SE102:Multivariable Calculus

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Lecture 04
Minima and Maxima

Theorem (Clairaut)

If the partial derivatives f_{xy}, f_{yx} are continuous at (x_0, y_0), then

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$$

Definition

Suppose that $f(x, y)$ has continuous second partial derivatives at (x_0, y_0). Then the polynomial

$$\begin{aligned} Q(x,y) = {} & f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) \\ & + f_y(x_0,y_0)(y-y_0) + \frac{1}{2} f_{xx}(x_0,y_0)(x-x_0)^2 \\ & + f_{xy}(x_0,y_0)(x-x_0)(y-y_0) \\ & + \frac{1}{2} f_{yy}(x_0,y_0)(y-y_0)^2 \end{aligned}$$

is called the **Taylor polynomial of second degree** of f at (x_0, y_0).

Remark

If all n-th order partial derivatives of a function $f(x, y)$ are continuous, then

- f is differentiable, and
- the formulae of n-order partial derivatives does not depend on the order of partial derivatives.

Remark

Let

$$\Delta\mathbf{x} = (\Delta x, \Delta y) = (x - x_0, y - y_0)$$

and denote the *gradient operator* ∇ in vector notation:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

Let us define the *multiplication* of differential operators as follows:

$$\left(\frac{\partial}{\partial x}\frac{\partial}{\partial x}\right) f = \frac{\partial^2 f}{\partial x^2}, \quad \left(\frac{\partial}{\partial x}\frac{\partial}{\partial y}\right) f = \frac{\partial^2 f}{\partial x \partial y}$$

By the assumption in the definition of Taylor polynomial, we have $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$. Thus we can write

$$Q(x, y) = \sum_{n=0}^{2} \frac{1}{n!} (\Delta \mathbf{x} \cdot \nabla)^n f(x_0, y_0)$$

We can generalize this to the k-th order Taylor polynomial.

$$P_k f(\mathbf{x}) = \sum_{n=0}^{k} \frac{1}{n!} (\Delta \mathbf{x} \cdot \nabla)^n f(\mathbf{x}_0)$$

This is a generalization of the Taylor polynomial for single variable function: since $(\Delta x \cdot \nabla)^n f(x_0) = \Delta x^n \nabla^n f(x_0)$,

$$P_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(x_0) \Delta x^n}{n!}$$

Theorem (Taylor)

Let $f(x, y)$ is a function whose third partial derivatives $f_{xxx}, f_{xyx}, \cdots, f_{yyy}$ are all continuous on a rectangular region

$$D = \{(x, y) \mid |x - x_0|, |y - y_0| \leq \epsilon\}$$

Then for each $(x, y) \in D$, there exists a constant $0 \leq c \leq 1$ satisfying

$$f(x, y) = Q(x, y) + R_2(x, y)$$

where

$$R_2(x, y) = \frac{1}{3!}(\Delta\mathbf{x} \cdot \nabla)^3 f(\mathbf{x}_0 + c\Delta\mathbf{x})$$

Remark

This theorem generalizes the Taylor theorem for single-variable function:

> *Let f be a $\mathcal{C}^{k+1}$-function on an interval $I = (x_0 - \epsilon, x_0 + \epsilon)$. Then for $x, c \in I$, there exists a constant ξ between x and c such that*
>
> $$f(x) = P_k(x) + \frac{f^{(k+1)}(\xi)}{(k+1)!}(x-c)^{k+1}$$

Here $x = x_0 + \Delta x$ and $\xi = x_0 + c\Delta x$ for some $0 \le c \le 1$. Note that the choice of c *depends* on the choice of x, x_0. We can approximate *any* function by $Q_2(x, y)$ if it has partial derivatives upto 3rd order. If (x, y) is sufficiently close to (x_0, y_0), the error $|R_n(x, y)|$ decreases to zero.

Example

Find $Q_2(x, y)$ at $(0, 0)$ for

- $f(x, y) = xy - x^2 - 5y^2 + y - 1$
- $f(x, y) = \cos x \cos y$

and compare the graphs of Q_2 and f near $(1, 0)$

Definition

Let $f(x, y)$ be a function $f(x, y)$ defined on a region D.

- A point (x_0, y_0) is said to be **local maximal (minimal**, respectively) at $\mathbf{x}_0 = (x_0, y_0)$ if there exists a (sufficiently small) $\epsilon > 0$ such that for all $\mathbf{x} = (x, y)$ satisfying $\|\mathbf{x} - \mathbf{x}_0\| < \epsilon$, we have $f(x_0, y_0) \geq f(x, y)$ ($f(x_0, y_0) \leq f(x, y)$, respectively). A local maximal / minimal is often called an **extremal**.
- A point (x_0, y_0) is called a **critical point** if it satisfies one of the following.
  1. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$;
  2. f_x or f_y does not exist at (x_0, y_0);
  3. f is discontinuous at (x_0, y_0).

  A critical point which is *not* an extremal is called a **saddle point**.

Example

Find the critical points of

$$f(x, y) = xy - x^2y - xy^2$$

and classify them. Also, find $Q_2(x, y)$ at each critical points and compare their graphs.

Remark

1. Let $f(x, y)$ is differentiable at (x_0, y_0). Suppose that (x_0, y_0) is a critical point $f(x, y)$ of the type 1 in the definition. Then the linear approximation (x_0, y_0) is the plane $z = f(x_0, y_0)$.
2. Suppose that (x_0, y_0) is a saddle point. Then there exists a curve $c : (-\epsilon, \epsilon) \to \mathbf{R}^2$, $c(0) = (x_0, y_0)$ such that composition $F(t) = (f \circ c)(t)$ is a inflection point.

Definition

Let $R \subset \mathbf{R}^2$ be a domain of $f(x,y)$ and $(x_0, y_0) \in R$. Suppose that all second partial derivatives of $f(x,y)$ are continuous on a region R. Then

$$\Delta_f = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$$

is called the **discriminant** of f.

Example

Graph the following function at $(0,0)$ and compare their discriminants.

$$z = -x^2 - y^2, \quad z = x^2 + y^2, \quad z = x^2 - y^2$$

Theorem (Hesse)

Let (x_0, y_0) be a critical point of the type 1 of $f(x, y)$.

- *If $\Delta > 0$ and $f_{xx}(x_0, y_0) > 0$, then (x_0, y_0) is a local minimum point.*
- *If $\Delta > 0$ and $f_{xx}(x_0, y_0) < 0$, then (x_0, y_0) is a local maximum point.*
- *If $\Delta < 0$, then $f(x_0, y_0)$ is a saddle point.*
- *If $\Delta = 0$, then we cannot determine local extremity by this method.*

Remark

Let $\Delta x = x - x_0, \Delta y = y - y_0$. The degree-2 summands of $Q(x, y)$ can be written as

$$\frac{1}{2} \begin{bmatrix} \Delta x & \Delta y \end{bmatrix} \underbrace{\begin{bmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{xy}(x_0, y_0) & f_{yy}(x_0, y_0) \end{bmatrix}}_{=A} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

Note that $\Delta_f = \det A$. Using linear transformation of x, y, we can coorrespond the matrix A to one of three matrices below, wihout changing the classifications of extremals.

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \pm 1 & 0 \\ 0 & \mp 1 \end{bmatrix}$$

Example (Least square method)

Suppose that a set of data is given by

$$(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$$

We want to find a line $y = mx + b$ which approximates these data. If there are values m_0, b_0 such that the sum of squares of vertical distances between data and the line $y = m_0x + b_0$ is the minimum among all possible lines, then we say that $y = m_0x + b_0$ best approximates the data. In other words, we want to find m, b such that

$$d(m, b) = \sum_{i=1}^{n} (y_i - (mx_i + b))^2$$

is minimum.

Consider $d(m, b)$ as two-variable function on m, b. The critical point is

$$m_0 = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$b_0 = \frac{\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

The Hessian of $d(m, b)$ at (m_0, b_0) is $\begin{bmatrix} 2\sum_{i=1}^{n} x_i^2 & 2\sum_{i=1}^{n} x_i \\ 2\sum_{i=1}^{n} x_i & 2n \end{bmatrix}$

Since the discriminant of f is

$$\Delta_f(m_0, b_0) = 4n \sum_{i=1}^{n} x_i^2 - 4 \left(\sum_{i=1}^{n} x_i \right)^2 > 0$$

The point (m_0, b_0) is a local minimum point. (in fact a global minimum point, why?)

Theorem

Let (x_0, y_0, z_0) be a critical point of $f(x, y, z)$ where f_x, f_y, f_z are all zero. Let H be the 3×3 matrix defined by

$$\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

Let d_1, d_2, d_3 be the determinants of the 1×1, 2×2, 3×3 sub-matrices on the left-top corner of H.

- *If $d_i > 0$ for all i, then (x_0, y_0, z_0) is a local minimum point.*
- *if $d_1, d_3 < 0$ and $d_2 > 0$, then (x_0, y_0, z_0) is a local maximal point.*
- *In all other cases, (x_0, y_0, z_0) is a saddle point.*

Proposition

Let $L_c(f)$ be the level curve at $c = f(x_0, y_0)$. on the xy-plane. Then the gradient vector $\nabla f(x_0, y_0)$ is perpendicular to the curve $L_c(f)$ at (x_0, y_0)

Theorem (Lagrange multiplier)

Let $g(x, y)$, $f(x, y)$ be differentiable functions. Let $L_c(g)$ be a level curve at c. Let us retrict the domain of f onto $L_c(g)$. If (x_0, y_0) is an extremal point of f and $\nabla g(x_0, y_0) \neq \mathbf{0}$, there exists λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

Example

The Lagrange multiplier finds the maxima or minima of a **target** function $f(x, y)$ under the **constraint** $g(x, y) = c$. Find the point on the circle $x^2 + y^2 = 10$ where the function $f(x, y) = 3x + y$ attains maximal or minimal.

Corollary

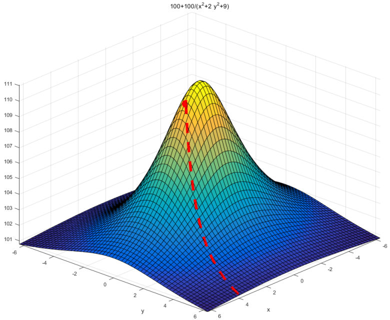
The gradient vector $\nabla f(x_0, y_0)$ has the direction where the value of function $f(x, y)$ increases the most from (x_0, y_0).

Example

Let

$$f(x,y) = 100 + \frac{100}{x^2 + 2y^2 + 9}$$

be a function whose graph represent the contour of a mountain. Suppose that a water flow from the point $(1, 0, 110)$ down the valley in the steepest direction. Find the trajectory of the water path.



Theorem

Let $g(x, y, z)$, $f(x, y, z)$ be differentiable functions. Suppose that (x_0, y_0, z_0) is a local extremal of $f(x, y, z)$ restricted the level set $L_c(g)$. If $\nabla g(x_0, y_0) \neq \mathbf{0}$, then there exists λ such that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

Example

Find the minimal and maximal value of $f(x, y, z) = x^3 + y^3 + z^3$ on the sphere $x^2 + y^2 + z^2 = 1$ on the first octant.

Problem

Find all critical points and classify them

1. $f(x, y) = xy + \dfrac{2}{x} + \dfrac{2}{y}$
2. $e^y(x^2 + y^2 - z^2)$

Problem

Find all local extremes of $f(x, y)$ with the give contraints.

1. $f(x, y) = 2x + y^2 + x^2, 2x^2 + y^2 = 2$
2. $f(x, y, z) = xy + yz, x^2 + y^2 = 1, yz = 1$

Problem

Find the local extremes of $f(x, y) = x^2 + xy + y^2$ on the disk $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$.

Problem

Find the point on the graph $xy^2z^3 = 2$ which is the closest to the origin.