

SE102:Multivariable Calculus

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Definition

Let D, R be regions in \mathbf{R}^n . A differentiable one-to-one function $T : R \rightarrow D$ is called a **transformation**. For $T(u_1, \dots, u_n) = (x_1, \dots, x_n)$, the **Jacobian** of T is defined as the determinant of the differential of T :

$$J_T = \det \mathbf{dT}.$$

We also denoted J_T as

$$J_T = \frac{\partial(x_1, \dots, x_n)}{\partial(u_1, \dots, u_n)}$$

Theorem (Integration by substitution)

Let $T : R \rightarrow D$ be a transformation, and $f(x, y)$ be a continuous function defined on D . Then

$$\iint_D f(x, y) dx dy = \iint_R (f \circ T)(u, v) |J_T| du dv.$$

Remark

It is useful to remember the substitution rule as:

$$\iint_D f(x, y) dx dy = \iint_R (f \circ T)(u, v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

Also, the Jacobian of the inverse $T^{-1}(x, y) = (u(x, y), v(x, y))$ is

$$J_{T^{-1}} = \left| \frac{\partial(u, v)}{\partial(x, y)} \right| = \frac{1}{J_T} = \frac{1}{\left| \frac{\partial(x, y)}{\partial(u, v)} \right|}$$

Example

Compute

$$\iint_D |x| e^{-x^2-y^2} dx dy$$

where $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$.

Theorem (Integration by substitution)

Let V, W be regions in \mathbf{R}^3 . A differentiable one-to-one function $T : W \rightarrow V$

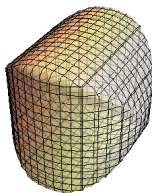
$$T(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))$$

is called a **transformation**. Let $f(x, y, z)$ be a continuous function on V . Then

$$\iiint_V f(x, y, z) dx dy dz = \iiint_W (f \circ T)(u, v, w) |J_T| du dv dw.$$

Example

Compute the volume between two cylinders $x^2 + y^2 \leq 1$, $y^2 + z^2 \leq 1$.



Example

Compute

$$\iiint_V \sqrt{x^2 + y^2 + z^2} e^{x^2 + y^2 + z^2} dx dy dz$$

where V is the region between the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$ ($0 < a < b$).

Definition

A **vector field** $\mathbf{F} : \mathbf{R}^n \rightarrow V$ is a map which assign a vector in a vector space V to each point in the space \mathbf{R}^n . (Usually we take V as n -dimensional vector space \mathbf{R}^n .)

Definition

Given a vector field

$\mathbf{F}(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$, The **curl** $\nabla \times F$ and **divergence** $\nabla \cdot \mathbf{F}$ is defined by

$$\nabla \times \mathbf{F} = (R_y - Q_z, P_z - R_x, Q_x - P_y) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

$$\nabla \cdot \mathbf{F} = P_x + Q_y + R_z$$

Theorem

Let f, g be 3-dimensional functions and \mathbf{F}, \mathbf{G} be 3-dimensional vector fields. The following properties hold.

1. $\nabla \times (\nabla f) = 0$
2. $\nabla \cdot (\nabla \times \mathbf{F}) = 0$
3. $\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$
4. $\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$
5. $\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f$
6. $\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + \nabla f \times \mathbf{F}$
7. $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$
8. $\nabla \cdot (\nabla f \times \nabla g) = 0$
9. Denote $\nabla^2 = \nabla \cdot \nabla$. Then

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \cdot \mathbf{F}$$

Definition

Let $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$ be a 2-dimensional vector field.
The **curl** of \mathbf{F} is

$$\text{curl}\mathbf{F} = Q_x - P_y.$$

The **divergence** of \mathbf{F} is

$$\text{div}\mathbf{F} = P_x + Q_y.$$

Definition

Let C be a curve in \mathbf{R}^n and $c : [a, b] \rightarrow \mathbf{R}^n$ be a parametrization of C . Given a n -dimensional vector field \mathbf{F} defined on C , the **line integral** of \mathbf{F} is defined by

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(c(t)) \cdot c'(t) dt$$

For 2-dimensional vector field $\mathbf{F} = (P, Q)$, the line integral is denoted by

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C P dx + Q dy.$$

For 3-dimensional vector field $\mathbf{F} = (P, Q, R)$, the line integral is denoted by

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C P dx + Q dy + R dz.$$

Definition

Let $c : [a, b] \rightarrow \mathbf{R}^n$ be a parametrization of a curve C . If $c(a) = c(b)$, the curve is said to be **closed**. The line integral over a closed curve is denoted by \oint_C .

Example

Let $\mathbf{A} = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$. Compute the line integral

$\oint_C \mathbf{A} \cdot d\mathbf{s}$ where C is a unit circle parametrized by counter-clockwise direction.

Definition

Let $X : D \rightarrow S$ be a parametrization of S . If the vector field

$$\mathbf{n} = \frac{\mathbf{X}_u \times \mathbf{X}_v}{\|\mathbf{X}_u \times \mathbf{X}_v\|}$$

is continuous, we say S **has an orientation** and \mathbf{n} is called an **orientation**.

Definition

Let \mathbf{F} be a 3-dimensional vector field defined on a parametrized surface S . The **surface integral** of \mathbf{F} over S is defined by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (\mathbf{F} \circ X) \cdot \mathbf{n} dS$$

where \mathbf{n} is an orientation of S .

Example

Pick an orientation of a unit sphere and compute the surface integral of $\mathbf{F} = \frac{(x, y, z)}{x^2 + y^2 + z^2}$.

Problem

Let \mathbf{F} , \mathbf{G} be 3-dimensional vector fields. Show that the following properties hold.

1. $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$
2. $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla \cdot \nabla \cdot \mathbf{F}$
3. $\nabla \times (\mathbf{F} \times \mathbf{G}) = (\nabla \cdot \mathbf{G})\mathbf{F} - (\nabla \cdot \mathbf{F})\mathbf{G} + (\mathbf{G} \cdot \nabla) \cdot \mathbf{F} - (\mathbf{F} \cdot \nabla) \cdot \mathbf{G}$

Problem

Explain geometric meanings of $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$.

Problem

Find the area of the region bounded by three cylinders $x^2 + y^2 \leq 1$, $y^2 + z^2 \leq 1$, and $x^2 + z^2 \leq 1$.

