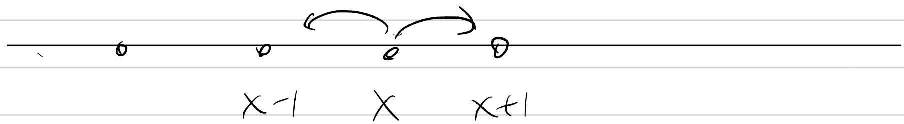
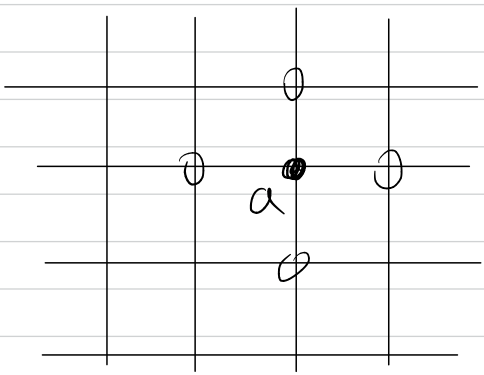
One dim RW:



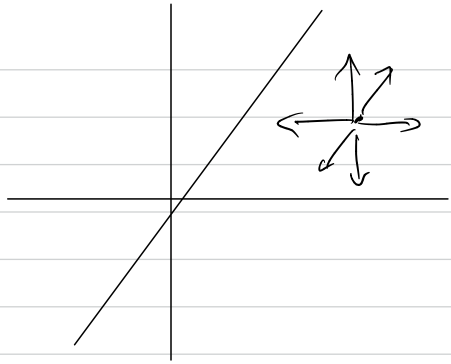
Two dim RW: $S = \mathbb{Z}^2$



from a move to neighbour. Prob $\frac{1}{4}$ each.

(X_n, Y_n) one changes by ± 1 other stays fixed.

3-dim:



6 possible moves, equal probab.

d-dimensional random walk:

$S = \mathbb{Z}^d$

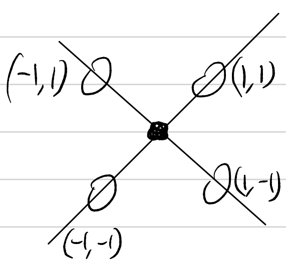
$P_{xy} = \frac{1}{2d}$ if $x \sim y$

$x \sim y$: only coord. differ, by ± 1

Thm (Polya): d-dim random walk is recurrent if $d \leq 2$ transient if $d > 2$.

Proof $d=1$: seen

$d=2$ rotate by 45° scale by $\sqrt{2}$



now in each step each coord. changes by ± 1

and X, Y changes are indep.

(X_n, Y_n) = loc. at time n after rotation.

$$P(X_{2n}=0 \mid X_0=0) = \binom{2n}{n} 2^{-2n} \sim \frac{1}{\sqrt{\pi n}}$$

If $(X_0, Y_0) = (0,0)$, then

Prob. that $X_{2n} = Y_{2n} = 0$ is

$$\left(\binom{2n}{n} 2^{-2n}\right)^2 \sim \frac{1}{\pi n}$$

Let $\vec{o} = (0,0)$

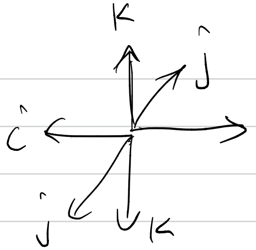
$$P^{2n}_{\vec{o},\vec{o}} \sim \frac{1}{\pi n}$$

So $$\sum_n P^{2n}_{\vec{o}\,\vec{o}} = \infty$$

Since $\sum \frac{1}{n} = \infty$.

So $E N_{\vec{o}} = \infty$ and MC

is recurrent. $\boxtimes$

$d>2$: 

$\vec{0} = (0,0,0 \cdots 0)$

To return to $\vec{0}$ after $2n$ steps need to make same number of $+1$ and -1 steps in every coord.

$$\text{Consider} \quad \left.\begin{array}{ll} \hat{i} \text{ steps} & (1,0,0) \\ \hat{i} & (-1,0,0) \\ \hat{j} & (0,1,0) \\ \hat{j} & (0,-1,0) \\ \hat{k} & (0,0,1) \\ \hat{k} & (0,0,-1) \end{array}\right\} A_{ijk}$$

If $i+j+k=n$ then

$$P(A_{ijk}) = \left(\frac{1}{6}\right)^{2n} \frac{(2n)!}{i!^2\, j!^2\, k!^2}$$

$$= \binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \left(\frac{n!}{i!\,j!\,k!}\right)^2 \cdot \left(\frac{1}{3}\right)^{2n}$$

$$P^{2n}_{\vec{0}\,\vec{0}} = \sum_{i+j+k=n} \binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \binom{n}{i,j,k}^2 \left(\frac{1}{3}\right)^{2n}$$

$$= \binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \sum_{i,j,k} \left[\binom{n}{i,j,k}\left(\frac{1}{3}\right)^{n}\right]^2$$

trick: $\sum a_i^2 \leq \sum a_i (\max a_i)$

for $a_i \geq 0$

$$P^{2n}_{\vec{0}\vec{0}}$$

$$\leq \binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \sum_{ijk} \binom{n}{i,j,k}\left(\frac{1}{3}\right)^n \max_{ijk} \binom{n}{i\,j\,k}\left(\frac{1}{3}\right)^n$$

$\binom{n}{i,j,k}\left(\frac{1}{3}\right)^n$ = Prob of i 1's, j 2's, k 3's when

rolling 3-sided die n times

So $\sum \binom{n}{i,j,k} 3^{-n} = 1$

$$P^{2n}_{\vec{0}\vec{0}} \leq \binom{2n}{n} 2^{-2n} \max_{i+j+k=n} \binom{n}{i,j,k} 3^{-n}$$

max is when $i, j, k \sim \frac{n}{3}$

By stirling, in that case

$$\frac{n!}{i!\,j!\,k!} \sim \frac{\sqrt{2\pi n}}{\left(\sqrt{2\pi \frac{n}{3}}\right)^3} = \frac{3^{3/2}}{2\pi n}$$

$$\binom{2n}{n} 2^{-2n} \sim \frac{1}{\sqrt{\pi n}}$$

$$P^{2n}_{\vec{0}\,\vec{0}} \lesssim \frac{1}{\sqrt{\pi n}} \cdot \frac{3^{3/2}}{2\pi n} \leq \frac{C}{n^{3/2}}$$

$\frac{3}{2} > 1$ so $\sum P^{2n}_{\vec{0},\vec{0}} < \infty$

so 3-dim walk is trans.

In $d>3$ same method gives

$$P^{2n}_{\vec{0}\vec{0}} \leq \frac{C(d)}{n^{d/2}}.$$

$$\sum P^{2n}_{\vec{0}\vec{0}} < \infty \quad \text{for } d>2$$

Can also deduce transience in $d>3$ by ignoring all but 3 coord.

$\square$