SE328:Topology

Hyosang Kang[1]

[1]Division of Mathematics
School of Interdisciplinary Studies
DGIST

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Definition

Two continuous maps $f_0, f_1 : X \to Y$ are said to be **homotopic** if there exists a continuous map

$$F : X \times [0, 1] \to Y$$

such that $F(x, 0) = f_0(x)$, $F(x, 1) = f_1(x)$, and denoted by $f_0 \simeq f_1$. If f_1 is a constant map, then f_0 is called **null homotopic**.

Definition

Two paths $f_0, f_1 : [0, 1] \to X$ are said to be **path homotopic** if $f_0(0) = f_1(0)$, $f_0(1) = f_1(1)$ and $f_0 \simeq f_1$.

Proposition

The homotopy relation is an equivalence relation.

Definition

The set of all homotopy classes of maps $X \to Y$ is denoted by $[X, Y]$. A space X is called **contractible** if the identity map $i_X : X \to X$ is null homotopic.

Definition

Let $f, g : [0, 1] \to X$ be two path such that $f(1) = g(0)$. Then the **product** of f, g is the path

$$h(t) = \begin{cases} f(2t) & t \in [0, 1/2] \\ g(2t-1) & t \in [1/2, 1] \end{cases}$$

and is denoted by $f * g$.

Proposition

Let $[f]$ denote the path homotopy class of continuous path containing f. For continuous path $g : [0,1] \to X$ satisfying $g(0) = f(1)$, define a product

$$[f] * [g] = [f * g]$$

Then the product satisfies the following properties:

- For three paths $f, g, h : [0,1] \to X$ satisfying $f(1) = g(0)$ and $g(1) = h(0)$,

$$[f] * ([g] * [h]) = ([f] * [g]) * [h]$$

- For $x_0 = f(0)$, let e_{x_0} be the constant map. Then

$$[e_{x_0}] * [f] = [f]$$

  Conversely, for $x_1 = f(1)$,

$$[f] * [e_{x_1}] = [f]$$

- Let $\bar{f} : [0,1] \to X$ be a path defined by $\bar{f}(t) = f(1-t)$. Then

$$[f] * [\bar{f}] = [e_{x_0}], \quad [\bar{f}] * [f] = [e_{x_1}]$$

Definition

A set G with an operation

$$\cdot : G \times G \to G$$

is called a **group** if it satisfies the following:

- For $g, h, r \in G$,
$$(g \cdot h) \cdot r = g \cdot (h \cdot r)$$
- There exists an **identity** $e \in G$ such that
$$g \cdot e = e \cdot g = g \text{ for all } g \in G$$
- For any $g \in G$, there exists $g^{-1} \in G$ such that
$$g \cdot g^{-1} = g^{-1} \cdot g = e$$

Example

There are many sets with familiar operations which becomes groups.

1. The set of all integers $\mathbf{Z}$ with additive operation is a group.
2. The set of all $n \times n$ *invertible* matrices with matrix multiplication is a group. Such group is denoted by $\mathrm{GL}(n, \mathbf{R})$.
3. The set of finite integers $S_n = \{0, 1, \cdots, n-1\}$ under the modular addition is a group: for any $a, b \in \mathbf{Z}$, let $[a] \in S_n$ be the element satisfying $a \equiv [a] \mod n$. Then

   $$[a] + [b] = [a + b]$$

   If n is a prime, then it is a group with modular multiplication:

   $$[a] \cdot [b] = [ab]$$

   Such group is denoted by $\mathbf{Z}_n$, or $\mathbf{Z}/n\mathbf{Z}$.

Example

We must be careful when a set does not become a group even if an operation is well-defined.

1. The set of integers $\mathbf{Z}$ with multiplication is not a group. There is no inverse of $n \neq 1$.
2. The set of all $n \times n$ matrices $M(n, \mathbf{R})$ with matrix multiplication is not a group. If $A \in M(n, \mathbf{R})$ is not invertible, it has no inverse. However, it is a group with additive operation.

Definition

Let $x_0 \in X$ be a fixed point in X. The set $\pi_1(X, x_0)$ of all path-homotopy classes of loops based at x_0 is called the **fundamental group**.

Theorem

Let X be path connected space. Let α be a path connecting x_0 to x_1 in X. Then the map $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ defined by

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha]$$

is a group isomorphism.

Example

The fundamental group of a circle is $\pi_1(\mathbf{S}^1) \cong \mathbf{Z}$. The fundamental group of 2-sphere is trivial: $\pi_1(\mathbf{S}^2) \cong 0$.

Definition

A path connected space X is called **simply connected** if $\pi_1(X) = 0$.

Proposition

Let $h : (X, x_0) \to (Y, y_0)$ *be a continuous map. Then*

$$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

is a group homomorphism defined by

$$h_*([f]) = [h \circ f]$$

If h *is a homeomorphism, then* h_* *is an isomorphism.*

Definition

Let $p : E \to B$ be a surjective continuous map. Suppose that for each $p \in B$ there exists a neighborhood $p \in U \subset B$ such that

$$p^{-1}(U) = \bigcup_{\alpha} V_\alpha$$

where V_α are all disjoint, for each α, and $p_{V_\alpha} : V_\alpha \to U$ is a homeomorphism. Then map p is called a **covering map**, and E is called a **covering space** of B.

Example

1. The map
$$p : \mathbf{R} \to \mathbf{S}^1; \quad t \mapsto (\cos 2\pi t, \sin 2\pi t)$$
is a covering map.
2. The map
$$p \times p : \mathbf{R} \times \mathbf{R} \to \mathbf{S}^1 \times \mathbf{S}^1$$
is a covering map. The base space $\mathbf{S}^1 \times \mathbf{S}^1$ is topologically homeomorphic to a torus.
3. Let $E = \mathbf{R} \times \mathbf{Z} \cup \mathbf{Z} \times \mathbf{R}$ and $B = \mathbf{S}^1 \times b_0 \times b_0 \times \mathbf{S}^1$ for a fixed point $b_0 \in \mathbf{S}^1$. Then the map $p \times p|_E : E \to B$ is a covering map.