SE328:Topology

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Week 01

Definition

Given a statement of the form $P \Rightarrow Q$, its contrapositive is $\sim Q \Rightarrow P$, and its converse is $Q \Rightarrow P$, where $\sim P$ is the negation of P.

Example

Determine whether the following statements are true.

1. $A \subset B$ and $A \subset C \Leftrightarrow A \subset (B \cap C)$
2. $A - (A - B) = B$
3. $(A \cap B) \cup (A - B) = A$

Example

1. Write the contrapositive and converse of $x < 0 \Rightarrow x^2 - x > 0$
2. Write the negation of $\forall a \in A, a^2 \in B$

Definition

Given sets A and B, the cartesian product $A \times B$ is the set

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\}$$

Example

Determine whether each of the following sets is a cartesian product of two subsets of $\mathbb{R}$.

1. $\{(x, y) \mid x \in \mathbb{Z}\}$
2. $\{(x, y) \mid y > x\}$

Definition

A function f is a subset of the cartesian product $A \times B$ of two sets, with the property that each elemetn in C appears as the first coordinate of at most one ordered pair. In other words,

$$(a, b), (a, b') \in f \Rightarrow b = b'$$

Definition

Given a function $f : A \to B$ and a subsets $A_0 \subset A$, the restriction of f to A_0 is

$$\{(a, f(a)) \mid a \in A_0\}$$

Definition

A function $f : A \to B$ is injective if

$$\forall a \in A, f(a) = f(a') \Rightarrow a = a'$$

and surjective if

$$\forall b \in B, \exists a \in A f(a) = b$$

Example

Let $f : A \to B$ and $g : B \to C$ be functions and $A_0 \subset A$, $B_0, B_1 \subset B$.

1. $A_0 \subset f^{-1}(f(A_0))$.
2. $B_0 \subset B_1 \Rightarrow f^{-1}(B_0) \subset f^{-1}(B_1)$.
3. If $C_0 \subset C$, then

$$(g \circ f)^{-1}(C_0) = f^{-1}(g^{-1}(C_0))$$

Definition

A equivalence relation $\sim$ on a set A is a subset $\subset A \times A$ such that

1. $\forall x \in A, (x, x) \in \sim$
2. $(x, y) \in C \Rightarrow (y, x) \in \sim$
3. $(x, y), (y, z) \in \sim \Rightarrow (x, z) \in \sim$

We denote $x \sim y$ if $(x, y) \in \sim$.

Definition

The equivalence class of $x \in A$ is the set

$$[x] = \{y \mid y \sim x\}$$

The collection of all equivalence classes for $\sim$ becomes a partition of A, i.e. the collection of disjoint nonempty subsets of A.

Definition

An order relation $<$ on a set A is a subset of $A \times A$ such that

1. $x, y \in A, x \neq y \Rightarrow (x, y) \in <$ or $(y, x) \in <$
2. $\nexists x \in A, (x, x) \in <$
3. $(x, y) \in <, (y, z) \in < \Rightarrow (x, z) \in <$.

We denote $x < y$ if $(x, y) \in <$.

Definition

If $<$ is an order relation on a set A, and if $a < b$, an open interval (a, b) is a subset defined by

$$(a, b) = \{x \in A \mid a < x < b\}$$

If $(a, b) = \emptyset$, then a is called the immediate predecessor of b, and b is called the immediate successor of a.

Definition

Suppose that A, B are two sets with order relations $<_A, <_B$. The dictionary order relation $<$ on $A \times B$ is defined by

$$a_1 \times b_1 < a_2 \times b_2$$

if $a_1 <_A a_2$, or if $a_1 = a_2$ and $b_1 <_B b_2$.

Definition

An ordered set A is said to have the least upper bound property if every nonempty subset $A_0 \subset A$ that is bounded above has a least upper bound. The greatest lower bound property is similary defined.

Example

1. Let $f : A \to B$ is a surjective function. Define $a_0 \sim a_1$ if $f(a_0) = f(a_1)$. Show that $\sim$ is an equivalence relation.
2. If an ordered set A has the least upper bound property, then it has the greatest lower bound property.
3. Assuming that the real line has the least upper bound property, show that $[0, 1] = \{x \mid 0 \leq x \leq 1\}$ also has the least upper bound property.
4. Does $[0, 1] \times [0, 1]$ in the dictionary order relation have the least upper bound property?

Definition

The set of real numbers, denoted by $\mathbb{R}$, is a set with two operations $+$(addition), $\cdot$(multiplication), and an order relation $<$. It contains two special elements, 1(one) and 0(zero). All elementary algebraic properties hold including the following statements.

1. $x < y \Rightarrow x + z < y + z$
2. $x < y, 0 < z \Rightarrow x \cdot z < y \cdot z$
3. $<$ has the least upper bound property
4. $x < y \Rightarrow \exists z \in \mathbb{R} x < z < y$

Definition

The subset $A \subset \mathbb{R}$ is called inductive if it contains 1 and if $x \in A$ then $x + 1 \in A$. The set of all positive integers, denoted by $\mathbb{Z}_+$ is the smallest among all inductive subsets.

Theorem

Every nonempty subset of $\mathbb{Z}_+$ has a smallest element.

Theorem

Let A be a set of positive integers containing 1. For each positive integer $n \in \mathbb{Z}_+$, $S_n = \{1, \cdots, n\} \subset A \Rightarrow n + 1 \in A$. If this is true for all n, then $A = \mathbb{Z}_+$.

Definition

Let m be a positive integer. An m-tuple of elements of X is a function $\mathbf{x} : \{1, \cdots, m\} \to X$. The ω-tuple of elements of X is a function $\mathbf{x} : \mathbb{Z}_+ \to X$.

Definition

Let $A_1, A_2, \cdots$ be a family of sets, indexed by $\mathbb{Z}_+$. The cartesian product of A_i, denoted by $\prod_{i \in \mathbb{Z}_+} A_i$, is the set of all ω-tuples of elements of $\bigcup_{i \in \mathbb{Z}_+} A_i$ such that $x_i \in A_i$. If $A_i = X$ for all i, then the cartesian product is denoted by X^ω.

Example

Find a bijective map $f : X^\omega \times X^\omega \to X^\omega$

Definition

A set is called finite if there is a bijection between the set and S_n for some positive integer n. A set is called infinite if it is not finite. It is called countably infinite if there is a bijection between the set and $\mathbb{Z}_+$. A infinite set which is not countable is called uncountable.

Theorem

Let A be a set. The followings are equivalent:

1. *A is countable.*
2. *There is a surjective function $f : \mathbb{Z}_+ \to A$.*
3. *There is an injective function $f : A \to \mathbb{Z}_+$.*

Lemma

If A is an infinite subset of $\mathbb{Z}_+$, then A is countably infinite.

Corollary

1. A subset of a countable set is countable.
2. The set $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countably infinite.
3. A countable union of countable sets is countable.
4. A finite product of countable sets is countable.

Theorem

Let X be the two element set $\{0, 1\}$. Then X^ω is uncountable.

Definition

Two sets A and B have the same cardinality if there is a bijection between A and B.

Example

1. Show that if $B \subset A$ and there is a injection $f : A \to B$, then A and B have the same cardinality.
2. If there are injection $f : A \to C$ and $g : C \to A$, then A and C have the same cardinality.
3. Let X be the two element set $\{0, 1\}$, and $\mathcal{B}$ be the set of all countable subsets of X^ω. Then X^ω and $\mathcal{B}$ have the same cardinality.

Theorem

Let A be a set. The followings are equivalent.

1. *There is an injective function $f : \mathbb{Z}_+ \to A$.*
2. *There is a bijection between A and a proper subset of A.*
3. *A is infinite.*

Axiom of choice

Given a collection $\mathcal{A}$ of disjoint nonempty sets, there exists a set C consisting of exactly one element from each element of $\mathcal{A}$.

Lemma

Given a collection $\mathcal{B}$ of nonempty sets, there exists a function $c : \mathcal{B} \to \displaystyle\bigcup_{B \in \mathcal{B}} B$ such that $c(B)$ is an element in B, for each $B \in \mathcal{B}$

The allowable methods for specifying sets:

1. Defining a set by listing its elements, or by taking a given set A and specifying a subset B of it by giving a property that the elements of B are to satisfy
2. Taking unions or intersections of the elements of a given collection of sets, or taking the difference of two sets
3. Taking the set of all subsets of given set
4. taking cartesian products of sets

Example

Define an injective map $f : \mathbb{Z}_+ \to X^\omega$ where $X = \{0, 1\}$ without using axiom of choice.

Definition

A set A with an order relation $<$ is called well-ordered if every nonempty subset of A has a smallest element.

Definition

Two ordered sets A and B have the same order type if there is a bijection between A and B preserving order relations.

Theorem

Every nonempty finite ordered set has the order type of a section $\{1, \cdots, n\}$ *of* $\mathbb{Z}_+$

Example

1. Show that $\{1,2\} \times \mathbb{Z}_+$ in dictionary order is well-ordered.
2. Do $\{1,2\} \times \mathbb{Z}_+$ and $\mathbb{Z}_+ \times \{1,2\}$ have the same order type?