Continuous-Time Fourier Transform (CTFT) & Discrete-Time Fourier Transform (DTFT)

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Lesson Objectives

At the end of this lesson, you should be able to:

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1. convert signals from time-domain to frequency-domain
2. analyse systems using Fourier Transform

Continuous-Time Fourier Transform (CTFT)

Definition

Time to Frequency:

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

Frequency to Time:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

$e^{-2(t-1)} \cdot u(t-1)$

$\delta(t)$ $\quad X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$

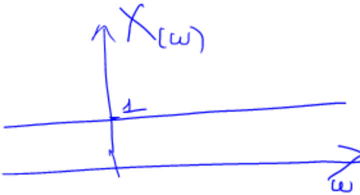
$$\delta(t) = \begin{cases} \neq 0 & \text{if } t = 0 \\ 0 & \text{else} \end{cases} \qquad \int_{-\infty}^{+\infty} \delta(t)\, dt = 1$$

Because $\delta(t) = 0$ everywhere except $t = 0$

$$X(\omega) = \int_{-\infty}^{+\infty} \delta(t) \cdot e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{+\infty} \delta(t) \cdot \underbrace{e^{-j\omega(t=0)}}_{1}\, dt$$

$$= \int_{-\infty}^{+\infty} \delta(t)\, dt = 1$$

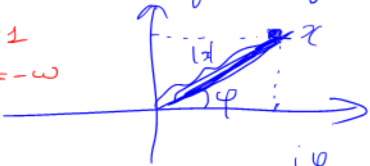


$X(\omega) = e^{-j\omega} = 1 \cdot e^{j(-\omega)}$

Complex plane

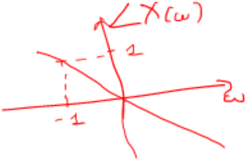
$|X(\omega)| \cdot e^{j\angle X(\omega)}$

$\Rightarrow \begin{cases} |X(\omega)| = 1 \\ \angle X(\omega) = -\omega \end{cases}$



$x = |x| e^{j\varphi}$

$\angle x = \varphi$



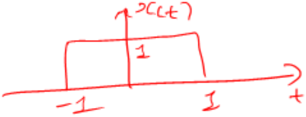
Time-Shift Property

if $x(t) \longleftrightarrow X(\omega)$ then:

$$x(t-t_0) \longleftrightarrow e^{-j\omega t_0}X(\omega)$$

Rectangular Signal

$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$



$$x(t) = \begin{cases} 1 & -1 \to 1 \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow X(\omega) = \int_{-1}^{1} e^{-j\omega t} dt$$

$$\mathrm{sinc}(t) = \frac{\sin(t)}{t}$$

$$= \frac{e^{-j\omega t}}{-j\omega}\Big|_{-1}^{1} = \frac{1}{-j\omega}\left(e^{-j\omega} - e^{j\omega}\right)$$

$$e^{j\omega} = \cos\omega + j\sin\omega$$

$$= 2\frac{\sin\omega}{\omega}$$

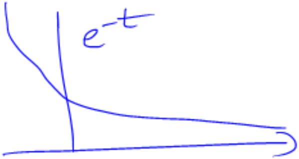
$e^{-t}u(t)$ $\quad X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$

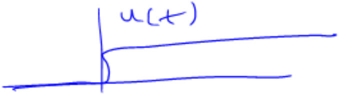
$$X(\omega) = \int_0^{+\infty} e^{-t} \cdot e^{-j\omega t} dt$$

$$= \int_0^{+\infty} e^{-(1+j\omega)t} dt = \left. \frac{e^{-(1+j\omega)t}}{-(1+j\omega)} \right|_0^{+\infty}$$

$$= \frac{\left. e^{-t} \cdot e^{-j\omega t} \right|_{t=\infty}}{-(1+j\omega)} - \frac{1}{-(1+j\omega)}$$

$$= \frac{1}{1+j\omega}$$





Formular

$$x(t) = e^{-at}u(t) \longleftrightarrow X(\omega) = \frac{1}{a + j\omega}$$

Fourier Transform for Periodic Signals

Until Now

- Fourier Series only exists with periodic signals.
- Fourier Transform is for aperiodic signals.

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From Now on

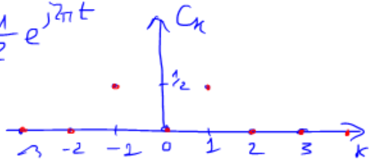
- Fourier Series only exists with periodic signals.
- Fourier Transform is for all signals.

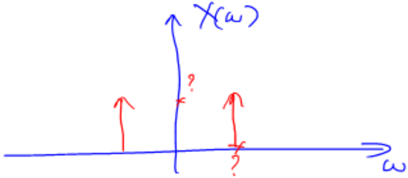
[Consult section 4.2 for more information.]

Fourier Transform of $cos(2\pi t)$

$$\cos(2\pi t) = \frac{e^{j2\pi t} + e^{-j2\pi t}}{2} \quad \begin{cases} c_{-1} = \frac{1}{2} \\ c_1 = \frac{1}{2} \end{cases}$$

$$= \frac{1}{2} \cdot e^{-j2\pi t} + \frac{1}{2} e^{j2\pi t}$$





Convolution Property

Convolution & Fourier

$$y(t) = x(t) * h(t) \longleftrightarrow Y(\omega) = X(\omega)H(\omega)$$

$y(t)$

$x(t) \xrightarrow{FT} Y(\omega)$

$$h(t) \xleftarrow{IFT} H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

Convolution Property

Convolution & Fourier

$$y(t) = x(t) * h(t) \longleftrightarrow Y(\omega) = X(\omega)H(\omega)$$

Differentiation Property

if $x(t) \longleftrightarrow X(\omega)$ then:

$$\frac{dx(t)}{dt} \longleftrightarrow j\omega X(\omega)$$

Example

Given a LTI system by the differential equation:

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = x(t)$$

Find the impulse of this system

Guide: $x(t) \to X(\omega)$

$y(t) \to Y(\omega)$

$(\ldots)Y(\omega) = X(\omega)$

$\Rightarrow H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{(\ldots)}$

$\downarrow$

$h(t)$

$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = x(t)$

Definition

Time to Frequency:

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n}$$

Frequency to Time:

$$x(n) = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega$$

$\delta(n)$

$$X(\omega) = \sum x(n)e^{-j\omega n}$$

$\delta(n-1)$ $\quad X(\omega) = \sum x(n) e^{-j\omega n}$

Rectangular Signal

$$X(\omega) = \sum x(n) e^{-j\omega n}$$

Homework

CTFT - Chapter 4

4.1, 4.2, 4.3, 4.7, 4.19, 4.21, 4.22

DTFT - Chapter 5

5.1, 5.2, 5.6, 5.19, 5.20, 5.22, 5.29

Mini-Test

$e^{-2(t-1)} u(t-1)$