SE328:Topology

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Week 01

Definition

Given a statement of the form $P \Rightarrow Q$, its contrapositive is $\sim Q \Rightarrow P$, and its converse is $Q \Rightarrow P$, where $\sim P$ is the negation of P.

Example

1. $A \subset B$ and $A \subset C \Leftrightarrow A \subset (B \cap C)$
2. $A - (A - B) = B$
3. $(A \cap B) \cup (A - B) = A$

Example

1. Write the contrapositive and converse of $x < 0 \Rightarrow x^2 - x > 0$
2. Write the negation of $\forall a \in A, a^2 \in B$

Definition

Given sets A and B, the cartesian product $A \times B$ is the set

$$A \times B = \{(a, b) \mid a \in A,\, b \in B\}$$

Example

Determine whether each of the following sets is a cartesian product of two subsets of $\mathbb{R}$.

1. $\{(x, y) \mid x \in \mathbb{Z}\}$
2. $\{(x, y) \mid y > x\}$

Definition

A function f is a subset of the cartesian product $A \times B$ of two sets, with the property that each elemetn in C appears as the first coordinate of at most one ordered pair. In other words,

$$(a, b), (a, b') \in f \Rightarrow b = b'$$

Definition

Given a function $f : A \to B$ and a subsets $A_0 \subset A$, the restriction of f to A_0 is

$$\{(a, f(a)) \mid a \in A_0\}$$

Definition
A function $f : A \to B$ is injective if

$$\forall a \in A, f(a) = f(a') \Rightarrow a = a'$$

and surjective if

$$\forall b \in B, \exists a \in A f(a) = b$$

Example
Let $f : A \to B$ and $g : B \to C$ be functions and $A_0 \subset A$, $B_0, B_1 \subset B$.

1. $A_0 \subset f^{-1}(f(A_0))$.
2. $B_0 \subset B_1 \Rightarrow f^{-1}(B_0) \subset f^{-1}(B_1)$.
3. If $C_0 \subset C$, then

$$(g \circ f)^{-1}(C_0) = f^{-1}(g^{-1}(C_0))$$

Definition

A equivalence relation $\sim$ on a set A is a subset $\subset A \times A$ such that

1. $\forall x \in A, (x, x) \in \sim$
2. $(x, y) \in C \Rightarrow (y, x) \in \sim$
3. $(x, y), (y, z) \in \sim \Rightarrow (x, z) \in \sim$

We denote $x \sim y$ if $(x, y) \in \sim$.

Definition

The equivalence class of $x \in A$ is the set

$$[x] = \{y \mid y \sim x\}$$

The collection of all equivalence classes for $\sim$ becomes a partition of A, i.e. the collection of disjoint nonempty subsets of A.

Definition

An order relation $<$ on a set A is a subset of $A \times A$ such that

1. $x, y \in A, x \neq y \Rightarrow (x, y) \in <$ or $(y, x) \in <$
2. $\nexists x \in A, (x, x) \in <$
3. $(x, y) \in <, (y, z) \in < \Rightarrow (x, z) \in <.$

We denote $x < y$ if $(x, y) \in <.$

Definition

If $<$ is an order relation on a set A, and if $a < b$, an open interval (a, b) is a subset defined by

$$(a, b) = \{x \in A \mid a < x < b\}$$

If $(a, b) = \emptyset$, then a is called the immediate predecessor of b, and b is called the immediate successor of a.

Definition

Suppose that A, B are two sets with order relations $<_A, <_B$. The dictionary order relation $<$ on $A \times B$ is defined by

$$a_1 \times b_1 < a_2 \times b_2$$

if $a_1 <_A a_2$, or if $a_1 = a_2$ and $b_1 <_B b_2$.

Definition

An ordered set A is said to have the least upper bound property if every nonempty subset $A_0 \subset A$ that is bounded above has a least upper bound. The greatest lower bound property is similary defined.

Example

1. Let $f : A \to B$ is a surjective function. Define $a_0 \sim a_1$ if $f(a_0) = f(a_1)$. Show that $\sim$ is an equivalence relation.
2. If an ordered set A has the least upper bound property, then it has the greatest lower bound property.
3. Showt that $[0, 1] = \{x \mid 0 \leq x \leq 1\}$ has the least upper bound property.
4. Does $[0, 1] \times [0, 1]$ in the dictionary order relation have the least upper bound property?

Definition

The set of real numbers, denoted by $\mathbb{R}$, is a set with two operations $+$(addition), $\cdot$(multiplication), and an order relation $<$. It contains two special elements, 1(one) and 0(zero). All elementary algebraic properties hold including the following statements.

1. $x < y \Rightarrow x + z < y + z$
2. $x < y, 0 < z \Rightarrow x \cdot z < y \cdot z$
3. $<$ has the least upper bound property
4. $x < y \Rightarrow \exists z \in \mathbb{R} x < z < y$

Definition

The subset $A \subset \mathbb{R}$ is called inductive if it contains 1 and if $x \in A$ then $x + 1 \in A$. The set of all positive integers, denoted by $\mathbb{Z}_+$ is the smallest among all inductive subsets.

Theorem

Every nonempty subset of $\mathbb{Z}_+$ has a smallest element.

Theorem

Let A be a set of positive integers. For each positive integer $n \in \mathbb{Z}_+$, $S_n \subset A \Rightarrow n \in A$. If this is true for all n, then $A = \mathbb{Z}_+$.

Definition

Let m be a positive integer. An m-tuple of elements of X is a function $\mathbf{x} : \{1, \cdots, m\} \to X$. The ω-tuple of elements of X is a function $\mathbf{x} : \mathbb{Z}_+ \to X$.

Definition

Let $A_1, A_2, \cdots$ be a family of sets, indexed by $\mathbb{Z}_+$. The cartesian product of A_i, denoted by $\prod_{i \in \mathbb{Z}_+} A_i$, is the set of all ω-tuples of elements of $\bigcup_{i \in \mathbb{Z}_+} A_i$ such that $x_i \in A_i$. If $A_i = X$ for all i, then the cartesian product is denoted by X^ω.

Example

Find a bijective map $f : X^\omega \times X^\omega \to X^\omega$

Definition

A set is called finite if there is a bijection between the set and S_n for some positive integer n. A set is called infinite if it is not finite. It is called countably infinite if there is a bijection between the set and $\mathbb{Z}_+$. A infinite set which is not countable is called uncountable.

Theorem

Let A be a set. The followings are equivalent:

1. *A is countable.*
2. *There is a surjective function $f : \mathbb{Z}_+ \to A$.*
3. *There is an injective function $f : A \to \mathbb{Z}_+$.*

Lemma

If A is an infinite subset of $\mathbb{Z}_+$, then A is countably infinite.

Corollary

1. A subset of a countable set is countable.
2. The set $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countably infinite.
3. A countable union of countable sets is countable.
4. A finite product of countable sets is countable.

Theorem

Let X be the two element set $\{0, 1\}$. Then X^ω is uncountable.

Definition
Two sets A and B have the same cardinality if there is a bijection between A and B.

Example
1. Show that if $B \subset A$ and there is a injection $f : A \to B$, then A and B have the same cardinality.
2. If there are injection $f : A \to C$ and $g : C \to A$, then A and C have the same cardinality.
3. Let X be the two element set $\{0, 1\}$, and $\mathcal{B}$ be the set of all countable subsets of X^ω. Then X^ω and $\mathcal{B}$ have the same cardinality.

Theorem

Let A be a set. The followings are equivalent.

1. *There is an injective function $f : \mathbb{Z}_+ \to A$.*
2. *There is a bijection between A and a proper subset of A.*
3. *A is infinite.*

Axiom of choice

Given a collection $\mathcal{A}$ of disjoint nonempty sets, there exists a set C consisting of exactly one element from each element of $\mathcal{A}$.

Example

Define an injective map $f : \mathbb{Z}_+ \to X^\omega$ where $X = \{0, 1\}$ with (or without) using axiom of choice.

Definition

A set A with an order relation $<$ is called well-ordered if every nonempty subset of A has a smallest element.

Definition

Two ordered sets A and B have the same order type if there is a bijection between A and B preserving order relations.

Theorem

Every nonempty finite ordered set has the order type of a section $\{1, \cdots, n\}$ *of* $\mathbb{Z}_+$

Example

1. Show that $\{1, 2\} \times \mathbb{Z}_+$ in dictionary order is well-ordered.
2. Do $\{1, 2\} \times \mathbb{Z}_+$ and $\mathbb{Z}_+ \times \{1, 2\}$ have the same order type?