

SE102:Multivariable Calculus

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Lecture 02 Continuity and Differentiability

Definition

The **polar coordinate** is a system of coordinate system which describes the Cartesian coordinate $P = (x, y)$ as (r, θ) where r is the length of \overline{OP} and θ is the angle between \overline{OP} and positive x -axis.

Example

The vector \vec{r} and $\vec{\theta}$ is the unit vector to the direction where r and θ increases at unit rate. That is,

$$\vec{r} = (x, y)/\sqrt{x^2 + y^2}, \quad \vec{\theta} = (-y, x)/\sqrt{x^2 + y^2}.$$

Definition

The **cylindrical coordinates** (r, θ, z) is

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases} .$$

The **spherical coordinates** (ρ, ϕ, θ) is

$$\begin{cases} x = \rho \sin \phi \cos \theta \\ y = \rho \sin \phi \sin \theta \\ z = \rho \cos \phi \end{cases} .$$

Example

In the cylindrical coordinates, the vectors $\vec{r}, \vec{\theta}, \vec{z}$ are given by

$$\vec{r} = (x, y, 0)/\sqrt{x^2 + y^2}, \quad \vec{\theta} = (-y, x, 0)/\sqrt{x^2 + y^2}, \quad \vec{z} = (0, 0, 1)$$

Example

The vectors $\vec{\rho}, \vec{\phi}, \vec{\theta}$ in the spherical coordinates are given by

$$\vec{\rho} = (x, y, z)/\sqrt{x^2 + y^2 + z^2}$$

$$\vec{\phi} = (xz, yz, -(x^2 + y^2))/\sqrt{x^2 + y^2}\sqrt{x^2 + y^2 + z^2}$$

$$\vec{\theta} = (-y, x, 0)/\sqrt{x^2 + y^2}$$

Remark

For n -dimensional space \mathbb{R}^n , there is a hyperspherical coordinate system $(\rho, \phi_1, \dots, \phi_{n-2}, \theta)$, defined by

$$\begin{cases} x_1 = \rho \sin \phi_1 \cdots \sin \phi_{n-2} \cos \theta \\ x_2 = \rho \sin \phi_1 \cdots \sin \phi_{n-2} \sin \theta \\ x_3 = \rho \sin \phi_1 \cdots \cos \phi_{n-2} \\ \vdots \\ x_n = \rho \cos \phi \end{cases}$$

Definition

A **function** is a collection of three types of sets: (1) a domain X , (2) a codomain Y , (3) and a rule of assignment that associates each element $x \in X$ with a unique element $y \in Y$.

Remark

One may wonder why we use the term *set* to describe the rule of assignment of a set. This is because we can actually define the assignment rule as a set:

$$\{(x, y) \in X \times Y \mid \text{no two } y_1, y_2 \in R \text{ appears in pairs} \\ (x, y_1), (x, y_2) \text{ with the same } x \in D\}$$

Definition

The range R of a function $f : X \rightarrow Y$ is the set of all elements in the codomain that are assigned by the function with an element in X . That is, $R = \{f(x) \mid x \in D\}$.

Definition

A function $f : X \rightarrow Y$ is called **onto** (or **surjective**) if $Y = R$. A function is called **one-to-one** (or **injective**) if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for all $x_1, x_2 \in X$. A function is called **one-to-one correspondence** (or **bijective**) if it is one-to-one and onto.

Definition

A function is called **multivariable** if it consists of more than two independent or dependent variables.

In general a multivariable function f consists of n independent variables and m dependent variables.

$$f(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_m) \quad (1)$$

The variable y_j is the dependent variable of a function f_j with n independent variables x_1, x_2, \dots, x_n . Thus we can also write the function f as m -tuple of real-valued function f_j 's.

$(j = 1, \dots, m)$

$$y_j = f_j(x_1, x_2, \dots, x_n) \quad (2)$$

Definition

Let $f(x, y) = z$ be a function defined on a set $D \subset \mathbf{R}^2$. The **graph** of f is the set in \mathbf{R}^3 defined by

$$G(f) = \{(x, y, f(x, y)) \mid (x, y) \in D\}$$

Example

Draw the graph of

1. $f(t) = (t^2, t^3)$
2. $c(t) = (\cos(t), t, \sin(t))$
3. $z = x^2 - y^2$

The set of all points satisfying a quadratic equation (i.e. a polynomial of degree 2) is called a quadric surface. The following are some examples of quadric surfaces.

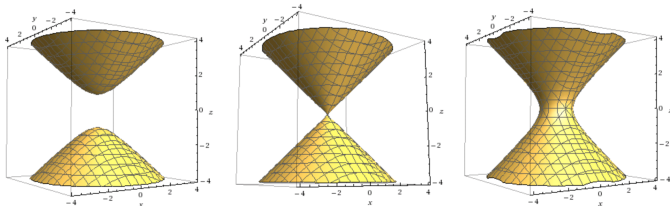
1. Ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{2} = 1$
2. (Elliptic) paraboloid $z = x^2 + y^2$
3. Saddle (or elliptic paraboloid) $z = x^2 - y^2$
4. (Elliptic) cone $z^2 = x^2 + y^2$
5. Hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$
6. Hyperboloid of two sheets $x^2 + y^2 - z^2 = -1$

Example

We cannot draw the graph of a function $w = f(x, y, z)$ with three independent variables since we would need 4-dimensional space. Thus we will use the **level set** instead: The level set of f at c , denoted by $L_c(f)$ is a set defined by

$$L_c(f) = \{(x, y, z) \in \mathbf{R}^3 \mid f(x, y, z) = c\}$$

The followings are the level sets of $f(x, y, z) = x^2 + y^2 - z^2$ at $c = -1, 0, 1$.



Definition

Let V be a vector space and D be a open subset of \mathbf{R}^n . A **vector field** $\mathbf{F} : D \rightarrow V$ is a function which assigns each point $(x_1, \dots, x_n) \in D$ a vector $\mathbf{F}(x_1, x_2, \dots, x_n) \in V$.

Definition

Let v, w be vector spaces. A function $t : v \rightarrow w$ is called a **linear transformation** if it satisfies the following.

1. $t(v_1 + v_2) = t(v_1) + t(v_2)$ for all $v_1, v_2 \in v$.
2. $t(cv) = ct(v)$ for all $v \in v$ and $c \in \mathbf{R}$.

Example

Every linear transformation can be represented by a matrix, and the composition of two linear transformation is represented by the multiplication of corresponding matrices.

Definition

Let $f(x, y)$ be a two-variable function defined on the entire plane \mathbf{R}^2 . A constant L is called the **limit of f at (x_0, y_0)** if for any (arbitrary small) $\epsilon > 0$, there exists a (small) $\delta > 0$ such that whenever the distance between (x, y) and (x_0, y_0) is less than δ , the inequality

$$|f(x, y) - L| < \epsilon$$

holds. In such case, we simply write

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = L$$

Definition

We say a function $f(x, y)$ is **continuous at** (x_0, y_0) if

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = f(x_0, y_0)$$

Remark

We say $f(x)$ is continuous at x_0 if $\lim_{x \rightarrow x_0} f(x) = f(x_0)$. The limit $\lim_{x \rightarrow x_0}$ assumes that both right and left limits exists and equals to each other. Since there are only two paths approaching x_0 in one-dimensional space, \mathbf{R} , the concept of the limit is intuitively clear.

On a two-dimensional plane, there are infinitely many directions and passes for (x, y) to approach the limit point (x_0, y_0) . The limit exists only when the limit coincides regardless of the choice of passes (and direction) of points approaching the limit point. This is why we need a rigorous definition of limit for the multivariable function.

Example

Show that

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases} \quad (3)$$

is continuous at $(0, 0)$.

Proposition

A function $f(x, y)$ is not continuous at (x_0, y_0) if there is a path $c(t) = (x(t), y(t))$ which converges to (x_0, y_0) while $f \circ c(t)$ does not converge to $f(x_0, y_0)$.

Example

Let us prove that the function

$$f(x, y) = \begin{cases} \frac{x^4 - 4y^2}{x^2 + 2y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

is not continuous at $(0, 0)$. Draw the graph and explain the discontinuity on the graph.

Let us explore the geometric meaning of limit (and continuity) of multi-variable function.

Suppose a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ satisfies $\lim_{\mathbf{x} \rightarrow \mathbf{a}} f(\mathbf{x}) = \mathbf{L}$. Here, the value \mathbf{L} is a point in \mathbb{R}^m and $\mathbf{a} \in \mathbb{R}^n$. The definition of the limit tells us that for any (small) $\varepsilon > 0$, we can choose (sufficiently small) $\delta > 0$ so that as long as \mathbf{x} lies in the ball $B_\delta(\mathbf{a})$ of radius δ centered at \mathbf{a} , the value $f(\mathbf{x})$ lies in the ball $B_\varepsilon(\mathbf{L})$ of radius ε centered at \mathbf{L} . This implies that $f(B_\delta) \subset B_\varepsilon$.

In Topology, we say that a function has a limit \mathbb{L} at \mathbf{a} if however small *neighborhood* V of \mathbf{L} we choose, we can always find a (very small) *neighborhood* U of \mathbf{a} whose image $f(U)$ is completely contained in V .

Definition

Let $f(x, y)$ be a function defined on a region $D \subset \mathbf{R}^2$ and $(x_0, y_0) \in D$. Let $\mathbf{u} = (a, b)$ a *unit* vector. If the limit

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{t \rightarrow 0} \frac{f(x_0 + at, y_0 + bt) - f(x_0, y_0)}{t}$$

exists, then it is called the **u-directional derivative** of f at (x_0, y_0) .

Example

Let $f(x, y) = x^2 + y^2$. The $(1, 0)$ -directional derivative of f at $(\frac{1}{2}, 0)$ is

$$\begin{aligned} D_{(1,0)} f \left(\frac{1}{2}, 0 \right) &= \lim_{t \rightarrow 0} \frac{f(\frac{1}{2} + t, 0) - f(\frac{1}{2}, 0)}{t} \\ &= \lim_{t \rightarrow 0} \frac{(\frac{1}{2} + t)^2 - (\frac{1}{2})^2}{t} = 1 \end{aligned}$$

Remark

For a single-variable function $y = f(x)$, the differential $f'(x_0)$ represents rate of change of $f(x)$ as x approaches to x . The directional derivative extends this concept. Let

$c(t) = (x_0, y_0) + t\mathbf{u}$. $c(t)$ approaches to (x_0, y_0) as $t \rightarrow 0$. The composition $f \circ c(t)$ is a single-variable function. The differential $(f \circ c)'(0)$ is

$$(f \circ c)'(0) = D_{\mathbf{u}}f(x_0, y_0)$$

The directional derivative is the *rate of change of f along straight line passing through (x_0, y_0) parallel to $\mathbf{u} = (u_1, u_2)$* . Let P be the plane parallel to both $(u_1, u_2, 0)$ and \mathbf{k} containing the point $(x_0, y_0, 0)$. Then $D_{\mathbf{u}}f(x_0, y_0)$ is the slope of the intersection curve between the graph of f and the plane P .

Remark

$f : U \rightarrow \mathbf{R}$ be a two-variable function defined on a region $U \subset \mathbf{R}^2$. Let \mathbf{u}, \mathbf{v} be 2-dimensional vectors. Suppose that the $D_{\mathbf{u}}f(x, y)$ exists for all (x, y) in a region U . Then we can define a new function g on U as

$$g(x, y) = D_{\mathbf{u}}f(x, y).$$

Suppose that $g(x, y)$ has \mathbf{v} -directional derivative $D_{\mathbf{v}}g(x_0, y_0)$. We call this as the *second* directional derivative of f . That is,

$$D_{\mathbf{v}}D_{\mathbf{u}}f(x_0, y_0) = D_{\mathbf{v}}g(x_0, y_0).$$

The value $D_{\mathbf{u}}f$ is the slope of the graph of f along the \mathbf{u} -direction. Thus there is a unique vector tangent to the graph of f at $(x_0, y_0, f(x_0, y_0))$ whose projection onto \mathbf{R}^2 is parallel to \mathbf{u} . The value $D_{\mathbf{v}}D_{\mathbf{u}}f$ measures how much such tangent vector *changes* along v -direction at (x_0, y_0) .

For example, $D_{\mathbf{u}}D_{\mathbf{u}}f$ is the acceleration f in the \mathbf{u} -direction.

Example

Let $f(x, y) = x^3 + 5x^2y + y^3$ and $\mathbf{u} = (\frac{3}{5}, \frac{4}{5})$. Find $D_{\mathbf{u}}f$ and $D_{\mathbf{u}}D_{\mathbf{u}}f$.

Definition

Let $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ be the orthonormal vectors in \mathbf{R}^2 . We denote D_x, D_y for the $\mathbf{e}_1, \mathbf{e}_2$ -directional derivatives with respect to respectively. The $D_x f(x_0, y_0), D_y f(x_0, y_0)$ are called the **partial derivatives** of $f(x, y)$ at (x_0, y_0) . Conventionally, the partial derivatives are denoted by

$$D_x f(x_0, y_0) = f_x(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)$$

$$D_y f(x_0, y_0) = f_y(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0)$$

We write $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ for the partial derivatives as two-variable functions.

The second partial derivatives are denoted as

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = D_{xx}f = D_x(D_x f)$$

$$\frac{\partial^2 f}{\partial x \partial y} = f_{yx} = D_{xy}f = D_x(D_y f)$$

$$\frac{\partial^2 f}{\partial y \partial x} = f_{xy} = D_{yx}f = D_y(D_x f)$$

$$\frac{\partial^2 f}{\partial y^2} = f_{yy} = D_{yy}f = D_y(D_y f)$$

Note that f_{xy} and f_{yx} are *not* the same function in general.

Example

Let

$$f(x, y) = \begin{cases} xy \frac{x^2 - y^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Compute $f_{xy}f(0, 0)$ and $f_{yx}(0, 0)$.

Definition

Let $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ be the partial derivatives of $f(x, y)$.
Then the function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

is called the **linear approximation** of $f(x, y)$ at (x_0, y_0) .

Remark

The graph of $z = L(x, y)$ is the tangent plane to the graph of $z = f(x, y)$.

Definition

Let $L(x, y)$ be the linear approximation of $f(x, y)$ at (x_0, y_0) .

We say the function $f(x, y)$ is **differentiable** at (x_0, y_0) if the following holds.

$$\lim_{(x,y) \rightarrow (x_0,y_0)} \frac{|f(x, y) - L(x, y)|}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$$

Example

The existence of the linear approximation $L(x, y)$ does not imply that $f(x, y)$ is differentiable. Find an example of a function $f(x, y)$ which has the linear approximation at $(0, 0)$, but not differentiable at $(0, 0)$.

Example

Show that the function

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

is not differentiable at $(0, 0)$.

Remark

Recall that we defined the differentiability of single variable function as

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

By taking $f'(x_0)$ to the left-hand side, equation becomes

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = 0$$

which is the special case of the previous definition.

Theorem

Suppose that there are two function $\epsilon_1 = \epsilon_1(x, y)$, $\epsilon_2 = \epsilon_2(x, y)$ satisfying

$$\begin{aligned} f(x, y) - f(x_0, y_0) &= f_x(x_0, y_0)(x - x_0) \\ &\quad + f_y(x_0, y_0)(y - y_0) + \epsilon_1(x - x_0) + \epsilon_2(y - y_0). \end{aligned}$$

If $\epsilon_1, \epsilon_2 \rightarrow 0$ as $(x, y) \rightarrow (x_0, y_0)$, then $f(x, y)$ is differentiable at (x_0, y_0) .

Definition

Let $f : \mathbf{R}^n \rightarrow \mathbf{R}$ be a real-valued function. The coordinates for the point $\mathbf{x} \in \mathbf{R}^n$ is denoted by

$$\mathbf{x} = (x^0, \dots, x^{n-1}).$$

The **gradient** of f at \mathbf{x} is the vector defined by

$$\nabla f = (f_{x^0}, \dots, f_{x^{n-1}})$$

Let us consider the gradient as a $n \times 1$ matrix and the point \mathbf{x} as a $1 \times n$ matrix. Then we say f is **differentiable at** $\mathbf{x}_0 \in \mathbf{R}^n$ if

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{\|\mathbf{x} - \mathbf{x}_0\|} = 0.$$

Definition

Let $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ be a function whose coordinate functions are differentiable. The differential of f at \mathbf{x}_0 is the matrix

$$Df(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}_0) \end{bmatrix}$$

Then f is differentiable at \mathbf{x}_0 if

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - Df(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)}{\|\mathbf{x} - \mathbf{x}_0\|} = \vec{0}$$

Problem

Draw the graph of

1. $z = x(x^2 - y^2)$ (This surface is called Monkey's saddle.)

$$2. f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Problem

Show that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{|xy|}{x^2 + y^2}$$

does not exist.

Problem

Show that the function

$$f(x, y) = \begin{cases} \frac{y^2}{|x - y|} & x \neq y \\ 0 & x = y \end{cases}$$

is discontinuous at $(0, 0)$.

Problem

Consider the following statements.

1. The partial derivatives f_x, f_y are continuous at (x_0, y_0) .
2. The function f is differentiable at (x_0, y_0) .
3. The directional derivative $D_{\mathbf{u}}f(x_0, y_0)$ exists for every \mathbf{u} .
4. The function f is continuous at (x_0, y_0) .

Show that the following implications hold.

- ▶ $1 \Rightarrow 2$
- ▶ $2 \Rightarrow 3$
- ▶ $2 \Rightarrow 4$

Find the counter-examples for

- ▶ $2 \Rightarrow 1$
- ▶ $3 \Rightarrow 2$
- ▶ $4 \Rightarrow 2$