SE102:Multivariable Calculus

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Lecture 08
More on Integrals

Definition

The integral $\iint_D f(x, y)dxdy$ is called **improper** if it satisfies one of the following.

- the region D is unbounded, or
- the function diverges at some point in D.

Example

Let $D = \mathbf{R}^2$ be the entire 2-dimensional plane. Let us compute

$$\iint_{\mathbf{R}^2} e^{-x^2-y^2}\, dxdy$$

By polar coordinate $T(r,\theta) = (r\cos\theta, r\sin\theta)$,

$$T^{-1}(D) = [0,\infty) \times [0, 2\pi].$$

Thus by change of coordinates,

$$\iint_{\mathbf{R}^2} e^{-x^2-y^2}\, dxdy = \int_0^\infty \int_0^{2\pi} e^{-r^2} r\, d\theta dr = 2\pi \left(\frac{-1}{2} e^{-r^2}\right)\Big|_0^\infty = \pi$$

Definition

The **gamma function** $\Gamma : \mathbf{R} \to \mathbf{R}$ is defined by

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx$$

Example

1. $\Gamma(n) = (n-1)!$ for $n \geq 1$.
2. $\Gamma(x)$ diverges at each non-positive integer x.
3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Definition

The **beta function** $B(x, y) : \mathbf{R}^2 \to \mathbf{R}$ is defined by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$$

Proposition

1. $B(x, y) = B(y, x)$
2. $B(x, y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

Definition

Let $V = [0,1]^n$ be a n-dimensional cube with side length 1. Then the volume of V is 1.

Remark

Let $dx_1 \wedge \cdots \wedge dx_n$ be a n-form. The definition above can be written as

$$\int_{[0,1]^n} dx_1 \wedge \cdots \wedge dx_n = 1.$$

The volume of n-dimensional region V is defined by

$$\int_V dx_1 \wedge \cdots \wedge dx_n.$$

What does anti-commutivity of the wedge product (i.e. $dx \wedge dy = -dy \wedge dx$) imply?

Definition

The n-dimensional ball of radius r is the set of points in 4-dimensional space defined by

$$B_n(r) = \{(x_1, \cdots, x_n) \mid x_1^2 + \cdots + x_n^2 \leq r^2\}.$$

The $n-1$-dimensional sphere of radius r is the boundary of $B_n(r)$, defined by

$$S_{n-1}(r) = \{(x_1, \cdots, x_n) \mid x_1^2 + \cdots + x_n^2 = 1\}.$$

There are 3 ways to compute the volume of 4-dimensional ball. (The volume of n-dimensional ball and $n-1$-dimensional sphere can be computed similarly.)

1. Using spherical coordinate.
2. Integrating sections.
3. Finding recursive formula.

Proposition

Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a transformation (i.e. one-to-one, differentiable) such that

$$T(u_1, \cdots, u_n) = (x_1, \cdots, x_n).$$

If U be a region in $\mathbf{R}^n$ and $V = T(U)$. Then

$$\int_V dx_1 \wedge \cdots \wedge dx_n = \int_U \frac{\partial(x_1, \cdots, x_n)}{\partial(u_1, \cdots, u_n)} du_1 \wedge \cdots \wedge du_n$$

Example

Let $T : [0, 1] \times [0, \pi] \times [0, \pi] \times [0, 2\pi] \to \mathbf{R}^4$ be a transformation defined by

$$T(r, \theta_1, \theta_2, \phi) = (r \sin\theta_1 \sin\theta_2 \cos\phi, r \sin\theta_1 \sin\theta_2 \sin\phi,$$
$$r \sin\theta_1 \cos\theta_2, r \cos\theta_1)$$

Such T is called a 4-spherical transformation and the Jacobian is

$$J_T = r^3 \sin^2\theta_1 \sin\theta_2$$

Thus the volume of $B_4(1)$ is

$$\int_0^1 \int_0^\pi \int_0^\pi \int_0^{2\pi} r^3 \sin^2\theta_1 \sin\theta_2 d\phi d\theta_2 d\theta_1 dr = \frac{\pi^2}{2}$$

Example

As we slice the 4-dimensional ball $B_4(1)$ at each w-coordinate, we obtain 3-dimensional ball of radius $\sqrt{1-w^2}$. Thus the volume of $B_4(1)$ is

$$\int_{-1}^{1} \mathrm{vol} B_3(\sqrt{1-w^2}) dV dw$$

Since we know $\mathrm{vol} B_3(r) = \frac{4\pi}{3} r^3$, we can compute the integral using Gamma and Beta functions.

Example

The ball $B_4(1)$ is the union of 3-dimensional spheres $S_3(r)$ for $0 \leq r \leq 1$. Thus the volume of $B_4(1)$ is

$$\int_0^1 S_3(r)dr = \int_0^1 \mathrm{vol} S_3(1) r^3 dr = \mathrm{vol} S_3(1)/4$$

Meanwhile, the 3-dimensional sphere $S_3(1)$ is the union of product of two circles $S_1(r) \times S_1(r')$ where $r^2 + r'^2 = 1$. Thus the volume of $S_3(1)$ is

$$\int_0^{\pi/2} \mathrm{vol} S_1(r) \mathrm{vol} S_1(r') d\theta = 2\pi^2$$

Problem

Compute the improper integral $\int_{-\infty}^{\infty} e^{-x^2} dx$.

Problem

Find the volume of $B_5(1)$ using

1. spherical coordinates on 5-dimensional space.
2. Gamma and Beta functions.
3. recursive formula on volunme of n-dimensional balls.