SE328:Topology

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Week 01

Given a statement of the form $P \Rightarrow Q$, its contrapositive is $\sim Q \Rightarrow P$, and its converse is $Q \Rightarrow P$, where $\sim P$ is the negation of P.

Example

Determine whether the following statements are true.

- 1. $A \subset B$ and $A \subset C \Leftrightarrow A \subset (B \cap C)$
- 2. A (A B) = B
- 3. $(A \cap B) \cup (A B) = A$

- 1. Write the contrapositive and converse of $x < 0 \Rightarrow x^2 x > 0$
- 2. Write the negation of $\forall a \in A, a^2 \in B$

Given sets A and B, the cartesian product $A \times B$ is the set

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

Example

Determine whether each of the following sets is a cartesian product of two subsets of \mathbb{R} .

- 1. $\{(x,y) \mid x \in \mathbb{Z}\}$
- 2. $\{(x,y) \mid y > x\}$

A function f is a subset of the cartesian product $A \times B$ of two sets, with the property that each elemeth in C appears as the first coordinate of at most one ordered pair. In other words,

$$(a,b),(a,b') \in f \Rightarrow b=b'$$

Definition

Given a function $f: A \to B$ and a subsets $A_0 \subset A$, the restriction of f to A_0 is

$$\{(a, f(a)) \mid a \in A_0\}$$

A function $f: A \to B$ is injective if

$$\forall a \in A, f(a) = f(a') \Rightarrow a = a'$$

and surjective if

$$\forall b \in B, \exists a \in Af(a) = b$$

Example

Let $f: A \to B$ and $g: B \to C$ be functions and $A_0 \subset A$, $B_0, B_1 \subset B$.

- 1. $A_0 \subset f^{-1}(f(A_0))$.
- 2. $B_0 \subset B_1 \Rightarrow f^{-1}(B_0) \subset f^{-1}(B_1)$.
- 3. If $C_0 \subset C$, then

$$(g \circ f)^{-1}(C_0) = f^{-1}(g^{-1}(C_0))$$

A equivalence relation \sim on a set A is a subset $\subset A \times A$ such that

- 1. $\forall x \in A, (x, x) \in \sim$
- 2. $(x,y) \in C \Rightarrow (y,x) \in \sim$
- 3. $(x,y),(y,z) \in \sim \Rightarrow (x,z) \in \sim$

We denote $x \sim y$ if $(x, y) \in \sim$.

Definition

The equivalence class of $x \in A$ is the set

$$[x] = \{y \mid y \sim x\}$$

The collection of all equivalence classes for \sim becomes a partition of A, i.e. the collection of disjoint nonempty subsets of A.

An order relation < on a set A is a subset of $A \times A$ such that

- 1. $x, y \in A, x \neq y \Rightarrow (x, y) \in \langle x, y \rangle \in \langle x, y \rangle \in \langle x, y \rangle \in \langle x, y \rangle$
- $2. \ \nexists x \in A, (x, x) \in <$
- 3. $(x,y) \in \langle (y,z) \in \langle \Rightarrow (x,z) \in \langle ... \rangle$

We denote x < y if $(x, y) \in <$.

Definition

If < is an order relation on a set A, and if a < b, an open interval (a, b) is a subset defined by

$$(a, b) = \{ x \in A \mid a < x < b \}$$

If $(a, b) = \emptyset$, then a is called the immediate predecessor of b, and b is called the immediate successor of a.

Suppose that A, B are two sets with order relations $<_A, <_B$. The dictionary order relation < on $A \times B$ is defined by

$$a_1 \times b_1 < a_2 \times b_2$$

if $a_1 <_A a_2$, or if $a_1 = a_2$ and $b_1 <_B b_2$.

Definition

An ordered set A is said to have the least upper bound property if every nonempty subset $A_0 \subset A$ that is bounded above has a least upper bound. The greatest lower bound property is similarly defined.

- 1. Let $f: A \to B$ is a surjective function. Define $a_0 \sim a_1$ if $f(a_0) = f(a_1)$. Show that \sim is an equivalence relation.
- 2. If an ordered set A has the least upper bound property, then it has the greatest lower bound property.
- 3. Assuming that the real line has the least upper bound property, show that $[0,1] = \{x \mid 0 \le x \le 1\}$ also has the least upper bound property.
- 4. Does $[0,1] \times [0,1]$ in the dictionary order relation have the least upper bound property?

The set of real numbers, denoted by \mathbb{R} , is a set with two operations +(addition), $\cdot(multiplication)$, and an order relation <. It contains two special elements, 1(one) and 0(zero). All elementary algebraic properties hold including the following statements.

- 1. $x < y \Rightarrow x + z < y + z$
- 2. $x < y, 0 < z \Rightarrow x \cdot z < y \cdot z$
- 3. < has the least upper bound property
- 4. $x < y \Rightarrow \exists z \in \mathbb{R} x < z < y$

The subset $A \subset \mathbb{R}$ is called inductive if it contains 1 and if $x \in A$ then $x + 1 \in A$. The set of all positive integers, denoted by \mathbb{Z}_+ is the smallest among all inductive subsets.

Theorem

Every nonempty subset of \mathbb{Z}_+ has a smallest element.

Theorem

Let A be a set of positive integers containing 1. For each positive integer $n \in \mathbb{Z}_+$, $S_n = \{1, \dots, n\} \subset A \Rightarrow n+1 \in A$. If this is true for all n, then $A = \mathbb{Z}_+$.

Let m be a positive integer. An m-tuple of elements of X is a function $\mathbf{x}: \{1, \dots, m\} \to X$. The ω -tuple of elements of X is a function $\mathbf{x}: \mathbb{Z}_+ \to X$.

Definition

Let A_1, A_2, \cdots be a family of sets, indexed by \mathbb{Z}_+ . The cartesian product of A_i , denoted by $\prod_{i \in \mathbb{Z}_+} A_i$, is the set of all

 ω -tuples of elements of $\bigcup_{i \in \mathbb{Z}_+} A_i$ such that $x_i \in A_i$. If $A_i = X$ for

all i, then the cartesian product is denoted by X^{ω} .

Example

Find a bijective map $f: X^{\omega} \times X^{\omega} \to X^{\omega}$

A set is called finite if there is a bijection between the set and S_n for some positive integer n. A set is called infinite if it is not finite. It is called countably infinite if there is a bijection between the set and \mathbb{Z}_+ . A infinite set which is not countable is called uncountable.

Theorem

Let A be a set. The followings are equivalent:

- 1. A is countable.
- 2. There is a surjective function $f: \mathbb{Z}_+ \to A$.
- 3. There is an injective function $f: A \to \mathbb{Z}_+$.

Lemma

If A is an infinite subset of \mathbb{Z}_+ , then A is countably infinite.

Corollary

- 1. A subset of a countable set is countable.
- 2. The set $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countably infinite.
- 3. A countable union of countable sets is countable.
- 4. A finite product of countable sets is countable.

Theorem

Let X be the two element set $\{0,1\}$. Then X^{ω} is uncountable.

Two sets A and B have the same cardinality if there is a bijection between A and B.

- 1. Show that if $B \subset A$ and there is a injection $f: A \to B$, then A and B have the same cardinality.
- 2. If there are injection $f:A\to C$ and $g:C\to A$, then A and C have the same cardinality.
- 3. Let X be the two element set $\{0,1\}$, and \mathcal{B} be the set of all countable subsets of X^{ω} . Then X^{ω} and \mathcal{B} have the same cardinality.

Theorem

Let A be a set. The followings are equivalent.

- 1. There is an injective function $f: \mathbb{Z}_+ \to A$.
- 2. There is a bijection between A and a proper subset of A.
- 3. A is infinite.

Axiom of choice

Given a collection \mathcal{A} of disjoint nonempty sets, there exists a set C consisting of exactly one element from each element of \mathcal{A} .

Lemma

Given a collection \mathcal{B} of nonempty sets, there exists a function $c: \mathcal{B} \to \bigcup_{B \in \mathcal{B}} B$ such that c(B) is an element in B, for each $B \in \mathcal{B}$

The allowable methods for specifying sets:

- 1. Defining a set by listing its elements, or by taking a given set A and specifying a subset B of it by giving a property that the elements of B are to satisfy
- 2. Taking unions or intersections of the elements of a given collection of sets, or taking the difference of two sets
- 3. Taking the set of all subsets of given set
- 4. taking cartesian products of sets

Example

Define an injective map $f: \mathbb{Z}_+ \to X^{\omega}$ where $X = \{0, 1\}$ without using axiom of choice.

A set A with an order relation < is called well-ordered if every nonempty subset of A has a smallest element.

Definition

Two ordered sets A and B have the same order type if there is a bijection between A and B preserving order relations.

Theorem

Every nonempty finite ordered set has the order type of a section $\{1, \dots, n\}$ of \mathbb{Z}_+

- 1. Show that $\{1,2\} \times \mathbb{Z}_+$ in dictionary order is well-ordered.
- 2. Do $\{1,2\} \times \mathbb{Z}_+$ and $\mathbb{Z}_+ \times \{1,2\}$ have the same order type?