Lecture 2 2022-01-12

Recall: Markov Property:

$$P(X_{n+1}=x_{n+1} \mid X_0=x_0 \cdots X_n=x_n) =$$

$$= P(X_{n+1}=x_{n+1} \mid X_n=x_n)$$

also does not depend on n.

Def: The transition probab. are $P_{ij} = P(X_{n+1}=j \mid X_n=i)$

The Transition Matrix is

$$P = (P_{ij})$$

rows/cols indexed by S

Properties: $0 \le P_{ij} \le 1$ for all i, j.

$$\sum_{j \in S} P_{ij} = 1 \quad \forall i$$

Def: A stochastic matrix is P with $P_{ij} \ge 0$ and $\sum_j P_{ij} = 1$.

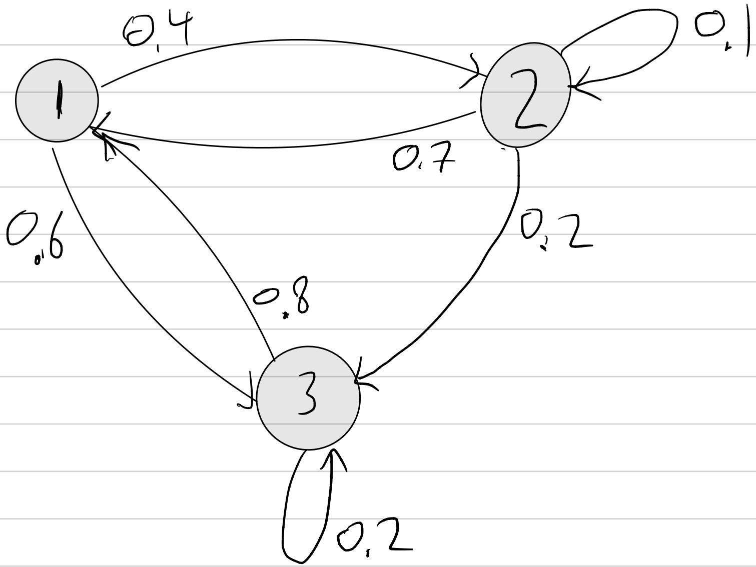
e.g.
$$P = \begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ & \left(\begin{array}{cc} 1-p & p \\ q & 1-q \end{array}\right) \begin{array}{c} 0 \\ 1 \end{array} \end{array} \Bigg\} S = \{0, 1\}$$

e.g.
$$P = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left(\begin{array}{ccc} 0 & 0.4 & 0.6 \\ 0.7 & 0.1 & 0.2 \\ 0.8 & 0 & 0.2 \end{array}\right) \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

Transition Diagram

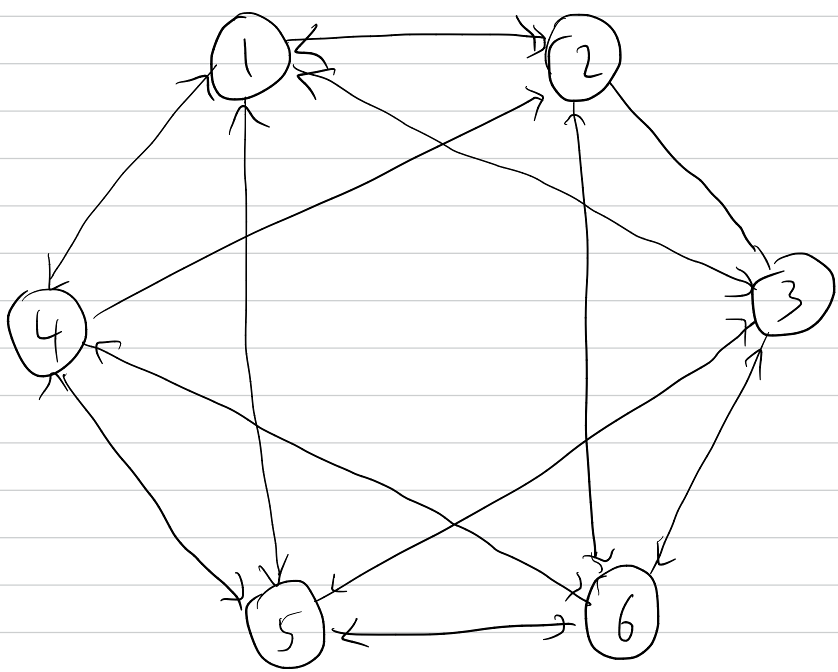
nodes for states

arrow $i \to j$ if $P_{ij} \neq 0$



e.g. 6-sided die

Step: push over in one of 4 directions.



Assume X_0 has distrib. μ

$\mu_i = P(X_0 = i)$. What is the distrib. of X_1?

<u>Ans</u>:

$$P(X_1 = j) = \sum_i P(X_1 = j,\ X_0 = i)$$

$$= \sum_i P(X_0 = i)\, P(X_1 = j \mid X_0 = i)$$

$$= \sum_i \mu_i \cdot P_{ij}$$

Special case: can we have X_1 with the same dist. as X_0?

If yes then μ is a stationary distrib. for the M.C.

In that case every i, X_i has distrib. μ.

Note: If μ is a $1\times n$ vector then the dist. of X_1 is the vector $\mu \cdot P$

Proposition: If X_n has distrib. μ then X_{n+1} has distrib. $\mu \cdot P$.

X_{n+2} has distrib.

$$(\mu \cdot P) \cdot P = \mu \cdot (P \cdot P) = \mu \cdot P^2$$

matrix product

Chapman-Kolmogorov equation

Recall P^n_{ij} is the n-step trans. prob.:

$$P^n_{ij} = P(X_{m+n} = j \mid X_m = i)$$

Proposition: $P^{n+m}_{ij} = \sum_k P^m_{ik} P^n_{kj}$

i.e. m+n step transitions are matrix product of m-step and n-step.

<u>Proof</u>: $P(X_{m+n}=j \mid X_0=i) =$

$$= \sum_k P(X_{n+m}=j, X_m=k \mid X_0=i)$$

$$= \sum_k P(X_m=k \mid X_0=i) \cdot P\left(X_{n+m}=j \,\middle|\, \begin{matrix} X_0=i \\ X_m=k \end{matrix}\right)$$

$$= \sum_k P_{ik}^m \cdot P(X_{n+m}=j \mid X_m=k)$$

[By Markov property]

$$= \sum_k P_{ik}^m P_{kj}^n$$

$\square$

<u>Corollary</u>: P^n is the nth power of P as a matrix.

If X_m has dist. μ, then X_{m+n} has dist. μP^n