**Ngày: 11/09/2021**

**ĐÁP ÁN**

**LUYỆN TẬP MỘT SỐ KIẾN THỨC TOÁN CƠ SỞ CỦA MÔN HỌC**

**Bài 1**: Cho số phức $z = \frac{1}{2}e^{j\frac{\pi}{4}}$. Tính:

a. $Re\{z\}$
b. $Im\{z\}$
c. $|z|$
d. $\measuredangle z$
e. $z^*$
f. $z + z^*$

**Đáp án: Mỗi ý 0,25 điểm, tổng 1,5 điểm**

**(a)** Using Euler's formula,

$$e^{j\pi/4} = \cos\frac{\pi}{4} + j\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}$$

Since $z = \frac{1}{2}e^{j\pi/4}$,

$$Re\{z\} = \frac{1}{2}Re\left\{\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right\} = \frac{\sqrt{2}}{4}$$

**(b)** Similarly,

$$Im\{z\} = \frac{1}{2}Im\left\{\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right\} = \frac{\sqrt{2}}{4}$$

**(c)** The magnitude of $z$ is the product of the magnitudes of $\frac{1}{2}$ and $e^{j\pi/4}$. However, $|\frac{1}{2}| = \frac{1}{2}$, while $|e^{j\theta}| = 1$ for all $\theta$. Thus,

$$|z| = |\tfrac{1}{2}e^{j\pi/4}| = |\tfrac{1}{2}|\,|e^{j\pi/4}| = \tfrac{1}{2}\cdot 1 = \tfrac{1}{2}$$

**(d)** The argument of $z$ is the sum of the arguments of $\frac{1}{2}$ and $e^{j\pi/4}$. Since $\measuredangle\frac{1}{2} = 0$ and $\measuredangle e^{j\theta} = \theta$ for all $\theta$,

$$\measuredangle z = \measuredangle\left(\frac{1}{2}e^{j\pi/4}\right) = \measuredangle\frac{1}{2} + \measuredangle e^{j\pi/4} = 0 + \frac{\pi}{4} = \frac{\pi}{4}$$

**(e)** The complex conjugate of $z$ is the product of the complex conjugates of $\frac{1}{2}$ and $e^{j\pi/4}$. Since $\frac{1}{2}^* = \frac{1}{2}$ and $(e^{j\theta})^* = e^{-j\theta}$ for all $\theta$,

$$z^* = (\tfrac{1}{2}e^{j\pi/4})^* = \tfrac{1}{2}^*(e^{j\pi/4})^* = \tfrac{1}{2}e^{-j\pi/4}$$

**(f)** $z + z^*$ is given by

$$z + z^* = \frac{1}{2}e^{j\pi/4} + \frac{1}{2}e^{-j\pi/4} = \frac{e^{j\pi/4} + e^{-j\pi/4}}{2} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

Alternatively,

$$Re\{z\} = \frac{z + z^*}{2}, \qquad \text{or} \qquad z + z^* = 2Re\{z\} = 2\frac{\sqrt{2}}{4} = \frac{\sqrt{2}}{2}$$

**Bài 2**: Cho $z$ là số phức tùy ý. Chứng minh rằng:

a. $Re\{z\} = \frac{z+z^*}{2}$

b. $jIm\{z\} = \frac{z-z^*}{2}$

**Đáp án: Mỗi ý 0,25 điểm, tổng 0,5 điểm**

**(a)** Express $z$ as $z = \sigma + j\Omega$, where $Re\{z\} = \sigma$ and $Im\{z\} = \Omega$. Recall that $z^*$ is the complex conjugate of $z$, or $z^* = \sigma - j\Omega$. Then

$$\frac{z + z^*}{2} = \frac{(\sigma + j\Omega) + (\sigma - j\Omega)}{2} = \frac{2\sigma + 0}{2} = \sigma$$

**(b)** Similarly,

$$\frac{z - z^*}{2} = \frac{(\sigma + j\Omega) - (\sigma - j\Omega)}{2} = \frac{0 + 2j\Omega}{2} = j\Omega$$

**Bài 3**: Từ công thức Euler suy ra:

a. $\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$

b. $\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$

**Đáp án: mỗi ý 0,25 điểm, tổng 0,5 điểm**

**(a)** Euler's relation states that $e^{j\theta} = \cos\theta + j\sin\theta$. Therefore, $e^{-j\theta} = \cos(-\theta) + j\sin(-\theta)$. But, $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$. Thus, $e^{-j\theta} = \cos\theta - j\sin\theta$. Substituting,

$$\frac{e^{j\theta} + e^{-j\theta}}{2} = \frac{(\cos\theta + j\sin\theta) + (\cos\theta - j\sin\theta)}{2} = \frac{2\cos\theta}{2} = \cos\theta$$

**(b)** Similarly,

$$\frac{e^{j\theta} - e^{-j\theta}}{2j} = \frac{(\cos\theta + j\sin\theta) - (\cos\theta - j\sin\theta)}{2j} = \frac{2j\sin\theta}{2j} = \sin\theta$$

**Bài 4**:

a. Cho biểu diễn số phức $z = re^{j\theta}$. Diễn đạt trong hệ tọa độ cực các hàm sau của $z$.
   i. $z^*$
   ii. $z^2$
   iii. $jz$
   iv. $zz^*$
   v. $\frac{z}{z^*}$
   vi. $\frac{1}{z}$

b. Vẽ các vector trong phần a trên mặt phẳng phức với $r = \frac{2}{3}$ và $\theta = \pi/6$.

**Đáp án: mỗi ý 0,25 điểm, tổng 3 điểm**

**(a)** (i) We first find the complex conjugate of $z = re^{j\theta}$. From Euler's relation, $re^{j\theta} = r\cos\theta + jr\sin\theta = z$. Thus,

$$z^* = r\cos\theta - jr\sin\theta = r\cos\theta + jr(-\sin\theta)$$

But $\cos\theta = \cos(-\theta)$ and $-\sin\theta = \sin(-\theta)$. Thus,

$$z^* = r\cos(-\theta) + jr\sin(-\theta) = re^{-j\theta}$$

(ii) $z^2 = (re^{j\theta})^2 = r^2(e^{j\theta})^2 = r^2e^{j2\theta}$

(iii) $jz = e^{j\pi/2}re^{j\theta} = re^{j[\theta+(\pi/2)]}$

(iv) $zz^* = (re^{j\theta})(re^{-j\theta}) = r^2e^{j(\theta-\theta)} = r^2 \cdot 1$

(v) $\dfrac{z}{z^*} = \dfrac{re^{j\theta}}{re^{-j\theta}} = e^{j(\theta+\theta)} = e^{j2\theta}$

(vi) $\dfrac{1}{z} = \dfrac{1}{re^{j\theta}} = \dfrac{1}{r}e^{-j\theta}$

**(b)**

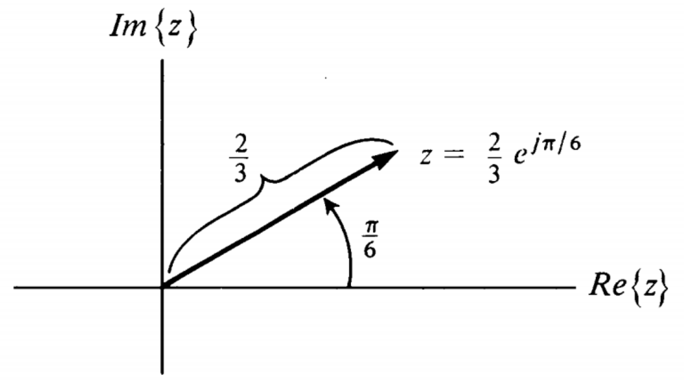


**Figure S1.4-1**

(i)

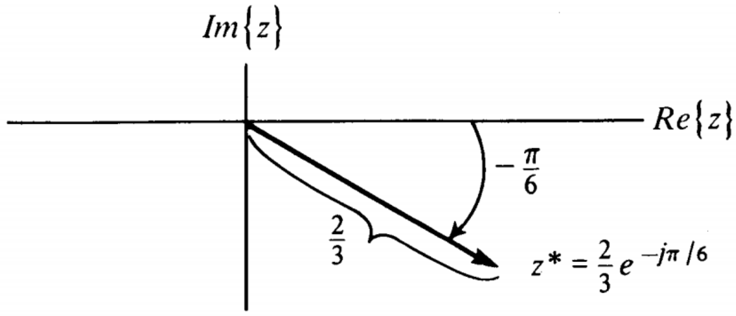


**Figure S1.4-2**

(ii)

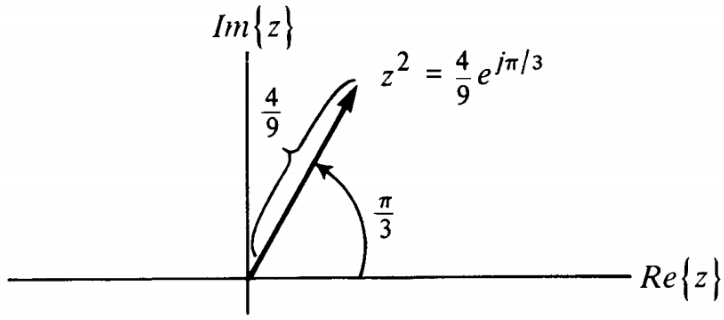


**Figure S1.4-3**

(iii)

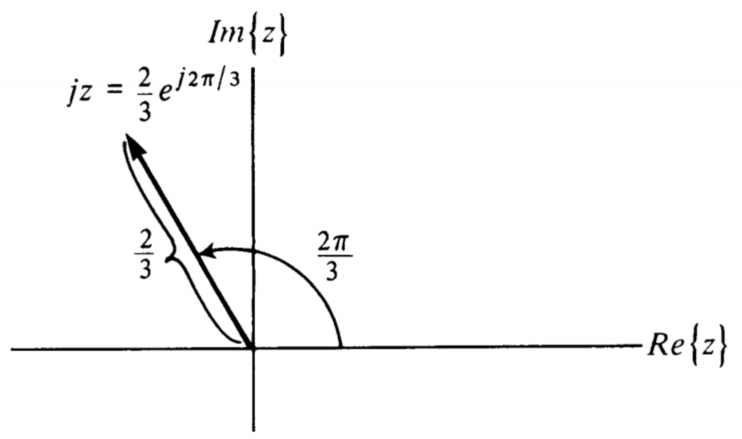


**Figure S1.4-4**

(iv)

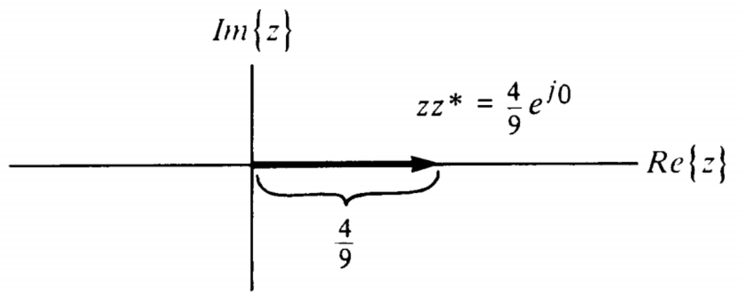


**Figure S1.4-5**

(v)

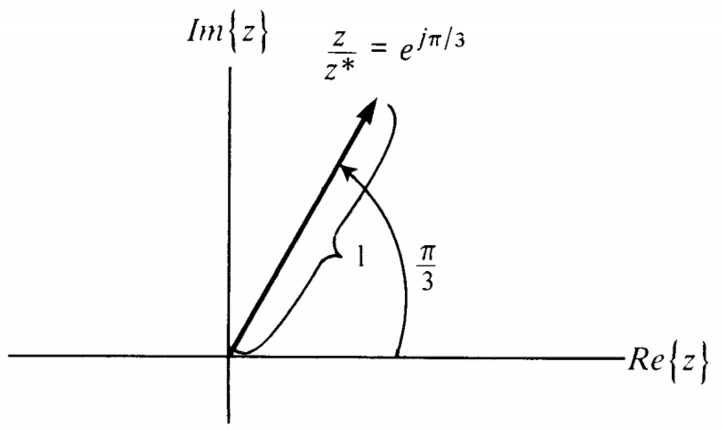


**Figure S1.4-6**

(vi)

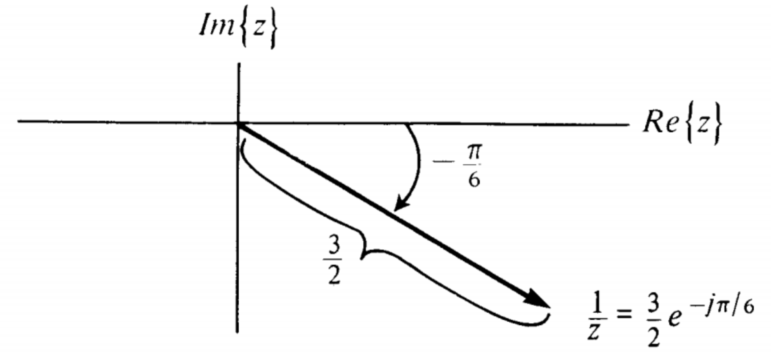


**Figure S1.4-7**

**Bài 5**: Chứng minh rằng

$$(1 - e^{j\alpha}) = 2\sin\left(\frac{\alpha}{2}\right)e^{j[(\alpha-\pi/2)]}$$

**Đáp án: 1,5 điểm**

This problem shows a useful manipulation. Multiply by $e^{+j\alpha/2}e^{-j\alpha/2} = 1$, yielding

$$e^{+j\alpha/2}e^{-j\alpha/2}(1 - e^{j\alpha}) = e^{j\alpha/2}(e^{-j\alpha/2} - e^{j\alpha/2})$$

Now we note that $2j\sin(-x) = -2j\sin x = e^{-x} - e^{x}$. Therefore,

$$1 - e^{j\alpha} = e^{j\alpha/2}\left(-2j\sin\frac{\alpha}{2}\right)$$

Finally, we convert $-j$ to complex exponential notation, $-j = e^{-j\pi/2}$. Thus,

$$1 - e^{j\alpha} = e^{j\alpha/2}\left(2e^{-j\pi/2}\sin\frac{\alpha}{2}\right) = 2\sin\frac{\alpha}{2}e^{j[(\alpha-\pi)/2]}$$

**Bài 6**: Tính các tích phân sau

a. $\int_0^a e^{-2t}dt$

b. $\int_2^\infty e^{-3t}dt$

**Đáp án: mỗi ý 0,5 điểm, tổng 1 điểm**

**(a)** $$\int_0^a e^{-2t}\,dt = -\tfrac{1}{2}e^{-2t}\Big|_0^a = -\tfrac{1}{2}e^{-2a} - [-\tfrac{1}{2}e^{-2(0)}]$$

$$= \tfrac{1}{2} - \tfrac{1}{2}e^{-2a}$$

**(b)** $$\int_2^\infty e^{-3t}\,dt = -\tfrac{1}{3}e^{-3t}\Big|_2^\infty = \lim_{t\to\infty}(-\tfrac{1}{3}e^{-3t}) + \tfrac{1}{3}e^{-3(2)}$$

Therefore,

$$\int_2^\infty e^{-3t}\,dt = 0 + \tfrac{1}{3}e^{-6} = \tfrac{1}{3}e^{-6}$$

**<u>Bài 7</u>**: Phân tích các biểu thức sau thành các thành phần đơn (viết đầy đủ các bước tính)

a. $Y(s) = \frac{1}{(s+1)(s+2)}$

b. $Y(z) = \frac{1+2z}{(1-\frac{1}{2}z)(1-\frac{1}{3}z)}$

c. $Y(j\omega) = \frac{j\omega}{(1+j\omega)(1+2j\omega)}\frac{1}{(2+j\omega)}$

d. $Y(s) = \frac{4s^2+12s+3}{8s^4+12s^3+6s^2+3s+1}$

**Đáp án: Mỗi ý 0,5 điểm, tổng 2 điểm**

a. $Y(s) = \frac{1}{(s+1)} - \frac{1}{(s+2)}$

b. $Y(z) = \frac{15}{(1-\frac{1}{2}z)} - \frac{14}{(1-\frac{1}{3}z)}$

c. $Y(j\omega) = \frac{1}{(1+j\omega)} - \frac{2/3}{(1+2j\omega)} - \frac{2/3}{(2+j\omega)}$

d. $Y(s) = \frac{1}{s+1} - \frac{2}{2s+1} - \frac{j}{s-\frac{j}{2}} + \frac{j}{s+\frac{j}{2}}$