I. **Biểu diễn Fourier của tín hiệu liên tục tuần hoàn (FS)/không tuần hoàn (FT)**

Bài 1:

(a) $T = \frac{2\pi}{\omega_0}$, ω_0

$$x(t) = cos\,\omega_0 t = \frac{1}{2}e^{i\omega_0 t} + \frac{1}{2}e^{-i\omega_0 t}$$

$$X[k] = \begin{cases} \frac{1}{2}, & k = \pm 1 \\ 0, & k \neq \pm 1 \end{cases}$$

(b) $T = \frac{2\pi}{\omega_0}$, ω_0

$$x(t) = \sin\omega_0 t = \frac{1}{2j}e^{i\omega_0 t} - \frac{1}{2j}e^{-i\omega_0 t})$$

$$X[k] = \begin{cases} \dfrac{1}{2j}, & k = 1 \\ -\dfrac{1}{2j}, & k = -1 \\ 0, & k \neq \pm 1 \end{cases}$$

(c) $T = \pi$, $\omega_0 = 2$

$$x(t) = \cos\left(2t + \frac{\pi}{4}\right) = \frac{1}{2}\left[e^{i\left(2t+\frac{\pi}{4}\right)} + e^{-i\left(2t+\frac{\pi}{4}\right)}\right] = \frac{1}{2}e^{i\frac{\pi}{4}}.e^{i2t} + \frac{1}{2}e^{-i\frac{\pi}{4}}.e^{(-1)i2t}$$

$$X[k] = \begin{cases} \dfrac{1}{2}e^{i\frac{\pi}{4}}, & k = 1 \\ \dfrac{1}{2}e^{-i\frac{\pi}{4}}, & k = -1 \\ 0, & k \neq \pm 1 \end{cases}$$

(d) $T = \pi$, $\omega_0 = 2$

$$x(t) = \cos 4t + \sin 6t = \frac{1}{2}\left(e^{i4t} + e^{-i4t}\right) + \frac{1}{2j}\left(e^{i6t} - e^{-i6t}\right)$$

$$= \frac{1}{2}e^{(2)i2t} + \frac{1}{2}e^{(-2)i2t} + \frac{1}{2j}e^{(3)i2t} - \frac{1}{2j}e^{(-3)i2t}$$

$$X[k] = \begin{cases} \dfrac{1}{2}, & k = \pm 2 \\ \dfrac{1}{2j}, & k = 3 \\ -\dfrac{1}{2j}, & k = -3 \\ 0, & otherwise \end{cases}$$

(e) $x(t) = sin^2(t) = \frac{1}{2} - \frac{1}{2}\cos(2t)$ $\rightarrow T = \pi$, $\omega_0 = 2$

$$x(t) = \frac{1}{2} - \frac{1}{4}e^{i2t} - \frac{1}{4}e^{i(-1)2t}$$

$$X[k] = \begin{cases} -\dfrac{1}{4}, & k = \pm 1 \\ \dfrac{1}{2}, & k = 0 \\ 0, & otherwise \end{cases}$$

Bài 2:

$$C_k = \frac{1}{T_0} \int_0^d A e^{-jk\omega_0 t} dt$$

(a) $d = \dfrac{T_0}{4}$

$\Rightarrow C_k = \dfrac{1}{T_0} \int_0^{\frac{T_0}{4}} A e^{-jk\omega_0 t} dt = \dfrac{A}{2\pi k j} - \dfrac{A}{2\pi k j} e^{\frac{jk\pi}{2}}$

(b) $d = \dfrac{T_0}{8}$

$\Rightarrow C_k = \dfrac{1}{T_0} \int_0^{\frac{T_0}{8}} A e^{-jk\omega_0 t} dt = \dfrac{A}{2\pi k j} - \dfrac{A}{2\pi k j} e^{\frac{jk\pi}{4}}$

Bài 3:

(a) $X[k] = -j\delta[k-2] + j\delta[k+2] + 2\delta[k-3] + 2\delta[k+3]$

$\Rightarrow x(t) = -je^{j(2)\pi t} + je^{j(-2)\pi t} + 2e^{j(3)\pi t} + 2e^{j(-3)\pi t}$

$= 4\cos(3\pi t) + 2\sin(2\pi t)$

(b) $x(t) = e^{-j(2\pi t - 2\pi)} + e^{-j\left(\frac{3}{2}\pi t - \frac{3}{2}\pi\right)} + e^{-j(\pi t - \pi)} + e^{-j\left(\frac{\pi}{2}t - \frac{\pi}{2}\right)} + 1 + e^{j\left(\frac{\pi}{2}t - \frac{\pi}{2}\right)} +$

$$+ e^{j(\pi t - \pi)} + e^{j\left(\frac{3}{2}\pi t - \frac{3}{2}\pi\right)} + e^{j(2\pi t - 2\pi)}$$

$$= 2\cos(2\pi t - 2\pi) + 2\cos\left(\frac{3}{2}\pi t - \frac{3}{2}\pi\right) + 2\cos(\pi t - \pi) + 2\cos\left(\frac{\pi}{2}t - \frac{\pi}{2}\right) + 1$$

Bài 4:

(a) $X(\omega) = \pi\delta(\omega) - \dfrac{1}{j\omega}$

(b) Đặt $v(t) = x(-t) = e^{-at}u(t)$

$$V(\omega) = \frac{1}{a + j\omega}$$

$$\rightarrow X(-\omega) = V(\omega) = \frac{1}{a + j\omega}$$

$\Rightarrow X(\omega) = \dfrac{1}{a - j\omega}$

(c) $X(\omega) = \sqrt{\dfrac{\pi}{a}} \cdot e^{-\frac{\omega^2}{4a}}$

Bài 5:

(a) $y(t) = x(t)\cos(\omega_0 t)$

$$= x(t).\frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

$$= \frac{1}{2}[x(t)e^{j\omega_0 t} + x(t)e^{-j\omega_0 t}]$$

$$= \frac{1}{2}[X(\omega - \omega_0) + X(\omega + \omega_0)]$$

(b) $y(t) = x(t)\sin(\omega_0 t)$

$$= x(t).\frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})$$

$$= \frac{1}{2j}[x(t)e^{j\omega_0 t} - x(t)e^{-j\omega_0 t}]$$

$$= \frac{1}{2j}[X(\omega - \omega_0) - X(\omega + \omega_0)]$$

Bài 6:

$$x(t) = \frac{1}{2}.e^{j\omega_0 t}.\frac{Sin(at)}{\pi t} + \frac{1}{2}.e^{-j\omega_0 t}.\frac{Sin(at)}{\pi t}$$

$$= \frac{Sin(at)}{\pi t}.\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2} = \frac{Sin(at)}{\pi t}Cos(\omega_0 t)$$

$$x(t) = \frac{Cos(\omega_0 t)Sin(at)}{\pi t}$$

Bài 7:

(a) $x(t) = \sin(2\pi t) e^{-t}u(t) = \frac{1}{2j}\left(e^{j2\pi t} - e^{-j2\pi t}\right)e^{-t}u(t) = \frac{1}{2j}e^{j2\pi t}e^{-t}u(t) - \frac{1}{2j}e^{-j2\pi t}e^{-t}u(t)$

Ta có: $e^{-t}u(t) \xrightarrow{FT} \frac{1}{1+j\omega}$

$\qquad e^{j2\pi t}s(t) \xrightarrow{FT} S(j\omega - j2\pi)$

$\Rightarrow$ $X(\omega) = \frac{1}{2j}\left[\frac{1}{1+j(\omega-2\pi)} - \frac{1}{1+j(\omega+2\pi)}\right]$

(b) $x(t) = te^{-3|t-1|}$

Ta có: $e^{-3|t|} \xrightarrow{FT} \frac{6}{9+\omega^2}$

$\qquad s(t-1) \xrightarrow{FT} e^{-j}\ S(j\omega)$

$\qquad t.v(t) \rightarrow j\frac{d}{d\omega}W(\omega)$

$\Rightarrow$ $X(\omega) = j\frac{d}{d\omega}\left[e^{-j\omega}\frac{6}{9+\omega^2}\right] = \frac{6e^{-j\omega}}{9+\omega^2} - \frac{12je^{-j\omega}}{(9+\omega^2)^2}$

(c) $x(t) = \left[\frac{2\sin(3\pi t)}{\pi t}\right]\left[\frac{\sin(2\pi t)}{\pi t}\right]$

$\qquad \frac{\sin(W(t))}{\pi t} \rightarrow \begin{cases} 1 & \omega \le W \\ 0, & otherwise \end{cases}$

$\qquad s_1(t)s_2(t) \rightarrow \frac{1}{2\pi}S_1(\omega)S_2(\omega)$

$\Rightarrow$ $X(\omega) = \begin{cases} 5 - \frac{|\omega|}{\pi}, \pi < |\omega| \le 5\pi \\ 4, |\omega| < \pi \\ 0, otherwise \end{cases}$

(d) $x(t) = \frac{d}{dt}\left(te^{-2t}\sin(t)\,u(t)\right) = \frac{d}{dt}\left(te^{-2t}\frac{e^{jt}-e^{-jt}}{2j}u(t)\right)$

$\quad = \frac{1}{2j}\frac{d}{dt}\left(e^{jt}te^{-2t}u(t) - e^{-j}\,te^{-2t}u(t)\right)$

$\quad te^{-2t}u(t) \rightarrow \frac{1}{(2+j\omega)^2}$

$\quad e^{jt}s(t) \rightarrow S\big(j(\omega-1)\big)$

$\quad \frac{d}{dt}s(t) \rightarrow j\omega S(\omega)$

$\Rightarrow X(\omega) = j\omega\frac{1}{2j}\left(\frac{1}{(2+j(\omega-1))^2} - \frac{1}{(2+j(\omega+1))^2}\right)$

(e) $x(t=) \int_{-\infty}^{t}\frac{\sin(2\pi\ )}{\pi\tau}d\tau$

$\quad \frac{\sin(2\pi\tau)}{\pi\tau} \rightarrow \begin{cases} 1, & \omega \le 2\pi \\ 0, & otherwise \end{cases}$

$\quad \int_{-\infty}^{t} s(\tau)d\tau \rightarrow \frac{S(\omega)}{j\omega} + \pi S(0)S(\omega)$

$\Rightarrow X(\omega) = \begin{cases} \pi\delta(\omega), & \omega = 0 \\ \frac{1}{j\omega}, & |\omega| \le 2\pi, \omega \ne 0 \\ 0, & otherwise \end{cases}$

(f) $x(t) = e^{-t+2}u(t-2) = e^{-(t-2)}u(t-2)$

$\quad e^{-t}u(t) \rightarrow \frac{1}{1+j\omega}$

$\quad s(t-2) \rightarrow e^{-j2\omega}S(\omega)$

$\Rightarrow X(\omega) = e^{-j2\omega}\frac{1}{1+j\omega}$

(g) $x(t) = \left(\frac{\sin(t)}{\pi t}\right) * \frac{d}{dt}\left[\left(\frac{\sin(2t)}{\pi t}\right)\right]$

$\quad \frac{\sin(Wt)}{\pi t} \rightarrow \begin{cases} 1, & \omega \le W \\ 0, & otherwise \end{cases}$

$\quad \frac{d}{dt}s(t) \rightarrow j\omega S(\omega)$

$\Rightarrow X(\omega) = \begin{cases} j\omega, & \omega \le 1 \\ 0, & otherwise \end{cases}$

Bài 8:

(a) $X(\omega) = \frac{j\omega}{(1+j\omega)^2}$

$\quad \frac{1}{(1+j\omega)^2} \rightarrow te^{-t}u(t)$

$\quad j\omega S(\omega) \rightarrow \frac{d}{dt}s(t)$

$\Rightarrow x(t) = \frac{d}{dt}[te^{-t}u(t)] = (1-t)e^{-t}u(t)$

(b) $X(\omega) = 2.\frac{2\sin(2(\omega-2))}{2(\omega-2)} - 2.\frac{2\sin(2(\omega+2))}{2(\omega+2)}$

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$$2\frac{\sin(\omega)}{\omega} \to rect(t) = \begin{cases} 1, & |t| \le 1 \\ 0, & otherwise \end{cases}$$

$$S(2\omega) \to \frac{1}{2}s\left(\frac{t}{2}\right)$$

$$S(\omega - 2) \to e^{j2t}s(t)$$

$\Rightarrow$ $x(t) = rect\left(\frac{t}{2}\right)e^{j2t} - rect\left(\frac{t}{2}\right)e^{-j2t} = 2jrect\left(\frac{t}{2}\right)\sin(2t)$

(c) $X(\omega) = \frac{1}{j\omega(j\omega+2)} - \pi\delta(\omega) = \frac{1}{2}\cdot\left(\pi\delta(\omega) + \frac{1}{j\omega}\right) - \frac{1}{2}\cdot\frac{1}{j\omega+2} - \frac{3}{2}\pi\delta(\omega)$

$$\pi\delta(\omega) + \frac{1}{j\omega} \to u(t)$$

$$\frac{1}{j\omega + 2} \to e^{-2t}u(t)$$

$$2\pi\delta(\omega) \to 1$$

$\Rightarrow$ $x(t) = \frac{1}{2}u(t) - \frac{1}{2}e^{-2t}u(t) - \frac{3}{4}$

(d) $X(\omega) = \frac{d}{dt}\left(4\sin(4\omega)\frac{\sin(2\omega)}{\omega}\right)$

$$2\frac{\sin(2\omega)}{\omega} \to rect(t) = \begin{cases} 1, & |t| \le 2 \\ 0, & otherwise \end{cases}$$

$$2\sin(4\omega)S(\omega) \to js(t+4) + js(t-4)$$

$$\frac{d}{dt}S_1(\omega) \to -jts_1(t)$$

$\Rightarrow$ $x(t) = -jt.rect(t+4) + jt.rect(t-4)$

(e) $X(\omega) = \frac{2\sin(\omega)}{\omega(j\omega+2)}$

$$\frac{2\sin(\omega)}{\omega} \to \begin{cases} 1, & |t| \le 1 \\ 0, & otherwise \end{cases}$$

$$\frac{1}{j\omega + 2} \to e^{-2t}u(t)$$

$\Rightarrow$ $x(t) = \begin{cases} 0, & t < -1 \\ \frac{1}{2}\left[1 - e^{-2(t+1)}\right], & -1 \le 1 < 1 \\ \frac{e^{-2t}}{2}[e^2 - e^{-2}], & t \ge 1 \end{cases}$

(f) $X(\omega) = \frac{4\sin^2(\omega)}{\omega^2}$

$$\frac{2\sin(\omega)}{\omega} \to s(t) = \begin{cases} 1, & |t| \le 1 \\ 0, & otherwise \end{cases}$$

$$x(t) = s(t) * s(t)$$

$\Rightarrow$ $x(t) = \begin{cases} 2 - |t|, & |t| \le 2 \\ 0, & otherwise \end{cases}$

II. **Đáp ứng tần số**

Bài 1:

$$\frac{d}{dt}y(t) + 2y(t) = x(t)$$

$$\xrightarrow{F} j\omega Y(\omega) + 2Y(\omega) = X(\omega)$$

$\Rightarrow\ H = \frac{Y(\omega)}{X(\omega)} = \frac{1}{2+j\omega}$

(a) $x(t) = e^{-t}u(t)$

$\Rightarrow\ X\omega = \frac{1}{1+j\omega}$

$\Rightarrow\ Y(\omega) = \frac{1}{(1+j\ )(2+j\omega)} = \frac{1}{1+j\omega} - \frac{1}{2+j\omega}$

$\Rightarrow\ y(t) = (e^{-t} - e^{-2t})u(t)$

(b) $x(t) = u(t)$

$\Rightarrow\ X(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$

$\Rightarrow\ Y(\omega) = \frac{\pi\delta(\omega)}{2+j\omega} + \frac{1}{j\omega(2+j\omega)} = \frac{1}{2}\cdot\frac{2\pi\ (\omega)}{2+j\omega} - \frac{1}{2}\left(\frac{1}{j\omega} + \pi\delta(\omega)\right) + \frac{1}{4}\cdot2\pi\delta(\omega) + \frac{1}{2}\cdot\frac{1}{2+j\omega}$
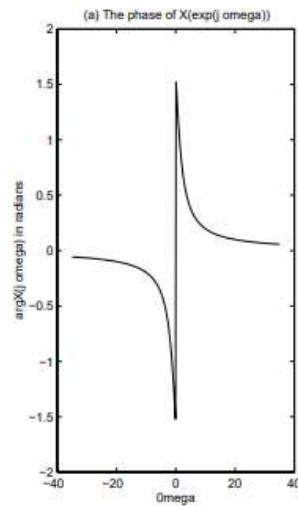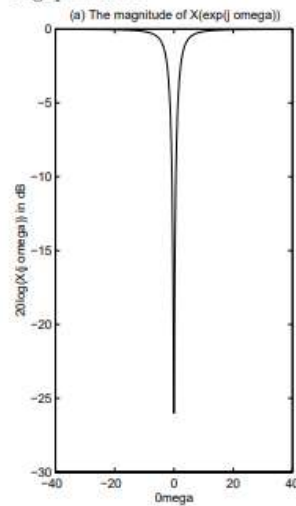
$\Rightarrow\ y(t) = \left(e^{-2t} - \frac{1}{2}\right)u(t) + \frac{1}{4}$

Bài 3:

(a) $h(t) = \delta(t) - 2e^{-2t}u(t)$

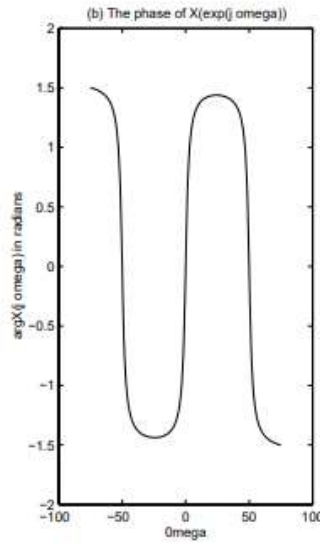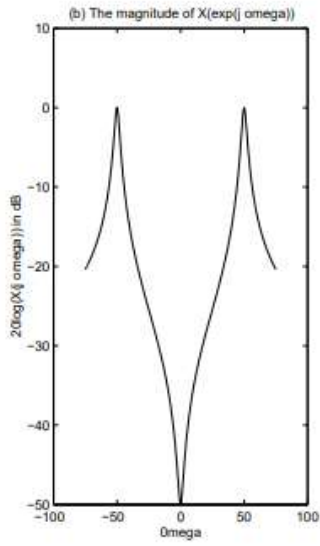$$H(\omega) = 1 - \frac{2}{2+j\omega} = \frac{j\omega}{2+j\omega}$$

Highpass filter

(b) $h(t) = 4e^{-2t}\cos(50t)u(t)$
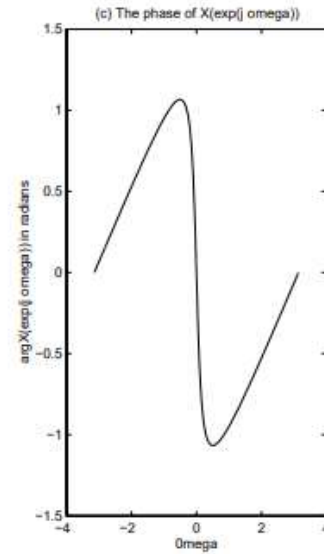
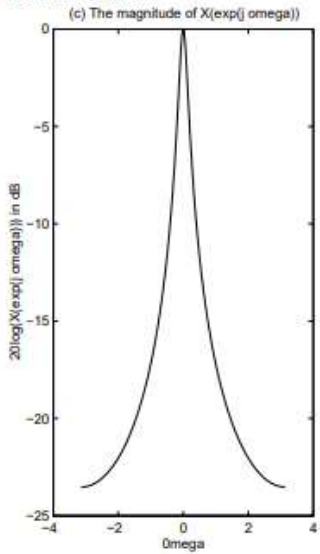$e^{-2t}u(t) \rightarrow \dfrac{1}{2+j\omega}$

$4\cos(50t)\,s(t) \rightarrow 2S(\omega - 50) + 2S(\omega + 50)$

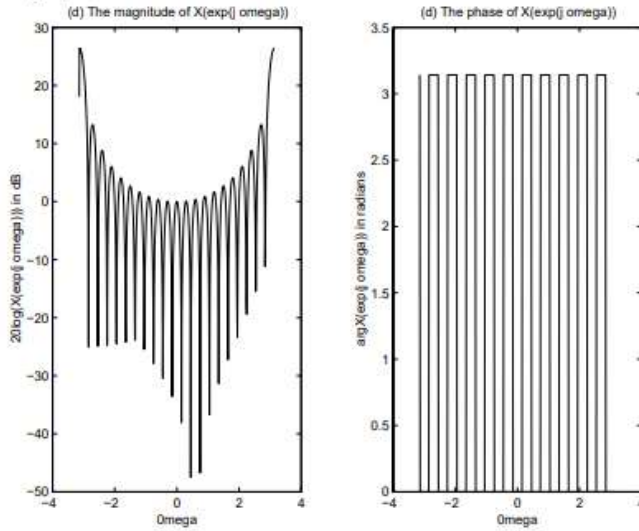$\Rightarrow\ H(\omega) = \frac{2}{2+j(\omega-50)} + \frac{2}{2+j(\omega+50)}$

(b) The magnitude of X(exp(j omega))

(b) The phase of X(exp(j omega))

(c) $h[n] = \frac{1}{8}\left(\frac{7}{8}\right)^n u[n]$

$H(e^{j\Omega}) = \dfrac{1}{8 - 7e^{-j\Omega}}$

Low Pass filter.

(c) The magnitude of X(exp(j omega))

(c) The phase of X(exp(j omega))

(d) $h[n] = \begin{cases} (-1)^n, & |n| \le 10 \\ 0, & otherwise \end{cases}$

$h[n] = \begin{cases} e^{j\pi n}, & |n| \le 10 \\ 0 & otherwise \end{cases}$

$H(e^{j\Omega}) = \dfrac{\sin\left(\dfrac{21}{2}(\Omega - \pi)\right)}{\sin\left(\dfrac{1}{2}(\Omega - \pi)\right)}$

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High pass filter.

Bài 4:

(a) $Y(j\omega) = H(j\omega)X(j\omega)$

$\Rightarrow Y(j\omega) = \frac{2+3j\omega - (j\omega)^2}{1+2j\omega} X(j\omega)$

$\Rightarrow Y(j\omega)(1+2j\omega) = [2+3j\omega - 3(j\omega)^2]X(j\omega)$

$\Leftrightarrow Y(j\omega) + 2j\omega Y(j\omega) = 2X(j\omega) + 3j\omega X(j\omega) - 3(j\omega)^2 X(j\omega)$

$\Rightarrow y(t) + 2\frac{d}{dt}y(t) = 2x(t) + 3\frac{d}{dt}x(t) - 3\frac{d^2}{dt^2}x(t)$

(b) $Y(j\omega) = H(j\omega)X(j\omega)$

$\Rightarrow Y(j\omega) = \frac{1-j\omega}{-(j\omega)^2 - 4} X(j\omega)$

$\Rightarrow Y(j\omega)(-(j\omega)^2 - 4) = (1-j\omega)X(j\omega)$

$\Leftrightarrow -(j\omega)^2 Y(j\omega) - 4Y(j\omega) = X(j\omega) - j\omega X(j\omega)$

$\Rightarrow -\frac{d^2}{dt^2}y(t) - 4y(t) = x(t) - \frac{d}{dt}x(t)$

(c) $Y(j\omega) = H(j\omega)X(j\omega)$

$\Rightarrow Y(j\omega) = \frac{1+j\omega}{(j\omega+2)(j\omega+1)} X(j\omega)$

$\Rightarrow (j\omega)^2 Y(j\omega) + 3j\omega Y(j\omega) + 2Y(j\omega) = X(j\omega) + j\omega X(j\omega)$

$\Rightarrow \frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = x(t) + \frac{d}{dt}x(t)$

Bài 5:

(a) $x(t) = e^{-t}u(t) \rightarrow X(\omega) = \frac{1}{1+j\omega}$

$y(t) = e^{-2t}u(t) + e^{-3t}u(t) \rightarrow Y(\omega) = \frac{1}{2+j\omega} + \frac{1}{3+j\omega}$

$\Rightarrow$ Đáp ứng tần số: $H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{(5+2j\omega)(1+j\ )}{(2+j\omega)(3+j\omega)} = 2 - \frac{1}{2+j\omega} - \frac{2}{3+j\omega}$

$\Rightarrow$ Đáp ứng xung: $h(t) = 2\delta(t) - (e^{-2t} + 2e^{-3t})u(t)$

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(b) $x(t) = e^{-3t}u(t) \rightarrow X(\omega) = \frac{1}{3+j\omega}$

$y(t) = e^{-3(t-2)}u(t-2) \rightarrow Y(\omega) = e^{-2j\omega} \cdot \frac{1}{3+j\omega}$

$\Rightarrow$ Đáp ứng tần số: $H(\omega) = \frac{Y(\omega)}{X(\omega)} = e^{-2j\omega}$

$\Rightarrow$ Đáp ứng xung: $h(t) = \delta(t-2)$

(c) $x(t) = e^{-2t}u(t) \rightarrow X(\omega) = \frac{1}{2+j\omega}$

$y(t) = 2te^{-2t}u(t) \rightarrow Y(\omega) = \frac{2}{(2+j\omega)^2}$

$\Rightarrow$ Đáp ứng tần số: $H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{2}{2+j\omega}$

$\Rightarrow$ Đáp ứng xung: $h(t) = 2te^{-2t}u(t)$

Bài 6:

(a) $\frac{d}{dt}y(t) + 3y(t) = x(t)$

$\rightarrow j\omega Y(\omega) + 3Y(\omega) = X(\omega)$

$\Rightarrow$ Đáp ứng tần số: $H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{3+j\omega}$

$\Rightarrow$ Đáp ứng xung: $h(t) = e^{-3t}u(t)$

(b) $\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = -\frac{d}{dt}x(t)$

$\rightarrow (j\omega)^2 Y(\omega) + 5j\omega Y(\omega) + 6Y(\omega) = -j\omega X(\omega)$

$\Rightarrow$ Đáp ứng tần số: $H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{-j\omega}{(j\omega)^2 + 5j\omega + 6} = -\frac{3}{3+j\omega} + \frac{2}{2+j\omega}$

$\Rightarrow$ Đáp ứng xung: $h(t) = (-e^{-3t} + 2e^{-2t})u(t)$