SE328:Topology

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Definition

A space X admits two open subsets A, B such that

$$X = A \cup B,$$

then the pair A, B is is called the **separation** of X, and X is called *disconnected*. If the space X does not admit any separation, it is called a **connected** space. A subset $U \subset X$ is called **connected** if it is connected as a subspace topology of X, i.e. there is not open subsets A, B of X such that

$$U = (A \cap U) \cup (B \cap U)$$

Example

1. Any finite subset S of $\mathbf{R}$ is disconnected.
2. The set of all rationals, $\mathbf{Q}$, is disconnected in $\mathbf{R}$.

Lemma

If X admits a separation $X = A \cup B$ and $Y \subset X$ is a connected subset, then either $Y \subset A$ or $Y \subset B$.

Proposition

Let $U_\alpha \subset X$ *be a connected subspace and suppose that there is a point* $x \in X$ *such that*

$$x \in \bigcap_\alpha U_\alpha$$

Then the union $\bigcup_\alpha U_\alpha$ *is connected.*

Proposition

If A is connected subset of X and $A \subset B \subset \overline{A}$. Then B is connected.

Proposition

Let $f : X \to Y$ be a continuous map. If $U \subset X$ is connected, then $f(U)$ is connected.

Proposition

Let $X_1, \cdots, X_n$ *be connected spaces. Then* $\prod_{i=1}^{n} X_i$ *is connected space in product topology.*

Example

The space $\mathbf{R}^\omega$ with the product topology is connected. To show this, we follow two steps below.

1. Define $\tilde{\mathbf{R}}^n = \{(x_1, \cdots, x_n, 0, \cdots) \mid x_i \in \mathbf{R}\}$ and show that $\mathbf{R}^\infty = \bigcup_{n=1}^{\infty} \tilde{\mathbf{R}}^n$ is connected.
2. Show that $\overline{\mathbf{R}^\infty} = \mathbf{R}^\omega$.

Theorem

The real line and its intervals (a, b) *and rays* (a, ∞) *are connected.*

Proof.

Note that the real line $\mathbf{R}$ satisfies the two axioms:

1. any subset of $\mathbf{R}$ admits the least upper bound;
2. if $x < y$, then there exists z such that $x < z < y$.

Let L be a *convex* subset of $\mathbf{R}$, i.e. for any $a, b \in L$, $[a, b] \subset L$. Suppose that L admits a separation $L = A \cup B$. Let $a \in A$ and $b \in B$. Then by the convexity of L, $[a, b] \subset L$. Let $A_0 = [a, b] \cap A$ and $B_0 = [a, b] \cap B$. Let $c = \sup A_0$. Since c lies in $[a, b]$, it must be contained in either A_0 or B_0.

Proof.

Suppose $c \in A_0$. Then $c \neq b$. Thus $a \leq c < b$. Since A_0 is open in $[a, b]$, there is an interval $[c, c + \varepsilon)$ contained in A_0, which contradicts the assumption that c is the supremum of A_0.
Suppose $c \in B_0$. Then $c \neq a$. Thus $a < c \leq b$. Since B_0 is open, there is an interval $(c - \varepsilon, c]$ contained in B_0. If $c = b$, then this contradicts to the assumption. If $c \neq b$, then $(c, b] \cap A_0 = \emptyset$ and

$$(c - \varepsilon, b] = (c - \varepsilon, c] \cap (c, b]$$

is not contained in A_0. This also contradicts to the assumption.

Theorem (Intermediate value theorem)

Let $f : X \to \mathbf{R}$ be a continuous function. If X is connected, then for any two $a, b \in X$ and $r \in [f(a), f(b)]$, there exists $c \in X$ such that $f(c) = r$.

Proof.

Let $A = f(X) \cap (-\infty, r)$ and $B = f(X) \cap (r, \infty)$. If there is no $c \in X$ such that $f(c) = r$, then $f(X) = A \cap B$. Since A, B are open subset in the subspace $f(X) \subset X$, this implies that $f(X)$ admits a separation. This is a contradiction. $\square$

Definition

A continuous map $f : [a, b] \to X$ is called a **path** from $f(a)$ to $f(b)$. A space X is called **path-connected** if any two points $x, y \in X$ can be joined by a path.

Proposition

A path-connected space is connected.

Proof.

Suppose X is path-connected, but not connected. Let $X = A \cap B$ be a separation. Take $a \in A, b \in B$, and a path $c : [0, 1] \to X$ joining $c(0) = a$ to $c(1) = b$. Since c is continuous, so is $c([0, 1])$. However, $c([0, 1]) = (c([0, 1]) \cap A) \cup (c([0, 1]) \cap B)$ is a separation of $c([0, 1])$, a contradiction. □

Example

Not all connected space is path-connected. For example, let

$$S = \{(x, \sin(1/x)) \,|\, 0 < x \leq 1\}$$

Then $\overline{S}$ is connected, but not path-connected. However, the set $\overline{S} = S \cup \{0\} \times [0, 1]$ is not path-connected. This set is called **topologist's sine curve**.

Definition

Given a topological space X, let us define $x \sim y$ if x, y lies in the same connected subset of X. Under this equivalence relation, each equivalence class of X is called the (connected) **component** of X. Similarly, if we define $x \sim y$ when x, y lies in the same path-connected subset of X, the equivalence classes are called the **path component** of X.

Proposition

The space X is disjoint union of (path) components of X. Every (path) connected subset of X intersects only one of (path-connected) components.

Example

Here are some examples of components of spaces.

1. The set $\mathbf{Q}$ is not connected in $\mathbf{R}$ and every components of $\mathbf{Q}$ is a one-point set.
2. The topologist's sine curve $\overline{S}$ has one component, and two path components.

Definition

A space X is called **locally connected at** x if for any neighborhood U of x, there is a connected neighborhood U_1 of x such that $U_1 \subset U$. If X is locally connected at every point $x \in X$, we say X is **locally connected**. Similarly, we say X is **locally path-connected** if for any neighborhood U of x, there is a path-connected neighborhood U_1 of x such that $U_1 \subset U$ for any $x \in X$.

Proposition

Let X be locally path connected. Then every connected open set is path connected.

Proposition

A space X is locally (path) connected if and only if each (path) component is open.

Proof.

Suppose X is locally connected. Let U be a component of X. For any $x \in C$, we can choose a neighborhood U_1 of x such that $U_1 \subset U$. Thus U is open. Conversely, suppose that every component if open. Then for any $x \in X$, let V be a component containing x. Then for any open neighborhood U of x, $U_1 = U \cap V$ is open and contained in U. Thus X is locally connected. □

Example

What are the components and path components of $\mathbf{R}_l$ (the lower limit topology). Which functions $f : \mathbf{R} \to \mathbf{R}_l$ are continuous?

Example

Consider $\mathbf{R}^\omega$ with the uniform topology. Then $\mathbf{x} = (x_1, \cdots)$, $\mathbf{y} = (y_1, \cdots)$ are in the same component if and only if the sequence $x_i - y_i$ is bounded. Moreover, if $\mathbf{R}^\omega$ has box topology, then $\mathbf{x}$, $\mathbf{y}$ are in the same component if and only if the limit of $x_i - y_i$ is zero.

Definition

A space X is said to be **weakly connected** at x if for every neighborhood U of x admits a connected subset which is contained in U and contains a neighborhood of x.

Proposition

If X is weakly connected at every point, then X is locally connected.

Example

The *infinite bloom* is not locally connected, but weakly locally connected.