SE328:Topology

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Definition

A collection $\mathcal{A}$ of subsets of X is called a **cover** of X, or **covering** of X, if

$$X = \bigcup_{A \in \mathcal{A}} A.$$

If a cover $\mathcal{A}$ consists of open subsets, then $\mathcal{A}$ is called a **open covering** of X. A space X is called **compact** if any open covering of X admits a finite subcovering. A subset $C \subset X$ is called **compact** if its subspace topology is compact.

Example

1. The finite subset of $\mathbf{R}$ is compact.
2. Let $K = \{1, 1/2, \cdots, 1/n, \cdots\}$. The set $\overline{K}$ is compact.
3. Any closed interval $[a, b]$ is compact in $\mathbf{R}$.

Example

1. The real line $\mathbf{R}$ is not compact.
2. The half-interval $(a, b]$ is not compact.
3. Any open interval (a, b) is not compact.

Proposition

A subspace $Y \subset X$ is compact if and only if every covering of Y by open subsets of X admits a finite subcovering of Y.

(This seems a repeat of the definition, but it is not.)

Proposition

Any closed subset C of compact space X is compact.

Proposition

Every compact subspace C of a Hausdorff space X is closed.

Example

Any subset in $\mathbf{R}$ with finite complement topology is compact.

Proposition

Let $f : X \to Y$ be a continuous map. If $C \subset X$ is compact in X, then so is $f(C)$ in Y.

Theorem

Let $f : X \to Y$ be a bijective continuous map. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Lemma (The tube lemma)

Let N be an open subset in $X \times Y$ containing a slice $\{x_0\} \times Y$. If Y is compact, then there is a neighborhood W of x_0 in X such that

$$W \times Y \subset N$$

Proposition

The product of finitely many compact spaces is compact.

Theorem

A collection $\mathcal{C}$ of subsets in X is said to have the ***finite intersection property*** *if for every finite subcollection of $\mathcal{C}$, the intersection of all elements is nonempty. A topological space X is compact if and only if for every collection $\mathcal{C}$ of closed subsets which has the finite intersection property, the intersection of* <u>*all*</u> *element in $\mathcal{C}$ is nonempty.*

Theorem

Let Y be a compact Hausdorff space. A function $f : X \to Y$ is continuous if and only if the graph $G(f) = \{(x, f(x)) \mid x \in X\}$ is closed in $X \times Y$.

Theorem

A subset $A \subset \mathbf{R}^n$ *is compact if and only if* A *is closed and bounded.*

Example

Let $f : X \to \mathbf{R}$ be continuous map. If X is compact, then there is $x_0, x_1 \in X$ such that

$$f(x_0) \leq f(x) \leq f(x_1)$$

for all $x \in X$.

Theorem

Let X be a metric space with metric d. If X is compact, then for every open covering $\mathcal{A}$, there is $\delta > 0$ such that any subset whose diameter less than δ is contained in an element of $\mathcal{A}$.

Example

Let $f : X \to Y$ be a continuous map between metric spaces. If X is compact, then f is uniformly continuous.