

# SE102:Multivariable Calculus

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Lecture 05  
Multiple Integrals

## Definition (Double integral)

Let  $f(x, y)$  be a function defined on a rectangular region  $D = [a, b] \times [c, d]$ . Let us subdivide the intervals  $[a, b]$  and  $[c, d]$  by  $n$  and  $m$  subintervals:

$$a = x_0 < x_1 < \cdots < x_n = b, \quad c = y_0 < y_1 < \cdots < y_m = d.$$

This subdivides  $D$  into  $nm$  subregions

$$D_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]. \quad (i = 1, \dots, n, j = 1, \dots, m)$$

Denote  $\Delta x_i = x_i - x_{i-1}$ ,  $\Delta y_j = y_j - y_{j-1}$ . The **Riemann sum** of  $f(x, y)$  with respect to the subdivision of  $D$  is

$$\sum_{i=1}^n \sum_{j=1}^m f(x_i^*, y_j^*) \Delta x_i \Delta y_j$$

if  $(x_i^*, y_j^*) \in D_{ij}$ .

If the limit exists when  $n, m \rightarrow \infty$  regardless of choice of  $x_i^*$  and  $y_i^*$ , we say  $f$  is **integrable** on  $D$  and the limit is called the **double integral**. We denote the limit as

$$\iint_D f dA = \lim \sum_{i=1}^n \sum_{j=1}^m f(x_i^*, y_j^*) \Delta x_i \Delta y_j.$$

### Theorem

*If  $f$  is continuous on the rectangular region  $D = [a, b] \times [c, d]$ , then  $f$  is integrable on  $D$ .*

## Example

Compute

$$\iint_D xy dx dy$$

for  $D = [0, 1] \times [0, 1]$ .

## Example

Show that the function  $f(x, y)$  on  $[0, 1] \times [0, 1]$  defined by

$$f(x, y) = \begin{cases} 1 & x \text{ or } y \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$$

is not integrable on  $[0, 1] \times [0, 1]$ .

## Definition

Let  $D \subset \mathbf{R}^2$  be bounded region contained in a rectangular box  $[a, b] \times [c, d]$ . We say  $f$  is integrable on  $D$  if the function  $F$  below is integrable on  $[a, b] \times [c, d]$ .

$$F(x, y) = \begin{cases} f(x, y) & (x, y) \in D \\ 0 & (x, y) \in [a, b] \times [c, d] \end{cases}$$

Moreover, the integral of  $f$  on  $D$  is defined by

$$\iint_D f dx dy = \iint_{[a,b] \times [c,d]} F dx dy.$$

## Theorem

*Let  $D$  be a bounded region. If a function  $f(x, y)$  is continuous on  $D$ , then  $f$  is integrable.*

## Definition (Iterated integrals)

Let  $f(x, y)$  be a two variable function define on a rectangular domain  $D = [a, b] \times [c, d]$ . The **iterated integral**

$\int_c^d \int_a^b f(x, y) dx dy$  on  $D$  is defined as follows.

$$\int_c^d \underbrace{\left[ \int_a^b f(x, y) dx \right]}_{\text{consider } y \text{ as a constant}} dy.$$

## Example

Compute  $\int_0^1 \int_0^1 \frac{y}{1+xy} dx dy$ .

## Theorem (Fubini I)

Let  $f(x, y)$  be a continuous function defined on  $D = [a, b] \times [c, d]$ . Then

$$\iint_D f dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

## Remark

The *continuity* condition in the theorem is crucial. For example, let

$$f(x, y) = \begin{cases} \frac{x^2 - y^2}{(x^2 + y^2)^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Let us compute  $\int_0^1 \int_0^1 f(x, y) dy dx$  first.

$$\begin{aligned} \int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dy dx &= \int_0^1 \left. \frac{y}{x^2 + y^2} \right|_{y=0}^1 dx \\ &= \int_0^1 \frac{1}{1 + x^2} dx = \tan^{-1} x \Big|_0^1 = \frac{\pi}{4} \end{aligned}$$



Next, the iterated integral  $\int_0^1 \int_0^1 f(x, y) dx dy$  is

$$\begin{aligned} \int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dx dy &= \int_0^1 \left. \frac{-x}{x^2 + y^2} \right|_{x=0}^1 dx \\ &= \int_0^1 \frac{-1}{1 + y^2} dy = -\tan^{-1} y \Big|_0^1 = -\frac{\pi}{4} \end{aligned}$$

and it does not coincide with  $\int_0^1 \int_0^1 f(x, y) dy dx$ .

## Theorem (Fubini II)

Let  $f(x, y)$  be a continuous function defined on  $D$ . If  $D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$ , then

$$\iint_D f(x, y) dx dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

Similarly, if  $D = \{(x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$ , then

$$\iint_D f(x, y) dx dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

## Example

Let us compute  $\iint_D e^{-y^2} dx dy$  where  $D$  is the triangular region whose vertices are  $(0, 0)$ ,  $(0, 1)$ , and  $(1, 1)$ . By Fubini's theorem,

$$\iint_D e^{-y^2} dx dy = \int_0^1 \left[ \int_x^1 e^{-y^2} dy \right] dx = \int_0^1 \left[ \int_0^y e^{-y^2} dx \right] dy.$$

Find which integration works.

## Definition (Multiple integral)

Let  $f(\mathbf{x})$  be a real-valued function defined on a boxed region

$$[x_0^0, x_1^0] \times [x_0^1, x_1^1] \times \cdots \times [x_0^{n-1}, x_1^{n-1}]$$

Then the triple integral is defined as

$$\iiint_V f(x, y, z) dx dy dz = \iiint_{[a,b] \times [c,d] \times [e,f]} F(x, y, z) dx dy dz$$

## Remark

Let  $f(x, y, z)$  be a function defined on the boxed region  $[a, b] \times [c, d] \times [e, f] \subset \mathbf{R}^3$ . The integral of  $f$  over  $[a, b] \times [c, d] \times [e, f]$  is called the **triple integral** of  $f$  and denoted by

$$\iiint_{[a,b] \times [c,d] \times [e,f]} f(x, y, z) dx dy dz$$

## Theorem

If  $f(x, y, z)$  is defined on a bounded region  $V \subset \mathbf{R}^3$  and  $V$  is contained in a boxed region  $[a, b] \times [c, d] \times [e, f]$ , then  $f$  is integrable if  $F$  below is integrable on  $[a, b] \times [c, d] \times [e, f]$ .

$$F(x, y, z) = \begin{cases} f(x, y, z) & \text{if } (x, y, z) \in V \\ 0 & \text{otherwise.} \end{cases}$$

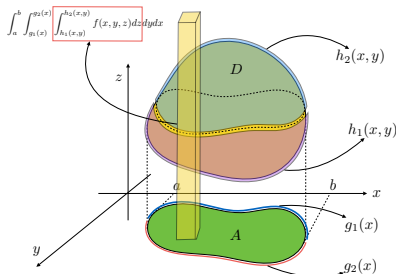
## Theorem (Fubini III)

Let  $f(x, y, z)$  be a continuous function defined on the region  $V$ .

$$V = \{(x, y, z) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x), h_1(x, y) \leq z \leq h_2(x, y)\}$$

Then the following holds.

$$\iiint_V f(x, y, z) dx dy dz = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x, y)}^{h_2(x, y)} f(x, y, z) dz dy dx$$



## Example

Let  $V$  be a parallelepiped region bounded by 6 planes :  $2x = y$ ,  $2x = y + 2$ ,  $y = 0$ ,  $y = 4$ ,  $z = 0$ ,  $z = 3$ . Compute

$$\iiint_V \frac{2x - y}{2} + \frac{z}{3} dx dy dz$$

## Definition

Given an interval  $I \subset \mathbf{R}$ , a curve  $C$  parametrized by  $c : I \rightarrow \mathbf{R}^n$

$$c(t) = (x_1(t), x_2(t), \dots, x_n(t))$$

is **piecewise differentiable** if all coordinate function  $x_i(t)$  are  $C^n$  on the interval  $I$  except for finitely many points.

## Definition

Let  $C$  be a piecewise differentiable curve on  $\mathbf{R}^2$  parametrized by  $c : [a, b] \rightarrow \mathbf{R}^2$ . Let  $c'(t) = (x'(t), y'(t))$  be the velocity vector at the point  $c(t)$ . Let  $f(x, y)$  be a function defined on the curve  $C$ . Then the **line integral** of  $f(x, y)$  along  $C$  is defined as

$$\int_C f ds = \int_a^b f(c(t)) |c'(t)| dt.$$



## Proposition

The line integral  $\int_C f ds$  does not depend on the parametrization of  $C$ .

## Proof.

Let  $c : [a, b] \rightarrow \mathbf{R}^2$  be a parametrization of  $C$ . Let  $h : [c, d] \rightarrow [a, b]$  be a one-to-one correspondence which gives a re-parametrization  $c \circ h$  of  $C$ . Let us write  $t = h(\tau)$ . Then

$$\begin{aligned} \int_c^d f(c \circ h(\tau)) \cdot |(c \circ h)'(\tau)| d\tau &= \int_c^d f(c(t)) \cdot |c'(t)| \cdot |h'(\tau)| d\tau \\ &= \int_a^b f(c(t)) \cdot |c'(t)| dt \end{aligned}$$

□

## Example

Let us compute the line integral

$$\int_C (2 + x^2 y) ds$$

where  $C$  is the unit circle centered at the origin with counter clockwise orientation.

## Definition

Let  $D$  be a region in  $\mathbf{R}^2$  and  $S$  a surface in  $\mathbf{R}^3$ . A map  $X : D \rightarrow S$

$$X(u, v) = (x(u, v), y(u, v), z(u, v))$$

is called the **parametrization** of  $S$  if it is one-to-one correspondence and every partial derivative of  $x, y, z$  is continuous on  $D$ .

## Example

Find a parametrization of  $x^2 + y^2 - z^2 = 1$ .

## Definition

Let  $f(x, y, z)$  be a function defined on a surface  $S$ . Let  $X : D \rightarrow S$  be a parametrization of  $S$ . The **surface integral** of  $f$  on  $S$  is defined by

$$\iint_S f dS = \iint_D (f \circ X)(u, v) \|X_u \times X_v\| du dv \quad (1)$$

## Example

Let  $S$  be the surface defined by the graph of  $z = \sqrt{x^2 + y^2}$  over the disk  $x^2 + y^2 \leq 1$ . Evaluate

$$\iint_S z dS$$

## Problem

For  $f(x, y) = x^2 - y^2$ , compute  $\iint_S x + z dS$ .

## Problem

Let  $f(x, y)$  is a density at the point  $(x, y)$  on domain  $D$ . Let

$$\bar{x} = \frac{\iint_D x f(x, y) dA}{\iint_D f(x, y) dA}, \quad \bar{y} = \frac{\iint_D y f(x, y) dA}{\iint_D f(x, y) dA}$$

Explain why  $(\bar{x}, \bar{y})$  is the center of mass of  $D$ .

## Problem

Let  $V = \{0 \leq x \leq 2, 0 \leq y \leq x, 0 \leq z \leq y\}$ . Write  $\iiint_V f(x, y, z)$  in six different iterated integrals.

## Problem

Let  $f(x, y)$  be a function defined on a curve  $C$ . Explain the meaning of the line integral  $\int_C f ds$  in the following senses:

1. when  $f(x, y)$  is a density at  $(x, y)$  on the curve  $C$ ;
2. the section of the graph  $z = f(x, y)$  over the curve  $C$ .