SE102:Multivariable Calculus

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Lecture 05
Multiple Integrals

Definition (Double integral)

Let $f(x, y)$ be a function defined on a rectangular region $D = [a, b] \times [c, d]$. Let us subdivide the intervals $[a, b]$ and $[c, d]$ by n and m subintervals:

$$a = x_0 < x_1 < \cdots < x_n = b, \quad c = y_0 < y_1 < \cdots < y_m = d.$$

This subdivides D into nm subregions $D_{ij} = [x_{i-1}, x_i] times [y_{i-1}, y_i]$. $(i = 1, \cdots, n,\ j = 1, \cdots, m)$ Denote $\Delta x_i = x_i - x_{i-1}, \quad \Delta y_j = y_j - y_{j-1}$. The **Riemann sum** of $f(x, y)$ with respect to the subdivision of D is

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i^*, y_j^*) \Delta x_i \Delta y_j$$

if $(x_i^*, y_j^*) \in D_{ij}$.

If the limit exists when $n, m \to \infty$ regardless of choice of x_i^* and y_i^*, we say f is **integrable** on D and the limit is called the **double integral**. We denote the limit as

$$\iint_D f dA = \lim \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i^*, y_i^*) \Delta x_i \Delta y_j.$$

Theorem

If f is continuous on the rectangular region $D = [a, b] \times [c, d]$, then f is integrable on D.

Example

Coumpute

$$\iint_D xy dx dy$$

for $D = [0,1] \times [0,1]$.

Example

Show that the function $f(x,y)$ on $[0,1] \times [0,1]$ defined by

$$f(x,y) = \begin{cases} 1 & x \text{ or } y \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$$

is not integrable on $[0,1] \times [0,1]$.

Definition

Let $D \subset \mathbf{R}^2$ be bounded region contained in a rectangular box $[a, b] \times [c, d]$. We say f is integrable on D if the function F below is integrable on $[a, b] \times [c, d]$.

$$F(x, y) = \begin{cases} f(x, y) & (x, y) \in D \\ 0 & (x, y) \in [a, b] \times [c, d] \end{cases}$$

Moreover, the integral of f on D is defined by

$$\iint_D f dxdy = \iint_{[a,b]\times[c,d]} F dxdy.$$

Theorem

Let D be a bounded region. If a function $f(x, y)$ is continuous on D, then f is integrable.

Definition (Iterated integrals)

Let $f(x, y)$ be a two variable function define on a rectangular domain $D = [a, b] \times [c, d]$. The **iterated integral** $\displaystyle\int_c^d \int_a^b f(x, y)dxdy$ on D is defined as follows.

$$\int_c^d \underbrace{\left[\int_a^b f(x, y)dx\right]}_{\text{consider } y \text{ as a constant}} dy.$$

Example

Compute $\displaystyle\int_0^1 \int_0^1 \frac{y}{1 + xy} dxdy.$

Theorem (Fubini I)

Let $f(x, y)$ be a continuous function defined on $D = [a, b] \times [c, d]$. Then

$$\iint_D f dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

Remark

The *continuity* condition in the theorem is crucial. For example, let

$$f(x,y)=\begin{cases}\dfrac{x^2-y^2}{(x^2+y^2)^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

Let us compute $\displaystyle\int_0^1\int_0^1 f(x,y)dydx$ first.

$$\begin{aligned}\int_0^1\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dydx &= \int_0^1 \frac{y}{x^2+y^2}\bigg|_{y=0}^1 dx\\ &= \int_0^1 \frac{1}{1+x^2}dx = \tan^{-1}x\Big|_0^1 = \frac{\pi}{4}\end{aligned}$$

Next, the iterated integral $\int_0^1 \int_0^1 f(x,y)dxdy$ is

$$\begin{aligned}\int_0^1 \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dxdy &= \int_0^1 \frac{-x}{x^2+y^2}\bigg|_{x=0}^{1} dx \\ &= \int_0^1 \frac{-1}{1+y^2}dy = -\tan^{-1} y\Big|_0^1 = -\frac{\pi}{4}\end{aligned}$$

and it does not coincide with $\int_0^1 \int_0^1 f(x,y)dydx$.

Theorem (Fubini II)

Let $f(x,y)$ be a continuous function defined on D. If $D = \{(x,y) \,|\, a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$, then

$$\iint_D f(x,y)dxdy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

Similarly, if $D = \{(x,y) \,|\, c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}$, then

$$\iint_D f(x,y)dxdy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

Example

Let us compute $\iint_D e^{-y^2}\,dxdy$ where D is the triangular region whose vertices are $(0,0)$, $(0,1)$, and $(1,1)$. By Fubini's theorem,

$$\iint_D e^{-y^2}\,dxdy = \int_0^1 \left[\int_x^1 e^{-y^2}\,dy\right] dx = \int_0^1 \left[\int_0^y e^{-y^2}\,dx\right] dy.$$

Find which integration works.

Definition (Multiple integral)

Let $f(\mathbf{x})$ be a real-valued function defind on a boxed region

$$V = [a_0^0, a_1^0] \times [a_0^1, a_1^1] \times \cdots \times [a_0^{n-1}, a_1^{n-1}].$$

We say f is **integrable** if the limit

$$\lim_{N\to\infty} \sum_{j_0,\ldots,j_{n-1}=1}^{N} f(\mathbf{x}_j^*)\Delta x^0 \Delta x^{n-1}$$

exists for arbitrary choice of $\mathbf{x}_j^* \in \prod_{i=0}^{n-1} [x_{j-1}^i, x_j^i]$ where $x_j^i = (a_1^i - a_0^i)/N$. The limit is called the **multiple integral** of f on V.

The multiple integral of f on 3-dimensional boxed region V is called the *triple integral*:

$$\iiint_V f(x,y,z)dxdydz = \iiint_{[a,b]\times[c,d]\times[e,f]} F(x,y,z)dxdydz$$

Definition

Let $f(x,y,z)$ be a function defined on a bounded region $V \subset \mathbf{R}^3$ which is contained in a boxed region $[a,b] \times [c,d] \times [e,f]$. We say f is integrable if the function $F(x,y,z)$ defined on $[a,b] \times [c,d] \times [e,f]$ as below is integrable:

$$F(x,y,z) = \begin{cases} f(x,y,z) & (x,y,z) \in V \\ 0 & \text{otherwise.} \end{cases}$$

Theorem

Let $f(x,y,z)$ be a continuous function defined on a bounded region $V \subset \mathbf{R}^3$. Then f is integrable.

Theorem (Fubini III)

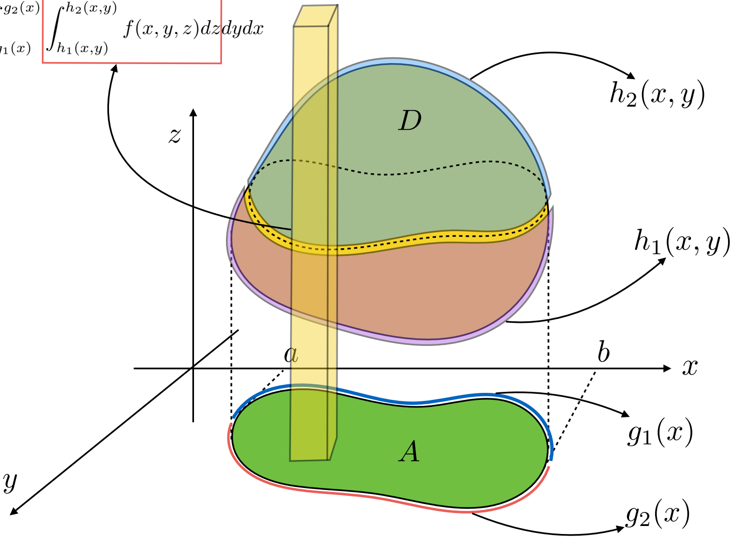
Let $f(x,y,z)$ be a continuous function defined on the region V.

$$V = \{(x,y,z) \,|\, a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x),\ h_1(x,y) \leq z \leq h_2(x,y)\}$$

Then the following holds.

$$\iiint_V f(x,y,z)dxdydz = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)dzdydx$$

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \boxed{\int_{h_1(x,y)}^{h_2(x,y)}} f(x,y,z)\,dz\,dy\,dx$$



Example

Let V be a parallelopiped region bounded by 6 planes : $2x = y$, $2x = y + 2$, $y = 0$, $y = 4$, $z = 0$, $z = 3$. Compute

$$\iiint_V \frac{2x - y}{2} + \frac{z}{3} dxdydz$$

Problem

For $f(x, y) = x^2 - y^2$, compute $\displaystyle\iint_S x + z dS$.

Problem

Let $f(x,y)$ is a density at the point (x,y) on domain D. Let

$$\bar{x} = \frac{\iint_D xf(x,y)dA}{\iint_D f(x,y)dA}, \quad \bar{y} = \frac{\iint_D yf(x,y)dA}{\iint_D f(x,y)dA}.$$

Explain why $(\bar{x}, \bar{y})$ is the center of mass of D.

Problem

Let $V = \{0 \le x \le 2, 0 \le y \le x, 0 \le z \le y\}$. Write $displaystyle \iiint_V f(x, y, z)$ in six different iterated integrals.