2022-01-10

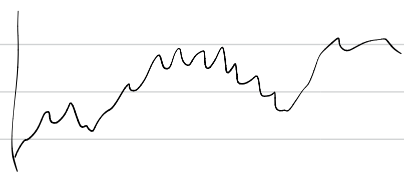
# Markov Chains

## Stochastic Process:

Sequence $(X_n)_{n\geq 0}$ : $\underbrace{X_0 \quad X_1 \quad X_2 \cdots}_{\text{Rand. Var.s}}$

$(X_t)_{t\in\mathbb{R}}$ : cont. time.

e.g. stock prices



## Markov chain def. (Markov Property): $\forall n, x_0 \cdots x_{n+1}$

$$P(X_{n+1}=x_{n+1} \mid X_0=x_0 \cdots X_n=x_n)$$

$$= P(X_{n+1}=x_{n+1} \mid X_n=x_n)$$

(Trivial) example: $(X_n)$ are all indep. Here, both are just $P(X_{n+1} = x_{n+1})$

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$X_n$ take values in set $S$

$S$ = state space.

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e.g. $S = \{0, 1\}$

e.g. $X_n = \begin{cases} 0 & \text{day } n \text{ sunny} \\ 1 & \text{rainy} \end{cases}$

$$P(X_{n+1}=1 \mid X_n=0)=p$$

$$\Rightarrow \quad P(X_{n+1}=0 \mid X_n=0)=1-p$$

$$P(X_{n+1}=0 \mid X_n=1)=q$$

$$\Rightarrow P(X_{n+1}=1 \mid X_n=1)=1-q$$

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Suppose $X_0=1$. what is prob that $X_1 \dots X_7$ all 1 ?

Ans : $(1-q)^7$

$$P(ABCD) = P(A) \cdot P(B|A) \cdot P(C|AB) \cdot P(D|ABC)$$

$$P(X_1 = 1 | X_0 = 1) = 1 - q$$

$$P(X_2 = 1 | X_0 = X_1 = 1) = P(X_2 = 1 | X_1 = 1) = 1 - q$$

$$P(X_3 = 1 | X_0 = X_1 = X_2 = 1) = 1 - q.$$

$$\vdots$$

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If $X_0 = 0$ what is prob that $X_1 \sim X_5$ are $0, 0, 1, 1, 0$.

<u>Ans</u>: $(1-p)(1-p) \cdot p \cdot (1-q) \cdot q$

Proposition: given $X_0$ and transition probabilities, we can compute prob. of any sequence of values.

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The transition prob.

$P(X_{n+1}=j \mid X_n=i)$ above

do not depend on $n$.

This is a time homogeneous M.C.

In a homog. M.C. transition probab. are denoted $P_{ij}$

$$P_{ij} = P(X_{n+1} = j \mid X_n = i)$$

$$Y_n = \begin{pmatrix} X_n \\ X_{n-1} \end{pmatrix} \in \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

Given $Y_n$, know $P(X_{n+1} = 1)$

given $Y_n \; X_{n+1} \Longrightarrow Y_{n+1}$

$$\begin{tikzcd} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ Y_n = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \arrow[ur, "0.3"] \arrow[dr, "0.7"'] & \\ & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{tikzcd}$$

$$P\left(Y_{n+1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \middle| Y_n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = 0$$

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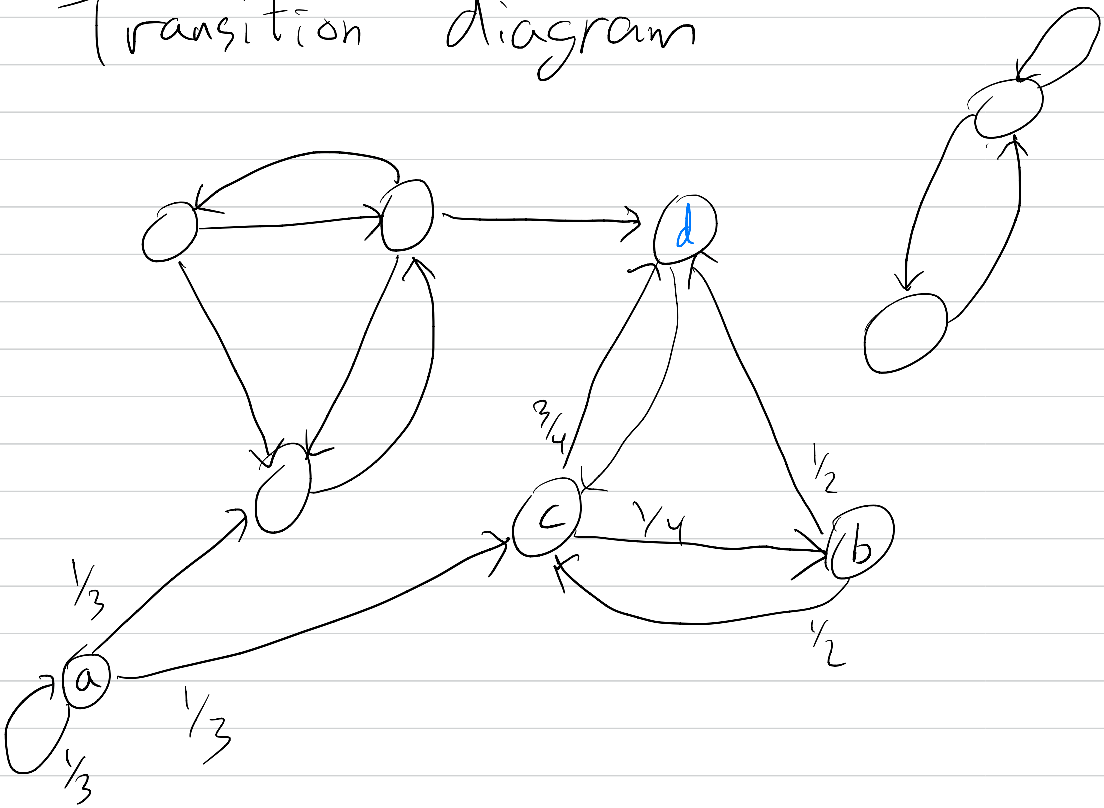
# Lecture 2

## Transition diagram




$$P^2_{ab} = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$$

$$P^3_{ac} =$$

$$P^3_{ac} = P(a \to c \to b \to c) + P(a \to c \to d \to c)$$

$$+ P(a \to a \to a \to c)$$

$$= \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{3}{4} \cdot 1 + \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}$$

$$= \frac{1}{24} + \frac{1}{4} + \frac{1}{27}$$

## Chapman-Kolmogorov eq.n

$P^n_{xy}$ = n-step trans. prob.

$$= P(X_{m+n} = y \mid X_m = x)$$

Thm: $\forall n, m, i, j$

$$P^{m+n}_{ij} = \sum_{k \in S} P^m_{ik} \cdot P^n_{kj}$$

As matrices $P^{m+n} = P^m \cdot P^n$

By induction: $P^n$ is nth power of $P$

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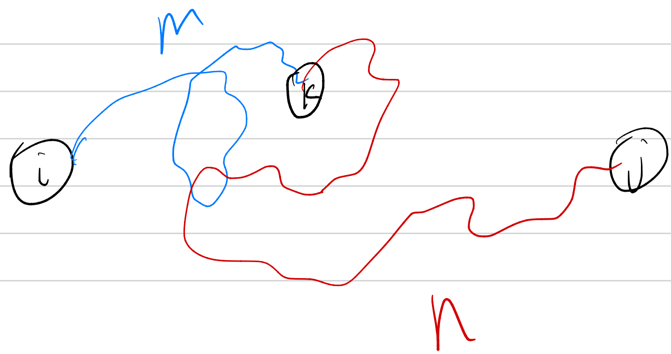
idea



$$P_{ij}^{m+n+s} = \sum_{k} P_{ik}^{m} P_{kj}^{n+s}$$

$$= \sum_{k} P_{ik}^{m} \sum_{l} P_{kl}^{n} P_{lj}^{s}$$

$$= \sum_{k,l} P_{ik}^{m} P_{kl}^{n} P_{lj}^{s}$$

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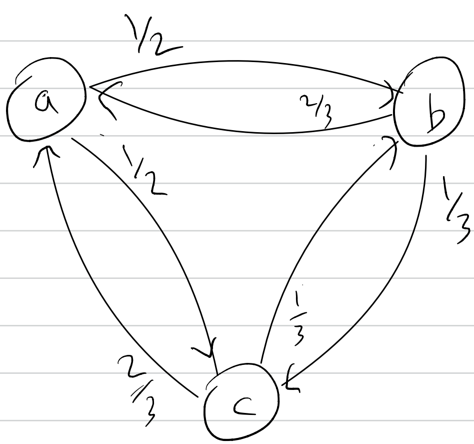

$$P^2_{ab} = \sum_k P^1_{ak} \cdot P^1_{kb}$$

$$= 0 \cdot P^1_{ab} + P^1_{ab} \cdot 0 + P^1_{ac} \cdot P^1_{cb}$$

$$= 0 + 0 + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$

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$$P = \begin{array}{c} \\ \\ \end{array} \begin{array}{cc} \begin{array}{ccc} 0 & 1 & 2 \end{array} & \\ \begin{pmatrix} 0.7 & 0.1 & 0.2 \\ 0.4 & 0.4 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{pmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \end{array}$$

Observe: $P^n_{ij} \xrightarrow[n \to \infty]{} \pi_j$

$$\pi = \left( \frac{22}{49}, \frac{13}{49}, \frac{14}{49} \right)$$

Assume $P^n \xrightarrow[n\to\infty]{} Q$ ← some matrix.

$$P^{n+1} = P^n \cdot P$$

$$\begin{array}{ccc} P^{n+1} & = & P^n \cdot P \\ \downarrow {\scriptstyle n\to\infty} & & \downarrow \\ Q & = & Q \cdot P \end{array}$$

so $Q = Q \cdot P$

First row of $Q$ is $(1,0,0,\dots)Q$

Let $\pi$ = first row of $Q$.

$$(1,0,0,0)Q = (1,0,0,0,0)Q \cdot P$$

$$\pi = \pi P$$

[$\pi$ is left eigenvector of $P$]

left e. vector : $v$ such that

$$v \cdot P = \lambda v$$

rows of $P^n$ converge to stationary dist. of the M.C.

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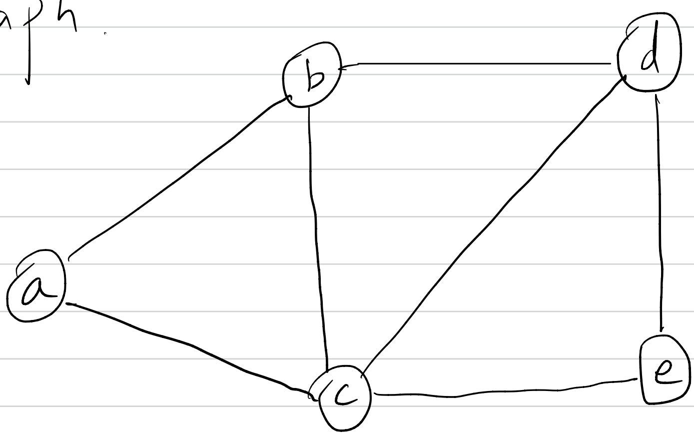
simple random walk on a graph



$S = \{\text{vertices}\}$

from $x$ jump to a uniform neighbourr of $x$

$P_{ab} = \frac{1}{2} = P_{ac}$ $\quad$ $P_{aa} = P_{ad} = P_{ae} = 0$

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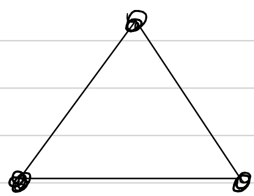
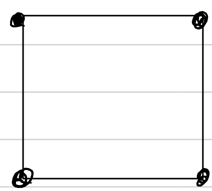
exercise:





what happens to random walk? $\quad$ $P^n_{xy} = ?$

## Lecture 3 Gambler's ruin

hitting probabilities

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M.C. on $\{0 \ldots N\}$

$0, N$ are absorbing states (sinks):

state $x$ s.t. $P_{xx} = 1$

Goal: find

$q_k := P(\text{reach } N \mid X_0 = k)$

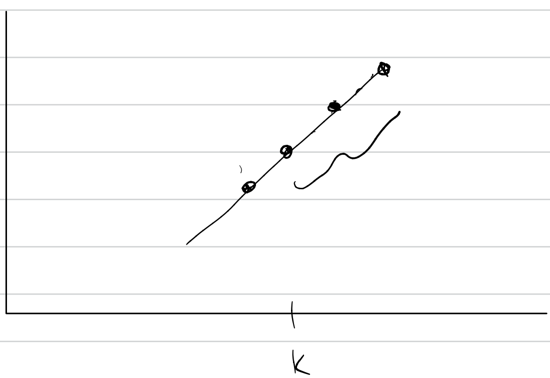
$x := \text{expr.}$ or $\text{expr.} =: x$

defines $x$

for $0 < k < N$:

$$q_k = \frac{1}{2} q_{k-1} + \frac{1}{2} q_{k+1} \qquad (*)$$

Know $q_0 = 0$ $\quad q_N = 1$




$q_k$ are arithmetic prog.

$$(*) \Rightarrow q_k = a \cdot k + b$$

$$\left. q_0 = 0 \text{ and } q_N = 1 \right\} \Rightarrow \begin{array}{l} a = \frac{1}{N} \\ b = 0 \end{array}$$

$$\boxed{q_K = \frac{K}{N}}$$

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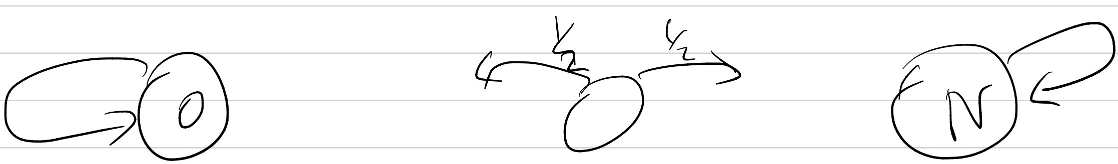
Martingale approach

$$EX_n = EX_0 \quad \text{for all } n.$$

If $X_0 = K$ then $EX_n = K$.



eventually $X_n = \begin{cases} 0 & \text{w.p. } 1-q_k \\ N & \text{w.p. } q_k \end{cases}$

$$E X_n = (1-q_k)\cdot 0 + N\cdot q_k = k$$

so $q_k = \frac{k}{N}$

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## Biased bets

win each round w.p. $p \neq \frac{1}{2}$

$$q_k = p\, q_{k+1} + (1-p)\, q_{k-1} \qquad (*)$$

This is linear in the $q_k$'s

$\Rightarrow$ Linear combinations of sol to $(*)$ also satisfy $(*)$.

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Guess $q_k = s^k$ for some $s$.

$$\Rightarrow \quad s = ps^2 + (1-p)$$

$$\Rightarrow \quad s = 1 \quad \text{or} \quad s = \alpha := \frac{1-p}{p}$$

$q_k = 1$ or $q_k = \alpha^k$ solve (*)

$q_k = a \cdot \alpha^k + b$ also works

for any $a, b$.

$$\left.\begin{array}{l} q_0 = 0 \\ q_N = 1 \end{array}\right\} \Rightarrow \begin{array}{l} a+b=0 \\ a\alpha^N = 1 \end{array} \Rightarrow \text{find } a, b$$

$$\Rightarrow \boxed{q_k = \frac{\alpha^k - 1}{\alpha^N - 1}}$$

$$q_k = \frac{1}{\alpha^N - 1} \cdot \alpha^k + \frac{-1}{\alpha^N - 1} \cdot 1^k$$

If $(a_k)$ solves $(*)$

$(b_k)$ also does.

$$a_k = p a_{k+1} + (1-p) a_{k-1}$$

$$b_k = p b_{k+1} + (1-p) b_{k-1}$$

$$(a_k + b_k) = p(a_{k+1} + b_{k+1}) + (1-p)(a_{k-1} + b_{k-1})$$

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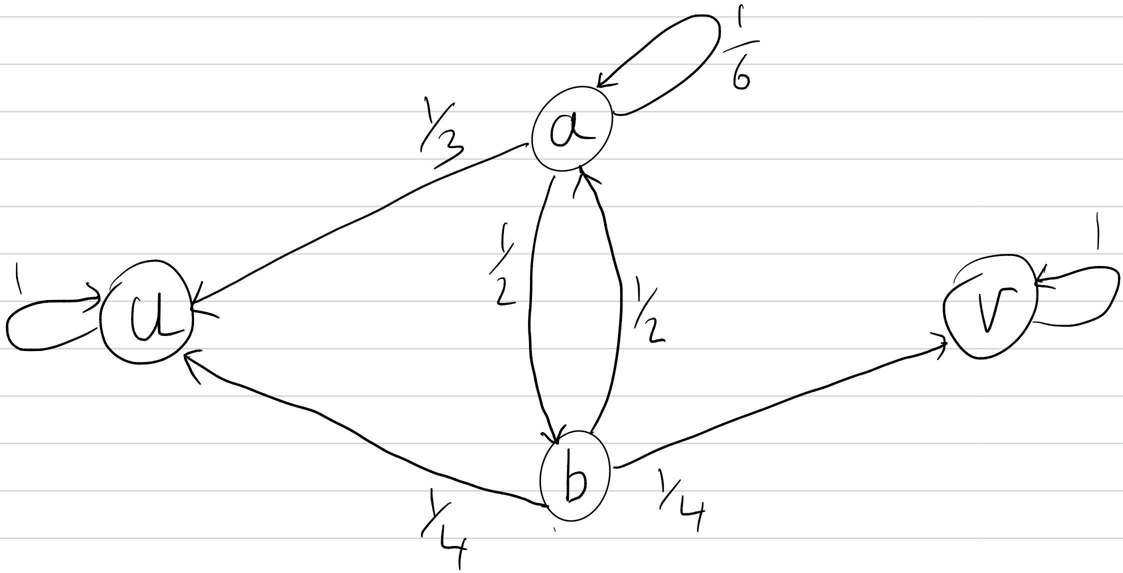
$p < 1/2$ then $\alpha = \frac{1-p}{p} > 1$ so

$$\alpha^k \gg 1$$

$$q_k \approx \frac{\alpha^k}{\alpha^N} = \alpha^{k-N}$$

# hitting probabilities




<u>Qn</u>: Find $P(\text{reach } u \mid X_0 = a)$

Idea: Find for all $i$

$$q_i = P(\text{reach } u \mid X_0 = i)$$

$$q_a = \sum_i P(\text{reach } u, X_1 = i \mid X_0 = a)$$

$$= \sum_i P(\text{reach } u \mid X_1 = i) \cdot P_{ai}$$

$$= \sum_i P_{ai} q_i$$

$$q_a = \frac{1}{6} \cdot q_a + \frac{1}{3} q_u + \frac{1}{2} q_b$$

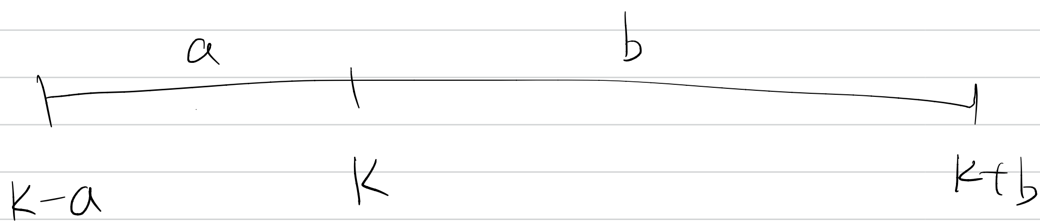
$$q_b = \frac{1}{2} q_a + \frac{1}{4} q_u + \frac{1}{4} q_v$$

$$q_u = 1 \qquad q_v = 0$$

$$q_a = \frac{1}{6} q_a + \frac{1}{2} q_b + \frac{1}{3}$$

$$q_b = \frac{1}{2} q_a + \frac{1}{4}$$


$$\frac{K}{N}$$


$$P(\text{reach } K+b) = \frac{a}{a+b}$$

$$p\alpha^{k+1} + (1-p)\alpha^{k-1}$$

$$= (p\alpha^2 + 1 - p)\alpha^{k-1}$$

$$= \alpha \cdot \alpha^{k-1}$$

$$= \alpha^k \qquad \alpha = p\alpha^2 + 1 - p$$

$$q_k = p q_{k+1} + (1-p) q_{k-1}$$

$$q_1 = \frac{\alpha - 1}{\alpha^N - 1} \qquad q_2 = \frac{\alpha^2 - 1}{\alpha^N - 1}$$

$$p q_k + (1-p) q_k$$

$$\underbrace{p(q_k - q_{k+1})}_{} = \underbrace{(1-p)(q_{k-1} - q_k)}_{}$$

$$\frac{q_k - q_{k+1}}{q_{k-1} - q_k} = \frac{1-p}{p} = \alpha$$

$$q_{k+1} - q_k = c \cdot \alpha^k$$

$$q_k = q_0 + c + c\alpha + c\alpha^2 + \dots c\alpha^{k-1}$$

$$q_N = 1$$

# Lecture 4

## Definitions

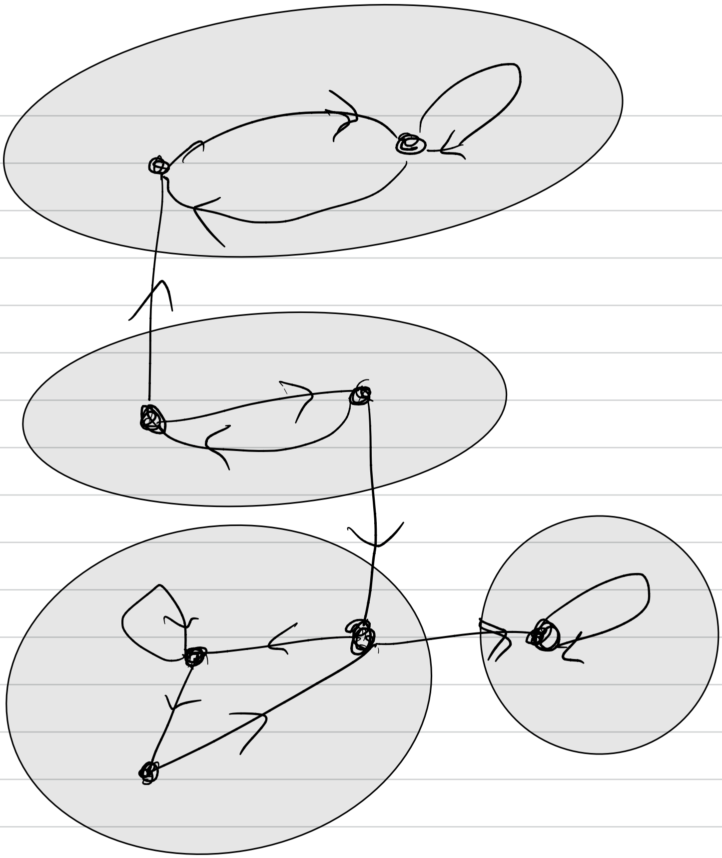
- $i \to j$ : $j$ accessible from $i$

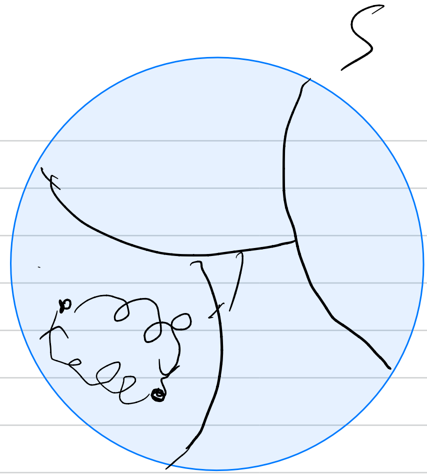
  $\exists n$ s.t. $P^n_{ij} \neq 0$

- $i \leftrightarrow j$ : $i$ communicates with $j$

  $i \to j$ $j \to i$

is an Equivalence relation



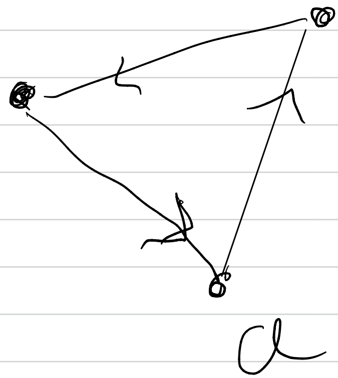



communic classes

- MC is <u>irreducible</u> if

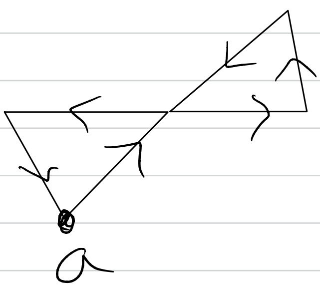
  $\forall i,j \quad i \longleftrightarrow j$

## Periodicity:




$P^n_{aa} \neq 0$ for

$n = 3, 6, 9, \ldots$




Same n's

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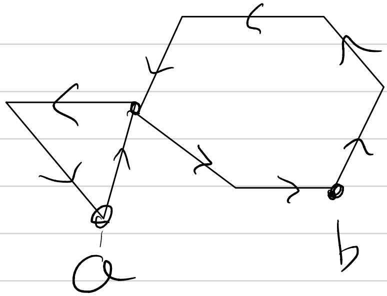



$P^n_{aa} \neq 0$ for

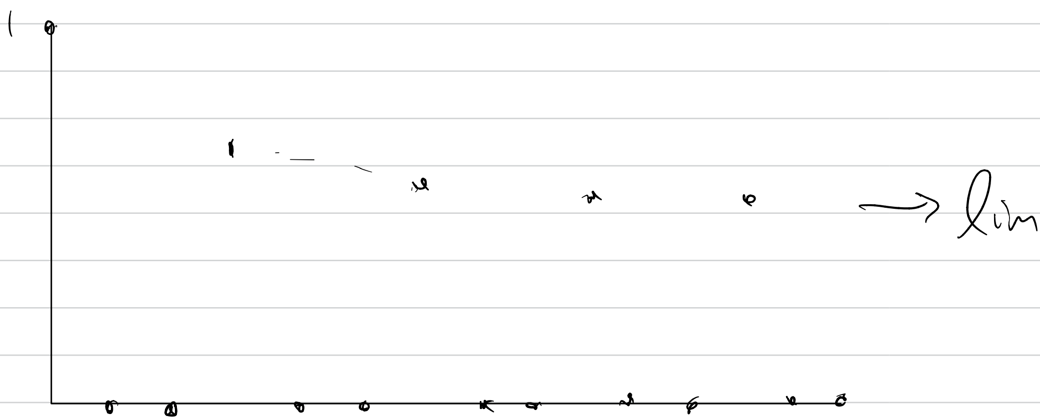
$n = 3, 6, 9, 12 \ldots$

$GCD = 3$

$P^{n}_{bb} \neq 0$ for $n = 6, 9, 12, \ldots$

$GCD = 3$.

If $X_0 = a$ $\quad P^{n}_{ai} = P(X_n = i \mid X_0 = a)$




no limit.

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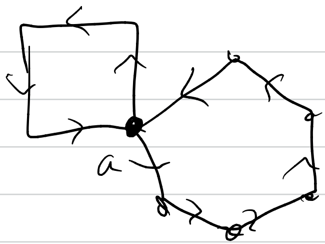
<u>note</u>: $i \Leftrightarrow j$ then they have same period.




$P^n_{aa} \neq 0$ for

$n = 4, 6, 8, 10, \ldots$

$GCD = 2$

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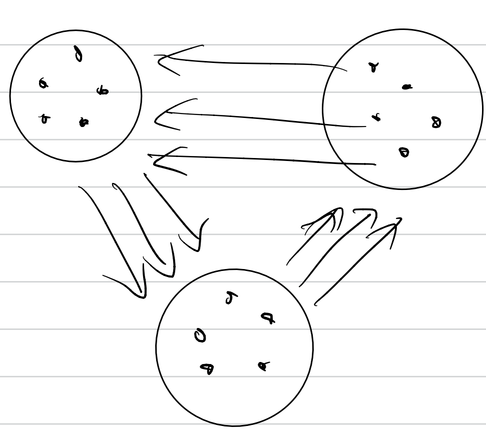
Period 3



- <u>Aperiodic</u> = period 1

# Recurrence + Transience

- $i$ is recurrent if

$$P(\text{return to } i \mid X_0 = i) = 1$$

Transient if

$$P(\text{return to } i \mid X_0 = i) < 1$$

Thm If $i \leftrightarrow j$ then both recur or both trans e.g. queuing theory

$X_n$ = size of queue at time $n$.

If $X_n$ transient then $X_n \to \infty$

If recurrent the Queue is stable

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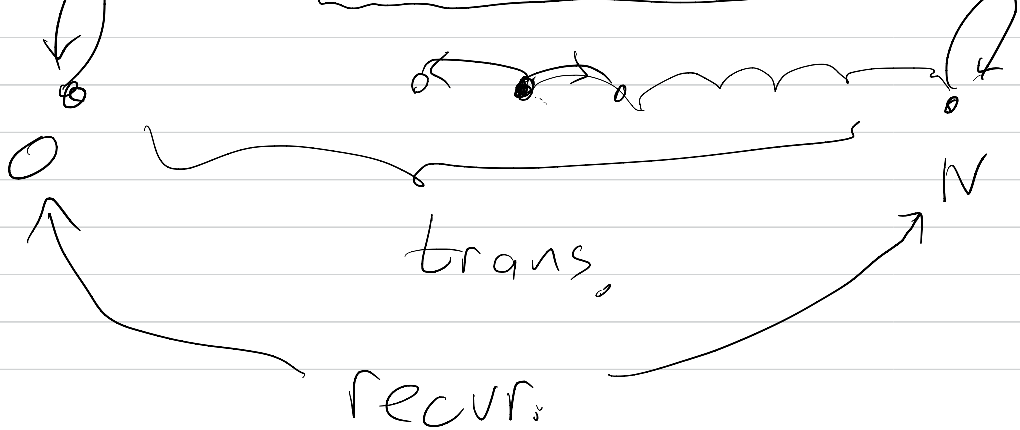
$\boxed{\text{Gambler's ruin}}$

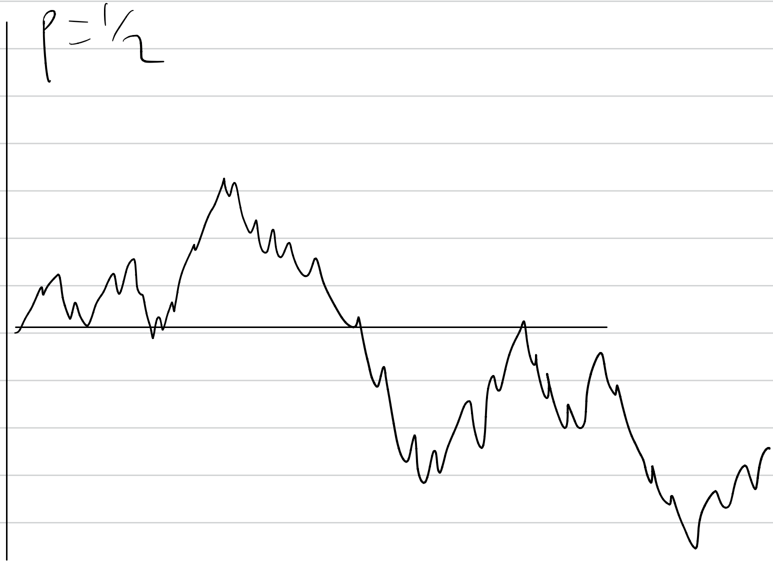


<u>Thm</u>: Random walk on $\mathbb{Z}$

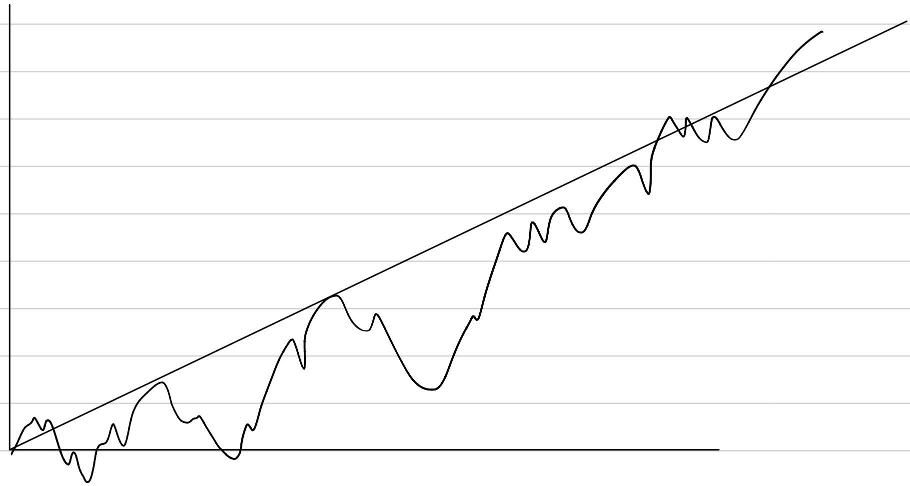
is recurrent if $p = 1/2$

transient if $p \neq 1/2$

$p = P_{n,n+1}$ $\quad$ $1-p = P_{n,n-1}$



$p > 1/2$



<u>Thm</u>: Assume $X_i$ are random integers, indep. all same dist.

$S_n = \sum_{i \le n} X_i$. Then $S_n$ is

recurrent if $EX_i = 0$

transient if $EX_i \neq 0$

LLN: $\frac{S_n}{n} \xrightarrow[n\to\infty]{} EX_i$

$\Rightarrow S_n \longrightarrow \pm\infty$ if $EX \neq 0$

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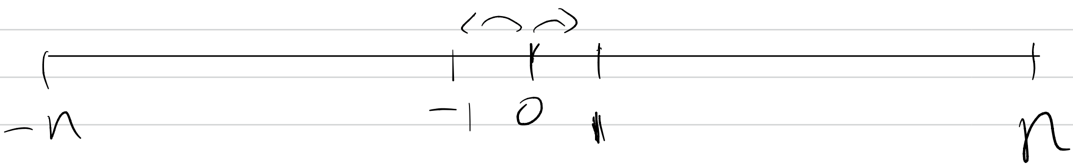



from 1, until reach 0 or n

Random walk same as

Gambler's ruin

$$P(\text{reach } 0 \text{ before } n \mid X_1 = 1) = 1 - \frac{1}{n}$$

$$\underbrace{P(\text{reach } 0 \mid X_1 = 1)}_{\text{no } n} \geq 1 - \frac{1}{n}$$

$$P(\text{reach } 0 \mid X_1 = -1) \geq 1 - \frac{1}{n}$$

# Lecture 5

## recurrence + transience

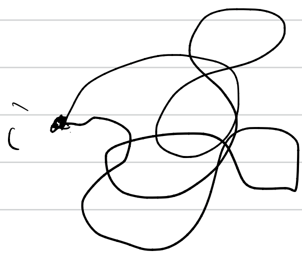
**Recall**: $f_i$ is prob of return to $i$.

$\mathrm{Ret}(i,0) = \{\exists n > 0 \text{ s.t. } X_n = i\}$

$$\boxed{f_i = P(\mathrm{Ret}(i,0) \mid X_0 = i)}$$

If $T$ is first return to $i$ then after time $T$ the

M.C. is same as if just starting at $i$.




Given $T<\infty$,

$$P\big(\mathrm{Ret}(i,T) \,\big|\, T<\infty\big) = f_i$$

Def: state $i$ is

recurrent if $f_i = 1$

transient if $f_i < 1$

If $i$ recurr, $X_0 = i$ then return to $i$, again and again

$N_i = \#$ visits to $i$

$= \#\{n \text{ s.t. } X_n = i\}$

$= \sum_n 1_{\{X_n = i\}}$

---

If $f_i = 1$ then $N_i = \infty$

$P(N_i = 1 \mid X_0 = i) = 1 - f_i$

$$P(N_i = 2 \mid X_0 = i) = f_i(1-f_i)$$

$$P(N_i = 3 \mid X_0 = i) = f_i^2(1-f_i)$$

$$P(N_i = k \mid X_0 = i) = f_i^{k-1} \cdot (1-f_i)$$

---

If $f_i < 1$ and $X_0 = i$ then

$N_i$ is $\text{Geom}(1-f_i)$

$$E(N_i \mid X_0 = i) = \frac{1}{1-f_i}$$

note $E(N_i \mid X_0 = i) = \sum_{n \geq 0} P^n_{ii}$

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<u>Thm</u>: If $i \leftrightarrow j$ then both rec. or both trans.

$P^k_{ij} > 0 \qquad P^l_{ji} > 0$ for some $k, l$

$$P^{k+n+l}_{ii} \geq P^k_{ij} P^n_{jj} P^l_{ji}$$

$$\sum_{n \geq 0} P^{k+n+l}_{ii} \geq \sum_{n \geq 0} P^k_{ij} P^n_{jj} P^l_{ji}$$

$$\sum_{n\geq 0} P_{ii}^{k+n+l} \geq P_{ij}^{k} P_{ji}^{l} \sum_{n\geq 0} P_{jj}^{n}$$

If $j$ recurrn. RHS $=\infty$

$$\text{LHS} = \sum_{n\geq k+l} P_{ii}^{n} = \infty$$

So $i$ also recur.

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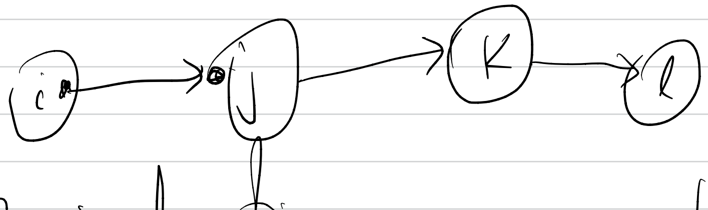
$$P_{ii}^{k+l} + P_{ii}^{k+l+1} + P_{ii}^{k+2+l} + \cdots$$

Another way: Assume $i$ recurr. and $P_{ij} > 0$. Start at $i$.

Visit $i$ infinitely often

so $\infty$ often M.C. jumps $i \to j$.

So $N_j = \infty$




By induction, any state accessible from $i$ is REC.

$i \leftrightarrow j$, One REC $\Rightarrow$ other REC

$\square$

For Random walk on $\mathbb{Z}$

with $P_{n\,n+1} = p$ $\qquad p \in [0,1]$

$P_{n,n-1} = 1-p$

Is recurrent if $p = 1/2$

transient if $p \neq 1/2$

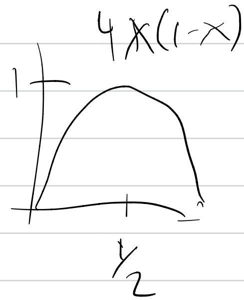
$$\sum_{n \geq 0} P_{00}^{n} = \sum_{n} P_{0,0}^{2n}$$

$$\sum P_{ii}^{n} = \sum_{n} \binom{2n}{n} p^{n}(1-p)^{n}$$

Stirling: If $p = \frac{1}{2}$ $\quad P_{00}^{2n} \sim \frac{1}{\sqrt{\pi n}}$

If $p \neq \frac{1}{2}$ : $\binom{2n}{n} \leq 4^{n}$

$$P_{ii}^{2n} \leq \left(4p(1-p)\right)^{n}$$



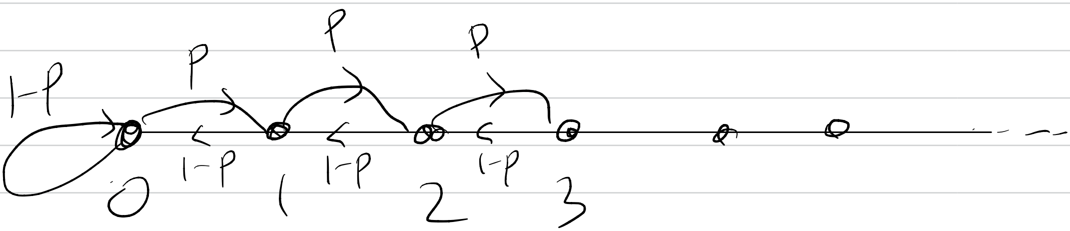

So $\sum P_{ii}^{2n} < \infty$

Queue: Each step $X_n$

increase w.p. $p$

decrease w.p. $1-p$

[If $X_n \neq 0$]



$P_{n\,n+1} = p$ for all $n$

$P_{00} = P_{n,n-1} = 1-p$ for $n \geq 1$

Guess: rec. $\iff$ $p \leq \frac{1}{2}$

Proof

If $p > 1/2$ by LLN $X_n$ increased

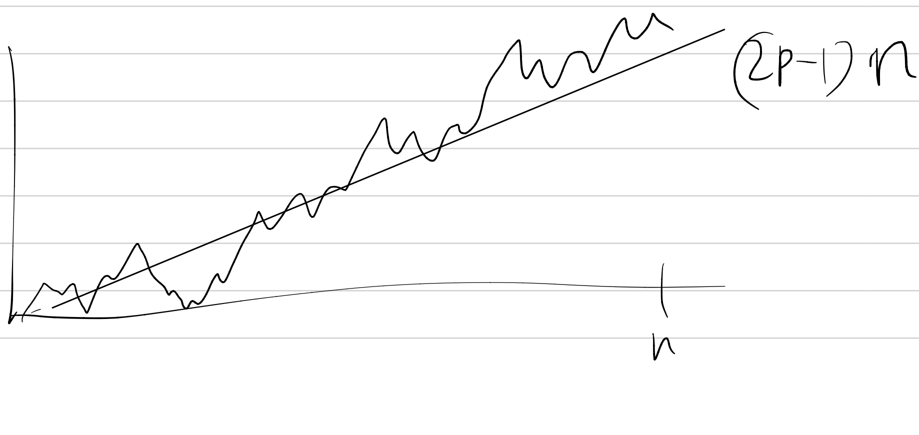
$\sim pn$ times

decreased $\sim (1-p)n$ times

$$\lim \frac{X_n}{n} = (2p-1) > 0$$

so $X_n \longrightarrow \infty$

and M.C. is TRANS.



If $p \leq 1/2$


$$P(\text{Ret}(0) \text{ before reach } N \mid X_1 = 1)$$

$$= \begin{cases} 1 - \dfrac{1}{N} & p = 1/2 \\ 1 - \dfrac{\alpha^1 - 1}{\alpha^N - 1} & p < 1/2 \end{cases}$$

$$\alpha = \frac{1-p}{p} > 1$$

If $N$ large,

$$P(\text{Ret}(0) \text{ before reach } N) \approx 1$$

note:

$$P(\text{return at time } t) = \begin{cases} 1-p & t=1 \\ \frac{1}{t+1}\binom{t}{t/2}\left(p(1-p)\right)^{t/2} & t>1 \text{ even} \end{cases}$$

example: infectious disease

$X_n$ = # infected people.

$X_n$ ways to dec. $X_n$

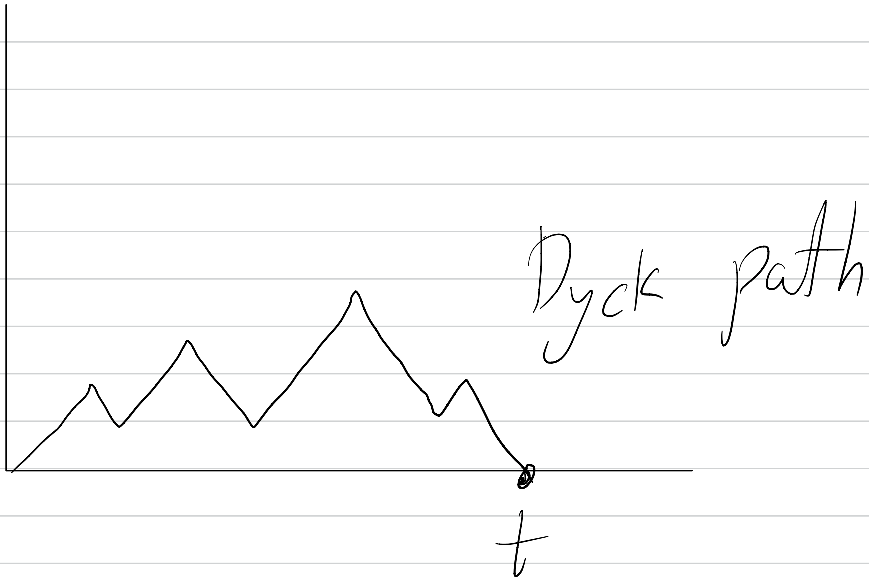
$rX_n$ ways to increase $X_n$

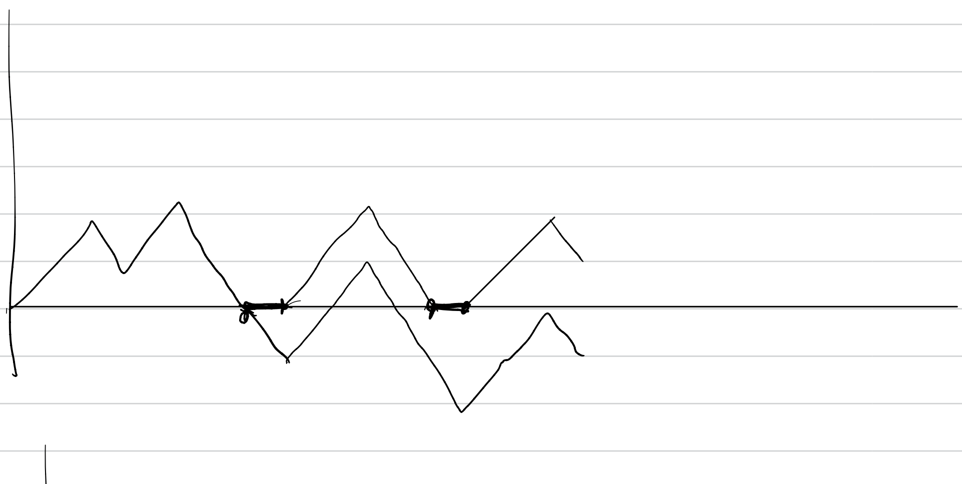
$$X_{n+1} = \begin{cases} X_n - 1 & \text{w.p. } \dfrac{X_n}{X_n + rX_n} \\ X_n + 1 & \text{w.p. } \dfrac{rX_n}{X_n + rX_n} \end{cases}$$

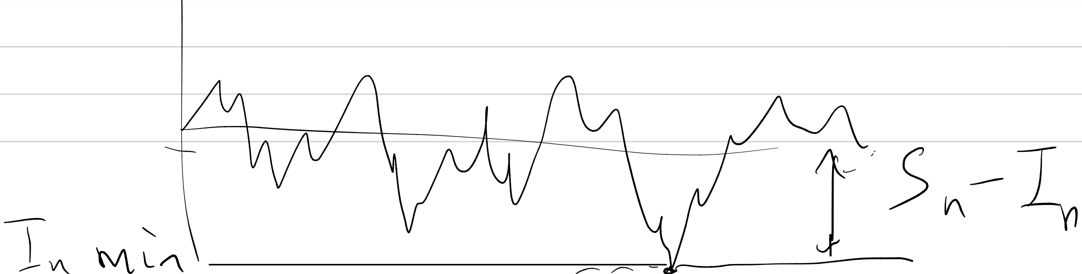
Same as before with

$$p = \frac{r}{1+r}$$

epidemic $\iff p > 1/2 \iff r > 1$


Dyck path
t

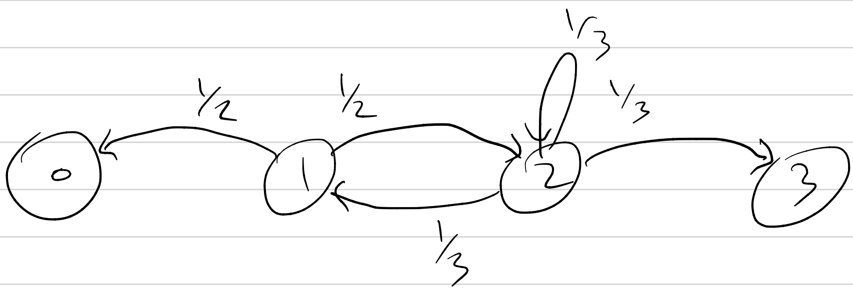



$S_n - I_n$
$I_n$ min

$p < 1/2$

Lecture 6

note on hitting prob. + times




Qn: Start at 1.

a) $P(\text{hit } 3 \text{ before } 0) = ?$

b) $E(\text{Time to hit } 0 \text{ or } 3) = ?$

---

a) $q_x := P(\text{hit } 3 \text{ before } 0 \mid X_0 = x)$

$q_0 = 0 \qquad q_3 = 1$

$q_1 = \frac{1}{2} \cdot q_0 + \frac{1}{2} q_2 = \frac{1}{2} q_2$

$q_2 = \frac{1}{3} \cdot q_2 + \frac{1}{3} q_3 + \frac{1}{3} q_1$

$= \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot q_1 + \frac{1}{3} \cdot q_2$

Solve to find $q_1$ $q_2$

---

b) Let $m_x := E(T_{\{0,3\}} | X_0 = x)$

$T_{\{0,3\}} = \min\{n : X_n \in \{0,3\}\}$

$$m_0 = m_3 = 0$$

$$m_1 = \underset{\substack{\uparrow \\ \text{first step}}}{1} + \underbrace{\frac{1}{2} \cdot m_0 + \frac{1}{2} \cdot m_2}_{\text{additional steps}}$$

$$m_1 = 1 + \frac{1}{2} m_2$$

$$m_2 = 1 + \frac{1}{3} m_1 + \frac{1}{3} m_2 + \frac{1}{3} m_3$$

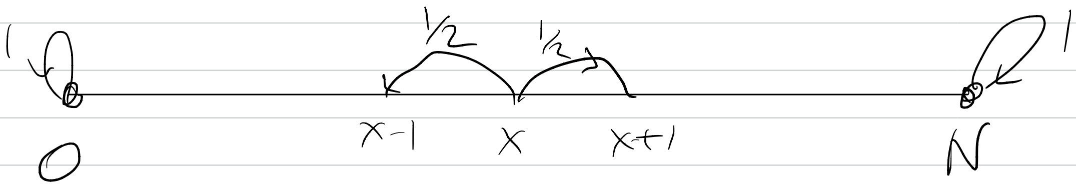
$$= 1 + \frac{1}{3} m_1 + \frac{1}{3} m_2$$

---

$(2, 2, 1, 2, 1, 0)$ 5 steps

$X_0$ ... $X_5$

# Gambler's ruin time




**Qn:** Find $m_k := E(T_{\{0,N\}} \mid X_0 = k)$

$$m_0 = m_N = 0$$

for other $k$:

$$m_k = 1 + \frac{1}{2} m_{k-1} + \frac{1}{2} m_{k+1} \qquad (*)$$

$$\Rightarrow \quad m_k = k(N-k)$$

(*) rewritten:

$$(m_{k+1} - m_k) = (m_k - m_{k-1}) - 2$$

so differences are arith. prog.

$$a = m_1 - m_0 = m_1$$

$$m_2 = (m_2 - m_1) + (m_1 - m_0)$$

$$= (a - 2) + a = 2a - 2$$

$$m_3 = (m_3 - m_2) + m_2$$

$$= a - 4 + m_2 = 3a - 6$$

$$\vdots$$

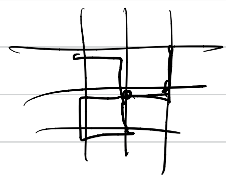
$$m_N = N \cdot a - N(N-1)$$

$m_N = 0$ so $a = N - 1$

# Recurrence + transience

<u>Thm</u>: Simple random walk on $\mathbb{Z}^d$ is recurrent if $d=1,2$. Transient if $d>2$.

Key: estimate $p^n_{\vec{0}\vec{0}}$



$\vec{0} = (0,0,\ldots 0)$, find if

$$\sum_{n=0}^{\infty} p^n_{\vec{0}\vec{0}} \quad (=\infty) \text{ or } (<\infty)$$

In dim=1: $P_{00}^{2n} = \binom{2n}{n} 2^{-2n} \sim \frac{1}{\sqrt{\pi n}}$

so $\sum P_{00}^{n} = \infty$

In <u>dim=2</u>: $P_{\vec{0}\vec{0}}^{2n} = \left[\binom{2n}{n} 2^{-2n}\right]^2 \sim \frac{1}{\pi n}$

Let $(X_n, Y_n)$ be the walk.

Let $U_n = X_n + Y_n$ $\quad V_n = Y_n - X_n$

$U, V$ are indep 1-dim random walks.

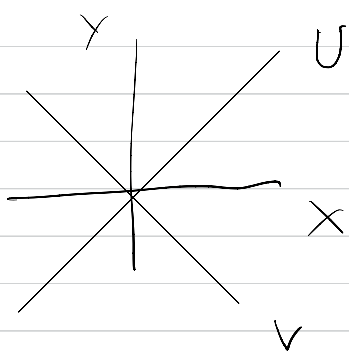
$P(X_n = Y_n = 0) = P(U_n = V_n = 0)$

$$= \mathbb{P}(U_n = 0) \cdot \mathbb{P}(V_n = 0) = \left[\binom{2n}{n} 2^{-2n}\right]^2$$

Change in $(U, V)$ is one of

$(1, 1)$ $\quad (1, -1)$ $\quad (-1, 1)$ $\quad (-1, -1)$

w.p. $1/4$ each.



## In $d=3$

If make $2i$ steps in $x$-coor
$2j$ $y$-coord.
$2k$ $z$-coord

Cond. on that, $S=(x,y,z)$

$$P(S_{2n}=0)=\binom{2i}{i}2^{-2i}\binom{2j}{j}2^{-2j}\binom{2k}{k}2^{-2k}$$

typically, $2i, 2j, 2k \approx \frac{2n}{3}$

$$\binom{2i}{i}2^{-2i} \sim \frac{1}{\sqrt{\pi i}}$$

then prod. is $\approx \left(\frac{1}{\sqrt{\pi n/3}}\right)^3$

Conclusion: $P^{2n}_{\vec{0}\vec{0}} \sim \frac{c_d}{n^{d/2}}$

$$\sum \frac{1}{n^{d/2}} < \infty \quad \text{if } d > 2$$

$$= \infty \quad \text{if } d \leq 2$$

$$P^{2n}_{\vec{0}\vec{0}} = \sum_{i+j+k=n} \binom{2i}{i} 2^{-2i} \binom{2j}{j} 2^{-2j} \binom{2k}{k} 2^{-2k} \times$$

$$\times P\left(\begin{array}{c}2i \text{ x-steps, } 2j\text{-y-steps} \\ 2k\text{-z-steps}\end{array}\right)$$

Recall characteristic func.:

for R.V. $X$

$$\varphi_X(t) = \mathbb{E}\, e^{itX} \qquad i = \sqrt{-1}$$

If $X$ is integer valued,

then $\varphi_X(t)$ has period $2\pi$

$$\varphi(t+2\pi) = \varphi(t) \qquad [e^{2\pi i} = 1]$$

If $X, Y$ are indep. then

$$\varphi_{X+Y}(t) = \varphi_X(t) \cdot \varphi_Y(t)$$

e.g. $X = \pm 1$ w.p. $1/2$

$$\varphi_X(t) = \frac{1}{2}e^{it} + \frac{1}{2}e^{-it} = \cos(t)$$

For $X_1, \ldots, X_n$ iid: indep + identic. dist.

$S_n = \sum_{j \leq n} X_j$ has ch.f.

$$\varphi_{S_n}(t) = \varphi_X(t)^n$$

Lecture 7

Inverse Fourier formula

$$P(X=0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_X(t)\, dt$$

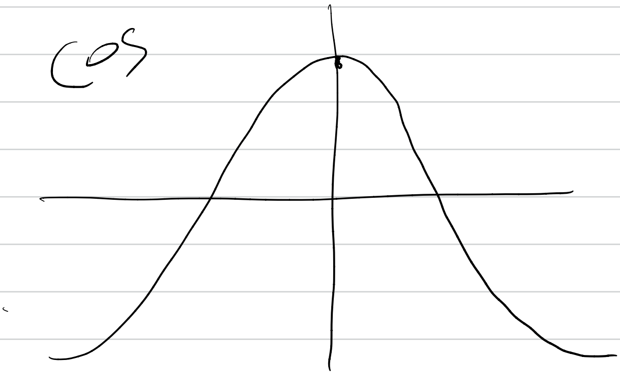
$$\sum_n P(S_n=0) = \sum_n \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_X(t)^n\, dt$$

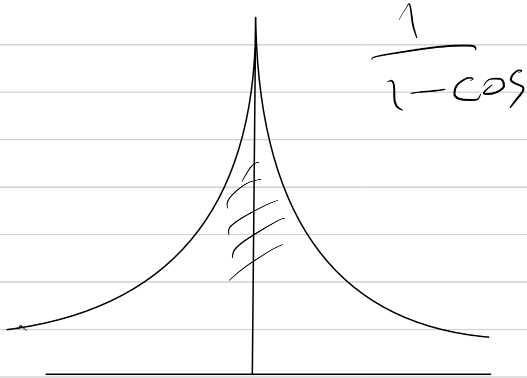
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_n \varphi_X(t)^n\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{1-\varphi_X(t)}\, dt$$

If $X = \pm 1$: $\displaystyle\int_{-\pi}^{\pi} \frac{dt}{1-(\cos t)} = \infty$

$1 - \cos t \sim \frac{1}{2} t^2$




$$\frac{1}{1-\cos t} \sim \frac{2}{t^2}$$

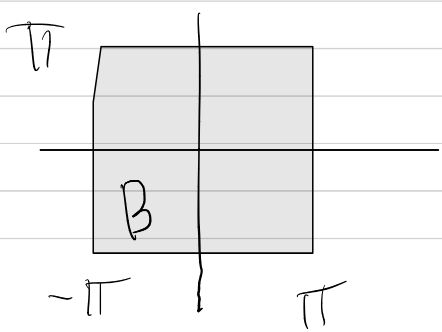
so $S_n$ is recurrent

If $\vec{X} \in \mathbb{R}^d$ is a random vector,

$$\varphi_X(t) = \mathbb{E}\, e^{i\langle \vec{t}, \vec{X}\rangle}$$

for $t \in \mathbb{R}^d$

If $X$ has integer coord. then $\varphi_X$ is periodic $2\pi$




$$B = [-\pi, \pi]^d$$

Inverse formula:

$$P(\vec{X}=\vec{0}) = \text{avg of } \varphi_X \text{ on } B$$

$$= \frac{1}{(2\pi)^d} \iint_B \varphi_X(\vec{t})\, d\vec{t}$$

e.g. X is uniformly one of

$(0,1)$ $(0,-1)$ $(1,0)$ $(-1,0)$

$$\varphi_X(\vec{t}) = \frac{1}{2}\left(\cos(t_1) + \cos(t_2)\right)$$

In d-dim if $X$ is one of $\pm e_i$

$$e_i = (0 \cdots 0 \underset{i}{1} 0 \cdots 0)$$

$$\varphi_X = \frac{1}{d} \sum_{j=1}^{d} \cos(t_j)$$

If $S_n$ = sum of $n$ indep. copies of $X$

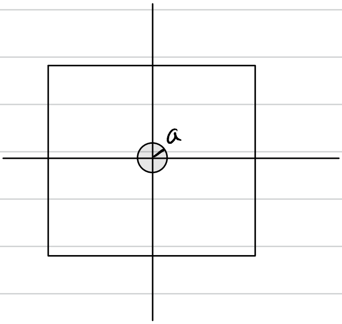
$$P(S_n = 0) = \frac{1}{(2\pi)^d} \iint_B \varphi_X(\vec{t})^n \, d\vec{t}$$

$$\sum_n P(S_n = 0) = \frac{1}{(2\pi)^d} \iint_B \frac{1}{1-\varphi_X(\vec{t})} \, d\vec{t}$$

For recur./trans.: need to check if $\iint$ is $\infty$ or finite.

For the random walk

$$\varphi = \frac{1}{d} \sum \cos(t_i)$$




$\frac{1}{1-\varphi}$ is bdd except near $\vec{t} = 0$

near 0, Taylor expansion gives $\varphi(t) = \frac{1}{d} \sum 1 - \frac{t_i^2}{2} + o(t_i^2)$

$$= 1 - \frac{1}{2d} \sum t_i^2 + o(t^2)$$

$$\approx 1 - \frac{1}{2d} \|t\|^2$$

$$\frac{1}{1-\varphi} \approx \frac{2d}{\|t\|^2}$$

Recurrent $\Leftrightarrow \iint_{Ball} \frac{2d}{\|t\|^2} d\vec{t} = \infty$

Polar coord:

$$\iint_{Ball} \frac{1}{\|t\|^2} d\vec{t} = \int_{\theta}\int_{r=0}^{a} \frac{1}{r^2} r^{d-1} \, dr \, d\theta$$

$$= C\int_0^a r^{d-3} \, dr$$

$$= \begin{cases} \infty & d \leq 2 \\ <\infty & d > 2 \end{cases}$$

Assume $i$ is recurrent.

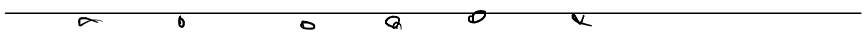
Let $N_i(M) = \#\{n \leq M : X_n = i\}$

$=$ visits to $i$ up to time $M$.

$N_i = N_i(\infty) = \infty$

Is there an asymptotic rate $\frac{N_i(M)}{M} \xrightarrow[M\to\infty]{} \pi_i$

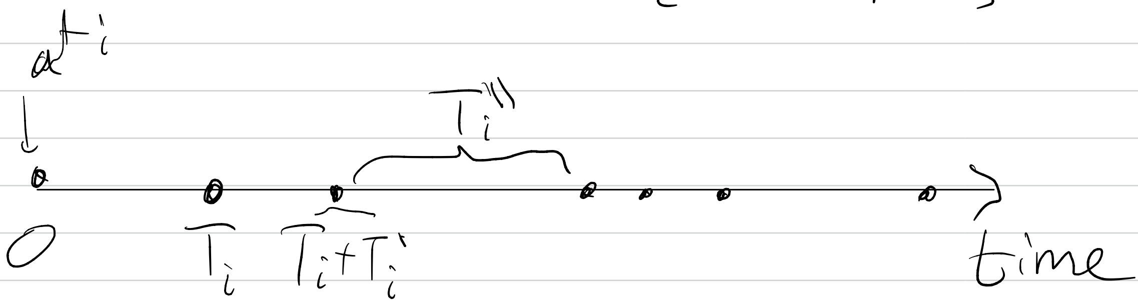
On $\mathbb{Z}$, $\frac{N_i(M)}{M} \longrightarrow 0$



$\mathbb{Z}$

Let $T_i$ = return time to $i$

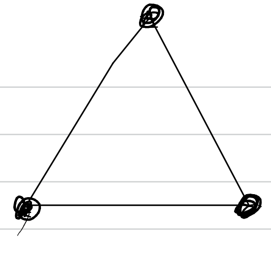
$= \min\{n > 0 : X_n = i\}$




$T_i'$ = same dist. as $T_i$, indep.

Times between visits are all indep. with same dist. as $T_i$

e.g.




$T_i = 1 + \text{Geom}(\tfrac{1}{2})$

$E(T_i \mid X_0 = i) = 3$

$$0\,\overbrace{\underbrace{121212}_{\text{Geom}(\frac{1}{2})}0}^{1+\text{Geom}(\frac{1}{2})}$$


indep. gaps.

<u>LLN</u>: $k$th return to $i$

is at time $\approx k\,E(T_i \mid X_0 = i)$

Let $M_k$ = time of $k$th visit to $i$.

$N_i(M_k) = k$ by def.

$M_k \approx k \cdot m_i$ by LLN

$$\frac{N_i(M_k)}{M_k} \approx \frac{k}{k \cdot m_i} = \frac{1}{m_i}$$

Summary: If $m_i = E(T_i \mid X_0 = i)$ is finite then fraction of time the chain stays at $i$ is $\frac{1}{m_i}$

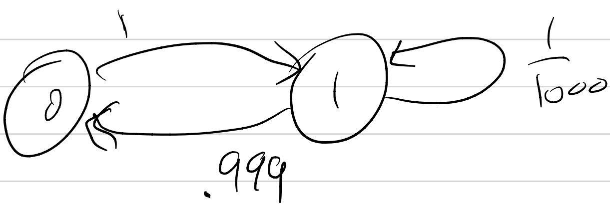
$i$ is pos.-recurrent if $m_i < \infty$

$i$ is null-recurrent if $m_i = \infty$

# Lecture 8

Some times have $\alpha P^n \xrightarrow[n\to\infty]{} \pi$ for any $\alpha$, with $\pi$ being a stat. dist.:

$\pi P = \pi$ and $\sum_i \pi_i = 1$




$$\pi = \left( \frac{q}{p+q}, \frac{p}{p+q} \right)$$

$p = 1$

$q = .999$

<u>Qn</u> what is the fraction of time in any state i?

- Does this converge?

  Limit must be a stat. dist.

- Is there always a stat. dist.

- Is it unique?

Assume $S$ is finite

## Perron Frobenius Theorem

Let $M$ be an $n \times n$ matrix with $M_{ij} \geq 0$ $\forall i,j$ and $M$ is irreducible (for any $i,j$ there is some $n$ with $M^n_{ij} > 0$). Then the maximal e.value $\lambda$ has an e.vector $v$ with all $v_i \underset{\neq}{>} 0$. Any other e.vector has some neg. entries.

---

$\underline{1} = (1\,1\,1 \ldots 1)^T$ is right e.vector of stochastic $P$

$[P\underline{1} = \underline{1}]$ with e.value 1.

So max e.value is 1.

There is a left e.vector $\pi P = \pi$. Can normalize to get $\sum \pi_i = 1$.

<u>Conclusion</u>: $S$ finite and $P$ irreducible then there exists unique $\pi$ with $\pi P = \pi$ and $\sum \pi_i = 1$

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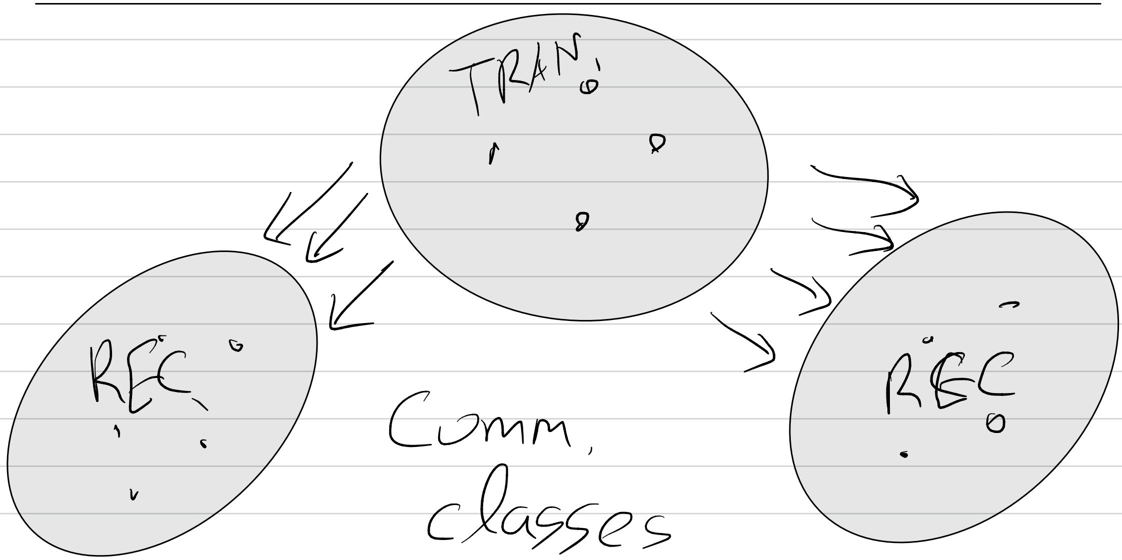
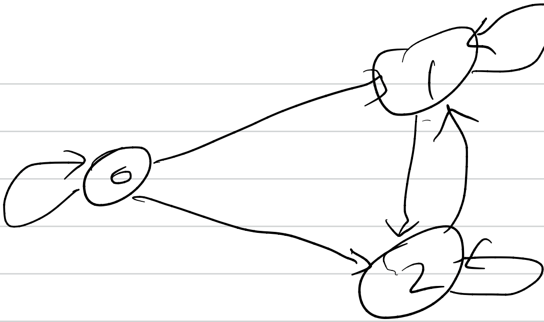


Claim: For any recurr. communicating class there is a stat dist. with $\pi_i \neq 0$ only on that class.

Idea: $P$ restricted to such a class is stochastic irreducible.




$$
\begin{array}{c|ccc}
 & 0 & 1 & 2 \\
0 & \multicolumn{3}{c}{\multirow{3}{*}{}} \\
\end{array}
\qquad
\begin{array}{c}
 \\ 0 \\ 1 \\ 2
\end{array}
\begin{array}{c}
\begin{array}{ccc} 0 & 1 & 2 \end{array} \\
\left(\begin{array}{c|cc}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\hline
0 & \frac{1}{2} & \frac{1}{2} \\
0 & \frac{1}{3} & \frac{2}{3}
\end{array}\right)
\end{array}
$$

restrict to $\{1,2\}$

$$
\begin{array}{c}
0 \\ 1 \\ 2 \\ 3 \\ 4
\end{array}
\left(\begin{array}{c|cc|cc}
\times & \times & \times & \times & \times \\
\hline
0 & \times & \times & 0 & 0 \\
0 & \times & \times & 0 & 0 \\
\hline
0 & 0 & 0 & \times & \times \\
0 & 0 & 0 & \times & \times
\end{array}\right)
\qquad
\begin{array}{l}
\{1,2\} \\ \{3,4\}
\end{array}
$$

For reducible M.C. can have multiple S.D.

<u>note</u>: In irreducible case, $\pi_i \neq 0$ for all $i$.

<u>Proof</u>: If $\pi_i \neq 0$ and $P_{ij} \neq 0$

then $(\pi P)_j = \sum_x \pi_x P_{ij} \geq \underbrace{\pi_i P_{ij}}_{\neq 0}$

So $\pi_j \neq 0$.

Any $j$ accesible from $i$ has $\pi_j \neq 0$

Infinite state space $S$.

$$m_i = \mathbb{E}(T_i \mid X_0 = i)$$

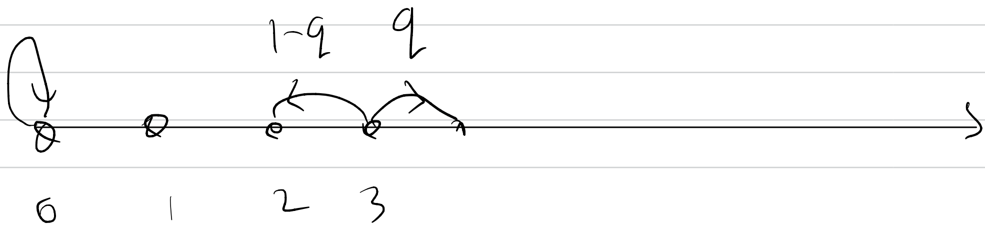
$$T_i = \min\{n > 0 : X_n = i\}$$

fraction of time at $i$ is

$\frac{1}{m_i}$.

$m_i < \infty$ then $i$ is pos-rec.

$m_i = \infty$ null-rec.



$$P_{n\,n+1} = q \quad \leq \tfrac{1}{2}$$

$$P_{n\,n-1} = 1-q \qquad n > 0$$

$$P_{00} = 1-q$$

recurrent $\Longleftrightarrow$ $q \leq \tfrac{1}{2}$

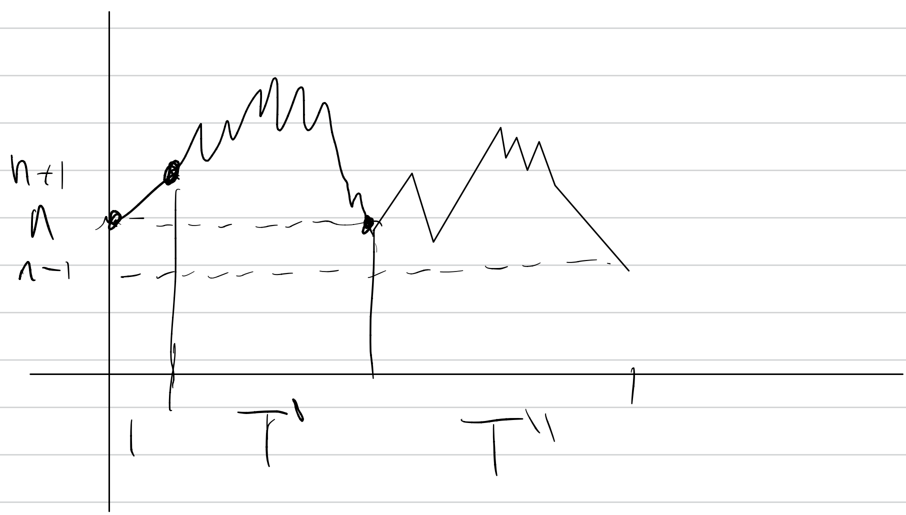
$m_0$ = avg return time to 0

= ?

Let $Y = E(\underbrace{\text{time to } n-1}_{T} \mid X_0 = n)$

$$T = \begin{cases} 1 & \text{w.p. } 1-q \\ 1 + T' + T'' & \text{w.p. } q \end{cases}$$

$n+1 \to n$ $\qquad$ $n \to n-1$




$T, T', T''$ have same dist.

$$Y = 1(1-q) + (1+Y+Y)\,q$$

$$= 1 + 2qY$$

$$Y = \frac{1}{1-2q}$$

$$m_0 = q(1+Y) + (1-q)\cdot 1$$

$$= 1 + qY = \frac{1-q}{1-2q}$$

$$q = .49 \quad = \frac{.51}{.02} = 25.5$$

$$\pi_0 = \frac{1}{25.5}$$

pos-rec. if $q < 1/2$
null-rec. if $q = 1/2$
transient if $q > 1/2$

$$\pi_i \sim (1-\beta)^i \cdot \beta \xrightarrow[i\to\infty]{} 0$$

$$m_i = \frac{1}{\beta(1-\beta)^i} \xrightarrow[i\to\infty]{} \infty$$

# Lecture 9

Recall: If $i$ is pos. rec.

$m_i = \mathbb{E}(T_i \mid X_0 = i)$ then

the fraction of time

at $i$ tends to $\frac{1}{m_i}$

$\pi_i = \frac{1}{m_i}$ is a stat.

dist. : $\pi P = \pi$ $\sum \pi_i = 1$

for any ergodic

ergodic = pos.-rec. and irreducible

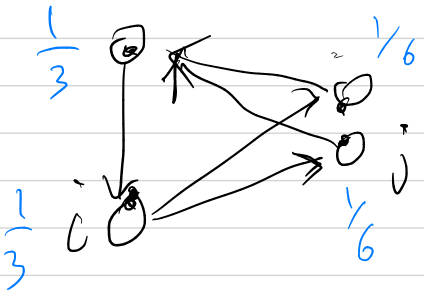
<u>Thm</u> If M.C. is ergodic

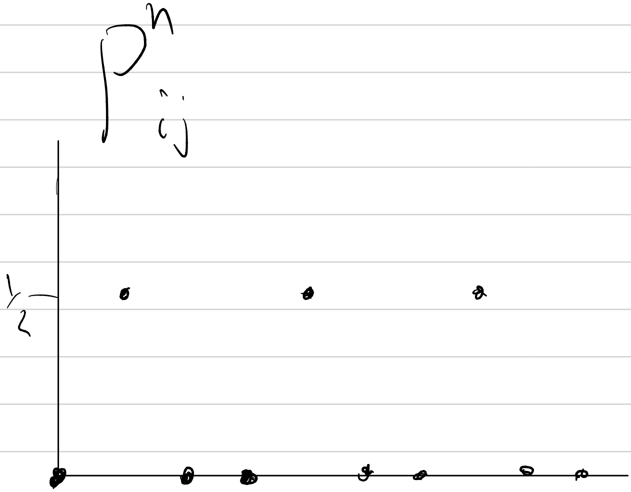
* then there is a unique stat. dist. $\pi$.
* Fraction of time at $i$ conv. to $\pi_i$

If M.C is also aperiodic then for all $i,j$:

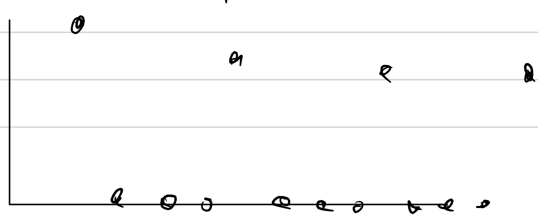
$$P_{ij}^{n} \xrightarrow[n\to\infty]{} \pi_j$$

If M.C. is periodic:




In general, if Period is K then $P^n_{ij}$ "converges" to a periodic pattern.



# Finding stat. dist. (methods)

1) solve eqn $\pi P = \pi$ $\sum \pi_i = 1$ (exactly or numerically)

2) guess and verify.

3) Find $m_i$, $\pi_i = 1/m_i$

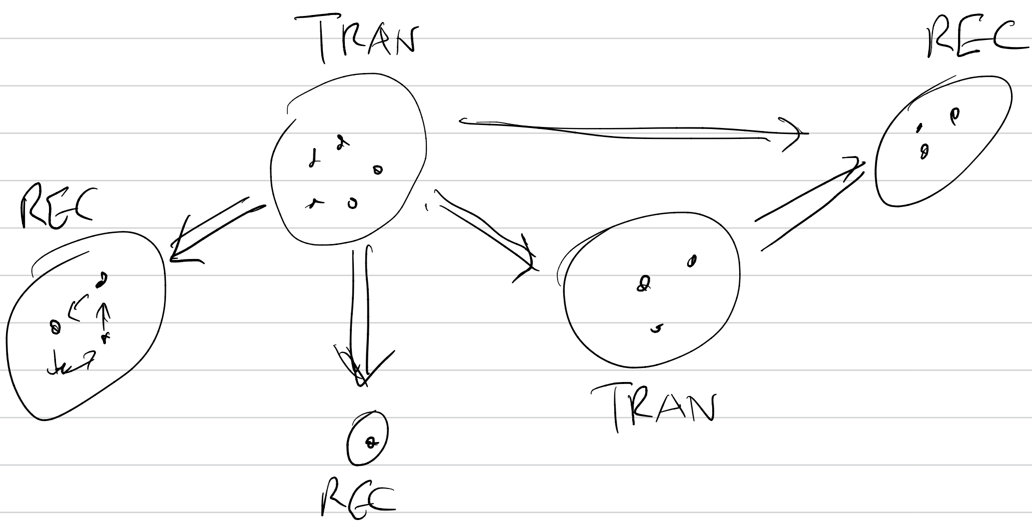
4) special cases

   * doubly stoch. matrix $P$
   * Symmetry

5) reversibility (if applies)

# reducible M.C.

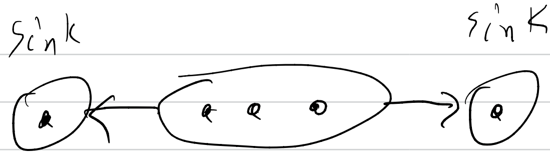
Comm. classes




In any pos.-recurrent comm. class there is a stat. dist.

If there are several such classes: multiple stat. dist.

$$\begin{array}{c} 0 \\ \\ \\ \\ 4 \end{array}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ x & x & x & & \\ & x & x & x & \\ & & x & x & x \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$




stat. dist.s are $(1,0,0,0,0)$

$(0,0,0,0,1)$

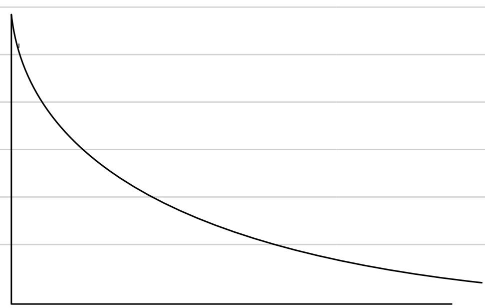
also $(p,0,0,0,1-p)$

---

e.g. $P_{n\,n+1} = q$ $\quad q < 1/2$

$P_{n\,n-1} = 1-q \quad n > 0$

$P_{00} = 1-q$



$\boxed{\begin{array}{l} \text{Birth + Death chain:} \\ S = \mathbb{N} = \{0, 1, 2, \ldots\} \\ P_{n,n+1} = p_n \quad P_{n,n-1} = q_n \quad P_{n,n} = r_n \end{array}}$

$\underline{\pi = \pi P}$: $\pi_j = \sum_i \pi_i P_{ij}$

For $j > 0$: $\pi_j = \pi_{j-1} \cdot q + \pi_{j+1}(1-q)$

$\pi_0 = \pi_0(1-q) + \pi_1(1-q)$

<u>Solve:</u> $\pi_1 = \frac{q}{1-q} \cdot \pi_0$

$$\pi_1 = \pi_0 q + \pi_2 \cdot (1-q)$$

$$\pi_2 = \frac{1}{1-q}(\pi_1 - q\pi_0) = \left(\frac{\frac{q}{1-q} - q}{1-q}\right)\pi_0$$

$$\pi_2 = \left(\frac{q}{1-q}\right)^2 \pi_0$$

$$\boxed{\alpha = \frac{q}{1-q}}$$

By induction : $\boxed{\pi_n = \alpha^n \cdot \pi_0}$

$$1 = \sum_0^\infty \pi_n = \sum_0^\infty \alpha^n \pi_0 = \frac{1}{1-\alpha} \cdot \pi_0$$

$\pi_0$ must be $1-\alpha$

$$\pi_n = (1-\alpha)\alpha^n$$

If $q < \frac{1}{2}$: found a stat. dist.

If $q \geq \frac{1}{2}$: no stat. dist.

**Thm** If an irreducible M.C. has a stat. dist. then it is pos.-recurrent.

---

A M.C. is <u>reversible</u> if for all $i, j$

$$\pi_i P_{ij} = \pi_j P_{ji} \qquad \text{(DB)}$$

Detailed Balance

If (DB) holds and $\sum \pi_i = 1$

then $\pi$ is a stat. dist.

for $P$.

$\binom{n}{2}$ eqn.s for $n$ variables

# Lecture 10

Reversible w.r.t. $\pi$ if

(DB)

<u>note</u>: DB $\Rightarrow$ $\pi P = \pi$.

<u>note</u>: can have reversible

M.C. where $\sum \pi_i = \infty$

e.g. random walk on $\mathbb{Z}$

$\pi_i P_{i,i+1} = \pi_{i+1} P_{i+1,i} \Rightarrow \pi_i = \pi_{i+1}$

So $\pi$ must be constant.

If $|i-j|>1$ then $P_{ij}=P_{ji}=0$

So random walk is reversible w.r.t. $\pi_j \equiv 1$

---

e.g. $P_{i\,i+1}=q$

$P_{i\,i-1}=1-q$

reversible w.r.t. $\pi_i=\left(\frac{q}{1-q}\right)^i$

Indeed: $\pi_i\, q=\pi_{i+1}(1-q)$

i.e. $\pi_{i+1}=\left(\frac{q}{1-q}\right)\pi_i$

Pos.-rec + Reversible $\Rightarrow$

$\Rightarrow$ $\pi$ is the stat. dist.

reversible w.r.t. $\pi$ and $\sum \pi < \infty$

$\Longrightarrow$ pos.-rec.

Rev. and $\sum \pi_i = \infty$

$\Longrightarrow$ transient or null-rec.

## Birth + Death chains

States $S = \mathbb{N} = \{0, 1, 2 \dots\}$

Transitions: $P_{n,n+1} = p_n$

$P_{n,n-1} = q_n$

$P_{n,n} = r_n$

$p_n + q_n + r_n = 1$

<u>Thm</u>: Any B+D chain with $p_n, q_n \neq 0$ (except $q_0$) is reversible.

If $S = \mathbb{Z}$ then also true.

Proof: we find $\pi$ so that (PB) holds. Let $j = i+1$.

$$\pi_i P_{i,i+1} = \pi_{i+1} \cdot P_{i+1,i}$$

$$\pi_i p_i = \pi_{i+1} \cdot q_{i+1}$$

$$\pi_{i+1} = \frac{p_i}{q_{i+1}} \cdot \pi_i$$

$$\pi_1 = \frac{p_0}{q_1} \cdot \pi_0$$

$$\pi_2 = \frac{p_0 \cdot p_1}{q_1 \cdot q_2} \cdot \pi_0$$

$$\pi_n = \frac{p_0 \cdots p_{n-1}}{q_1 \cdots q_n} \cdot \pi_0 \qquad \square$$

Given $(p_n)$ $(q_n)$ can find $\pi$. If $\sum \pi_n < \infty$ can get stat. dist. and so M.C. is pos.-rec.

---

e.g. $p_n = \frac{1}{n+1}$ $\quad q_n = \frac{n}{n+1}$

$$\pi_n = \frac{\frac{1}{1} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdots \frac{1}{n}}{\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{n}{n+1}} \pi_0$$

$$\pi_n = \frac{1/n!}{1/(n+1)} \pi_0 = \frac{n+1}{n!} \pi_0$$

$$= \left( \frac{1}{(n-1)!} + \frac{1}{n!} \right) \pi_0$$

$$\pi_0, \ (1+1)\pi_0, \ \left(1+\frac{1}{2}\right)\pi_0, \ \left(\frac{1}{2}+\frac{1}{3!}\right)\pi_0 \ldots$$

$$\sum \pi_n = 2e\pi_0$$

If $\pi_0 = \frac{1}{2e}$ $\sum \pi_n = 1$

So stat. dist. is $\pi_n = \dfrac{n+1}{2e\,n!}$

Imagine mass $m_i$ at state $i$

If $\sum m_i = 1$ this is a distrib.

At each step $i$ sends to $j$

$m_i P_{ij}$ mass.

Total mass received is

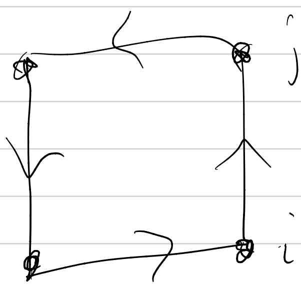
$$\sum_j m_j P_{ji} = (mP)_i$$

If $m = mP$, then system is in equilibrium.

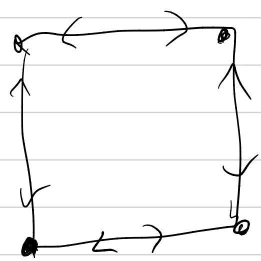
If (DB) holds w.r.t. $m$,

$$m_i P_{ij} = m_j P_{ji}$$

mass $i \to j$ mass $j \to i$



non-revers.



reversible

If MC is reversible, much easier to find a stat. dist. (or show none exists)

---

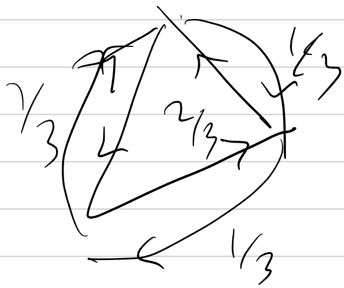
$$\pi P = \pi \qquad \sum \pi_i = 1 \qquad \boxed{\begin{array}{c} n+1 \\ \text{eqns} \end{array}}$$

If P is reversible:

$$\pi_i P_{ij} = \pi_j P_{ji} \quad , \qquad \binom{n}{2} \text{ eqns}$$

$$\sum \pi_i = 1 \qquad +1$$

Simpler eqns, but not always have a sol.




$$\begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/3 & 0 & 2/3 \\ 2/3 & 1/3 & 0 \end{pmatrix}$$

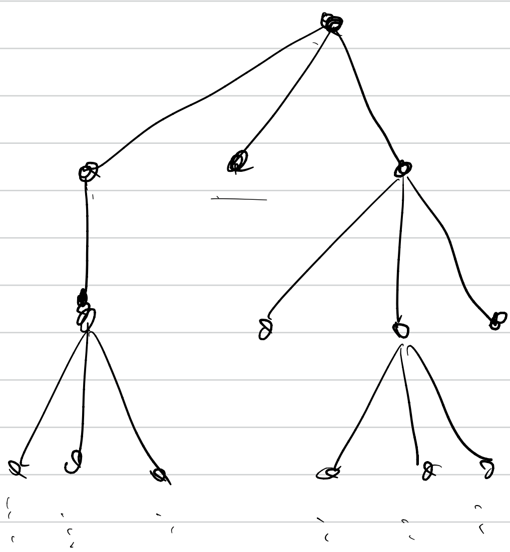
$$\left.\begin{aligned} \pi_0 \cdot \frac{2}{3} &= \pi_1 \cdot \frac{1}{3} \\ \pi_1 \cdot \frac{2}{3} &= \pi_2 \cdot \frac{1}{3} \\ \pi_2 \cdot \frac{2}{3} &= \pi_0 \cdot \frac{1}{3} \end{aligned}\right\} \text{ no non-zero sol.}$$

So this chain is not reversible.

Lecture 11

# Branching Processes

models population, one generation / step.



$Z_0 = 1$

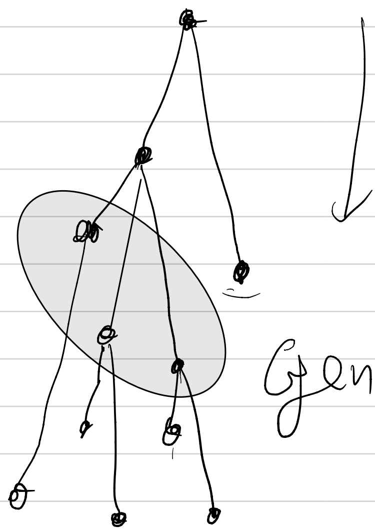
$Z_1 = 3$

$Z_2 = 4$

$Z_3 = 6$

$P_3 P_1 P_0 P_3 P_3^2 P_0^2$

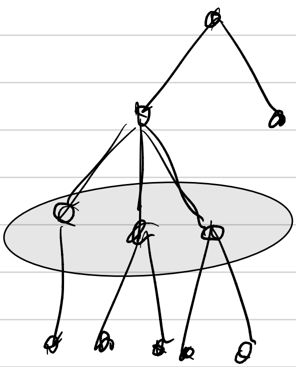
Patient 0 →



time

Gen. 2

tree ordered by gen:



gen. 2

Key question:

does $Z_n \to \infty$ survival

$Z_n \to 0$ extinction

# Bienaymé – Galton – Watson

Each indiv. has indep. # of children,

$P(k \text{ children}) = P(\xi = k) = p_k$

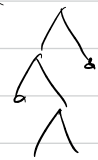
e.g. $\xi = \begin{cases} 0 & 1-q \\ 2 & q \end{cases}$

$1-q$ $\quad q(1-q)^2$ $\quad 2 \quad 2 \quad 4$

$$P(Z_2=4) = q^3$$

$$P(Z_2=2) = 2q^2(1-q)$$

$$P(Z_2=0) = 1-q+q(1-q)^2$$

$P(Z_3=2) = P($  $) + P($  $)$

$+ P($  $) + \ldots$

---

Variations:

- Population size $N$ matters once $Z_k \approx N$

- Types of individual vary
- Policy/Behaviour effects

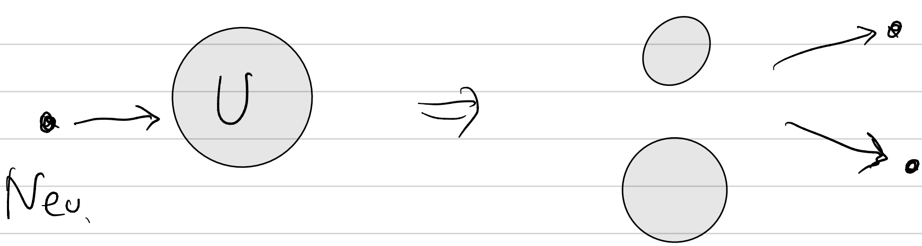
---

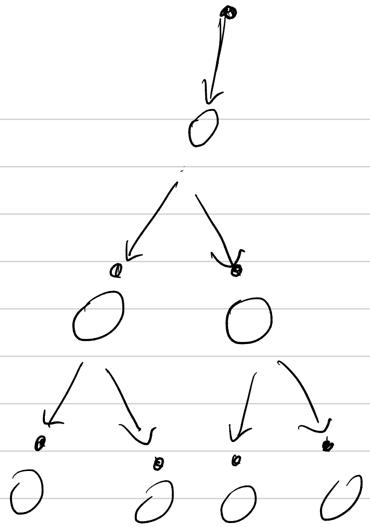
Note! 0 is absorbing.

$$Z_n = 0 \Rightarrow Z_{n+1} = 0$$

---

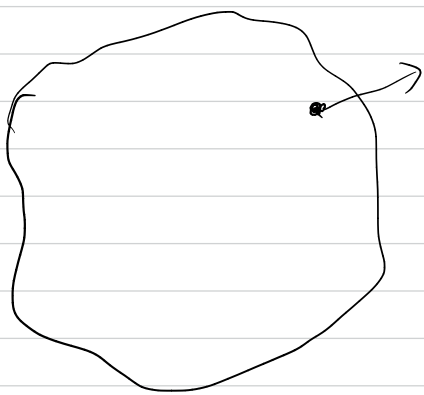
Nuclear chain reaction?





Avogadro: $\frac{\text{gram}}{\text{atomic wt.}} \approx 6 \cdot 10^{23}$

reach all atoms in $< 100$



Turns out that the key parameter is

$$m = \mathbb{E}\,\xi$$

$$= \text{avg \# of children}$$

(R-factor in epidemiology).

$\boxed{\underline{\text{Note}}:\ Z_n \text{ is a M.C.}}$

State space $= \mathbb{N}$

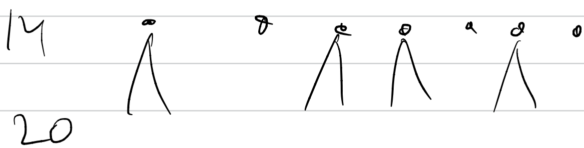
If $Z_n = k$ then

$$Z_{n+1} = X_1 + X_2 + \cdots + X_k$$

Each $X_i$ has dist of $\xi$.

In the 0-2 case:

$$P(Z_{n+1}=20 \mid Z_n=14)=\binom{14}{10}\cdot q^{10}(1-q)^4$$




$$P(Z_{n+1}=2m \mid Z_n=k)=\binom{k}{m}q^m(1-q)^{k-m}$$

$Z_{n+1}$ is $2\mathrm{Bin}(Z_n, q)$.

## Probab. generating func

(related to moment GF)

most useful for integer R.V.

$$G_X(s) = \mathbb{E}\, s^X$$

e.g. $X$ 6-sided die

$$G_X(s) = \frac{1}{6}s + \frac{1}{6}s^2 + \frac{1}{6}s^3 \cdots + \frac{1}{6}s^6$$

$$= \frac{s + s^2 + \cdots + s^6}{6}$$

$G_X$ always defined for $s \in [-1,1]$

sometimes more.

e.g. $X = \text{Geom}(q)$

$$\mathbb{E} s^X = \sum_n s^n q (1-q)^{n-1}$$

$$= sq \sum_{n=1}^{\infty} (s(1-q))^{n-1} = \frac{sq}{1-s(1-q)}$$

for $s < \frac{1}{1-q}$

MGF $M_X(t) = \mathbb{E}(e^t)^X = G_X(e^t)$

related by c.o.v. $s = e^t$

moments related to deriv. at $s=1$. $\square$

Lecture 12

Recall: $0^0 = 1$ so $G_X(0) = P(X=0)$

$G_X(1) = 1$ always.

derivatives at 1 relate to moments: $G'(1) = \mathbb{E}X$

$$G''(s) = \sum_n n(n-1)s^{n-2} P(X=n)$$

$$G''(1) = \mathbb{E}(X^2 - X)$$
$$= \mathrm{Var}(X) - \mathbb{E}X + \mathbb{E}X^2$$

---

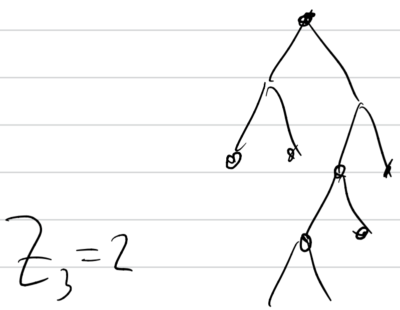
PGF of a sum is the product of PGFs

If $N$ is random, $X_1, X_2, \ldots$ iid, and $T = \sum_{i \leq N} X_i$,

then $G_T = G_N \circ G_X$

$$G_T(s) = G_N(G_X(s))$$

---



Each indiv. has $\xi$ children

$$Z_{n+1} \overset{d}{=} \sum_{i=1}^{Z_n} \xi_i$$

PGF for $Z_n$ is

$$F_n := \underbrace{G_\xi \circ G_\xi \circ \cdots \circ G_\xi}_{n}$$

Proved by induction.

---

e.g. If $\xi = \text{Geom}(1/2)$

$$G_\xi = \frac{s/2}{1 - s/2} = \frac{s}{2-s}$$

$$G_{Z_2}(s) = \frac{s/2-s}{2 - s/2-s} = \frac{s}{4-3s}$$

$$\mathbb{E} Z_n = F_n'(1) = \mu^n \qquad \boxed{\mu = \mathbb{E}\xi}$$

$$\mathrm{Var}(Z_n) = \sigma^2 \mu^{n-1} \frac{\mu^n - 1}{\mu - 1}$$

where $\sigma^2 = \mathrm{Var}(\xi)$

---

Qn : does $Z_n \longrightarrow \infty$

$Z_n \longrightarrow 0$ ?

Let $\rho = P(\underbrace{\text{extinction}}_{\substack{Z_n = 0 \\ \text{eventually}}})$

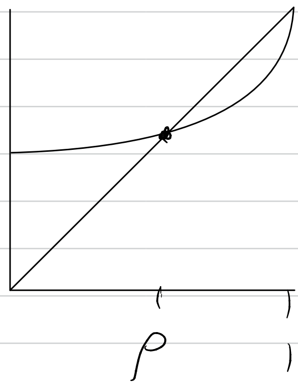
$$\rho = \lim_{n\to\infty} P(Z_n = 0)$$

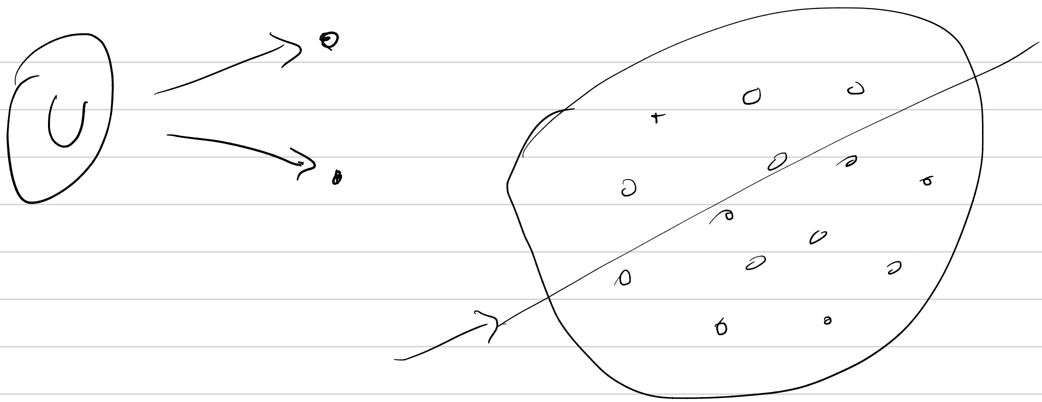
$\underline{\text{Thm}}^{(1)}$ $\rho$ is the smallest pos. sol. to $\rho = G_\xi(\rho)$

(1 is always a sol)

$\underline{\text{Thm}}^{(2)}$: If $\mu \le 1$ and $\mathbb{P}(\xi = 1) \neq 1$ then $\rho = 1$.

Otherwise $\rho < 1$






Let $q = P(\text{Neut. hits another } U)$

$\mu = 2q$

Chain reaction $\iff q > 1/2$

(Critical mass)

## Proof (Thm 1)

$$P(Z_n = 0) = \underbrace{\overbrace{G \circ G \circ \cdots \circ G}^{F_n}}_{n}(0)$$

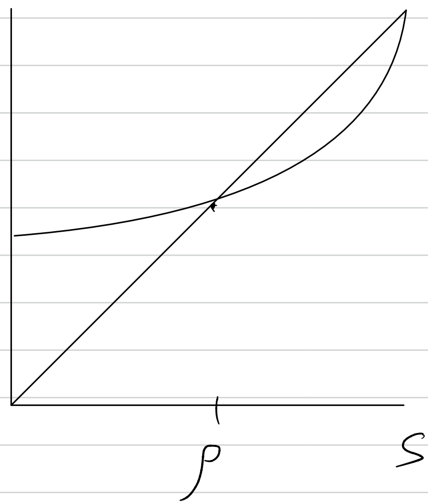
$G = G_\xi$

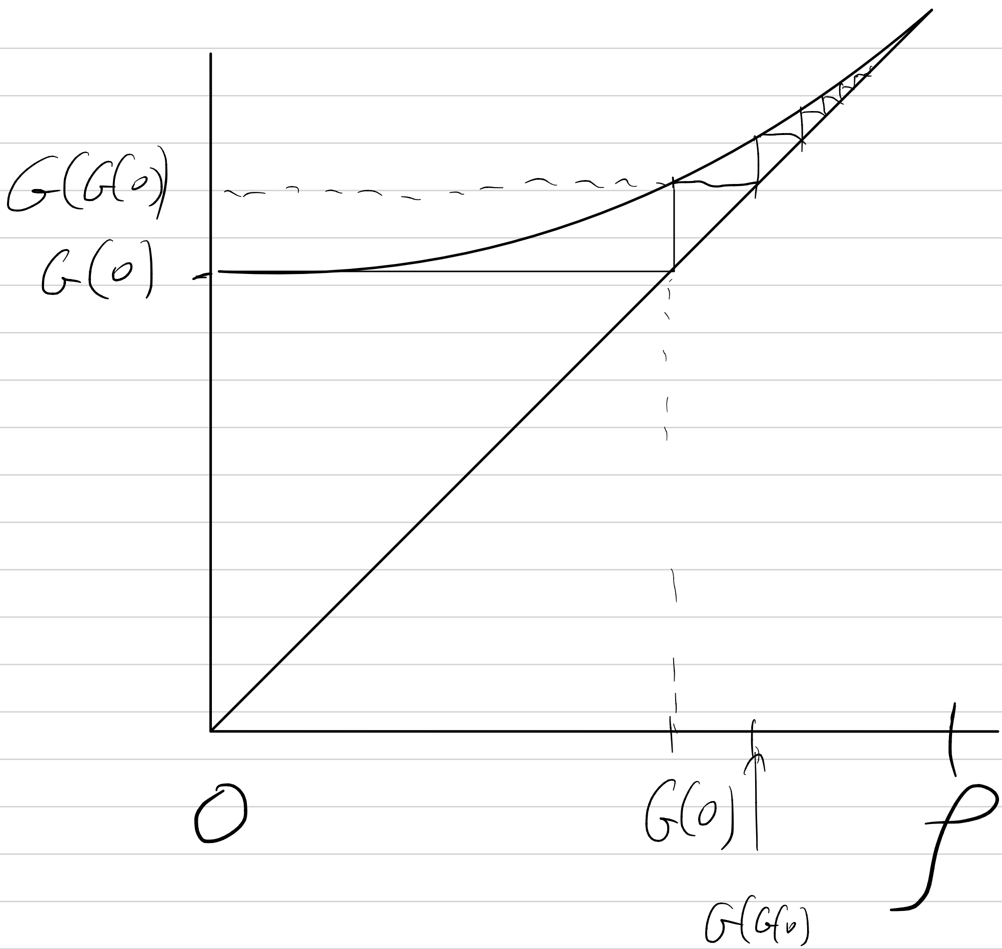
$G$ is inc.

$G(s) = \sum p_n s^n$

$G(1) = 1$

$G$ is convex






On $[0,\rho]$ $G(x) \geq x$

so $F_n(0)$ increasing,

$F_n(0) \leq \rho$ so there is a limit.

$$F_{n+1}(0) = G(F_n(0))$$

so the limit $L$ must have

$$L = G(L) \quad \text{so} \quad L = \rho.$$

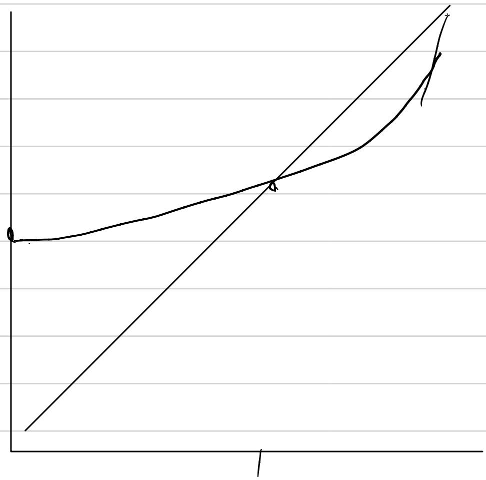
$\square$

Proof (Thm 2)

If $\mu > 1$

$G'(1) > 1$

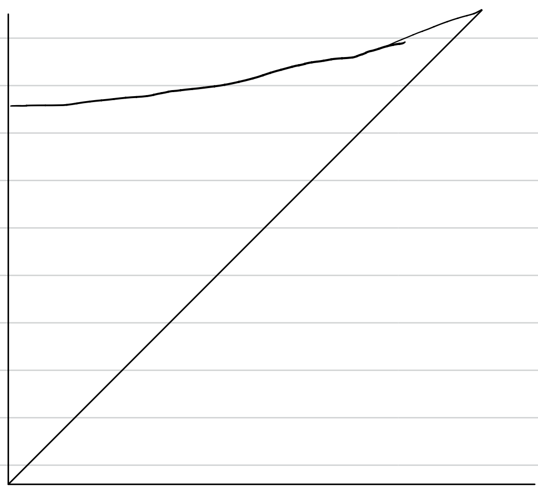
$G(0) \geq 0$



so IVT: There is some

$\rho < 1$ with $G(\rho) = \rho$.

If $\mu \le 1$



$G$ convex

so $G' \le 1$

everywhere

So no sol. $\rho = G(\rho)$

$\rho < 1$.

$\square$

$$Z_0 = 1$$

$$F_0(s) = G_{Z_0} = E s^1 = s$$