MỘT SỐ CÔNG THỨC TOÁN HỌC LIÊN QUAN

Review of Complex Numbers

C.1 REPRESENTATION OF COMPLEX NUMBERS

The complex number z can be expressed in several ways.

Cartesian or rectangular form:

$$z = a + jb \tag{C.1}$$

where $j = \sqrt{-1}$ and a and b are real numbers referred to the *real part* and the *imaginary part* of z. a and b are often expressed as

$$a = \operatorname{Re}\{z\} \qquad b = \operatorname{Im}\{z\} \tag{C.2}$$

where "Re" denotes the "real part of" and "Im" denotes the "imaginary part of."

Polar form:

$$z = re^{j\theta} \tag{C.3}$$

where $r > 0$ is the *magnitude* of z and θ is the *angle* or *phase* of z. These quantities are often written as

$$r = |z| \qquad \theta = \angle z \tag{C.4}$$

Figure C-1 is the graphical representation of z. Using Euler's formula,

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{C.5}$$

or from Fig. C-1 the relationships between the cartesian and polar representations of z are

$$a = r\cos\theta \qquad b = r\sin\theta \tag{C.6a}$$

$$r = \sqrt{a^2 + b^2} \qquad \theta = \tan^{-1}\frac{b}{a} \tag{C.6b}$$

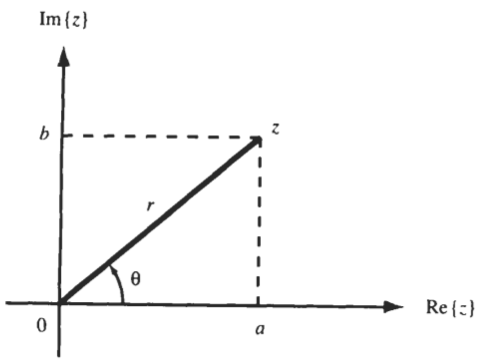

Fig. C-1

C.2 ADDITION, MULTIPLICATION, AND DIVISION

If $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$, then

$$z_1 + z_2 = (a_1 + a_2) + j(b_1 + b_2) \tag{C.7}$$

$$z_1 z_2 = (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + b_1 a_2) \tag{C.8}$$

$$\frac{z_1}{z_2} = \frac{a_1 + jb_1}{a_2 + jb_2} = \frac{(a_1 + jb_1)(a_2 - jb_2)}{(a_2 + jb_2)(a_2 - jb_2)}$$

$$= \frac{(a_1 a_2 + b_1 b_2) + j(-a_1 b_2 + b_1 a_2)}{a_2^2 + b_2^2} \tag{C.9}$$

If $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$, then

$$z_1 z_2 = (r_1 r_2) e^{j(\theta_1 + \theta_2)} \tag{C.10}$$

$$\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right) e^{j(\theta_1 - \theta_2)} \tag{C.11}$$

C.3 THE COMPLEX CONJUGATE

The *complex conjugate* of z is denoted by z^* and is given by

$$z^* = a - jb = re^{-j\theta} \tag{C.12}$$

Useful relationships:

1. $zz^* = r^2$
2. $\dfrac{z}{z^*} = e^{j2\theta}$
3. $z + z^* = 2\,\mathrm{Re}\{z\}$
4. $z - z^* = j2\,\mathrm{Im}\{z\}$
5. $(z_1 + z_2)^* = z_1^* + z_2^*$
6. $(z_1 z_2)^* = z_1^* z_2^*$
7. $\left(\dfrac{z_1}{z_2}\right)^* = \dfrac{z_1^*}{z_2^*}$

The nth root of a complex z is the number w such that

$$w^n = z = re^{j\theta} \tag{C.15}$$

Thus, to find the nth root of a complex number z we must solve

$$w^n - re^{j\theta} = 0 \tag{C.16}$$

which is an equation of degree n and hence has n roots. These roots are given by

$$w_k = r^{1/n} e^{j[\theta + 2(k-1)\pi]/n} \qquad k = 1, 2, \ldots, n \tag{C.17}$$

Useful Mathematical Formulas

D.1 SUMMATION FORMULAS

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} \dfrac{1-\alpha^N}{1-\alpha} & \alpha \neq 1 \\ N & \alpha = 1 \end{cases}$$

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \qquad |\alpha| < 1$$

$$\sum_{n=k}^{\infty} \alpha^n = \frac{\alpha^k}{1-\alpha} \qquad |\alpha| < 1$$

$$\sum_{n=0}^{\infty} n\alpha^n = \frac{\alpha}{(1-\alpha)^2} \qquad |\alpha| < 1$$

$$\sum_{n=0}^{\infty} n^2\alpha^n = \frac{\alpha^2+\alpha}{(1-\alpha)^3} \qquad |\alpha| < 1$$

D.2 EULER'S FORMULAS

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

$$\cos\theta = \tfrac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$

$$\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

D.3 TRIGONOMETRIC IDENTITIES

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta)$$

$$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta)$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\cos^2\theta$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \cos\alpha\cos\beta$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha - \beta) + \sin(\alpha + \beta)]$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha + \beta}{2}\cos\frac{\alpha - \beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha + \beta}{2}\cos\frac{\alpha - \beta}{2}$$

$$a\cos\alpha + b\sin\alpha = \sqrt{a^2 + b^2}\cos\left(\alpha - \tan^{-1}\frac{b}{a}\right)$$

D.4 POWER SERIES EXPANSIONS

$$e^{\alpha} = \sum_{k=0}^{\infty}\frac{\alpha^k}{k!} = 1 + \alpha + \frac{1}{2!}\alpha^2 + \frac{1}{3!}\alpha^3 + \cdots$$

$$(1 + \alpha)^n = 1 + n\alpha + \frac{n(n-1)}{2!}\alpha^2 + \cdots + \binom{n}{k}\alpha^k + \cdots + \alpha^n$$

$$\ln(1 + \alpha) = \alpha - \frac{1}{2}\alpha^2 + \frac{1}{3}\alpha^3 - \cdots + \frac{(-1)^{k+1}}{k}\alpha^k + \cdots \qquad |\alpha| < 1$$

D.5 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

$$e^{\alpha}e^{\beta} = e^{\alpha + \beta}$$

$$\frac{e^{\alpha}}{e^{\beta}} = e^{\alpha - \beta}$$

$$\ln(\alpha\beta) = \ln\alpha + \ln\beta$$

$$\ln\frac{\alpha}{\beta} = \ln\alpha - \ln\beta$$

$$\ln\alpha^{\beta} = \beta\ln\alpha$$

$$\log_b N = \log_a N \log_b a = \frac{\log_a N}{\log_a b}$$

D.6 SOME DEFINITE INTEGRALS

$$\int_0^{\infty} x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}} \qquad a > 0$$

$$\int_0^{\infty} e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \qquad a > 0$$

$$\int_0^{\infty} xe^{-ax^2}\,dx = \frac{1}{2a} \qquad a > 0$$