SE328:Topology

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Week 10

Definition

For two metric space X, Y, a function $f : X \to Y$ is said to be **uniformly continuous** if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x_1, x_2 \in X$,

$$d(x_1, x_2) < \delta \Rightarrow d(f(x_1), f(x_2)) < \varepsilon$$

Theorem (Uniform continuity theorem)

Let $f : X \to Y$ be a continuous function between metric spaces. If X is compact, then f is uniformly continuous.

Theorem

Let X be a metric space with metric d. If X is compact, then for every open covering $\mathcal{A}$, there is $\delta > 0$ such that any subset whose diameter less than δ is contained in an element of $\mathcal{A}$.

Example

Let $f : X \to Y$ be a continuous map between metric spaces. If X is compact, then f is uniformly continuous.

Definition

Let x_n be sequence in X and x_{n_i} be a subsequence. If every sequence in X has a convergent subsequence, then X is said to be **sequentially compact**. Meanwhile, X is said to be **limit point compact** if every infinite subset of X has a limit point.

Proposition

If X is a metric space, then the followings are equivalent:

1. *X is compact,*
2. *X is limit point compact.*
3. *X is sequenctially compact,*

Definition

A space X is said to have a **countable basis at** x if there is a countable collection $\mathcal{B}$ of neighborhoods of x such that each neighborhood contains at least one of the elements of $\mathcal{B}$. If a space X has a countable basis at every point in X, then we say X satisfies the **first countability axiom** or X is **first-countable**.

Example

Show that every metrizable space is first countable.

Definition

If a space have countable basis, then we say X satisfies the **second countability axiom**, or X is **second countable**.

Example

Show that the uniform topology on $\mathbf{R}^\omega$ satisfies the first countability axiom, but not satisfies the second countability axiom.

Example

Show that every compact metrizable space has a countable basis.

Definition

A subset A of a space X is said to be **dense** if $\overline{A} = X$.

Example

The set of rational numbers are dense in the lower-limit topology $\mathbf{R}_l$.

Definition

A space for which every open covering contains a countable subcovering is called a **Lindelöf** space. A space having a countable dense subset is said to be **separable**.

Theorem

Suppose that X is second countable. Then

1. *X is Lindelöf.*
2. *X is separable.*

Example

1. Show that the lower limit topology $\mathbf{R}_l$ is Lindelöf but not second countable.
2. Show that $\mathbf{R}_l \times \mathbf{R}_l$ is not Lindelöf.
3. Show that the order topology I^2 where $I = [0, 1]$ is Lindelöf but the subspace topology $I \times (0, 1) \subset I^2$ is not.

Theorem

Let X be a metrizable space. Then X is second countable if X is either Lindelöf or separable.

Definition

Suppose that one-point sets are closed in X. (We say X is T_1.) Then X is said to be **regular** if for each pair consisting of a point x and a closed set B disjoint from x, there exists disjoint open sets containing x and B, respectively. The space X is said to be **normal** if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets containing A and B, respectively.

Lemma

Let X be T_1.

1. X is regular if and only if given a point $x \in X$ and a neighborhood U of x, there is a neighborhood V of x such that $\overline{V} \subset U$.
2. X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A and $\overline{V} \subset U$.

Theorem

1. *A subspace of a Haudorff space is Hausdorff.*
2. *A product of Hausdorff spaces is Hausdorff.*
3. *A subspace of regular space is regular.*
4. *A product of regular spaces is regular.*

Example

1. Show that $\mathbf{R}_K$ is Hausdorff but not regular.
2. Show that $\mathbf{R}_l$ is normal but $\mathbf{R}_l^2$ is not.