SE102:Multivariable Calculus

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Lecture 02
Continuity and Differentiability

Definition

The **polar coordinate** is a system of coordinate system which describes the Cartesian coordinate $P = (x, y)$ as (r, θ) where r is the length of $\overline{OP}$ and θ is the angle between $\overline{OP}$ and positive x-axis.

Example

The vector $\vec{r}$ and $\vec{\theta}$ is the unit vector to the direction where r and θ increases at unit rate. That is,

$$\vec{r} = (x, y)/\sqrt{x^2 + y^2}, \quad \vec{\theta} = (-y, x)/\sqrt{x^2 + y^2}.$$

Definition

The **cylinderical coordinates** (r, θ, z) is

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}.$$

The **spherical coordinates** (ρ, ϕ, θ) is

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}.$$

Example

In the cylinderical coordinates, the vectors $\vec{r}, \vec{\theta}, \vec{z}$ are given by

$$\vec{r} = (x, y, 0)/\sqrt{x^2 + y^2}, \quad \vec{\theta} = (-y, x, 0)/\sqrt{x^2 + y^2}, \quad \vec{z} = (0, 0, 1)$$

Example

The vectors $\vec{\rho}, \vec{\phi}, \vec{\theta}$ in the spherical coordinates are given by

$$\begin{aligned}
\vec{\rho} &= (x, y, z)/\sqrt{x^2 + y^2 + z^2} \\
\vec{\phi} &= (xz, yz, -(x^2 + y^2))/\sqrt{x^2 + y^2}\sqrt{x^2 + y^2 + z^2} \\
\vec{\theta} &= (-y, x, 0)/\sqrt{x^2 + y^2}
\end{aligned}$$

Remark

For n-dimensional space $\mathbb{R}^n$, there is a hyperspherical coordinate system $(\rho, \phi_1, \cdots, \phi_{n-2}, \theta)$, defined by

$$\begin{cases} x_1 = \rho \sin\phi_1 \cdots \sin\phi_{n-2} \cos\theta \\ x_2 = \rho \sin\phi_1 \cdots \sin\phi_{n-2} \sin\theta \\ x_3 = \rho \sin\phi_1 \cdots \cos\phi_{n-2} \\ \vdots \\ x_n = \rho \cos\phi \end{cases}$$

Definition

A **function** is a collection of three types of sets: (1) a domain X, (2) a codomain Y, (3) and a rule of assignment that associates each element $x \in X$ with a unique element $y \in Y$.

Remark

One may wonder why we use the term *set* to describe the rule of assignment of a set. This is because we can actually define the assignment rule as a set:

$$\{(x, y) \in X \times Y \mid \text{no two } y_1, y_2 \in R \text{ appears in pairs } (x, y_1), (x, y_2) \text{ with the same } x \in D\}$$

Definition

The range R of a function $f : X \to Y$ is the set of all elements in the codomain that are assigned by the function with an element in X. That is, $R = \{f(x) \mid x \in D\}$.

Definition

A function $f : X \to Y$ is called **onto** (or **surjective**) if $Y = R$. A function is called **one-to-one** (or **injective**) if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for all $x_1, x_2 \in X$. A function is called **one-to-one correspondence** (or **bijective**) if it is one-to-one and onto.

Definition

A function is called **multivariable** if it consists of more than two independent or dependent variables.

In general a multivariable function f consists of n independent variables and m dependent variables.

$$f(x_1, x_2, \cdots, x_n) = (y_1, y_2, \cdots, y_m) \tag{1}$$

The variable y_j is the dependent variableof a function f_j with n independent variables $x_1, x_2, \cdots, x_n$. Thus we can also write the function f as m-tuple of real-valued function f_j's.
$(j = 1, \cdots, m)$

$$y_j = f_j(x_1, x_2, \cdots, x_n) \tag{2}$$

Definition

Let $f(x,y) = z$ be a function defined on a set $D \subset \mathbf{R}^2$. The **graph** of f is the set in $\mathbf{R}^3$ defined by

$$G(f) = \{(x, y, f(x,y)) \,|\, (x,y) \in D\}$$

Example

Draw the graph of

1. $f(t) = (t^2, t^3)$
2. $c(t) = (\cos(t), t, \sin(t))$
3. $z = x^2 - y^2$

The set of all points satisfying a quadratic equation (i.e. a polynomial of degree 2) is called a quadric surface. The following are some examples of quadric surfaces.

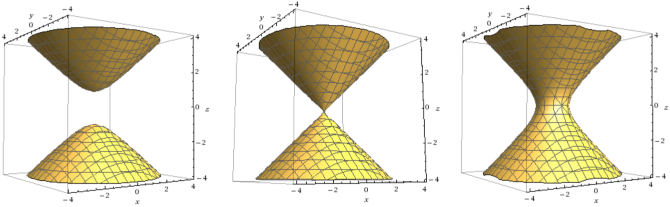
1. Ellipsoid $\dfrac{x^2}{4} + y^2 + \dfrac{z^2}{2} = 1$
2. (Elliptic) paraboloid $z = x^2 + y^2$
3. Saddle (or elliptic paraboloid) $z = x^2 - y^2$
4. (Elliptic) cone $z^2 = x^2 + y^2$
5. Hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$
6. Hyperboloid of two sheets $x^2 + y^2 - z^2 = -1$

Example

We cannot draw the graph of a function $w = f(x, y, z)$ with three independent variables since we would need 4-dimensional space. Thus we will use the **level set** instead: The level set of f at c, denoted by $L_c(f)$ is a set defined by

$$L_c(f) = \{(x, y, z) \in \mathbf{R}^3 \mid f(x, y, z) = c\}$$

The followings are the level sets of $f(x, y, z) = x^2 + y^2 - z^2$ at $c = -1, 0, 1$.



Definition

Let V be a vector space and D be a open subset of $\mathbf{R}^n$. A **vector field** $\mathbf{F}: D \to V$ is a function which assigns each point $(x_1, \cdots, x_n) \in D$ a vector $\mathbf{F}(x_1, x_2, \cdots, x_n) \in V$.

Definition

Let v, w be vector spaces. A function $t : v \to w$ is called a **linear transformation** if it satisfies the following.

1. $t(v_1 + v_2) = t(v_1) + t(v_2)$ for all $v_1, v_2 \in v$.
2. $t(cv) = ct(v)$ for all $v \in v$ and $c \in \mathbf{R}$.

Example

Every linear transformation can be represented by a matrix, and the composition of two linear transformation is represented by the multiplication of corresponding matrices.

Definition

Let $f(x, y)$ be a two-variable function defined on the entire plane $\mathbf{R}^2$. A constant L is called the **limit of f at** (x_0, y_0) if for any (arbitrary small) $\epsilon > 0$, there exists a (small) $\delta > 0$ such that whenever the distance between (x, y) and (x_0, y_0) is less than δ, the inequality

$$|f(x, y) - L| < \epsilon$$

holds. In such case, we simply write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

Definition

We say a function $f(x, y)$ is **continuous at** (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$$

Remark

We say $f(x)$ is continuous at x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$. The limit $\lim\limits_{x \to x_0}$ assumes that both right and left limits exists and equals to each other. Since there are only two paths approaching x_0 in one-dimensional space, $\mathbf{R}$, the concept of the limit is intuitively clear.

On a two-dimensional plane, there are infinitely many directions and passes for (x, y) to approach the limit point (x_0, y_0). The limit exists only when the limit coincides regardless of the choice of passes (and direction) of points approaching the limit point. This is why we need a rigorous definition of limit for the multivariable function.

Example

Show that

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} \tag{3}$$

is continuous at $(0,0)$.

Proposition

A function $f(x, y)$ is <u>not</u> continuous at (x_0, y_0) if there is a path $c(t) = (x(t), y(t))$ which converges to (x_0, y_0) while $f \circ c(t)$ does not converges to $f(x_0, y_0)$.

Example

Let us prove that the function

$$f(x,y) = \begin{cases} \dfrac{x^4 - 4y^2}{x^2 + 2y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is not continuous at $(0, 0)$. Draw the graph and explain the discontinuity on the graph.

Let us explore the geometric meaning of limit (and continuity) of multi-variable function.

Suppose a function $f : \mathbb{R}^n \to \mathbb{R}^m$ satisfies $\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = \mathbf{L}$. Here, the value $\mathbf{L}$ is a point in $\mathbb{R}^m$ and $\mathbf{a} \in \mathbb{R}^n$. The definition of the limit tells us that for any (small) $\varepsilon > 0$, we can choose (sufficiently small) $\delta > 0$ so that as long as $\mathbf{x}$ lies in the ball $B_\delta(\mathbf{a})$ of radius δ centered at $\mathbf{a}$, the value $f(\mathbf{x})$ lies in the ball $B_\varepsilon(\mathbf{L})$ of radius ε centered at $\mathbf{L}$. This implies that $f(B_\delta) \subset B_\varepsilon$.

In Topology, we say that a function has a limit $\mathbb{L}$ at $\mathbf{a}$ if however small *neighborhood* V of $\mathbf{L}$ we choose, we can always find a (very small) *neighborhood* U of $\mathbf{a}$ whose image $f(U)$ is completely contained in V.

Definition

Let $f(x, y)$ be a function defined on a region $D \subset \mathbf{R}^2$ and $(x_0, y_0) \in D$. Let $\mathbf{u} = (a, b)$ a *unit* vector. If the limit

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{t \to 0} \frac{f(x_0 + at, y_0 + bt) - f(x_0, y_0)}{t}$$

exists, then it is called the **u-directional derivative** of f at (x_0, y_0).

Example

Let $f(x,y) = x^2 + y^2$. The $(1,0)$-directional derivative of f at $(\frac{1}{2}, 0)$ is

$$\begin{aligned} D_{(1,0)} f\left(\tfrac{1}{2}, 0\right) &= \lim_{t \to 0} \frac{f(\frac{1}{2} + t, 0) - f(\frac{1}{2}, 0)}{t} \\ &= \lim_{t \to 0} \frac{\left(\frac{1}{2} + t\right)^2 - (\frac{1}{2})^2}{t} = 1 \end{aligned}$$

Remark

For a single-variable function $y = f(x)$, the differential $f'(x_0)$ represents rate of change of $f(x)$ as x approaches to x. The directional derivative extends this concept. Let $c(t) = (x_0, y_0) + t\mathbf{u}$. $c(t)$ approaches to (x_0, y_0) as $t \to 0$. The composition $f \circ c(t)$ is a single-variable function. The differential $(f \circ c)'(0)$ is

$$(f \circ c)'(0) = D_{\mathbf{u}} f(x_0, y_0)$$

The directional derivative is the *rate of chage of f along straight line passing through* (x_0, y_0) *parallel to* $\mathbf{u} = (u_1, u_2)$. Let P be the plane parallel to both $(u_1, u_2, 0)$ and $\mathbf{k}$ containing the point $(x_0, y_0, 0)$. Then $D_{\mathbf{u}} f(x_0, y_0)$ is the slope of the intersection curve between the graph of f and the plane P.

Remark

$f : U \to \mathbf{R}$ be a two-variable function defined on a region $U \subset \mathbb{R}^2$. Let $\mathbf{u}, \mathbf{v}$ be 2-dimensional vectors. Suppose that the $D_{\mathbf{u}} f(x, y)$ exists for all (x, y) in a region U. Then we can define a new function g on U as

$$g(x, y) = D_{\mathbf{u}} f(x, y).$$

Suppose that $g(x, y)$ has $\mathbf{v}$-direnctional derivative $D_{\mathbf{v}} g(x_0, y_0)$. We call this as the *second* directional derivative of f. That is,

$$D_{\mathbf{v}} D_{\mathbf{u}} f(x_0, y_0) = D_{\mathbf{v}} g(x_0, y_0).$$

The value $D_{\mathbf{u}} f$ is the slope of the graph of f along the $\mathbf{u}$-direction. Thus there is a unique vector tagent to the graph of f at $(x_0, y_0, f(x_0, y_0))$ whose projection onto $\mathbf{R}^2$ is parallel to $\mathbf{u}$. The value $D_{\mathbf{v}} D_{\mathbf{u}} f$ measures how much such tangent vector *changes* along v-direction at (x_0, y_0).

For example, $D_{\mathbf{u}} D_{\mathbf{u}} f$ is the acceleration f in the $\mathbf{u}$-direction.

Example

Let $f(x,y) = x^3 + 5x^2y + y^3$ and $\mathbf{u} = (\frac{3}{5}, \frac{4}{5})$. Find $D_{\mathbf{u}}f$ and $D_{\mathbf{u}}D_{\mathbf{u}}f$.

Definition

Let $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ be the orthonormal vectors in $\mathbf{R}^2$. We denote D_x, D_y for the $\mathbf{e}_1$, $\mathbf{e}_2$-directional derivatives with respectively. The $D_x f(x_0, y_0)$, $D_y f(x_0, y_0)$ are called the **partial derivatives** of $f(x,y)$ at (x_0, y_0). Conventionally, the partial derivatives are denoted by

$$D_x f(x_0, y_0) = f_x(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)$$

$$D_y f(x_0, y_0) = f_y(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0)$$

We write $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ for the partial derivatives as two-variable functions.

The second partial derivatives are denoted as

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = D_{xx} f = D_x(D_x f)$$

$$\frac{\partial^2 f}{\partial x \partial y} = f_{yx} = D_{xy} f = D_x(D_y f)$$

$$\frac{\partial^2 f}{\partial y \partial x} = f_{xy} = D_{yx} f = D_y(D_x f)$$

$$\frac{\partial^2 f}{\partial y^2} = f_{yy} = D_{yy} f = D_y(D_y f)$$

Note that f_{xy} and f_{yx} are *not* the same function in general.

Example

Let

$$f(x,y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Compute $f_{xy}f(0,0)$ and $f_{yx}(0,0)$.

Definition

Let $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ be the partial derivatives of $f(x, y)$. Then the function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

is called the **linear approximation** of $f(x, y)$ at (x_0, y_0).

Remark

The graph of $z = L(x, y)$ is the tangent plane to the graph of $z = f(x, y)$.

Definition

Let $L(x,y)$ be the linear approximation of $f(x,y)$ at (x_0,y_0). We say the function $f(x,y)$ is **differentiable** at (x_0,y_0) if the following holds.

$$\lim_{(x,y)\to(x_0,y_0)} \frac{|f(x,y)-L(x,y)|}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$$

Example

The existence of the linear approximation $L(x, y)$ does not imply that $f(x, y)$ is differentiable. Find an example of a function $f(x, y)$ which has the linear approximation at $(0, 0)$, but not differentiable at $(0, 0)$.

Example

Show that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is not differentiable at $(0, 0)$.

Remark

Recall that we defined the differentiability of single variable function as

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

By taking $f'(x_0)$ to the left-hand side, equation becomes

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = 0$$

which is the special case of the previous definition.

Theorem

Suppose that there are two function $\epsilon_1 = \epsilon_1(x, y)$, $\epsilon_2 = \epsilon_2(x, y)$ *satisfying*

$$
\begin{aligned}
f(x, y) - f(x_0, y_0) = {} & f_x(x_0, y_0)(x - x_0) \\
& + f_y(x_0, y_0)(y - y_0) + \epsilon_1(x - x_0) + \epsilon_2(y - y_0).
\end{aligned}
$$

If $\epsilon_1, \epsilon_2 \to 0$ *as* $(x, y) \to (x_0, y_0)$, *then* $f(x, y)$ *is differentiable at* (x_0, y_0).

Definition

Let $f : \mathbf{R}^n \to \mathbf{R}$ be a function whose coordinate functions f_i are differentiable. The **gradient** is the vector

$$\nabla f = (f_{x_1}, \cdots, f_{x_n})$$

Then f is differentiable at $\mathbf{x}_0 \in \mathbf{R}^n$ if

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{\|\mathbf{x} - \mathbf{x}_0\|} = 0$$

Definition

Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be a function whose coordinate functions are differentiable. The differential of f at x_0 is the matrix

$$Df(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial f_1}{x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_1}{x_n}(\mathbf{x}_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_m}{x_n}(\mathbf{x}_0) \end{bmatrix}$$

Then f is differentiable at $\mathbf{x}_0$ if

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - Df(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)}{\|\mathbf{x}-\mathbf{x}_0\|} = \vec{0}$$

Problem

Draw the graph of

1. $z = x(x^2 - y^2)$ (This surface is called Monkey's saddle.)
2. $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Problem

Show that

$$\lim_{(x,y)\to(0,0)} \frac{|xy|}{x^2+y^2}$$

does not exist.

Problem

Show that the function

$$f(x,y) = \begin{cases} \dfrac{y^2}{|x-y|} & x \neq y \\ 0 & x = y \end{cases}$$

is discontinuous at $(0,0)$.

Problem

Consider the following statements.

1. The partial derivatives f_x, f_y are continuous at (x_0, y_0).
2. The function f is differentiable at (x_0, y_0).
3. The directional derivative $D_{\mathbf{u}}f(x_0, y_0)$ exists for every $\mathbf{u}$.
4. The function f is continuous at (x_0, y_0).

Show that the following implications hold.

- $1 \Rightarrow 2$
- $2 \Rightarrow 3$
- $2 \Rightarrow 4$

Find the counter-examples for

- $2 \Rightarrow 1$
- $3 \Rightarrow 2$
- $4 \Rightarrow 2$