Lecture 9

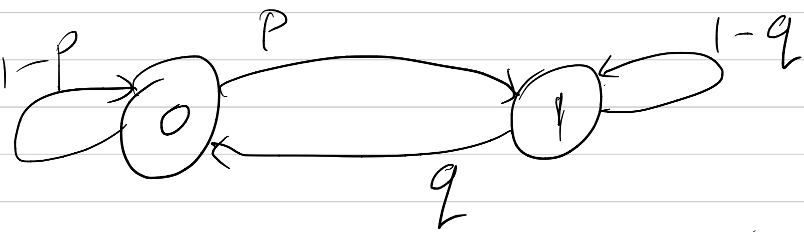
reversible M.C.s

Recall: a M.C. with trans. matrix P is reversible if for some π we have detailed balance

$$\forall i,j \qquad \pi_i P_{ij} = \pi_j P_{ji} \qquad \text{(DB)}$$

(Also say that the M.C. is reversible w.r.t. π.)

e.g. 2 state M.C.



stat. meas. is $\pi = \left(\frac{q}{p+q} \quad \frac{p}{p+q} \right)$

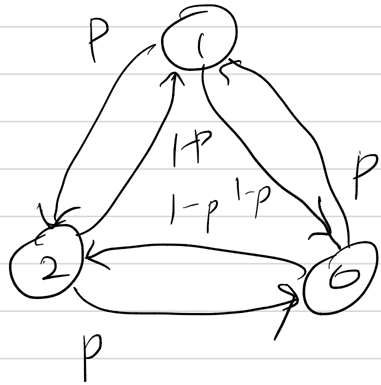
need to check (DB) $\forall i j$

$i=j$ always satis. (DB)

$i=0 \quad j=1$: need

$$\frac{q}{p+q} \cdot p = \frac{p}{p+q} \cdot q \qquad \checkmark$$

3-states:



$$P = \begin{pmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{pmatrix}$$

By symmetry $\pi = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

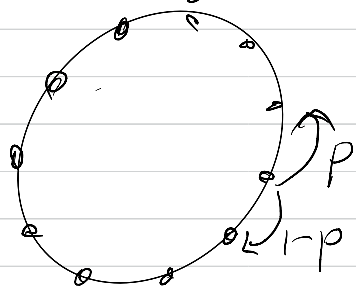
e.g. $i=0$ $j=1$: need

$$\pi_0 P_{01} = \pi_1 P_{10}$$

$$\frac{1}{3} p = \frac{1}{3}(1-p)$$

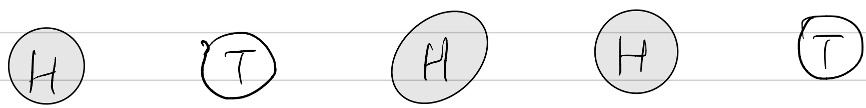
(DB) holds iff $p = 1/2$.

Similarly for any cycle reversible iff $p = 1/2$



Ehrenfest Urn

M coins on a table



Step: pick a random coin and flip it over.

Q: stat. meas $\pi = ?$

$X_n = \#$ Heads showing.

$P_{ij} = 0$ unless $j = i \pm 1$

$$P_{i,i+1} = \frac{M-i}{M}$$

(i heads, $M-i$ tails)

$$P_{i,i-1} = \frac{i}{M}.$$

$\pi = \pi P$:

$$\pi_j = \sum_i \pi_i P_{ij}$$

$$\pi_j = \pi_{j-1} \cdot \frac{M-(j-1)}{M} + \pi_{j+1} \cdot \frac{j+1}{M}$$

Check if DB holds for some π:

[Recall: DB holds for π then π is the stat. dist.]

$$\pi_i P_{ij} = \pi_j P_{ji}$$

If $j \neq i \pm 1$ $P_{ij} = P_{ji} = 0$

so (DB) holds.

If $j = i+1$:

$$\pi_i \frac{M-i}{M} = \pi_{i+1} \cdot \frac{i+1}{M} \quad (*)$$

Sol. : $\pi_i = \binom{M}{i} \cdot 2^{-M}$

one way: guess + check.

To solve $(*)$

$$\pi_{i+1} = \pi_i \frac{M-i}{i+1}$$

$$\pi_1 = \pi_0 \cdot \frac{M}{1}$$

$$\pi_2 = \pi_1 \cdot \frac{M-1}{2} = \pi_0 \cdot \frac{M}{1} \cdot \frac{(M-1)}{2}$$

$$\pi_3 = \pi_0 \cdot \frac{M}{1} \cdot \frac{M-1}{2} \cdot \frac{M-2}{3}$$

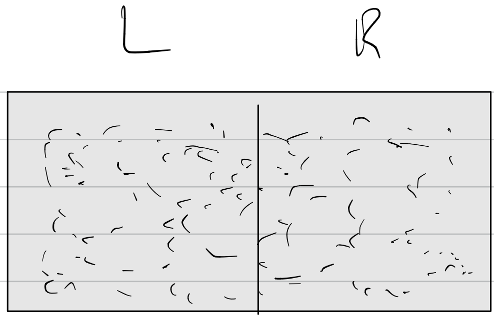
$$\pi_k = \pi_0 \frac{M(M-1) \cdots (M-k+1)}{1 \cdot 2 \cdots k}$$

$$= \pi_0 \cdot \frac{M! / (M-k)!}{k!} = \pi_0 \binom{M}{k}$$

If $\sum \pi_i = 1$

$$1 = \sum \pi_i = \sum \pi_0 \binom{M}{i} = \pi_0 \cdot 2^M$$

so $\pi_0 = 2^{-M}$



molecules in a room.

At each step one molecule switches sides.

X_n = # of mol. on left

so stat. dist. is $\mathrm{Bin}(M, \frac{1}{2})$