Lecture 8

time reversal + reversibility

X_n is a M.C. fix some N

Let $Y_n = X_{N-n}$

<u>Thm</u>: If X_n is a M.C. with stat. dist. π and X_0 has dist. π, then Y is also a M.C. with trans. prob. $Q_{ij} = \dfrac{P_{ji} \cdot \pi_j}{\pi_i}$

Proof: for any seq. of states

$x_0\, x_1 \cdots x_N$, the

$P(X_i = x_i \text{ for } i = 0 \ldots N)$

$= \pi_{x_0} P_{x_0 x_1} P_{x_1 x_2} \cdots P_{x_{N-1} x_N}$

This is $P(Y_i = x_{N-i} \text{ for all } i)$

Claim: This equals

$\pi_{x_N} Q_{x_N x_{N-1}} Q_{x_{N-1} x_{N-2}} \cdots Q_{x_1 x_0}$

need:

$$\pi_{x_0} \prod P_{x_i x_{i+1}} = \pi_{x_N} \prod Q_{x_{i+1} x_i}$$

If plug in $Q_{ij} = \dfrac{P_{ji} \pi_j}{\pi_i}$

$$\pi_{x_N} \cdot \prod Q_{x_{i+1} x_i} = \pi_{x_N} \cdot \prod \frac{P_{x_i x_{i+1}} \cdot \pi_{x_i}}{\pi_{x_{i+1}}}$$

$$= \prod P_{x_i x_{i+1}} \cdot \frac{\pi_{x_0}}{\pi_{x_1}} \frac{\pi_{x_1}}{\pi_{x_2}} \cdots \frac{\pi_{x_{N-1}}}{\pi_{x_N}} \pi_{x_N}$$

$$= \pi_{x_0} \prod P_{x_i x_{i+1}} \qquad \square$$

Proposition: Q has same stat. dist. as P, namely π.

Proof: need $\pi Q = \pi$.

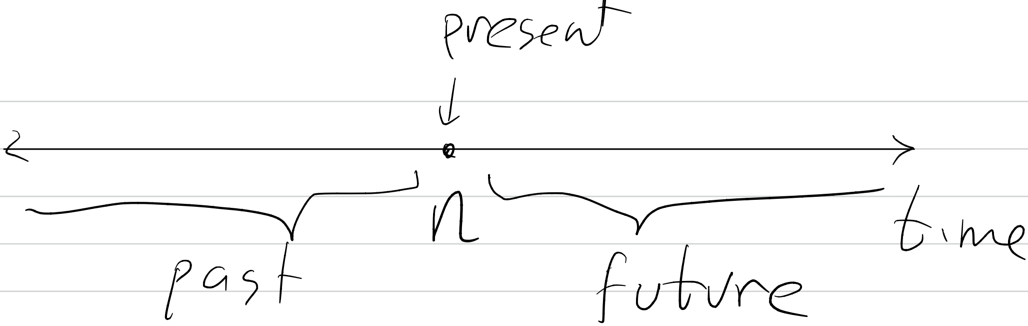
$$(\pi Q)_j = \sum_i \pi_i Q_{ij} = \sum_i \pi_i \frac{P_{ji}\,\pi_j}{\pi_i}$$

$$= \left(\sum_i P_{ji}\right) \cdot \pi_j = \pi_j \qquad \square$$

Thm: If $(X_i)_{i=0..N}$ is any M.C. then $Y_n = X_{N-n}$ is also a M.C.

Note: If X is not stationary then Y might not be time homog.

Proof idea: Markov property: given present, future is indep. of past.



$\square$

In general $Q \neq P$.

<u>Def</u>: A Markov chain is <u>reversible</u> if $Q = P$

i.e. for any $i\,j$ $\; P_{ij} = Q_{ij}$

$$\forall i\,j \quad \pi_i P_{ij} = \pi_j P_{ji} \qquad (*)$$

(*) is called <u>detailed balance</u> equation

<u>Thm</u>: If P is an irreducible stoch. matrix and π has $\sum \pi_i = 1$ and detailed balance then π is the stat. dist. for P.

<u>Pf</u> need $\pi P = \pi$

$$(\pi P)_j = \sum_i \pi_i P_{ij}$$

$$= \sum_i \pi_j P_{ji} \qquad \text{(def. bal.)}$$

$$= \pi_j \sum_i P_{ji} = \pi_j \qquad \square$$