Lecture 12

# Branching Processes (cont.)

Bienayme-Galton-Watson:

each indiv has indep. # of children. $p_k = P(\xi = k)$

$Z_n = |\text{generation } n|$

fix $Z_0 = 1$.

Prob. gen func:

$$G_X(s) = \mathbb{E}\, s^X$$

Prop

If $N$ is random, $X_1, \dots$ indep. same dist.

$$T = \sum_{i \le N} X_i$$ then

$$G_T(s) = G_N(G_X(s))$$

Thm: $$G_{Z_n}(s) = \underbrace{G_\xi(G_\xi(\dots G_\xi(s)))}_{n \text{ times}}$$

Proof: By induction.

$n=1$: clear.

$Z_2$ is a sum of $Z_1$ copies of $\xi$. By Prop,

$$G_{Z_2}(s) = G_{Z_1}(G_\xi(s)) = G_\xi \circ G_\xi(s)$$

$$\boxed{f \circ g(x) = f(g(x))}$$

By induction:

$$G_{Z_{n+1}} = G_{Z_n}(G_\xi(s))$$

so $$= \underbrace{G_\xi \circ G_\xi \circ \cdots G_\xi}_{G_{Z_n}} \circ G_\xi(s)$$

e.g. If $\xi = \begin{cases} 0 & 1-q \\ 2 & q \end{cases}$

$$G_\xi(s) = (1-q)\cdot s^0 + q s^2$$
$$= 1-q+qs^2$$

<u>Prop</u>: Let $\mu = \mathbb{E}\xi$ $\sigma^2 = Var(\xi)$

① $\mathbb{E} Z_n = \mu^n$

② $Var(Z_n) = \begin{cases} n\sigma^2 & \mu = 1 \\ \dfrac{\sigma^2(\mu^n - 1)\mu^{n-1}}{\mu - 1} & \mu \neq 1 \end{cases}$

Proof

Recall $\mathbb{E}X = G_X'(1)$.

For ① need $(\underbrace{G \circ G \circ \cdots \circ G}_{n})'(1)$

By chain rule:

$$G'(1) \cdot (\underbrace{G \circ G \circ \cdots \circ G}_{n-1})'(G(1))$$

$G(1) = 1$ so

$$\mathbb{E}Z_n = \underbrace{G'(1)}_{\mu = \mathbb{E}\xi} \mathbb{E}Z_{n-1} = \mu^n$$

② Use $\mathrm{Var}(X) = G_X''(1) + \mathbb{E}X - (\mathbb{E}X)^2$

$\mathbb{E} Z_n = \mu^n$ so need $G''_{Z_n}(1)$

$$G''_{Z_n}(s) = \frac{d}{ds} G'_{Z_n}(s)$$

$$= \frac{d}{ds} \left( G \circ G_{Z_{n-1}} \right)'(s)$$

$$= \frac{d}{ds} \left[ G'_{Z_{n-1}}(s) \cdot G'\left( G_{Z_{n-1}} \right) \right]$$

$$= G''_{Z_{n-1}} \cdot G'\left( G_{Z_{n-1}} \right) +$$

$$+ G''\left( G_{Z_{n-1}} \right) G'_{Z_{n-1}}$$

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Set $s=1$ !

$$G''_{Z_n}(1) = \underbrace{G'(1)}_{\mu} \cdot G''_{Z_{n-1}}(1) +$$

$$+ G''(1) \cdot \underbrace{G'_{Z_{n-1}}(1)}_{\mu^{n-1}}$$

$$G''_{Z_n} = \mu G''_{Z_{n-1}} + \mu^{n-1}(\sigma^2 - \mu + \mu^2)$$

Finish by induction, $n=0$ or $1$

as base case.

Case $\mu = 1$: $G''_{Z_n} = G''_{Z_{n-1}} + \sigma^2$

so $G''_{Z_n} = n\sigma^2$

$$\mathrm{Var}(Z_n) = G''_{Z_n} + \cancel{\mu^n} - \cancel{(\mu^n)^2} = n\sigma^2$$

$\square$

**Note:** If $\mu < 1$ $\quad \mathbb{E} Z_n \longrightarrow 0$

$\mu > 1$ $\quad \mathbb{E} Z_n \longrightarrow \infty$

Let $\rho = P(\underbrace{\text{extinction}}_{Z_n = 0 \text{ for some } n.})$

$$\boxed{\begin{array}{l} \textbf{Thm } \rho \text{ is smallest positive solution of } s = G_\xi(s). \\ \circledast \text{ If } \mu < 1 \text{ then } \rho = 1 \\ \circledast \text{ If } \mu > 1 \text{ then } \rho < 1 \\ \circledast \text{ If } \mu = 1 \quad P_1 \neq 1 \quad \rho = 1 \end{array}}$$

$P(X=0) = G_X(0)$

$$\boxed{0^0 = 1}$$

so $P(Z_n = 0) = \underbrace{G \circ G \circ \cdots G}_{n}(0)$