SE102:Multivariable Calculus

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Week 09

Definition (Iterated integrals)

Let $f(x,y)$ be a two variable function define on a rectangular domain $D = [a,b] \times [c,d]$. The **iterated integral** $\int_c^d \int_a^b f(x,y)dxdy$ on D is defined as follows.

$$\int_c^d \underbrace{\left[\int_a^b f(x,y)dx\right]}_{\text{consider } y \text{ as a constant}} dy.$$

Definition (Double integral)

Let $f(x,y)$ be a function defined on a rectangular region $D = [a,b] \times [c,d]$. Let us subdivide the intervals $[a,b]$ (respectively $[c,d]$) by n (m, respectively) intervals.

$$a = x_0 < x_1 < \cdots < x_n = b, \quad c = y_0 < y_1 < \cdots < y_m = d$$

The region D is subdivided by nm rectangular regions $D_{ij} = [x_{i-1}, x_i] \times [y_{i-1}, y_i]$. For each $i = 1, \cdots, n$ and $j = 1, \cdots, m$, let us choose a point $(x_i^*, y_j^*) \in D_{ij}$. Denote $\Delta x_i = x_i - x_{i-1}, \quad \Delta y_j = y_j - y_{j-1}$. The sum

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i^*, y_j^*) \Delta x_i \Delta y_j$$

is called the **Riemann sum** of $f(x, y)$ with respect to the subdivision D_{ij}'s. If the limit exists when $n, m \to \infty$, we denote

$$\iint_D f dA = \lim \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i^*, y_i^*) \Delta x_i \Delta y_j$$

This is called the **double integral** of f over D.

Theorem

If f is continuous on the region $D = [a, b] \times [c, d]$, then the double integral $\iint_D f\,dA$ exists.

Example

The function $f(x, y) = \dfrac{y}{1 + xy}$ is continuous on $D = [0, 1] \times [0, 1]$. Find the double integral $\iint_D f\,dxdy$.

Example

Show that the function $f(x, y)$ on $[0, 1] \times [0, 1]$ defined by

$$f(x, y) = \begin{cases} 1 & x \text{ or } y \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$$

is not integrable on $[0, 1] \times [0, 1]$.

Theorem (Fubini I)

Let $f(x, y)$ be a continuous function defined on $D = [a, b] \times [c, d]$. Then

$$\iint_D f dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

Example

Compute

$$\iint\limits_{[0,1]\times[0,1]} \frac{y}{1+xy} dx dy$$

using Fubini's theorem.

Remark

The *continuity* condition in the theorem is crucial. For example, let

$$f(x,y) = \begin{cases} \dfrac{x^2 - y^2}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Let us compute $\displaystyle\int_0^1 \int_0^1 f(x,y)\,dydx$ first.

$$\begin{aligned} \int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2} dydx &= \int_0^1 \left. \frac{y}{x^2+y^2} \right|_{y=0}^1 dx \\ &= \int_0^1 \frac{1}{1+x^2} dx = \tan^{-1} x \Big|_0^1 = \frac{\pi}{4} \end{aligned}$$

Next, the iterated integral $\int_0^1 \int_0^1 f(x,y)dxdy$ is

$$\int_0^1 \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dxdy = \int_0^1 \frac{-x}{x^2+y^2}\Big|_{x=0}^1 dx$$
$$= \int_0^1 \frac{-1}{1+y^2}dy = -\tan^{-1} y\Big|_0^1 = -\frac{\pi}{4}$$

and it does not coincide with $\int_0^1 \int_0^1 f(x,y)dydx$.

Definition

Let $f(x, y)$ be defined on a <u>bounded</u> region D in $\mathbf{R}^2$. Suppose that D lies on a large rectangular domain, say $D \subset [a, b] \times [c, d]$. Let us define a new function $F(x, y)$ as follows.

$$F(x,y) = \begin{cases} f(x,y) & (x,y) \in D \\ 0 & (x,y) \notin D \end{cases}$$

Then the **definite integral** of f over the domain D is defined as

$$\iint_D f(x,y)dxdy = \iint\limits_{[a,b]\times[c,d]} F(x,y)dxdy$$

Theorem (Fubini II)

Let $f(x,y)$ be a continuous function defined on D. If $D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$, then

$$\iint_D f(x,y)dxdy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

Similarly, if $D = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$, then

$$\iint_D f(x,y)dxdy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

Example

Let us compute $\displaystyle\iint_D e^{-y^2}\,dxdy$ where D is the triangular region whose vertices are $(0,0)$, $(0,1)$, and $(1,1)$. First, let us identify the domain for Equation (**??**).

$$D = \{(x,y)\,|\,0 \le x \le 1,\; x \le y \le 1\}$$

Then we need to compute $\int_0^1 \left[\int_x^1 e^{-y^2}\,dy\right] dx$. However, the first integration $\displaystyle\int_x^1 e^{-y^2}\,dy$ not easy to compute. Thus let us identify D as

$$D = \{(x,y)\,|\,0 \le y \le 1,\; 0 \le x \le y\}$$

Then we can compute the integration.

Definition

Let $f(x,y,z)$ be a function defined on the boxed domain $D=[a,b]\times[c,d]\times[e,f]$. The **triple integral** of $f(x,y,z)$ over D is denoted by

$$\iiint\limits_{[a,b]\times[c,d]\times[e,f]} f(x,y,z)dxdydz$$

If f is defined on a region $V\subset[a,b]\times[c,d]\times[e,f]$, Then define

$$F(x,y,z)=\begin{cases} f(x,y,z) & \text{if } (x,y,z)\in V \\ 0 & \text{otherwise}\end{cases}$$

Then the triple integral is defined as

$$\iiint_V f(x,y,z)dxdydz = \iiint\limits_{[a,b]\times[c,d]\times[e,f]} F(x,y,z)dxdydz$$

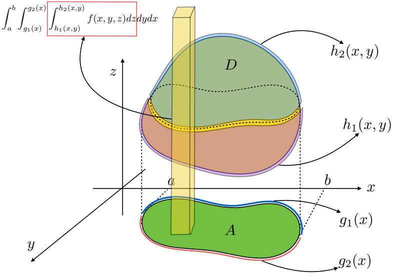
Theorem (Fubini III)

Let $f(x,y,z)$ be a continuous function defined on the region V.

$$V = \{(x,y,z)\,|\,a \le x \le b,\ g_1(x) \le y \le g_2(x),\ h_1(x,y) \le z \le h_2(x,y)\}$$

Then the following holds.

$$\iiint_V f(x,y,z)dxdydz = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)dzdydx$$



Example

Let V be a parallelopiped region bounded by 6 planes : $2x = y$, $2x = y + 2$, $y = 0$, $y = 4$, $z = 0$, $z = 3$. Compute

$$\iiint_V \frac{2x - y}{2} + \frac{z}{3} dxdydz$$

Definition

Given an interval $I \subset \mathbf{R}$, a curve C parametrized by $c : I \to \mathbf{R}^n$

$$c(t) = (x_1(t), x_2(t), \cdots, x_n(t))$$

is **piecewise differentiable** if all coordinate function $x_i(t)$ are $\mathcal{C}^n$ on the interval I except for finitely many points.

Definition

Let C be a piecewise differentiable curve on $\mathbf{R}^2$ parametrized by $c : [a, b] \to \mathbf{R}^2$. Let $c'(t) = (x'(t), y'(t))$ be the velocity vector at the point $c(t)$. Let $f(x, y)$ be a function defined on the curve C. Then the **line integral** of $f(x, y)$ along C is defined as

$$\int_C f ds = \int_a^b f(c(t))|c'(t)|dt.$$

Proposition

The line integral $\int_C f ds$ *does not depend on the parametrization of* C*.*

Proof.

Let $c : [a, b] \to \mathbf{R}^2$ be a parametrization of C. Let $h : [c, d] \to [a, b]$ be a one-to-one correspondence which gives a *re-parametrization* $c \circ h$ of C. Let us write $t = h(\tau)$. Then

$$\begin{aligned}\int_c^d f(c \circ h(\tau)) \cdot |(c \circ h)'(\tau)| d\tau &= \int_c^d f(c(t)) \cdot |c'(t)| \cdot |h'(\tau)| d\tau \\ &= \int_a^b f(c(t)) \cdot |c'(t)| dt\end{aligned}$$

$\square$

Example

Let us compute the line integral

$$\int_C (2 + x^2 y) ds$$

where C is the unit circle centered at the origin with counter clockwise orientation.