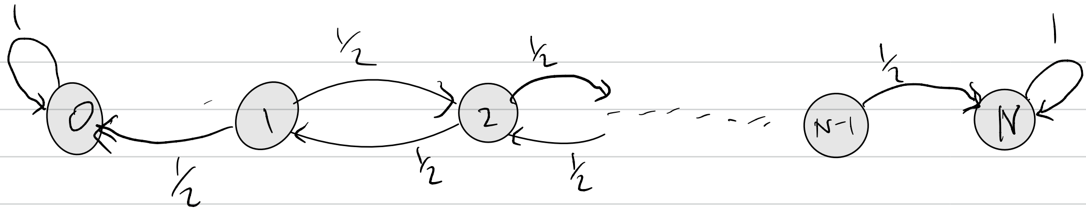
Lecture 3 2022-01-14

# Gambler's Ruin

- Start with $K$ dollars.
- Each step win or lose 1 w.p. $\frac{1}{2}$ each, all indep.
- Stop when reach $0$ or $N$

Qn: what is $P$(reach $N$, not $0$)?

Note: If $X_t$ = money after $t$ steps then $(X_t)$ is a Markov chain.




$$P = \begin{array}{c} 0 \\ \\ \\ \\ \\ N-1 \\ N \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ & \frac{1}{2} & 0 & \frac{1}{2} & & & 0 \\ & & \ddots & 0 & \ddots & & \\ & & & \ddots & & & \\ & 0 & & & \frac{1}{2} & 0 & \frac{1}{2} \\ & & & & 0 & 0 & 0 & 1 \end{pmatrix}$$

**Note**: must reach $0$ or $N$ at some point

**Idea**: Let

$$q_k = P(\text{reach } N \mid X_0 = k)$$

Solve for all $k$ at once

(1) get eq.ns for the $\{q_k\}$

(2) solve these.

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(1) use the Markov property.

If $0 < k < N$

win = "reach N before 0"

$$P(\text{win} \mid X_0 = k) = P(\text{win}, X_1 = k+1 \mid X_0 = k) + P(\text{win}, X_1 = k-1 \mid X_0 = k)$$

$$= \frac{1}{2} q_{k+1} + \frac{1}{2} q_{k-1}$$

$$P(\text{win}, X_1 = k+1 \mid X_0 = k) =$$

$$= \underbrace{P(X_1 = k+1 \mid X_0 = k)}_{P_{k,k+1} = 1/2} \cdot \underbrace{P(\text{win} \mid X_1 = k+1, X_0 = k)}_{q_{k+1}}$$

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For $k = 1 \dots N-1$ :

$$q_k = \frac{1}{2}\left(q_{k-1} + q_{k+1}\right)$$

$$q_0 = 0 \qquad q_N = N$$

(2) solving: $q_{k+1} - q_k = q_k - q_{k-1}$

so diff $q_{k+1} - q_k$ is constant.

$q_k$ is arithmetic prog.

$q_k = a \cdot k + b$ for some $a, b$.

$q_0 = 0$, $q_N = N$ imply $b = 0$

$a = 1/N$

$$\boxed{q_k = \frac{k}{N}}$$

**Note**: all bets are fair, so

$EX_t = K$ for all $t$.

In the limit $X_\infty = \begin{cases} N & \text{w.p. } q_K \\ 0 & \text{w.p. } 1-q_K \end{cases}$

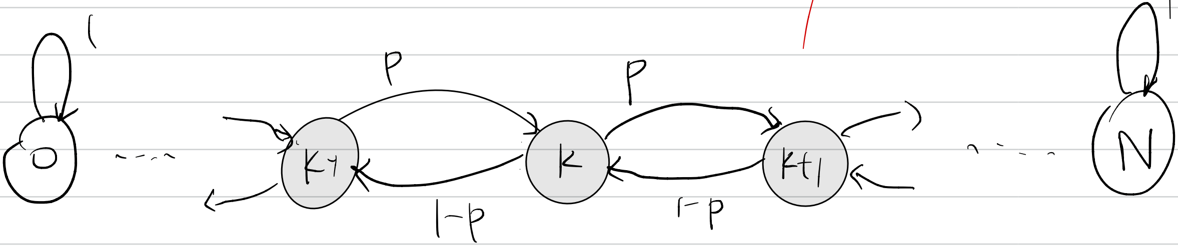
so $EX_\infty = N \cdot q_K$

so $q_K = K/N$.

Qn: what if each round is won w.p $p \neq 1/2$ ?

e.g. roullette, bet on red/black

win each round w.p. $\cancel{\frac{19}{40}}$ $\frac{18}{38}$




Qn: what is $q_k = P(\text{win} \mid X_0 = k)$

Same method:

$$q_k = P(X_1 = k+1 \mid X_0 = k) \cdot q_{k+1} +$$
$$+ P(X_1 = k-1 \mid X_0 = k) \cdot q_{k-1}$$

$$q_k = p\, q_{k+1} + (1-p)\, q_{k-1} \qquad (*)$$

Try $q_k = x^k$

$$x^k = p\, x^{k+1} + (1-p)\, x^{k-1}$$

$$(p x^2 - x + 1 - p)\, x^{k-1} = 0$$

$$x^{k-1}(x-1)(px + p - 1) = 0$$

$\Rightarrow x = 1$ or $x = \frac{1-p}{p} = \alpha$ work.

Since (*) are linear, any combination of 1 and $\alpha^k$ also works.

$$q_k = a \cdot \alpha^k + b \quad \text{for some } a, b.$$

$q_0 = 0 \Rightarrow a + b = 0$

$q_N = 1 \Rightarrow a\alpha^N + b = 1$

So $\boxed{b = -a}$ and $\boxed{a = \frac{1}{\alpha^N - 1}}$

So $$\boxed{q_k = \frac{\alpha^k - 1}{\alpha^N - 1}}$$

e.g. $k = 100$, $N = 1000$ $p = \frac{18}{38}$

so $\alpha = \frac{1-p}{p} = \frac{20}{18}$

$$q_{100} \simeq 6.58 \cdot 10^{-42}$$