Lecture 5: Recurrence + Transience

Recall: A state x is

recurrent if $P(\text{return to } x \mid X_0 = x) = 1$

transient if $P(\text{return to } x \mid X_0 = x) < 1$

Let $f_x = P(\text{return to } x \mid X_0 = x)$

return to x : $\exists\, n > 0$ s.t. $X_n = x$

Let N_x = number of visits to x

$$= \{ n \geq 0 : X_n = x \}$$

Thm: i is recurrent if and only if $E(N_i \mid X_0 = i) = \infty$

In that case $N_i = \infty$.

Proof: If i recurrent then there is some $n_1 > 0$ s.t. $X_{n_1} = i$. After n_1 the M.C. is same as if started at i,

So w.p. 1 return again

$\exists n_2 > n_1$ s.t. $X_{n_2} = i$

$\exists n_3 > n_2$ s.t. $X_{n_3} = i$

$\vdots$

so: return ∞ many times.

If i transient then return

w.p. $f_i < 1$.

w.p. $1-f_i$ never return.

If return to i, then

never return again w.p. $1-f_i$

The number of returns is $\text{Geom}(1-f_i)$,

$$E(N_i \mid X_0 = i) = \frac{1}{1-f_i} < \infty.$$

$\square$

<u>Thm</u>: $E(N_i \mid X_0 = i) = \sum_{n=0}^{\infty} P_{ii}^n$,

so i recurrent $\iff$

$$\sum_{n=0}^{\infty} P_{ii}^n = \infty.$$

Proof: Let $A_n = \begin{cases} 1 & X_n = i \\ 0 & X_n \neq i \end{cases}$

$$P(A_n \mid X_0 = i) = p_{ii}^n$$

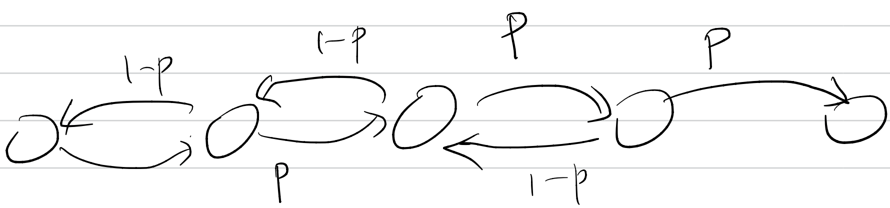
$$E(A_n \mid X_0 = i) = p_{ii}^n$$

$$N_i = \sum_{n=0}^{\infty} A_n .$$

$$E N_i = \sum_{n=0}^{\infty} p_{ii}^n .$$

Thm: the random walk on $\mathbb{Z}$ with no bias is recurr.

if $\left.\begin{array}{l} P_{n,n+1} = p \neq 1/2 \\ P_{n,n-1} = 1-p \end{array}\right\}$ transient



Proof: $P_{ij}^{2n+1} = 0$ since after odd number of steps parity.

$$P_{ii}^{2n} = \binom{2n}{n} p^n (1-p)^n .$$

$$E(N_i \mid X_0 = i) = \sum_n P_{ii}^n = \sum_n P_{ii}^{2n}$$

$$= \sum_n \binom{2n}{n} p^n (1-p)^n .$$

Stirling's formula:

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n} = 1$$

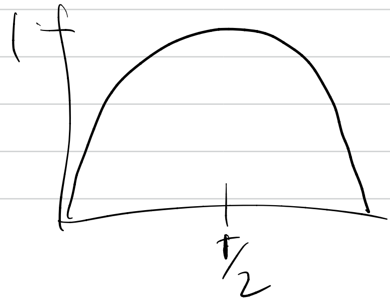
$$\binom{2n}{n} = \frac{2n!}{n!^2} \approx \frac{\sqrt{2\pi\cdot 2n}\left(\frac{2n}{e}\right)^{2n}}{\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}\right)^2}$$

$$= \frac{2^{2n}}{\sqrt{\pi n}}$$

$$P_{ii}^{2n} = \binom{2n}{n} p^n (1-p)^n \sim \frac{\left(4p(1-p)\right)^n}{\sqrt{\pi n}}$$

If $p \neq 1/2$ then $4p(1-p) < 1$

then $\sum P_{ii}^{2n} < \infty$



If $p = \frac{1}{2}$ then $P_{ii}^{2n} \approx \frac{1}{\sqrt{\pi n}}$

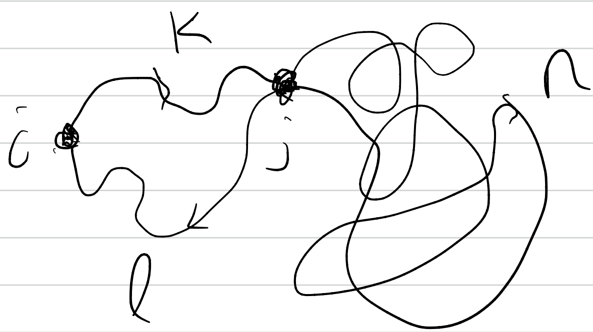
$\sum P_{ii}^{2n} = \infty$. $\square$

<u>Thm</u>: If $i \longleftrightarrow j$ then both are recurrent or both transient.

<u>Conseq.</u>: each commun. equiv. class is recurrent or transient.

Proof: There is some $k, l > 0$ s.t. $p^k_{ij} > 0$ and $p^l_{ji} > 0$

Claim: $P^{k+n+l}_{ii} \geq p^k_{ij}\, p^n_{jj}\, p^l_{ji}$



$$\sum_n p^{k+n+l}_{ii} \geq p^k_{ij}\, p^l_{ji} \sum_n p^n_{jj}$$

If j recurrent then $\sum p_{jj}^{n} = \infty$

$$\sum_n p_{ii}^{n} \geq \sum_n p_{ii}^{k+n+l} = \infty$$

so i also transient.

Same method: i transient

so is j. $\square$

Recall: a M.C. is irreducible if $\forall i j$

$i \longleftrightarrow j$

In this case say that the M.C. is recurrent if all states recurr.

<u>Note</u>: If S finite and M.C. is irreducible, then it must be recurrent. (Since some state must be visited ∞ many times and so is recurrent.)