SE328:Topology

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Week 03

Definition

Let X, Y be two topological spaces. The **product topology** on the product $X \times Y$ is the topology generated by the subset of the form

$$U \times V = \{(u,v) \mid u \in U, v \in V\}.$$

where U, V are open subsets in X, Y respectively.

Theorem

If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the toplogy of Y, then the collection

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B}, C \in \mathcal{C}\}$$

is a basis for the topology of $X \times Y$.

product top.

Def of prod. top.

NTS $\mathcal{C} = \{ U \times V \mid \underset{\text{open}}{\mathcal{U} \subset X}, \underset{\text{open}}{V \subset Y} \}$ is a basis.

① $$\bigcup_{\substack{U \in \mathcal{T}_X \\ V \in \mathcal{T}_Y}} U \times V = \bigcup_{\mathcal{U} \in \mathcal{T}_X} \left(\bigcup_{V \in \mathcal{T}_Y} \mathcal{U} \times V \right) = \bigcup_{U \in \mathcal{T}_X} \mathcal{U} \times \bigcup_{V \in \mathcal{T}_Y} V = \left(\bigcup_{\mathcal{U} \in \mathcal{T}_X} \mathcal{U} \right) \times Y$$

$$= X \times Y.$$

② $\mathcal{U}_1 \times V_1, \mathcal{U}_2 \times V_2 \in \mathcal{C}$, $(x, y) \in \mathcal{U}_1 \times V_1 \cap \mathcal{U}_2 \times V_2$

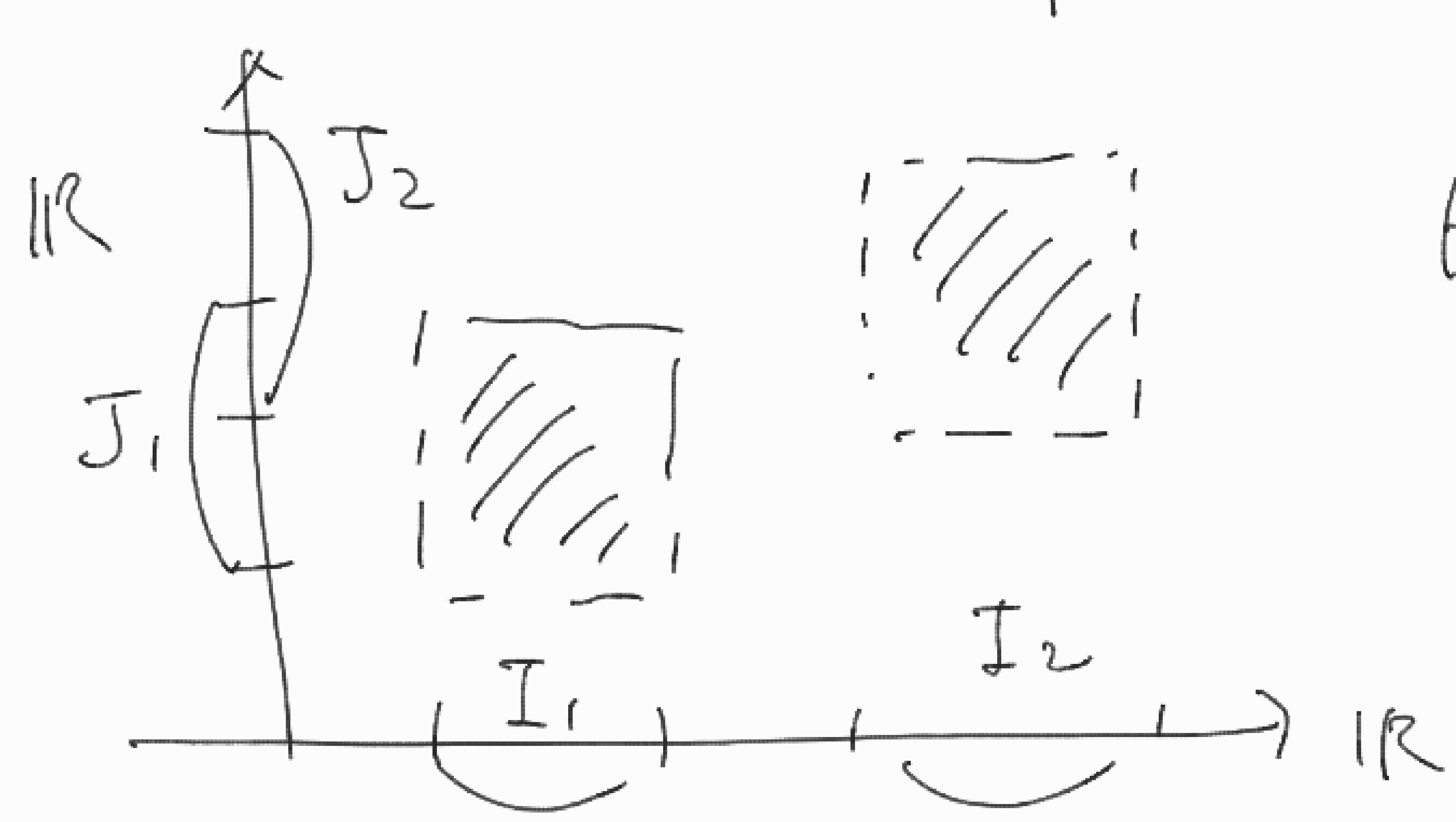
choose $\mathcal{U}_3 \in \mathcal{T}_X$ s.t. $x \in \mathcal{U}_3 \subset \mathcal{U}_1 \cap \mathcal{U}_2$

$V_3 \in \mathcal{T}_Y$ s.t. $y \in V_3 \subset V_1 \cap V_2$

Thus $(x, y) \in \mathcal{U}_3 \times V_3 \subset \mathcal{U}_1 \times V_1 \cap \mathcal{U}_2 \times V_2$.

$A \subset X \times Y$ is open if and only if

$\forall x \in A$, $\exists \underset{\text{open}}{\mathcal{U} \subset X}$. $\underset{\text{open}}{V \subset Y}$ s.t. $x \in U \times V \subset A$.



$A = I_1 \times J_1 \cup I_2 \times J_2$.

Thm tells us that it suffices to choose $\mathcal{U}, V$ from the basis.

pf of thm NTS: $\forall \underset{\text{open}}{\mathcal{U} \subset X \times Y}$ $\forall x \in \mathcal{U}$, $\exists B \times C \in \mathcal{D}$ s.t. $x \in B \times C \subset \mathcal{U}$.

$x = (a, b)$ $\exists \underset{\text{open}}{V \subset X}, \underset{\text{open}}{W \subset Y}$ s.t. $x \in V \times W \subset \mathcal{U}$, $a \in V$, $b \in W$.

$\exists B \in \mathcal{B}, C \in \mathcal{C}$ s.t. $a \in B \subset V$. $b \in C \subset W$.

$x = (a, b) \in B \times C \subset V \times W \subset \mathcal{U}$. ☒

Definition

Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be defined by

$$\pi_1(x, y) = x, \quad \pi_2(x, y) = y.$$

The maps π_1, π_2 are called the projections of $X \times Y$ onto its factors.

Theorem

The collection

$$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ is open in } X\}$$

is a subbasis for the topology $X \times Y$.

prod.

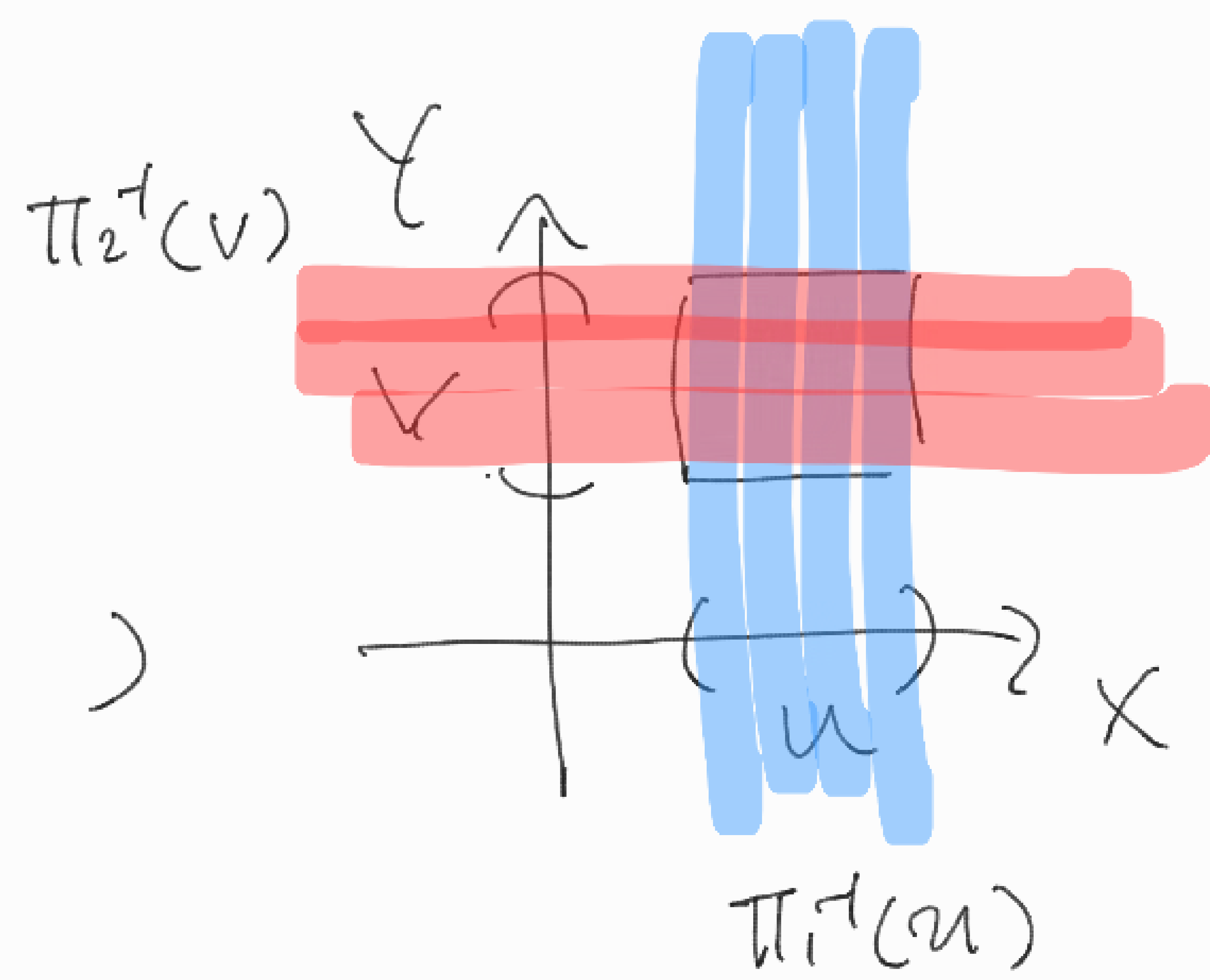
$$\bigcup_{U \in \mathcal{T}_X} \pi_1^{-1}(U) \cup \bigcup_{V \in \mathcal{T}_Y} \pi_2^{-1}(V) = \bigcup_{U \in \mathcal{T}_X} U \times Y \cup \bigcup_{V \in \mathcal{T}_Y} X \times V$$

$$= (X \times Y) \cup (X \times Y) = X \times Y.$$

Suffices to show.

① $\forall\, U \times V \underset{\text{open}}{\subset} X \times Y$, $U \times V = \bigcap_{i=1}^{n} S_i$ (finite intersect of $S_i \in \mathcal{S}$)

② $\forall S \in \mathcal{S}$ is open in $X \times Y$



(① implies that $\mathcal{T}_{prod} \subset \mathcal{T}_{\mathcal{S}}$

② " $\mathcal{T}_{\mathcal{S}} \subset \mathcal{T}_{prod}$.)

① $U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$

② $\pi_1^{-1}(U) = U \times Y \underset{\text{open}}{\subset} X \times Y$. $\pi_2^{-1}(V) = X \times V \underset{\text{open}}{\subset} X \times Y$.

Rmk $A \subset Y$ is open in the subspace top iff $\exists\, U \underset{\text{open}}{\subset} X$ s.t. $A = Y \cap U$.

Definition

Let X be a topological space with the topology $\mathcal{T}$. Let $Y \subset X$ be a subset. The collection

$$\mathcal{T}' = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology of Y, called the **subspace topology**. With this topology, Y is called a subspace of X.

Lemma

If $\mathcal{B}$ is a basis for the topology of X, then the collection

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

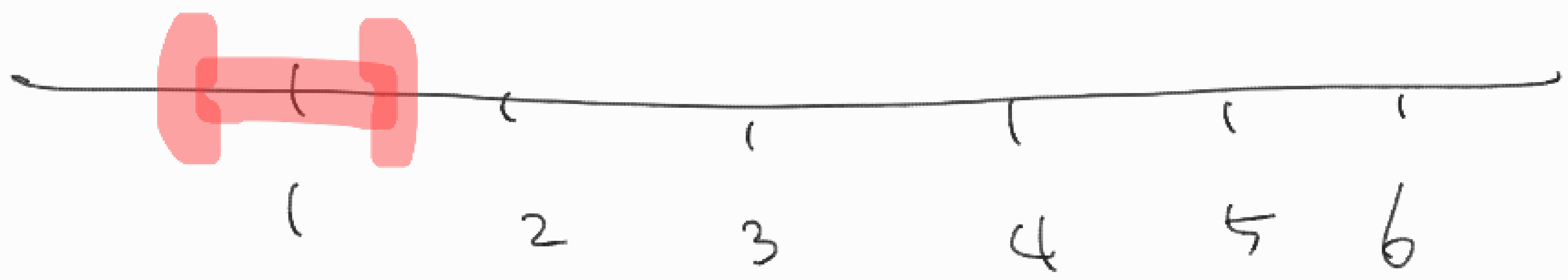
is a basis for the subspace topology of Y.

Ex $\mathbb{R}$: standard top.

$Y = \mathbb{Z}_+ = \{1, 2, 3, \ldots\} \subset \mathbb{R}$

$\Rightarrow$ the subspace top on $\mathbb{Z}_+$ is the discrete top.

$I = (1 - \frac{1}{2}, 1 + \frac{1}{2})$



$1 \in I$, $\quad Y \cap I = \{1\}$: open in $Y = \mathbb{Z}_+$.

<u>pf of lem</u>

$\underset{\text{open}}{\mathcal{U} \subset Y}$ (i.e. $\mathcal{U} = Y \cap V$, $\underset{\text{open}}{V \subset X}$). $\quad x \in \mathcal{U}$.

$x \in V \Rightarrow \exists B \in \mathcal{B}$ s.t. $\quad x \in B \subset V$.

$x \in \underline{\underline{B \cap Y}} \subset V \cap Y = \mathcal{U}$.

$B \cap Y$ is open in Y. ($\because B \in \mathcal{B}$ is open in X)

$\mathcal{U} = Y \cap V$, $V \underset{\text{open}}{\subset} X$. , $Y \underset{\text{open}}{\subset} X$. $\exists$ $\mathcal{U} \underset{\text{open}}{\subset} X$.

Lemma

Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Theorem

If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same topology $A \times B$ inherits as a subspace of $X \times Y$.

The "canonical" top. on $A \times B$ is well-defined.

Thm $\mathcal{T}_{\text{prod}} = \mathcal{T}_{\text{subspace}}$

pf) "$\supset$" $\underset{\text{open}}{\mathcal{U} \subset A \times B} \iff \mathcal{U} = A \times B \cap V$. $V \underset{\text{open}}{\subset} X \times Y$.

let $x \in \mathcal{U}$ $(x \in V)$. $\exists$ $\underset{\text{open}}{\mathcal{U}_1 \subset X}$ $\underset{\text{open}}{V_1 \subset Y}$ s.t.

$$x \in \underline{\mathcal{U}_1} \times \underline{V_1} \subset V.$$

$$x \in \underbrace{\underbrace{(\mathcal{U}_1 \cap A)}_{\text{open in } A} \times \underbrace{(V_1 \cap B)}_{\text{open in } B}}_{\text{basis of prod top. of } A \times B} \subset A \times B \cap V = \mathcal{U}.$$

"$\subset$" $\mathcal{U} \underset{\text{open}}{\subset} A \times B$ i.e. $\forall x \in \mathcal{U}$. $\exists$ $\mathcal{U}_1 \underset{\text{open}}{\subset} A$, $V_1 \underset{\text{open}}{\subset} B$ s.t.

$$x \in \mathcal{U}_1 \times V_1 \subset \mathcal{U}.$$

NTS $\exists$ $V \underset{\text{open}}{\subset} X \times Y$ s.t. $x \in \underline{(A \times B) \cap V} \subset \mathcal{U}$.

$\mathcal{U}_1 = A \cap \mathcal{U}_2$, $\mathcal{U}_2 \underset{\text{open}}{\subset} X$. $V_1 = B \cap V_2$, $V_2 \underset{\text{open}}{\subset} Y$.

$$\underline{V = \mathcal{U}_2 \times V_2} \Rightarrow x \in \underline{(A \times B) \cap (\mathcal{U}_2 \times V_2)} = (A \cap \mathcal{U}_2) \times (B \cap V_2)$$

$$= \mathcal{U}_1 \times V_1 \subset \underline{\mathcal{U}}. \quad \blacksquare$$

Example

Let X be a set with order relation. Assume $|X| > 1$. Let $\mathcal{B}$ a collection of subsets of the following types:

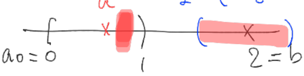
1. all open intervals (a, b) in X;
2. all half-intervals $[a_0, b)$ where a_0 is the smallest element (if any) of X;
3. all half-interval $(a, b_0]$ where b_0 is the largest element (if any) of X.

Then the collection $\mathcal{B}$ is a basis for a topology of X, called the order toplogy.

Suppose that $Y = [0, 1) \cup \{2\}$. Show that the one-point set $\{2\}$ is open in the subspace topology $Y \subset \mathbb{R}$, while it is not open in the order topology, $\{2\}$ is not open.

Standard top.

$I = (2-\frac{1}{2}, 2+\frac{1}{2})$



order top

$(a, 2] \subset Y$.

open

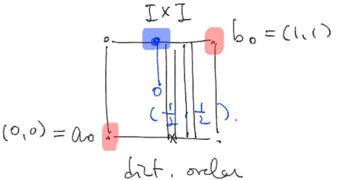
$\{2\} = I \cap Y \subset Y$

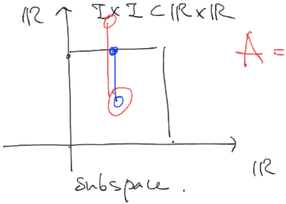
open subspace

subspace

$(a, 2] = Y \cap (a, 2+\frac{1}{2}) \subset Y$

open





$A = I \times I \cap \underbrace{((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{3}{2}))}_{\text{open int.}}$

Example

Let $I = [0,1]$. The dictionary order on $I \times I$ is the restriction of dictionary order on $\mathbb{R} \times \mathbb{R}$. Show that dictionary order topology on $I \times I$ is not the same as the subspace topology obtained from dictionary order topology on $\mathbb{R} \times \mathbb{R}$.

$A = \{\frac{1}{2}\} \times (\frac{1}{2}, 1] \subset I \times I$ NOT open in order top.

(If it is open, at $(\frac{1}{2}, 1) \in A$, $\exists$ basis elt B s.t.

$(\frac{1}{2}, 1) \in B \subset A$. $B = \underbrace{(a, b)}_{\text{open interval}}$, $a = (a_1, a_2)$, $b = (b_1, b_2)$

$b_1 > \frac{1}{2} \Rightarrow B \not\subset A$.)

$A \underset{\text{open}}{\subset} I \times I$ as subspace top of $\mathbb{R} \times \mathbb{R}$

Definition

Given an ordered set X, a subset $Y \subset X$ is called convex if for each pair of points $a < b$, the interval (a, b) lies in Y.

✩ Theorem

Let X be an ordered set in the order topology. Let Y be a subset of X that is convex in X. Then the order topology on Y is the same as the topology Y as the subspace of X.

(Rmk $I \times I \subset \mathbb{R} \times \mathbb{R}$ in dict order is NOT convex! why).

Thm $\mathcal{T}_{ord} = \mathcal{T}_{subspace}$

"$\subset$" $(a,b) \in \mathcal{T}_{ord}$, $a,b \in X \Rightarrow (a,b) \subset Y$.

by convexity of Y

$\underbrace{(a,b)}_{\text{intv in } Y} = Y \cap \underbrace{(a,b)}_{\text{intv in } X}$.

Do same with $(a,b_0]$, $[a_0,b) \in \mathcal{T}_{ord}$

(if such a_0, b_0 exist).

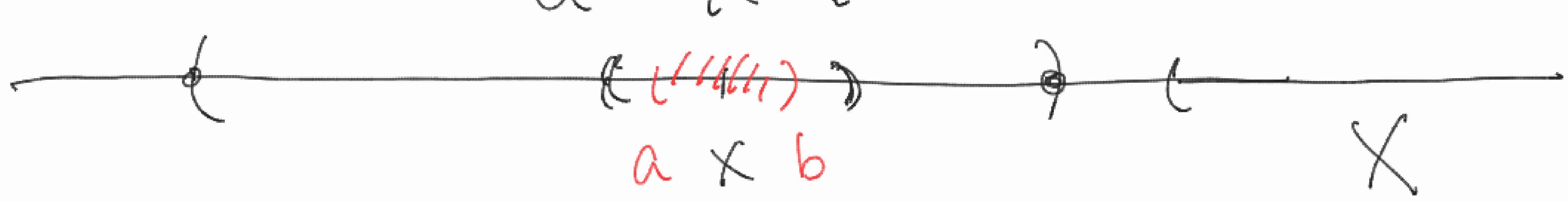
"$\supset$" $U \in \mathcal{T}_{subspace} \Rightarrow U = Y \cap V$, $V \underset{\text{open}}{\subset} X$

$x \in U \Rightarrow x \in V \Rightarrow \exists\ (a,b),\ [a_0,b),\ (a,b_0] = B$ intv. in X

s.t. $x \in B \subset V$.

Y.

$U = Y \cap V$.



$\exists\ a_1, b_1 \in Y$. s.t. $x \in (a_1, b_1) \subset U$.

Divide the case of x

- x lies "interior" of $(a,b) \cap Y$.
- x is the max. / min. of $(a,b) \cap Y$
- x could be " of X.

topological spaces

Example

A map $f : X \to Y$ is an **open map** if for every open set $U \subset X$ the image $f(U)$ is open in $\cancel{X}$ Y. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

pf: $A \underset{\text{open}}{\subset} X \times Y$. $\pi_1(A) = \{ x \in X \mid \exists\, y \in Y \text{ s.t. } (x,y) \in A \}$.

Let $x \in \pi_1(A)$. Need to find $U \underset{\text{open}}{\subset} X$ s.t. $x \in U \subset \pi_1(A)$.

Let $y \in Y$ s.t. $(x,y) \in A$. Then exists $U \underset{\text{open}}{\subset} X$, $V \underset{\text{open}}{\subset} Y$ s.t. $(x,y) \in \underline{U \times V} \subset A$. Then $x \in U \subset A$

Therefore $\pi_1(A) \underset{\text{open}}{\subset} X$.

Similar for $\pi_2(A)$. ∎

Definition

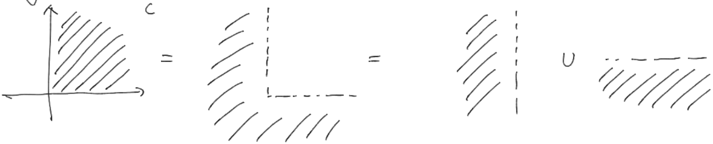
A subset A of a topological space X is **close** if $X - A$ is open.

Example

closed boundary

$[a,b] = \mathbb{R} \setminus ((-\infty, a) \cup (b, \infty))$.

1. A subset $[a, b] \subset \mathbb{R}$ is closed in the standard topology.
2. A subset $\{(x, y) \mid x, y \geq 0\} \subset \mathbb{R}^2$ is closed in the standard topology.
3. In the finite complement topology of X, every closed set is either X itself or finite subset.



Ex3. ① If $C \underset{\text{closed}}{\subset} X$, then $|C| < +\infty$ or $C = X$.

$X\setminus C$: open $\Rightarrow$ $(X\setminus C)^c = C$ is either X or finite.

② $C = X$ or $|C| < +\infty$ $\Rightarrow$ $C \underset{\text{closed}}{\subset} X$.

NTS : $X\setminus C \underset{\text{open}}{\subset} X$ $\Leftrightarrow$ $(X\setminus C)^c = C$ finite or X.

Thm ②. C_α, $\alpha \in J$, closed. $\Leftrightarrow$ $C_\alpha^c = X\setminus C_\alpha \underset{\text{open}}{\subset} X$.

$$\left(\bigcap_\alpha C_\alpha\right)^c = \bigcup_\alpha C_\alpha^c = \bigcup_\alpha X - C_\alpha \underset{\text{open}}{\subset} X.$$

③ C_i $i = 1, \dots, n$. closed.

$$\left(\bigcup_{i=1}^n C_i\right)^c = \bigcap_{i=1}^n C_i^c = \bigcap_{i=1}^n X - C_i \underset{\text{open}}{\subset} X.$$

Ex $\bigcup_{i=1}^\infty \underbrace{[0, 1-\frac{1}{n}]}_{\text{closed in the stand top of } \mathbb{R}} = [0,1)$ NOT closed.

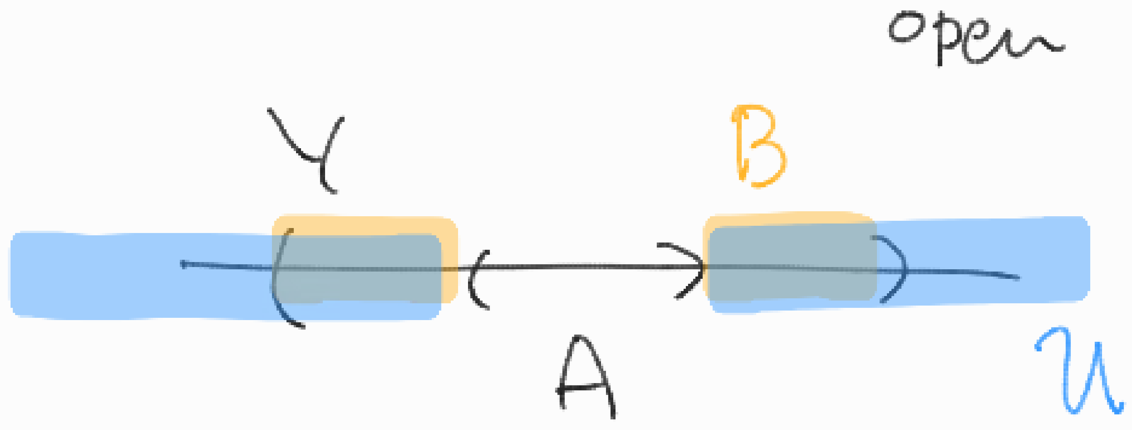
($[0,1)^c = \underline{(-\infty, 0) \cup [1, \infty)} \ni 1$

$1 \in (a,b) \Rightarrow a < 1$, $\Rightarrow \exists x_0 \in (a,b)$ s.t. $x < 1$.

$(a,b) \not\subset (-\infty, 0) \cup [1, \infty)$.)

Thm $A \underset{\text{closed}}{\subset} Y$ $\Leftrightarrow$ $A = A \cap C$, $C \underset{\text{closed}}{\subset} X$

pf) "$\Rightarrow$". $Y\setminus A = B \underset{\text{open}}{\subset} Y$. $\Rightarrow$ $B = B \cap U$, $U \underset{\text{open}}{\subset} X$.



$\Rightarrow$ $A = A \cap \underbrace{U^c}_{\substack{\shortparallel \\ C}}$.

"$\Leftarrow$" NTS : $Y\setminus A \underset{\text{open}}{\subset} Y$ i.e. $\exists U \underset{\text{open}}{\subset} X$ s.t. $Y\setminus A = (Y\setminus A) \cap U$.

Let $U = X\setminus C$, $(Y\setminus A) \cap U = Y\setminus A$ ∎

Theorem

Let X be a topological space. Then

1. *$\emptyset$ and X are closed.* → ϕ, X are both open and closed.
2. *Arbitrary intersection of closed sets is closed.*
3. *Finite union of closed sets is closed.*

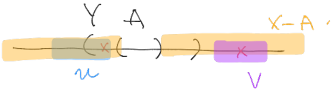
Theorem

Let Y be a subspace of X. A subset $A \subset Y$ is closed (in Y) if and only if $A = A \cap C$ for some closed subset $C \subset X$.

as a subset of X

Theorem

Let Y be a subspace of X. If $A \subset Y$ is closed in Y and Y is closed in X, then A is closed in X.



pf. $Y \setminus A = (Y \setminus A) \cap \mathcal{U}$, $\mathcal{U} \underset{\text{open}}{\subset} X$.

NTS: $X - A \underset{\text{open}}{\subset} X$. Let $x \in X \setminus A$. Then $x \in Y$ or $x \in Y^c$

If $x \in Y$. then $x \in Y \setminus A$ since $\mathcal{U} \subset Y \setminus A$. and $x \in \mathcal{U}$.

therefore $x \in \underset{=}{\mathcal{U}} \subset X \setminus A$.

If $x \notin Y$. since $\underset{\text{closed}}{Y \subset X}$, $\exists \underset{\text{open}}{V \subset X}$ s.t. $x \in V \subset Y^c$

then $x \in V \subset X - A$. ∎

<u>Thm</u> $cl_Y(A) = cl_X(A) \cap Y$.

pf). "$\subset$" $cl_Y(A)$ = the smallest closed set in Y containing A.

i.e. $\forall \underset{\text{closed}}{C \subset Y}$, $A \subset C \Rightarrow cl_Y(A) \subset C$.

let $\underset{\text{closed}}{C' \subset X}$ s.t. $C = Y \cap C'$, $A \subset C'$

$cl_X(A)$ is a closed set in X containing A.

Choose $C' = cl_X(A)$. Then define $C = Y \cap cl_X(A)$

Thus $cl_Y(A) \subset C = Y \cap cl_X(A)$

"$\supset$" $cl_X(A)$ = the smallest closed set in X containing A.

i.e $\forall \underset{\text{closed}}{C \subset X}$ s.t. $A \subset C$, then $\underline{cl_X(A) \subset C}$.

NTS: $\forall \underset{\text{closed}}{C' \subset Y}$ s.t. $A \subset C'$, then $\underline{\underline{cl_X(A) \cap Y}} \subset \underset{=}{C'}$.

$\Downarrow$ $\exists \underset{\text{closed}}{C \subset X}$ s.t. $C' = C \cap Y$.

Since $cl_X(A) \subset C$, $\underline{cl_X(A)} \cap Y \subset \underline{C} \cap Y = C'$ ∎

Int(A) ⊂ A ⊂ $\overline{A}$ = cl(A) The largest open set contained in A

Definition

Let A be a subset of topological space X. The **interior** of A is the union of all open sets contained in A. The **closure** of A is the intersection of all closed sets containing A. The interior of A is denoted by Int$A$. The closure of A is denoted by $\overline{A}$.

The smallest closed set containing A

Theorem

Let Y be a subspace of X. Let A be a subset of Y. Let $\overline{A}$ is the closure of A in X. Then the closure of A in Y equals $\overline{A} \cap Y$.

Some properties of subset in the ambient space are not preserved in the subspace top.

Theorem

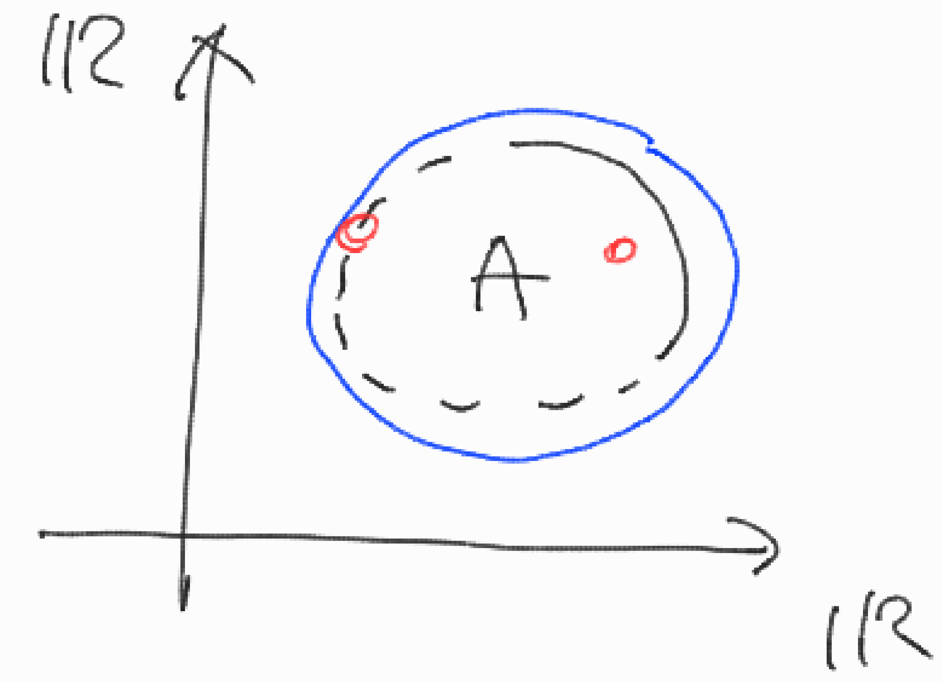
Let A be a subset of a topological space X.

1. $x \in \overline{A}$ *if and only if every open set U containing x intersects A.* (i.e. $U \cap A \neq \emptyset$)
2. *Let $\mathcal{B}$ be a basis of X. $x \in \overline{A}$ if and only if $x \in B \in \mathcal{B}$ implies $B \cap A \neq \emptyset$.*

Remark

The phrase "U is open set containing x" equal to the statement "U is (open) neighborhood of x."

Thm ① $x\in\bar{A} \iff$ "$x\in U \underset{\text{open}}{\subset} X \Rightarrow U\cap A\neq\emptyset$"



"$\Rightarrow$" Suppose $\exists\, U \underset{\text{open}}{\subset} X$ s.t. $\underline{x\in U}$.

but $\underline{U}\cap\underline{A}=\emptyset$.

$C = X\setminus U \underset{\text{closed}}{\subset} X$, $A\subset C$. $\Rightarrow \underset{\in x}{\bar{A}}\subset \underset{\not\ni x}{C}$. contrad

"$\Leftarrow$". Suppose $x\notin\bar{A}$. (implies $x\notin A$).

$\exists\, C\underset{\text{closed}}{\subset}X$ s.t. $A\subset C$ but $\underline{x\notin C}$.

$U = X\setminus C \underset{\text{open}}{\subset} X$. $x\in U$. $\Rightarrow$ $\underline{U\cap A\neq\emptyset}$. contradiction

$\Rightarrow x\in\bar{A}$. ■

Thm $\bar{A} = A\cup A'$.

"$\supset$" $A\subset\bar{A}$, NTS: $A'\subset\bar{A}$.

$x\in A'$. $\forall\, U\ni x$ nbhd $U\cap A\neq\emptyset$. $\Rightarrow x\in\bar{A}$.

"$\subset$" $x\in\bar{A}$ if $x\in A$ done

if $x\notin A$. then $\forall\, U\ni x$ $\underset{\not\ni x}{U\cap A}\neq\emptyset$.

$\Rightarrow \exists\, y\neq x$ s.t. $y\in U\cap A$ ■

Definition

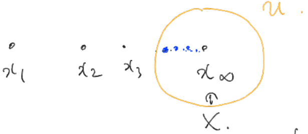
Let A be a subset of a topological space X. An element $x \in X$ is a **limit point** of A if every neighborhood of x intersects A in some point other than x itself.

Ex $\{1\} = A \subset \mathbb{R}$ st. top., 1 is NOT a limit point of A.

Theorem

Let A be a subset of a topological space X. Let A' be the set of all limit points of A. Then

$$\overline{A} = A \cup A'$$



$x_n = \frac{1}{n} \to x_\infty = 0 \quad \in \mathbb{R}.$

$x_n = n \not\to$?

limit point of A is a limit of a sequence $x_1, \cdots, x_n \cdots \in A$.

Definition

a limit.

Given a topological space X, a sequence $x_1, x_2, \cdots$ **converges to** $x_\infty \in X$ if for each neighborhood U of x_∞, there exists an integer $N > 0$ such that $x_i \in U$ for all $i > N$.

Definition

$x_n = (-1)^n \frac{1}{2n} \to 0$. (0 is NOT a limit point of $(-1,1)$)

$x_n = 1 - \frac{1}{n} \to 1$ (1 is a limit point)

A topological space X is **Hausdorff** if for each pair x_1, x_2 of distinct points of X, there exist neighborhoods U_1, U_2 of x_1, x_2 respectively which are disjoint.

Theorem

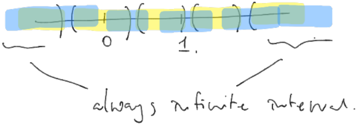
Every finite set in a Hausdorff space is closed.

Definition

A topological space X has the T_1 **axiom** if finite point set is closed.

Example

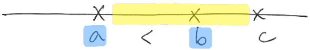
The finite complement topology on $\mathbb{R}$ has the T_1 axiom but it is not Hausdorff.



Theorem

Every sequence in a Haudorff space converges at most one point.

Example



(a_0, b)
(a, b_0). (a, b).

1. Every ordered set is Hausdorff in the order topology.
2. The product of two Hausdorff space is Hausdorff.
3. A subspace of Hausdorff space is Hausdorff.

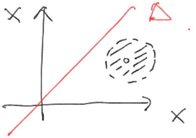
$Y \subset X$. $y_1, y_2 \in Y$. $\mathcal{U}_i \ni y_i$, $\mathcal{U}_i \underset{\text{open}}{\subset} X$. $\Rightarrow$ $V_i = \mathcal{U}_i \cap Y \underset{\text{open}}{\subset} Y$.

$\mathcal{U}_1 \cap \mathcal{U}_2 = \phi$ $\quad V_1 \cap V_2 = \phi$.

Example

X is Hausdorff if and only if the **diagonal** $\Delta = \{(x,x) \mid x \in X\}$ is closed in $X \times X$.

If $\Delta \underset{\text{closed}}{\subset} X \times X$, then X is Hausdorff.



p.f) $(x,y) \notin \Delta$ i.e. $x \neq y$. $\exists\, \mathcal{U} \underset{\text{open}}{\subset} X \times X$.

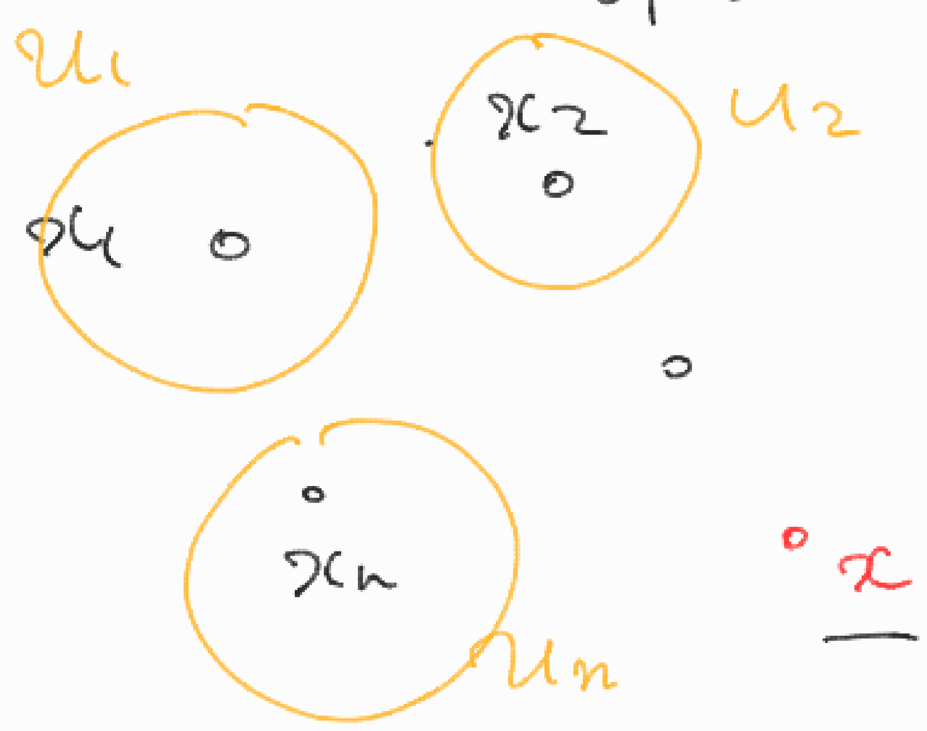
s.t. $(x,y) \in \mathcal{U} \subset (X \times X) \setminus \Delta$.

$\exists\, \mathcal{U}_1, \mathcal{U}_2 \underset{\text{open}}{\subset} X \times X$ s.t. $(x,y) \in \mathcal{U}_1 \times \mathcal{U}_2 \subset \mathcal{U}$

$\Rightarrow$ $x \in \mathcal{U}_1$, $y \in \mathcal{U}_2$, $\mathcal{U}_1 \cap \mathcal{U}_2 = \phi$.

Thm X: Hausdorff. $A \subset X$. $|A| < +\infty \Rightarrow A \underset{\text{closed}}{\subset} X$.

pf) NTS: $X - A \underset{\text{open}}{\subset} X$. i.e. $x \in X - A$, $\exists\, U \underset{\text{open}}{\subset} X$ $x \in U \subset X - A$.



For $i = 1, \cdots, n$, pick U_i' s.t.

$$U_i' \cap U_i = \phi.$$

by the def. of Hausdorff.

Then $U' = \bigcap_{i=1}^{n} U_i' \underset{\text{open}}{\subset} X$. $\quad U' \cap U_i = \phi$.

$x \in U' \subset X - A \Rightarrow X - A \underset{\text{closed}}{\subset} X$.

Thm. $x_1, \cdots, x_n, \cdots \in X$: Hausdorff space

"convergence" is well-defined.

pf) Suppose NOT. i.e. $\exists$ convergent seq $x_1, \cdots, x_n, \cdots$

which has two limits x_∞, x_∞'.

Let $U_1 \ni x_\infty$, $U_2 \ni x_\infty'$, $U_1 \cap U_2 = \phi$.

$\exists\, N_1, N_2$ s.t. $i > N_i \Rightarrow x_i \in U_i$

$N = \max\{N_1, N_2\} \Rightarrow i > N \Rightarrow x_i \in U_1 \cap U_2 = \phi$ cont.

Example

The **boundary** of a subset $A \subset X$, denoted by BdA or ∂A, is defined by

$$\partial A = \overline{A} \cap \overline{X - A}$$

1. Show that $\overline{A} \cap \partial A = \emptyset$.
2. Show that $\overline{A} = \partial A \cup \mathrm{Int} A$
3. Show that $\partial A = \emptyset$ if and only if A is both open and closed.
4. Show that U is open if and only If $\partial U = \overline{U} - U$.
5. Prove or disprove: $U = \mathrm{Int}(\overline{U})$.