Discrete time Markov Chains

Collection of Random Var.

$X_0 \; X_1 \; X_2 \; \ldots \qquad (X_n)_{n \geq 0}$

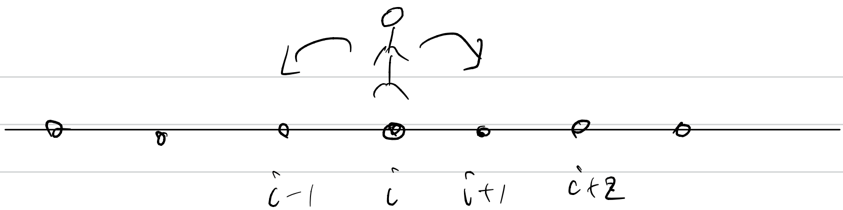
taking values in S

S = state space.

[finite or countable]

<u>examples</u>: Simple random walk

$S = \mathbb{Z}$



X_n = position at time n
(after n steps)

each step is right or left
w.p. $1/2$ each.

<u>Example</u> $S = \{0, 1\}$

$$X_n = \begin{cases} 0 & \text{day } n \text{ is sunny} \\ 1 & \text{day } n \text{ is rainy} \end{cases}$$

(not exactly but modeled
by a M.C.)

Idea: (X_n) is a Markov chain if future: X_{n+1} ----
may depend on present: X_n
but given X_n, not on past.

Def: (X_n) taking values in S is a Markov chain if it satisfies the Markov property

$\forall n \ \forall x_0 \cdots x_{n+1} \in S$

$$P(X_{n+1} = x_{n+1} \mid X_0 = x_0 \ldots X_n = x_n) = P(X_{n+1} = x_{n+1} \mid X_n = x_n)$$

Homogeneous Markov Chain:

$P(X_{n+1}=j \mid X_n=i)$ does not depend on n.

In this case $P_{ij}=P(X_{n+1}=j \mid X_n=i)$ are called transition probabilities e.g. In SRW:

$$P(X_{n+1}=j \mid X_n=i)=\begin{cases} 1/2 & j=i+1 \\ 1/2 & j=i-1 \\ 0 & \text{otherwise} \end{cases}$$

<u>Example</u>: (2 state Markov ch.)

$S=\{0,1\}$,

$P_{00}=P(X_{n+1}=0 \mid X_n=0)=0.8$

$P_{01}=P(X_{n+1}=1 \mid X_n=0)=0.2$ [follows]

$P_{11}=P(X_{n+1}=1 \mid X_n=1)=0.3$

$P_{10}= \qquad =0.7$

Key question: what happens after many steps?

n-step transition probability:

$$P^n_{ij} = P(X_n = j \mid X_0 = i)$$

$$= P(X_{m+n} = j \mid X_m = i) \quad \forall m$$

In SRW: If $X_0 = 0$ what is $P(X_{100} = 10 \mid X_0 = 0) = P^{100}_{0,10}$

end at 10 if first 100 coins

have 55 heads

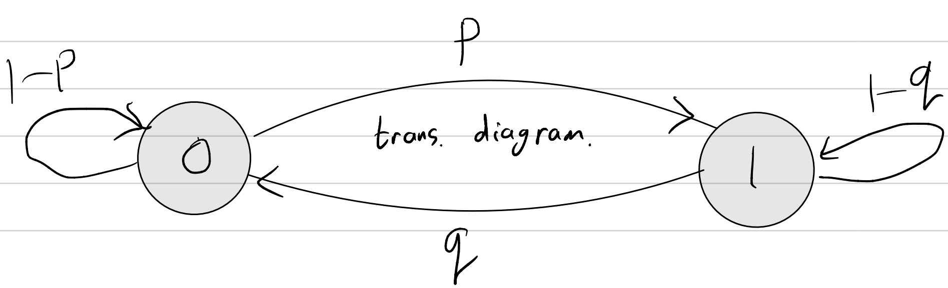
45 tails

$$P^{100}_{0,10} = \binom{100}{55} \cdot 2^{-100}$$

Assume $S = \{0, 1\}$

$P_{01} = p$ $\qquad$ $P_{10} = q$

$\Rightarrow P_{00} = 1 - p$ $\qquad$ $P_{11} = 1 - q$



Qn: what is $\lim_{n\to\infty} P(X_n = i)$

(does limit exist?)

Let $\alpha_i = P(X_0 = i)$

$$P(X_1=0) = P(X_1=0, X_0=0) + P(X_1=0, X_0=1)$$

$$= P(X_0=0)\cdot P(X_1=0 \mid X_0=0) +$$

$$+ P(X_0=1)\cdot P(X_1=0 \mid X_0=1)$$

$$= \alpha_0 \cdot (1-p) + \alpha_1 \cdot q$$

Let $a_n = P(X_n = 0)$

$$a_{n+1} = P(X_n=0)\cdot P(X_{n+1}=0 \mid X_n=0) + P(X_n=1)\cdot P(X_{n+1}=0 \mid X_n=1)$$

$$= a_n \cdot (1-p) + (1-a_n)\,q$$

$$a_{n+1} = q + a_n(1-p-q) = q + \beta a_n$$

where $\beta = 1-p-q$

$a_0 = \alpha_0$

$a_1 = q + \beta\alpha_0 = q + \beta a_0$

$$a_2 = q + \beta(q + \beta\alpha_0) = q + \beta q + \beta^2 \alpha_0$$

$$a_3 = q + \beta q + \beta^2 q + \beta^{\cancel{2}\,3}\alpha_0$$

$$\vdots$$

$$a_n = q\left(1 + \beta + \beta^2 + \cdots + \beta^{n-1}\right) + \beta^{n\cancel{-1}} \cdot \alpha_0$$

$$= q\,\frac{1-\beta^n}{1-\beta} + \beta^{n\cancel{-1}} \cdot \alpha_0$$

assume β, q not both 0 or 1

$|\beta| < 1$ so $\beta^n \xrightarrow[n\to\infty]{} 0$

In this case, $\lim\limits_{n\to\infty} a_n = \dfrac{q}{1-\beta}$

$$\frac{q}{1-\beta} = \frac{q}{p+q}$$

This is called the stationary distribution.

<u>note</u>: If $\alpha_0 = \frac{q}{p+q}$

then $a_n = \frac{q}{p+q}$ for all n.

Qn: Assume that the Probab. of rain tomorrow depends on both today and yesterday.

$$P(X_{n+1}=1 \mid X_n=0, X_{n-1}=0) = 0.2$$

$$P(X_{n+1}=1 \mid X_n=1, X_{n-1}=0) = 0.4$$

$$P(X_{n+1}=1 \mid X_n=0, X_{n-1}=1) = 0.3$$

$$P(X_{n+1}=1 \mid X_n=1, X_{n-1}=1) = 0.6$$

Assume no dep. on earlier days.

Can this be made a M.C.?