SE328:Topology

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Proposition

Let X, Y be a metric space with metric topology. A function $f : X \to Y$ is continuous if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f(B_\delta(x)) \subset B_\varepsilon(f(x))$$

for every $x \in X$.

Definition

We say a sequence x_n in a topological space **converges** to $x_\infty \in X$ if every open neighborhood U of x_∞ contains all but finitely many elements from the sequence. In such case, we denote

$$x_n \to x_\infty.$$

Lemma

Let X be a topological space. If a sequence x_n in a subset $A \subset X$ converges to $x_\infty \in X$, then $x_\infty \in A$. The converse is true if X is a metric space.

Proof.

Suppose $x_\infty \notin A$, then there exists $\varepsilon > 0$ such that

$$B_\varepsilon(x_\infty) \cap A = \emptyset$$

This contradicts to $x_n \to x_\infty$.
Conversely, assume that X is a metric space and $x_\infty \in A$. For each $n = 1, 2, \cdots$, Choose a point x_n satisfying

$$x_n \in B_{1/n}(x_\infty).$$

Then $x_n \to x_\infty$. $\square$

Proposition

If a function $f : X \to Y$ is continuous then for every convergent sequence $x_n \to x_\infty$,

$$f(x_n) \to f(x_\infty).$$

The converse holds if X is a metric space.

Proof.

Suppose that the function f is continuous. Let U be an open neighborhood of $f(x_\infty)$. Then $f^{-1}(U)$ is open and contains x_∞. Thus there exists $\varepsilon > 0$ such that

$$B_\varepsilon(x_\infty) \subset f^{-1}(U).$$

Since $B_\varepsilon(x_\infty)$ contains all but finitely many x_n's, so does U for $y_n = f(x_n)$.

Conversely, assume that $f(x_n) \to f(x_\infty)$ for any convergent sequence $x_n \to x_\infty$. We only need to show $f(\overline{U}) \subset \overline{f(U)}$, which follows from the lemma. $\square$

Definition

The collection $\mathcal{C}$ of pairs $(\{x_n\}_{n=0}^{\infty}, x_\infty)$ is called **convergence class of sequences** if it satisfies the following conditions: if $(\{x_n\}_{n=0}^{\infty}, x_\infty) \in \mathcal{C}$, let us denote

$$\lim_{n\to\infty} x_n \overset{\mathcal{C}}{=} x_\infty, \text{ or simply } x_n \xrightarrow{\mathcal{C}} x_\infty$$

1. If $x_n = x_\infty$ for all $n = 0, 1, \cdots$, then $x_n \xrightarrow{\mathcal{C}} x_\infty$.
2. If $x_n \to x_\infty$, then for every subsequence x_{n_i} satisfies $x_{n_i} \xrightarrow{\mathcal{C}} x_\infty$.
3. If $x_n \overset{\mathcal{C}}{\not\to} x_\infty$, then there is a subsequence x_{n_i} such that no further subsequence converges to x_∞.
4. If $x_{n,m}$ is a double sequence such that for each $n = 0, 1, \cdots$, the sequence $x_{n,m} \xrightarrow{\mathcal{C}} x_{\infty,m}$ and $x_{\infty,m} \xrightarrow{\mathcal{C}} x_{\infty,\infty}$. Then for any increasing map $i : \mathbf{N} \to \mathbf{N}$, $x_{i(m),m} \xrightarrow{\mathcal{C}} x_{\infty,\infty}$.

Definition

Given a convergence class $\mathcal{C}$ of sequence in X, the **closure operator** $^-$ is defined by

$$\overline{A} = \{x_\infty \in X \mid x_n \xrightarrow{\mathcal{C}} x_\infty,\ x_n \in A\}.$$

Proposition

Given a set X and a closure operator $^-$, there is a unique topology on X such that the closure of a subset $A \subset X$ is $\overline{A}$.

Proof.

It is clear that the topology exists because we can define a collection of closed sets:

$$\mathcal{T}_{\text{closed}} = \{A \subset X \mid A = \overline{A}\}.$$

This is the minimal topology among all topologies admitting the closure operator. □

Proposition

Let X be a topological space defined by a convergence class of sequence $\mathcal{C}$. Then for every pair $(\{x_n\}_{n=0}^{\infty}, x_\infty) \in \mathcal{C}$, the sequence x_n converges to x_∞ in the topology of X.

Proof.

Suppose $x_n \not\to x_\infty$. Let U be an open neighborhood of x_∞ such that there is an infinite subsequence x'_n which are not contained in U. Then $x'_n \to x_\infty$, thus $x_\infty \in X \setminus U$, which is a contradiction. $\square$

Proposition

Let X and Y be topological spaces defined by a convergence class of sequences. A function $f : X \to Y$ is continuous if and only if $f(x_n) \to f(x_\infty)$ for any $x_n \to x_\infty$.

Remark

One may observe the proposition does not assume that X is a *metric* space.

- We do not need X to be a metric space to prove the converse of the proposition. It only requires that there is a basis $\mathcal{B}$ of X such that for each $x \in X$ there are countably many elements in $\mathcal{B}$. This is called the **first countability axiom**.
- The topology generated by the convergence class of sequence already satisfies the first countability axiom, because every sequence is countable.

Remark

The description of a topology is simpler in terms of convergence class of sequence. For example, let

$$\mathbf{S}^2 = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

and define a map

$$f : \mathbf{S}^2 \to \mathbf{R}^2 \cup \{\infty\}$$
$$(x, y, z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z} \right)$$

where $f(0, 0, 1) = \infty$. Let us give a subspace topology on $\mathbf{S}^2 \subset \mathbf{R}^3$ and define a topology on $\mathbf{R}^2 \cup \{\infty\}$ as follows: $U \subset \mathbf{R}^2 \cup \{\infty\}$ is open if $f^{-1}(U)$ is open in $\mathbf{S}^2$. The description of open subsets of $\mathbf{R}^2$ is complicate. However, the convergence class of sequence $\mathcal{C}$ consists of all convergent sequence $x_n \to x_\infty$ in $\mathbf{R}^2$ together with any unbounded sequence x_n which converges to ∞.