Lecture 11

# Branching Processes

model for population size one generation at a time

Chain: some individuals in each generation: $Z_n$

Given a distrib. for the # of children of each indiv.

Assume all indep.

$\xi$ = # of children!

$P(x \text{ has } n \text{ children}) = P(\xi = n)$

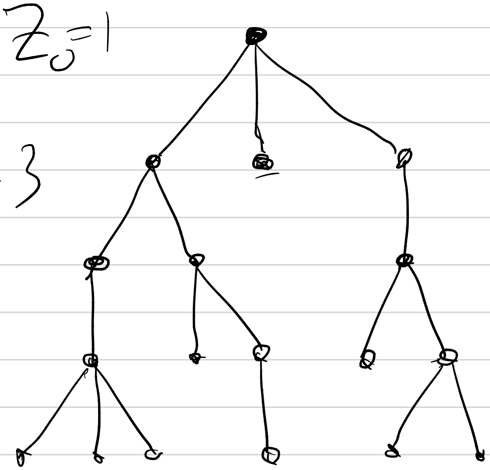
$=: p_n$

$p_n$: pmf of $\xi$ on $\mathbb{N} = \{0, 1, 2, \ldots\}$

e.g. $\xi$ is uniform in $\{0, 1, 2, 3\}$

$Z_0 = 1$

$Z_1 = 3$



$Z_2 = 3$

$Z_3 = 5$

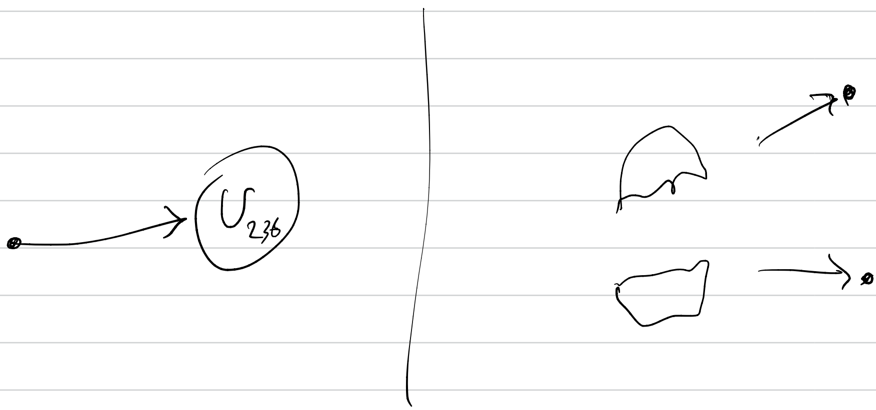
$Z_4 = 6$

Galton – Watson (1873)

Bienayme (earlier)

Each indiv. is infected.

childeren of x : those infected by x.




0, 1, or 2 children.

Qn: Is $Z_n \xrightarrow[n\to\infty]{} \infty$

Is $Z_n = 0$ from some $n$?

Note: $Z_n$ is a M.C.

State space is $\mathbb{N} = \{0, 1, --\}$

$0$ is an absorbing state.

$i$ for $i > 0$ is transient.

(unless $\xi = 1$ always)

Proof: If $p_0 \neq 0$ then

$P_{i0} = p_0^i \neq 0$

$P(i \to 0) \neq 0$ so might never return.

If $p_0 = 0$ then $Z_{n+1} \geq Z_n$.

Here $Z_n \to \infty$. $\square$

Let $E$ = extinction event

$= \{ Z_n = 0 \text{ for some } n \}$

$= \{ \lim Z_n = 0 \}$

Key tool: <u>Probability generating function</u>

For a R.V. $\xi \in \{0,1,2,\dots\}$ the PGF is defined by

$$G_\xi(s) := \mathbb{E}\, s^\xi = \sum_n s^\xi P(\xi = n)$$

e.g. if $\xi$ is unif. on $\{0,1,2,3\}$

then $G_\xi(s) = \frac{1}{4} \cdot s^0 + \frac{1}{4} s^1 + \frac{1}{4} s^2 + \frac{1}{4} s^3$

Properties of PGF:

1) $G_\xi$ is defined for all $|s| \leq 1$

(might be $\infty$ for $|s| > 1$)

2) If $X, Y$ are indep. then

$$G_{X+Y}(s) = G_X(s) \cdot G_Y(s)$$

Pf: $\mathbb{E} s^{X+Y} = \mathbb{E} s^X s^Y = (\mathbb{E} s^X)(\mathbb{E} s^Y)$

3) related to the MGF

$$M_X(t) = \mathbb{E} e^{tX} = G_X(e^t)$$

If $s = e^t$ $\quad G_X(s) = M_X(t)$

4) Let $N \in \{0, 1, \ldots\}$ be random

$X_1, X_2, \ldots$ are indep. with

dist. of $\xi$. Let $T = \sum_{i=1}^{N} X_i$.

<u>Claim</u>: $G_T(s) = G_N(G_\xi(s))$

e.g. If $N = \begin{cases} 1 & \text{w.p. } 1/2 \\ 3 & \text{w.p. } 1/2 \end{cases}$

$$G_N = \frac{s + s^3}{2}$$

If $X_i = \begin{cases} 0 & \text{w.p. } 1/3 \\ 1 & \text{w.p. } 1/3 \\ 2 & \text{w.p. } 1/3 \end{cases}$

$$G_X(s) = \frac{1 + s + s^2}{3}$$

$$G_T(s) = \frac{\left(\frac{1+s+s^2}{3}\right) + \left(\frac{1+s+s^2}{3}\right)^3}{2}$$

Pf: $G_T(s) = \mathbb{E} s^T = \mathbb{E}\left[\mathbb{E}(s^T | N)\right]$

$$= \sum_k P(N=k) \cdot \mathbb{E}(s^T | N=k)$$

$$= \sum_k P(N=k)\, \mathbb{E} s^{X_1 + X_2 + \cdots + X_k}$$

$$= \sum_k P(N=k) \cdot \underbrace{\left(\mathbb{E} s^{\xi}\right)}_{G_\xi(s)}{}^k$$

$$= G_N\left(G_\xi(s)\right) \qquad \square$$

5) relation to moments:

$$G_X(1) = \sum 1^n P(X=n) = 1$$

$$G_X'(s) = \sum_n n s^{n-1} \cdot P(X=n)$$

$$G_X'(1) = \sum n P(X=n) = \mathbb{E}X$$

$$G_X''(s) = \sum_n n(n-1) s^{n-2} P(X=n)$$

$$G_X''(1) = \mathbb{E}(X^2 - X) = \mathbb{E}X^2 - \mathbb{E}X$$

$$\mathrm{Var}(X) = G_X''(1) + G_X'(1) - \left(G_X'(1)\right)^2$$