SE102:Multivariable Calculus

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 $\begin{array}{c} {\rm Lecture}~08 \\ {\rm More~on~Integrals} \end{array}$

The integral $\iint_D f(x,y)dxdy$ is called **improper** if it satisfies one of the following.

- \blacktriangleright the region D is unbounded, or
- \blacktriangleright the function diverges at some point in D.

Let $D = \mathbb{R}^2$ be the entire 2-dimensional plane. Let us compute

$$\iint_{\mathbf{R}^2} e^{-x^2 - y^2} dx dy$$

By polar coordinate $T(r, \theta) = (r \cos \theta, r \sin \theta)$,

$$T^{-1}(D) = [0, \infty) \times [0, 2\pi].$$

Thus by change of coordinates,

$$\iint_{\mathbf{R}^2} e^{-x^2 - y^2} dx dy = \int_0^\infty \int_0^{2\pi} e^{-r^2} r d\theta dr = 2\pi \left(\frac{-1}{2} e^{-r^2}\right) \Big|_0^\infty = \pi$$

The gamma function $\Gamma: \mathbf{R} \to \mathbf{R}$ is defined by

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx$$

Example

- 1. $\Gamma(n) = (n-1)!$ for $n \ge 1$.
- 2. $\Gamma(x)$ diverges at each non-positive integer x.
- 3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

The **beta function** $B(x,y): \mathbf{R}^2 \to \mathbf{R}$ is defined by

$$B(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt$$

Proposition

- 1. B(x,y) = B(y,x)
- 2. $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

Let $V = [0, 1]^n$ be a *n*-dimensional cube with side length 1. Then the volume of V is 1.

Remark

Let $dx_1 \wedge \cdots \wedge dx_n$ be a *n*-form. The definition above can be written as

$$\int_{[0,1]^n} dx_1 \wedge \dots \wedge dx_n = 1.$$

The volume of n-dimensional region V is defined by

$$\int_{V} dx_1 \wedge \cdots \wedge dx_n.$$

What does anti-commutativity of the wedge product (i.e. $dx \wedge dy = -dy \wedge dx$) imply?

The n-dimensional ball of radius r is the set of points in 4-dimensional space defined by

$$B_n(r) = \{(x_1, \dots, x_n) \mid x_1^2 + \dots + x_n^2 \le r^2\}.$$

The n-1-dimensional sphere of radius r is the boundary of $B_n(r)$, defined by

$$S_{n-1}(r) = \{(x_1, \dots, x_n) \mid x_1^2 + \dots + x_n^2 = 1\}.$$

There are 3 ways to compute the volume of 4-dimensional ball. (The volume of n-dimensional ball and n-1-dimensional sphere can be computed similarly.)

- 1. Using spherical coordinate.
- 2. Integrating sections.
- 3. Finding recursive formula.

Proposition

Let $T: \mathbf{R}^n \to \mathbf{R}^n$ be a transformation (i.e. one-to-one, differentiable) such that

$$T(u_1,\cdots,u_n)=(x_1,\cdots,x_n).$$

If U be a region in \mathbb{R}^n and V = T(U). Then

$$\int_{V} dx_{1} \wedge \cdots \wedge dx_{n} = \int_{U} \frac{\partial(x_{1}, \cdots, x_{n})}{\partial(u_{1}, \cdots, u_{n})} du_{1} \wedge \cdots \wedge du_{n}$$

Let $T:[0,1]\times[0,\pi]\times[0,\pi]\times[0,2\pi]\to\mathbf{R}^4$ be a transformation defined by

$$T(r, \theta_1, \theta_2, \phi) = (r \sin \theta_1 \sin \theta_2 \cos \phi, r \sin \theta_1 \sin \theta_2 \sin \phi,$$

$$r \sin \theta_1 \cos \theta_2, r \cos \theta_1)$$

Such T is called a 4-spherical transformation and the Jacobian is

$$J_T = r^3 \sin^2 \theta_1 \sin \theta_2$$

Thus the volume of $B_4(1)$ is

$$\int_{0}^{1} \int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{2\pi} r^{3} \sin^{2} \theta_{1} \sin \theta_{2} d\phi d\theta_{2} d\theta_{1} dr = \frac{\pi^{2}}{2}$$



As we slice the 4-dimensional ball $B_4(1)$ at each w-coordinate, we obtain 3-dimensional ball of radius $\sqrt{1-w^2}$. Thus the volume of $B_4(1)$ is

$$\int_{-1}^{1} \operatorname{vol}B_3(\sqrt{1-w^2})dVdw$$

Since we know $\operatorname{vol} B_3(r) = \frac{4\pi}{3}r^3$, we can compute the integral using Gamma and Beta functions.

The ball $B_4(1)$ is the union of 3-dimensional spheres $S_3(r)$ for $0 \le r \le 1$. Thus the volume of $B_4(1)$ is

$$\int_0^1 S_3(r)dr = \int_0^1 \text{vol}S_3(1)r^3dr = \text{vol}S_3(1)/4$$

Meanwhile, the 3-dimensional sphere $S_3(1)$ is the union of product of two circles $S_1(r) \times S_1(r')$ where $r^2 + r'^2 = 1$. Thus the volume of $S_3(1)$ is

$$\int_0^{\pi/2} \text{vol} S_1(r) \text{vol} S_1(r') d\theta = 2\pi^2$$

Problem

Compute the improper integral $\int_{-\infty}^{\infty} e^{-x^2} dx$.

Problem

Find the volume of $B_5(1)$ using

- 1. spherical coordinates on 5-dimensional space.
- 2. Gamma and Beta functions.
- 3. recursive formula on volumme of n-dimensional balls.