Lecture 10

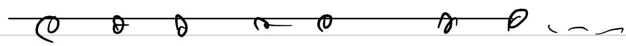
More on Reversible Chains

<u>Recall</u>: a M.C. is reversible (with respect to π) if it satisfies detailed balance condition

$$\forall i,j \quad \pi_i P_{ij} = \pi_j P_{ji} \qquad \text{(DB)}$$

<u>Thm</u> If $\sum \pi_i = 1$ then π must be a stat. dist. for P.

examples: random walk

on line $\mathbb{Z}$ 

$P_{i\,i+1} = P_{i\,i-1} = \frac{1}{2}$

If $|i-j| > 1$ $\quad P_{ij} = P_{ji} = 0$

If $j = i \pm 1$ $\quad P_{ij} = P_{ji} = \frac{1}{2}$

(DB) $\quad \pi_i \cdot \frac{1}{2} = \pi_j \cdot \frac{1}{2}$ so $\pi_i = \pi_j$

so all π_i are equal.

If $\pi_i = 1$ for all i, (DB) holds

Biased random walk:

$S = \mathbb{Z}$. $P_{i,i+1} = q$

$P_{i,i-1} = 1-q$

If $j = i+1$:

$$\pi_i P_{ij} = \pi_j P_{ji}$$

$$\pi_i \cdot q = \pi_{i+1} \cdot (1-q)$$

$$\pi_{i+1} = \frac{q}{1-q} \cdot \pi_i = \alpha \cdot \pi_i$$

with $\alpha = \frac{q}{1-q}$

$\pi_n = \alpha^n$ for all n,
so this is reversible.

<u>example</u> $S = \mathbb{N} = \{0, 1, 2, \ldots\}$

Fix some $\lambda > 0$.

$$P_{i,i+1} = \frac{\lambda}{i+\lambda} \quad P_{i,i-1} = \frac{i}{i+\lambda}$$

If (DB) holds with π,

$$\pi_i \cdot \frac{\lambda}{i+\lambda} = \pi_{i+1} \cdot \frac{i+1}{i+1+\lambda}$$

$$\pi_{i+1} = \pi_i \cdot \frac{\lambda}{i+1} \cdot \frac{i+1+\lambda}{i+\lambda}$$

If $\pi_0 = 1$

$$\pi_1 = \frac{\lambda}{1} \quad \frac{1+\lambda}{0+\lambda} = (1+\lambda)$$

$$\pi_2 = \pi_1 \, \frac{\lambda}{2} \cdot \frac{2+\lambda}{1+\lambda} = (1+\lambda)\frac{\lambda}{2}\,\frac{2+\lambda}{1+\lambda}$$

$$= \frac{\lambda}{2}(2+\lambda)$$

$$\pi_3 = \pi_2 \cdot \frac{\lambda}{3} \cdot \frac{3+\lambda}{2+\lambda} = \frac{\lambda^2}{3!}(3+\lambda)$$

$$\pi_n = \frac{\lambda^{n-1}}{n!}(n+\lambda) = \frac{\lambda^n}{n!} + \frac{\lambda^{n-1}}{(n-1)!}$$

$\sum \pi_n < \infty$ so can be normalized.

$$\pi_n = \frac{\lambda^n}{n!} + \frac{\lambda^{n-1}}{(n-1)!}$$

$$\sum_0^\infty \frac{\lambda^n}{n!} = e^\lambda \qquad \sum_0^\infty \frac{\lambda^{n-1}}{(n-1)!} = e^\lambda$$

To normalize, divide by

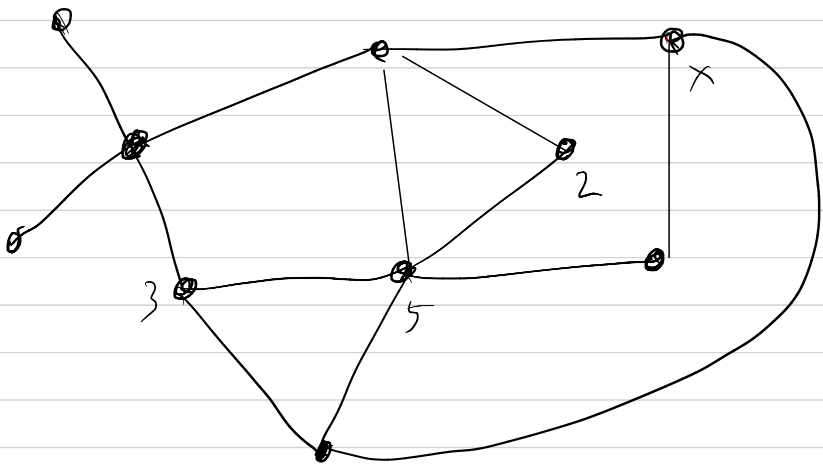
$2e^\lambda$

$$\boxed{\pi_n = e^{-\lambda}\frac{\lambda^n}{n!}\,\frac{n+\lambda}{2}}$$

this is the stat. dist.

example:

Random walk on a graph.



$\pi_x = \frac{3}{28}$

nodes + edges.

deg(v) = edges at v.

From v, move along each edge w.p. $\frac{1}{\deg(v)}$

$$P_{xy} = \begin{cases} \frac{1}{\deg(x)} & \text{if } (x,y) \in G \\ 0 & \text{if not} \end{cases}$$

If (DB) holds then:

If x, y not an edge

(DB): $0 = 0$ ✓

If (x,y) is an edge:

$$\pi_x \cdot \frac{1}{\deg(x)} = \pi_y \cdot \frac{1}{\deg(y)}$$

This holds if $\pi_x = c \cdot \deg(x)$

Sums to 1 if $c = (2E)^{-1}$

with $E = \#$ edges.

So stat. dist. is

$$\boxed{\pi_x = \frac{\deg(x)}{2E}}$$

In ∞ graph, cannot normalize, but still reversible w.r.t. $\pi_x = \deg(x)$