SE102:Multivariable Calculus

Hyosang Kang[1]

[1]Division of Mathematics
School of Interdisciplinary Studies
DGIST

Week 13

Theorem (Stokes)

Let S be an oriented surface with a piecewise continuous boundary C. For $\mathbf{F}$ be a continuous vector field defined on S. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$$

where C and S are positively[1] *oriented.*

[1] The boundary is **postively** oriented if the direction is counter-clockwise with $\mathbf{n}$ being *upward*.

Proof.

Let $\mathbf{F} = \langle P, Q, R \rangle$. Suppose that the surface S is given by the graph of a function $z = f(x, y)$ on a bounded domain $D \subset \mathbf{R}^2$. Let $X(x, y) = (x, y, f(x, y))$ be the parametrization of S. Then

$$\begin{aligned} \iint_S \nabla \times \mathbf{F} d\mathbf{S} &= \iint_D -(R_y - Q_z)f_x - (P_z - R_x)f_y + (Q_x - P_y)dA \\ &= \iint_D \frac{\partial}{\partial x}(Q + Rf_y) - \frac{\partial}{\partial y}(P + Rf_x)\, dA \end{aligned}$$

Let C' be a planar curve which bounds that area D. By Green's theorem, the last integral becomes

$$\oint_{C'} (P + Rf_x)dx + (Q + Rf_y)dy = \oint_C Pdx + Qdy + Rdz = \oint_C \mathbf{F} \cdot d\mathbf{s}$$

$\square$

Remark

The Stokes' theorem provides the meaning of curl $\nabla \times \mathbf{F}$ of a vector field $\mathbf{F}$. Suppose that the surface S is planar disk with sufficiently small radius r centered at (x_0, y_0, z_0). Then

$$\oint_C \mathbf{F} \cdot d\mathbf{s} \approx \text{area}S \cdot (\nabla \times \mathbf{F})(x_0, y_0) \circ \mathbf{n}$$

$$(\nabla \times \mathbf{F})(x_0, y_0) \circ \mathbf{n} \approx \frac{1}{\text{area}S} \oint_C \mathbf{F} \cdot d\mathbf{s}$$

where $\mathbf{n}$ is the orientation of S. Therefore, the curl of a vector field $\mathbf{F}$ at the point p on the surface S has a projection onto the orthogonal direction of a surface S equal to the work done by $\mathbf{F}$ along the neighboring boundary of the point p on S.

Example

Let S be the surface bounded by $z = 1 - x^2 - y^2$, $z \geq 0$ with upward orientation ($\mathbf{n} \cdot \mathbf{k} \geq 0$). Confirm that Stokes' theorem holds for $\mathbf{F} = (y, -x, 0)$.

Theorem (Divergence Theorem)

Let V be a region in $\mathbf{R}^3$ whose boundary $S = \partial V$ is a closed surface.[2] *For a vector field $\mathbf{F}$ defined on V,*

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{F} dV$$

where the orientation of S is the outward direction.

[2] A surface is called **closed** if it has no boundary curve.

Proof.

Note that for $\mathbf{F} = (P, Q, R)$,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S P\mathbf{i} \cdot \mathbf{n} dS + \iint_S Q\mathbf{j} \cdot \mathbf{n} dS + \iint_S R\mathbf{k} \cdot \mathbf{n} dS$$

Suppose that the volume V is given by

$$V = \{(x, y, z) \,|\, h_1(x, y) \leq z \leq h_2(x, y),\, (x, y) \in D\}.$$

where D is the region in $\mathbf{R}^2$ on which the volume V is defined. Then

$$\begin{aligned} \iint_S R\mathbf{k} \cdot \mathbf{n} dS &= \iint_{S_1} R\mathbf{k} \cdot \mathbf{n} dS + \iint_{S_2} R\mathbf{k} \cdot \mathbf{n} dS \\ &= \iint_D R(x, y, h_2(x, y)) - R(x, y, h_1(x, y)) dx dy \\ &= \iiint_V R_z dV \end{aligned}$$

Example

Compute the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = (z^2, \frac{1}{3}x^3 + \tan z, z + y^2)$ and S is the closed surface $x^2 + y^2 + z^2 = 1$.

Example

Let S be a parabola $x^2 + y^2 + z = 2$ above the plane $z = 1$. Find the flux of $\mathbf{F} = (z\tan^{-1}(y^2), z^3\ln(x^2+1), z)$ to the upward direction of S.

Example

Let $\mathbf{F} = \dfrac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}$ and S be the surface $z = 4 - x^2 - y^2$, $z \geq 0$ with upward orientation $\mathbf{n} \cdot \mathbf{k} \geq 0$. Use divergence theorem to compute the flux $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$. (How should we choose the volume V?)

Multivariable Calculus summerizes in two sentences:

- Derivative is a linear transformation.
  - Derivative $Df(\mathbf{a})$ of a multivariable function is a linear map between tangent spaces at $\mathbf{a}$ and $f(\mathbf{a})$.
- Divergence theorem is a stokes theorem.
  - The general form of stokes theorem is

$$\int_V d\mu = \int_{\partial V} \mu$$

where ν is a differential $(k-1)$-form and V is a k-dimensional space. The differential $d\mu$ is a k-form. The (special) Stokes theorem is when

$$\mu = Pdx + Qdy = Rdz$$

and divergence theorem is when

$$\mu = Pdx \wedge dy + Qdy \wedge dz + Rdz \wedge dx.$$

Problem

Let

$$\mathbf{A} = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}$$

Let S be a surface bounded by $z = 4 - x^2 - y^2$, $z \geq 0$, orieted upward. Find

$$\iint_S \mathbf{A} \cdot d\mathbf{S}$$

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Problem

Let S be the surface bounded by $z = e^{-x^2-y^2}$ and $z \geq 1/e$. Let $\mathbf{n}$ be the orientation of S satisfying $\mathbf{n} \cdot \mathbf{k} \geq 0$. Find the flux of

$$\mathbf{F} = (e^{x+y} - xe^{y+z}, e^{y+z} - e^{x+y}, 2)$$

on S to the direction of $\mathbf{n}$.

Problem

Let C be the intersection of $z = 1 - 2(x^2 + y^2)$ and $z = x^2 - y^2$ oriented counter-clockwise. Find $\oint_C \mathbf{F} \cdot d\mathbf{s}$ where

$$\mathbf{F} = (y\cos(x) - yz, \sin x, e^z)$$

Problem

Let $S = \partial V$ be a closed surface. Prove the following.

1. For any constant vector field $\mathbf{C}$,

$$\iint_S \mathbf{C} \cdot d\mathbf{S} = 0$$

2. For any vector field $\mathbf{F}$,

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = 0$$