SE328:Topology

Hyosang Kang[1]

[1]Division of Mathematics
School of Interdisciplinary Studies
DGIST

Week 01

Definition

Given a statement of the form $P \Rightarrow Q$, its contrapositive is $\sim Q \Rightarrow \sim P$, and its converse is $Q \Rightarrow P$, where $\sim P$ is the negation of P.

Example

Determine whether the following statements are true.

1. $\boxed{A \subset B \text{ and } A \subset C} \Leftrightarrow \boxed{A \subset (B \cap C)}$ "$\Rightarrow$" & "$\Leftarrow$"
2. $A - (A - B) = B$ $A - B = A \cap B^c$
3. $(A \cap B) \cup (A - B) = A$ "

Example

1. Write the contrapositive and converse of
   "$x < 0 \Rightarrow x^2 - x > 0$"
2. Write the negation of "$\forall a \in A, a^2 \in B$" : "$\exists a \in A, a^2 \notin B$".

"for all", "for every"

" $A \subset B$ and $A \subset C$ " $\Leftrightarrow$ " $A \subset B \cap C$ "

($\Rightarrow$) Need to prove $x \in A$ then $x \in B \cap C$

$x \in B \cap C$ if and only if $x \in B$ and $x \in C$.

Thus if $x \in A$, then $x \in B$ and $x \in C$

Thus $A \subset B$ and $A \subset C$.

Brain-storming

" $x \in A \Rightarrow x \in B$ and $x \in A \Rightarrow x \in C$ "

$\Rightarrow$ " $x \in A \Rightarrow x \in B$ and $x \in C$ "

$\Rightarrow$ " $x \in A \Rightarrow x \in B \cap C$ "

$\Rightarrow$ " $A \subset B \cap C$. "

($\Leftarrow$) " $x \in A \Rightarrow x \in B \cap C$ "

$\Rightarrow$ " $x \in A \Rightarrow x \in B \cap C \subset B$ and $x \in B \cap C \subset C$ "

$\Rightarrow$ " $x \in A \Rightarrow x \in B$ and $x \in A \Rightarrow x \in C$ "

$\Rightarrow$ $A \subset B$ and $A \subset C$.

" $A - (A - B) \neq B$ "

$$A - (A - B) = A \cap (A - B)^c = A \cap (A \cap B^c)^c$$

$$= A \cap (A^c \cup B) = (A \cap A^c) \cup (A \cap B) = \phi \cup (A \cap B) = A \cap B$$

Need a counter-example when $A \cap B \neq B$.

$A = \{1, 2\}$, $B = \{2, 3\}$.

" $x < 0 \Rightarrow x^2 - x > 0$ " (" $P \Rightarrow Q$ " $P : x < 0$, $Q : x^2 - x > 0$)

contrapositive : " $x^2 - x < 0 \Rightarrow x > 0$ "

converse : " $x^2 - x > 0 \Rightarrow x < 0$ ".

Definition

Given sets A and B, the cartesian product $A \times B$ is the set

$$A \times B = \{\underbrace{(a,b)}_{\text{ordered pair (can define by } \{\{a\},\{a,b\}\}\text{.}} \mid a \in A,\, b \in B\}$$

Example

Determine whether each of the following sets is a cartesian product of two subsets of $\mathbb{R}$.

1. $\{(x,y) \mid x \in \mathbb{Z}\} \subset \mathbb{R} \times \mathbb{R}$ $\quad$ $\mathbb{Z} \times \mathbb{R}$.
2. $\boxed{\{(x,y) \mid y > x\}} \subset \mathbb{R} \times \mathbb{R}$ $\quad$ $\times$.

"$\{(x,y) \mid x\in\mathbb{Z}\} = \mathbb{Z}\times\mathbb{R}$"

① $\{(x,y) \mid x\in\mathbb{Z}\} \subset \mathbb{Z}\times\mathbb{R}$

$x\in\mathbb{Z}$, then $(x,y)\in\mathbb{Z}\times\mathbb{R}$

② $\{(x,y) \mid x\in\mathbb{Z}\} \supset \mathbb{Z}\times\mathbb{R}$

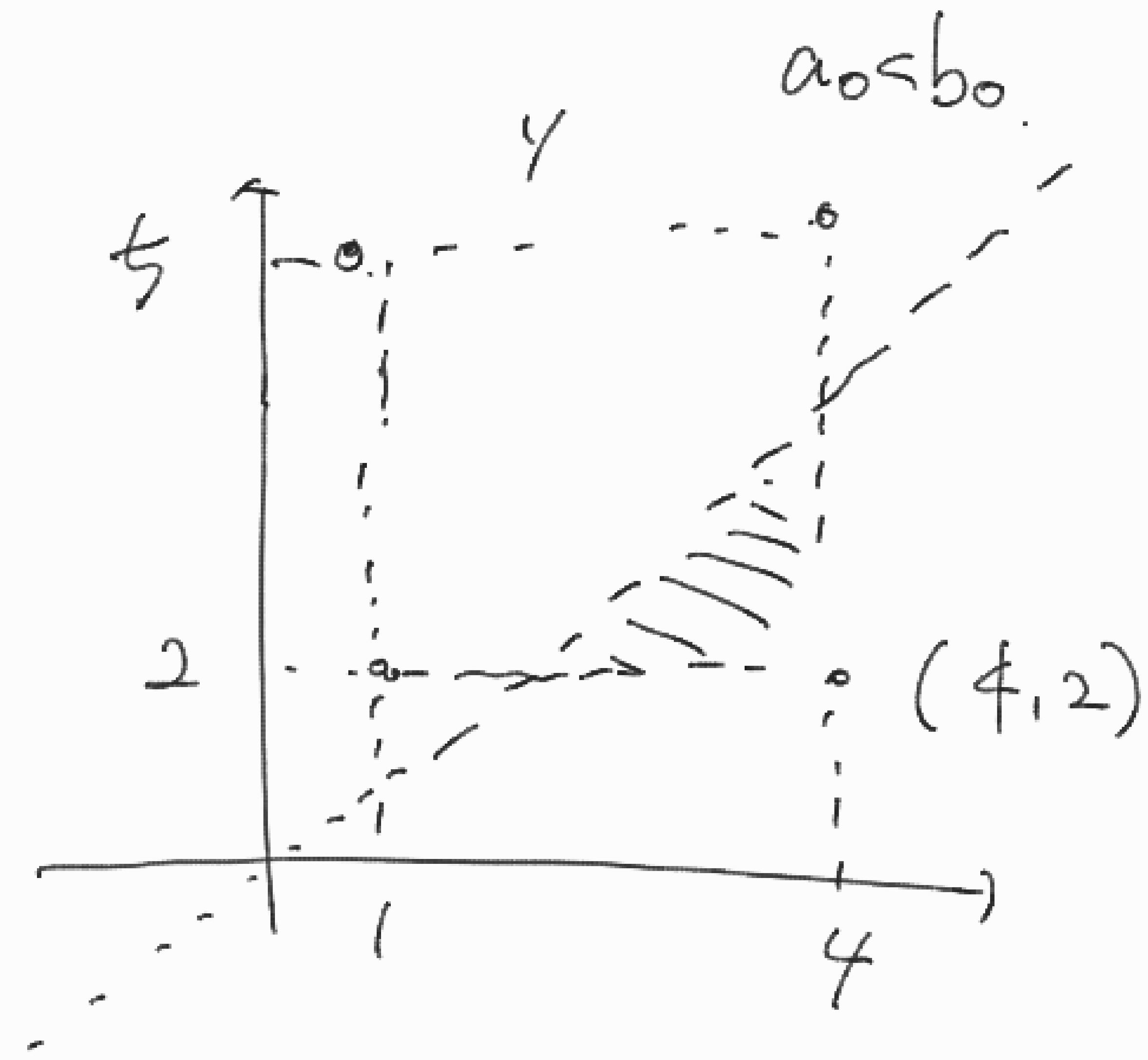
Let $(x,y)\in\mathbb{Z}\times\mathbb{R}$. Then $\underline{\underline{x\in\mathbb{Z}}}, y\in\mathbb{R}$

Thus $(x,y)\in\{(x,y) \mid x\in\mathbb{Z}\}$

"There is no $A, B\subset\mathbb{R}$ such that $\{(x,y) \mid y>x\} = A\times B$".

Suppose that there are $\underline{A}, \underline{B}\subset\mathbb{R}$ such that

$C = \{(x,y) \mid y>x\} = A\times B$.



C contains $(1,2)$ and $(4,5)$.

That means A contains $1, 4$.

B 〃 $2, 5$.

Thus $A\times B$ must contain $(4,2)$

But $(4,2)\notin C$. contradiction.

Definition

A function f is a subset of the cartesian product $A \times B$ of two sets, with the property that each elemetn in C appears as the first coordinate of at most one ordered pair. In other words,

$$(a,b),\ (a,b') \in f \Rightarrow b = b'.$$

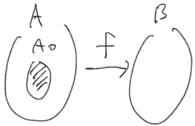
$f(a) = b$

$\Leftrightarrow$ If the $f(a) = b$ and $f(a) = b'$, then $b = b'$.

Definition

Given a function $f : A \to B$ and a subsets $A_0 \subset A$, the restriction of f to A_0 is

$$\{(a, f(a)) \mid a \in A_0\}$$

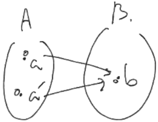


$f(A_0) \subset f(A)$.

Definition

A function $f : A \to B$ is injective if

$$\forall a \in A, f(a) = f(a') \Rightarrow a = a'$$



$\Rightarrow a = a'$.

and surjective if

$$\forall b \in B, \exists a \in A f(a) = b$$

bijective $\Leftrightarrow$ inj + surj

Example

Let $f : A \to B$ and $g : B \to C$ be functions and $A_0 \subset A$, $B_0, B_1 \subset B$.

1. $A_0 \subset f^{-1}(f(A_0))$.
2. $B_0 \subset B_1 \Rightarrow f^{-1}(B_0) \subset f^{-1}(B_1)$.
3. If $C_0 \subset C$, then

$$(g \circ f)^{-1}(C_0) = f^{-1}(g^{-1}(C_0))$$

" $A_0 \subset f^{-1}(f(A_0))$ ".

Need to show $x \in A_0$, then $x_0 \in f^{-1}(f(A_0))$.

Note that $x_0 \in f^{-1}(f(A_0))$ if $f(x_0) \in f(A_0)$.

$x \in A_0 \Rightarrow f(x) \in f(A_0) \Rightarrow x_0 \in f^{-1}(f(A_0))$.

" $B_0 \subset B_1 \Rightarrow f^{-1}(B_0) \subset f^{-1}(B_1)$ ".

Let $x \in f^{-1}(B_0)$. Then $f(x) \in B_0$. Since $B_0 \subset B_1$, $f(x) \in B_1$.

Therefore $x \in f^{-1}(B_1)$ Thus $f^{-1}(B_0) \subset f^{-1}(B_1)$.

Definition

A equivalence relation $\sim$ on a set A is a subset $\subset A \times A$ such that

1. $\forall x \in A, (x, x) \in \sim$: $x \sim x$
2. $(x, y) \in C \Rightarrow (y, x) \in \sim$: $x \sim y \Rightarrow y \sim x$
3. $(x, y), (y, z) \in \sim \Rightarrow (x, z) \in \sim$: $x \sim y,\ y \sim z \Rightarrow x \sim z$

We denote $x \sim y$ if $(x, y) \in \sim$.

Definition

$1 \equiv_3 4$, $0 \equiv_3 3 \equiv_3 6, \cdots$

The equivalence class of $x \in A$ is the set

$1 \sim 4$, $4 \sim 7, \cdots$

$$[x] = \{y \mid y \sim x\}$$

$[1] = \{n \in \mathbb{Z} \mid 1 \equiv_3 n\}$.

$[0] = [3]$

$[2] = [5]$

The collection of all equivalence classes for $\sim$ becomes a partition of A, i.e. the collection of disjoint nonempty subsets of A.

Definition

An order relation $<$ on a set A is a subset of $A \times A$ such that

1. $\boxed{x, y} \in A, x \neq y \Rightarrow (x, y) \in <$ or $(y, x) \in <$: $x \neq y \Rightarrow x < y$ or $y < x$
2. $\nexists x \in A, (x, x) \in <$: $\nexists x \in A, x < x$
3. $(x, y) \in <, (y, z) \in < \Rightarrow (x, z) \in <.$: $x < y, y < z \Rightarrow x < z$

We denote $x < y$ if $(x, y) \in <$.

Definition

If $<$ is an order relation on a set A, and if $a < b$, an open interval (a, b) is a subset defined by

$$(a, b) = \{x \in A \mid a < x < b\}$$

If $(a, b) = \emptyset$, then a is called the immediate predecessor of b, and b is called the immediate successor of a.

Definition

Suppose that A, B are two sets with order relations $<_A, <_B$. The dictionary order relation $<$ on $A \times B$ is defined by

$$(a_1, b_1) = a_1 \times b_1 < a_2 \times b_2 = (a_2, b_2)$$

if $a_1 <_A a_2$, or if $a_1 = a_2$ and $b_1 <_B b_2$.

Definition

$(0,1) \subset \mathbb{R}$

An ordered set A is said to "have the least upper bound property" if every nonempty subset $A_0 \subset A$ that is bounded above has a least upper bound. The greatest lower bound property is similary defined.

Example

1. Let $f : A \to B$ is a surjective function. Define $a_0 \sim a_1$ if $f(a_0) = f(a_1)$. Show that $\sim$ is an equivalence relation.
2. If an ordered set A has the least upper bound property, then it has the greatest lower bound property.
3. Assuming that the real line has the least upper bound property, show that $[0,1] = \{x \mid 0 \leq x \leq 1\}$ also has the least upper bound property.
4. Does $[0,1] \times [0,1]$ in the dictionary order relation have the least upper bound property?

" $a_0 \sim a_1 \Leftrightarrow f(a_0) = f(a_1)$ " is an e.r.

pf) ① Need to show $a \sim a$ for all $a \in A$.

$f(a) = f(a)$. Thus true.

② Need to show $a \sim b$ then $b \sim a$ for all $a, b \in A$.

$a \sim b \Leftrightarrow f(a) = f(b) \Leftrightarrow f(b) = f(a) \Leftrightarrow b \sim a$

③ Need to show $a \sim b$, $b \sim c \Rightarrow b \sim c$

$a \sim b \Leftrightarrow f(a) = f(b)$ $\quad b \sim c \Leftrightarrow f(b) = f(c)$

$a \sim b$, $b \sim c \Rightarrow f(a) = f(b) = f(c) \Rightarrow a \sim c$

"If A has l.u.b.p, then A has g.l.b.p"

Suppose that there is a subset $A_0 \subset A$ which is bound below, but does NOT have a greatest lower bound.

Let B_0 be the set of all lower bounds of A_0.

$B_0 \neq \phi$ because A_0 is bounded below (there exists at least one lower bound of A_0)

$B_0 \subset A$ and A has l.u.b.p., B_0 is bound above by an element $a \in A_0$ (any element a will do.)

Thus B_0 has least upper bound say b_0

If $b_0 \in B_0$, then b_0 is a lower bound of A and it is the greatest lower bound. contradiction.

If $b_0 \notin B_0$, then there is $a \in A_0$ such that $a < b_0$.

Note that a is a upper bound of B_0 less than b_0.

contradicts to the def. of b_0

Thus there is No such A_0, and All nonempty subsets of A having lower bound admits greatest lower bound.

Definition

The set of real numbers, denoted by $\mathbb{R}$, is a set with two operations $+$(addition), $\cdot$(multiplication), and an order relation $<$. It contains two special elements, 1(one) and 0(zero). All elementary algebraic properties hold including the following statements.

1. $x < y \Rightarrow x + z < y + z$
2. $x < y, 0 < z \Rightarrow x \cdot z < y \cdot z$
3. $<$ has the least upper bound property
4. $x < y \Rightarrow \exists z \in \mathbb{R} x < z < y$

Definition

The subset $A \subset \mathbb{R}$ is called inductive if it contains 1 and if $x \in A$ then $x+1 \in A$. The set of all positive integers, denoted by $\mathbb{Z}_+$ is the smallest among all inductive subsets.

$\mathbb{Z}_+ = \bigcap_{A \subset \mathbb{R},\ A:\text{inductive}} A$.

Theorem

Every nonempty subset of $\mathbb{Z}_+$ has a smallest element.

Theorem

Let A be a set of positive integers containing 1. For each positive integer $n \in \mathbb{Z}_+$, $S_n = \{1, \cdots, n\} \subset A \Rightarrow n+1 \in A$. If this is true for all n, then $A = \mathbb{Z}_+$.

$$A = \{ n \in \mathbb{Z}_+ \mid P(n) \}$$

$$\left. \text{“}S_n \subset A \Rightarrow n+1 \in A\text{”} \Rightarrow A = \mathbb{Z}_+ \right\} \text{“Induction method”}$$

Thm Every nonempty subset of $\mathbb{Z}_+$ has the least element.

pf) First, consider a special case when $S_n = \{1, \cdots, n\}$.

Let A be a subset of S_n.

We want to use induction method.

$n=1$. $A = \{1\}$ has the least element, 1.

Suppose that any subset of S_k has the least element for $k \le n$.

Let $A \subset S_{n+1}$. If $A = \{n+1\}$, then A has the least element $n+1$.

If $n+1 \notin A$, then $A \subset S_n$. Therefore by assumption, A has the least elt.

If $A' = A \setminus \{n+1\} \neq \phi$, then $A' \subset S_n$, thus A' has the least elt.

and this elt is the least elt of A.

Let $A \subset \mathbb{Z}_+$ and $n \in A$. Then $A' = A \cap S_n \subset S_n$

Thus A' has the least element. This elt is the least elt of A. $\blacksquare$

Thm —//—

pf) Suppose $A \neq \mathbb{Z}_+$, that is, there is $n \notin A$. $n \in \mathbb{Z}_+$.

Choose the smallest n $\in \mathbb{Z}_+$ such that $n \notin A$.

Then $S_{n-1} \subset A$. $(n \neq 1)$. Thus by assumption, $n \in A$. contradiction.

Therefore the assumption is false. That means $A = \mathbb{Z}_+$ $\blacksquare$

Definition

$\mathbb{x} = (x_1, \cdots, x_m)$

Let m be a positive integer. An m-tuple of elements of X is a function $\mathbf{x} : \{1, \cdots, m\} \to X$. The ω-tuple of elements of X is a function $\mathbf{x} : \mathbb{Z}_+ \to X$.

$\mathbb{x} = (x_1, x_2, \cdots, x_n, \cdots)$

Definition

Let $A_1, A_2, \cdots$ be a family of sets, indexed by $\mathbb{Z}_+$. The cartesian product of A_i, denoted by $\prod_{i \in \mathbb{Z}_+} A_i$, is the set of all ω-tuples of elements of $\bigcup_{i \in \mathbb{Z}_+} A_i$ such that $x_i \in A_i$. If $A_i = X$ for all i, then the cartesian product is denoted by X^ω.

$A_i = X$, $A_1 \times A_2 = X \times X = X^2$

$$f(\mathbb{x}, \mathbb{y})(i) = \begin{cases} \mathbb{x}\left(\frac{i+1}{2}\right) & i: \text{odd} \\ \mathbb{y}\left(\frac{i}{2}\right) & i: \text{even} \end{cases}$$

Example

Find a bijective map $f : X^\omega \times X^\omega \to X^\omega$

example.

$(\mathbb{x}, \mathbb{y}) \longmapsto (x_1, y_1, x_2, y_2, \cdots) = f(\mathbb{x}, \mathbb{y})$

Definition

A set is called finite if there is a bijection between the set and S_n for some positive integer n. A set is called infinite if it is not finite. It is called countably infinite if there is a bijection between the set and $\mathbb{Z}_+$. A infinite set which is not countable is called uncountable.

Theorem

Let A be a set. The followings are equivalent:

1. *A is countable.*
2. *There is a surjective function $f : \mathbb{Z}_+ \to A$.*
3. *There is an injective function $f : A \to \mathbb{Z}_+$.*

ex) $A = \{2n \mid n \in \mathbb{Z}_+\}$.

$f : \mathbb{Z}_+ \to A$ $\quad f(n) = 2n$ $\quad$: surj

$f : A \to \mathbb{Z}_+$ $\quad f(m) = m$ $\quad$: inj

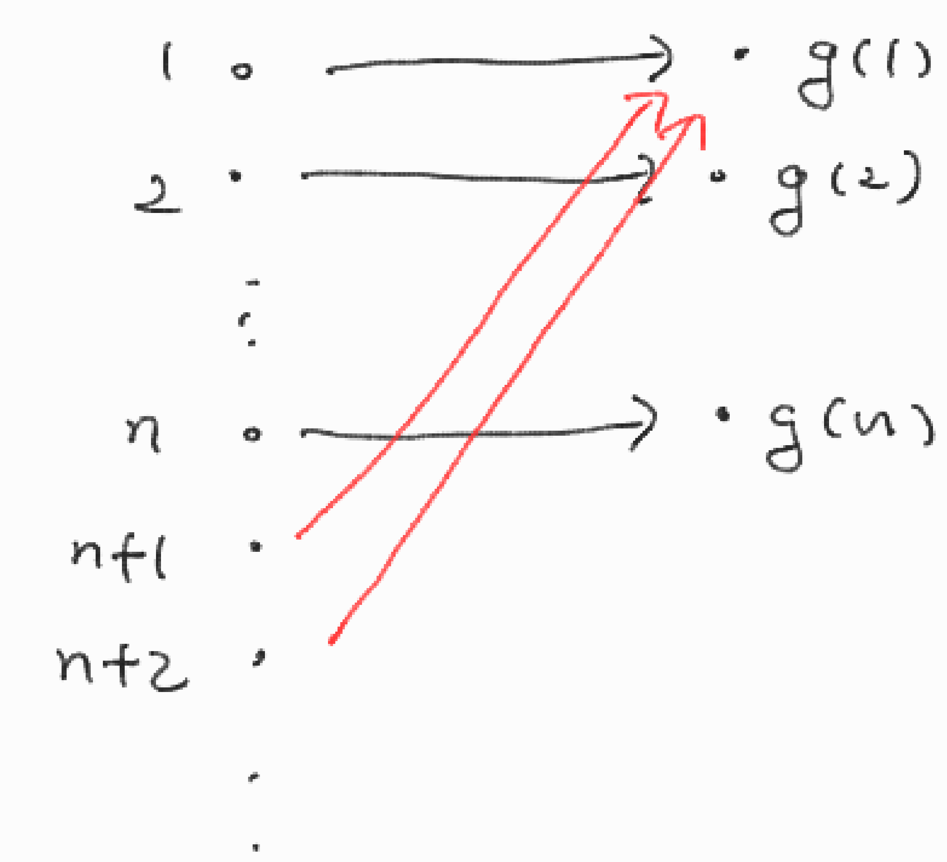
Thm —||—

① ⇒ ②

If A is finite, then $g : S_n \to A$ bijection.

Thus we can define $f : \mathbb{Z}_+ \to A$ as

$$f(k) = \begin{cases} g(k) & \text{if } k \in S_n \\ g(1) & \text{if } k \notin S_n. \end{cases}$$



② ⇒ ③

Let $g : \mathbb{Z}_+ \to A$. Define $f : A \to \mathbb{Z}_+$ as

$$f(a) = \text{smallest in } g^{-1}(a).$$

f is injective because if $a_0 \neq a_1 \in A$, then $g^{-1}(a_0) \cap g^{-1}(a_1) = \emptyset$

Thus $f(a_0) \neq f(a_1)$.

③ ⇒ ①

prove that every subset of $\mathbb{Z}_+$ is countable

Let $C \subset \mathbb{Z}_+$ inf   Let us define $h : \mathbb{Z}_+ \to C$ as

$$h(1) = \text{smallest in } C$$
$$h(2) = \text{smallest in } C \setminus \{h(1)\}$$
$$\vdots$$
$$h(n) = \text{smallest in } C \setminus \{h(1), \cdots, h(n-1)\}.$$

h is injective, because if $i < j$ then

since $h(i) \in \{h(1), \cdots h(j)\}$ $h(j) \in C \setminus \{h(1), \cdots, h(j-1)\}$.

$h(i) \neq h(j)$.

h is surjective. To show this, pick $c \in C$

Let $m \in \mathbb{Z}_+$ be the smallest element in $\mathbb{Z}_+$ s.t. $h(m) \geq c$.

If $i < m$, then $h(i) < c$ (why? if not, say $h(i) \geq c$, then i is a smaller elt than m satisfying ___, and this contradicts the assumption.)

Thus $c \notin \{h(1), \cdots, h(m-1)\}$

Thus $h(m)$ = smallest elt in $(C \setminus \{h(1), \dots, h(m-1)\}) \leq c$.

does not contain c.

contains c

Thus $h(m) = c$. (i.e. m is the preimage of c).

Supp $f: A \to \mathbb{Z}_+$ is injective. Then $f: A \to f(A)$ is bijective.

Then $f(A)$ is a subset of $\mathbb{Z}_+$, thus $f(A)$ is countable by the claim

Thus there is a bijection $h: f(A) \to \mathbb{Z}_+$ and by composition,

$$h \circ f : A \to \mathbb{Z}_+ \quad : \text{bijection.}$$

∎

Lemma ← Is proved in the theorem.

If A is an infinite subset of $\mathbb{Z}_+$, then A is countably infinite.

Corollary

1. A subset of a countable set is countable.
2. The set $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countably infinite.
3. A countable union of countable sets is countable.
4. A finite product of countable sets is countable.

Theorem

Let X be the two element set $\{0, 1\}$. Then X^ω is uncountable.

Corollary 1. $A \subset B$, B: countable. $\Rightarrow f : B \to \mathbb{Z}_+$ inj.

$\Rightarrow f|_A : A \to \mathbb{Z}_+$ inj. $\Rightarrow A$: countable

Corollary 2. $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{Z}_+$, $f(n,m) = 2^n \cdot 3^m \in \mathbb{Z}_+$ is injective.

Corollary 3. $A_1, \cdots, A_n \cdots$ countable $\Leftrightarrow f_n : \mathbb{Z}_+ \to A_n$ surjective.

$J \subset \mathbb{Z}_+$: countable subset $\Leftrightarrow g : \mathbb{Z}_+ \to J$ surjection

NTS : $\bigcup_{i \in J} A_i$ is countable. Suffice to show $\exists$ surj $f : \mathbb{Z}_+ \to \bigcup_{i \in J} A_i$

$h : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \bigcup_{i \in J} A_i$; $h(n,m) = f_{g(n)}(m)$

h is surjective b/c. $x \in A_i$, then $f_{g(n)}(m) = x$ where $g(n) = i$

$f_i(m) = x$.

(by the surjectivity of f_n, g).

$\mathbb{Z}_+ \xrightarrow{1-1} \mathbb{Z}_+ \times \mathbb{Z}_+ \xrightarrow{\text{surj.}} \bigcup_{i \in J} A_i$

b/c $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable (proved in Coro 2)

Corollary 4. Suff. to show that A, B : countable $\Rightarrow A \times B$: countable

A, B : countable $\Rightarrow f : \mathbb{Z}_+ \to A$, $g : \mathbb{Z}_+ \to B$ surj.

$\Rightarrow h : \mathbb{Z}_+ \times \mathbb{Z}_+ \to A \times B$, $h(n,m) = (f(n), g(m))$

is surjective.

<u>Thm</u> $\{0,1\}^\omega$ is uncountable

pf) Suppose $\{0,1\}^\omega$ is countable. That is there is 1-1.

$f : \mathbb{Z}_+ \to \{0,1\}^\omega$

$1 \longrightarrow 001001 \cdots x_1$ $\quad y = (y_i)$, $\quad y_i = \begin{cases} 1 & \text{if } x_{ii} = 0 \\ 0 & \text{if } x_{ii} = 1. \end{cases}$

$2 \longrightarrow 101101 \cdots x_2$

$3 \longrightarrow 111100 \cdots x_3$.

$\vdots$

$x_i = (x_{i1}, x_{i2}, \cdots)$

$110 \cdots\cdots \quad y$

f is NOT surjective

$\Downarrow$

Then y has no preimage in f $\Rightarrow \{0,1\}^\omega$ is NOT countable $\blacksquare$

Definition

Two sets A and B have the same cardinality if there is a bijection between A and B.

Example

$A = \mathbb{Z}_+$, $B = \{2n \mid n \in \mathbb{Z}_+\} = 2\mathbb{Z}_+$ $f : \mathbb{Z}_+ \to 2\mathbb{Z}_+ : f(n) = 2n$.

1. Show that if $B \subset A$ and there is a injection $f : A \to B$, then A and B have the same cardinality.
2. If there are injection $f : A \to C$ and $g : C \to A$, then A and C have the same cardinality.
3. Let X be the two element set $\{0, 1\}$, and $\mathcal{B}$ be the set of all countable subsets of X^ω. Then X^ω and $\mathcal{B}$ have the same cardinality.

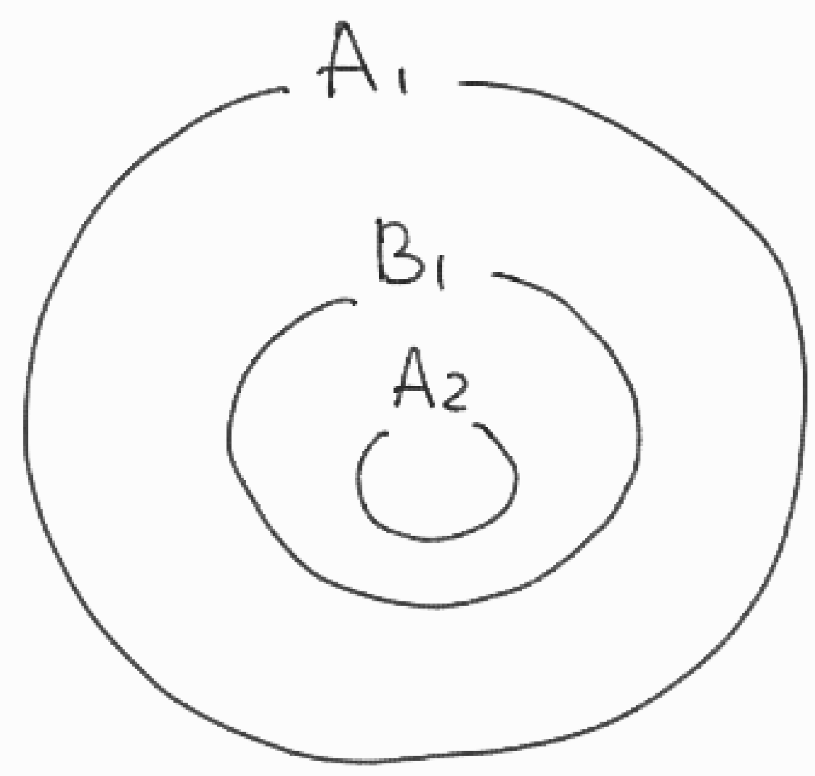
NTS: $X^\omega \xrightarrow[f]{1-1} \mathcal{B}$

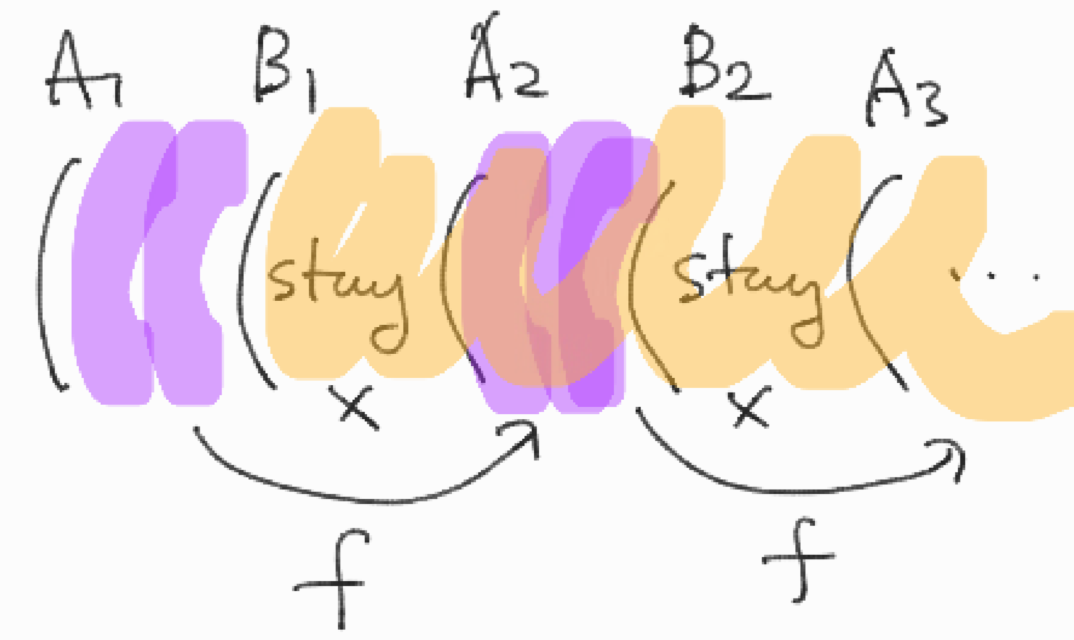
#1. $A_1 = A, \quad B_1 = B.$

$f(A_1) = A_2 \subset B_1.$

$f(B_1) = B_2 \subset A_2$

$\vdots$





Define a bijection $h : A \to B.$

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_i - B_i \quad (i = 1, 2, \cdots) \\ x & \text{otherwise} \end{cases}$$

h : inj. because if $x_0 \in A_i - B_i$, $x_1 \in A_j - B_j$ $\quad x_0 \neq x_1$

then $h(x_0) = f(x_0) \neq h(x_1) = f(x_1)$ because f is inj.

and if $x_i \notin A_i - B_i$, then $h(x_i) = x_i$

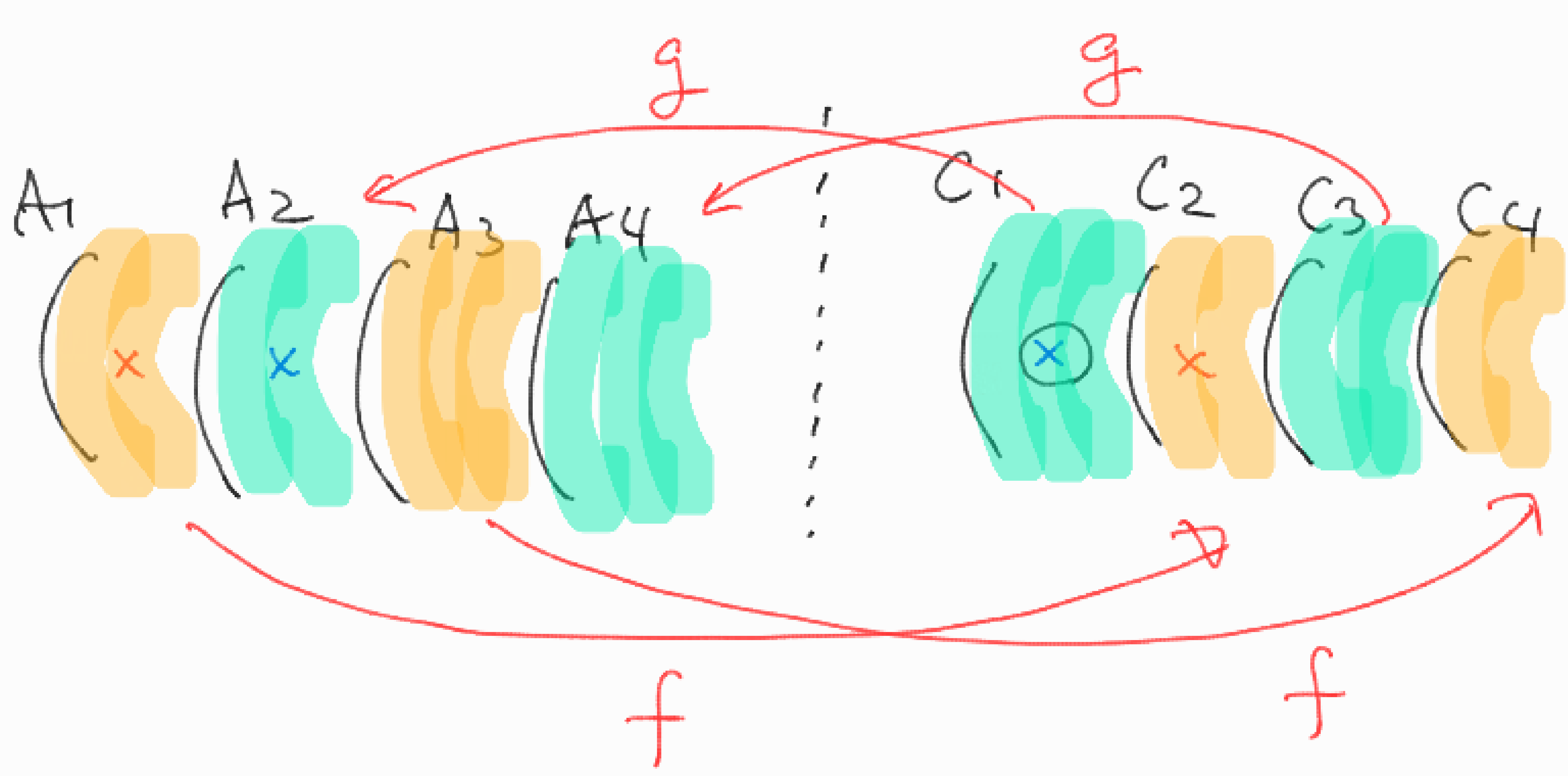
h : surj. b/c any element in the set B always has a preimage.

#2. $A_1 = A, \quad C_1 = C.$

$f(A_1) = C_2, \quad g(C_1) = A_2.$

$f(A_2) = C_3 \quad g(C_2) = A_3$

$\vdots \qquad \vdots$



$C_2 \subset C_1 \Rightarrow g(C_2) = A_3 \subset g(C_1) = A_2.$

$h : A \to C$ define by $h(x) = \begin{cases} f(x) & \text{if } x \in A_{2n-1} - A_{2n} \\ g^{-1}(x) & \text{if } x \in A_{2n} - A_{2n+1}. \end{cases}$

is bijective. ($\because$ injectivity follows from inj. of f, g &

surjectivity follows from the def. of f, g).

Need an ellaboration! (using def of inj. surj.)

Theorem

Let A be a set. The followings are equivalent.

1. *There is an injective function $f : \mathbb{Z}_+ \to A$.*
2. *There is a bijection between A and a proper subset of A.*
3. *A is infinite.*

if is NOT entire A.

☆ Axiom of choice

Given a collection $\mathcal{A}$ of disjoint nonempty sets, there exists a set C consisting of exactly one element from each element of $\mathcal{A}$.

Lemma

NOT necessarily disjoint.

Given a collection $\mathcal{B}$ of nonempty sets, there exists a function $c : \mathcal{B} \to \bigcup_{B \in \mathcal{B}} B$ such that $c(B)$ is an element in B, for each $B \in \mathcal{B}$

Thm 4

① ⇒ ②

$f(1) = a_1, \cdots, f(n) = a_n, \cdots$

$h : A \to A \setminus \{a_1\}$ defined by $h(a_n) = a_{n+1}$.
$h(a) = a$ if $a \notin f(\mathbb{Z}_+)$.

h is inj. b/c if a_i, a_j are taken. $(i \neq j)$, then $h(a_i) = a_{i+1} \neq h(a_j) = a_{j+1}$.
and if $a \notin f(\mathbb{Z}_+)$, then $h(a) = a \notin f(\mathbb{Z}_+)$, thus $h(a) \neq h(a_i)$ $\forall i$.
and if $a, a' \notin f(\mathbb{Z}_+)$, $a \neq a'$, then $h(a) = a \neq h(a') = a'$.

② ⇒ ③

Suppose A is finite. $\exists A \xrightarrow{1-1} S_n$

$$\begin{tikzcd} A_0 \subsetneq A \arrow[r, "1-1"] \arrow[d, "1-1"'] & A_0 \arrow[dl, dashed, "1-1"] \\ S_n & \end{tikzcd}$$

contr.

③ ⇒ ① NTS: $\exists$ inj $f : \mathbb{Z}_+ \to A$.

$f(1) = a_1 \in A$. (pick an elt a_1 in A)

$f(2) = a_2 \in A$ (pick an elt a_2 in $A \setminus \{a_1\}$)

$\vdots$

$f(n) = a_n \in A$, a_n : some elt in $A \setminus \{a_1, \cdots, a_{n-1}\}$.

$\vdots$

"disturbing".

Lem $\exists c : \mathcal{B} \to \bigcup_{B \in \mathcal{B}} B$, $c(B) \in B$.

pf) $\forall B \in \mathcal{B}$, define $B' = \{(B, x) \mid x \in B\} \subset \mathcal{B} \times \bigcup_{B \in \mathcal{B}} B$.

If $B_1, B_2 \in \mathcal{B}$, then $B_1' \cap B_2' = \emptyset$.

Let $\mathcal{A} = \{B' \mid B \in \mathcal{B}\}$, $\exists C$ from A.C. $\Rightarrow$ $c(B) = x$ if $(B, x) \in C$. ▨

Modified version of ③ ⇒ ①.

$\mathcal{B}$ = the collection of nonempty subsets of A.

$\overset{?}{\Rightarrow}$ $\exists\, c : \mathcal{B} \to \bigcup_{B\in\mathcal{B}} B = A$

Lem

Define $f(1) = c(A)$.

$f(2) = c(A - \{f(1)\})$

$\vdots$

$f(n) = c(A - \{f(1), \dots, f(n-1)\})$.

$\vdots$

NOT a recursive definition.

From now on, we may assume A.C. whenever we need to define a function (or set) in this manner.

The allowable methods for specifying sets:

1. Defining a set by listing its elements, or by taking a given set A and specifying a subset B of it by giving a property that the elements of B are to satisfy
2. Taking unions or intersections of the elements of a given collection of sets, or taking the difference of two sets
3. Taking the set of all subsets of given set
4. taking cartesian products of sets

Example

Define an injective map $f : \mathbb{Z}_+ \to X^\omega$ where $X = \{0, 1\}$ without using axiom of choice.

$\{x \mid x = (x_1, x_2, \cdots)\}$.

$f(1) = (1, 0, \cdots)$

$f(2) = (0, 1, 0, \cdots)$

$f(n) = (0, \cdots, 1, 0 \cdots)$ nth place.

Definition

A set A with an order relation $<$ is called well-ordered if every nonempty subset of A has a smallest element.

ex. $\mathbb{Z}_+$ is well-ordered. $\mathbb{Z}_- \supset (-\infty, 0]$ has no smallest

ex. $\mathbb{Z}$ is not well-ordered.

Definition

Two ordered sets A and B have the same order type if there is a bijection between A and B preserving order relations.

$f : A \xrightarrow{1-1} B$ $\quad f(a_1) > f(a_2)$ iff $a_1 > a_2$.

Theorem

Every nonempty finite ordered set has the order type of a section $\underbrace{\{1, \cdots, n\}}_{= S_n}$ *of* $\mathbb{Z}_+$

Example

1. Show that $\{1, 2\} \times \mathbb{Z}_+$ in dictionary order is well-ordered.
2. Do $\{1, 2\} \times \mathbb{Z}_+$ and $\mathbb{Z}_+ \times \{1, 2\}$ have the same order type?

Thm #

pf) Let A be a finite ordered set.

If $A = \{a\}$. Then $A \cong S_1$. Done.

Suppose statement holds for $|A| < n$.

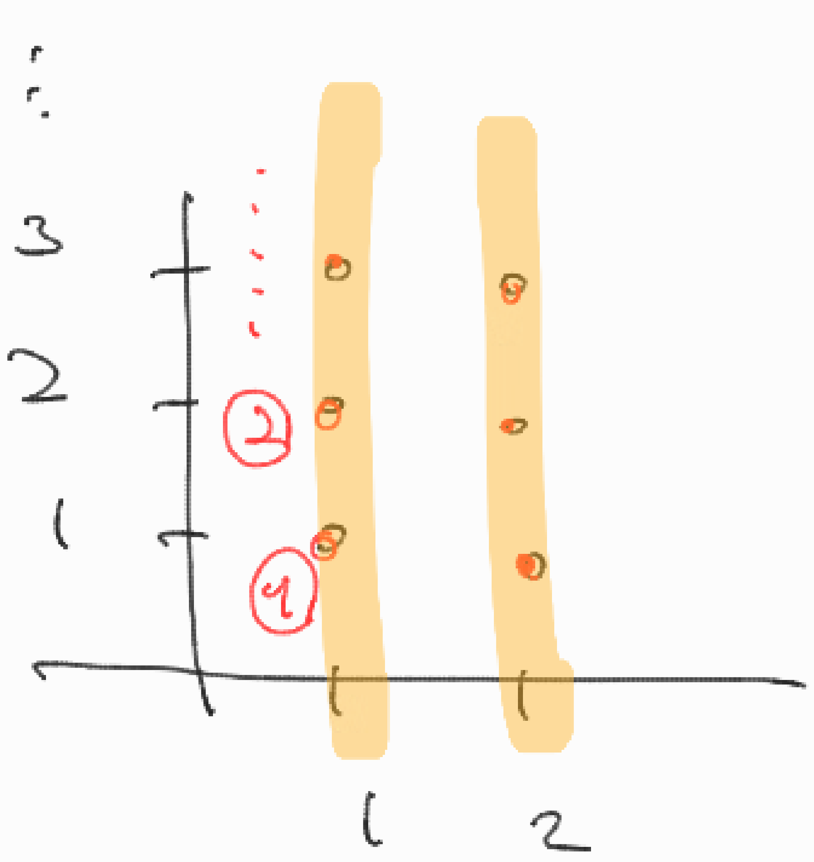
Let $|A| = n$. Then there is a largest element $b \in A$.

Then $|A \setminus \{b\}| = n-1 < n$. Thus $A \setminus \{b\} \cong S_{n-1}$. i.e.

$$f : S_{n-1} \longrightarrow A \setminus \{b\}$$

Define $f' : S_n \to A$ as $f'(i) = f(i)$ for $i < n$
$f'(n) = b$. ∎

Ex



$A \subset \{1,2\} \times \mathbb{Z}_+$.

NTS: A has the smallest elt.

$A = A_1 \cup A_2$ where $A_1 = \{(1,x) \mid (1,x) \in A\}$
$A_2 = \{(2,x) \mid (2,x) \in A\}$.

If $A_1 = \emptyset$, then need to find smallest elt in A_2

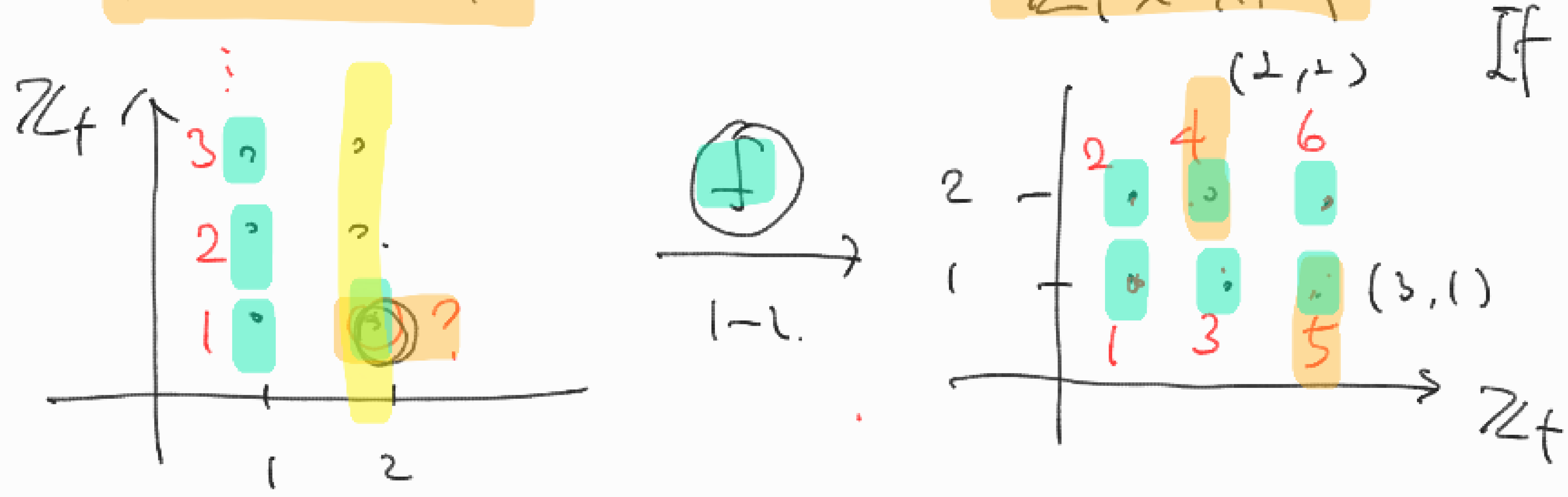
$$A' = \{x \in \mathbb{Z}_+ \mid (2,x) \in A\} \subset \mathbb{Z}_+$$

Since $\mathbb{Z}_+$ is well-ordered, A' has the smallest elt, which is the smallest elt in A.

If $A_1 \neq \emptyset$, then the smallest elt lies in A_1.

$A' = \{x \in \mathbb{Z}_+ \mid (1,x) \in A\} \subset \mathbb{Z}_+$ has the smallest elt. ∎

Ex $\{1,2\} \times \mathbb{Z}_+$ do not have the same order type. $\mathbb{Z}_+ \times \{1,2\}$



If $f(2,1) = (n,m) \in \mathbb{Z}_+ \times \{1,2\}$.
(n,m) has immediate predecessor in $\mathbb{Z}_+ \times \{1,2\}$.

For example, if $(n,m) = (3,1)$, then $(2,2)$ is the immediate predecessor. In general,

$$\begin{cases} (n-1, m+1) & \text{if } m=1 \\ (n, m-1) & \text{if } m=2 \end{cases}$$

is the immediate predecessor of $(n,m) \in \mathbb{Z}_+ \times \{1,2\}$.

But $(2,1) = f^{-1}(n,m)$ Do NOT have the immediate predecessor. If it has one, say $(1,n)$, then

$$(1,n) < (1,n+1) < (2,1)$$

Thus $(1,n)$ is NOT the immediate predecessor.