosing $k \geq \max\{K', L\}$, we have $|x_k| - |x_k'| \leq |x_k - x_k'|$. Thus,
$|x_k'| \geq |x_k| - \frac{1}{4M} \geq \frac{1}{2M} - \frac{1}{4M} = \frac{1}{4M}$.

Setting $N = \cancel{\ } 4M$ finishes the proof.

Pf. of ⑥: Let $x \neq 0$ be in $\mathbb{R}$ & $[(x_k)]$ represent x. Note from Bubble Lemma, $\exists N \in \mathbb{N}$ ~~&~~ $\exists K \in \mathbb{N}$ s.t. $k \geq K \Rightarrow |x_k| \geq \frac{1}{N}$.
Define (y_k) by $y_k = \frac{1}{x_k}$ if $x_k \neq 0$ & $y_k = 0$ if $x_k = 0$. Show y_k is Cauchy.

Fix $n \in \mathbb{N}$, choose K' so that $k, l \geq K'$ imply $|x_k - x_l| < \frac{1}{n}$.
Let ~~k≥max~~ $k, l \geq \max\{K, K'\}$. Then $|y_k - y_l| = \left|\frac{1}{x_k} - \frac{1}{x_l}\right| = \frac{1}{|x_k x_l|}|x_l - x_k|$

$\left(|x_k| \geq \frac{1}{N} \Rightarrow \frac{1}{|x_k|} \leq N \Rightarrow \frac{1}{|x_l x_k|} \leq N^2\right.$.
Thus (y_k) is Cauchy. Clearly $(x_k y_k) \sim (1)$.