Study on differentiation & Integration.

SE102: Multivariable Calculus

$f(x, y) = z$

Hyosang Kang[1]

[1]Division of Mathematics
School of Interdisciplinary Studies
DGIST

Week 01

Definition

A (n-dimensional) **vector** is a n-tuple of real numbers

$$\mathbf{a} = (a_1, a_2, \cdots, a_n)$$

with two operations.

- (vector sum) For a vector $\mathbf{b} = (b_1, b_2, \cdots, b_n)$,

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n).$$

- (scalar multiplication) For $k \in \mathbf{R}$, we define a vector $k\mathbf{a}$ as

$$k\mathbf{a} = (ka_1, ka_2, \cdots, ka_n).$$

The (n-dimensional) **vector space** is the set of all (n-dimensional) vectors, and we denote it as $\mathbf{R}^n$.

Rmk $\mathbb{R}^n$ is also used for n-dim'l SPACE:

Example

Let $O = (0, 0, 0)$ be the origin and $P = (a_1, a_2, a_3)$ a point in 3-dimensional space. A **position vector** from O to P is a vector

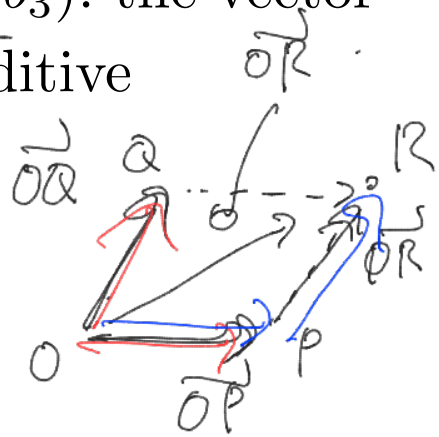
$$\overrightarrow{OP} = (a_1, a_2, a_3).$$

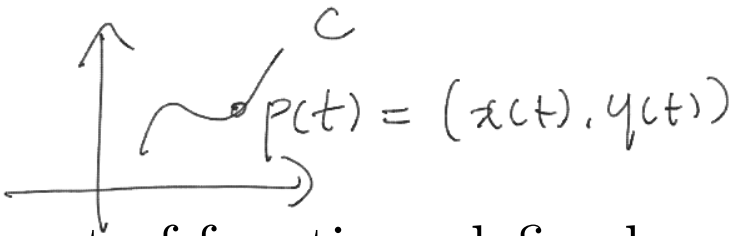
$\langle a_1, a_2, a_3 \rangle = \overrightarrow{OP}$

The scalar multiplication of a position vector is a dilation. The vector sum of two position vectors is a superposition.

Let $Q = (b_1, b_2, b_3)$ and $R = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$. the vector $\overrightarrow{PR}$ is the same as $\overrightarrow{OQ}$. We have the following additive operation.

$$\overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR}$$





Example

A parametrization of a curve is a set of functions defined on a common interval, each indicates a coordinate of a point on the curve. Let $c : (-\varepsilon, \varepsilon) \to \mathbf{R}^2$ be a parametrization of a curve on a plane.

open interval. ($\varepsilon > 0$)

the 2-dim'l space.

t : ind. variable

x, y : dep. variable.

$$c(t) = (x(t), y(t))$$

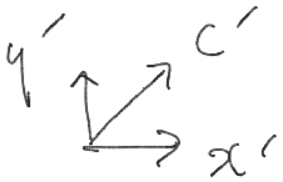
The differntial $c'(t) \overset{\text{defn}}{=} (x'(t), y'(t))$ is a vector which represents the **velocity** of the parametrization at t. The x, y-components of the vector $c'(t)$ represent the projection of the speed of $c(t)$ in x, y-directions respectively. We can decompose the velocity vector into the sum of horizontal and vertical velocity of $c(t)$.

$$c'(t) = (x'(t), 0) + (0, y'(t))$$

$c(t)$: point.

$c'(t)$: vector.

t : time.



$f : \mathbb{R}^2 \to \mathbb{R}^3$

Example

Let $\mathbb{R}^n$ be the set of all n-dimensional ~~space~~ point. Let $\boxed{f : \mathbb{R}^2 \to \mathbb{R}^3}$ be a function:

ind. dep.

$$f(u, v) = (x(u, v), y(u, v), z(u, v)) \tag{1}$$

The variables x, y, z are dependent to the variables u, v. The functions f (and also x, y, z as functions) is called a **multivariable function** since it contains two or more independent or dependent variables. If we write $\mathbf{a} = (u, v)$ and $\mathbf{b} = (x, y, z)$, the equation (1) can be written as $f(\mathbf{a}) = \mathbf{b}$.

Rmk: the "graph" of f is a "surface" in $\mathbb{R}^3$.

Definition

Let $\mathbf{a} = (a_1, a_2, \cdots, a_n)$ be a (n-dimensional) vector. The **norm** of $\mathbf{a}$ is the value

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}.$$

Euclidean norm.

We call a vector $\mathbf{a}$ a unit vector if $\|a\| = 1$. The **zero vector** $\mathbf{0}$ is the vector satisfying

$$\boxed{\mathbf{0} + \mathbf{a} = \mathbf{a} + \mathbf{0} = \mathbf{a}.}$$

for any vector $\mathbf{a}$. Note that $\|\mathbf{a}\| = 0$ if and only if $\mathbf{a} = \mathbf{0}$. The **normalization** of $\mathbf{a}$ is the vector

$$\mathbf{u} = \mathbf{a}/\|\mathbf{a}\|$$

has the same direction as $\vec{a}$.
but the length is 1.

Example

For each $i = 1, \cdots, n$, the (unit) **basis vector** is

ith position

$$\mathbf{e}_i = (0, \cdots, 1, \cdots, 0).$$

$\vec{a} = (a_1, \cdots, a_n) = a_1\vec{e_1} + \cdots + a_n\vec{e_n} = (a_1, 0, \cdots, 0) + \cdots + (0, \cdots, a_n)$

(1 is at the i th place.) In particular, we denote 3-dimensional basis vectors as

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1)$$

We can decompose any vector $\mathbf{a} = (a_1, a_2, \cdots, a_n)$ as a linear sum of basis vectors:

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n$$

Thus, we call the set $\boxed{\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}}$ as a **basis** of $\mathbf{R}^n$.

Definition

$\langle \vec{a}, \vec{b} \rangle = \vec{a} \cdot \vec{b}$

The **inner product** (also called **dot product**) of two vectors $\mathbf{a}$, $\mathbf{b}$ is an operation defined by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

If $\boxed{\mathbf{a} \cdot \mathbf{b} = 0}$, then we say $\mathbf{a}$, $\mathbf{b}$ are **orthogonal**.

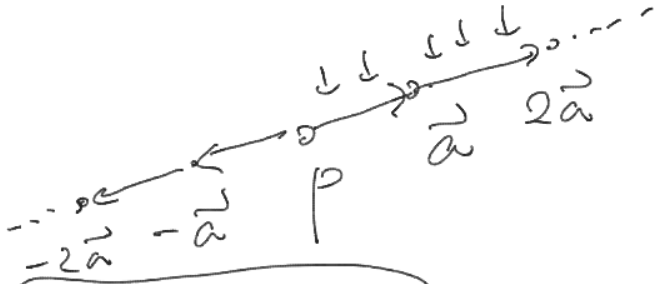
Proposition

The inner product satisfies the following.

1. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
2. $\mathbf{a} \cdot (k\mathbf{b}) = k(\mathbf{a} \cdot \mathbf{b})$
3. $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$

cf. $\vec{0} \cdot \vec{a} = 0$.

Example



Let us parametrize a line l in $\mathbf{R}^3$.

1. Suppose that l passes through $P = (x_0, y_0, z_0)$ and parallel to $\mathbf{a} = (a_1, a_2, a_3)$. Then the paramtric equations of l is

$$(x, y, z) = l(t) = P + t\mathbf{a} \tag{2}$$

$$= (x_0, y_0, z_0) + (ta_1, ta_2, ta_3)$$

Equivalently, equation (2) can be written as a *symmetric form* as follows.

$$(x - x_0, y - y_0, z - z_0) = (ta_1, ta_2, ta_3)$$

$$x - x_0 = ta_1$$
$$y - y_0 = ta_2$$
$$z - z_0 = ta_3$$

$$t = \frac{x - x_0}{a_1} = \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3} \qquad a_1, a_2, a_3 \neq 0 \tag{3}$$

2. Suppoe that l passes through two points P, Q. By subsitute $\mathbf{a} = \overrightarrow{OP} - \overrightarrow{OQ}$ to (2) or (3), we get a parametric equation of l.

$$\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ} \Rightarrow \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$
$$= \vec{a}.$$

Theorem

Let $\mathbf{a}$, $\mathbf{b}$ *be nonzero* $\overset{n}{2}$*-dimensional vectors. If* θ *is the angle between* $\mathbf{a}$ *and* $\mathbf{b}$*, then*

$$\boxed{\mathbf{a} \cdot \mathbf{b}} = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot \cos\theta$$

Definition

For nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ with the same dimension, the vector defined by

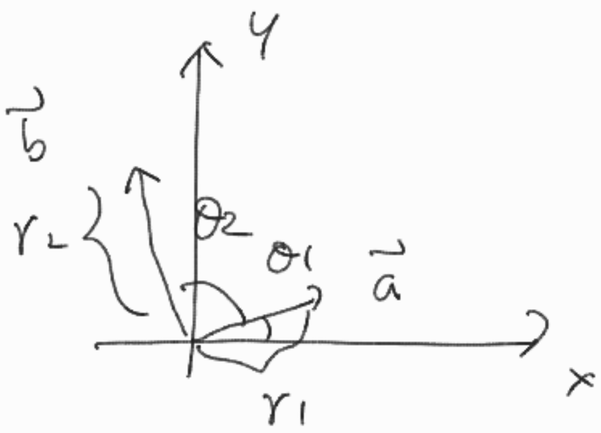
$$\text{proj}_{\mathbf{b}}\mathbf{a} = \boxed{\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\right)\mathbf{b}} = \left(\frac{\vec{a} \cdot \vec{b}}{\|b\|}\right) \cdot \underbrace{\left(\frac{\vec{b}}{\|b\|}\right)}_{\text{unit direction of } \vec{b}}.$$

is called the **projection of a onto b**.

proof of $\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot\cos\theta$.

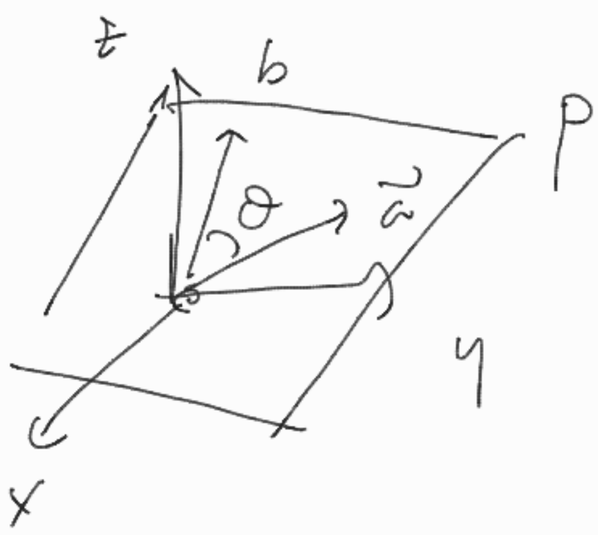
$$\vec{a} = r_1(\cos\theta_1, \sin\theta_1)$$
$$\vec{b} = r_2(\cos\theta_2, \sin\theta_2)$$

$$\vec{a}\cdot\vec{b} = r_1 r_2 \cos\theta_1\cos\theta_2 + r_1 r_2 \sin\theta_1 \sin\theta_2$$
$$= r_1 r_2 (\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$$
$$= r_1 r_2 \cos(\theta_2-\theta_1) = r_1 r_2 \cos(\theta_1-\theta_2)$$
$$= |\vec{a}|\cdot|\vec{b}|\cdot\cos\theta.$$

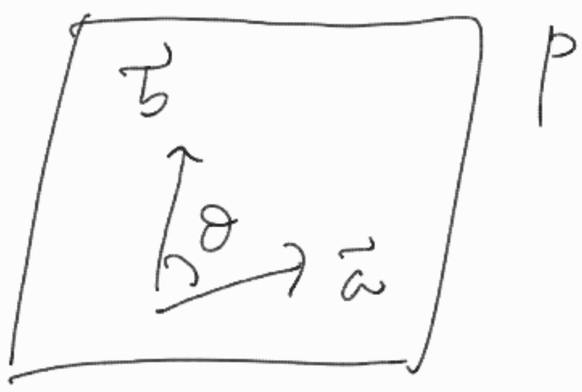


Rmk $\theta = \pi/2 \Rightarrow \vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a}, \vec{b}$ are "perpendicular".

orthogonal.
$\Updownarrow$



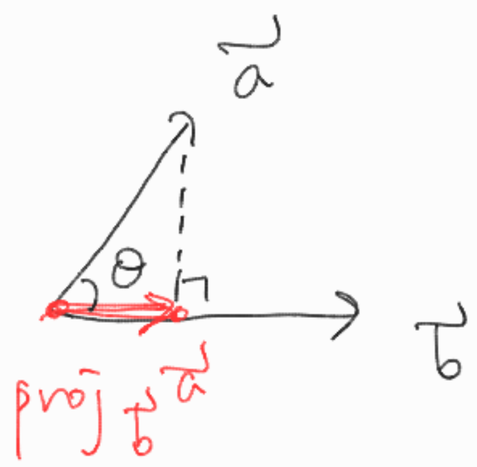
There must be a plane P, $O \in P$ containing $\vec{a}, \vec{b}$.



Even in n-dim'l space there is a plane P containing $\vec{a}, \vec{b}$, passing through O.

$$|\text{proj}_{\vec{b}}\vec{a}| = \boxed{|\vec{a}|\cdot\cos\theta}.$$

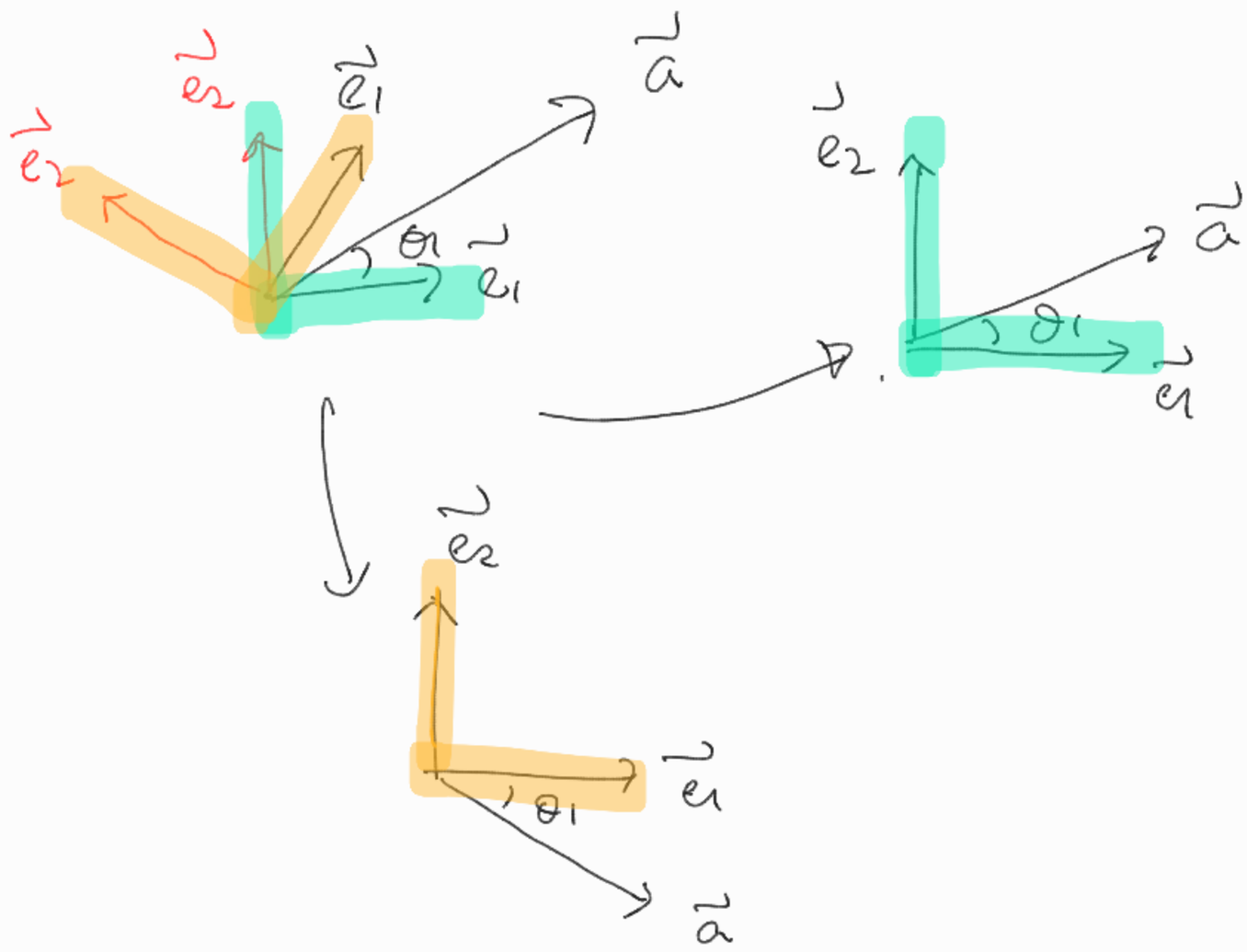
direct of $\text{proj}_{\vec{b}}\vec{a} = \dfrac{b}{|\vec{b}|}$.



$$\Rightarrow \text{proj}_{\vec{b}}\vec{a} = |\vec{a}|\cdot\cos\theta\cdot\frac{b}{|\vec{b}|}.$$

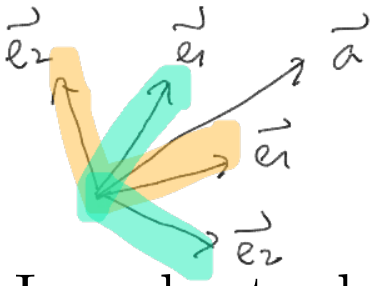
$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cdot\cos\theta \Rightarrow \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$$

$$\Rightarrow \text{proj} = \cancel{|\vec{a}|}\cdot\frac{\vec{a}\cdot\vec{b}}{\cancel{|\vec{a}|}\cdot|\vec{b}|}\cdot\frac{b}{|\vec{b}|} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$


$\vec{e}_2$
$\vec{e}_1$
$\vec{a}$
θ_1
$\vec{e}_1$
$\vec{e}_2$
$\vec{a}$
θ_1
$\vec{e}_1$
$\vec{e}_2$
$\vec{e}_1$
θ_1
$\vec{a}$

Remark

As norm measures the *size* of a vector, the inner product measure the *direction*. For example, the direction of a 2-dimensional vector $\mathbf{a}$ is determined by $0 \le \theta_1, \theta_2 \le \pi$ satisfying

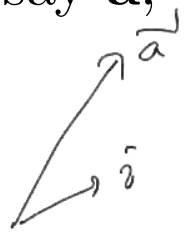


$$\mathbf{a} \cdot \mathbf{e}_1 = \|\mathbf{a}\| \cos\theta_1$$

$$\frac{\vec{a} \cdot \vec{e}_1}{\|\vec{a}\|} = \boxed{\cos\theta_1}$$

$$\mathbf{a} \cdot \mathbf{e}_2 = \|\mathbf{a}\| \cos\theta_2$$

In order to determine the direction of a 3-dimensional vector, say $\mathbf{a}$, we need three angles $0 \le \alpha, \beta, \gamma \le \pi$ satisfying



$$\cos\alpha = \frac{\mathbf{a} \cdot \mathbf{i}}{\|\mathbf{a}\|}, \quad \cos\beta = \frac{\mathbf{a} \cdot \mathbf{j}}{\|\mathbf{a}\|}, \quad \cos\gamma = \frac{\mathbf{a} \cdot \mathbf{k}}{\|\mathbf{a}\|} \tag{4}$$

Such quantities are called the **direction cosines**.

Definition

A $n \times m$ (n-by-m) **matrix** is a collection of nm numbers (or functions) arranged in the following way.

$$A = (a_{ij})_{n\times m} = \begin{array}{c} \overbrace{\hphantom{a_{11} \quad a_{12} \quad \cdots \quad a_{1m}}}^{m} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \end{array} \left.\vphantom{\begin{matrix} a \\ a \\ \vdots \\ a \end{matrix}}\right\} n$$

nm-tuple of real #s.

The indices i, j of an entry a_{ij} represents the row and column indices respectively.

Example

1. A $n \times m$ matrix is called **square** matrix if $n = m$.
2. If A is a square matrix and $a_{ij} = 0$ for all $i \neq j$, then A is called **diagonal**.

$$A = \begin{pmatrix} a_{11} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & a_{nn} \end{pmatrix}$$

3. If the diagonal entries of a diagonal matrix are all 1, then it is called the **identity matrix**, and denoted by I_n.

$$I_n = \begin{pmatrix} 1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & 1 \end{pmatrix}$$

4. If A is a square matrix and $a_{ij} = 0$ for all $i < j$ ($i < j$, respectively), then A is called **lower triangle** (**upper**

Definition

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $n \times m$ matrix. Then we define

$$A + B = (a_{ij} + b_{ij})$$

a matrix is a vector

$$k \cdot A = (ka_{ij})$$

Let C be a $m \times l$ matrices, then $A \cdot C$ is a $n \times l$ matrix whose entries are

$A : n \times m$

i,j entry of $A \cdot B$.

$$A \cdot B = \boxed{(a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj})}_{1 \le i \le n, 1 \le j \le l}$$

jth col.

ith row

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{im} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1l} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2l} \\ \vdots & & \vdots & & \vdots \\ b_{m1} & \cdots & b_{mj} & \cdots & b_{ml} \end{pmatrix}$$

Example

Suppose Bob, Larry, and Joanna worked in a fruits store for three days. Table 1 shows *how many* fruits each sold in total represented by the matrix A, and Table 2 shows *how much* was the fruits on each day represented by the matrix B.

	Apple	Orange	Banana
Bob	38	25	10
Larry	15	22	15
Joanna	8	70	27

← # of items sold by person.

Table: The volumn of sales of each person per items

	Day1	Day2	Day3
Apple	\$1.19	\$1.45	\$.99
Orange	\$1.70	\$0.99	\$2.1
Banana	\$2.19	\$3.5	\$1.29

← price of items on each day.

Table: The prices of items per day

Example

The i, j-entries of the $A \cdot B$ represents the total revenue sold by the person i at the day j.

$$\overset{A}{\overset{3\times 3}{\begin{pmatrix} 38 & 25 & 10 \\ 15 & 22 & 15 \\ 8 & 70 & 27 \end{pmatrix}}} \quad \overset{B}{\overset{3\times 3}{\begin{pmatrix} 1.19 & 1.45 & 0.99 \\ 1.70 & 0.99 & 2.1 \\ 2.19 & 3.5 & 1.29 \end{pmatrix}}}$$

A deeper, and more mathematical, thus important meaning of matrix multiplication will be discovered when we visit in the *chain rule* in later sections.

amount of \$ that Bob sold on day 1.

$$\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

$$= \begin{pmatrix} \boxed{38\times 1.19 + 25\times 1.70 + 10\times 2.19} & * & * \\ * & * & * \\ \boxed{8\cdot 1.19 + 70\cdot 1.70 + 27\cdot 2.19} & * & * \end{pmatrix}$$

amount of \$ Joanna sold on day 1.

If you want to get $A \cdot B$, # of col of A = # row of B.

Proposition

Let A, B, C be matrices. Whenever the operations are valid, the following holds.

1. $(A \cdot B) \cdot C = A \cdot (B \cdot C)$ ← associativity rule.
2. $A \cdot (B + C) = A \cdot B + A \cdot C$ ← distribution rule.
3. $(B + C) \cdot A = B \cdot A + C \cdot A$ ✓
4. $k \cdot (A \cdot B) = (k \cdot A) \cdot B = A \cdot (k \cdot B)$

   $k \in \mathbb{R}$

Proposition

The ***transpose*** *of a matrix* $A = (a_{ij})$ *defined by*

$$A^T = (a_{ji})$$ ← You switch row & col.

and it satisfies the following.

5. $(A^T)^T = A$
6. $k \cdot A^T = (k \cdot A)^T$
7. $(A+B)^T = A^T + B^T$ ← dist. rule.
8. $(AB)^T = B^T A^T$ Important because $A \cdot B \neq B \cdot A$ (even if $A \cdot B$, $B \cdot A$ are well-defined)

$\neq$

$A^T \cdot B^T$

(This may not be well-define)

$A : n \times m$

$B : m \times l$

$\Rightarrow A^T : m \times n$, $B^T : l \times m$

Definition

The **determinant** of 2×2 matrix A is defined as follows.

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

The **determinant** of 3×3 matrix B is defined as follows.

$$\det B = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} = b_{11}b_{22}b_{33} + b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32}$$

$$- b_{13}b_{22}b_{31} - b_{11}b_{23}b_{32} - b_{12}b_{21}b_{33}$$

Definition

Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ be two 3-dimensional vectors. The **cross-product** of $\mathbf{a}, \mathbf{b}$ is a vector defined by

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$$

An easy way to remember the formula is the following.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= i \cdot a_2 \cdot b_3 + j \cdot a_3 \cdot b_1 + k \cdot a_1 b_2 - k \cdot b_1 \cdot a_2 - i \cdot b_2 \cdot a_3 - j \cdot a_1 \cdot b_3$$

$$= (a_2 b_3 - a_3 b_2) i + (a_3 \cdot b_1 - a_1 b_3) j + (a_1 b_2 - b_1 a_2) k.$$

$i = (1, 0, 0)$

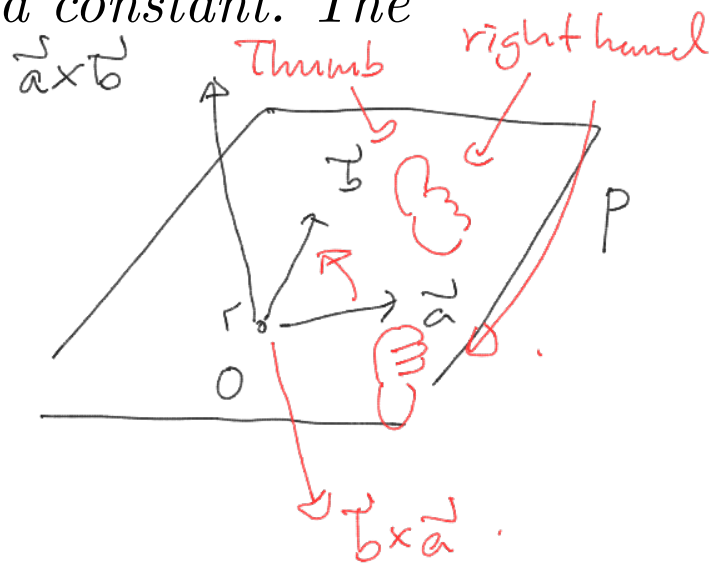
$j = (0, 1, 0)$

$k = (0, 0, 1)$

Rmk: If $\vec{a}, \vec{b}$ are not parallel then $\vec{a}\times\vec{b}$ is a normal (perpendicular) direction to the plane containing $\vec{a}, \vec{b}$.

Proposition

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *be 3-dimensional vectors and* k *a constant. The following identity holds.*

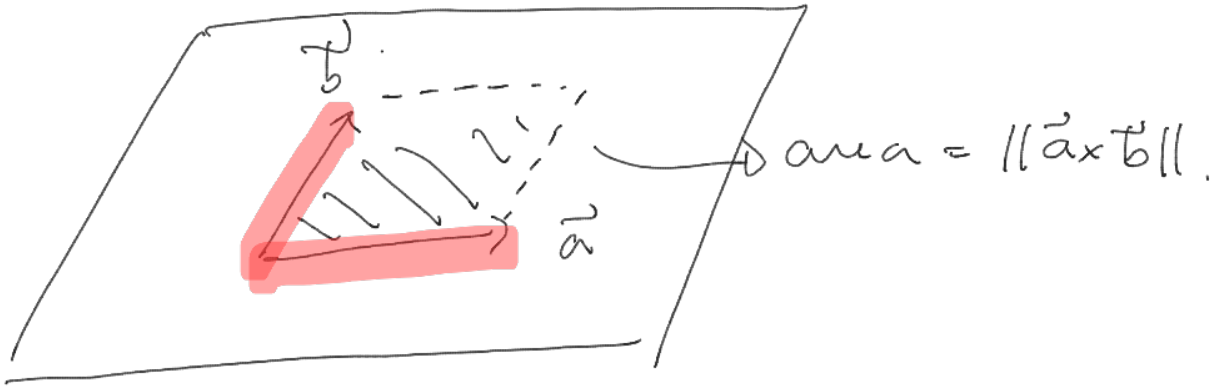


1. $\mathbf{a} \times \mathbf{0} = \mathbf{0} \times \mathbf{a} = \mathbf{0}$
2. $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ ← If two rows of the matrix is the same then det = 0.
3. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
4. $\mathbf{a} \times (k\mathbf{b}) = (k\mathbf{a}) \times \mathbf{b} = k(\mathbf{a} \times \mathbf{b})$
5. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
6. $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$ *(This shows that the cross product* $\mathbf{a} \times \mathbf{b}$ *is normal to the plane spanned by* $\mathbf{a}$ *and* $\mathbf{b}$*.)*
7. $\underbrace{(\mathbf{a} \times \mathbf{b})}_{\text{3-dim'l vec}} \times \mathbf{c} = \underbrace{(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}}_{\text{3-dim'l vec}}$

$$\vec{a}\circ(\vec{a}\times\vec{b}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$$

$$\vec{a}\times\vec{b} = (A)i + (B)j + (C)k = (A, B, C)$$

$$\vec{a}\circ(\vec{a}\times\vec{b}) = (a_1, a_2, a_3) \circ (A, B, C) = A\cdot a_1 + B\cdot a_2 + C\cdot a_3$$



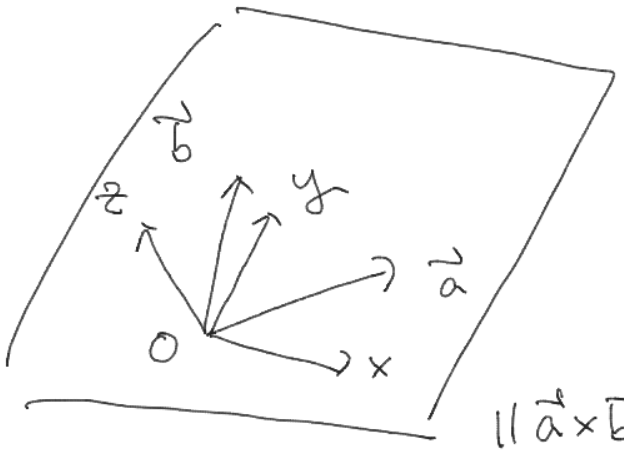
Theorem

Let $\mathbf{a}, \mathbf{b}$ *be two 3-dimensional vectors. Then*

$$\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\cdot\|\mathbf{b}\|\cdot|\sin\theta|$$

where θ *is the angle between* $\mathbf{a}$ *and* $\mathbf{b}$.

consider P as xy-plane.



P Then $\vec{a} = |\vec{a}|\cdot(\cos\theta_1, \sin\theta_1, 0)$

$\vec{b} = |\vec{b}|\cdot(\cos\theta_2, \sin\theta_2, 0)$

$$\vec{a}\times\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot\begin{vmatrix} i & j & k \\ c(\theta_1) & s(\theta_1) & 0 \\ c(\theta_2) & s(\theta_2) & 0 \end{vmatrix} = \big(c(\theta_1)s(\theta_2) - c(\theta_2)s(\theta_1)\big)\vec{k}$$

$$\|\vec{a}\times\vec{b}\| = |\vec{a}|\cdot|\vec{b}|\cdot\big|c(\theta_1)s(\theta_2) - s(\theta_1)c(\theta_2)\big| = |\vec{a}|\cdot|\vec{b}|\,|s(\theta_1-\theta_2)|$$

Proposition

Let us denote $|A|$ by the determinant of a matrix A. Then the following holds

1. *If A has a row (or a column) whose entries are all zero, then $|A| = 0$.*
2. *Let B be the matrix obtained by interchanging two rows (or columns) of A. Then $|B| = -|A|$.*
3. *Let B be a matrix obtained by multiply c on a row (or column) followed by adding it to another row (or column). Then $|B| = |A|$.*

$$A = \begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \vdots & \vdots & \vdots \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array} \times c, \quad B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \vdots & \vdots & \vdots \\ a_{31} + c a_{11} & a_{32} + c a_{12} & a_{33} + c \cdot a_{13} \end{bmatrix}$$

Definition

Let A be a $n \times n$ matrix. A matrix B satisfying

$$A \cdot B = B \cdot A = I_n$$

is called the **inverse of** A, denoted by $B = A^{-1}$. If an inverse matrix A^{-1} exists, then A is said to be **non-singular**. Otherwise, it is called **singular**.

Theorem

A matrix A is singular if and only if $\det A = 0$.

Proposition

If A is a 2×2 matrix, then A^{-1} is

① switch diagonal entries

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{\cancel{21}\,12} \\ -a_{\cancel{12}\,21} & a_{11} \end{pmatrix}$$

② negative sign to the off-diagonal entries.

For a 3×3-matrix B, the inverse is given by

2nd row, 1st col.

$$B^{-1} = \frac{1}{\det B} \begin{pmatrix} c_{11} & -c_{21} & c_{31} \\ -c_{12} & c_{22} & -c_{32} \\ c_{13} & -c_{23} & c_{33} \end{pmatrix} \tag{5}$$

*where each c_{ij}, called the **cofactor**, is the determinant of 2×2-matrix obtained by deleting ith row and jth column. For example,*

$$c_{21} = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ \cancel{b_{21}} & \cancel{b_{22}} & \cancel{b_{23}} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} = \begin{vmatrix} b_{12} & b_{13} \\ b_{32} & b_{33} \end{vmatrix}$$

Notice that the row and column indices are switched in (5).

Definition

A **vector space** V is a set of element called **vectors** satisfying the following properties:

1. (Zero vector) V contains the **zero vector** $\mathbf{0}$, which is a unique vector satisfying $\mathbf{0} + \mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
2. (Vector sum) For any two vectors $\mathbf{v}$, $\mathbf{w} \in V$, the vector $\mathbf{v} + \mathbf{w}$ lies in V.
3. (Scalar multiplication) For any $k \in \mathbf{R}$ and $v \in V$, the vector $k\mathbf{v}$ lies in V.

A subset $V \subset \mathbf{R}^n$ is called a **vector subspace** if it is a vector space itself.

Example

$C^1(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuously diff.} \}$.

Let $\mathcal{C}^1(\mathbf{R})$ be the set of all differentiable functions on $\mathbf{R}$ whose derivatives are continuous on $\mathbf{R}$. Since f, g are such functions so is the function $h = f + g$. Also for any $k \in \mathbf{R}$, the function kf is differentiable and its derivative is continuous. Thus $\mathcal{C}^1(\mathbf{R})$ is a vector space. Likewise, we can define vectors spaces $\mathcal{C}^n(\mathbf{R})$, $\mathcal{C}^\infty(\mathbf{R})$

$x \in \mathbb{R} \quad (f+g)(x) = f(x) + g(x)$

$(f+g)' = f' + g'$

$k \in \mathbb{R} \quad k \cdot f$: conti. diff.

$f \equiv 0$ conti. diff.

Example

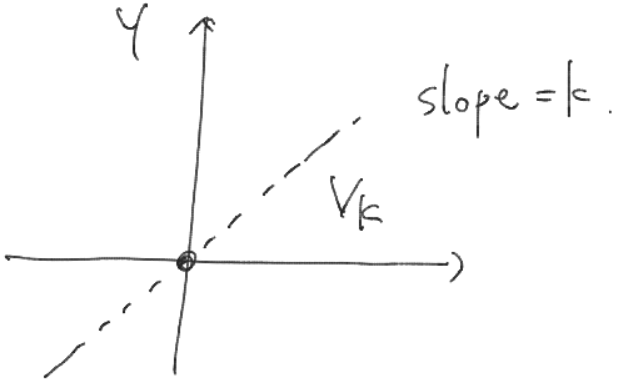
For each constant $k \in \mathbf{R}$, let V_k be the set of all points on the line $y = kx$ in $\mathbf{R}^2$:

$$V_k = \{(x, y) \mid y = kx\}$$

k is fixed.

Then V_k is a vector subspace of $\mathbf{R}^2$. Let V_∞ be the vertical line $V_\infty = \{(0, y) \mid y \in \mathbf{R}\}$. Then V_∞ is also a vector subspace of $\mathbf{R}^2$

vector space



$(0,0) \in V_k$

$(x_0, y_0), (x_1, y_1) \in V_k \Rightarrow (x_0 + x_1, y_0 + y_1) \in V_k$

$(\because k(x_0 + x_1) = kx_0 + k \cdot x_1 = y_0 + y_1)$.

$l \in \mathbb{R}, (x_0, y_0) \in V_k \Rightarrow l(x_0, y_0) = (lx_0, ly_0) \in V_k$

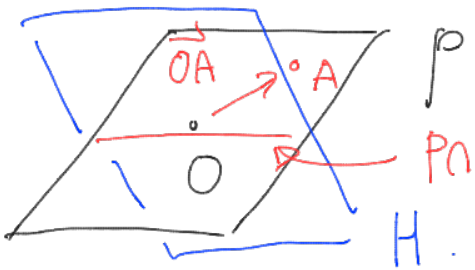
$k \cdot lx_0 = l \cdot (kx_0) = l \cdot y_0$

Example

Let P be the set of all point on the plane

$$P = \{(x, y, z) \mid ax + by + cz = 0\}$$

in $\mathbf{R}^3$. By identifying points in P as position vectors, we can say P is a vector subspace of $\mathbf{R}^3$, orthogonal to $\mathbf{n} = (a, b, c)$
In particular, let V, W be vector subspace of $\mathbf{R}^n$. Then so is $V \cap W$. For example, let V, W be vector subspace for two planes in $\mathbf{R}^3$ passing through the origin $\mathbf{0}$. Then $V \cap W$ is either a line (if V, W are transversal) or a plane (if $V = W$).



$P \cap H$: line passing through O.
This is 1-dim'l vector subspace of $\mathbb{R}^3$.

Definition

Let V a vector space and $\mathbf{v}_1, \cdots, \mathbf{v}_m \in V$. We say the vector space subspace

$$W = \{a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m \,|\, a_1, \cdots, a_m \in \mathbf{R}\}$$

is **spanned** by $\mathbf{v}_1, \cdots, \mathbf{v}_m$, and denote by $W = \mathrm{span}\langle \mathbf{v}_1, \cdots, \mathbf{v}_m \rangle$.

Definition

We say a vector space V is **spanned by** the set of vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$, if every $v \in V$ can be expressed a linear combination of v_i's:

$$v = a_1\mathbf{v}_1 + \cdots a_n\mathbf{v}_n$$

The set $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is said to be **linearly independent** if the coefficients satisfying

$$a_1\mathbf{v}_1 + \cdots a_n\mathbf{v}_n = \mathbf{0}$$

is the trivial ones, namely $a_1 = \cdots = a_n = 0$.

Definition

Let $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ be the set of linear independent vectors which spans the vector space V. Such set is called the **basis** of V, and its element is called the **basis vector**. The **dimension** of V is the number of basis vectors which spans V.

#1. $(\|b\|\cdot\vec{a} + \|\vec{a}\|\cdot\vec{b}) \circ (\|\vec{b}\|\vec{a} - \|\vec{a}\|\cdot\vec{b})$

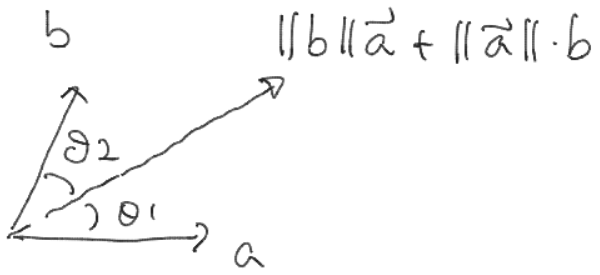
$$= \|\vec{b}\|^2\, \vec{a}\circ\vec{a} + \cancel{\|\vec{a}\|\cdot\|\vec{b}\|\cdot\vec{b}\circ\vec{a}} - \cancel{\|\vec{b}\|\cdot\|\vec{a}\|\cdot\vec{a}\circ\vec{b}} - \|\vec{a}\|^2\cdot\vec{b}\circ\vec{b}$$

$$= \|\vec{b}\|^2\cdot\|\vec{a}\|^2 - \|\vec{a}\|^2\cdot\|\vec{b}\|^2 = 0.$$

Problem

1. Show that the vectors $\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b}$ and $\|\mathbf{b}\|\mathbf{a} - \|\mathbf{a}\|\mathbf{b}$ are orthogonal.
2. Show that $\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

#2.



$$\cos\theta_1 = \frac{(\|\vec{b}\|\vec{a} + \|\vec{a}\|\cdot\vec{b})\circ\vec{a}}{\cancel{\|\vec{a}\|}\,\|\,\|\vec{b}\|\vec{a} + \|\vec{a}\|\cdot\vec{b}\,\|} = \frac{\|\vec{b}\|\cdot\|\vec{a}\|^{\cancel{2}} + \cancel{\|\vec{a}\|}\cdot\vec{a}\circ\vec{b}}{''}$$

$$\cos\theta_2 = \frac{(\|\vec{b}\|\cdot\vec{a} + \|\vec{a}\|\cdot\vec{b})\circ\vec{b}}{\cancel{\|\vec{b}\|}\cdot\|\ \text{——}''\ \|} = \frac{\cancel{\|\vec{b}\|}\cdot\vec{a}\circ\vec{b} + \|\vec{a}\|\cdot\|\vec{b}\|^{\cancel{2}}}{''}$$

Problem

1. Let $\mathbf{a} = (1, 2, 1)$, $\mathbf{b} = (2, 1, 2)$, and $\mathbf{u} = (0, 1, -1)$. Suppose that the vector $\mathbf{u}$ can be decomposed by

$$\underset{3\times 1}{\vec{u}'} = \underbrace{[\vec{a}\ \vec{b}]}_{3\times 2} \underset{2\times 1}{\begin{bmatrix} s \\ t \end{bmatrix}}$$

← Enough to use only the first two components

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$$
$$= s\,\vec{a} + t\,\vec{b} + r\,\vec{c} = \star$$

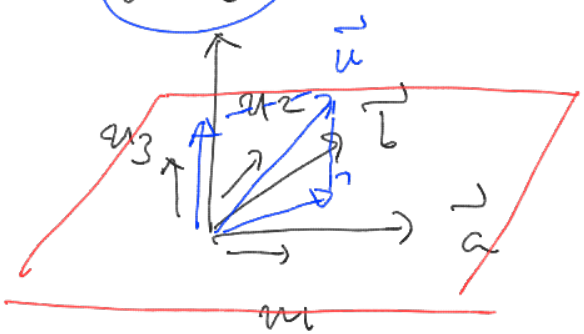
where $\mathbf{u}_1$ is parallel to $\mathbf{a}$, $\mathbf{u}_2$ is parallel to $\mathbf{b}$, and $\mathbf{u}_3$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. Find the vector $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ explicitly.

$$\vec{u} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = [\vec{a}\ \vec{b}\ \vec{c}] \begin{bmatrix} s \\ t \\ r \end{bmatrix} = \star$$

col. vec.

$= 3\times 3$ matrix.

$$\vec{a} \times \vec{b} = \vec{c}$$



$$\vec{u}_3 = \mathrm{proj}_{\vec{a}\times\vec{b}}(\vec{u}).$$

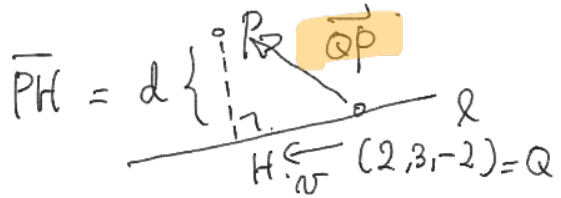
$$\vec{u}' = \vec{u} - \vec{u}_3 = s\cdot\vec{a} + t\,\vec{b}.$$

Find s, t.

$$\begin{bmatrix} s \\ t \\ r \end{bmatrix} = [\vec{a}\ \vec{b}\ \vec{c}]^{-1} \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

#1. $\text{proj}_{\vec{v}} \overrightarrow{QP} = \cdots = \overrightarrow{QH}$.

$\overline{PH} = \|\overrightarrow{PH}\| = \|\overrightarrow{QP} - \overrightarrow{QH}\|$

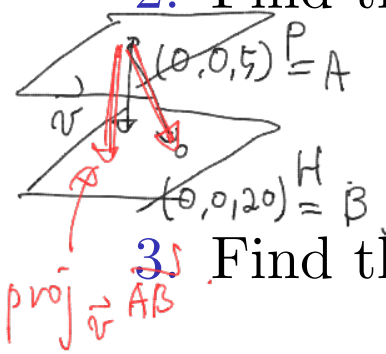


Problem

1. Find the distance between $P = (2, 1, 3)$ and the line $l(t) = (2, 3, -2) + t(-1, 1, -2)$.

$\vec{v} = (-1, 1, 2)$.

2. Find the distance between two parallel planes



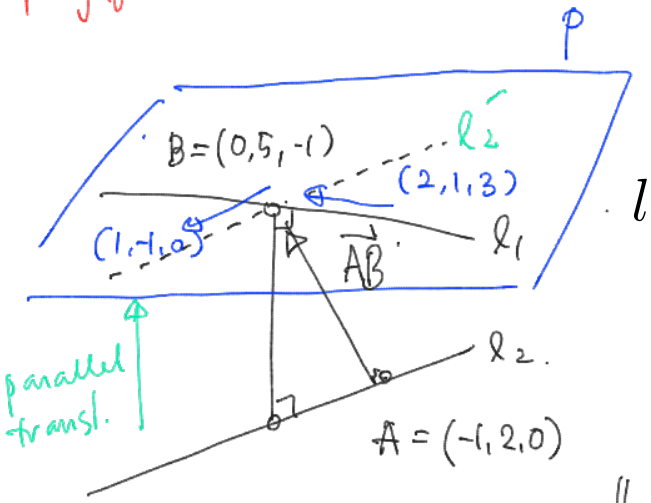
$$2x - 2y + z = 5, \quad 2x - 2y + z = 20$$

P H

$\|\text{proj}_{\vec{v}} \overrightarrow{AB}\|$

$\vec{v} = (2, -2, 1)$.

3. Find the distance between two skew lines



$$l_1(t) = (0, 5, -1) + t(2, 1, 3)$$

$$l_2(t) = (-1, 2, 0) + t(1, -1, 0)$$

dist. P and ℓ_2 = dist. ℓ_1 and ℓ_2.

$\vec{v}$ = normal vect. to P = $(2,1,3) \times (1,-1,0)$.

$\|\text{proj}_{\vec{v}} \overrightarrow{AB}\|$ = dist. between ℓ_1, ℓ_2

Problem

Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, $\mathbf{c} = (c_1, c_2, c_3)$. Verify that

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

prop 6 on slide 21

$$\vec{c} \cdot (\vec{a} \times \vec{b}) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = - \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

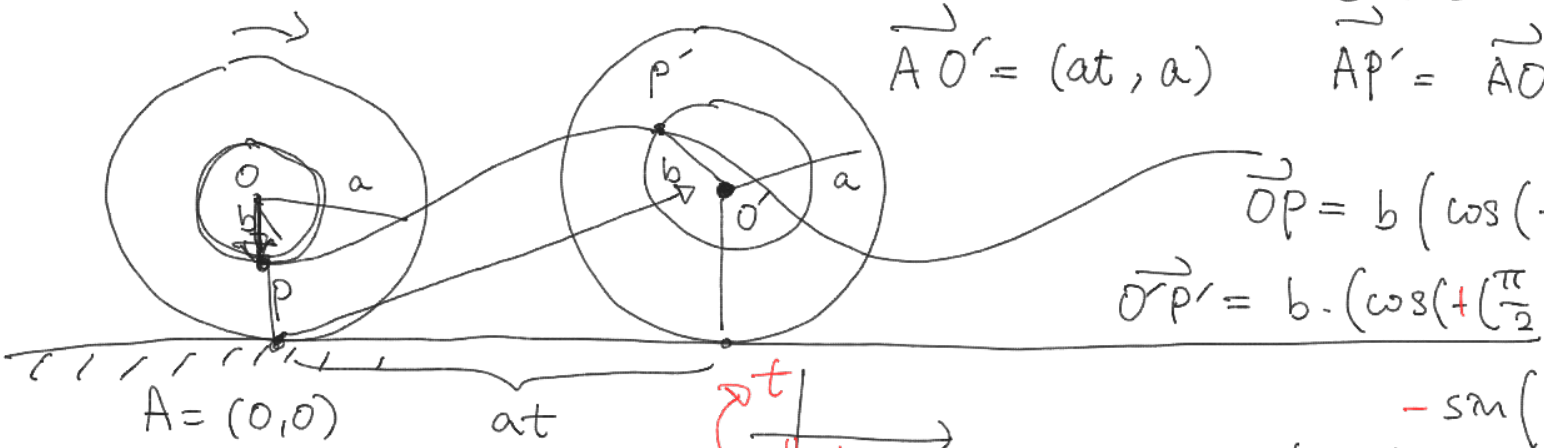
prop 2 on slide 23

Problem

Imagine two concentric circles with radius a, b $(b < a)$ which rolls on flat line with the same angular velocity. A curtate cycloid is a trajectory of a point on the circle of radius b. Find a set of parametric equation for the curtate cycloid with $a = 3$, $b = 2$.

angular vel $= t$ (rad/sec). $(x(t), y(t)) = \underbrace{(at, a) + b(-\sin t, -\cos t)}$

$\vec{AO'} = (at, a)$ $\quad \vec{AP'} = \vec{AO'} + \vec{O'P'}$



$\vec{OP} = b\left(\cos\left(-\frac{\pi}{2}\right), \sin\left(-\frac{\pi}{2}\right)\right)$

$\vec{O'P'} = b \cdot \left(\cos\left(+\left(\frac{\pi}{2} + t\right)\right),\right.$

$\left.-\sin\left(+\left(\frac{\pi}{2} + t\right)\right)\right).$

$P = (0, a-b)$

$$\begin{cases} x(t) = at - b\sin t \\ y(t) = a - b\cos t \end{cases}$$