SE102:Multivariable Calculus

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Lecture 07
Stokes Theorem

Definition

Let $\varphi(x, y)$ be a function defined on a region $D \subset \mathbf{R}^2$. The vector field $\nabla\varphi$ is called the **gradient vector field** of φ. Conversely, let $\mathbf{F} : D \to \mathbf{R}^2$ be a vector field defined on D. A function $\varphi(x, y)$ satisfying

$$\nabla\varphi = (\varphi_x, \varphi_y) = \mathbf{F}$$

is called the **potential function** of $\mathbf{F}$.

Definition

Let $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$. If $Q_x - P_y = 0$, then $\mathbf{F}$ is called a **closed** vector field.

Theorem

If a vector field $\mathbf{F}$ *admits a potential function, then it is closed.*

Proof.

Suppose that $\mathbf{F} = (P, Q) = \nabla\varphi$. Then

$$Q_x - P_y = \varphi_{yx} - \varphi_{xy} = 0.$$

□

Definition

A vector field $\mathbf{F}$ defined on $D \subset \mathbf{R}^2$ is called **conservative** if the line integral $\int_C \mathbf{F} \cdot d\mathbf{s}$ only depends on the start and end point of the curve $C \subset D$. In other words, the vector field is conservative if

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = 0$$

for any closed curve $C \subset D$.

Theorem

If a vector field $\mathbf{F}$ *admits a potential function, then it is conservative.*

Proof.

Let $c(t) = (x(t), y(t))$, $a \leq t \leq b$ be a parametrization of C from $p_0 = c(a)$ to $p_1 = c(b)$. Note that

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C \varphi_x dx + \varphi_y dy = \int_a^b \frac{\partial \varphi}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \varphi}{\partial y}\frac{\partial y}{\partial t} dt$$

By the Chain rule, we getn

$$\int_a^b \frac{\partial \varphi}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \varphi}{\partial y}\frac{\partial y}{\partial t} dt = \int_a^b \frac{d}{dt}\varphi(x(t), y(t)) dt = \varphi(p_1) - \varphi(p_0).$$

□

Corollary

If a vector field $\mathbf{F}$ admits a potential function on D, then

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = 0$$

for any closed curve C in D.

Example

Let $D = \{(x, y) \mid x, y > 0\}$ be the first quadrant on $\mathbf{R}^2$. The function

$$\theta(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$$

is a potential function of the vector field

$$\mathbf{A} = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right)$$

on D.

Example

Not every closed vector field admits a potential function. Consider the vector field

$$\mathbf{A} = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

defined on $D = \mathbf{R}^2 \setminus \{(0,0)\}$. is closed, but it does not admit any potential function on D. (Think carefully why this does not contradict to the previous example.) To show this, suppose that $\mathbf{a}$ has a potential function on $\mathbf{R}^2 \setminus \{(0,0)\}$. Then $\oint_C \mathbf{a} \cdot d\mathbf{s}$ must be 0 for any closed curve C. However we can show that the line integral of this vector field over a circle around the origin is not zero.

Theorem (Green)

Let D be a connected region in $\mathbf{R}^2$ bounded by piecewise differentiable curve C. Let $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$ be a vector field defined on D. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_D Q_x - P_y dA$$

where the curve C is positively oriented.

Proof.

Suppose that the region D is given by

$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

Then

$$\begin{aligned}\oint_C P dx &= \int_a^b P(x, g_1(x)) dx + \int_b^a P(x, g_2(x)) dx \\ &= \int_a^b P(x, g_1(x)) - P(x, g_2(x)) dx = -\iint_D P_y dA\end{aligned}$$

Similarly, $\int_C Q dy = \iint_D Q_x dA$. $\square$

Corollary

Let D be a simply connected region in $\mathbf{R}^2$. A vector field $\mathbf{F}$ defined on D is conservative if it is closed.

Theorem (Poincare lemma)

Let D be a simply connected region in $\mathbf{R}^2$ and $\mathbf{F}$ a vector field defined on D. If $\mathbf{F}$ is closed, then $\mathbf{F}$ admits a potential function. Furthermore, if D is connected, then the potential function is unique up to constant.

Proof.

Let p_0 be a point in D. For $p = (x, y)$ in D, let us define a function $\varphi(x, y)$ as follows.

$$\varphi(x, y) = \int_{p_0}^{p} \mathbf{F} \cdot d\mathbf{s} = \int_{p_0}^{p} P dx + Q dy$$

The function $\varphi(x, y)$ is well-defined. Suppose that the path in the integral near p is given by $c(t) = (x + t, y)$, $\epsilon \in (-\epsilon, 0]$. Then

$$\varphi_x = \frac{\partial}{\partial x} \int_{(x-\Delta x, y)}^{(x,y)} P dx = P.$$

We can show $\varphi_y = Q$ in a similar way. □

Example

Let C be the semicircular arc from $(0, 2)$ to $(0, -2)$ oriented counter-clockwise. Evaluate

$$\int_C xy^2 dx - xy dy$$

Example

Let C be the circle $(x-2)^2 + (y-3)^2 = 1$ oriented counter-clockwise. Evaluate

$$\int_C \left(y - \log(x^2 + y^2)\right) dx + \left(2 \tan^{-1}(y/x)\right) dy$$

Example

Let (a_1, b_1), (a_2, b_2), $\cdots$, (a_n, b_n) be the successive vertices of n-polygon. Show that the area inside the polygon is

$$\frac{1}{2}\left(\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} + \cdots + \begin{vmatrix} a_{n-1} & b_{n-1} \\ a_n & b_n \end{vmatrix} + \begin{vmatrix} a_n & b_n \\ a_1 & b_1 \end{vmatrix}\right)$$

Problem

Let $\mathbf{F} = \langle 3y, -4x \rangle$. Find $\displaystyle\oint_{\partial D} \mathbf{F} \cdot d\mathbf{s}$ oriented counter-clockwise for the following region D.

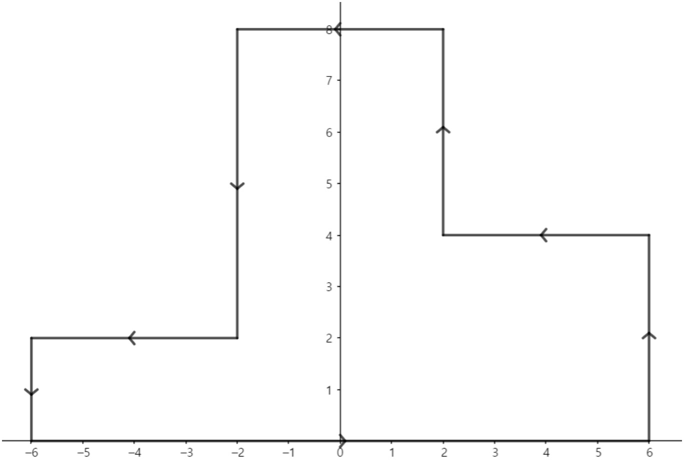
1. $D = \{(x, y) \,|\, x^2 + y^2 \leq 4\}$
2. $D = \{(x, y,) \,|\, x^2 + 2y^2 \leq 4\}$

Problem

Evaluate $\oint_C 5y dx - 3x dy$ where C is the cardioid $r = 1 - \sin\theta$ oriented counter-clockwise.

Problem

Evaluate $\oint_C (x^4y^5 - 2y)dx + (3x + x^5y^4)dy$ where C is as shown below.



Problem

Determine the value of

$$\oint_C \frac{xdx + ydy}{x^2 + y^2}$$

where C is a simple closed curve enclosing the origin oriented counter-clockwise.

Theorem (Stokes)

Let S be an oriented surface with a piecewise continuous boundary C. For $\mathbf{F}$ be a continuous vector field defined on S. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$$

where C and S are positively[1] oriented.

[1] The boundary is **postively** oriented if the direction is counter-clockwise with $\mathbf{n}$ being *upward*.

Proof.

Let $\mathbf{F} = \langle P, Q, R \rangle$. Suppose that the surface S is given by the graph of a function $z = f(x, y)$ on a bounded domain $D \subset \mathbf{R}^2$. Let $X(x, y) = (x, y, f(x, y))$ be the parametrization of S. Then

$$\begin{aligned}\iint_S \nabla \times \mathbf{F} d\mathbf{S} &= \iint_D -(R_y - Q_z) f_x - (P_z - R_x) f_y + (Q_x - P_y) dA \\ &= \iint_D \frac{\partial}{\partial x}(Q + R f_y) - \frac{\partial}{\partial y}(P + R f_x)\, dA\end{aligned}$$

Let C' be a planar curve which bounds that area D. By Green's theorem, the last integral becomes

$$\oint_{C'} (P + R f_x) dx + (Q + R f_y) dy = \oint_C P dx + Q dy + R dz = \oint_C \mathbf{F} \cdot d\mathbf{s}$$

$\square$

Remark

The Stokes' theorem provides the meaning of curl $\nabla \times \mathbf{F}$ of a vector field $\mathbf{F}$. Suppose that the surface S is planar disk with sufficiently small radius r centered at (x_0, y_0, z_0). Then

$$\oint_C \mathbf{F} \cdot d\mathbf{s} \approx \text{area}S \cdot (\nabla \times \mathbf{F})(x_0, y_0) \circ \mathbf{n}$$

$$(\nabla \times \mathbf{F})(x_0, y_0) \circ \mathbf{n} \approx \frac{1}{\text{area}S} \oint_C \mathbf{F} \cdot d\mathbf{s}$$

where $\mathbf{n}$ is the orientation of S. Therefore, the curl of a vector field $\mathbf{F}$ at the point p on the surface S has a projection onto the orthogonal direction of a surface S equal to the work done by $\mathbf{F}$ along the neighboring boundary of the point p on S.

Example

Let S be the surface bounded by $z = 1 - x^2 - y^2$, $z \geq 0$ with upward orientation ($\mathbf{n} \cdot \mathbf{k} \geq 0$). Confirm that Stokes' theorem holds for $\mathbf{F} = (y, -x, 0)$.

Theorem (Divergence Theorem)

Let V be a region in $\mathbf{R}^3$ whose boundary $S = \partial V$ is a closed surface.[2] *For a vector field $\mathbf{F}$ defined on V,*

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{F} dV$$

where the orientation of S is the outward direction.

[2] A surface is called **closed** if it has no boundary curve.

Proof.
Note that for $\mathbf{F} = (P, Q, R)$,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S P\mathbf{i} \cdot \mathbf{n} dS + \iint_S Q\mathbf{j} \cdot \mathbf{n} dS + \iint_S R\mathbf{k} \cdot \mathbf{n} dS$$

Suppose that the volume V is given by

$$V = \{(x, y, z) \,|\, h_1(x, y) \le z \le h_2(x, y),\ (x, y) \in D\}.$$

where D is the region in $\mathbf{R}^2$ on which the volume V is defined. Then

$$\begin{aligned} \iint_S R\mathbf{k} \cdot \mathbf{n} dS &= \iint_{S_1} R\mathbf{k} \cdot \mathbf{n} dS + \iint_{S_2} R\mathbf{k} \cdot \mathbf{n} dS \\ &= \iint_D R(x, y, h_2(x, y)) - R(x, y, h_1(x, y)) dx dy \\ &= \iiint_V R_z dV \end{aligned}$$

Example

Compute the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = (z^2, \frac{1}{3}x^3 + \tan z, z + y^2)$ and S is the closed surface $x^2 + y^2 + z^2 = 1$.

Example

Let S be a parabola $x^2 + y^2 + z = 2$ above the plane $z = 1$. Find the flux of $\mathbf{F} = (z \tan^{-1}(y^2), z^3 \ln(x^2 + 1), z)$ to the upward direction of S.

Example

Let $\mathbf{F} = \dfrac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}$ and S be the surface $z = 4 - x^2 - y^2$, $z \geq 0$ with upward orientation $\mathbf{n} \cdot \mathbf{k} \geq 0$. Use divergence theorem to compute the flux $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$. (How should we choose the volume V?)

Multivariable Calculus summerizes in two sentences:

- Derivative is a linear transformation.
  - Derivative $Df(\mathbf{a})$ of a multivariable function is a linear map between tangent spaces at $\mathbf{a}$ and $f(\mathbf{a})$.
- Divergence theorem is a stokes theorem.
  - The general form of stokes theorem is

$$\int_V d\mu = \int_{\partial V} \mu$$

where ν is a differential $(k-1)$-form and V is a k-dimensional space. The differential $d\mu$ is a k-form. The (special) Stokes theorem is when

$$\mu = Pdx + Qdy = Rdz$$

and divergence theorem is when

$$\mu = Pdx \wedge dy + Qdy \wedge dz + Rdz \wedge dx.$$

Problem

Let

$$\mathbf{A} = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}$$

Let S be a surface bounded by $z = 4 - x^2 - y^2$, $z \geq 0$, orieted upward. Find

$$\iint_S \mathbf{A} \cdot d\mathbf{S}$$

.

Problem

Let S be the surface bounded by $z = e^{-x^2-y^2}$ and $z \geq 1/e$. Let $\mathbf{n}$ be the orientation of S satisfying $\mathbf{n} \cdot \mathbf{k} \geq 0$. Find the flux of

$$\mathbf{F} = (e^{x+y} - xe^{y+z}, e^{y+z} - e^{x+y}, 2)$$

on S to the direction of $\mathbf{n}$.

Problem

Let C be the intersection of $z = 1 - 2(x^2 + y^2)$ and $z = x^2 - y^2$ oriented counter-clockwise. Find $\oint_C \mathbf{F} \cdot d\mathbf{s}$ where

$$\mathbf{F} = (y\cos(x) - yz, \sin x, e^z)$$

Problem

Let $S = \partial V$ be a closed surface. Prove the following.

1. For any constant vector field $\mathbf{C}$,

$$\iint_S \mathbf{C} \cdot d\mathbf{S} = 0$$

2. For any vector field $\mathbf{F}$,

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = 0$$