

# SE102:Multivariable Calculus

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## Definition

A ( $n$ -dimensional) **vector** is a  $n$ -tuple of real numbers

$$\mathbf{a} = (a_1, a_2, \dots, a_n)$$

with two operations.

- ▶ (vector sum) For a vector  $\mathbf{b} = (b_1, b_2, \dots, b_n)$ ,

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n).$$

- ▶ (scalar multiplication) For  $k \in \mathbf{R}$ , we define a vector  $k\mathbf{a}$  as

$$k\mathbf{a} = (ka_1, ka_2, \dots, ka_n).$$

The ( $n$ -dimensional) **vector space** is the set of all ( $n$ -dimensional) vectors, and we denote it as  $\mathbf{R}^n$ .

## Example

Let  $O = (0, 0, 0)$  be the origin and  $P = (a_1, a_2, a_3)$  a point in 3-dimensional space. A **position vector** from  $O$  to  $P$  is a vector

$$\overrightarrow{OP} = (a_1, a_2, a_3).$$

The scalar multiplication of a position vector is a dilation. The vector sum of two position vectors is a superposition.

Let  $Q = (b_1, b_2, b_3)$  and  $R = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$ . the vector  $\overrightarrow{PR}$  is the same as  $\overrightarrow{OQ}$ . We have the following additive operation.

$$\overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR}$$

## Example

A parametrization of a curve is a set of functions defined on a common interval, each indicates a coordinate of a point on the curve. Let  $c : (-\varepsilon, \varepsilon) \rightarrow \mathbf{R}^2$  be a parametrization of a curve on a plane.

$$c(t) = (x(t), y(t))$$

The differential  $c'(t) = (x'(t), y'(t))$  is a vector which represents the **velocity** of the parametrization at  $t$ . The  $x, y$ -components of the vector  $c'(t)$  represent the projection of the speed of  $c(t)$  in  $x, y$ -directions respectively. We can decompose the velocity vector into the sum of horizontal and vertical velocity of  $c(t)$ .

$$c'(t) = (x'(t), 0) + (0, y'(t))$$

## Example

Let  $\mathbb{R}^n$  be the set of all  $n$ -dimensional space. Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be a function:

$$f(u, v) = (x(u, v), y(u, v), z(u, v)) \quad (1)$$

The variables  $x, y, z$  are dependent to the variables  $u, v$ . The functions  $f$  (and also  $x, y, z$  as functions) is called a **multivariable function** since it contains two or more independent or dependent variables. If we write  $\mathbf{a} = (u, v)$  and  $\mathbf{b} = (x, y, z)$ , the equation (1) can be written as  $f(\mathbf{a}) = \mathbf{b}$ .

## Definition

Let  $\mathbf{a} = (a_1, a_2, \dots, a_n)$  be a ( $n$ -dimensional) vector. The **norm** of  $\mathbf{a}$  is the value

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}.$$

We call a vector  $\mathbf{a}$  a unit vector if  $\|\mathbf{a}\| = 1$ . The **zero vector**  $\mathbf{0}$  is the vector satisfying

$$\mathbf{0} + \mathbf{a} = \mathbf{a} + \mathbf{0} = \mathbf{a}.$$

for any vector  $\mathbf{a}$ . Note that  $\|\mathbf{a}\| = 0$  if and only if  $\mathbf{a} = \mathbf{0}$ . The **normalization** of  $\mathbf{a}$  is the vector

$$\mathbf{u} = \mathbf{a} / \|\mathbf{a}\|$$

## Example

For each  $i = 1, \dots, n$ , the (unit) **basis vector** is

$$\mathbf{e}_i = (0, \dots, 1, \dots, 0).$$

(1 is at the  $i$  th place.) In particular, we denote 3-dimensional basis vectors as

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1)$$

We can decompose any vector  $\mathbf{a} = (a_1, a_2, \dots, a_n)$  as a linear sum of basis vectors:

$$\mathbf{a} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + \dots + a_n \mathbf{e}_n$$

Thus, we call the set  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$  as a **basis** of  $\mathbf{R}^n$ .

## Definition

The **inner product** (also called **dot product**) of two vectors  $\mathbf{a}$ ,  $\mathbf{b}$  is an operation defined by

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

If  $\mathbf{a} \cdot \mathbf{b} = 0$ , then we say  $\mathbf{a}$ ,  $\mathbf{b}$  are **orthogonal**.

## Proposition

*The inner product satisfies the following.*

1.  $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
2.  $\mathbf{a} \cdot (k\mathbf{b}) = k(\mathbf{a} \cdot \mathbf{b})$
3.  $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$



## Example

Let us parametrize a line  $l$  in  $\mathbf{R}^3$ .

1. Suppose that  $l$  passes through  $P = (x_0, y_0, z_0)$  and parallel to  $\mathbf{a} = (a_1, a_2, a_3)$ . Then the parametric equations of  $l$  is

$$l(t) = P + t\mathbf{a} \quad (2)$$

Equivalently, equation (2) can be written as a *symmetric form* as follows.

$$\frac{x - x_0}{a_1} = \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3} \quad (3)$$

2. Suppose that  $l$  passes through two points  $P, Q$ . By substituting  $\mathbf{a} = \overrightarrow{OP} - \overrightarrow{OQ}$  to (2) or (3), we get a parametric equation of  $l$ .

## Theorem

Let  $\mathbf{a}$ ,  $\mathbf{b}$  be nonzero 2-dimensional vectors. If  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$ , then

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot \cos \theta$$

## Definition

For nonzero vectors  $\mathbf{a}$  and  $\mathbf{b}$  with the same dimension, the vector defined by

$$\text{proj}_{\mathbf{b}} \mathbf{a} = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2} \right) \mathbf{b}$$

is called the **projection of  $\mathbf{a}$  onto  $\mathbf{b}$** .

## Remark

As norm measures the *size* of a vector, the inner product measure the *direction*. For example, the direction of a 2-dimensional vector  $\mathbf{a}$  is determined by  $0 \leq \theta_1, \theta_2 \leq \pi$  satisfying

$$\mathbf{a} \cdot \mathbf{e}_1 = \|\mathbf{a}\| \cos \theta_1$$

$$\mathbf{a} \cdot \mathbf{e}_2 = \|\mathbf{a}\| \cos \theta_2$$

In order to determine the direction of a 3-dimensional vector, say  $\mathbf{a}$ , we need three angles  $0 \leq \alpha, \beta, \gamma \leq \pi$  satisfying

$$\cos \alpha = \frac{\mathbf{a} \cdot \mathbf{i}}{\|\mathbf{a}\|}, \quad \cos \beta = \frac{\mathbf{a} \cdot \mathbf{j}}{\|\mathbf{a}\|}, \quad \cos \gamma = \frac{\mathbf{a} \cdot \mathbf{k}}{\|\mathbf{a}\|} \quad (4)$$

Such quantities are called the **direction cosines**.

## Definition

A  $n \times m$  ( $n$ -by- $m$ ) **matrix** is a collection of  $nm$  numbers (or functions) arranged in the following way.

$$A = (a_{ij})_{n \times m} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

The indices  $i, j$  of an entry  $a_{ij}$  represents the row and column indices respectively.

## Example

1. A  $n \times m$  matrix is called **square** matrix if  $n = m$ .
2. If  $A$  is a square matrix and  $a_{ij} = 0$  for all  $i \neq j$ , then  $A$  is called **diagonal**.

$$A = \begin{pmatrix} a_{11} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & a_{nn} \end{pmatrix}$$

3. If the diagonal entries of a diagonal matrix are all 1, then it is called the **identity matrix**, and denoted by  $I_n$ .

$$I_n = \begin{pmatrix} 1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & 1 \end{pmatrix}$$

4. If  $A$  is a square matrix and  $a_{ij} = 0$  for all  $i < j$  ( $i > j$ , respectively), then  $A$  is called **lower triangle** (**upper**).

## Definition

Let  $A = (a_{ij})$  and  $B = (b_{ij})$  be  $n \times m$  matrix. Then we define

$$A + B = (a_{ij} + b_{ij})$$

$$k \cdot A = (ka_{ij})$$

Let  $C$  be a  $m \times l$  matrices, then  $A \cdot C$  is a  $n \times l$  matrix whose entries are

$$\begin{aligned} A \cdot B &= (a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj})_{1 \leq i \leq n, 1 \leq j \leq l} \\ &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{im} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} b_{11} \cdots b_{1j} \cdots b_{1l} \\ b_{21} \cdots b_{2j} \cdots b_{2l} \\ \vdots & \vdots & \vdots \\ b_{m1} \cdots b_{mj} \cdots b_{ml} \end{pmatrix} \end{aligned}$$

## Example

Suppose Bob, Larry, and Joanna worked in a fruits store for three days. Table 1 shows *how many* fruits each sold in total represented by the matrix  $A$ , and Table 2 shows *how much* was the fruits on each day represented by the matrix  $B$ .

	Apple	Orange	Banana
Bob	38	25	10
Larry	15	22	15
Joanna	8	70	27

Table: The column of sales of each person per items

	Day1	Day2	Day3
Apple	\$1.19	\$1.45	\$0.99
Orange	\$1.70	\$0.99	\$2.1
Banana	\$2.19	\$3.5	\$1.29

Table: The prices of items per day

## Example

The  $i, j$ -entries of the  $A \cdot B$  represents the total revenue sold by the person  $i$  at the day  $j$ .

$$\begin{pmatrix} 38 & 25 & 10 \\ 15 & 22 & 15 \\ 8 & 70 & 27 \end{pmatrix} \begin{pmatrix} 1.19 & 1.45 & 0.99 \\ 1.70 & 0.99 & 2.1 \\ 2.19 & 3.5 & 1.29 \end{pmatrix}$$

A deeper, and more mathematical, thus important meaning of matrix multiplication will be discovered when we visit in the *chain rule* in later sections.



## Proposition

*Let  $A, B, C$  be matrices. Whenever the operations are valid, the following holds.*

1.  $(A \cdot B) \cdot C = A \cdot (B \cdot C)$
2.  $A \cdot (B + C) = A \cdot B + A \cdot C$
3.  $(B + C) \cdot A = B \cdot A + C \cdot A$
4.  $k \cdot (A \cdot B) = (k \cdot A) \cdot B = A \cdot (k \cdot B)$

## Proposition

The **transpose** of a matrix  $A = (a_{ij})$  defined by

$$A^T = (a_{ji})$$

and it satisfies the following.

5.  $(A^T)^T = A$
6.  $k \cdot A^T = (k \cdot A)^T$
7.  $(A + B)^T = A^T + B^T$
8.  $(AB)^T = B^T A^T$

## Definition

The **determinant** of  $2 \times 2$  matrix  $A$  is defined as follows.

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

The **determinant** of  $3 \times 3$  matrix  $B$  is defined as follows.

$$\begin{aligned} \det B = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} &= b_{11}b_{22}b_{33} + b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32} \\ &\quad - b_{13}b_{22}b_{31} - b_{11}b_{23}b_{32} - b_{12}b_{21}b_{33} \end{aligned}$$

## Definition

Let  $\mathbf{a} = (a_1, a_2, a_3)$ ,  $\mathbf{b} = (b_1, b_2, b_3)$  be two 3-dimensional vectors. The **cross-product** of  $\mathbf{a}, \mathbf{b}$  is a vector defined by

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

An easy way to remember the formula is the following.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

## Proposition

Let  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  be 3-dimensional vectors and  $k$  a constant. The following identity holds.

1.  $\mathbf{a} \times \mathbf{0} = \mathbf{0} \times \mathbf{a} = \mathbf{0}$
2.  $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
3.  $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
4.  $\mathbf{a} \times (k\mathbf{b}) = (k\mathbf{a}) \times \mathbf{b} = k(\mathbf{a} \times \mathbf{b})$
5.  $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
6.  $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$  (This shows that the cross product  $\mathbf{a} \times \mathbf{b}$  is normal to the plane spanned by  $\mathbf{a}$  and  $\mathbf{b}$ .)
7.  $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$

## Theorem

Let  $\mathbf{a}, \mathbf{b}$  be two 3-dimensional vectors. Then

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot |\sin \theta|$$

where  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$ .

## Proposition

*Let us denote  $|A|$  by the determinant of a matrix  $A$ . Then the following holds*

- 1. If  $A$  has a row (or a column) whose entries are all zero, then  $|A| = 0$ .*
- 2. Let  $B$  be the matrix obtained by interchanging two rows (or columns) of  $A$ . Then  $|B| = -|A|$ .*
- 3. Let  $B$  be a matrix obtained by multiply  $c$  on a row (or column) followed by adding it to another row (or column). Then  $|B| = |A|$ .*

## Definition

Let  $A$  be a  $n \times n$  matrix. A matrix  $B$  satisfying

$$A \cdot B = B \cdot A = I_n$$

is called the **inverse of**  $A$ , denoted by  $B = A^{-1}$ . If an inverse matrix  $A^{-1}$  exists, then  $A$  is said to be **non-singular**. Otherwise, it is called **singular**.

## Theorem

*A matrix  $A$  is singular if and only if  $\det A = 0$ .*



## Proposition

If  $A$  is a  $2 \times 2$  matrix, then  $A^{-1}$  is

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix}$$

For a  $3 \times 3$ -matrix  $B$ , the inverse is given by

$$B^{-1} = \frac{1}{\det B} \begin{pmatrix} c_{11} & -c_{21} & c_{31} \\ -c_{12} & c_{22} & -c_{32} \\ c_{13} & -c_{23} & c_{33} \end{pmatrix} \quad (5)$$

where each  $c_{ij}$ , called the **cofactor**, is the determinant of  $2 \times 2$ -matrix obtained by deleting  $i$ th row and  $j$ th column. For example,

$$c_{21} = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ \cancel{b_{21}} & \cancel{b_{22}} & \cancel{b_{23}} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} = \begin{vmatrix} b_{12} & b_{13} \\ b_{32} & b_{33} \end{vmatrix}$$

Notice that the row and column indices are switched in (5).

## Definition

A **vector space**  $V$  is a set of element called **vectors** satisfying the following properties:

1. (Zero vector)  $V$  contains the **zero vector**  $\mathbf{0}$ , which is a unique vector satisfying  $\mathbf{0} + \mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v}$  for all  $\mathbf{v} \in V$ .
2. (Vector sum) For any two vectors  $\mathbf{v}, \mathbf{w} \in V$ , the vector  $\mathbf{v} + \mathbf{w}$  lies in  $V$ .
3. (Scalar multiplication) For any  $k \in \mathbf{R}$  and  $v \in V$ , the vector  $k\mathbf{v}$  lies in  $V$ .

A subset  $V \subset \mathbf{R}^n$  is called a **vector subspace** if it is a vector space itself.

## Example

Let  $\mathcal{C}^1(\mathbf{R})$  be the set of all differentiable functions on  $\mathbf{R}$  whose derivatives are continuous on  $\mathbf{R}$ . Since  $f, g$  are such functions so is the function  $h = f + g$ . Also for any  $k \in \mathbf{R}$ , the function  $kf$  is differentiable and its derivative is continuous. Thus  $\mathcal{C}^1(\mathbf{R})$  is a vector space. Likewise, we can define vector spaces  $\mathcal{C}^n(\mathbf{R})$ ,  $\mathcal{C}^\infty(\mathbf{R})$

## Example

For each constant  $k \in \mathbf{R}$ , let  $V_k$  be the set of all points on the line  $y = kx$  in  $\mathbf{R}^2$ :

$$V_k = \{(x, y) \mid y = kx\}$$

Then  $V_k$  is a vector subspace of  $\mathbf{R}^2$ . Let  $V_\infty$  be the vertical line  $V_\infty = \{(0, y) \mid y \in \mathbf{R}\}$ . Then  $V_\infty$  is also a vector subspace of  $\mathbf{R}^2$

## Example

Let  $P$  be the set of all point on the plane

$$P = \{(x, y, z) \mid ax + by + cz = 0\}$$

in  $\mathbf{R}^3$ . By identifying points in  $P$  as position vectors, we can say  $P$  is a vector subspace of  $\mathbf{R}^3$ , orthogonal to  $\mathbf{n} = (a, b, c)$ . In particular, let  $V, W$  be vector subspace of  $\mathbf{R}^n$ . Then so is  $V \cap W$ . For example, let  $V, W$  be vector subspace for two planes in  $\mathbf{R}^3$  passing through the origin  $\mathbf{0}$ . Then  $V \cap W$  is either a line (if  $V, W$  are transversal) or a plane (if  $V = W$ ).

## Definition

Let  $V$  a vector space and  $\mathbf{v}_1, \dots, \mathbf{v}_m \in V$ . We say the vector space subspace

$$W = \{a_1\mathbf{v}_1 + \dots + a_m\mathbf{v}_m \mid a_1, \dots, a_m \in \mathbf{R}\}$$

is **spanned** by  $\mathbf{v}_1, \dots, \mathbf{v}_m$ , and denote by  $W = \text{span}\langle \mathbf{v}_1, \dots, \mathbf{v}_m \rangle$ .

## Definition

We say a vector space  $V$  is **spanned by** the set of vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ , if every  $v \in V$  can be expressed a linear combination of  $v_i$ 's:

$$v = a_1 \mathbf{v}_1 + \dots a_n \mathbf{v}_n$$

The set  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is said to be **linearly independent** if the coefficients satisfying

$$a_1 \mathbf{v}_1 + \dots a_n \mathbf{v}_n = \mathbf{0}$$

is the trivial ones, namely  $a_1 = \dots = a_n = 0$ .

## Definition

Let  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  be the set of linear independent vectors which spans the vector space  $V$ . Such set is called the **basis** of  $V$ , and its element is called the **basis vector**. The **dimension** of  $V$  is the number of basis vectors which spans  $V$ .

## Problem

1. Show that the vectors  $\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b}$  and  $\|\mathbf{b}\|\mathbf{a} - \|\mathbf{a}\|\mathbf{b}$  are orthogonal.
2. Show that  $\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b}$  bisects the angle between  $\mathbf{a}$  and  $\mathbf{b}$ .



## Problem

1. Let  $\mathbf{a} = (1, 2, 1)$ ,  $\mathbf{b} = (2, 1, 2)$ , and  $\mathbf{u} = (0, 1, -1)$ . Suppose that the vector  $\mathbf{u}$  can be decomposed by

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$$

where  $\mathbf{u}_1$  is parallel to  $\mathbf{a}$ ,  $\mathbf{u}_2$  is parallel to  $\mathbf{b}$ , and  $\mathbf{u}_3$  is orthogonal to both  $\mathbf{a}$  and  $\mathbf{b}$ . Find the vector  $\mathbf{u}_1$ ,  $\mathbf{u}_2$ ,  $\mathbf{u}_3$  explicitly.

## Problem

1. Find the distance between  $P = (2, 1, 3)$  and the line  $l(t) = (2, 3, -2) + t(-1, 1, -2)$ .
2. Find the distance between two parallel planes

$$2x - 2y + z = 5, \quad 2x - 2y + z = 20$$

3. Find the distance between two skew lines

$$l_1(t) = (0, 5, -1) + t(2, 1, 3)$$

$$l_2(t) = (-1, 2, 0) + t(1, -1, 0)$$

## Problem

Let  $\mathbf{a} = (a_1, a_2, a_3)$ ,  $\mathbf{b} = (b_1, b_2, b_3)$ ,  $\mathbf{c} = (c_1, c_2, c_3)$ . Verify that

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

## Problem

Imagine two concentric circles with radius  $a, b$  ( $b < a$ ) which rolls on flat line with the same angular velocity. A curtate cycloid is a trajectory of a point on the circle of radius  $b$ . Find a set of parametric equation for the curtate cycloid with  $a = 3$ ,  $b = 2$ .