SE328:Topology

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Definition

Let X, Y be two topological spaces. The **product topology** on the product $X \times Y$ is the topology generated by the subset of the form

$$U \times V$$

where U, V are open subsets in X, Y respectively.

Theorem

If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the toplogy of Y, then the collection

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B} C \in \mathcal{C}\}$$

is a basis for the topology of $X \times Y$.

Definition
Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be defined by

$$\pi_1(x, y) = x, \quad \pi_2(x, y) = y.$$

The maps π_1, π_2 are called the projections of $X \times Y$ onto its factors.

Theorem
The collection

$$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ is open in } X\}$$

is a subbasis for the topology $X \times Y$.

Definition

Let X be a topological space with the topology $\mathcal{T}$. Let $Y \subset X$ be a subset. The collection

$$\mathcal{T}' = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology of Y, called the **subspace topology**. With this topology, Y is called a subspace of X.

Lemma

If $\mathcal{B}$ is a basis for the topology of X, then the collection

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

is a bsis for the subspace topology of Y.

Lemma

Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Theorem

If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same topology $A \times B$ inherits as a subspace of $X \times Y$.

Example

Let X be a set with order relation. Assume $|X| > 1$. Let $\mathcal{B}$ a collection of subsets of the following types:

1. all open intervals (a, b) in X;
2. all half-intervals $[a_0, b)$ where a_0 is the smallest element (if any) of X;
3. all half-interval $(a, b_0]$ where b_0 is the largest element (if any) of X.

Then the collection $\mathcal{B}$ is a basis for a topology of X, called the order toplogy.
Suppose that $Y = [0, 1) \cup \{2\}$. Show that the one-point set $\{2\}$ is open in the subspace topology $Y \subset \mathbb{R}$, while it is not open in the order topology, $\{2\}$ is not open.

Example

Let $I = [0,1]$. The dictionary order on $I \times I$ is the restriction of dictionary order on $\mathbb{R} \times \mathbb{R}$. Show that dictionary order topology on $I \times I$ is not the same as the subspace topology obtained from dictionary order topology on $\mathbb{R} \times \mathbb{R}$.

Definition

Given an ordered set X, a subset $Y \subset X$ is called convex if for each pair of points $a < b$, the interval (a, b) lies in Y.

Theorem

Let X be an ordered set in the order topology. Let Y be a subset of X that is convex in X. Then the order topology on Y is the same as the topology Y as the subspace of X.

Example

A map $f : X \to Y$ is an **open map** if for every open set $U \subset X$ the image $f(U)$ is open in X. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

Definition

A subset A of a topological space X is **close** if $X - A$ is open.

Example

1. A subset $[a, b] \subset \mathbb{R}$ is closed in the standard topology.
2. A subset $\{(x, y) \mid x, y \geq 0\} \subset \mathbb{R}^2$ is closed in the standard topology.
3. In the finite complement topology of X, every closed set is either X itself or finite subset.

Theorem

Let X be a topological space. Then

1. *$\emptyset$ and X are closed.*
2. *Arbitrary intersection of closed sets is closed.*
3. *Finite union of closed sets is closed.*

Theorem

Let Y be a subspace of X. A subset $A \subset Y$ is closed (in Y) if and only if $A = A \cap C$ for some closed subset $C \subset X$.

Theorem

Let Y be a subspace of X. If $A \subset Y$ is closed in Y and Y is closed in X, then A is closed in X.

Definition

Let A be a subset of topological space X. The **interior** of A is the union of all open sets contained in A. The **closure** of A is the intersection of all closed sets containing A. The interior of A is denoted by $\mathrm{Int}A$. The closure of A is denoted by $\overline{A}$.

Theorem

Let Y be a subspace of X. Let A be a subset of Y. Let $\overline{A}$ is the closure of A in X. Then the closure of A in Y equals $\overline{A} \cap Y$.

Theorem

Let A be a subset of a topological space X.

1. $x \in \overline{A}$ *if and only if every open set* U *containing* x *intersects* A*.*
2. *Let* $\mathcal{B}$ *be a basis of* X*.* $x \in \overline{A}$ *if and only if* $x \in B \in \mathcal{B}$ *implies* $B \cap A \neq \emptyset$*.*

Remark

The phrase "U is open set containing x" equal to the statement "U is (open) neighborhood of x."

Definition

Let A be a subset of a topological space X. An element $x \in X$ is a **limit point** of A if every neighborhood of x intersects A in some point other than x itself.

Theorem

Let A be a subset of a topological space X. Let A' be the set of all limit points of A. Then

$$\overline{A} = A \cup A'$$

Definition

Given a topological space X, a sequence $x_1, x_2, \cdots$ **converges to** $x_\infty \in X$ if for each neighborhood U of x_∞, there exists an integer $N > 0$ such that $x_i \in U$ for all $i > N$.

Definition

A topological space X is **Hausdorff** if for each pair x_1, x_2 of distinct points of X, there exist neighborhoods U_1, U_2 of x_1, x_2 respectively which are disjoint.

Theorem

Every finite set in a Hausdorff space is closed.

Definition

A topological space X has the T_1 **axiom** if finite point set is closed.

Example

The finite complement topology on $\mathbb{R}$ has the T_1 axiom but it is not Hausdorff.

Theorem

Every sequence in a Haudorff space converges at most one point.

Example

1. Every ordered set is Hausdorff in the order topology.
2. The product of two Hausdorff space is Hausdorff.
3. A subspace of Hausdorff space is Hausdorff.

Example

X is Hausdorff if and only if the **diagonal** $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X \times X$.

Example

The **boundary** of a subset $A \subset X$, denoted by BdA or ∂A, is defined by

$$\partial A = \overline{A} \cap \overline{X - A}$$

1. Show that $\overline{A} \cap \partial A = \emptyset$.
2. Show that $\overline{A} = \partial A \cup \mathrm{Int} A$
3. Show that $\partial A = \emptyset$ if and only if A is both open and closed.
4. Show that U is open if and only If $\partial U = \overline{U} - U$.
5. Prove or disprove: $U = \mathrm{Int}(\overline{U})$.