

SE102:Multivariable Calculus

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Lecture 05
Multiple Integrals

Definition (Iterated integrals)

Let $f(x, y)$ be a two variable function define on a rectangular domain $D = [a, b] \times [c, d]$. The **iterated integral**

$\int_c^d \int_a^b f(x, y) dx dy$ on D is defined as follows.

$$\int_c^d \underbrace{\left[\int_a^b f(x, y) dx \right]}_{\text{consider } y \text{ as a constant}} dy.$$

Definition (Double integral)

Let $f(x, y)$ be a function defined on a rectangular region $D = [a, b] \times [c, d]$. Let us subdivide the intervals $[a, b]$ (respectively $[c, d]$) by n (m , respectively) intervals.

$$a = x_0 < x_1 < \cdots < x_n = b, \quad c = y_0 < y_1 < \cdots < y_m = d$$

The region D is subdivided by nm rectangular regions $D_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$. For each $i = 1, \dots, n$ and $j = 1, \dots, m$, let us choose a point $(x_i^*, y_j^*) \in D_{ij}$. Denote $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$. The sum

$$\sum_{i=1}^n \sum_{j=1}^m f(x_i^*, y_j^*) \Delta x_i \Delta y_j$$

is called the **Riemann sum** of $f(x, y)$ with respect to the subdivision D_{ij} 's. If the limit exists when $n, m \rightarrow \infty$, we denote

$$\iint_D f dA = \lim_{n, m \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^m f(x_i^*, y_j^*) \Delta x_i \Delta y_j$$

This is called the **double integral** of f over D .

Theorem

If f is continuous on the region $D = [a, b] \times [c, d]$, then the double integral $\iint_D f dA$ exists.

Example

The function $f(x, y) = \frac{y}{1 + xy}$ is continuous on

$D = [0, 1] \times [0, 1]$. Find the double integral $\iint_D f dx dy$.

Example

Show that the function $f(x, y)$ on $[0, 1] \times [0, 1]$ defined by

$$f(x, y) = \begin{cases} 1 & x \text{ or } y \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$$

is not integrable on $[0, 1] \times [0, 1]$.

Theorem (Fubini I)

Let $f(x, y)$ be a continuous function defined on $D = [a, b] \times [c, d]$. Then

$$\iint_D f dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

Example

Compute

$$\iint_{[0,1] \times [0,1]} \frac{y}{1+xy} dx dy$$

using Fubini's theorem.

Remark

The *continuity* condition in the theorem is crucial. For example, let

$$f(x, y) = \begin{cases} \frac{x^2 - y^2}{(x^2 + y^2)^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Let us compute $\int_0^1 \int_0^1 f(x, y) dy dx$ first.

$$\begin{aligned} \int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dy dx &= \int_0^1 \left. \frac{y}{x^2 + y^2} \right|_{y=0}^1 dx \\ &= \int_0^1 \frac{1}{1 + x^2} dx = \tan^{-1} x \Big|_0^1 = \frac{\pi}{4} \end{aligned}$$

Next, the iterated integral $\int_0^1 \int_0^1 f(x, y) dx dy$ is

$$\begin{aligned} \int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dx dy &= \int_0^1 \left. \frac{-x}{x^2 + y^2} \right|_{x=0}^1 dx \\ &= \int_0^1 \frac{-1}{1 + y^2} dy = -\tan^{-1} y \Big|_0^1 = -\frac{\pi}{4} \end{aligned}$$

and it does not coincide with $\int_0^1 \int_0^1 f(x, y) dy dx$.

Definition

Let $f(x, y)$ be defined on a bounded region D in \mathbf{R}^2 . Suppose that D lies on a large rectangular domain, say $D \subset [a, b] \times [c, d]$. Let us define a new function $F(x, y)$ as follows.

$$F(x, y) = \begin{cases} f(x, y) & (x, y) \in D \\ 0 & (x, y) \notin D \end{cases}$$

Then the **definite integral** of f over the domain D is defined as

$$\iint_D f(x, y) dx dy = \iint_{[a, b] \times [c, d]} F(x, y) dx dy$$

Theorem (Fubini II)

Let $f(x, y)$ be a continuous function defined on D . If $D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$, then

$$\iint_D f(x, y) dx dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

Similarly, if $D = \{(x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$, then

$$\iint_D f(x, y) dx dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

Example

Let us compute $\iint_D e^{-y^2} dx dy$ where D is the triangular region whose vertices are $(0, 0)$, $(0, 1)$, and $(1, 1)$. By Fubini's theorem,

$$\iint_D e^{-y^2} dx dy = \int_0^1 \left[\int_x^1 e^{-y^2} dy \right] dx = \int_0^1 \left[\int_0^y e^{-y^2} dx \right] dy.$$

Find which integration works.

Definition

Let $f(x, y, z)$ be a function defined on the boxed domain $D = [a, b] \times [c, d] \times [e, f]$. The **triple integral** of $f(x, y, z)$ over D is denoted by

$$\iiint_{[a,b] \times [c,d] \times [e,f]} f(x, y, z) dx dy dz$$

If f is defined on a region $V \subset [a, b] \times [c, d] \times [e, f]$, Then define

$$F(x, y, z) = \begin{cases} f(x, y, z) & \text{if } (x, y, z) \in V \\ 0 & \text{otherwise} \end{cases}$$

Then the triple integral is defined as

$$\iiint_V f(x, y, z) dx dy dz = \iiint_{[a,b] \times [c,d] \times [e,f]} F(x, y, z) dx dy dz$$

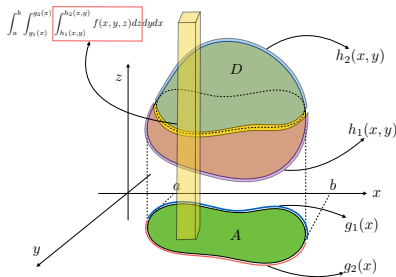
Theorem (Fubini III)

Let $f(x, y, z)$ be a continuous function defined on the region V .

$$V = \{(x, y, z) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x), h_1(x, y) \leq z \leq h_2(x, y)\}$$

Then the following holds.

$$\iiint_V f(x, y, z) dx dy dz = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x, y)}^{h_2(x, y)} f(x, y, z) dz dy dx$$



Example

Let V be a parallelepiped region bounded by 6 planes : $2x = y$, $2x = y + 2$, $y = 0$, $y = 4$, $z = 0$, $z = 3$. Compute

$$\iiint_V \frac{2x - y}{2} + \frac{z}{3} dx dy dz$$

Definition

Given an interval $I \subset \mathbf{R}$, a curve C parametrized by $c : I \rightarrow \mathbf{R}^n$

$$c(t) = (x_1(t), x_2(t), \dots, x_n(t))$$

is **piecewise differentiable** if all coordinate function $x_i(t)$ are C^n on the interval I except for finitely many points.

Definition

Let C be a piecewise differentiable curve on \mathbf{R}^2 parametrized by $c : [a, b] \rightarrow \mathbf{R}^2$. Let $c'(t) = (x'(t), y'(t))$ be the velocity vector at the point $c(t)$. Let $f(x, y)$ be a function defined on the curve C . Then the **line integral** of $f(x, y)$ along C is defined as

$$\int_C f ds = \int_a^b f(c(t)) |c'(t)| dt.$$

Proposition

The line integral $\int_C f ds$ does not depend on the parametrization of C .

Proof.

Let $c : [a, b] \rightarrow \mathbf{R}^2$ be a parametrization of C . Let $h : [c, d] \rightarrow [a, b]$ be a one-to-one correspondence which gives a re-parametrization $c \circ h$ of C . Let us write $t = h(\tau)$. Then

$$\begin{aligned} \int_c^d f(c \circ h(\tau)) \cdot |(c \circ h)'(\tau)| d\tau &= \int_c^d f(c(t)) \cdot |c'(t)| \cdot |h'(\tau)| d\tau \\ &= \int_a^b f(c(t)) \cdot |c'(t)| dt \end{aligned}$$



Example

Let us compute the line integral

$$\int_C (2 + x^2 y) ds$$

where C is the unit circle centered at the origin with counter clockwise orientation.

Definition

Let D be a region in \mathbf{R}^2 and S a surface in \mathbf{R}^3 . A map $X : D \rightarrow S$

$$X(u, v) = (x(u, v), y(u, v), z(u, v))$$

is called the **parametrization** of S if it is one-to-one correspondence and every partial derivative of x, y, z is continuous on D .

Example

Find a parametrization of $x^2 + y^2 - z^2 = 1$.

Definition

Let $f(x, y, z)$ be a function defined on a surface S . Let $X : D \rightarrow S$ be a parametrization of S . The **surface integral** of f on S is defined by

$$\iint_S f dS = \iint_D (f \circ X)(u, v) \|X_u \times X_v\| du dv \quad (1)$$

Example

Let S be the surface defined by the graph of $z = \sqrt{x^2 + y^2}$ over the disk $x^2 + y^2 \leq 1$. Evaluate

$$\iint_S z dS$$

Problem

For $f(x, y) = x^2 - y^2$, compute $\iint_S x + z dS$.

Problem

Let $f(x, y)$ is a density at the point (x, y) on domain D . Let

$$\bar{x} = \frac{\iint_D x f(x, y) dA}{\iint_D f(x, y) dA}, \quad \bar{y} = \frac{\iint_D y f(x, y) dA}{\iint_D f(x, y) dA}$$

Explain why (\bar{x}, \bar{y}) is the center of mass of D .

Problem

Let $V = \{0 \leq x \leq 2, 0 \leq y \leq x, 0 \leq z \leq y\}$. Write $\iiint_V f(x, y, z)$ in six different iterated integrals.

Problem

Let $f(x, y)$ be a function defined on a curve C . Explain the meaning of the line integral $\int_C f ds$ in the following senses:

1. when $f(x, y)$ is a density at (x, y) on the curve C ;
2. the section of the graph $z = f(x, y)$ over the curve C .