SE328:Topology

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Week 02

Definition

A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. $\emptyset$ and X are in $\mathcal{T}$.
2. An arbitrary union of elements is in $\mathcal{T}$.
3. A finite intersection of elements is in $\mathcal{T}$.

A element of $\mathcal{T}$ is called an **open** set of X. If X admits such collection $\mathcal{T}$, we say X a topological space.

Example

If $\mathcal{T}$ contains all subsets of X, then we call $\mathcal{T}$ the **discrete** topology on X. Define a discrete topology on $\mathbb{Z}$.

Example

Let $\mathcal{T}$ be the collection of all subsets of X such that $X - U$ is finite. Then $\mathcal{T}$ is called a **finite complement** topology. (Why is it a topology?)

Definition

Let $\mathcal{T}, \mathcal{T}'$ be two topology on X. If $\mathcal{T} \subset \mathcal{T}'$, then we say $\mathcal{T}$ is coarser than $\mathcal{T}$, and $\mathcal{T}'$ is finer than $\mathcal{T}$.

Definition

A basis of the topology $\mathcal{T}$ is a collection $\mathcal{B}$ of subsets of X satisfying

1. For each $x \in X$, there is at least one basis element containing x.
2. If $x \in B_1 \cap B_2$ for $B_1, B_2 \in \mathcal{B}$, then there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

If a collection of subsets of X satisfies the above conditions, then the **topology generated by** $\mathcal{B}$ is the collection of subset U satisfying that if $x \in U$ then there is $B \in \mathcal{B}$ such that $B \subset U$.

Example

The collection of one-point sets is the basis of the discrete topology.

Example

Let $B_r(x)$ be the ball of radius r centered at $x \in \mathbb{R}^n$, namely,

$$B_r(x) = \{y \in \mathbb{R} \mid |y - x| < r\}$$

Then two topology generated by

$$\mathcal{B}_1 = \{B_r(x) \mid r \in \mathbb{R}, x \in \mathbb{R}^n\}$$

$$\mathcal{B}_2 = \{(a_1, b_1) \times \cdots \times (a_n, b_n) \mid a_i, b_i \in \mathbb{R},\ a_i < b_i, \}$$

are the same topology on $\mathbb{R}^n$.

Lemma

Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$. Then $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$.

Lemma

Let $\mathcal{C}$ be a collection of open sets of X such that for each open set U of X and each $x \in U$, there is an element $C \in \mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.

Lemma

Let $\mathcal{B}, \mathcal{B}'$ be bases for the topology $\mathcal{T}$, $\mathcal{T}'$. The followings are equivalent:

1. $\mathcal{T}'$ is finer than $\mathcal{T}$.
2. For each $x \in X$ and a basis element $B \in \mathcal{B}$ containing x, there is $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Example

1. Let $\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{R}\}$. The topology generated by $\mathcal{B}$ is called the **standard topology** on $\mathbb{R}$.
2. Let $\mathcal{B}' = \{[a, b) \mid a, b \in \mathbb{R}\}$. The topology generated by $\mathcal{B}'$ is called the **lower limit topology** on $\mathbb{R}$.
3. Let $K = \{1/n \mid n \in \mathbb{Z}_+\}$, and $\mathcal{B}'' = \{(a, b) - K \mid a, b \in \mathbb{R}\}$. The toplogy generated by $\mathcal{B}''$ is called the **K- toplogy** on $\mathbb{R}$.

Lemma

The lower limit topology and K-topology are strictly finer than the standard topology, but not comparable with each other.

Definition

A subbasis $\mathcal{S}$ for a topology of X is a collection of subsets of X whose union equals X. The topology generated by the subbasis $\mathcal{S}$ is the collection of all union of finite intersections of elements of $\mathcal{S}$.

Example

Let $\mathcal{A}$ be a (sub)basis for a topology on X. Show that the topology generated by $\mathcal{A}$ is the intersection of all topologies on X that contains $\mathcal{A}$.

Example

Consider the following topologies on $\mathbb{R}$.

1. $\mathcal{T}_1$: the standard topology.
2. $\mathcal{T}_2$: the K-topology
3. $\mathcal{T}_3$: the finite complement topology.
4. $\mathcal{T}_4$: the upper limit topology, having all sets $(a, b]$ as basis.
5. $\mathcal{T}_5$: the topology having all sets $(-\infty, a)$ as a basis.

Determine which topolgy contains the other.