

# SE102:Multivariable Calculus

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## Lecture 02 Continuity and Differentiability

## Definition

The **polar coordinate** is a system of coordinate system which describes the Cartesian coordinate  $P = (x, y)$  as  $(r, \theta)$  where  $r$  is the length of  $\overline{OP}$  and  $\theta$  is the angle between  $\overline{OP}$  and positive  $x$ -axis.

## Example

The vector  $\vec{r}$  and  $\vec{\theta}$  is the unit vector to the direction where  $r$  and  $\theta$  increases at unit rate. That is,

$$\vec{r} = (x, y) / \sqrt{x^2 + y^2}, \quad \vec{\theta} = (-y, x) / \sqrt{x^2 + y^2}.$$

## Definition

The **cylindrical coordinates**  $(r, \theta, z)$  is

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases} .$$

The **spherical coordinates**  $(\rho, \phi, \theta)$  is

$$\begin{cases} x = \rho \sin \phi \cos \theta \\ y = \rho \sin \phi \sin \theta \\ z = \rho \cos \phi \end{cases} .$$

### Example

In the cylindrical coordinates, the vectors  $\vec{r}, \vec{\theta}, \vec{z}$  are given by

$$\vec{r} = (x, y, 0)/\sqrt{x^2 + y^2}, \quad \vec{\theta} = (-y, x, 0)/\sqrt{x^2 + y^2}, \quad \vec{z} = (0, 0, 1)$$

### Example

The vectors  $\vec{\rho}, \vec{\phi}, \vec{\theta}$  in the spherical coordinates are given by

$$\vec{\rho} = (x, y, z)/\sqrt{x^2 + y^2 + z^2}$$

$$\vec{\phi} = (xz, yz, -(x^2 + y^2))/\sqrt{x^2 + y^2}\sqrt{x^2 + y^2 + z^2}$$

$$\vec{\theta} = (-y, x, 0)/\sqrt{x^2 + y^2}$$

## Remark

For  $n$ -dimensional space  $\mathbb{R}^n$ , there is a hyperspherical coordinate system  $(\rho, \phi_1, \dots, \phi_{n-2}, \theta)$ , defined by

$$\begin{cases} x_1 = \rho \sin \phi_1 \cdots \sin \phi_{n-2} \cos \theta \\ x_2 = \rho \sin \phi_1 \cdots \sin \phi_{n-2} \sin \theta \\ x_3 = \rho \sin \phi_1 \cdots \cos \phi_{n-2} \\ \vdots \\ x_n = \rho \cos \phi \end{cases}$$

## Definition

A **function** is a collection of three types of sets: (1) a domain  $X$ , (2) a codomain  $Y$ , (3) and a rule of assignment that associates each element  $x \in X$  with a unique element  $y \in Y$ .

## Remark

One may wonder why we use the term *set* to describe the rule of assignment of a set. This is because we can actually define the assignment rule as a set:

$$\{(x, y) \in X \times Y \mid \text{no two } y_1, y_2 \in R \text{ appears in pairs} \\ (x, y_1), (x, y_2) \text{ with the same } x \in D\}$$

## Definition

The range  $R$  of a function  $f : X \rightarrow Y$  is the set of all elements in the codomain that are assigned by the function with an element in  $X$ . That is,  $R = \{f(x) \mid x \in D\}$ .

## Definition

A function  $f : X \rightarrow Y$  is called **onto** (or **surjective**) if  $Y = R$ . A function is called **one-to-one** (or **injective**) if  $f(x_1) = f(x_2)$  implies  $x_1 = x_2$  for all  $x_1, x_2 \in X$ . A function is called **one-to-one correspondence** (or **bijective**) if it is one-to-one and onto.

## Definition

A function is called **multivariable** if it consists of more than two independent or dependent variables.

In general a multivariable function  $f$  consists of  $n$  independent variables and  $m$  dependent variables.

$$f(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_m) \quad (1)$$

The variable  $y_j$  is the dependent variable of a function  $f_j$  with  $n$  independent variables  $x_1, x_2, \dots, x_n$ . Thus we can also write the function  $f$  as  $m$ -tuple of real-valued function  $f_j$ 's.

$(j = 1, \dots, m)$

$$y_j = f_j(x_1, x_2, \dots, x_n) \quad (2)$$



## Definition

Let  $f(x, y) = z$  be a function defined on a set  $D \subset \mathbf{R}^2$ . The **graph** of  $f$  is the set in  $\mathbf{R}^3$  defined by

$$G(f) = \{(x, y, f(x, y)) \mid (x, y) \in D\}$$

## Example

Draw the graph of

1.  $f(t) = (t^2, t^3)$
2.  $c(t) = (\cos(t), t, \sin(t))$
3.  $z = x^2 - y^2$

The set of all points satisfying a quadratic equation (i.e. a polynomial of degree 2) is called a quadric surface. The following are some examples of quadric surfaces.

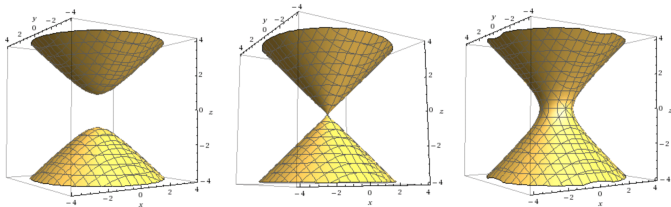
1. Ellipsoid  $\frac{x^2}{4} + y^2 + \frac{z^2}{2} = 1$
2. (Elliptic) paraboloid  $z = x^2 + y^2$
3. Saddle (or elliptic paraboloid)  $z = x^2 - y^2$
4. (Elliptic) cone  $z^2 = x^2 + y^2$
5. Hyperboloid of one sheet  $x^2 + y^2 - z^2 = 1$
6. Hyperboloid of two sheets  $x^2 + y^2 - z^2 = -1$

## Example

We cannot draw the graph of a function  $w = f(x, y, z)$  with three independent variables since we would need 4-dimensional space. Thus we will use the **level set** instead: The level set of  $f$  at  $c$ , denoted by  $L_c(f)$  is a set defined by

$$L_c(f) = \{(x, y, z) \in \mathbf{R}^3 \mid f(x, y, z) = c\}$$

The followings are the level sets of  $f(x, y, z) = x^2 + y^2 - z^2$  at  $c = -1, 0, 1$ .



## Definition

Let  $V$  be a vector space and  $D$  be an open subset of  $\mathbf{R}^n$ . A **vector field**  $\mathbf{F} : D \rightarrow V$  is a function which assigns each point  $(x_1, \dots, x_n) \in D$  a vector  $\mathbf{F}(x_1, x_2, \dots, x_n) \in V$ .

## Definition

Let  $v, w$  be vector spaces. A function  $t : v \rightarrow w$  is called a **linear transformation** if it satisfies the following.

1.  $t(v_1 + v_2) = t(v_1) + t(v_2)$  for all  $v_1, v_2 \in v$ .
2.  $t(cv) = ct(v)$  for all  $v \in v$  and  $c \in \mathbf{R}$ .

## Example

Every linear transformation can be represented by a matrix, and the composition of two linear transformation is represented by the multiplication of corresponding matrices.

## Definition

Let  $f(x, y)$  be a two-variable function defined on the entire plane  $\mathbf{R}^2$ . A constant  $L$  is called the **limit of  $f$  at  $(x_0, y_0)$**  if for any (arbitrary small)  $\epsilon > 0$ , there exists a (small)  $\delta > 0$  such that whenever the distance between  $(x, y)$  and  $(x_0, y_0)$  is less than  $\delta$ , the inequality

$$|f(x, y) - L| < \epsilon$$

holds. In such case, we simply write

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = L$$

## Definition

We say a function  $f(x, y)$  is **continuous at**  $(x_0, y_0)$  if

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = f(x_0, y_0)$$

## Remark

We say  $f(x)$  is continuous at  $x_0$  if  $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ . The limit  $\lim_{x \rightarrow x_0}$  assumes that both right and left limits exists and equals to each other. Since there are only two paths approaching  $x_0$  in one-dimensional space,  $\mathbf{R}$ , the concept of the limit is intuitively clear.

On a two-dimensional plane, there are infinitely many directions and passes for  $(x, y)$  to approach the limit point  $(x_0, y_0)$ . The limit exists only when the limit coincides regardless of the choice of passes (and direction) of points approaching the limit point. This is why we need a rigorous definition of limit for the multivariable function.

## Example

Show that

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases} \quad (3)$$

is continuous at  $(0, 0)$ .



## Proposition

A function  $f(x, y)$  is not continuous at  $(x_0, y_0)$  if there is a path  $c(t) = (x(t), y(t))$  which converges to  $(x_0, y_0)$  while  $f \circ c(t)$  does not converge to  $f(x_0, y_0)$ .

## Example

Let us prove that the function

$$f(x, y) = \begin{cases} \frac{x^4 - 4y^2}{x^2 + 2y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

is not continuous at  $(0, 0)$ . Draw the graph and explain the discontinuity on the graph.

Let us explore the geometric meaning of limit (and continuity) of multi-variable function.

Suppose a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  satisfies  $\lim_{\mathbf{x} \rightarrow \mathbf{a}} f(\mathbf{x}) = \mathbf{L}$ . Here, the value  $\mathbf{L}$  is a point in  $\mathbb{R}^m$  and  $\mathbf{a} \in \mathbb{R}^n$ . The definition of the limit tells us that for any (small)  $\varepsilon > 0$ , we can choose (sufficiently small)  $\delta > 0$  so that as long as  $\mathbf{x}$  lies in the ball  $B_\delta(\mathbf{a})$  of radius  $\delta$  centered at  $\mathbf{a}$ , the value  $f(\mathbf{x})$  lies in the ball  $B_\varepsilon(\mathbf{L})$  of radius  $\varepsilon$  centered at  $\mathbf{L}$ . This implies that  $f(B_\delta) \subset B_\varepsilon$ .

In Topology, we say that a function has a limit  $\mathbf{L}$  at  $\mathbf{a}$  if however small *neighborhood*  $V$  of  $\mathbf{L}$  we choose, we can always find a (very small) *neighborhood*  $U$  of  $\mathbf{a}$  whose image  $f(U)$  is completely contained in  $V$ .

## Definition

Let  $f(x, y)$  be a function defined on a region  $D \subset \mathbf{R}^2$  and  $(x_0, y_0) \in D$ . Let  $\mathbf{u} = (a, b)$  a *unit* vector. If the limit

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{t \rightarrow 0} \frac{f(x_0 + at, y_0 + bt) - f(x_0, y_0)}{t}$$

exists, then it is called the **u-directional derivative** of  $f$  at  $(x_0, y_0)$ .

### Example

Let  $f(x, y) = x^2 + y^2$ . The  $(1, 0)$ -directional derivative of  $f$  at  $(\frac{1}{2}, 0)$  is

$$\begin{aligned} D_{(1,0)} f \left( \frac{1}{2}, 0 \right) &= \lim_{t \rightarrow 0} \frac{f(\frac{1}{2} + t, 0) - f(\frac{1}{2}, 0)}{t} \\ &= \lim_{t \rightarrow 0} \frac{(\frac{1}{2} + t)^2 - (\frac{1}{2})^2}{t} = 1 \end{aligned}$$

## Remark

For a single-variable function  $y = f(x)$ , the differential  $f'(x_0)$  represents rate of change of  $f(x)$  as  $x$  approaches to  $x$ . The directional derivative extends this concept. Let  $c(t) = (x_0, y_0) + t\mathbf{u}$ .  $c(t)$  approaches to  $(x_0, y_0)$  as  $t \rightarrow 0$ . The composition  $f \circ c(t)$  is a single-variable function. The differential  $(f \circ c)'(0)$  is

$$(f \circ c)'(0) = D_{\mathbf{u}}f(x_0, y_0)$$

The directional derivative is the *rate of change of  $f$  along straight line passing through  $(x_0, y_0)$  parallel to  $\mathbf{u} = (u_1, u_2)$* . Let  $P$  be the plane parallel to both  $(u_1, u_2, 0)$  and  $\mathbf{k}$  containing the point  $(x_0, y_0, 0)$ . Then  $D_{\mathbf{u}}f(x_0, y_0)$  is the slope of the intersection curve between the graph of  $f$  and the plane  $P$ .

## Remark

$f : U \rightarrow \mathbf{R}$  be a two-variable function defined on a region  $U \subset \mathbf{R}^2$ . Let  $\mathbf{u}, \mathbf{v}$  be 2-dimensional vectors. Suppose that the  $D_{\mathbf{u}}f(x, y)$  exists for all  $(x, y)$  in a region  $U$ . Then we can define a new function  $g$  on  $U$  as

$$g(x, y) = D_{\mathbf{u}}f(x, y).$$

Suppose that  $g(x, y)$  has  $\mathbf{v}$ -directional derivative  $D_{\mathbf{v}}g(x_0, y_0)$ . We call this as the *second* directional derivative of  $f$ . That is,

$$D_{\mathbf{v}}D_{\mathbf{u}}f(x_0, y_0) = D_{\mathbf{v}}g(x_0, y_0).$$

The value  $D_{\mathbf{u}}f$  is the slope of the graph of  $f$  along the  $\mathbf{u}$ -direction. Thus there is a unique vector tangent to the graph of  $f$  at  $(x_0, y_0, f(x_0, y_0))$  whose projection onto  $\mathbf{R}^2$  is parallel to  $\mathbf{u}$ . The value  $D_{\mathbf{v}}D_{\mathbf{u}}f$  measures how much such tangent vector *changes* along  $v$ -direction at  $(x_0, y_0)$ .

For example,  $D_{\mathbf{u}}D_{\mathbf{u}}f$  is the acceleration  $f$  in the  $\mathbf{u}$ -direction.

### Example

Let  $f(x, y) = x^3 + 5x^2y + y^3$  and  $\mathbf{u} = (\frac{3}{5}, \frac{4}{5})$ . Find  $D_{\mathbf{u}}f$  and  $D_{\mathbf{u}}D_{\mathbf{u}}f$ .

## Definition

Let  $\mathbf{e}_1 = (1, 0)$  and  $\mathbf{e}_2 = (0, 1)$  be the orthonormal vectors in  $\mathbf{R}^2$ . We denote  $D_x, D_y$  for the  $\mathbf{e}_1, \mathbf{e}_2$ -directional derivatives with respect to  $x, y$  respectively. The  $D_x f(x_0, y_0), D_y f(x_0, y_0)$  are called the **partial derivatives** of  $f(x, y)$  at  $(x_0, y_0)$ . Conventionally, the partial derivatives are denoted by

$$D_x f(x_0, y_0) = f_x(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)$$

$$D_y f(x_0, y_0) = f_y(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0)$$

We write  $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$  for the partial derivatives as two-variable functions.



The second partial derivatives are denoted as

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = D_{xx}f = D_x(D_x f)$$

$$\frac{\partial^2 f}{\partial x \partial y} = f_{yx} = D_{xy}f = D_x(D_y f)$$

$$\frac{\partial^2 f}{\partial y \partial x} = f_{xy} = D_{yx}f = D_y(D_x f)$$

$$\frac{\partial^2 f}{\partial y^2} = f_{yy} = D_{yy}f = D_y(D_y f)$$

Note that  $f_{xy}$  and  $f_{yx}$  are *not* the same function in general.

## Example

Let

$$f(x, y) = \begin{cases} xy \frac{x^2 - y^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Compute  $f_{xy}f(0, 0)$  and  $f_{yx}(0, 0)$ .

## Definition

Let  $f_x(x_0, y_0)$ ,  $f_y(x_0, y_0)$  be the partial derivatives of  $f(x, y)$ .  
Then the function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

is called the **linear approximation** of  $f(x, y)$  at  $(x_0, y_0)$ .

## Remark

The graph of  $z = L(x, y)$  is the tangent plane to the graph of  $z = f(x, y)$ .

## Definition

Let  $L(x, y)$  be the linear approximation of  $f(x, y)$  at  $(x_0, y_0)$ .

We say the function  $f(x, y)$  is **differentiable** at  $(x_0, y_0)$  if the following holds.

$$\lim_{(x,y) \rightarrow (x_0,y_0)} \frac{|f(x, y) - L(x, y)|}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$$

## Example

The existence of the linear approximation  $L(x, y)$  does not imply that  $f(x, y)$  is differentiable. Find an example of a function  $f(x, y)$  which has the linear approximation at  $(0, 0)$ , but not differentiable at  $(0, 0)$ .

## Example

Show that the function

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

is not differentiable at  $(0, 0)$ .

## Remark

Recall that we defined the differentiability of single variable function as

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

By taking  $f'(x_0)$  to the left-hand side, equation becomes

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = 0$$

which is the special case of the previous definition.

## Theorem

*Suppose that there are two function  $\epsilon_1 = \epsilon_1(x, y)$ ,  $\epsilon_2 = \epsilon_2(x, y)$  satisfying*

$$\begin{aligned} f(x, y) - f(x_0, y_0) &= f_x(x_0, y_0)(x - x_0) \\ &+ f_y(x_0, y_0)(y - y_0) + \epsilon_1(x - x_0) + \epsilon_2(y - y_0). \end{aligned}$$

*If  $\epsilon_1, \epsilon_2 \rightarrow 0$  as  $(x, y) \rightarrow (x_0, y_0)$ , then  $f(x, y)$  is differentiable at  $(x_0, y_0)$ .*

## Definition

Let  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  be a function whose coordinate functions  $f_i$  are differentiable. The **gradient** is the vector

$$\nabla f = (f_{x_1}, \dots, f_{x_n})$$

Then  $f$  is differentiable at  $\mathbf{x}_0 \in \mathbf{R}^n$  if

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{\|\mathbf{x} - \mathbf{x}_0\|} = 0$$



## Definition

Let  $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$  be a function whose coordinate functions are differentiable. The differential of  $f$  at  $\mathbf{x}_0$  is the matrix

$$Df(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}_0) \end{bmatrix}$$

Then  $f$  is differentiable at  $\mathbf{x}_0$  if

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - Df(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)}{\|\mathbf{x} - \mathbf{x}_0\|} = \vec{0}$$

## Problem

Draw the graph of

1.  $z = x(x^2 - y^2)$  (This surface is called Monkey's saddle.)

2. 
$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

## Problem

Show that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{|xy|}{x^2 + y^2}$$

does not exist.

## Problem

Show that the function

$$f(x, y) = \begin{cases} \frac{y^2}{|x - y|} & x \neq y \\ 0 & x = y \end{cases}$$

is discontinuous at  $(0, 0)$ .

## Problem

Consider the following statements.

1. The partial derivatives  $f_x, f_y$  are continuous at  $(x_0, y_0)$ .
2. The function  $f$  is differentiable at  $(x_0, y_0)$ .
3. The directional derivative  $D_{\mathbf{u}}f(x_0, y_0)$  exists for every  $\mathbf{u}$ .
4. The function  $f$  is continuous at  $(x_0, y_0)$ .

Show that the following implications hold.

- ▶  $1 \Rightarrow 2$
- ▶  $2 \Rightarrow 3$
- ▶  $2 \Rightarrow 4$

Find the counter-examples for

- ▶  $2 \Rightarrow 1$
- ▶  $3 \Rightarrow 2$
- ▶  $4 \Rightarrow 2$