$$\underbrace{(y_1, y_2, \cdots, y_m)}_{\mathbf{y}} = f(\underbrace{x_1, \cdots, x_n}_{\mathbf{x}}) \quad \Rightarrow \quad f(\mathbf{x}) = \mathbf{y}.$$

SE102:Multivariable Calculus

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Week 02

Definition

The polar coordinate is a system of coordinate system which describes the Cartesian coordinate $P = (x, y)$ as (r, θ) where r is the length of $\overline{OP}$ and θ is the angle between $\overline{OP}$ and positive x-axis.

$$\begin{cases} \sqrt{x^2+y^2} = r. \\ \tan\theta = \frac{y}{x}. \end{cases}$$

Example

The vector $\vec{r}$ and $\vec{\theta}$ is the unit vector to the direction where r and θ increases at unit rate. That is,

$$\hat{r} = \vec{r} = (x, y)/\sqrt{x^2 + y^2}, \quad \vec{\theta} = (-y, x)/\sqrt{x^2 + y^2}. = \hat{\theta}$$

$\hat{r} = \vec{OP}/\|\vec{OP}\|$ $\quad \vec{OP} = (x,y)$.

$\hat{x} = \hat{e}_1$, $\hat{y} = \hat{e}_2$

$$\hat{\theta} = \left(r\cos\left(\tfrac{\pi}{2}+\theta\right), r\sin\left(\tfrac{\pi}{2}+\theta\right)\right)/r$$
$$= \left(-r\sin(\theta), r\cos(\theta)\right)/r$$
$$= (-y, x)/\sqrt{x^2+y^2}$$

space of points

$\hat{r}, \hat{\theta} : \mathbb{R}^2 \to \mathbb{R}^2$ ← 2-dim'l vector space

$\hat{r}, \hat{\theta}$: are vector fields.

Definition

Given the Cartesian coordinates (x, y, z) of a point in $\mathbb{R}^3$, the cylinderical coordinates (r, θ, z) and the spherical coordinates (ρ, ϕ, θ) is given by

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}, \qquad \begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$$

Example

In the cylinderical coordinates, the vectors $\vec{r}, \vec{\theta}, \vec{z}$ are given by

$$\vec{r} = (x, y, 0)/\sqrt{x^2+y^2}, \quad \vec{\theta} = (-y, x, 0)/\sqrt{x^2+y^2}, \quad \vec{z} = (0,0,1) = \hat{k}$$

vector fields

Example

The vectors $\vec{\rho}, \vec{\phi}, \vec{\theta}$ in the spherical coordinates are given by

$$
\begin{aligned}
* \quad \vec{\rho} &= (x, y, z)/\sqrt{x^2+y^2+z^2} = \overrightarrow{OP}/\|\overrightarrow{OP}\|. \\
\vec{\phi} &= (xz, yz, -(x^2+y^2))/\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2} = \frac{\hat{\theta}\times\hat{\rho}}{\|\hat{\theta}\times\hat{\rho}\|} \\
* \quad \vec{\theta} &= (-y, x, 0)/\sqrt{x^2+y^2}
\end{aligned}
$$

Remark

For n-dimensional space $\mathbb{R}^n$, there is a hyperspherical coordinate system $(\rho, \phi_1, \cdots, \phi_{n-2}, \theta)$, defined by

$$
\begin{cases}
x_1 = \rho \sin\phi_1 \cdots \sin\phi_{n-2} \cos\theta \\
x_2 = \rho \sin\phi_1 \cdots \sin\phi_{n-2} \sin\theta \\
x_3 = \rho \sin\phi_1 \cdots \cos\phi_{n-2} \\
\vdots \\
x_n = \rho \cos\phi
\end{cases}
$$

Definition

A function is called **multivariable** if it consists of more than two independent or dependent variables.

In general a multivariable function f consists of n independent variables and m dependent variables.

$y_j = f_j(x_1, \cdots, x_n)$: multi-v. function.

$$f(x_1, x_2, \cdots, x_n) = (y_1, y_2, \cdots, y_m) \tag{1}$$

The variable y_j is the dependent variableof a function f_j with n independent variables $x_1, x_2, \cdots, x_n$. Thus we can also write the function f as m-tuple of real-valued function f_j's.
$(j = 1, \cdots, m)$

$$y_j = f_j(x_1, x_2, \cdots, x_n) \tag{2}$$

$y = f(x)$ $\quad G(f) = \{(x, f(x)) \mid x \in I\}$. $I \subset \mathbb{R}$ interval of domain of f.

Definition

Let $f(x, y) = z$ be a function defined on a set $D \subset \mathbf{R}^2$. The **graph of** f is the set in $\mathbf{R}^3$ defined by

$$G(f) = \{(x, y, f(x, y)) \mid (x, y) \in D\}$$

Example

Draw the graph of

1. $f(t) = (t^2, t^3)$
2. $c(t) = (\cos(t), t, \sin(t))$
3. $z = x^2 - y^2$ $= f(x,y)$

(x, y, z)

$(x, y) \in D$

$z = f(x, y)$

$D = [-1, 1] \times [-1, 1]$

Example

$G(f) = \{(x,y,z,w) \mid w = f(x,y,z)\} \subset \mathbb{R}^4$

We cannot draw the graph of a function $w = f(x, y, z)$ with three independent variables since we would need 4-dimensional space. Thus we will use the **level set** instead: The level set of f at c, denoted by $L_c(f)$ is a set defined by

$$L_c(f) = \{(x, y, z) \in \mathbf{R}^3 \mid f(x, y, z) = c\}$$

The followings are the level sets of $f(x, y, z) = x^2 + y^2 - z^2$ at $c = -1, 0, 1$.

$x^2 + y^2 - z^2 = -1.$

$x^2 + y^2 = z^2 - 1 \Rightarrow z \geq 1,\ z \leq -1.$

point

$\mathbb{F}(x) = v$ vector.

space of points

Definition

Let V be a vector space and D be a open subset of $\mathbf{R}^n$. A **vector field** $\mathbf{F} : D \to V$ is a function which assigns each point $(x_1, \cdots, x_n) \in D$ a vector $\mathbf{F}(x_1, x_2, \cdots, x_n) \in V$.

vector spaces

$t : V \to W$.

Definition

Let v, w be vector spaces. A function $t : v \to w$ is called a **linear transformation** if it satisfies the following.

1. $t(v_1 + v_2) = t(v_1) + t(v_2)$ for all $v_1, v_2 \in v$. (preserves the vector sum)
2. $t(cv) = ct(v)$ for all $v \in v$ and $c \in \mathbf{R}$. (preserves the scalar multiplication)

Example

Every linear transformation can be represented by a matrix, and the composition of two linear transformation is represented by the multiplication of corresponding matrices.

$t(x,y) = (x+2y,\ 2x)$: linear trans. $\rightarrow t = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$

$= \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad = \begin{bmatrix} x+2y \\ 2x \end{bmatrix}$

$$\boxed{\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ 2x \end{bmatrix}}$$

$s(u,v) = u+v \rightarrow s = \begin{bmatrix} 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} u \\ v \end{bmatrix}$ $\quad = [u+v]$

$$\boxed{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = [u+v]}$$

$s \circ t : \mathbb{R}^2 \xrightarrow{t} \mathbb{R}^2 \xrightarrow{s} \mathbb{R}$ $\qquad (s \circ t)(x,y) = s(x+2y,\ 2x) = 3x+2y$

$t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$s : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$s \circ t = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \end{bmatrix}$$

$$(s \circ t) \cdot (x,y) \rightarrow \begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = [3x+2y]$$

Definition

Let $f(x, y)$ be a two-variable function defined on the entire plane $\mathbf{R}^2$. A constant L is called the **limit of f at** (x_0, y_0) if for any (arbitrary small) $\epsilon > 0$, there exists a (small) $\delta > 0$ such that whenever the distance between (x, y) and (x_0, y_0) is less than δ, the inequality

(x,y) is in the "ball" of radius δ centered at (x_0,y_0)

$$|f(x, y) - L| < \epsilon$$

$f(x,y) \in (L-\varepsilon, L+\varepsilon)$

holds. In such case, we simply write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

"If (x,y) is close to (x_0,y_0), then $f(x,y)$ is close to L".

$f: \mathbb{R}^1 \to \mathbb{R}^1$: single var func.

$f: \mathbb{R}^n \to \mathbb{R}^m$, $n>1$ or $m>1$: multi. var. func.

Definition

We say a function $f(x, y)$ is **continuous at** (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$$

The main difference between continuity of $f(x)$ and $f(x, y)$ is the following. We say $f(x)$ is continuous at x_0 if $\lim_{x\to x_0} f(x) = f(x_0)$. The limit $\lim\limits_{x\to x_0}$ assumes that both right and left limits exists and equals to each other. Since there are only two paths approaching to x_0 in one-dimensional space $\mathbf{R}$, the concept of the limit is intuitively clear. However, the limit $\lim\limits_{(x,y)\to(x_0,y_0)}$ is not intuitively clear because there are infinitely many paths approaching to (x_0, y_0) in $\mathbf{R}^2$. Thus taking some example paths is not enough to show the limit of $f(x, y)$.

$$\|(x,y)-(x_0,y_0)\| = \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$$

Example

Show that

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} \tag{3}$$

is continuous at $(0,0)$.

Need to show $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = f(0,0) = 0.$

Let $\varepsilon > 0$. want to find $\delta > 0$ such that

if $\|(x,y)-(0,0)\| = \boxed{\sqrt{x^2+y^2} < \delta}$, then $\left|\dfrac{xy}{\sqrt{x^2+y^2}} - 0\right| < \varepsilon$.

$$\frac{x^2+y^2}{2} \geq \sqrt{x^2y^2} = |xy| \Rightarrow \frac{|xy|}{\sqrt{x^2+y^2}} \leq \frac{\sqrt{x^2+y^2}}{2}$$

Let $\delta = \varepsilon$ Then $\frac{\sqrt{x^2+y^2}}{2} < \frac{\delta}{2} = \frac{\varepsilon}{2} < \varepsilon$

Thus $\frac{|xy|}{\sqrt{x^2+y^2}} \leq \frac{\sqrt{x^2+y^2}}{2} < \varepsilon$

Therefore $f(x,y)$ is continuous.

Proposition

A function $f(x,y)$ is <u>*not*</u> *continuous at (x_0,y_0) if there is a path $c(t)=(x(t),y(t))$ which converges to (x_0,y_0) while $f\circ c(t)$ does not converges to $f(x_0,y_0)$.*

Example

Let us prove that the function

$$t \xrightarrow{c} (x,y) \xrightarrow{f} z$$

$$f\circ c$$

$$f(x,y)=\begin{cases}\dfrac{x^4-4y^2}{x^2+2y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

is not continuous at $(0,0)$. Draw the graph and explain the discontinuity on the graph.

Find a path $c(t)\to(0,0)$ $t\to 0$. Such that $f\circ c(t)\not\to 0$.

Take 1. $c(t) = (t, t)$

$$f \circ c(t) = \begin{cases} \dfrac{t^{\cancel{4}\,2} - 4\cancel{t^2}}{3\cancel{t^2}} = \dfrac{t^2-4}{3} \\ 0 \end{cases}$$

$$\lim_{t\to 0} f \circ c(t) = -\frac{4}{3} \neq f(0,0) = 0$$

Take 2. $c(t) = (t, 2t)$

$$f \circ c(t) = \begin{cases} \dfrac{t^4 - 4\cdot 4\cdot t^2}{t^2 + 2\cdot 4t^2} = \dfrac{t^4 - 16t^2}{9t^2} = \dfrac{t^2-16}{9} \\ 0 \end{cases}$$

$$\lim_{t\to 0} f \circ c(t) = -\frac{16}{9}.$$

Take a limit of $(x, y, f(x,y))$ when $(x,y) \longrightarrow (x_0, y_0)$

If it stops at (x_0, y_0, L) then L is the limit of $f(x,y)$ at (x_0, y_0)

How to find a limit (knowing that the func "has" a limit)

ex. $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{t\to 0} f \circ c(t) = \lim_{t\to 0} \frac{t^2}{\sqrt{2t^2}}$$

$c(t) = (t, t)$

$$= \lim_{t\to 0} \frac{t^2}{\sqrt{2}|t|} = \lim_{t\to 0} \frac{|t|}{\sqrt{2}} = 0.$$

<u>Rmk</u>

You can check conti. / disconti. by

① ε-δ method (for conti.)

② finding special path (for disconti)

③ drawing the graph (find a hole or cliff.)

$f'(x_0)$ = slope of the tangent line at x_0.

Definition

Let $f(x, y)$ be a function defined on a region $D \subset \mathbf{R}^2$ and $(x_0, y_0) \in D$. Let $\mathbf{u} = (a, b)$ a *unit vector*. If the limit

the ratio bet. diff of f and diff of para. t.

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{t \to 0} \frac{f(x_0 + at, y_0 + bt) - f(x_0, y_0)}{t}$$

the slope of tangent at (x_0, y_0)

exists, then it is called the **u-directional derivative** of f at (x_0, y_0).

P ← the plane ⊥ xy-plane // $\vec{u}$

The intersection of G(f) and P

$C(t) = (x_0, y_0) + t\vec{u}$

← the intersection

⇒ $D_u f(x_0, y_0)$

$$D_u f(x_0, y_0) = \lim_{t \to 0} \frac{f(c(t)) - f(c(0))}{t} = \lim \frac{(f \circ c)(t) - (f \circ c)(0)}{t}$$

$$= \frac{d}{dt}\Big|_{t=0} (f \circ c) = (f \circ c)'(0)$$

Example

Let $f(x, y) = x^2 + y^2$. The $(1, 0)$-directional derivative of f at $(\frac{1}{2}, 0)$ is

$c(t) = (\frac{1}{2} + t, 0)$

$$D_{(1,0)} f\left(\tfrac{1}{2}, 0\right) = \lim_{t \to 0} \frac{f(\frac{1}{2} + t, 0) - f(\frac{1}{2}, 0)}{t}$$

$c(t) = (\frac{1}{2}, 0) + t(1, 0)$

$$= \lim_{t \to 0} \frac{\left(\frac{1}{2} + t\right)^2 - \left(\frac{1}{2}\right)^2}{t} = 1$$

$(f \circ c)(t) = (\frac{1}{2} + t)^2 + 0^2 = \frac{1}{4} + t + t^2$

$(f \circ c)'(0) = 1$

Since the value of f is scalar ($z \in \mathbb{R}$), $D_u f$ is scalar. (If the value f is a vector, then $D_u f$ is a vector.)

For a single-variable function $y = f(x)$, the differential $f'(x_0)$ represents rate of change of $f(x)$ as x approaches to x_0. The directional derivative extends this concept. Let $c(t) = (x_0, y_0) + t\mathbf{u}$. $c(t)$ approaches to (x_0, y_0) as $t \to 0$. The composition $f \circ c(t)$ is a single-variable function. The differential $(f \circ c)'(0)$ is

$$(f \circ c)'(0) = D_{\mathbf{u}} f(x_0, y_0)$$

The directional derivative is the *rate of chage of f along straight line passing through* (x_0, y_0) *parallel to* $\mathbf{u} = (u_1, u_2)$. Let P be the plane parallel to both $(u_1, u_2, 0)$ and $\mathbf{k}$ containing the point $(x_0, y_0, 0)$. Then $D_{\mathbf{u}} f(x_0, y_0)$ is the slope of the intersection curve between the graph of f and the plane P. Lastly, the directional derivative does not depends on the shape of a curve.

$f : U \to \mathbf{R}$ be a two-variable function defined on a region $U \subset \mathbb{R}^2$. Let $\mathbf{u}, \mathbf{v}$ be 2-dimensional vectors. Suppose that the $D_{\mathbf{u}} f(x, y)$ exists for all (x, y) in a region D. Then we can define a new two-variable function

$$g(x, y) = D_{\mathbf{u}} f(x, y)$$

is again a two-variable function defined on U. Suppose that $g(x, y)$ has $\mathbf{v}$-direnctional derivative $D_{\mathbf{v}} g(x_0, y_0)$. We call this as the *second* directional derivative of f. That is,

$$D_{\mathbf{v}} D_{\mathbf{u}} f(x_0, y_0) = D_{\mathbf{v}} g(x_0, y_0).$$

The value $D_{\mathbf{u}} f$ is the slope of the graph of f along the $\mathbf{u}$-direction. Thus there is a unique vector tagent to the graph of f at $(x_0, y_0, f(x_0, y_0))$ whose projection onto $\mathbf{R}^2$ is parallel to $\mathbf{u}$. The value $D_{\mathbf{v}} D_{\mathbf{u}} f$ measures how much such tangent vector *changes* along v-direction at (x_0, y_0). For example, $D_{\mathbf{u}} D_{\mathbf{u}} f$ is the acceleration f in $\mathbf{u}$-direction.

$$D_u f(x,y) = \lim_{t\to 0} \frac{f(x+\frac{3}{5}t,\, y+\frac{4}{5}t) - f(x,y)}{t}$$

$$= \lim_{t\to 0} \frac{1}{t}\cdot\Big((x+\tfrac{3}{5}t)^3 + 5\cdot(x+\tfrac{3}{5}t)^2(y+\tfrac{4}{5}t) + (y+\tfrac{4}{5}t)^3 - x^3 - 5x^2y - y^3 \Big)$$

$$= \lim_{t\to 0} \frac{1}{t}\cdot\Big(3\cdot x^2\cdot(\tfrac{3}{5}t) + \cdots + 5\big(x^2 + 2\cdot x(\tfrac{3}{5}t) + (\tfrac{3}{5}t)^2\big)(y+\tfrac{4}{5}t) - 5x^2y + 3\cdot y^2\cdot(\tfrac{4}{5}t) + \cdots \Big)$$

$$= \lim_{t\to 0} \frac{1}{\cancel{t}}\Big(3x^2\cdot\frac{3}{5}\cancel{t} + 5\cdot 2\cdot x(\tfrac{3}{5}\cancel{t})\cdot y + 3\cdot y^2(\tfrac{4}{5}\cancel{t}) \Big)$$

$$= \frac{9}{5}x^2 + 6xy + \frac{12}{5}y^2 = D_u f$$

Example

Let $f(x,y) = x^3 + 5x^2y + y^3$ and $\mathbf{u} = (\frac{3}{5}, \frac{4}{5})$. Find $D_{\mathbf{u}}f$ and $D_{\mathbf{u}}D_{\mathbf{u}}f$.

Definition

Let $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ be the orthonormal vectors in $\mathbf{R}^2$. We denote D_x, D_y for the $\mathbf{e}_1$, $\mathbf{e}_2$-directional derivatives with respectively. The $D_x f(x_0, y_0)$, $D_y f(x_0, y_0)$ are called the **partial derivatives** of $f(x, y)$ at (x_0, y_0). Conventionally, the partial derivatives are denoted by

$$D_x f(x_0, y_0) = f_x(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)$$

$$D_y f(x_0, y_0) = f_y(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0)$$

We write $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ for the partial derivatives as two-variable functions.

$\dfrac{df}{dx}$ $\dfrac{\partial f}{\partial x}$ ← f has x as an independent variable. but f has more.

← It means f is a single variable func on x

The second partial derivatives are denoted as

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = D_{xx} f = \underline{D_x(D_x f)}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx} = D_{xy} f = \underline{D_x(D_y f)}$$

$$= (f_y)_x$$

$$\frac{\partial^2 f}{\partial y \partial x} = f_{xy} = D_{yx} f = D_y(D_x f)$$

$$\frac{\partial^2 f}{\partial y^2} = f_{yy} = D_{yy} f = D_y(D_y f)$$

Note that f_{xy} and f_{yx} are *not* the same function in general.

Rule If f_x is defined at (x_0, y_0), then f_x can be easily obtained by taking the derivative of x as if y is const.

$$f_x(x,y) = \begin{cases} \frac{\partial}{\partial x}\left(\frac{xy(x^2-y^2)}{x^2+y^2}\right) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Example

Let

$$f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Compute $f_{xy}(0,0)$ and $f_{yx}(0,0)$.

$$f_x(0,0) = D_{(1,0)} f(0,0)$$

$$= \lim_{t \to 0} \frac{1}{t} \cdot \left(f(t,0) - f(0,0)\right)$$

$$= \lim_{t \to 0} \frac{1}{t} \cdot (0 - 0) = \boxed{0}$$

$$\frac{\left(y(x^2-y^2) + xy(2x)\right)(x^2+y^2) - xy(x^2-y^2)\cdot(2x)}{(x^2+y^2)^2}$$

$$= \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}$$

$$f_{xy}(0,0) = (f_x)_y(0,0) \qquad f_y(x,y) = \qquad (x,y) \neq (0,0)$$

$$= D_{(0,1)}(f_x)\cdot(0,0)$$

$$= \lim_{t\to 0} \frac{1}{t}\cdot\left(f_x(0,t) - \overset{0}{\overset{\shortparallel}{f_x(0,0)}}\right)$$

$$= \lim_{t\to 0} \frac{1}{\cancel{t}}\cdot\left(\frac{\cancel{t}\cdot(-\cancel{t^4})}{\cancel{t^4}} - 0\right) = -1$$

$$f_y(x,y) = \frac{-x\left(-x^4+4x^2y^2+y^4\right)}{(x^2+y^2)^2}$$

$$f_y(0,0) = D_{(0,1)}f(0,0)$$

$$= \lim_{t\to 0}\frac{1}{t}\cdot\left(f(0,t) - f(0,0)\right)$$

$$= \lim_{t\to 0}\frac{1}{t}\cdot(0-0) = 0$$

$$f_{yx}(0,0) = (f_y)_x(0,0) = D_{(1,0)}f_y(0,0)$$

$$= \lim_{t\to 0}\frac{1}{t}\cdot\left(f_y(t,0) - f_y(0,0)\right)$$

$$= \lim_{t\to 0}\frac{1}{\cancel{t}}\cdot\left(\frac{+\cancel{t}\cdot(+\cancel{t^4})}{\cancel{t^4}} - 0\right) = +1$$

<u>Rmk</u> If f_{xy}, f_{yx} are all continuous at (x_0,y_0)

then $f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0)$.

Definition

Let $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ be the partial derivatives of $f(x, y)$. Then the function

$\deg = 1$ $\deg = 1.$

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

is called the **linear approximation** of $f(x, y)$ at (x_0, y_0).

Remark

The graph of $z = L(x, y)$ is the tangent plane to the graph of $z = f(x, y)$.

Want to get two tangent vectors at P on $G(f)$

How? Take $\vec{e_1}, \vec{e_2}$ from (x_0, y_0)

$$c_1(t) = (x_0, y_0) + t \cdot \vec{e_1}$$
$$= (x_0 + t, y_0)$$

$$c_2(t) = (x_0, y_0 + t)$$

Consider two parametric curves

$$d_1(t) = (x_0 + t, y_0, (f \circ c_1)(t))$$

$$d_2(t) = (x_0, y_0 + \underline{\underline{t}}, (f \circ c_2)(t))$$

$$\boxed{\begin{aligned} d_1'(0) &= \left(1, 0, \frac{d}{dt}\Big|_{t=0} (f \circ c_1)(t)\right) = (1, 0, D_{\vec{e_1}} f(x_0, y_0)) = (1, 0, f_x(x_0, y_0)) \\ d_2'(0) &= (0, 1, f_y(x_0, y_0)) \end{aligned}}$$

are tangent vectors at (x_0, y_0) on $S = G(f)$

$$\Rightarrow \quad d_1'(0) \times d_2'(0) = \begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x i - f_y j + k$$

omitted (x_0, y_0) for simplicity.

$$= (-f_x(x_0, y_0), -f_y(x_0, y_0), 1)$$

$$P = \{(x, y, z) \mid \underbrace{(-f_x(x_0, y_0, -f_y(x_0, y_0), 1)}_{} \circ (x - x_0, y - y_0, z - f(x_0, y_0)) = 0\}$$

$$-f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + \boxed{z} - f(x_0, y_0) = 0.$$

$$\Rightarrow z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$
$$= L(x, y).$$

Thus $G(L)$ is the tangent plane at $(x_0, y_0, f(x_0, y_0))$ on $G(f)$

Assumes then $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ exist!

Definition

Let $L(x, y)$ be the linear approximation of $f(x, y)$ at (x_0, y_0). We say the function $f(x, y)$ is **differentiable** at (x_0, y_0) if the following holds.

$$\lim_{(x,y)\to(x_0,y_0)} \frac{|f(x,y) - L(x,y)|}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$$

$$\sqrt{(x-x_0)^2 + (y-y_0)^2} = \|(x,y) - (x_0,y_0)\|$$

The difference between $f(x,y)$ and $L(x,y)$ get smaller to 0 in the speed faster than (x,y) approaches to (x_0, y_0)

Rmk In single var. func. $y=f(x)$, we say $f(x)$ is differentiable at x_0 if

$$(*) \qquad c = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \quad \text{exists.}$$

Then we can re-write $(*)$ as follows.

$$0 = \lim_{x\to x_0} \frac{|f(x)-f(x_0)-c(x-x_0)|}{|x-x_0|}$$

func of tangent line

$$\boxed{\begin{aligned} &\lim_{x\to x_0} \rho(x) = 0 \\ &\Leftrightarrow \lim_{x\to x_0} |\rho(x)| = 0 \end{aligned}}$$

$$= \lim_{x\to x_0} \frac{|f(x) - [f(x_0)+c(x-x_0)]|}{|x-x_0|}$$

$$= \lim_{x\to x_0} \frac{|f(x)-L(x)|}{|x-x_0|}$$

Thus def of differentiablity of $f(x,y)$ is a generalization of differentiability of $f(x)$

Ex $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$

$$f'(0) = \lim_{t\to 0} \frac{f(t)-f(0)}{t} = \lim_{t\to 0} \frac{t^2\sin\left(\frac{1}{t}\right)-0}{t} = 0.$$

$$L(x) = f(0) + 0\,(x-0) = 0$$

$$\lim_{t\to 0} \frac{|f(t)-L(t)|}{t} = \lim_{t\to 0} \frac{t^2\sin\left(\frac{1}{t}\right)-0}{t} = 0$$

Ex $f(x,y) = \begin{cases} (x^2+y^2)\cdot\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

$f_x(0,0)$, $f_y(0,0)$

$$f_x(0,0) = D_{(1,0)} f(0,0) = \frac{d}{dt}\Big|_{t=0} f(t,0) = \frac{d}{dt}\Big|_{t=0} \begin{cases} t^2 \sin\frac{1}{t} & t \neq 0 \\ 0 & t = 0 \end{cases}$$

$$= 0$$

$$f_y(0,0) = 0.$$

$$\underline{L(x,y) = 0.}$$

Def. $\Rightarrow$ $$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - L(x,y)}{\sqrt{(x-0)^2 + (y-0)^2}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} - 0 = \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}\sin\frac{1}{\sqrt{x^2+y^2}}$$

$$= 0$$

Ex. $$f_x(0,0) = \lim_{t\to 0} \frac{1}{t}\left(f(t,0) - f(0,0)\right) = \lim_{t\to 0} 0 = 0.$$

$$f_y(0,0) = \quad '' \left(f(0,t) - f(0,0)\right) = 0$$

$$L(x,y) = 0 + 0.(x-0) + 0.(y-0) = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{|f(x,y) - L(x,y)|}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \overset{?}{=} 0.$$

the deg of num and denum are same $= 2$, and usually, the limit of such ratio does not exist!

$$c_1(t) = (t,t) \Rightarrow \lim_{t\to 0} \frac{t^2}{2t^2} = \frac{1}{2} \neq 0$$

$$c_2(t) = (t,2t) \Rightarrow \lim_{t\to 0} \frac{2t^2}{5t^2} = \frac{2}{5} \neq 0$$

$\Rightarrow f(x,y)$ is NOT diff. at $(0,0)$

Example

The existence of the linear approximation $L(x, y)$ does not imply that $f(x, y)$ is differentiable. Find an example of a function $f(x, y)$ which has the linear approximation at $(0, 0)$, but not differentiable at $(0, 0)$.

Example

Show that the function

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

is not differentiable at $(0, 0)$.

Recall that we defined the differentiability of single variable function as

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

By taking $f'(x_0)$ to the left-hand side, equation becomes

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = 0$$

which is the special case of the previous definition.

Theorem

Suppose that there are two function $\epsilon_1 = \epsilon_1(x, y)$, $\epsilon_2 = \epsilon_2(x, y)$ *satisfying*

$$\begin{aligned} f(x, y) - f(x_0, y_0) &= f_x(x_0, y_0)(x - x_0) \\ &\quad + f_y(x_0, y_0)(y - y_0) + \epsilon_1(x - x_0) + \epsilon_2(y - y_0). \end{aligned}$$

If $\epsilon_1, \epsilon_2 \to 0$ *as* $(x, y) \to (x_0, y_0)$, *then* $f(x, y)$ *is differentiable at* (x_0, y_0).

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{bmatrix}.$$

Definition

Let $f : \mathbf{R}^n \to \mathbf{R}$ be a function whose coordinate functions f_i are differentiable. The **gradient** is the vector

"nabla f" $$\nabla f = (f_{x_1}, \cdots, f_{x_n})$$ //.

$$\nabla f(p) = (f_{x_1}(p), \cdots f_{x_n}(p))$$

Then f is differentiable at $\mathbf{x}_0 \in \mathbf{R}^n$ if

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{f(\mathbf{x}) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) - f(x_0)}{\|\mathbf{x} - \mathbf{x}_0\|} = 0$$

linear approx. of f at x_0

$$= \lim_{\mathbf{x} \to \mathbf{x}_0} \frac{f(\mathbf{x}) - \left(f(\mathbf{x}_0) + f_{x_1}(\mathbf{x}_0) \cdot (x_1 - x_1^0) + \cdots + f_{x_n}(\mathbf{x}_0)(x_n - x_n^0)\right)}{\|\mathbf{x} - \mathbf{x}_0\|}$$

$f(x_1, \cdots, x_n) = (y_1, \cdots, y_m)$

$y_j = f_j(x_1, \cdots, x_n)$: differentiable.

Definition

Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be a function whose coordinate functions are differentiable. The differential of f at x_0 is the matrix

$$\begin{bmatrix} \nabla f_1(\mathbf{x}_0) \\ \vdots \\ \nabla f_m(\mathbf{x}_0) \end{bmatrix} = Df(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial f_1}{x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_1}{x_n}(\mathbf{x}_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_m}{x_n}(\mathbf{x}_0) \end{bmatrix}$$

$\nabla f_1(\mathbf{x}_0)$

$m \times n$ matrix

Then f is differentiable at $\mathbf{x}_0$ if

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{f(\mathbf{x}) - Df(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)}{\|\mathbf{x} - \mathbf{x}_0\|} = \vec{0}$$

vector. $-f(\mathbf{x}_0)$

linear approx. of f_i at $\mathbf{x}_0$ (combined with $f_i(\mathbf{x}_0)$)

$$\begin{bmatrix} \nabla f_1(\mathbf{x}_0) \circ (\mathbf{x} - \mathbf{x}_0) \\ \vdots \\ \nabla f(\mathbf{x}_0) \circ (\mathbf{x} - \mathbf{x}_0) \end{bmatrix} = \begin{bmatrix} \nabla f_1(\mathbf{x}_0) \\ \vdots \\ \nabla f_m(\mathbf{x}_0) \end{bmatrix} (\mathbf{x} - \mathbf{x}_0) = \underset{m \times n}{Df(\mathbf{x}_0)} \cdot \underset{n \times 1}{\begin{bmatrix} x_1 - x_1^0 \\ \vdots \\ x_n - x_n^0 \end{bmatrix}} = m \times 1 \text{ mat.}$$

In the single-variable case,

$c = f'(x_0)$ is a "slope" of a tangent line

$\underline{v_y = c \cdot v_x}$.

$c : V_x \to V_y$; $c(v_x) = c \cdot v_x = v_y$.

v.s. y-vec.

vector space of x-directional vector

$c : \mathbb{R} \to \mathbb{R}$; $x \mapsto c \cdot x$.

x-dir. vec.

y-dir. vec.

vector spaces

$A = m \times n$ matrix represents a linear map $\mathbb{R}^n \to \mathbb{R}^m$

$v \in \mathbb{R}^n$, $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \Rightarrow A \cdot v = m \times 1$ matrix $\in \mathbb{R}^m$.

Rmk. $f: \mathbb{R}^n \to \mathbb{R}^m$, $Df(\underline{x_0}) = m \times n$ matrix.

$Df: \mathbb{R}^n \to \{\text{matrix of } m \times n \text{ size}\} = M(m,n) = \mathbb{R}^{mn}$.

vector field b/c matrix is vector.

"The differential of a function is a matrix and therefore a linear transformation"

Problem

Draw the graph of

1. $z = x(x^2 - y^2)$ (This surface is called Monkey's saddle.)
2. $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Problem

Show that

$$\lim_{(x,y)\to(0,0)} \frac{|xy|}{x^2+y^2} = f(x,y)$$

does not exist.

try $C_1(t) = (t,t)$ and $C_2(t) = (t, 2t)$

and see $f \circ C_1$ and $f \circ C_2$ conv. to diff. val.

Problem

Show that the function

$$f(x,y) = \begin{cases} \dfrac{y^2}{|x-y|} & x \neq y \\ 0. & \boxed{x = y} \end{cases}$$

is discontinuous at $(0,0)$.

$c_1(t) = (t, kt) \qquad k \neq 1.$

$\Rightarrow f \circ c_1(t) = \dfrac{k \cdot t^2}{|1-k| \cdot |t|} \xrightarrow{t \to 0} 0$

$c_2(t) = (t, t^2) \qquad y = x^2.$

$|t^3| \approx \dfrac{t^4}{|t|} \approx \dfrac{(t^2)^2}{|t - t^2|} \xrightarrow[t \to 0]{} 0.$

$c_3(t) = (t + t^2, t).$?

Problem

$$\begin{tikzcd} ① \arrow[r, Rightarrow, shift left] & ② \arrow[l, Rightarrow, shift left] \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & ③ \\ & & ④ \end{tikzcd}$$

Consider the following statements.

1. The partial derivatives f_x, f_y are continuous at (x_0, y_0).
2. The function f is differentiable at (x_0, y_0).
3. The directional derivative $D_{\mathbf{u}}f(x_0, y_0)$ exists for every $\mathbf{u}$.
4. The function f is continuous at (x_0, y_0).

Show that the following implications hold.

- $1 \Rightarrow 2$ ← see textbook for rigorous proof
- $2 \Rightarrow 3$
- $2 \Rightarrow 4$

ex. $3 \not\Rightarrow 4$. $4 \not\Rightarrow 3$.

Find the counter-examples for

- $2 \Rightarrow 1$ ← see 28/42
- $3 \Rightarrow 2$
- $4 \Rightarrow 2$

$f(x,y) = |x|$.

$f_x(0,0)$ doesn't exist.

$L(x,y)$ doesn't exist.

Rmk Mean value theorem (평균값 정리)

$$f(a) - f(b) = f'(c)(a-b) \quad \text{for some } c \in (a,b)$$

Fix y_0.

$$f(b, y_0) - f(a, y_0) = f_x'(c, y_0)(b-a)$$

for some $c \in (a,b)$

($\because$ $z = f(x, y_0)$ 는 $z = f(x,y)$ 의 그래프를 $y = y_0$ 평면으로 잘라서 본 단면을 그래프로 갖고, 이 그래프는 일변수 함수의 그래프로 변수 하나 대응)

① ⇒ ② Need to show

$$\star = \lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - f_x(x_0,y_0)(x-x_0) - f_y(x_0,y_0)(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$$

$$f(x,y) - f(x_0,y_0) = \underbrace{f(x,y) - f(x,y_0)}_{①} + \overbrace{f(x,y_0) - f(x_0,y_0)}^{②}$$

$$① = f_y(x, y_1)(y - y_0)$$

for some $y_1 \in (y_0, y)$

$$② = f_x(x_1, y_0)(x - x_0)$$

$$\star = \lim \frac{1}{\sqrt{}}\Big(\big(f_y(x,y_1) - f_y(x_0,y_0)\big)\cdot(y-y_0) + \big(f_x(x_1,y_0) - f_x(x_0,y_0)\big)\cdot(x-x_0)\Big).$$

$$= \lim \left(\underbrace{\boxed{f_y(x_1,y_1)}}_{x_0,\ y_0} - \boxed{f(x_0,y_0)} \right) \frac{y-y_0}{\sqrt{\ }} + \underbrace{\left(f_x(x_1,y_0) - f_x(x_0,y_0) \right)}_{\to 0} \cdot \frac{x-x_0}{\sqrt{\ }}$$

$\to 0$

$$\left| \frac{y-y_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \right| \le 1. \quad \text{bounded}$$

bounded

$= 0$ ∎

② ⇒ ③

$$\lim_{(x,y)\to(x_0,y_0)} \frac{*}{*} = 0$$

take $c(t) = (x_0,y_0) + t\cdot\vec{u} = (\underbrace{x_0+ta}_{=x}, \underbrace{y_0+tb}_{=y})$

$\vec{u} = (a,b)$

$$0 = \lim_{t\to 0^+} \frac{f(x_0+ta, y_0+tb) - f(x_0,y_0) - f_x(x_0,y_0)ta - f_y(x_0,y_0)tb}{\sqrt{(ta)^2+(tb)^2}}$$

$$= \lim_{t\to 0^+} \frac{1}{t} \left(f(x_0+ta, y_0+tb) - f(x_0,y_0) \right) - \underbrace{\left(f_x(x_0,y_0)a + f_y(x_0,y_0)b \right)}_{= \text{const}}$$

$$\underbrace{f_x(x_0,y_0)a + f_y(x_0,y_0)b}_{= \nabla f \circ \vec{u}} = \lim_{t\to 0^+} \frac{1}{t} \left(f(x_0+ta, y_0+tb) - f(x_0,y_0) \right)$$

$$= D_u f(x_0,y_0)$$

∎

② ⇒ ④. $0 = \lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y) - f(x_0,y_0) - \boxed{}}{*}$

Since $* \to 0$, $\underbrace{f(x,y) - f(x_0,y_0)}_{\downarrow\, 0} - f_x(x_0,y_0)\cancelto{0}{(x-x_0)} - f_y(x_0,y_0)\cancelto{0}{(y-y_0)}$

$$\Rightarrow \lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0) \qquad \boxtimes$$

② $\not\Rightarrow$ ①.

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right) & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$

is diff. at $(0,0)$ (prove in #28(42)).

$$f_x(x,y) = \begin{cases} 2x\cdot\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right) + \underbrace{(x^2+y^2)\cdot\cos\left(\dfrac{1}{\sqrt{x^2+y^2}}\right)\cdot\dfrac{\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\cdot 2x}{x^2+y^2}}_{} \\ 0 \end{cases}$$

$$= \frac{x}{\sqrt{x^2+y^2}}\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

$$f_x(0,0) = \lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0}\frac{t^2\sin\left(\frac{1}{|t|}\right)}{t} = 0$$

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = \lim_{(x,y)\to(0,0)} \underbrace{2x\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)}_{0} + \underbrace{\frac{x}{\sqrt{x^2+y^2}}\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)}_{\text{no limit.}}$$

oscillate

has limit.

$u(t)=(t,t)$

$u(t)=(t,2t)$

Similar happens for f_y.

f_x, f_y NOT conti (in fact no limit) at (x_0,y_0). $\boxtimes$

③ $\not\Rightarrow$ ②

$$f(x,y) = \begin{cases} 1 & y=x^2,\ (x,y)\neq(0,0) \\ 0 & \text{otherwise} \end{cases}$$

$\lim f(x,y) =$ NOT exist. $\Rightarrow$ NOT conti $\Rightarrow$ NOT diff.

If $c(t) = t \cdot \vec{u}$, $f \circ c(t) = 0$ except for at most one point

$\Rightarrow$ $f \circ c(t) = 0$ $\forall t$. except that point. (if any).

$\Rightarrow$ $D_{\vec{u}} f(0,0) = 0$ $\forall_{\vec{u}}$.