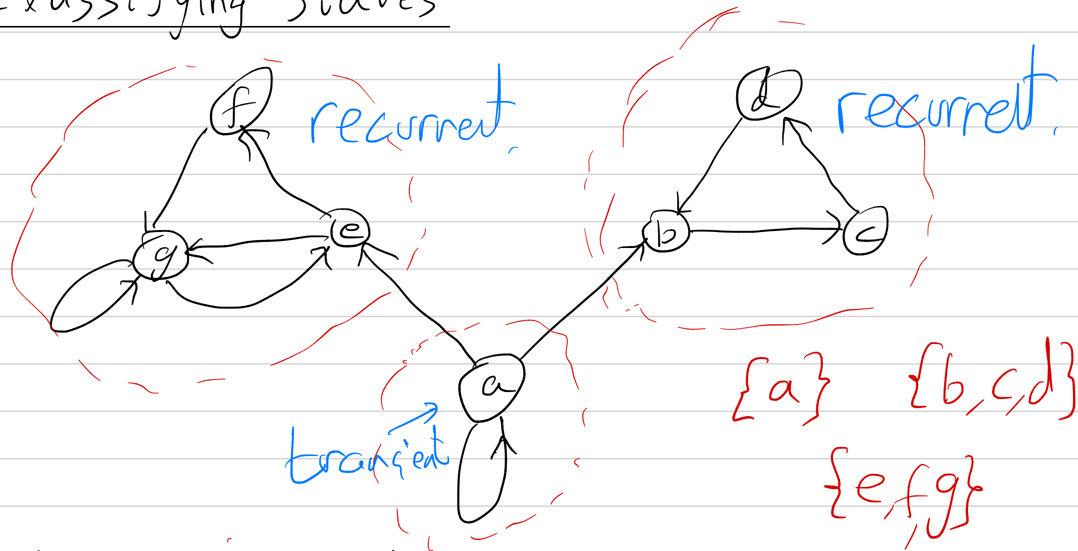
Lecture 4

Classifying states



def: states i, j communicate if there are m, n s.t.

$$\underbrace{P^n_{ij} \neq 0}_{j \text{ reachable from } i} \quad \text{and} \quad \underbrace{P^m_{ji} \neq 0}_{i \text{ reachable from } j}$$

$i \longrightarrow j$: j reachable from i

$i \longleftrightarrow j$: i, j communicate.

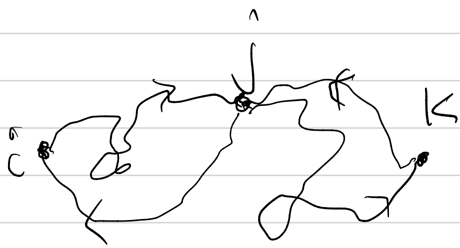
Thm $\longleftrightarrow$ is an equivalence relation, i.e.

- symmetric
- reflexive
- transitive

Proof
- $i \leftrightarrow j \iff j \leftrightarrow i$
- $i \leftrightarrow i$ holds since $P^0_{ii} = 1$

• assume $i \leftrightarrow j \leftrightarrow k$

idea:



formally: $p^n_{ij} > 0$ and $p^l_{jk} > 0$

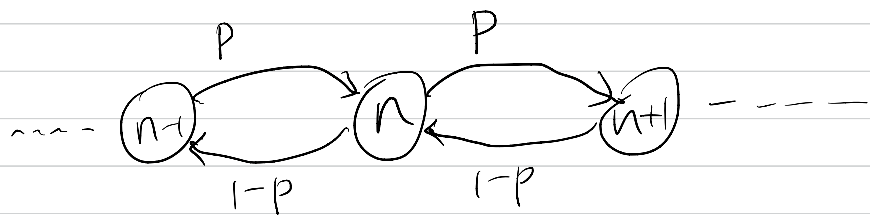
$$p^{n+l}_{ik} = \sum_x p^n_{ix} p^l_{xk} \geq p^n_{ij} p^l_{jk} > 0$$

similarly $p^{n'+l'}_{ki} > 0$ $\square$

<u>Def</u>: <u>communicating classes</u> are the equiv. classes.

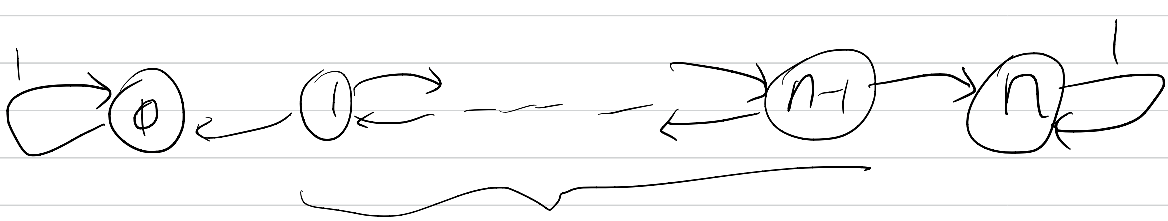
In first e.g., $\{a\}$ $\{b,c,d\}$, $\{e,f,g\}$

e.g. SRW with bias: $S=\mathbb{Z}$



all states comm. (if $p \neq 0,1$)

e.g. Gambler's ruin:



classes are $\{0\}$ $\{n\}$ $\{1,\dots,n-1\}$

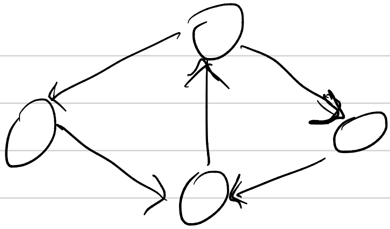
<u>Def</u>: A state i has period d if d is the GCD of $\{n \text{ such that } P_{ii}^n \neq 0\}$

<u>Rmk</u>: equiv. d is maximal integer s.t. $P_{ii}^n = 0$ when $d \nmid n$.

does not divide

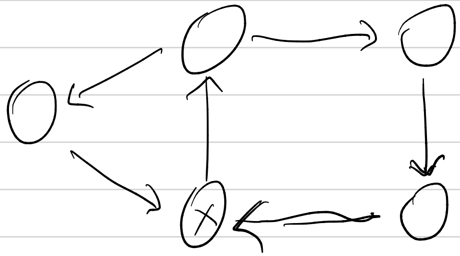
<u>Claim</u>: if $i \leftrightarrow j$ then they have same period.

e.g.



Period 3

e.g.



$p^3_{xx} > 0$ $\quad p^4_{xx}$ $\quad$ so $d \mid 3, 4$

so $d = 1$

a state is aperiodic if $d = 1$

periodic if $d > 1$.

<u>def</u>: a M.C. is <u>irreducible</u> if

$\forall i, j \quad i \longleftrightarrow j$.

def: A state i is recurrent if $P(\exists n>0\ X_n=i \mid X_0=i)=1$

i.e. surely return to i.

i is transient if

$$P(\exists n \text{ s.t. } X_n=i \mid X_0=i)<1$$

Thm: If $i \leftrightarrow j$ then both recurrent or both transient.

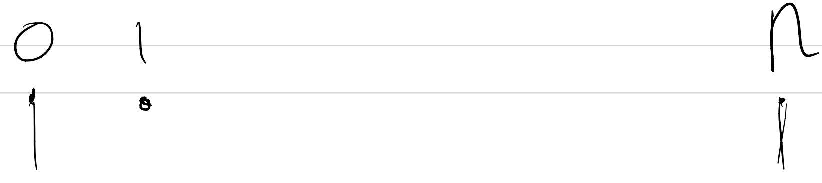
In gambler's ruin, $0, n$ recurrent, $\{1 \dots n-1\}$ are transient.

e.g. SRW with bias:

<u>Thm</u>: This is recurrent if $p=\frac{1}{2}$, transient if $p \neq \frac{1}{2}$.

<u>Proof</u>: Assume $X_0 = 0$.

$X_1 = \pm 1$. Consider case $X_1 = 1$.



By gambler's ruin, If $X_1 = 1$

$P(\text{reach } n \text{ before } 0) = \frac{1}{n}$

$P(\text{reach } 0 \text{ before } n) = 1 - \frac{1}{n}$

$P(\text{return to } 0) \geq 1 - \frac{1}{n}$

n arbitrary, so $P(\text{return to } 0) = 1$

If $p \neq \frac{1}{2}$: Case $p > \frac{1}{2}$.

$P(X_1 = 1) = p$.

$P(\text{reach } n \text{ before } 0 \mid X_1 = 1) = \frac{\alpha^1 - 1}{\alpha^n - 1}$

$\alpha = \frac{1-p}{p} < 1$

$$P(\text{reach } n \text{ before } 0) = \frac{1-\alpha}{1-\alpha^{n}}$$

$$\xrightarrow[n \to \infty]{} 1-\alpha > 0 .$$