Ngày: 21/11/2021

**LUYỆN TẬP MỘT SỐ KIẾN THỨC VỀ DTFS và DTFT**

**Bài 1**: Cho hệ thống LTI rời rạc có đáp ứng xung

$$h[n] = (\tfrac{1}{2})^n u[n]$$

Xác định đáp ứng của hệ thống tương ứng với các lối vào:

**(a)** $x[n] = (-1)^n = e^{j\pi n}$ for all $n$

**(b)** $x[n] = e^{j(\pi n/4)}$ for all $n$

**(c)** $x[n] = \cos\left(\dfrac{\pi n}{4} + \dfrac{\pi}{8}\right)$ for all $n$

**Đáp án: 0,5 điểm/ý x 3 ý = 1,5 điểm**

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k],$$

**(a)** $$y[n] = \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k e^{j\pi(n-k)} = e^{j\pi n}\sum_{k=0}^{\infty}\left(\frac{e^{-j\pi}}{2}\right)^k$$

$$= \frac{e^{j\pi n}}{1-\frac{1}{2}e^{-j\pi}} = \frac{2}{3}(-1)^n$$

**(b)** $$y[n] = \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k e^{j[\pi(n-k)/4]} = e^{j(\pi n/4)}\sum_{k=0}^{\infty}\left[\frac{e^{-j(\pi/4)}}{2}\right]^k$$

$$= \frac{e^{j(\pi n/4)}}{1-\frac{1}{2}e^{-j(\pi/4)}}$$

**(c)** $$y[n] = \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k\left[\frac{1}{2}e^{j(\pi/8)}e^{j[\pi(n-k)/4)]} + \frac{1}{2}e^{-j(\pi/8)}e^{-j[\pi(n-k)/4]}\right],$$ where we have used Euler's relation

$$= \frac{1}{2}e^{j(\pi/8)}e^{j(\pi n/4)}\sum_{k=0}^{\infty}\left[\frac{e^{-j(\pi/4)}}{2}\right]^k + \frac{1}{2}e^{-j(\pi/8)}e^{-j(\pi n/4)}\sum_{k=0}^{\infty}\left[\frac{e^{j(\pi/4)}}{2}\right]^k$$

$$= \frac{\frac{1}{2}e^{j[(\pi/8)+(\pi n/4)]}}{1-\frac{1}{2}e^{-j(\pi/4)}} + \frac{\frac{1}{2}e^{-j[(\pi/8)+(\pi n/4)]}}{1-\frac{1}{2}e^{j(\pi/4)}}$$

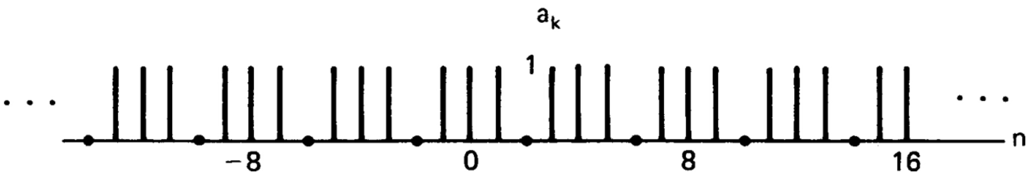
$$= \frac{\cos\left(\frac{\pi}{4}n+\frac{\pi}{8}\right) - \frac{1}{2}\cos\left(\frac{\pi}{4}n+\frac{3\pi}{8}\right)}{\frac{5}{4}-\cos\left(\frac{\pi}{4}\right)}$$

**Bài 2**: Xác định x(n) trong các trường hợp sau:

a) $$a_k = \cos\left(k\frac{\pi}{4}\right) + \sin\left(3k\frac{\pi}{4}\right)$$

b) $$a_k = \begin{cases} \sin\left(\dfrac{k\pi}{3}\right), & 0 \le k \le 6 \\ 0, & k = 7 \end{cases}$$

c)




**<u>Đáp án:</u> 0,5 điểm/ý x 3 ý = 1,5 điểm**

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk(2\pi/N)n}$$

For $N = 8$,

$$a_k = \frac{1}{8}\sum_{n=0}^{7} x[n]e^{-jk(\pi/4)n}$$

a)

$$a_k = \cos\left(\frac{\pi k}{4}\right) + \sin\left(\frac{3\pi k}{4}\right),$$

$$a_k = \frac{1}{2}e^{j(\pi k/4)} + \frac{1}{2}e^{-j(\pi k/4)} + \frac{1}{2j}e^{j(3\pi k/4)} - \frac{1}{2j}e^{-j(3\pi k/4)}$$

$$x[n] = 4\delta[n-1] + 4\delta[n-7] - 4j\delta[n-3] + 4j\delta[n-5], \quad 0 \le n \le 7$$

$$\textbf{(b)}\;\; x[n] = \sum_{k=0}^{7} a_k e^{jk(2\pi/8)} = \sum_{k=0}^{7} a_k e^{jk(\pi/4)n}$$

$$= \sum_{k=0}^{6}\left[\frac{1}{2j}e^{j(k\pi/3)} - \frac{1}{2j}e^{-j(k\pi/3)}\right]e^{jk(\pi/4)n}$$

$$= \frac{1}{2j}\sum_{k=0}^{6} e^{jk\pi[(1/3)+(n/4)]} - \frac{1}{2j}\sum_{k=0}^{6} e^{-jk\pi[(1/3)-(n/4)]}$$

$$= \frac{1}{2j}\,\frac{1-e^{j[(7\pi n/4)+(7\pi/3)]}}{1-e^{j[(\pi n/4)+(\pi/3)]}} - \frac{1}{2j}\,\frac{1-e^{j[(7\pi n/4)-(7\pi/3)]}}{1-e^{j[(\pi n/4)-(\pi/3)]}}$$

$$= \frac{1}{2j}\left[\frac{1-e^{j[(7\pi n/4)+(7\pi/3)]}}{1-e^{j[(\pi n/4)+(\pi/3)]}} - \frac{1-e^{j[(7\pi n/4)-(7\pi/3)]}}{1-e^{j[(\pi n/4)-(\pi/3)]}}\right]$$

$$\textbf{(c)}\;\; x[n] = \sum_{k=0}^{7} a_k e^{jk(2\pi/8)n} = \sum_{k=0}^{7} a_k e^{jk(\pi/4)n}$$

$$= 1 + e^{j(\pi/4)n} + e^{j(3\pi/4)n} + e^{j\pi n} + e^{j(5\pi/4)n} + e^{j(7\pi/4)n}$$

$$= 1 + (-1)^n + 2\cos\left(\frac{\pi}{4}n\right) + 2\cos\left(\frac{3\pi}{4}n\right), \qquad 0 \le n \le 7$$

**<u>Bài 3</u>**:

a) Cho hệ LTI với đáp ứng xung

$$h[n] = \left(\tfrac{1}{2}\right)^{|n|}$$

Tìm FS lối ra của hệ thống biết các lối vào có dạng

(i) $\tilde{x}[n] = \sin\left(\frac{3\pi n}{4}\right)$

(ii) $\tilde{x}[n] = \sum_{k=-\infty}^{\infty} \delta[n - 4k]$

(iii) $\tilde{x}[n]$ is periodic with period 6, and

$$\tilde{x}[n] = \begin{cases} 1, & n = 0, \pm 1 \\ 0, & n = \pm 2, \pm 3, \pm 4 \end{cases}$$

(iv) $\tilde{x}[n] = j^n + (-1)^n$

b) Tương tự câu (a) với

$$h[n] = \begin{cases} 1, & 0 \le n \le 2 \\ -1, & -2 \le n \le -1 \\ 0 & \text{otherwise} \end{cases}$$

**<u>Đáp án:</u> 0,25 điểm/ý x 8 ý = 2 điểm**

a)

$$\begin{aligned} H(\Omega) &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\Omega n} + \sum_{n=-\infty}^{0} \left(\frac{1}{2}\right)^{-n} e^{-j\Omega n} - 1 \\ &= \frac{1}{1 - \frac{1}{2}e^{-j\Omega}} + \frac{1}{1 - \frac{1}{2}e^{j\Omega}} - 1 \\ &= \frac{3}{5 - 4\cos\Omega} \end{aligned}$$

(i)

$$x[n] = \sin\left(\frac{3\pi}{4}n\right) = \frac{1}{2j}e^{j(3\pi/4)n} - \frac{1}{2j}e^{-j(3\pi/4)n}$$

The period of $x[n]$ is

$$\sin\left(\frac{3\pi}{4}n\right) = \sin\left[\frac{3\pi}{4}(n + N)\right]$$

Thus

$$\sin\left(\frac{3\pi}{4}n\right) = \sin\left(\frac{3\pi}{4}n + \frac{3\pi}{4}N\right)$$

We set $3\pi N/4 = 2\pi m$ to get $N = 8$ ($m = 3$). Hence, the period is 8.

$$x[n] = \sum_{k=0}^{7} a_k e^{jk(2\pi/8)n}$$

$a_3 = \frac{1}{2j}; a_5 = -\frac{1}{2j};$

All other coefficients $a_k$ are zero. By the convolution property, the Fourier series representation of $y[n]$ is given by $b_k$, where

$$b_k = a_k H(\Omega)\Big|_{\Omega=(2\pi k)/8}$$

Thus

$$\begin{aligned} b_3 &= \frac{1}{2j}\frac{3}{5 - 4\cos(3\pi/4)} \\ &= b_5^* \end{aligned}$$

All other $b_k$ are zero in the range $0 \le k \le 7$.

(ii) $\tilde{x}[n] = \sum_{k=-\infty}^{\infty} \delta[n - 4k]$

The Fourier series of $\tilde{x}[n]$ is

$$a_k = \tfrac{1}{4}\sum_{n=0}^{3} \tilde{x}[n]e^{-jk(2\pi/4)n} = \tfrac{1}{4}, \qquad \text{for all } k$$

And the Fourier series of $\tilde{y}[n]$ is

$$\begin{aligned} b_k &= a_k H(\Omega)\Big|_{\Omega=\pi k/2} \\ &= \frac{1}{4}\frac{3}{5 - 4\cos[(\pi/2)k]} = \frac{3}{20} \qquad \text{for all } k \end{aligned}$$

(iv) $x[n] = j^n + (-1)^n$

The period of $\tilde{x}[n]$ is 4. $x[n]$ can be rewritten as

$$\begin{aligned} x[n] &= [e^{j(\pi/2)}]^n + (e^{j\pi})^n \\ &= \sum_{k=0}^{3} a_k e^{jk(2\pi/4)n} \end{aligned}$$

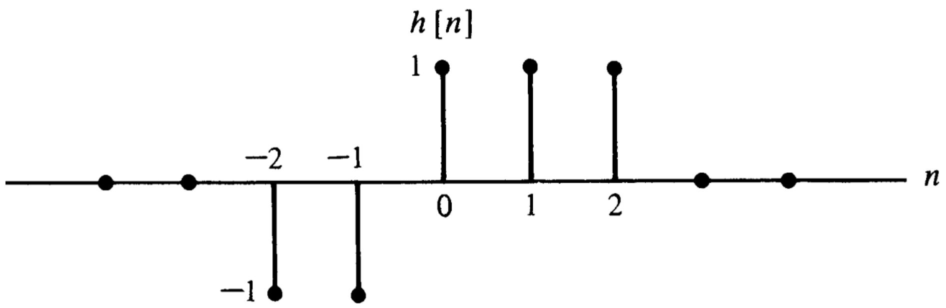
Hence,

$$\begin{aligned} a_0 &= 0, & a_1 &= 1, \\ a_2 &= 1, & a_3 &= 0 \end{aligned}$$

Therefore, $b_0 = b_3 = 0$ and

$$\begin{aligned} b_1 &= \frac{3}{5 - 4\cos(\pi/2)} = \frac{3}{5}, \\ b_2 &= \frac{3}{5 - 4\cos\pi} = \frac{3}{9} \end{aligned}$$

c)



$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n} = -e^{j2\Omega} - e^{j\Omega} + 1 + e^{-j\Omega} + e^{-j2\Omega},$$

$$H(\Omega) = 1 - 2j \sin \Omega - 2j \sin 2\Omega$$

(i) $$b_3 = \frac{1}{2j} H(\Omega) \Big|_{\Omega=3\pi/4} = \frac{1}{2j} - \sin \frac{3\pi}{4} - \sin \frac{3\pi}{2} = b_5^*$$

All other coefficients $b_k$ are zero, in the range $0 \le k \le 7$.

(ii) $$b_k = \tfrac{1}{4} H(\Omega) \Big|_{\Omega=\pi k/2}$$

$$= \frac{1}{4} - \frac{j}{2} \sin \frac{\pi k}{2} - \frac{j}{2} \sin \pi k = \frac{1}{4} - \frac{j}{2} \sin \frac{\pi k}{2}$$

(iii) $$b_k = \frac{1}{6} \left[ 1 + 2 \cos \left( \frac{\pi}{3} k \right) \right] H(\Omega) \Big|_{\Omega=\pi k/3}$$

(iv) $$b_0 = 0,$$

$$b_1 = H(\Omega) \Big|_{\Omega=\pi/2} = 1 - 2j,$$

$$b_2 = H(\Omega) \Big|_{\Omega=\pi} = 1,$$

$$b_3 = 0$$

**<u>Bài 3</u>**: Tìm hệ số FS và biểu diễn phổ tần số và phổ pha của các tín hiệu sau:

**(a)** $x[n] = \sin \left[ \frac{\pi(n-1)}{4} \right]$

**(b)** $x[n] = \cos \left( \frac{2\pi n}{3} \right) + \sin \left( \frac{2\pi n}{7} \right)$

**(c)** $x[n] = \cos \left( \frac{11\pi n}{4} - \frac{\pi}{3} \right)$

**<u>Đáp án:</u> 0,5 điểm/ý x 3 ý = 1,5 điểm**

**(a)** $\tilde{x}[n] = \sin \left[ \frac{\pi(n-1)}{4} \right]$

To find the period, we set $\tilde{x}[n] = \tilde{x}[n+N]$. Thus,

$$\sin \left[ \frac{\pi(n-1)}{4} \right] = \sin \left[ \frac{\pi(n+N-1)}{4} \right] = \sin \left[ \frac{\pi(n-1)}{4} + \frac{\pi N}{4} \right]$$

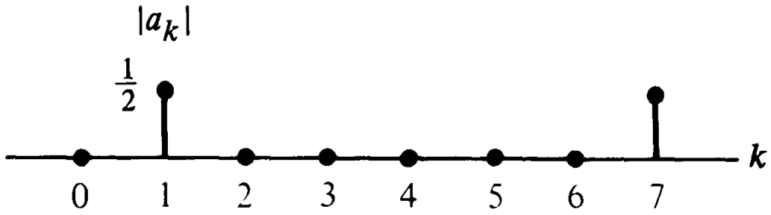
Let $(\pi N)/4 = 2\pi i$, when $i$ is an integer. Then $N = 8$ and

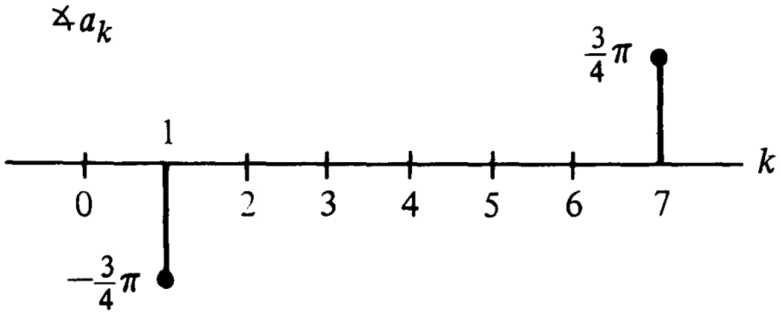
$$\tilde{x}[n] = \frac{1}{2j} e^{j[\pi(n-1)/4]} - \frac{1}{2j} e^{-j[\pi(n-1)/4]}$$
$$= \frac{1}{2j} e^{-j(\pi/4)} e^{j(\pi n/4)} - \frac{1}{2j} e^{j(\pi/4)} e^{-j(\pi n/4)}$$

Therefore,

$$a_1 = \frac{e^{-j(\pi/4)}}{2j}, \qquad a_7 = -\frac{e^{j(\pi/4)}}{2j}$$

All other coefficients $a_k$ are zero, in the range $0 \le k \le 7$.

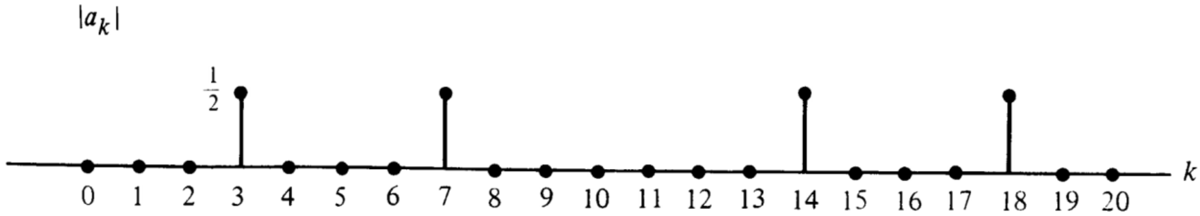


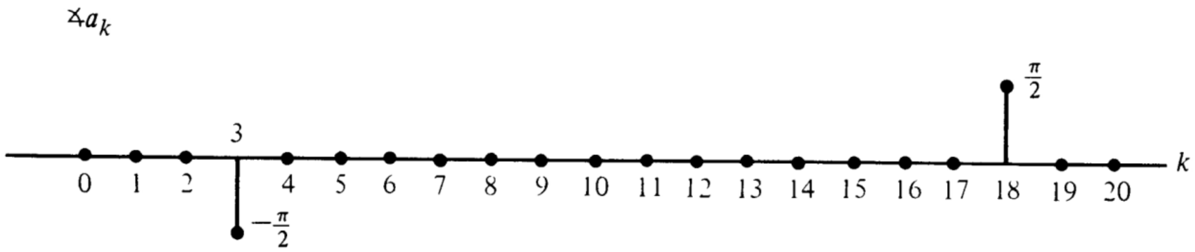




**(b)** The period $N = 21$ and the Fourier series coefficients are

$$a_7 = a_{14} = \frac{1}{2}, \qquad a_3 = a_{18}^* = \frac{1}{2j}$$

The rest of the coefficients $a_k$ are zero

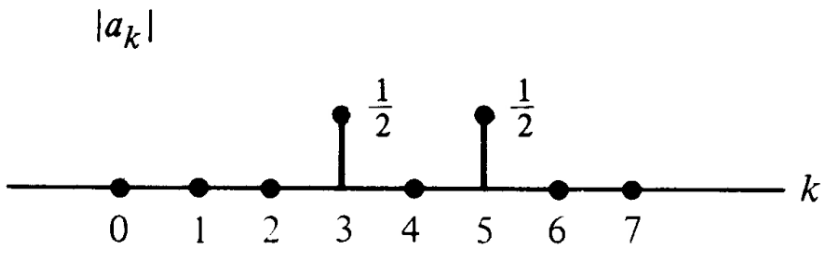


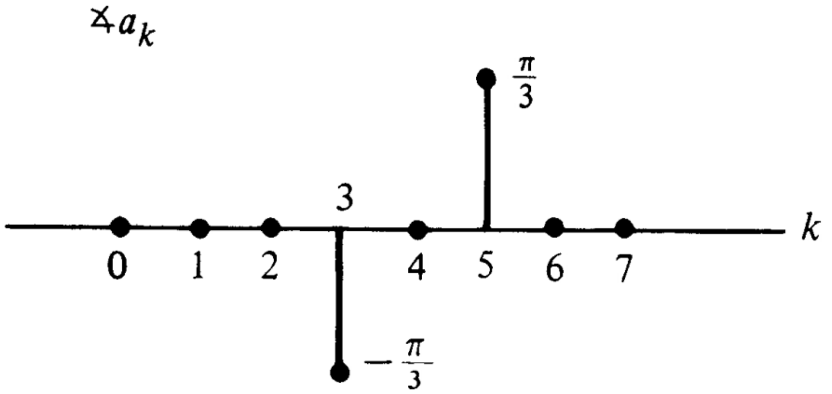




**(c)** The period $N = 8$.

$$a_3 = a_5^* = \tfrac{1}{2} e^{-j(\pi/3)}$$

The rest of the coefficients $a_k$ are zero



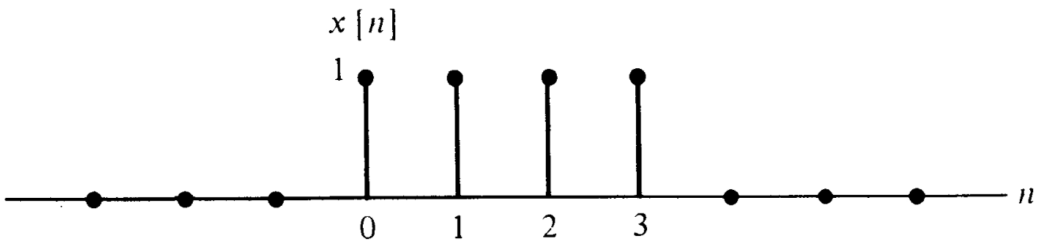




**Bài 4**: Tính DTFT cho các tín hiệu sau:

**(a)** $x[n] = (\frac{1}{4})^n u[n]$

**(b)** $x[n] = (a^n \sin \Omega_0 n)u[n], \qquad |a| < 1$

(c)




**(d)** $x[n] = (\frac{1}{4})^n u[n + 2]$

**Đáp án: 0,5 điểm/ý x 4 ý = 2 điểm**

**(a)**
$$\begin{aligned} X(\Omega) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} (\tfrac{1}{4})^n u[n] e^{-j\Omega n} \\ &= \sum_{n=0}^{\infty} (\tfrac{1}{4} e^{-j\Omega})^n \\ &= \frac{1}{1 - \frac{1}{4} e^{-j\Omega}} \end{aligned}$$

Here we have used the fact that

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \qquad \text{for } |a| < 1$$

**(b)** $x[n] = (a^n \sin \Omega_0 n)u[n]$

We can use the modulation property to evaluate this signal. Since

$$\sin \Omega_0 n \overset{\mathcal{F}}{\longleftrightarrow} \frac{2\pi}{2j}[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)],$$

periodically repeated, then

$$X(\Omega) = \frac{1}{2j}\left[\frac{1}{1 - ae^{-j(\Omega-\Omega_0)}} - \frac{1}{1 - ae^{-j(\Omega+\Omega_0)}}\right]$$

periodically repeated.

**(c)**
$$X(\Omega) = \sum_{n=0}^{3} e^{-j\Omega n} = \frac{1 - e^{-j4\Omega}}{1 - e^{-j\Omega}},$$

using the identity

$$\sum_{n=0}^{N-1} a^n = \frac{1 - a^N}{1 - a}$$

Alternatively, we can use the fact that $x[n] = u[n] - u[n - 4]$, so

$$X(\Omega) = \frac{1}{1 - e^{-j\Omega}} - \frac{e^{-j4\Omega}}{1 - e^{-j\Omega}} = \frac{1 - e^{-j4\Omega}}{1 - e^{-j\Omega}}$$

**(d)**
$$\begin{aligned} x[n] &= (\tfrac{1}{4})^n u[n + 2] \\ &= (\tfrac{1}{4})^{n+2}(\tfrac{1}{4})^{-2} u[n + 2] \\ &= 16(\tfrac{1}{4})^{n+2} u[n + 2] \end{aligned}$$

We know that

$$16\left(\frac{1}{4}\right)^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{16}{1 - \frac{1}{4}e^{-j\Omega}},$$

so

$$16\left(\frac{1}{4}\right)^{n+2} u[n + 2] \overset{\mathcal{F}}{\longleftrightarrow} \frac{16e^{j2\Omega}}{1 - \frac{1}{4}e^{-j\Omega}}$$

**<u>Bài 5</u>**:

a) Cho hệ thống LTI mô tả bởi phương trình sai phân với các điều kiện ban đầu bằng 0:

$$y[n] - \tfrac{1}{2}y[n - 1] = x[n]$$

Tính đáp ứng tần số của hệ thống.

b) Sử dụng phép biến đổi Fourier để tính y(n) nếu x(n) là:

(i) $\delta[n]$

(ii) $\delta[n - n_0]$

(iii) $(\tfrac{3}{4})^n u[n]$

**<u>Đáp án:</u> 0,5 điểm/ý x 4 ý = 2 điểm**

a)

$$Y(\Omega)(1 - \tfrac{1}{2}e^{-j\Omega}) = X(\Omega)$$

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1}{1 - (\frac{1}{2})^{-j\Omega}}$$

**(b)** (i) If $x[n] = \delta[n]$, then $X(\Omega) = 1$ and

$$Y(\Omega) = H(\Omega)X(\Omega) = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}},$$

so

$$y[n] = (\tfrac{1}{2})^n u[n]$$

(ii) $X(\Omega) = e^{-j\Omega n_0}$, so

$$Y(\Omega) = \frac{e^{-j\Omega n_0}}{1 - \frac{1}{2}e^{-j\Omega}}$$

and, using the delay property of the Fourier transform,

$$y[n] = (\tfrac{1}{2})^{n-n_0} u[n - n_0]$$

(iii) If $x[n] = (\frac{3}{4})^n u[n]$, then

$$X(\Omega) = \frac{1}{1 - \frac{3}{4}e^{-j\Omega}},$$

$$Y(\Omega) = \left(\frac{1}{1 - \frac{1}{2}e^{-j\Omega}}\right)\left(\frac{1}{1 - \frac{3}{4}e^{-j\Omega}}\right) = \frac{-2}{1 - \frac{1}{2}e^{-j\Omega}} + \frac{3}{1 - \frac{3}{4}e^{-j\Omega}},$$

so

$$y[n] = -2(\tfrac{1}{2})^n u[n] + 3(\tfrac{3}{4})^n u[n]$$

**<u>Bài 6</u>**: Hệ LTI được cho bởi đáp ứng xung:

$$h[n] = \left[\left(\frac{1}{2}\right)^n \cos\frac{\pi n}{2}\right] u[n]$$

a) Tính đáp ứng tần số của hệ thống.
b) Tính lối ra y(n) sử dụng đáp ứng tần số của hệ thống được tính trong câu (a) và tín hiệu vào $x(n) = \cos(\pi n/2)$

**<u>Đáp án:</u> 0,5 điểm/ý x 2 ý = 1 điểm**

a) $h(n) = \left[\left(\frac{1}{2}\right)^n \cos\frac{\pi n}{2}\right] u(n) = \left[\left(\frac{1}{2}\right)^n \frac{1}{2}\left(e^{j\frac{\pi n}{2}} + e^{-j\frac{\pi n}{2}}\right)\right] u(n)$

$$H(\omega) = FT\{h(n)\} = \frac{1}{2}\left[\frac{1}{1 - \frac{1}{2}e^{-j\left(\Omega - \frac{\pi}{2}\right)}} + \frac{1}{1 - \frac{1}{2}e^{-j\left(\Omega + \frac{\pi}{2}\right)}}\right]$$

b)

$$x[n] = \tfrac{1}{2}e^{j(\pi n/2)} + \tfrac{1}{2}e^{-j(\pi n/2)}$$

$$H(\Omega)\Big|_{\Omega=\pi/2} = \frac{1}{2}\left(\frac{1}{1-\frac{1}{4}} + \frac{1}{1+\frac{1}{2}}\right)$$

$$= \frac{1}{2}\left(2 + \frac{2}{3}\right) = \frac{4}{3},$$

$$H(\Omega)\Big|_{\Omega=-\pi/2} = H^*(\Omega)\Big|_{\Omega=\pi/2} = \frac{4}{3}$$

$$y[n] = \frac{2}{3}e^{j(\pi n/2)} + \frac{2}{3}e^{-j(\pi n/2)}$$

$$= \frac{4}{3}\cos\frac{\pi}{2}n$$

**<u>Bài 7</u>**: Cho hệ LTI được mô tả bởi phương trình sai phân sau:

$$y[n] + \tfrac{1}{4}y[n-1] - \tfrac{1}{8}y[n-2] = x[n] - x[n-1]$$

a) Tìm đáp ứng xung của hệ thống
b) Tính biên độ và pha của đáp ứng tần số tại $\Omega = 0$; $\Omega = \pi/4$; $\Omega = -\pi/4$ và $\Omega = 9\pi/4$.

**<u>Đáp án:</u> 0,5 điểm/ý x 2 ý = 1 điểm**

a)

$$y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] - x[n-1],$$

$$Y(\Omega)(1 + \frac{1}{4}e^{-j\Omega} - \frac{1}{8}e^{-j2\Omega}) = X(\Omega)(1 - e^{-j\Omega}),$$

$$\frac{Y(\Omega)}{X(\Omega)} = H(\Omega) = \frac{1 - e^{-j\Omega}}{(1 + \frac{1}{2}e^{-j\Omega})(1 - \frac{1}{4}e^{-j\Omega})}$$

$$H(\Omega) = \frac{2}{1 + \frac{1}{2}e^{-j\Omega}} + \frac{-1}{1 - \frac{1}{4}e^{-j\Omega}}$$

$$h[n] = 2(-\tfrac{1}{2})^n u[n] - (\tfrac{1}{4})^n u[n]$$

b)

At $\Omega = 0$, $H(\Omega) = 0$. At $\Omega = \pi/4$, $H(\Omega) = 0.65e^{j(1.22)}$. Since $h[n]$ is real, $H(\Omega) = H^*(-\Omega)$, so $H(-\Omega) = H^*(\Omega)$ and $H(-\pi/4) = 0.65e^{-j(1.22)}$. Since $H(\Omega)$ is periodic in $2\pi$,

$$H\left(\frac{9\pi}{4}\right) = H\left(\frac{\pi}{4}\right) = 0.65e^{j(1.22)}$$