SE102:Multivariable Calculus

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Lecture 06
Line and Surface Integrals

Definition

Given an interval $I \subset \mathbf{R}$, a curve C parametrized by $c : I \to \mathbf{R}^n$

$$c(t) = (x^0(t), x^1(t), \cdots, x^{n-1}(t))$$

is **piecewise differentiable** if all coordinate functions $x^i(t)$ are of $\mathcal{C}^n$ class on the interval I except for finitely many points.

Definition

Let C be a piecewise differentiable curve on $\mathbf{R}^2$ parametrized by $c : [a, b] \to \mathbf{R}^2$ and $c'(t) = (x'(t), y'(t))$ be the velocity vector. Given a function $f(x, y)$ defined on the curve C, the **line integral** of $f(x, y)$ along C is defined as

$$\int_C f ds = \int_a^b f(c(t)) \|c'(t)\| dt.$$

Proposition

The line integral $\int_C f ds$ *does not depend on the parametrization of* C*.*

Proof.

Let $c : [a, b] \to \mathbf{R}^2$ be a parametrization of C. Let $h : [c, d] \to [a, b]$ be a one-to-one correspondence which gives a *re-parametrization* $c \circ h$ of C. Let us write $t = h(\tau)$. Then

$$\begin{aligned}
\int_c^d f(c \circ h(\tau)) \cdot \|(c \circ h)'(\tau)\| d\tau &= \int_c^d f(c(t)) \cdot \|c'(t)\| \cdot \|h'(\tau)\| d\tau \\
&= \int_a^b f(c(t)) \cdot \|c'(t)\| dt
\end{aligned}$$

$\square$

Example

Compute the line integral

$$\int_C (2 + x^2 y) ds$$

where C is the unit circle centered at the origin with counter clockwise orientation.

Definition

Let S be a suface in $\mathbf{R}^3$ and D be the region in $\mathbf{R}^2$. A map $X : D \to S$

$$X(u, v) = (x(u, v), y(u, v), z(u, v))$$

is called the **parametrization** of the surface S if it is a one-to-one correspondence and every partial derivative of x, y, z is continuous on D.

Example

Find a parametrization of $S = \{(x, y, z) \mid x^2 + y^2 - z^2 = 1\}$.

Definition

Let $f(x, y, z)$ be a function defined on a surface S. which is parametrized by $X : D \to S$. The **surface integral** of f on S is defined by

$$\iint_S f dS = \iint_D (f \circ X)(u, v) \|X_u \times X_v\| dudv.$$

Example

Let S be the surface defined by the graph of $z = \sqrt{x^2 + y^2}$ over the disk $x^2 + y^2 \leq 1$. Compute $\displaystyle\iint_S z dS$.

Definition

Let D, R be regions in $\mathbf{R}^n$. A map $\mathbf{T} : R \to D$ is called a **transformation** if it is differentiable and one-to-one. The determinant of the differential of $\mathbf{T}$ is called the **Jacobian** of T and is denoted by

$$J_{\mathbf{T}} = \det \mathbf{dT}.$$

When the variables are explicitly presented as $T(u^0, \cdots, u^{n-1}) = (x^0, \cdots, x^{n-1})$, the Jacobian is also denoted as

$$J_{\mathbf{T}} = \det \frac{\partial(x^0, \cdots, x^{n-1})}{\partial(u^0, \cdots, u^{n-1})}.$$

Theorem (Integration by substitution)

Let $T : R \to D$ be a transformation, and $f(x, y)$ be a continuous function defined on D. Then

$$\iint_D f(x,y)dxdy = \iint_R (f \circ T)(u,v)|J_T|dudv.$$

Remark

It is useful to remember the subsitution rule as:

$$\iint_D f(x,y)dxdy = \iint_R (f \circ T)(u,v)\left|\frac{\partial(x,y)}{\cancel{\partial(u,v)}}\right|\cancel{dudv}$$

Also, the Jacobian of the inverse $\mathbf{T}^{-1}(x,y) = (u(x,y), v(x,y))$ is

$$J_{\mathbf{T}^{-1}} = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \frac{1}{J_{\mathbf{T}}} = \frac{1}{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}$$

Example

Compute

$$\iint_D |x| e^{-x^2-y^2} \, dx dy$$

where $D = \{(x, y) \,|\, x^2 + y^2 \leq 1\}$.

Remark

The integration by substitution works on triple integral as well. Let V, W be regions in $\mathbf{R}^3$. A differentiable one-to-one function $T : W \to V$

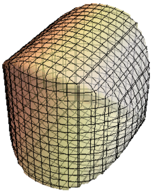
$$T(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))$$

is called a **transformation**. Let $f(x, y, z)$ be a continuous function on V. Then

$$\iiint_V f(x, y, z) dxdydz = \iiint_W (f \circ T)(u, v, w)|J_T| dudvdw.$$

Example

Compute the volume between two cylinders $x^2 + y^2 \leq 1$, $y^2 + z^2 \leq 1$.



Example

Compute

$$\iiint_V \sqrt{x^2+y^2+z^2}\, e^{x^2+y^2+z^2}\, dxdydz$$

where V is the region between the spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$ $(0<a<b)$.

Definition

Let V be a n-dimensional vector space. A **vector field** $\mathbf{F} : \mathbf{R}^n \to V$ is a map which assign a vector in a vector space V to each point in the space $\mathbf{R}^n$. For the simplicity, we call $\mathbf{F}$ as n-dimensional vector field.

Definition

Let $\mathbf{F} : \mathbf{R}^n \to V$ be a 3-dimensional vector field defined by

$$\mathbf{F}(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z)).$$

The **curl** $\nabla \times \mathbf{F}$ and the **divergence** $\nabla \cdot \mathbf{F}$ is defined by

$$\begin{aligned} \nabla \times \mathbf{F} &= (R_y - Q_z, P_z - R_x, Qx - P_y), \\ \nabla \cdot \mathbf{F} &= P_x + Q_y + R_z. \end{aligned}$$

Remark

The curl of a vector field is a vector field, while the divergence is a scalar function. It is convenient to remember the formula of the curl $\nabla \times \mathbf{F}$ as

$$\nabla \times \mathbf{F} = \begin{vmatrix} i & j & k \\ \partial/\partial_x & \partial/\partial_y & \partial/\partial_z \\ P & Q & R \end{vmatrix}.$$

Also, some defines the curl and divergence of 2-dimensional vector fields as well. For 2-dimensional vector field $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$, the curl is a *scalar*

$$\mathrm{curl}\mathbf{F} = Q_x - P_y,$$

and the divergence is

$$\mathrm{div}\mathbf{F} = P_x + Q_y.$$

Theorem

Let f, g be 3-dimensional functions and $\mathbf{F}$, $\mathbf{G}$ be 3-dimensional vector fields. The following properties hold.

1. $\nabla \times (\nabla f) = 0$
2. $\nabla \cdot (\nabla \times \mathbf{F}) = 0$
3. $\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$
4. $\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$
5. $\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f$
6. $\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + \nabla f \times \mathbf{F}$
7. $\nabla \cdot (\mathbf{F} + \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$
8. $\nabla \cdot (\nabla f \times \nabla g) = 0$
9. *Denote* $\nabla^2 = \nabla \cdot \nabla$. *Then*

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \cdot \mathbf{F}$$

Definition

Let C be a curve in $\mathbf{R}^n$ parametrized by $c : [a, b] \to \mathbf{R}^n$. The **line integral** of $\mathbf{F}$ on C, denoted by $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{s}$, is defined by

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}((c(t))) \cdot c'(t) dt.$$

When $c(a) = c(b)$, the curve is said to be **closed**. The line integral on a closed curve is denoted by

$$\oint_C \mathbf{F} \cdot d\mathbf{s}.$$

For 2-dimensional vector field $\mathbf{F} = (P, Q)$, the line integral is denoted by

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C P dx + Q dy.$$

For 3-dimensional vector field $\mathbf{F} = (P, Q, R)$, the line integral is denoted by

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C P dx + Q dy + R dz.$$

Example

Let $\mathbf{A} = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$. Compute the line integral $\oint_C \mathbf{A} \cdot d\mathbf{s}$ where C is a unit circle parametrized by counter-clockwise direction.

Definition

Let $X : D \to S$ be a parametrization of S. When the vector field $\mathbf{n}$ defined by

$$\mathbf{n} = \frac{\mathbf{X}_u \times \mathbf{X}_v}{\|\mathbf{X}_u \times \mathbf{X}_v\|}$$

is continuous, $\mathbf{n}$ is called an **orientation** of S. If a surface admits a parametrization with an orientation, then we say S is **orientable**.

Definition

Let $\mathbf{F}$ be a 3-dimensional vector field defined on an orientable surface S parametrized by $X : D \to S$. Then **surface integral** of $\mathbf{F}$ over S with respect to an orientation $\mathbf{n}$, denoted by $\iint_S \mathbf{F} \cdot d\mathbf{S}$, is defined by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D (\mathbf{F} \circ X) \cdot \mathbf{n} dA.$$

Example

Pick an orientation of a unit sphere and compute the surface integral of $\mathbf{F} = \dfrac{(x, y, z)}{x^2 + y^2 + z^2}$.

Problem

Let $f(x, y)$ be a function defined on a curve C. Explain the meaning of the line integral $\displaystyle\int_C f ds$ in the following senses:

1. when $f(x, y)$ is a density at (x, y) on the curve C, the integral $\displaystyle\int_C f ds$ is the mass of C;
2. the integral $\displaystyle\int_C f ds$ is the area of the section of the graph $z = f(x, y)$ over the curve C.

Problem

Let $\mathbf{F}$, $\mathbf{G}$ be 3-dimensional vector fields. Show that the following properties hold.

1. $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$
2. $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla \cdot \nabla \cdot \mathbf{F}$
3. $\nabla \times (\mathbf{F} \times \mathbf{G}) = (\nabla \cdot \mathbf{G})\mathbf{F} - (\nabla \cdot \mathbf{F})\mathbf{G} + (\mathbf{G} \cdot \nabla) \cdot \mathbf{F} - (\mathbf{F} \cdot \nabla) \cdot \mathbf{G}$

Problem

Explain geometric meanings of $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$.

Problem

Find the area of the region bounded by three cylinders $x^2 + y^2 \leq 1$, $y^2 + z^2 \leq 1$, and $x^2 + z^2 \leq 1$.