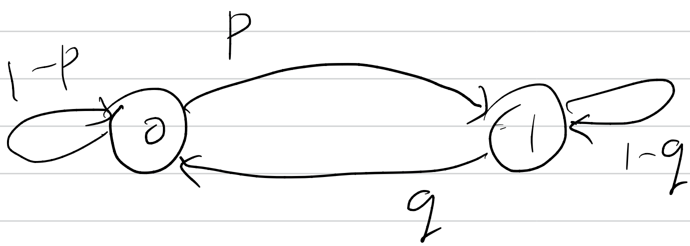
Lecture 7

Stationary Distributions

2-state M.C.



If $\alpha = (\alpha_0, \alpha_1)$ is the dist. of X_0

$$\left[\alpha_0 = P(X_0 = 0) \qquad \alpha_1 = P(X_1 = 1) \right]$$

then X_n has dist. αP^n

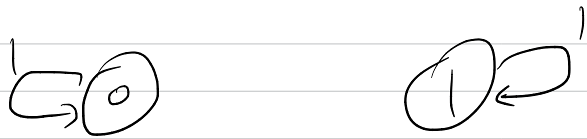
Seen: unless $p = q = 0$

or $p = q = 1$

$$\alpha P^n \xrightarrow[n \to \infty]{} \left(\frac{q}{p+q}, \frac{p}{p+q} \right)$$

Fails if $|1-p-q| = 1$

$p = q = 0$:



reducible

$q = p = 1$:

periodic

Many M.Cs converge to a stat. dist.

Def: A stat. dist. for a M.C. with trans matrix P is a π s.t. • $\pi_i \geq 0$

• $\sum_i \pi_i = 1$

• $\pi P = \pi$

i.e. π is a left e.vector of P.

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e.g. $\left(\frac{q}{p+q}, \frac{p}{p+q}\right)\begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} = \left(\frac{q}{p+q}, \frac{p}{p+q}\right)$

Qn: existence? uniqueness?
convergence?

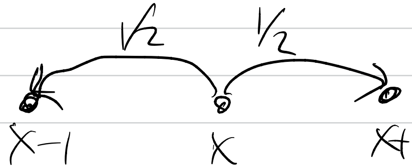
Def: For a recurrent state i let T_i be the return time: $T_i := \min\{n>0 : X_n = i\}$

$$m_i := \mathbb{E}(T_i \mid X_0 = i)$$

- i is positive-recurrent if $m_i < \infty$
- i is null-recurrent if $m_i = \infty$

null-recur. only possible in ∞ state spaces

e.g. simple walk on $\mathbb{Z}$



<u>Claim</u>: all states are null recurrent.

Assume $X_0 = 0$ $X_1 = 1$.

Let Y be the time to go from 1 to 0

If first step is $1 \to 0$: $Y=1$

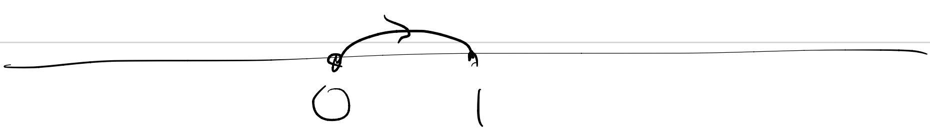
If first step is $1 \to 2$:

$$Y = 1 + (\text{time from } 2 \text{ to } 1) + (\text{time from } 1 \text{ to } 0)$$

$$\mathbb{E}Y = \frac{1}{2}\cdot 1 + \frac{1}{2}\left(1 + \mathbb{E}Y + \mathbb{E}Y\right)$$

$$= \frac{1}{2} + \frac{1}{2} + \mathbb{E}Y$$

So $\mathbb{E}Y = \infty$



So $m_0 = \infty$. $\square$

Prop.: null and pos. recur. are class properties. ($i \leftrightarrow j \Rightarrow$ same type)

Thm If a Markov chain is aperiodic, irreducible, and pos. recurrent then

① There is a unique stat. dist.

π s.t. $\pi P = \pi$ and $\sum \pi_i = 1$

$\pi_i \geq 0$

② for any initial dist. α

$$\alpha P^n \xrightarrow[n \to \infty]{} \pi$$

(regardless of α)

③ $\pi_i = \frac{1}{m_i}$ for every i

(so $\pi_i > 0$)

Note: If the M.C. is periodic ① and ③ still hold, but ② fails.

Note: $\pi P = \pi$ is singular.
include $\sum \pi_i = 1$

e.g. $P = \begin{pmatrix} .5 & .4 & .1 \\ .3 & .4 & .3 \\ .2 & .3 & .5 \end{pmatrix}$

$\pi P = \pi$ has sol. $(21, 23, 18)$

The stat. dist. $\pi = \left(\frac{21}{62}, \frac{23}{62}, \frac{18}{62} \right)$

<u>Def</u>: A M.C. is <u>ergodic</u> if it is irreducible and pos.-recurrent.

<u>Thm</u> If a M.C. is ergodic then π_i is the asympt. fraction of time at i

$$\frac{\#\{n < N : X_n = i\}}{N} \xrightarrow[N \to \infty]{} \pi_i$$

<u>note</u>: If i is transient or null-recurrent, then

$$\frac{\#\{n \leq N : X_n = i\}}{N} \xrightarrow[N \to \infty]{} 0$$