# SE102:Multivariable Calculus

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Week 01

## Definition

A ($n$-dimensional) **vector** is a $n$-tuple of real numbers

$$\mathbf{a} = (a_1, a_2, \cdots, a_n)$$

with the following operations.

- (vector sum) For a vector $\mathbf{b} = (b_1, b_2, \cdots, b_n)$,

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n).$$

- (scalar multiplication) For $k \in \mathbf{R}$, we define a vector $k\mathbf{a}$ as

$$k\mathbf{a} = (ka_1, ka_2, \cdots, ka_n).$$

The set of all ($n$-dimensional) vectors is called the ($n$-dimensional) **vector space**, and denoted by $\mathbf{R}^n$.

## Example

Let $O = (0, 0, 0)$ be the origin and $P = (a_1, a_2, a_3)$ a point in 3-dimensional space. Then the arrow $\overrightarrow{OP}$ can be represented by the **position vector**

$$\overrightarrow{OP} = (a_1, a_2, a_3)$$

The scalar multiplication is a dilation, and the vector sum is a superposition. For points $Q = (b_1, b_2, b_3)$ and
$R = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$, the vector $\overrightarrow{PR}$ is represented by the same vector as $\overrightarrow{OQ}$. Thus we have the following additive operation.

$$\overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR}$$

## Example

Let $c : (-\varepsilon, \varepsilon) \to \mathbf{R}^2$ be a parametrization of a curve on a plane. Suppose that each coordinate functions are given by differentiable function $x(t), y(t)$ defined on $(-\varepsilon, \varepsilon)$.

$$c(t) = (x(t), y(t))$$

Let us define $c'(t)$ as

$$c'(t) = (x'(t), y'(t))$$

We can consider $c'(t)$ as a vector, which represents the **velocity** at $c(t)$. The $x, y$-components of the vector $c'(t)$ represent the projective speed of $c(t)$ on $x, y$-axis respectively. We can decompose the velocity vector into the sum of horizontal and vertical velocity of $c(t)$.

$$c'(t) = (x'(t), 0) + (0, y'(t))$$

### Example

Let us consider a function $f$ sending every point in $\mathbf{R}^2$ to a point in $\mathbf{R}^3$ as follows.

$$f(u,v) = (x(u,v), y(u,v), z(u,v)) \tag{1}$$

The variables $x, y, z$ are dependent to the variables $u, v$. The functions $f$ (and also $x, y, z$ as functions) contain(s) two or more independent or dependent variables. Such functions are are called **multivariable functions**. If we write $\mathbf{a} = (u, v)$ and $\mathbf{b} = (x, y, z)$, the equation (1) can be written as $f(\mathbf{a}) = \mathbf{b}$. A multivariable function can be viewed as a *single variable* function which assigns a vector to another vector.

### Definition

For a ($n$-dimensional) vector $\mathbf{a} = (a_1, a_2, \cdots, a_n)$, the value

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

is called the **norm** of $\mathbf{a}$. A vector with norm 1 is called a **unit** vector. The norm $\|\mathbf{a}\|$ is 0 if and only if $\mathbf{a}$ is a **zero vector**, that is,

$$\mathbf{0} + \mathbf{a} = \mathbf{a} + \mathbf{0} = \mathbf{a}$$

for any vector $\mathbf{a}$. For nonzero vector $\mathbf{a}$,

$$\mathbf{u} = \mathbf{a}/\|\mathbf{a}\|$$

is called the **normalization** of $\mathbf{a}$.

## Example

For each $i = 1, \cdots, n$, the vector

$$\mathbf{e}_i = (0, \cdots, 1, \cdots, 0)$$

with 1 at $i$th place is called a (unit) **basis vector**. Especially, 3-dimensional basis vectors are denoted by

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1)$$

A vector $\mathbf{a} = (a_1, a_2, \cdots, a_n)$ can be decomposed as a linear sum of basis vectors.

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n$$

Thus the set

$$\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$$

is called the **basis**, and generates all $n$-dimensional vectors.

## Definition

The **inner product** (also called **dot product**) of two vectors $\mathbf{a}$, $\mathbf{b}$ is an operation defined by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

If $\mathbf{a} \cdot \mathbf{b} = 0$, then we say $\mathbf{a}$, $\mathbf{b}$ are **orthogonal**.

## Proposition

*The inner product satisfies the following.*

1. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
2. $\mathbf{a} \cdot (k\mathbf{b}) = k(\mathbf{a} \cdot \mathbf{b})$
3. $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$

### Example

Let us parametrize a line $l$ in $\mathbf{R}^3$.

1. Suppose that $l$ passes through $P = (x_0, y_0, z_0)$ and parallel to $\mathbf{a} = (a_1, a_2, a_3)$. Then the paramtric equations of $l$ is

$$l(t) = P + t\mathbf{a} \tag{2}$$

   Equivalently, equation (2) can be written as a *symmetric form* as follows.

$$\frac{x - x_0}{a_1} = \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3} \tag{3}$$

2. Suppoe that $l$ passes through two points $P, Q$. By subsitute $\mathbf{a} = \overrightarrow{OP} - \overrightarrow{OQ}$ to (2) or (3), we get a parametric equation of $l$.

### Theorem

*Let* $\mathbf{a}$, $\mathbf{b}$ *be nonzero* 2*-dimensional vectors. If* $\theta$ *is the angle between* $\mathbf{a}$ *and* $\mathbf{b}$, *then*

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot \cos\theta$$

### Definition

For nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ with the same dimension, the vector defined by

$$\mathrm{proj}_{\mathbf{b}}\mathbf{a} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\right)\mathbf{b}$$

is called the **projection of a onto b**.

## Remark

As norm measures the *size* of a vector, the inner product measure the *direction*. For example, the direction of a 2-dimensional vector $\mathbf{a}$ is determined by $0 \le \theta_1, \theta_2 \le \pi$ satisfying

$$\mathbf{a} \cdot \mathbf{e}_1 = \|\mathbf{a}\| \cos \theta_1$$

$$\mathbf{a} \cdot \mathbf{e}_2 = \|\mathbf{a}\| \cos \theta_2$$

In order to determine the direction of a 3-dimensional vector, say $\mathbf{a}$, we need three angles $0 \le \alpha, \beta, \gamma \le \pi$ satisfying

$$\cos \alpha = \frac{\mathbf{a} \cdot \mathbf{i}}{\|\mathbf{a}\|}, \quad \cos \beta = \frac{\mathbf{a} \cdot \mathbf{j}}{\|\mathbf{a}\|}, \quad \cos \gamma = \frac{\mathbf{a} \cdot \mathbf{k}}{\|\mathbf{a}\|} \tag{4}$$

Such quantities are called the **direction cosines**.

## Definition

A $n \times m$ ($n$-by-$m$) **matrix** is a collection of $nm$ numbers (or functions) arranged in the following way.

$$A = (a_{ij})_{n\times m} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

The indices $i$, $j$ of an entry $a_{ij}$ represents the row and column indices respectively.

## Example

1. A $n \times m$ matrix is called **square** matrix if $n = m$.
2. If $A$ is a square matrix and $a_{ij} = 0$ for all $i \neq j$, then $A$ is called **diagonal**.

$$A = \begin{pmatrix} a_{11} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & a_{nn} \end{pmatrix}$$

3. If the diagonal entries of a diagonal matrix are all 1, then it is called the **identity matrix**, and denoted by $I_n$.

$$I_n = \begin{pmatrix} 1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & 1 \end{pmatrix}$$

4. If $A$ is a square matrix and $a_{ij} = 0$ for all $i < j$ ($i < j$, respectively), then $A$ is called **lower triangle** (**upper**

## Definition

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $n \times m$ matrix. Then we define

$$A + B = (a_{ij} + b_{ij})$$

$$k \cdot A = (ka_{ij})$$

Let $C$ be a $m \times l$ matrices, then $A \cdot C$ is a $n \times l$ matrix whose entries are

$$A \cdot B = (a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj})_{1 \le i \le n, 1 \le j \le l}$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{im} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1l} \\ c_{21} & \cdots & c_{2j} & \cdots & c_{2l} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{ml} \end{pmatrix}$$

## Example

Suppose Bob, Larry, and Joanna worked in a fruits store for three days. Table 1 shows *how many* fruits each sold in total represented by the matrix $A$, and Table 2 shows *how much* was the fruits on each day represented by the matrix $B$.

| | Apple | Orange | Banana |
|---|---|---|---|
| Bob | 38 | 25 | 10 |
| Larry | 15 | 22 | 15 |
| Joanna | 8 | 70 | 27 |

Table: The volumn of sales of each person per items

| | Day1 | Day2 | Day3 |
|---|---|---|---|
| Apple | \$1.19 | \$1.45 | \$.99 |
| Orange | \$1.70 | \$0.99 | \$2.1 |
| Banana | \$2.19 | \$3.5 | \$1.29 |

Table: The prices of items per day

### Example

The $i, j$-entries of the $A \cdot B$ represents the total revenue sold by the person $i$ at the day $j$.

$$\begin{pmatrix} 38 & 25 & 10 \\ 15 & 22 & 15 \\ 8 & 70 & 27 \end{pmatrix} \begin{pmatrix} 1.19 & 1.45 & 0.99 \\ 1.70 & 0.99 & 2.1 \\ 2.19 & 3.5 & 1.29 \end{pmatrix}$$

A deeper, and more mathematical, thus important meaning of matrix multiplication will be discovered when we visit in the *chain rule* in later sections.

## Proposition

*Let $A, B, C$ be matrices. Whenever the operations are valid, the following holds.*

1. $(A \cdot B) \cdot C = A \cdot (B \cdot C)$
2. $A \cdot (B + C) = A \cdot B + A \cdot C$
3. $(B + C) \cdot A = B \cdot A + C \cdot A$
4. $k \cdot (A \cdot B) = (k \cdot A) \cdot B = A \cdot (k \cdot B)$

## Proposition

*The **transpose** of a matrix $A = (a_{ij})$ defined by*

$$A^T = (a_{ji})$$

*and it satisfies the following.*

5. $\left(A^T\right)^T = A$
6. $k \cdot A^T = (k \cdot A)^T$
7. $(A + B)^T = A^T + B^T$
8. $(AB)^T = B^T A^T$

Definition

The **determinant** of $2 \times 2$ matrix $A$ is defined as follows.

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

The **determinant** of $3 \times 3$ matrix $B$ is defined as follows.

$$\begin{aligned} \det B = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} &= b_{11}b_{22}b_{33} + b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32} \\ &\quad - b_{13}b_{22}b_{31} - b_{11}b_{23}b_{32} - b_{12}b_{21}b_{33} \end{aligned}$$

## Definition

Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ be two 3-dimensional vectors. The **cross-product** of $\mathbf{a}, \mathbf{b}$ is a vector defined by

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

An easy way to remember the formula is the following.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

## Proposition

*Let* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *be* 3*-dimensional vectors and* $k$ *a constant. The following identity holds.*

1. $\mathbf{a} \times \mathbf{0} = \mathbf{0} \times \mathbf{a} = \mathbf{0}$
2. $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
3. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
4. $\mathbf{a} \times (k\mathbf{b}) = (k\mathbf{a}) \times \mathbf{b} = k(\mathbf{a} \times \mathbf{b})$
5. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
6. $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$ *(This shows that the cross product* $\mathbf{a} \times \mathbf{b}$ *is normal to the plane spanned by* $\mathbf{a}$ *and* $\mathbf{b}$*.)*
7. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$

**Theorem**

*Let $\mathbf{a}, \mathbf{b}$ be two 3-dimensional vectors. Then*

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot |\sin\theta|$$

*where $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$.*

### Proposition

*Let us denote $|A|$ by the determinant of a matrix $A$. Then the following holds*

1. *If $A$ has a row (or a column) whose entries are all zero, then $|A| = 0$.*
2. *Let $B$ be the matrix obtained by interchanging two rows (or columns) of $A$. Then $|B| = -|A|$.*
3. *Let $B$ be a matrix obtained by multiply c on a row (or column) followed by adding it to another row (or column). Then $|B| = |A|$.*

### Definition

Let $A$ be a $n \times n$ matrix. A matrix $B$ satisfying

$$A \cdot B = B \cdot A = I_n$$

is called the **inverse of** $A$, denoted by $B = A^{-1}$. If an inverse matrix $A^{-1}$ exists, then $A$ is said to be **non-singular**. Otherwise, it is called **singular**.

### Theorem

*A matrix A is singular if and only if* $\det A = 0$.

## Proposition

*If $A$ is a $2 \times 2$ matrix, then $A^{-1}$ is*

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix}$$

*For a $3 \times 3$-matrix $B$, the inverse is given by*

$$B^{-1} = \frac{1}{\det B} \begin{pmatrix} c_{11} & -c_{21} & c_{31} \\ -c_{12} & c_{22} & -c_{32} \\ c_{13} & -c_{23} & c_{33} \end{pmatrix} \tag{5}$$

*where each $c_{ij}$, called the **cofactor**, is the determinant of $2 \times 2$-matrix obtained by deleting $i$th row and $j$th column. For example,*

$$c_{21} = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ \cancel{b_{21}} & \cancel{b_{22}} & \cancel{b_{23}} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} = \begin{vmatrix} b_{12} & b_{13} \\ b_{32} & b_{33} \end{vmatrix}$$

*Notice that the row and column indices are switched in (5).*

### Definition

A **vector space** $V$ is a set of element called **vectors** satisfying the following properties:

1. (Zero vector) $V$ contains the **zero vector** $\mathbf{0}$, which is a unique vector satisfying $\mathbf{0} + \mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
2. (Vector sum) For any two vectors $\mathbf{v}, \mathbf{w} \in V$, the vector $\mathbf{v} + \mathbf{w}$ lies in $V$.
3. (Scalar multiplication) For any $k \in \mathbf{R}$ and $v \in V$, the vector $k\mathbf{v}$ lies in $V$.

A subset $V \subset \mathbf{R}^n$ is called a **vector subspace** if it is a vector space itself.

## Example

Let $\mathcal{C}^1(\mathbf{R})$ be the set of all differentiable functions on $\mathbf{R}$ whose derivatives are continuous on $\mathbf{R}$. Since $f, g$ are such functions so is the function $h = f + g$. Also for any $k \in \mathbf{R}$, the function $kf$ is differentiable and its derivative is continuous. Thus $\mathcal{C}^1(\mathbf{R})$ is a vector space. Likewise, we can define vectors spaces $\mathcal{C}^n(\mathbf{R})$, $\mathcal{C}^\infty(\mathbf{R})$

## Example

For each constant $k \in \mathbf{R}$, let $V_k$ be the set of all points on the line $y = kx$ in $\mathbf{R}^2$:

$$V_k = \{(x, y) \,|\, y = kx\}$$

Then $V_k$ is a vector subspace of $\mathbf{R}^2$. Let $V_\infty$ be the vertical line $V_\infty = \{(0, y) \,|\, y \in \mathbf{R}\}$. Then $V_\infty$ is also a vector subspace of $\mathbf{R}^2$

## Example

Let $P$ be the set of all point on the plane

$$P = \{(x, y, z) \,|\, ax + by + cz = 0\}$$

in $\mathbf{R}^3$. By identifying points in $P$ as position vectors, we can say $P$ is a vector subspace of $\mathbf{R}^3$, orthogonal to $\mathbf{n} = (a, b, c)$ In particular, let $V, W$ be vector subspace of $\mathbf{R}^n$. Then so is $V \cap W$. For example, let $V, W$ be vector subspace for two planes in $\mathbf{R}^3$ passing through the origin $\mathbf{0}$. Then $V \cap W$ is either a line (if $V, W$ are transversal) or a plane (if $V = W$).

### Definition

Let $V$ a vector space and $\mathbf{v}_1, \cdots, \mathbf{v}_m \in V$. We say the vector space subspace

$$W = \{a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m \,|\, a_1, \cdots, a_m \in \mathbf{R}\}$$

is **spanned** by $\mathbf{v}_1, \cdots, \mathbf{v}_m$, and denote by $W = \mathrm{span}\langle \mathbf{v}_1, \cdots, \mathbf{v}_m \rangle$.

### Definition

We say a vector space $V$ is **spanned by** the set of vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$, if every $v \in V$ can be expressed a linear combination of $v_i$'s:

$$v = a_1\mathbf{v}_1 + \cdots a_n\mathbf{v}_n$$

The set $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is said to be **linearly independent** if the coefficients satisfying

$$a_1\mathbf{v}_1 + \cdots a_n\mathbf{v}_n = \mathbf{0}$$

is the trivial ones, namely $a_1 = \cdots = a_n = 0$.

### Definition

Let $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ be the set of linear independent vectors which spans the vector space $V$. Such set is called the **basis** of $V$, and its element is called the **basis vector**. The **dimension** of $V$ is the number of basis vectors which spans $V$.

## Problem

1. Show that the vectors $\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b}$ and $\|\mathbf{b}\|\mathbf{a} - \|\mathbf{a}\|\mathbf{b}$ are orthogonal.
2. Show that $\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

## Problem

1. Let $\mathbf{a} = (1,2,1)$, $\mathbf{b} = (2,1,2)$, and $\mathbf{u} = (0,1,-1)$. Suppose that the vector $\mathbf{u}$ can be decomposed by

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$$

where $\mathbf{u}_1$ is parallel to $\mathbf{a}$, $\mathbf{u}_2$ is parallel to $\mathbf{b}$, and $\mathbf{u}_3$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. Find the vector $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ explicitly.

## Problem

1. Find the distance between $P = (2, 1, 3)$ and the line $l(t) = (2, 3, -2) + t(-1, 1, -2)$.
2. Find the distance between two parallel planes

$$2x - 2y + z = 5, \quad 2x - 2y + z = 20$$

3. Find the distance between two skew lines

$$l_1(t) = (0, 5, -1) + t(2, 1, 3)$$

$$l_2(t) = (-1, 2, 0) + t(1, -1, 0)$$

## Problem

Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, $\mathbf{c} = (c_1, c_2, c_3)$. Verify that

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

## Problem

Imagine two concentric circles with radius $a, b$ ($b < a$) which rolls on flat line with the same angular velocity. A curtate cycloid is a trajectory of a point on the circle of radius $b$. Find a set of parametric equation for the curtate cycloid with $a = 3$, $b = 2$.