SE102:Multivariable Calculus

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Lecture 07
Stokes Theorem

From now on, we will assume that every function is differentiable and is of class $\mathcal{C}^\infty$. (We call such functions as *smooth* functions.)

Definition

Let $\varphi(x, y)$ be a function defined on a region $D \subset \mathbf{R}^2$. The vector field $\nabla\varphi$ is called the **gradient vector field** of φ. Conversely, let $\mathbf{F} : D \to \mathbf{R}^2$ be a vector field defined on D. A function $\varphi(x, y)$ satisfying

$$\nabla\varphi = (\varphi_x, \varphi_y) = \mathbf{F}$$

is called the **potential function** of $\mathbf{F}$.

Definition
Let $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$. If $Q_x - P_y = 0$, then $\mathbf{F}$ is called a **closed** vector field.

Theorem
If a vector field $\mathbf{F}$ *admits a potential function, then it is closed.*

Proof.
Suppose that $\mathbf{F} = (P, Q) = \nabla\varphi$. Then

$$Q_x - P_y = \varphi_{yx} - \varphi_{xy} = 0.$$

$\square$

Definition

A vector field $\mathbf{F}$ defined on $D \subset \mathbf{R}^2$ is called **conservative** if the value of line integral $\int_C \mathbf{F} \cdot d\mathbf{s}$ only depends on the start and end point of a parametrization of the curve $C \subset D$.

Equivalently, a vector field is conservative if

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = 0$$

for any closed curve $C \subset D$.

Theorem

If a vector field $\mathbf{F}$ *admits a potential function, then it is conservative.*

Proof.

Let $c(t) = (x(t), y(t))$, $a \leq t \leq b$, be a parametrization of C from $p_0 = c(a)$ to $p_1 = c(b)$. Note that

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C \varphi_x dx + \varphi_y dy = \int_a^b \frac{\partial \varphi}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \varphi}{\partial y}\frac{\partial y}{\partial t} dt$$

By the Chain rule, we get

$$\int_a^b \frac{\partial \varphi}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \varphi}{\partial y}\frac{\partial y}{\partial t} dt = \int_a^b \frac{d}{dt}\varphi(x(t), y(t))dt = \varphi(p_1) - \varphi(p_0).$$

$\square$

Example

Let $D = \{(x, y) \,|\, x, y > 0\}$ be the first quadrant on $\mathbf{R}^2$. The function

$$\theta(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$$

is a potential function of the vector field

$$\mathbf{A} = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right)$$

on D. Does $\mathbf{A}$ admits a potential function on $\mathbf{R}^2 - \{(0, 0)\}$?

Definition

A curve $C \subset \mathbf{R}^2$ is said to be **simple** if it has no self-intersection. We say a simple closed curve $C \subset \mathbf{R}^2$ is **positively oriented** if it is parametrized by $c(t)$ where the region by C always lies on the *left* side of the tangent vector $c'(t)$.

Theorem (Green)

Let D be a connected region in $\mathbf{R}^2$ bounded by piecewise differentiable simple closed curve C. Let $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$ be a vector field defined on D. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_D Q_x - P_y dA$$

where the curve C is positively oriented.

Proof.

Suppose that the region D is given by

$$D = \{(x, y) \,|\, a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

Then

$$\begin{aligned} \oint_C P dx &= \int_a^b P(x, g_1(x))dx + \int_b^a P(x, g_2(x))dx \\ &= \int_a^b P(x, g_1(x)) - P(x, g_2(x))dx = -\iint_D P_y dA \end{aligned}$$

Similarly, we can show that $\int_C Q dy = \iint_D Q_x dA$. $\square$

Definition

A region $D \subset \mathbf{R}^2$ is said to be **connected** if every pair of points in D can be connected by a path in D. The set D is said to be **simply connected** if every *loop* can be continuously *shrink* to a point.

Corollary

A vector field $\mathbf{F}$ defined on a simply connected region is conservative if it is a closed vector field.

Theorem (Poincare lemma)

Let $\mathbf{F}$ *a vector field defined on a simply connected region* $D \subset \mathbf{R}^2$. *If* $\mathbf{F}$ *is closed, then* $\mathbf{F}$ *admits a potential function. Furthermore, if* D *is connected, then the potential function is unique up to constant multiple.*

Proof.

Let p_0 be a point in D. For $p = (x, y)$ in D, let us define a function $\varphi(x, y)$ as follows.

$$\varphi(x, y) = \int_{p_0}^{p} \mathbf{F} \cdot d\mathbf{s} = \int_{p_0}^{p} P dx + Q dy$$

The function $\varphi(x, y)$ is well-defined. Suppose that the path in the integral near p is given by $c(t) = (x + t, y)$, $\epsilon \in (-\epsilon, 0]$. Then

$$\varphi_x = \frac{\partial}{\partial x} \int_{(x-\Delta x, y)}^{(x,y)} P dx = P.$$

We can show $\varphi_y = Q$ in a similar way. $\square$

Example

Let C be the semicircular arc from $(0, 2)$ to $(0, -2)$ oriented counter-clockwise. Evaluate

$$\int_C xy^2 dx - xy dy$$

Example

Let C be the circle $(x-2)^2 + (y-3)^2 = 1$ oriented counter-clockwise. Evaluate

$$\int_C \left(y - \log(x^2 + y^2)\right) dx + \left(2 \tan^{-1}(y/x)\right) dy$$

Example

Let (a_1, b_1), (a_2, b_2), $\cdots$, (a_n, b_n) be the successive vertices of a *convex* n-polygon. Show that the area inside the polygon is

$$\frac{1}{2}\left(\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} + \cdots + \begin{vmatrix} a_{n-1} & b_{n-1} \\ a_n & b_n \end{vmatrix} + \begin{vmatrix} a_n & b_n \\ a_1 & b_1 \end{vmatrix}\right)$$

Definition

Suppose that a simple closed curve C in $\mathbf{R}^3$ bounds a orientable surface S with a fixed orientation $\mathbf{n}$. We say C is **positively oriented** with respect to S if it is parametrized by $\mathbf{c}(t)$ where the surface S is on the *left* side of the tangent vector $\mathbf{c}'(t)$ when the *upward* direction is $\mathbf{n}$.

Theorem (Stokes)

Let S be an oriented surface with a piecewise continuous boundary C. For $\mathbf{F}$ be a continuous vector field defined on S. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$$

where C is positively orientated with respect to S.

Proof.

Let $\mathbf{F} = (P, Q, R)$ be a 3-dimensional vector field. Suppose that the surface S is given by the graph of $z = f(x, y)$ over a region $D \subset \mathbf{R}^2$. Let $X(x, y) = (x, y, f(x, y))$ be the parametrization of S. Then

$$\begin{aligned}\iint_S \nabla \times \mathbf{F} d\mathbf{S} &= \iint_D -(R_y - Q_z)f_x - (P_z - R_x)f_y + (Q_x - P_y)dA \\ &= \iint_D \frac{\partial}{\partial x}(Q + Rf_y) - \frac{\partial}{\partial y}(P + Rf_x)\, dA\end{aligned}$$

Let C' be a planar curve which bounds that area D. By Green's theorem, the last integral becomes

$$\oint_{C'} (P+Rf_x)dx+(Q+Rf_y)dy = \oint_C Pdx+Qdy+Rdz = \oint_C \mathbf{F}\cdot d\mathbf{s}$$

$\square$

Remark

Stokes' theorem provides the meaning of curl $\nabla \times \mathbf{F}$ of a vector field $\mathbf{F}$. Suppose that the surface S is planar disk with sufficiently small radius r centered at (x_0, y_0, z_0). Then

$$\oint_C \mathbf{F} \cdot d\mathbf{s} \approx \text{area}S \cdot (\nabla \times \mathbf{F})(x_0, y_0) \circ \mathbf{n}$$

When $\mathbf{n}$ is the orientation of S, we can say

$$(\nabla \times \mathbf{F})(x_0, y_0) \circ \mathbf{n} \approx \frac{1}{\text{area}S} \oint_C \mathbf{F} \cdot d\mathbf{s}$$

Example

Let S be the surface bounded by $z = 1 - x^2 - y^2$, $z \geq 0$ with upward orientation ($\mathbf{n} \cdot \mathbf{k} \geq 0$). Confirm that Stokes' theorem holds for $\mathbf{F} = (y, -x, 0)$.

Definition

A surface S in $\mathbf{R}^3$ is said to be **closed** it it has no boundary.

Theorem (Divergence Theorem)

Let V be a region in $\mathbf{R}^3$ whose boundary $S = \partial V$ is a closed orientable surface. For a vector field $\mathbf{F}$ defined on V,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{F} dV$$

where the orientation of S is set to be the outward direction.

Proof.

For $\mathbf{F} = (P, Q, R)$, we have

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S P\mathbf{i} \cdot \mathbf{n} dS + \iint_S Q\mathbf{j} \cdot \mathbf{n} dS + \iint_S R\mathbf{k} \cdot \mathbf{n} dS$$

Suppose that the volume V is given by

$$V = \{(x, y, z) \,|\, h_1(x, y) \leq z \leq h_2(x, y),\ (x, y) \in D\}.$$

Then

$$\begin{aligned}
\iint_S R\mathbf{k} \cdot \mathbf{n} dS &= \iint_{S_1} R\mathbf{k} \cdot \mathbf{n} dS + \iint_{S_2} R\mathbf{k} \cdot \mathbf{n} dS \\
&= \iint_D R(x, y, h_2(x, y)) - R(x, y, h_1(x, y)) dx dy \\
&= \iiint_V R_z dV
\end{aligned}$$

$\square$

Example

Compute the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = (z^2, \frac{1}{3}x^3 + \tan z, z + y^2)$ and S is the closed surface $x^2 + y^2 + z^2 = 1$.

Example

Let S be a parabola $x^2 + y^2 + z = 2$ above the plane $z = 1$. Find the flux of $\mathbf{F} = (z\tan^{-1}(y^2), z^3\ln(x^2+1), z)$ to the upward direction of S.

Example

Let $\mathbf{F} = \dfrac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}$ and S be the surface $z = 4 - x^2 - y^2$, $z \geq 0$ with upward orientation $\mathbf{n} \cdot \mathbf{k} \geq 0$. Use divergence theorem to compute the flux $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$. (How should we choose the volume V?)

Problem

Let $\mathbf{F} = \langle 3y, -4x \rangle$. Find $\displaystyle\oint_{\partial D} \mathbf{F} \cdot d\mathbf{s}$ oriented counter-clockwise for the following region.

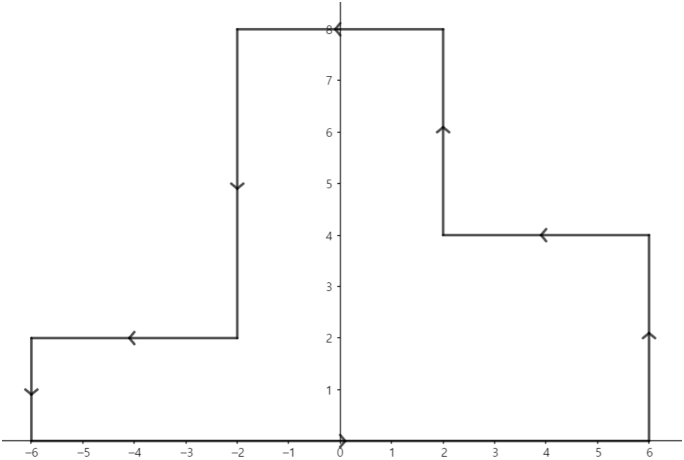
1. $D = \{(x, y) \mid x^2 + y^2 \leq 4\}$
2. $D = \{(x, y) \mid x^2 + 2y^2 \leq 4\}$

Problem

Evaluate $\oint_C 5y\,dx - 3x\,dy$ where C is the cardioid $r = 1 - \sin\theta$ oriented counter-clockwise.

Problem

Evaluate $\oint_C (x^4y^5 - 2y)dx + (3x + x^5y^4)dy$ where C is as shown below.



Problem

Determine the value of

$$\oint_C \frac{xdx + ydy}{x^2 + y^2}$$

where C is a simple closed curve enclosing the origin oriented counter-clockwise.

Problem

Let

$$\mathbf{A} = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}$$

Let S be a surface bounded by $z = 4 - x^2 - y^2$, $z \geq 0$, orieted upward. Find

$$\iint_S \mathbf{A} \cdot d\mathbf{S}$$

.

Problem

Let S be the surface bounded by $z = e^{-x^2-y^2}$ and $z \geq 1/e$. Let $\mathbf{n}$ be the orientation of S satisfying $\mathbf{n} \cdot \mathbf{k} \geq 0$. Find the flux of

$$\mathbf{F} = (e^{x+y} - xe^{y+z}, e^{y+z} - e^{x+y}, 2)$$

on S to the direction of $\mathbf{n}$.

Problem

Let C be the intersection of $z = 1 - 2(x^2 + y^2)$ and $z = x^2 - y^2$ oriented counter-clockwise. Find $\oint_C \mathbf{F} \cdot d\mathbf{s}$ where

$$\mathbf{F} = (y\cos(x) - yz, \sin x, e^z)$$

Problem

Let $S = \partial V$ be a closed surface. Prove the following statements.

1. For any constant vector field $\mathbf{C}$,

$$\iint_S \mathbf{C} \cdot d\mathbf{S} = 0$$

2. For any vector field $\mathbf{F}$,

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = 0$$

Problem

Explain geometric meanings of curl ($\nabla \times \mathbf{F}$) and divergence ($\nabla \cdot \mathbf{F}$) for 2 or 3-dimensional vector fields.