

SE328:Topology

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Definition

A collection \mathcal{A} of subsets of X is called a **cover** of X , or **covering** of X , if

$$X = \bigcup_{A \in \mathcal{A}} A.$$

If a cover \mathcal{A} consists of open subsets, then \mathcal{A} is called a **open covering** of X . A space X is called **compact** if any open covering of X admits a finite subcovering. A subset $C \subset X$ is called **compact** if its subspace topology is compact.

Example

1. The finite subset of \mathbf{R} is compact.
2. Let $K = \{1, 1/2, \dots, 1/n, \dots\}$. The set \overline{K} is compact.
3. Any closed interval $[a, b]$ is compact in \mathbf{R} .

Example

1. The real line \mathbf{R} is not compact.
2. The half-interval $(a, b]$ is not compact.
3. Any open interval (a, b) is not compact.

Proposition

A subspace $Y \subset X$ is compact if and only if every covering of Y by open subsets of X admits a finite subcovering of Y .

(This seems a repeat of the definition, but it is not.)

Proposition

Any closed subset C of compact space X is compact.

Proposition

Every compact subspace C of a Hausdorff space X is closed.

Example

Any subset in \mathbf{R} with finite complement topology is compact.

Proposition

Let $f : X \rightarrow Y$ be a continuous map. If $C \subset X$ is compact in X , then so is $f(C)$ in Y .

Theorem

Let $f : X \rightarrow Y$ be a bijective continuous map. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Lemma (The tube lemma)

Let N be an open subset in $X \times Y$ containing a slice $\{x_0\} \times Y$. If Y is compact, then there is a neighborhood W of x_0 in X such that

$$W \times Y \subset N$$

Proposition

The product of finitely many compact spaces is compact.

Theorem

A collection \mathcal{C} of subsets in X is said to have the **finite intersection property** if for every finite subcollection of \mathcal{C} , the intersection of all elements is nonempty. A topological space X is compact if and only if for every collection \mathcal{C} of closed subsets which has the finite intersection property, the intersection of all element in \mathcal{C} is nonempty.

Theorem

Let Y be a compact Hausdorff space. A function $f : X \rightarrow Y$ is continuous if and only if the graph $G(f) = \{(x, f(x)) \mid x \in X\}$ is closed in $X \times Y$.

Theorem

A subset $A \subset \mathbf{R}^n$ is compact if and only if A is closed and bounded.

Example

Let $f : X \rightarrow \mathbf{R}$ be continuous map. If X is compact, then there is $x_0, x_1 \in X$ such that

$$f(x_0) \leq f(x) \leq f(x_1)$$

for all $x \in X$.