SE102:Multivariable Calculus

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Lecture 06
Line and Surface Integrals

Definition

Let D, R be regions in $\mathbf{R}^n$. A differentiable one-to-one function $T : R \to D$ is called a **transformation**. For $T(u_1, \cdots, u_n) = (x_1, \cdots, x_n)$, the **Jacobian** of T is defined as the determinant of the differential of T:

$$J_T = \det \mathbf{d}T.$$

We also denoted J_T as

$$J_T = \frac{\partial(x_1, \cdots, x_n)}{\partial(u_1, \cdots, u_n)}$$

Theorem (Integration by substitution)

Let $T : R \to D$ be a transformation, and $f(x, y)$ be a continuous function defined on D. Then

$$\iint_D f(x,y)dxdy = \iint_R (f \circ T)(u,v)|J_T|dudv.$$

Remark

It is useful to remember the subsitution rule as:

$$\iint_D f(x,y)dxdy = \iint_R (f \circ T)(u,v) \left| \frac{\partial(x,y)}{\cancel{\partial(u,v)}} \right| \cancel{dudv}$$

Also, the Jacobian of the inverse $T^{-1}(x,y) = (u(x,y), v(x,y))$ is

$$J_{T^{-1}} = \left| \frac{\partial(u,v)}{\partial(x,y)} \right| = \frac{1}{J_T} = \frac{1}{\left| \frac{\partial(x,y)}{\partial(u,v)} \right|}$$

Example

Compute

$$\iint_D |x|e^{-x^2-y^2}\,dxdy$$

where $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$.

Theorem (Integration by substitution)

Let V, W be regions in $\mathbf{R}^3$. A differentiable one-to-one function $T : W \to V$

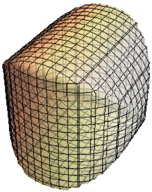
$$T(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))$$

*is called a **transformation**. Let $f(x, y, z)$ be a continuous function on V. Then*

$$\iiint_V f(x, y, z) dxdydz = \iiint_W (f \circ T)(u, v, w)|J_T| dudvdw.$$

Example

Compute the volume between two cylinders $x^2 + y^2 \leq 1$, $y^2 + z^2 \leq 1$.



Example

Compute

$$\iiint_V \sqrt{x^2 + y^2 + z^2}\, e^{x^2+y^2+z^2}\, dxdydz$$

where V is the region between the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$ $(0 < a < b)$.

Definition

A **vector field** $\mathbf{F} : \mathbf{R}^n \to V$ is a map which assign a vector in a vector space V to each point in the space $\mathbf{R}^n$. (Usually we take V as n-dimensional vector space $\mathbf{R}^n$.)

Definition

Given a vector field
$\mathbf{F}(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$, The **curl** $\nabla \times F$ and **divergence** $\nabla \cdot \mathbf{F}$ is defined by

$$\nabla \times \mathbf{F} = (R_y - Q_z, P_z - R_x, Qx - P_y) = \begin{vmatrix} i & j & k \\ \partial/\partial_x & \partial/\partial_y & \partial/\partial_z \\ P & Q & R \end{vmatrix}$$

$$\nabla \cdot \mathbf{F} = P_x + Q_y + R_z$$

Theorem

Let f, g be 3-dimensional functions and $\mathbf{F}$, $\mathbf{G}$ be 3-dimensional vector fields. The following properties hold.

1. $\nabla \times (\nabla f) = 0$
2. $\nabla \cdot (\nabla \times \mathbf{F}) = 0$
3. $\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$
4. $\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$
5. $\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f$
6. $\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + \nabla f \times \mathbf{F}$
7. $\nabla \cdot (\mathbf{F} + \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$
8. $\nabla \cdot (\nabla f \times \nabla g) = 0$
9. *Denote* $\nabla^2 = \nabla \cdot \nabla$. *Then*

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \cdot \mathbf{F}$$

Definition

Let $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$ be a 2-dimensional vector field.
The **curl** of $\mathbf{F}$ is

$$\mathrm{curl}\mathbf{F} = Q_x - P_y.$$

The **divergence** of $\mathbf{F}$ is

$$\mathrm{div}\mathbf{F} = P_x + Q_y.$$

Definition

Let C be a curve in $\mathbf{R}^n$ and $c : [a, b] \to \mathbf{R}^n$ be a parametrization of C. Given a n-dimensional vector field $\mathbf{F}$ defined on C, the **line integral** of $\mathbf{F}$ is defined by

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}((c(t))) \cdot c'(t)dt$$

For 2-dimensional vector field $\mathbf{F} = (P, Q)$, the line integral is denoted by

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C Pdx + Qdy.$$

For 3-dimensional vector field $\mathbf{F} = (P, Q, R)$, the line integral is denoted by

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C Pdx + Qdy + Rdz.$$

Definition

Let $c : [a, b] \to \mathbf{R}^n$ be a parametrization of a curve C. If $c(a) = c(b)$, the curve is said to be **closed**. The line integral over a closed curve is denoted by $\displaystyle\oint_C$.

Example

Let $\mathbf{A} = \left(-\dfrac{y}{x^2+y^2}, \dfrac{x}{x^2+y^2}\right)$. Compute the line integral $\displaystyle\oint_C \mathbf{A} \cdot d\mathbf{s}$ where C is a unit circle parametrized by counter-clockwise direction.

Definition

Let $X : D \to S$ be a parametrization of S. If the vector field

$$\mathbf{n} = \frac{\mathbf{X}_u \times \mathbf{X}_v}{\|\mathbf{X}_u \times \mathbf{X}_v\|}$$

is continuous, we say S **has an orientation** and $\mathbf{n}$ is called an **orientation**.

Definition

Let $\mathbf{F}$ be a 3-dimensional vector field defined on a parametrized surface S. The **surface integral** of $\mathbf{F}$ over S is defined by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (\mathbf{F} \circ X) \cdot \mathbf{n} dS$$

where $\mathbf{n}$ is an orientation of S.

Example

Pick an orientation of a unit sphere and compute the surface integral of $\mathbf{F} = \dfrac{(x, y, z)}{x^2 + y^2 + z^2}$.

Problem

Let $\mathbf{F}$, $\mathbf{G}$ be 3-dimensional vector fields. Show that the following properties hold.

1. $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$
2. $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla \cdot \nabla \cdot \mathbf{F}$
3. $\nabla \times (\mathbf{F} \times \mathbf{G}) = (\nabla \cdot \mathbf{G})\mathbf{F} - (\nabla \cdot \mathbf{F})\mathbf{G} + (\mathbf{G} \cdot \nabla) \cdot \mathbf{F} - (\mathbf{F} \cdot \nabla) \cdot \mathbf{G}$

Problem

Explain geometric meanings of $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$.

Problem

Find the area of the region bounded by three cylinders $x^2 + y^2 \leq 1$, $y^2 + z^2 \leq 1$, and $x^2 + z^2 \leq 1$.