SE102:Multivariable Calculus

Hyosang Kang[1]

[1]Division of Mathematics
School of Interdisciplinary Studies
DGIST

Lecture 08
Manifolds

Definition

The integral $\iint_D f(x,y)dxdy$ is called **improper** if it satisfies one of the following.

- the region D is unbounded, or
- the function diverges at some point in D.

Example

Let $D = \mathbf{R}^2$ be the entire 2-dimensional plane. Let us compute

$$\iint_{\mathbf{R}^2} e^{-x^2-y^2}\,dxdy$$

By polar coordinate $T(r,\theta) = (r\cos\theta, r\sin\theta)$,

$$T^{-1}(D) = [0,\infty) \times [0, 2\pi].$$

Thus by change of coordinates,

$$\iint_{\mathbf{R}^2} e^{-x^2-y^2}\,dxdy = \int_0^\infty \int_0^{2\pi} e^{-r^2} r\,d\theta dr = 2\pi \left(\frac{-1}{2} e^{-r^2} \right) \Big|_0^\infty = \pi$$

Definition

The **gamma function** $\Gamma : \mathbf{R} \to \mathbf{R}$ is defined by

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx$$

Example

1. $\Gamma(n) = (n-1)!$ for $n \geq 1$.
2. $\Gamma(x)$ diverges at each non-positive integer x.
3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Definition

The **beta function** $B(x, y) : \mathbf{R}^2 \to \mathbf{R}$ is defined by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$$

Proposition

1. $B(x, y) = B(y, x)$
2. $B(x, y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

Definition

The n-dimensional ball of radius r is the set of points in 4-dimensional space defined by

$$B_n(r) = \{(x_1, \cdots, x_n) \mid x_1^2 + \cdots + x_n^2 \leq r^2\}.$$

The $n-1$-dimensional sphere of radius r is the boundary of $B_n(r)$, defined by

$$S_{n-1}(r) = \{(x_1, \cdots, x_n) \mid x_1^2 + \cdots + x_n^2 = 1\}.$$

Proposition

Let $T : \mathbf{R}^n \to \mathbf{R}^n$ *be a transformation (i.e. one-to-one, differentiable) such that*

$$T(u_1, \cdots, u_n) = (x_1, \cdots, x_n).$$

If U *be a region in* $\mathbf{R}^n$ *and* $V = T(U)$. *Then*

$$\int_V dx_1 \cdots dx_n = \int_U \left| \det \frac{\partial(x_1, \cdots, x_n)}{\partial(u_1, \cdots, u_n)} \right| du_1 \cdots du_n$$

We define the **volume** of the cube $[0, 1]^n$ to be 1. There are 3 ways to compute the volume of 4-dimensional ball.

1. Using spherical coordinate.
2. Integrating sections.
3. Finding recursive formula.

Example

Let $T : [0,1] \times [0,\pi] \times [0,\pi] \times [0,2\pi] \to \mathbf{R}^4$ be a transformation defined by

$$T(r,\theta_1,\theta_2,\phi) = (r\sin\theta_1\sin\theta_2\cos\phi, r\sin\theta_1\sin\theta_2\sin\phi,$$
$$r\sin\theta_1\cos\theta_2, r\cos\theta_1)$$

Such T is called a 4-spherical transformation and the Jacobian is

$$J_T = r^3\sin^2\theta_1\sin\theta_2$$

Thus the volume of $B_4(1)$ is

$$\int_0^1\int_0^\pi\int_0^\pi\int_0^{2\pi} r^3\sin^2\theta_1\sin\theta_2 d\phi d\theta_2 d\theta_1 dr = \frac{\pi^2}{2}$$

Example

As we slice the 4-dimensional ball $B_4(1)$ at each w-coordinate, we obtain 3-dimensional ball of radius $\sqrt{1-w^2}$. Thus the volume of $B_4(1)$ is

$$\int_{-1}^{1} \mathrm{vol} B_3(\sqrt{1-w^2}) dV dw$$

Since we know $\mathrm{vol} B_3(r) = \dfrac{4\pi}{3} r^3$, we can compute the integral using Gamma and Beta functions.

Example

The ball $B_4(1)$ is the union of 3-dimensional spheres $S_3(r)$ for $0 \leq r \leq 1$. Thus the volume of $B_4(1)$ is

$$\int_0^1 S_3(r)dr = \int_0^1 \mathrm{vol}S_3(1)r^3 dr = \mathrm{vol}S_3(1)/4$$

Meanwhile, the 3-dimensional sphere $S_3(1)$ is the union of product of two circles $S_1(r) \times S_1(r')$ where $r^2 + r'^2 = 1$. Thus the volume of $S_3(1)$ is

$$\int_0^{\pi/2} \mathrm{vol}S_1(r)\mathrm{vol}S_1(r')d\theta = 2\pi^2$$

Multivariable Calculus summerizes in two sentences:

- Derivative is a linear transformation.
  - Derivative $Df(\mathbf{a})$ of a multivariable function is a linear map between tangent spaces at $\mathbf{a}$ and $f(\mathbf{a})$.
- Divergence theorem is a Stokes' theorem.
  - The general form of Stokes' theorem is

$$\int_V d\omega = \int_{\partial V} \omega$$

where ω is a differential $(k-1)$-form and V is a k-dimensional space. The differential $d\omega$ is a k-form. The Stokes theorem is when

$$\omega = Pdx + Qdy = Rdz$$

and divergence theorem is when

$$\omega = Pdx \wedge dy + Qdy \wedge dz + Rdz \wedge dx.$$

Definition

Let $\mathcal{C}^\infty(\mathbf{R}^n)$ be the set of all $\mathcal{C}^\infty$-class real-valued functions defined on $\mathbf{R}^n$. A tangent vector $\mathbf{v}$ at $\mathbf{p} \in \mathbf{R}^n$ is a map $\mathcal{C}^\infty(\mathbf{R}^n) \to \mathbf{R}$ such that there is a parametric curve $\mathbf{c}(t)$ of $\mathcal{C}^\infty$-class that satisfies

- $\mathbf{c}(0) = \mathbf{p}$, and
- $\mathbf{v}(f) = (f \circ \mathbf{c})'(0)$.

Remark

A tangent vector at $\mathbf{p}$ in $\mathbf{R}^n$ is usually denoted as

$$\mathbf{v} = a_0 \left.\frac{\partial}{\partial x^0}\right|_{\mathbf{p}} + \cdots + a_{n-1} \left.\frac{\partial}{\partial x^{n-1}}\right|_{\mathbf{p}}$$

for real coefficients $a_0, \ldots, a_{n-1}$.

Definition

A n-dimensional **vector field F** is a map assigns a tangent vector at $\mathbf{p}$ to each point $\mathbf{p}$ in $\mathbf{R}^n$. We can write $\mathbf{F}$ as

$$\mathbf{F}(\mathbf{p}) = a_0(\mathbf{p})\frac{\partial}{\partial x^0} + \cdots + a_{n-1}(\mathbf{p})\frac{\partial}{\partial x^{n-1}}$$

for real-valued function $\mathbf{a}_i$. We say $\mathbf{F}$ is of class $\mathcal{C}^\infty$ if all coefficient functions a_i are of class $\mathcal{C}^\infty$.

Remark

The set of all tangent vectors at $\mathbf{p}$ is denoted by $T_{\mathbf{p}}\mathbf{R}^n$, and it is a n-dimensional vector space. A $\mathcal{C}^\infty$-vector field $\mathbf{F}$ is a map that sends $\mathcal{C}^\infty$-functions to $\mathcal{C}^\infty$-functions. The set of all $\mathcal{C}^\infty$-vector fields on $\mathbf{R}^n$ is denoted by $\mathfrak{X}(\mathbf{R}^n)$. As a vector space $\mathfrak{X}(\mathbf{R}^n)$ is generated by $\dfrac{\partial}{\partial x^j}$, $j = 0, \ldots, n-1$.

Definition

Let V be a n-dimensional vector space. The **dual space** V^* of V is the vector space defined by

$$V^* = \{f : V \to \mathbf{R} \,|\, f \text{ is linear}\}.$$

Let $e_0, \ldots, e_{n-1}$ be a basis for V. The linear map $e_i^* : V \to \mathbf{R}$ defined by

$$e_i^*(e_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

is called a dual vector to e_i.

Definition

Let dx^j be the dual vector to the vector field $\dfrac{\partial}{\partial x^j}$. The n-dimensional 1-**form** ω is a linear combination of dx^j with coefficients in $\mathcal{C}^\infty(\mathbf{R}^n)$.

$$\omega = f_0 dx^0 + \cdots + f_n dx^{n-1}.$$

Definition

The **wedge** product between two n-dimensional 1-forms dx^i and dx^j, denoted by $dx^i \wedge dx^j$ satisfies

- $dx^i \wedge dx^j = -dx^j \wedge dx^i$;
- $dx^i \wedge dx^i = 0$.

A k**-form** is a linear combination of wedge products on k 1-forms. The n-dimensional n-form is called the **volume form**.

Remark

The integration of n-form $dx_1 \wedge \cdots \wedge dx_n$ on the cube $[0,1]^n$ is defined by

$$\int_{[0,1]^n} dx_1 \wedge \cdots \wedge dx_n = 1.$$

The volume of n-dimensional region $V \subset \mathbf{R}^n$ is defined by

$$\int_V dx_1 \wedge \cdots \wedge dx_n.$$

Anti-commutivity of the wedge product ($dx \wedge dy = -dy \wedge dx$) implies that the integration of n-form is a generalization of surface (and line) integrals of vector fields.

Definition

Let $\mathcal{T}^k(\mathbf{R}^n)$ be the set of all k-forms on $\mathbf{R}^n$. The d-operator is a linear map $d : \mathcal{T}^k(\mathbf{R}^n) \to \mathcal{T}^{k+1}(\mathbf{R}^n)$ defined by

$$d(f dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \sum_{j=0}^{n-1} \frac{\partial f}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge d^{i_k}.$$

Theorem (Stokes)

Let ω be a $k-1$-form defined a bounded region $V \subset \mathbf{R}^n$. Then

$$\int_V d\omega = \int_{\partial V} \omega.$$

Problem

Compute the improper integral $\int_{-\infty}^{\infty} e^{-x^2}\, dx$.

Problem

Find the volume of $B_5(1)$ using

1. spherical coordinates on 5-dimensional space.
2. Gamma and Beta functions.
3. recursive formula on volunme of n-dimensional balls.