

SE102:Multivariable Calculus

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Lecture 08
Manifolds

Definition

The integral $\iint_D f(x,y)dxdy$ is called **improper** if it satisfies one of the following.

- ▶ the region D is unbounded, or
- ▶ the function diverges at some point in D .

Example

Let $D = \mathbf{R}^2$ be the entire 2-dimensional plane. Let us compute

$$\iint_{\mathbf{R}^2} e^{-x^2-y^2} dx dy$$

By polar coordinate $T(r, \theta) = (r \cos \theta, r \sin \theta)$,

$$T^{-1}(D) = [0, \infty) \times [0, 2\pi].$$

Thus by change of coordinates,

$$\iint_{\mathbf{R}^2} e^{-x^2-y^2} dx dy = \int_0^\infty \int_0^{2\pi} e^{-r^2} r d\theta dr = 2\pi \left(\frac{-1}{2} e^{-r^2} \right) \Big|_0^\infty = \pi$$

Definition

The **gamma function** $\Gamma : \mathbf{R} \rightarrow \mathbf{R}$ is defined by

$$\Gamma(p) = \int_0^{\infty} x^{p-1} e^{-x} dx$$

Example

1. $\Gamma(n) = (n-1)!$ for $n \geq 1$.
2. $\Gamma(x)$ diverges at each non-positive integer x .
3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Definition

The **beta function** $B(x, y) : \mathbf{R}^2 \rightarrow \mathbf{R}$ is defined by

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt$$

Proposition

1. $B(x, y) = B(y, x)$
2. $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

Definition

The n -dimensional ball of radius r is the set of points in 4-dimensional space defined by

$$B_n(r) = \{(x_1, \dots, x_n) \mid x_1^2 + \dots + x_n^2 \leq r^2\}.$$

The $n - 1$ -dimensional sphere of radius r is the boundary of $B_n(r)$, defined by

$$S_{n-1}(r) = \{(x_1, \dots, x_n) \mid x_1^2 + \dots + x_n^2 = r^2\}.$$

Proposition

Let $T : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a transformation (i.e. one-to-one, differentiable) such that

$$T(u_1, \dots, u_n) = (x_1, \dots, x_n).$$

If U be a region in \mathbf{R}^n and $V = T(U)$. Then

$$\int_V dx_1 \cdots dx_n = \int_U \left| \det \frac{\partial(x_1, \dots, x_n)}{\partial(u_1, \dots, u_n)} \right| du_1 \cdots du_n$$

We define the **volume** of the cube $[0, 1]^n$ to be 1. There are 3 ways to compute the volume of 4-dimensional ball.

1. Using spherical coordinate.
2. Integrating sections.
3. Finding recursive formula.

Example

Let $T : [0, 1] \times [0, \pi] \times [0, \pi] \times [0, 2\pi] \rightarrow \mathbf{R}^4$ be a transformation defined by

$$T(r, \theta_1, \theta_2, \phi) = (r \sin \theta_1 \sin \theta_2 \cos \phi, r \sin \theta_1 \sin \theta_2 \sin \phi, \\ r \sin \theta_1 \cos \theta_2, r \cos \theta_1)$$

Such T is called a 4-spherical transformation and the Jacobian is

$$J_T = r^3 \sin^2 \theta_1 \sin \theta_2$$

Thus the volume of $B_4(1)$ is

$$\int_0^1 \int_0^\pi \int_0^\pi \int_0^{2\pi} r^3 \sin^2 \theta_1 \sin \theta_2 d\phi d\theta_2 d\theta_1 dr = \frac{\pi^2}{2}$$

Example

As we slice the 4-dimensional ball $B_4(1)$ at each w -coordinate, we obtain 3-dimensional ball of radius $\sqrt{1 - w^2}$. Thus the volume of $B_4(1)$ is

$$\int_{-1}^1 \text{vol} B_3(\sqrt{1 - w^2}) dV dw$$

Since we know $\text{vol} B_3(r) = \frac{4\pi}{3} r^3$, we can compute the integral using Gamma and Beta functions.

Example

The ball $B_4(1)$ is the union of 3-dimensional spheres $S_3(r)$ for $0 \leq r \leq 1$. Thus the volume of $B_4(1)$ is

$$\int_0^1 S_3(r) dr = \int_0^1 \text{vol} S_3(1) r^3 dr = \text{vol} S_3(1)/4$$

Meanwhile, the 3-dimensional sphere $S_3(1)$ is the union of product of two circles $S_1(r) \times S_1(r')$ where $r^2 + r'^2 = 1$. Thus the volume of $S_3(1)$ is

$$\int_0^{\pi/2} \text{vol} S_1(r) \text{vol} S_1(r') d\theta = 2\pi^2$$

Multivariable Calculus summarizes in two sentences:

- ▶ Derivative is a linear transformation.
 - ▶ Derivative $Df(\mathbf{a})$ of a multivariable function is a linear map between tangent spaces at \mathbf{a} and $f(\mathbf{a})$.
- ▶ Divergence theorem is a Stokes' theorem.
 - ▶ The general form of Stokes' theorem is

$$\int_V d\omega = \int_{\partial V} \omega$$

where ω is a differential $(k-1)$ -form and V is a k -dimensional space. The differential $d\omega$ is a k -form. The Stokes theorem is when

$$\omega = Pdx + Qdy = Rdz$$

and divergence theorem is when

$$\omega = Pdx \wedge dy + Qdy \wedge dz + Rdz \wedge dx.$$

Definition

Let $\mathcal{C}^\infty(\mathbf{R}^n)$ be the set of all \mathcal{C}^∞ -class real-valued functions defined on \mathbf{R}^n . A tangent vector \mathbf{v} at $\mathbf{p} \in \mathbf{R}^n$ is a map $\mathcal{C}^\infty(\mathbf{R}^n) \rightarrow \mathbf{R}$ such that there is a parametric curve $\mathbf{c}(t)$ of \mathcal{C}^∞ -class that satisfies

- ▶ $\mathbf{c}(0) = \mathbf{p}$, and
- ▶ $\mathbf{v}(f) = (f \circ \mathbf{c})'(0)$.

Remark

A tangent vector at \mathbf{p} in \mathbf{R}^n is usually denoted as

$$\mathbf{v} = a_0 \left. \frac{\partial}{\partial x^0} \right|_{\mathbf{p}} + \cdots + a_{n-1} \left. \frac{\partial}{\partial x^{n-1}} \right|_{\mathbf{p}}$$

for real coefficients a_0, \dots, a_{n-1} .

Definition

A n -dimensional **vector field** \mathbf{F} is a map assigns a tangent vector at \mathbf{p} to each point \mathbf{p} in \mathbf{R}^n . We can write \mathbf{F} as

$$\mathbf{F}(\mathbf{p}) = a_0(\mathbf{p})\frac{\partial}{\partial x^0} + \cdots + a_{n-1}(\mathbf{p})\frac{\partial}{\partial x^{n-1}}$$

for real-valued function \mathbf{a}_i . We say \mathbf{F} is of class \mathcal{C}^∞ if all coefficient functions a_i are of class \mathcal{C}^∞ .

Remark

The set of all tangent vectors at \mathbf{p} is denoted by $T_{\mathbf{p}}\mathbf{R}^n$, and it is a n -dimensional vector space. A \mathcal{C}^∞ -vector field \mathbf{F} is a map that sends \mathcal{C}^∞ -functions to \mathcal{C}^∞ -functions. The set of all \mathcal{C}^∞ -vector fields on \mathbf{R}^n is denoted by $\mathfrak{X}(\mathbf{R}^n)$. As a vector space $\mathfrak{X}(\mathbf{R}^n)$ is generated by $\frac{\partial}{\partial x^j}$, $j = 0, \dots, n-1$.

Definition

Let V be a n -dimensional vector space. The **dual space** V^* of V is the vector space defined by

$$V^* = \{f : V \rightarrow \mathbf{R} \mid f \text{ is linear}\}.$$

Let e_0, \dots, e_{n-1} be a basis for V . The linear map $e_i^* : V \rightarrow \mathbf{R}$ defined by

$$e_i^*(e_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

is called a dual vector to e_i .

Definition

Let dx^j be the dual vector to the vector field $\frac{\partial}{\partial x^j}$. The n -dimensional **1-form** ω is a linear combination of dx^j with coefficients in $\mathcal{C}^\infty(\mathbf{R}^n)$.

$$\omega = f_0 dx^0 + \dots + f_n dx^{n-1}.$$

Definition

The **wedge** product between two n -dimensional 1-forms dx^i and dx^j , denoted by $dx^i \wedge dx^j$ satisfies

▶ $dx^i \wedge dx^j = -dx^j \wedge dx^i$;

▶ $dx^i \wedge dx^i = 0$.

A **k -form** is a linear combination of wedge products on k 1-forms. The n -dimensional n -form is called the **volume form**.

Remark

The integration of n -form $dx_1 \wedge \cdots \wedge dx_n$ on the cube $[0, 1]^n$ is defined by

$$\int_{[0,1]^n} dx_1 \wedge \cdots \wedge dx_n = 1.$$

The volume of n -dimensional region $V \subset \mathbf{R}^n$ is defined by

$$\int_V dx_1 \wedge \cdots \wedge dx_n.$$

Anti-commutivity of the wedge product ($dx \wedge dy = -dy \wedge dx$) implies that the integration of n -form is a generalization of surface (and line) integrals of vector fields.

Definition

Let $\mathcal{T}^k(\mathbf{R}^n)$ be the set of all k -forms on \mathbf{R}^n . The d -operator is a linear map $d : \mathcal{T}^k(\mathbf{R}^n) \rightarrow \mathcal{T}^{k+1}(\mathbf{R}^n)$ defined by

$$d(f dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \sum_{j=0}^{n-1} \frac{\partial f}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Theorem (Stokes)

Let ω be a $k-1$ -form defined a bounded region $V \subset \mathbf{R}^n$. Then

$$\int_V d\omega = \int_{\partial V} \omega.$$

Problem

Compute the improper integral $\int_{-\infty}^{\infty} e^{-x^2} dx$.

Problem

Find the volume of $B_5(1)$ using

1. spherical coordinates on 5-dimensional space.
2. Gamma and Beta functions.
3. recursive formula on volume of n -dimensional balls.