SE102:Multivariable Calculus

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Week 03

Theorem (The chain rule)

Let $f(x,y)$ be a two variable function and $c(t) = (x(t), y(t))$ be a parametrization of a curve in $\mathbf{R}^2$ which is differentiable at t_0. If $f(x,y)$ is differentiable at $(x_0, y_0) = (x(t_0), y(t_0))$, then the composition

$$F(t) = (f \circ c)(t) = f(x(t), y(t))$$

is also differentiable at t_0 and its differential is

$$F'(t_0) = \underbrace{f_x(x_0, y_0)}_{\text{p.d.}} x'(t_0) + \underbrace{f_y(x_0, y_0)}_{\text{p.d.}} y'(t_0)$$

Enlarge surface looks like almost a plane

$$\Delta z = z_2 - z_0$$

$$= (z_2 - z_1) + (z_1 - z_0)$$

$$\approx f_x(x_0, y_0) \cdot \Delta x$$

$$+ f_y(x_0 + \Delta x, y_0) \cdot \Delta y.$$

Divide Δt.

$$\frac{\Delta z}{\Delta t} \approx f_x(x_0, y_0) \frac{\Delta x}{\Delta t} + f_y(x_0 + \Delta x, y_0) \frac{\Delta y}{\Delta t}.$$

$\Delta t \to 0$

$$\left.\frac{dz}{dt}\right|_{t=t_0} = f_x(x_0, y_0) \cdot \frac{dx}{dt} + f_y(x_0, y_0) \frac{dy}{dt}$$

$$z = f \circ c(t) \Rightarrow \frac{dz}{dt} = \frac{d}{dt}(f \circ c) = \frac{dF}{dt}$$

Proof.

Let us define $g_1(t), g_2(t)$ as follows.

$$g_1(t) = \frac{x(t) - x_0 - x'(t_0)(t - t_0)}{t - t_0}$$

$$g_2(t) = \frac{y(t) - y_0 - y'(t_0)(t - t_0)}{t - t_0}$$

Since $x(t)$, $y(t)$ are differentiable at t_0,

$$\lim_{t \to t_0} g_1(t) = \lim_{t \to t_0} g_2(t) = 0$$

. Thus $g_1(t)$, $g_2(t)$ are continuous. Define $F(x, y)$ as

$$F(x, y) = \frac{f(x, y) - L(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}}$$

Proof.

Since we assumed that $f(x, y)$ is differentiable at (x_0, y_0),

$$\lim_{(x,y)\to(x_0,y_0)} F(x, y) = 0.$$

Thus $F(x, y)$ is continuous on $\mathbf{R}^2$. Note that

$$x(t) - x_0 = (x'(t_0) + g_1(t))(t - t_0),$$

$$y(t) - y_0 = (y'(t_0) + g_2(t))(t - t_0),$$

$$\begin{aligned} &f(x, y) - f(x_0, y_0) \\ &\quad = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &\quad + F(x, y)\sqrt{(x - x_0)^2 + (y - y_0)^2}. \end{aligned}$$

Proof.

$$
\begin{aligned}
&\frac{f(x(t),y(t))-f(x_0,y_0)}{t-t_0}\\
&\quad=\frac{1}{t-t_0}\Big(f_x(x_0,y_0)(x(t)-x_0)+f_y(x_0,y_0)(y(t)-y_0)\\
&\qquad\qquad+F(x(t),y(t))\sqrt{(x(t)-x_0)^2+(y(t)-y_0)^2}\Big)\\
&\quad=f_x(x_0,y_0)(x'(t_0)+g_1(t))+f_y(x_0,y_0)(y'(t_0)+g_2(t))\\
&\qquad\qquad+F(x(t),y(t))\sqrt{\left(\frac{x(t)-x_0}{t-t_0}\right)^2+\left(\frac{y(t)-y_0}{t-t_0}\right)^2}\\
&\quad=f_x(x_0,y_0)x'(t_0)+f_y(x_0,y_0)y'(t_0)\\
&\qquad\qquad+f_x(x_0,y_0)g_1(t)+f_y(x_0,y_0)g_2(t)\\
&\qquad\qquad+F(x(t),y(t))\sqrt{\left(\frac{x(t)-x_0}{t-t_0}\right)^2+\left(\frac{y(t)-y_0}{t-t_0}\right)^2}
\end{aligned}
$$

$$(f\circ c)'(t_0) = f_x(c(t_0))\, x'(t_0) + f_y(c(t_0)) \cdot y'(t_0)$$

$$= (f_x, f_y)\Big|_{c(t_0)} \circ (x'(t_0), y'(t_0)) = \nabla f(c(t_0)) \circ c'(t_0)$$

Remark

We can write the chain rule using the gradient of $f(x, y)$.

$$(f \circ c)'(t_0) = \nabla f(c(t_0)) \cdot c'(t_0)$$

The rate of change of $f(x, y)$ at (x_0, y_0) to the direction of $c'(t_0)$ is given by the inner product. The gradient measures how much $f(x, y)$ changes to the given direction. In other words, the gradient $\nabla f(x_0, y_0)$ is a linear map

$$\nabla f(x_0, y_0) : \mathbf{R}^2 \to \mathbf{R}$$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \mapsto \begin{bmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= f_x v_1 + f_y v_2 \in \mathbb{R}.$$

such that

inner prod

$$\nabla f(x_0, y_0)(\mathbf{v} + \mathbf{w}) = \nabla f(x_0, y_0) \cdot \mathbf{v} + \nabla f(x_0, y_0) \cdot \mathbf{w}$$

for all $\mathbf{v}$, $\mathbf{w}$ in $\mathbf{R}^2$.

$$D_{\vec{u}} f(x_0, y_0) = \lim_{t \to 0} \frac{1}{t} \left(f \circ c(t) - f \circ c(t_0) \right)$$

$$c(t) = (x_0, y_0) + t\vec{u} \quad . \quad \left(\Rightarrow c'(t_0) = \vec{u} \right) .$$

$$= \frac{d}{dt}\Big|_{t=0} (f \circ c)(t) = (f \circ c)'(0)$$

$$\overset{\star}{=} \nabla f(c(0)) \circ c'(t) = \nabla f(x_0, y_0) \circ \vec{u}$$

If $\underline{c'(0)} = \underline{v} + \underline{w}$, and we know $\nabla f(c(0)) \circ v$, $\nabla f(c(0)) \circ w$

then $\nabla f(c(0)) \circ c'(0) = \underline{\nabla f(c(0)) \circ v} + \underline{\nabla f(c(0)) \circ w} \in \mathbb{R}$,

(This means sometimes we do not need to know entire ∇f to find the rate of change at certain direction.)

$$v = \underline{c_1} v_1 + \underline{c_2} v_2$$

$$\nabla f \circ v = \underline{c_1} \cdot \nabla f \circ v_1 + \underline{c_2} \nabla f \circ v_2$$

Theorem

Let $\mathbf{X} : \mathbf{R}^2 \to \mathbf{R}^2$ *be defined as*

$$\mathbf{X}(u, v) = (x(u, v), y(u, v)).$$

Let $x_0 = x(u_0, v_0)$, $y_0 = y(u_0, v_0)$. *Assume that* $x(u, v)$, $y(u, v)$ *are differentiable at* (u_0, v_0). *Given a function* $f : \mathbf{R}^2 \to \mathbf{R}$ *which is differentiable at* (x_0, y_0), *the composition* $z = f(x, y)$

$F(u, v) = f(x(u, v), y(u, v))$ *is differentiable at* (u_0, v_0), *and*

$= f \circ \mathbf{X}$

$$(1) \quad F_u(u_0, v_0) = f_x(x_0, y_0)x_u(u_0, v_0) + f_y(x_0, y_0)y_u(u_0, v_0), \quad \star$$

$$F_v(u_0, v_0) = f_x(x_0, y_0)x_v(u_0, v_0) + f_y(x_0, y_0)y_v(u_0, v_0)$$

(1) $\dfrac{\partial F}{\partial u} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial u}$ (2) $\dfrac{\partial F}{\partial v} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial v}$.

Proof.

$$F_u(u_0, v_0) = \frac{d}{du}\Big|_{u=u_0} f(x(u, v_0), y(u, v_0)),$$

$$F_v(u_0, v_0) = \frac{d}{dv}\Big|_{v=v_0} f(x(u_0, v), y(u_0, v)).$$

$$F_u(u_0,v_0) = \frac{d}{dt}\Big|_{t=0} F(u_0+t, v_0).$$

$$= \frac{d}{dt}\Big|_{t=0} f\circ X(u_0+t, v_0)$$

$$= \frac{d}{dt}\Big|_{t=0} f(\underbrace{x(u_0+t, v_0)}_{=a(t)}, \underbrace{y(u_0+t, v_0)}_{=b(t)}).$$

$$= \frac{d}{dt}\Big|_{t=0} (f\circ c)(t) = \nabla f(c(0)) \cdot c'(0)$$

$c(t) = (a(t), b(t))$. The "first" chain rule.

$$= f_x(a(0), b(0)) \cdot a'(0) + f_y(a(0), b(0)) \cdot b'(0)$$

$$= f_x(x_0, y_0)\, x_u(u_0, v_0) + f_y(x_0, y_0) \cdot y_u(u_0, v_0).$$

$x_0 = x(u_0, v_0)$

$y_0 = y(u_0, v_0)$

$$a'(0) = \frac{d}{dt}\Big|_{t=0} x(u_0+t, v_0) = \frac{\partial x}{\partial u}(u_0, v_0)$$

$$b'(0) = \frac{d}{dt}\Big|_{t=0} y(u_0+t, v_0) = \frac{\partial y}{\partial u}(u_0, v_0)$$

Remark

The "first".

We can write the chain rule using matrices. For

$F(t) = (f \circ c)(t)$,

$f(x,y) = z \rightsquigarrow Df : 1 \times 2$ mat.

$c(t) = (x(t), y(t)) \rightsquigarrow Dc : 2 \times 1$ mat.

$$(f \circ c)'(t) = F'(t) = \begin{bmatrix} f_x & f_y \end{bmatrix} \cdot \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \nabla f \cdot c'(t).$$

$1 \times 2 \quad 2 \times 1$

$(Df) = \nabla f$

For $F(u,v) = (f \circ X)(u,v)$,

DX

multi. of matrices

$$D(f \circ X) = DF = \begin{bmatrix} F_u & F_u \end{bmatrix} = \begin{bmatrix} f_x & f_y \end{bmatrix} \cdot \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = Df \cdot DX$$

Note that the gradient of F and $f \circ X$ can be viewed as 1×2 matrices. Thus the last equation can be written as

$$\nabla(f \circ X) = \nabla f \cdot DX$$

Example

$F : \mathbb{R}^2 \to \mathbb{R} ; (r,\theta) \mapsto F(r,\theta)$

Let $T(r,\theta) = (\overset{x}{r\cos\theta}, \overset{y}{r\sin\theta})$. Suppose that $F = f \circ T$. Then

$$F_r = f_x x_r + f_y y_r = f_x \cos\theta + f_y \sin\theta$$

$$F_\theta = f_x x_\theta + f_y y_\theta = -r f_x \sin\theta + r f_y \cos\theta$$

In matrix form,

$$DF = [F_r \;\; F_\theta] = \underbrace{[f_x \;\; f_y]}_{Df} \cdot \underbrace{\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}}_{DT} = Df \cdot DT.$$

$$F_r = \frac{\partial F}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = f_x \cos\theta + f_y \sin\theta$$

$$F_\theta = \frac{\partial F}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = f_x \, r\cdot(-\sin\theta) + f_y (r\cos\theta).$$

$$\begin{array}{ccccc} \mathbb{R}^n & \xrightarrow{f} & \mathbb{R}^m & \xrightarrow{g} & \mathbb{R}^l \\ \rotatebox{90}{$\in$} & & & & \\ a & \longmapsto & f(a) & \longmapsto & g(f(a)) \end{array}$$

Theorem (Chain rule)

Let $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^m$ *be a differentiable function at* $\mathbf{a} \in \mathbf{R}^n$ *and* $\mathbf{g} : \mathbf{R}^m \to \mathbf{R}^l$ *be a differentiable function at* $\mathbf{f}(\mathbf{a}) \in \mathbf{R}^m$. *Then* $\mathbf{g} \circ \mathbf{f}$ *is a differentiable function at* $\mathbf{a}$, *and*

$$\mathbf{d}(\mathbf{g} \circ \mathbf{f})(\mathbf{a}) = \mathbf{dg}(\mathbf{f}(\mathbf{a}))\mathbf{df}(\mathbf{a})$$

f, g : single-var func

$$\frac{d}{dt}(f \circ g) = \frac{df}{dt}\cdot\frac{dg}{dt}$$

eval at g(t)

eval at t.

- matrix multiplication
- order of multi must be preserved

$f : \mathbb{R}^n \to \mathbb{R}^m$ $\quad Df(\underline{a}) = m \times n$ matrix $= \begin{bmatrix} (f_1)_{x_1}(\underline{a}) & \cdots & (f_1)_{x_n}(\underline{a}) \\ \vdots & & \\ (f_m)_{x_1}(\underline{a}) & \cdots & (f_m)_{x_n}(\underline{a}) \end{bmatrix}$

$f = (f_1, \cdots, f_m)$ $\quad \underline{a} = (a_1, \cdots, a_n)$

$Df(\underline{a})$ is a linear transformation (선형변환)

$Df(\underline{a}) : T_{\underline{a}}\mathbb{R}^n \to T_{f(\underline{a})}\mathbb{R}^m$.

tangent vector space (접공간)

Definition

Let S be a subset of $\mathbf{R}^n$. The **tangent space** $T_{\mathbf{a}}S$ at $\mathbf{a} \in S$ is the vector space consists of all *tangent* vectors of S at $\mathbf{a}$.

$c'(0) = \underline{a}$ 를 지나는 순간속도벡터

$(a_1, a_2, a_3) = \underline{a} = c(0)$

$T_{\underline{a}}\mathbb{R}^n = \{c'(0) \mid c(0) = \underline{a}\}$.

$T_{\underline{a}}S = \{c'(0) \mid c(0) = \underline{a},\ c(t) \in S\}$

Remark

$T_{\underline{a}}\mathbb{R}^n \cong \mathbb{R}^n \qquad T_{f(\underline{a})}\mathbb{R}^m \cong \mathbb{R}^m$

The tangent space $T_{\mathbf{a}}\mathbf{R}^n$ is a n-dimensional vector space $\mathbf{R}^n$. Sicne $\mathbf{df}(\mathbf{a})$ is a matrix, one can view the differential $\mathbf{df}(\mathbf{a})$ as a linear map $\mathbf{df}(\mathbf{a}) : \mathbf{R}^n \to \mathbf{R}^m$ as follows: for each n-dimensional vector $\mathbf{v} = (v_1, \cdots, v_n)$ in $T_{\mathbf{a}}\mathbf{R}^n$, the vector $\mathbf{df}(\mathbf{a})\mathbf{v}$ is a m-dimensional vector defined by

$$d\mathbf{f}(\mathbf{a})\mathbf{v} = \overbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{bmatrix}}^{m \times \cancel{n}} \overbrace{\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}}^{\cancel{n} \times 1} = \underline{m \times 1} \text{ matrix}$$

(represent $\vec{w}$)

Rmk $df(\underline{a})$는 정의역 $\mathbb{R}^n$의 $\underline{a}$에서의 접공간 $T_{\underline{a}}\mathbb{R}^n$의 한 벡터 $\vec{v}$를, 치역 $\mathbb{R}^m$의 $f(\underline{a})$에서의 접공간 $T_{f(\underline{a})}\mathbb{R}^m$의 벡터 $\vec{w}$로 보내는 사상이고,

$\vec{w} = df(\underline{a})\vec{v}$

$$f : S \longrightarrow R$$
$$\begin{array}{ccc} \cap & & \cap \\ \mathbb{R}^n & & \mathbb{R}^m \end{array}$$

$$df(\underline{a}) : T_{\underline{a}}S \longrightarrow T_{f(\underline{a})}R$$
$$\begin{array}{ccc} \cup & & \cup \\ \vec{v} & \longmapsto & df(\underline{a})\vec{v} \end{array}$$

$$\vec{w} = df(\underline{a}) \cdot \vec{v} .$$

Example

$f: \mathbb{R}^3 \to \mathbb{R}$.

Let $f(x, y, z) = x + 2y + 3z$. Let S be the graph of $z = xy$. The differential $df(p)$ at $p = (1, 1, 1)$ on S is

$df = [f_x \quad f_y \quad f_z]$
$= [1 \quad 2 \quad 3]$

$$df(1, 1, 1) = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}.$$

The tangent plane of S at p is

$$x + y - z = 1$$

and the tangent space T_pS is spanned by

$$\mathbf{v}_1 = (1, 0, 1), \quad \mathbf{v}_2 = (0, 1, 1).$$

$\{z = xy\}$.

The parametric curve $c(t) = (t, 1, t)$ lies on S, Since $c(1) = p$ and $c'(1) = \mathbf{v}_1$, by the chain rule,

$(f \circ c)(t) = t + 2 + 3t = 2 + 4t$

$= (1, 0, 1)$

$(f \circ c)'(1) = 4$.

$$(f \circ c)'(1) = \underbrace{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}}_{= df(c(1))} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 4.$$

$c'(1)$

In other words, $df(p) \cdot \mathbf{v}_1 = 4$. Similarly, we can compute $df(p) \cdot \mathbf{v}_2 = 5$. Since given a tangent vector $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$ for suitable a, b, the rate of change of f in direction of $\mathbf{v}$ is

$$df(p) \cdot \mathbf{v} = 4a + 5b$$

$$= df(p) \cdot (a\vec{v}_1 + b \cdot \vec{v}_2) = df(p) \cdot (a\vec{v}_1) + df(p)(b\vec{v}_2)$$

$$= a \cdot df(p) \cdot \vec{v}_1 + b \cdot df(p) \cdot \vec{v}_2$$

☆ "미분계수는 선형변환이다".

☆ "그런데, 방향미분은 스칼라 정수이다.".

Problem

A cylinderical ice is melting in a room. When the radius of ice is 6(cm) and the height is 10(cm), the radius is decreasing at 0.1(cm/min) and the height is decreasing at 0.2(cm/min). How fast (cm^3/min) is the ice melting?

$$\frac{dV}{dt} = \frac{d}{dt}(\pi r^2 \cdot h) = \frac{\partial}{\partial r}(\pi r^2 \cdot h)\cdot\frac{dr}{dt} + \frac{\partial}{\partial h}(\pi r^2 \cdot h)\frac{dh}{dt}$$

$V = \pi r^2 \cdot h$ chain rule.

$$= 2\pi r h \cdot \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}$$

at $t = t_0$. $r = 6$. $h = 10$. $\frac{dr}{dt} = -0.1$ $\frac{dh}{dt} = -0.2$

$$\frac{dV}{dt} = \cdots = -\boxed{\cdots}$$

Problem

Let $f(x,y) = \sqrt{|xy|}$. Find the tangent plane to the graph at $(1,1,1)$.

$c_1(t) = (1+t,\ 1,\ \sqrt{1+t}) \qquad t \geq 0.$

$c_2(t) = (1,\ 1+t,\ \sqrt{1+t}) \qquad t \geq 0.$

$c_1'(0) = \left(1,\ 0,\ \frac{1}{2\sqrt{1+t}}\Big|_{t=0}\right) = \left(1, 0, \frac{1}{2}\right)$

$c_2'(0) = \left(0, 1, \frac{1}{2}\right).$

$c_1'(0) \times c_2'(0)$ = normal vector at $(1,1,1)$.

Problem

Let $f(x,y) = (\overbrace{x^2 - y^2}^{f_1 = u}, \overbrace{2xy}^{f_2 = v})$.

$$df = \begin{bmatrix} (f_1)_x & (f_1)_y \\ (f_2)_x & (f_2)_y \end{bmatrix} = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}.$$

1. Find the differential $df(1,1)$. $= \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} \to \det = 8$
2. Let $D = [1, 1+\varepsilon] \times [1, 1+\varepsilon]$. Compute the limit

$$\lim_{\varepsilon \to 0} \frac{\text{area} f(D)}{\underbrace{\text{area}(D)}_{= \varepsilon^2}} = 8$$

3. Find any relation between results in 1 and 2.

$y = 1 \quad f(x,1) = (\underbrace{x^2 - 1}_{u}, \underbrace{2x}_{v})$.

$u = \left(\frac{1}{2}v\right)^2 - 1$

$= \frac{1}{4}v^2 - 1$.

$(f \circ c)(t) = ((1+t)^2 - 1, 2(1+t)) = (t^2 + 2t, 2 + 2t)$

$c(t) = (1+t, 1)$

$(f \circ d)(t) = (1 - ((1+t)^2, 2(1+t)) = (-t^2 - 2t, 2 + 2t)$

$d(t) = (1, 1+t)$

$(f\circ c)'(0) = (2,2)$. $\quad (f\circ d)'(0) = (-2,2)$

$$\varepsilon\cdot(2,2,0) \times \varepsilon(-2,2,0)$$

$$= \varepsilon^2 \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 0 \\ -2 & 2 & 0 \end{vmatrix} = \varepsilon^2 \cdot (4+4)\cdot\vec{k}$$

$$\text{area}(f(D)) \approx 8\varepsilon^2.$$

$$\lim_{\varepsilon\to 0} \frac{\text{area}(f(D))}{\text{area}(D)} = \lim \frac{8\varepsilon^2}{\varepsilon^2} = 8$$

<u>Rule</u> For $f: \mathbb{R}^n \to \mathbb{R}^n$:

$\det(df(\underline{a})) \approx$ ratio between "small" area in the domain containing $\underline{a}$ AND the area of its image under f.

$$\frac{B}{A} \approx \det(df(\underline{a}))$$

Problem

Let $f(x, y)$ z be a function with continuous partial derivatives. Let $x = e^r \cos\theta$ and $y = e^r \sin\theta$. Show that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = e^{-2r}\left(\left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{\partial f}{\partial \theta}\right)^2\right)$$

view f as func of x, y

view f as $f(x(r,\theta), y(r,\theta))$
$= f(r,\theta)$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cdot e^r\cos\theta + \frac{\partial f}{\partial y}\cdot e^r\sin\theta$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\cdot e^r(-\sin\theta) + \frac{\partial f}{\partial y}\cdot e^r\cos\theta$$

$$\left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{\partial f}{\partial \theta}\right)^2 = e^{2r}\left(\cdots\right)$$

Problem

The *Laplacian* Δf of $f(x, y)$ is defined as

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

Show that

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 f}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}\right)$$

$$= \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\cdot\cos\theta + \frac{\partial f}{\partial y}\cdot\sin\theta\right) = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\cos\theta\right) + \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\sin\theta\right)$$

$$= \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\right)\cos\theta + \frac{\partial f}{\partial x}\cancelto{0}{\frac{\partial}{\partial r}(\cos\theta)} + \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\right)\sin\theta + \frac{\partial f}{\partial y}\cancelto{0}{\frac{\partial}{\partial r}(\sin\theta)}$$

$$= \left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\frac{\partial y}{\partial r}\right)\cos\theta + \left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\cdot\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\frac{\partial y}{\partial r}\right)\sin\theta$$

$$= \frac{\partial^2 f}{\partial x^2}\cos^2\theta + \frac{\partial^2 f}{\partial y\partial x}\sin\theta\cos\theta + \frac{\partial^2 f}{\partial x\partial y}\cos\theta\sin\theta + \frac{\partial^2 f}{\partial y^2}\sin^2\theta$$