SE328:Topology

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Week 02

Definition

A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. $\emptyset$ and X are in $\mathcal{T}$.
2. An arbitrary union of elements is in $\mathcal{T}$. $A_\alpha \in \mathcal{T}, \alpha \in J \Rightarrow \bigcup_\alpha A_\alpha \in \mathcal{T}$
3. A finite intersection of elements is in $\mathcal{T}$. $A_1, \cdots, A_n \in \mathcal{T} \Rightarrow \bigcap_{i=1}^{n} A_i \in \mathcal{T}$.

A element of $\mathcal{T}$ is called an **open** set of X. If X admits such collection $\mathcal{T}$, we say X a topological space.

Example

If $\mathcal{T}$ contains all subsets of X, then we call $\mathcal{T}$ the **discrete** topology on X. Define a discrete topology on $\mathbb{Z}$.

When we say $U \subset X$ is an open set, we assume

- X is a topological space, i.e. a topology $\mathcal{T}$ on X.
- $U \in \mathcal{T}$.

Ex. $\mathcal{T} = \{A \subset \mathbb{Z}\}$: discrete topology of $\mathbb{Z}$.

i.e. $\mathcal{T} = \mathcal{P}(\mathbb{Z})$: power "set" of $\mathbb{Z}$, i.e. the "set" of all subsets

Rmk. $\mathcal{P}(X)$ is always well-defined for any X. by the axiom.

Any topology $\mathcal{T}$ of X is subcollection $\mathcal{T} \subset \mathcal{P}(X)$.

Ex. ① $\phi, X \in \mathcal{T}$.

pf). $X - \phi = X$, $\phi \in \mathcal{T}$.

$X - X = \phi$, $|\phi| = 0 < +\infty$, $X \in \mathcal{T}$.

② $A_\alpha \in \mathcal{T}$, $\alpha \in J$ (J is called the "index set".)

$$X - \bigcup_\alpha A_\alpha = X \cap \left(\bigcup_\alpha A_\alpha\right)^c = \cancel{X \cap} \left(\bigcap_\alpha A_\alpha^c\right) = \bigcap_\alpha (X - A_\alpha)$$

$|X - A_\alpha| < +\infty$. Thus $\bigcap_\alpha (X - A_\alpha) \subset X - A_{\alpha'}$ for some $\alpha' \in J$.

Thus $\left|\bigcap_\alpha (X - A_\alpha)\right| \le |X - A_{\alpha'}| < +\infty$. Thus $\bigcup_\alpha A_\alpha \in \mathcal{T}$.

③ $A_1, \cdots, A_n \in \mathcal{T}$

$$X - \left(\bigcap_{i=1}^n A_i\right) = X \cap \left(\bigcap_{i=1}^n A_i\right)^c = \cancel{X \cap} \left(\bigcup_{i=1}^n (X - A_i)\right).$$

Since finite union of finite set is finite, $\bigcap_{i=1}^n A_i \in \mathcal{T}$.

Example

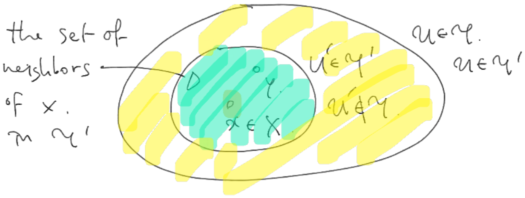
or X itself.

$U \in \mathcal{T}$ if

Let $\mathcal{T}$ be the collection of all subsets of X such that $X - U$ is finite. Then $\mathcal{T}$ is called a **finite complement** topology. (Why is it a topology?)

Definition

Let $\mathcal{T}, \mathcal{T}'$ be two topology on X. If $\mathcal{T} \subset \mathcal{T}'$, then we say $\mathcal{T}$ is coarser than $\mathcal{T}'$, and $\mathcal{T}'$ is finer than $\mathcal{T}$.



You cannot think of this small neighbors in $\mathcal{T}$

This is the neighbors of x that you can identify in $\mathcal{T}$

Definition

A basis of the topology $\mathcal{T}$ is a collection $\mathcal{B}$ of subsets of X satisfying

1. For each $x \in X$, there is at least one basis element containing x. $\forall x \in X.\ \exists B \in \mathcal{B}$ s.t. $x \in B$.
2. If $x \in B_1 \cap B_2$ for $B_1, B_2 \in \mathcal{B}$, then there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

If a collection of subsets of X satisfies the above conditions, then the **topology generated by $\mathcal{B}$** is the collection of subset U satisfying that if $x \in U$ then there is $B \in \mathcal{B}$ such that $B \subset U$.

$\boxed{x \in B \subset U}$

$\mathcal{T} \to \mathcal{B}$: basis (not unique!)

$\mathcal{B}$ (satisfying 1.2) $\to \mathcal{T}_{\mathcal{B}}$: generated by $\mathcal{B}$.

Example $\mathcal{B} = \{\{x\} \mid x \in X\}$, $\tau_{\mathcal{B}}$ = the discrete top.

The collection of one-point sets is the basis of the discrete topology.

Example

Let $B_r(x)$ be the ball of radius r centered at $x \in \mathbb{R}^n$, namely,

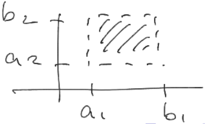
$$B_r(x) = \{y \in \mathbb{R} \mid |y - x| < r\}$$

Then two topology generated by

$$\mathcal{B}_1 = \{B_r(x) \mid r \in \mathbb{R}, x \in \mathbb{R}^n\}$$

$$\mathcal{B}_2 = \{(a_1, b_1) \times \cdots \times (a_n, b_n) \mid a_i, b_i \in \mathbb{R},\ a_i < b_i, \}$$

are the same topology on $\mathbb{R}^n$.



Ex Let $\mathcal{T}$: the discrete top. on X.

NTS: $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$

"$\supset$" $A \in \mathcal{T}_{\mathcal{B}} \Rightarrow A \in \mathcal{T} = \mathcal{P}(X)$.

"$\subset$" $A \in \mathcal{T}$, i.e. $A \subset X$.

For $x \in A$, let $B = \{x\} \in \mathcal{B}$. Then $x \in B \subset A$.

Thus $A \in \mathcal{T}_{\mathcal{B}}$. $\blacksquare$

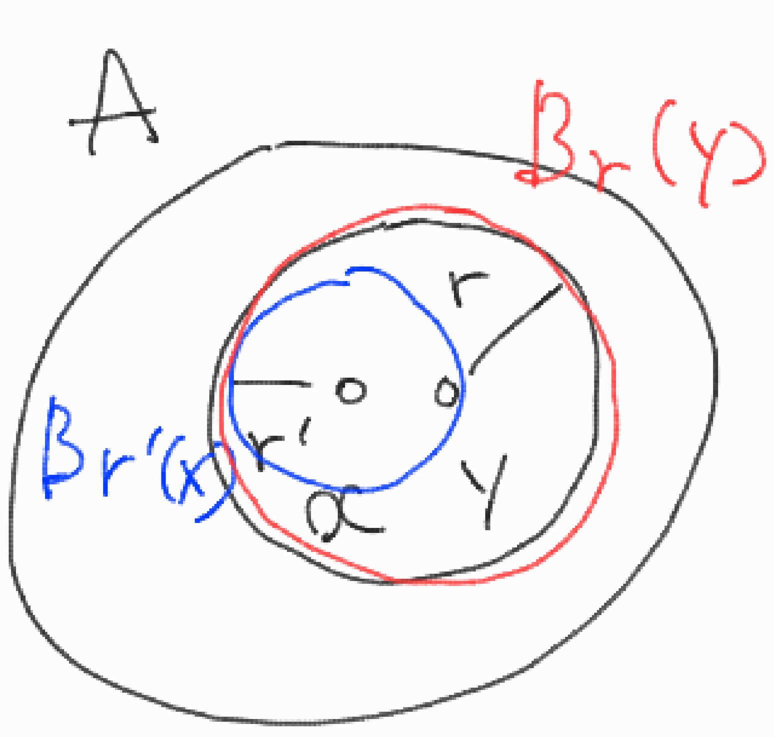
Ex $\mathcal{T}_1 = \mathcal{T}_2$

"$\subset$" Let $A \in \mathcal{T}_1$ WTS : $A \in \mathcal{T}_2$

NTS $\forall x \in A$ $\exists B \in \mathcal{B}_2$ s.t. $x \in B \subset A$.

rectangular domain.

Since $A \in \mathcal{T}_1$, $\exists B_r(y) \in \mathcal{B}_1$ s.t. $x \in B_r(y) \subset A$.

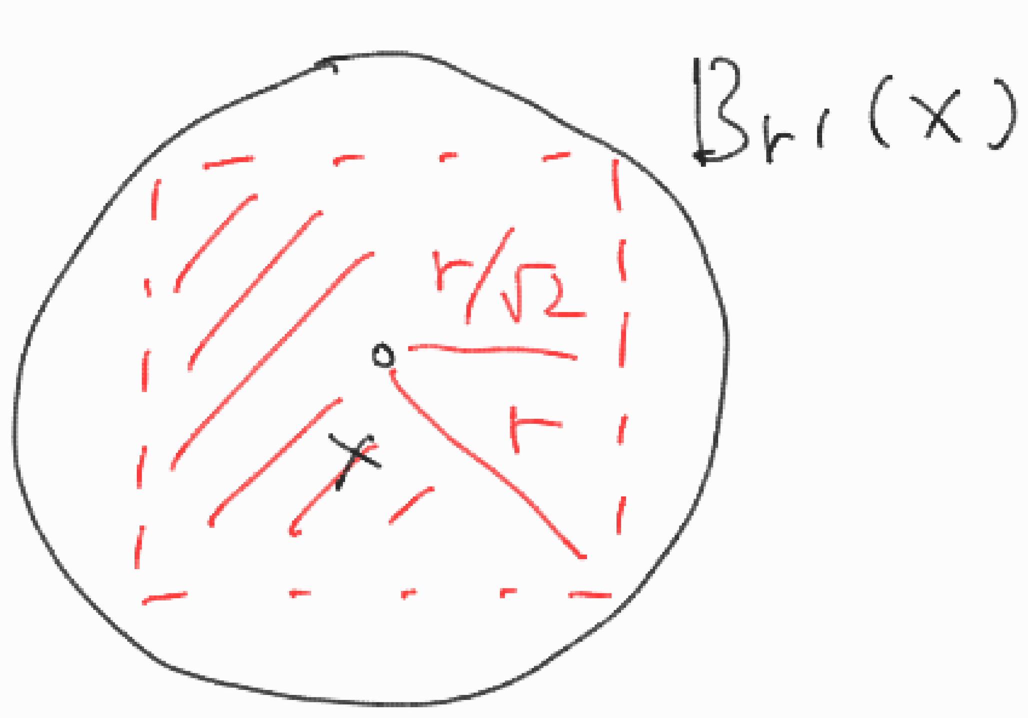


Choose $r' = r - |x-y|$, then

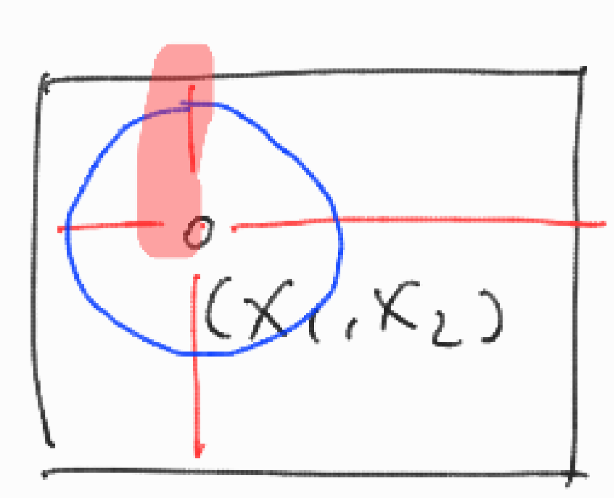
$B_{r'}(x) \subset B_r(y)$.

Let $x = (x_1, \dots, x_n)$ define $\varepsilon = \dfrac{r'}{\sqrt{2}}$.

Then $B = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_n - \varepsilon, x_n + \varepsilon) \subset B_{r'}(x)$.



Thus $x \in B \subset B_{r'}(x) \subset B_r(y) \subset A$



"$\supset$" $A \in \mathcal{T}_2$. NTS $A \in \mathcal{T}_1$

$\forall x \in A$, $\exists\, x \in \overbrace{(a_1, b_1) \times \cdots \times (a_n, b_n)}^{B} \in \mathcal{B}_2$

Let $\varepsilon = \min\{|x_i - a_i|, |x_i - b_i| \mid i = 1, \dots, n\}$.

Then $B_\varepsilon(x) \subset B$. Thus $x \in \underset{\substack{\Uparrow \\ \mathcal{B}_1}}{B_\varepsilon(x)} \subset B \subset A$ $\boxtimes$

Lemma

Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$. Then $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$. $\mathcal{T} = \{ \bigcup_\alpha B_\alpha \mid B_\alpha \in \mathcal{B} \}$.

Lemma

Assumes X is a top. space i.e. there is a top. of X.

Let $\mathcal{C}$ be a collection of open sets of X such that for each open set U of X and each $x \in U$, there is an element $C \in \mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.

$\mathcal{T}$: top. of X.

$\mathcal{T} = \mathcal{T}_\mathcal{C}$ ← top. generated by $\mathcal{C}$.

- $\mathcal{B} \to \mathcal{T}_\mathcal{B}$
- $\mathcal{T} \to \mathcal{B}$ s.t. $\mathcal{T}_\mathcal{B} = \mathcal{T}$. ← lem.

Lem $\mathcal{T}_{\mathcal{B}} = \{ \bigcup_\alpha B_\alpha \mid B_\alpha \in \mathcal{B} \}$.

pf) "$\supset$" $A = \bigcup_\alpha B_\alpha$. NTS: $A \in \mathcal{T}_{\mathcal{B}}$.

$\forall x \in A$, $\exists \alpha$ s.t. $x \in \underline{B_\alpha}$. Thus $x \in \underset{\mathcal{B}}{\underset{\cap}{B_\alpha}} \subset A$

"$\subset$" $A \in \mathcal{T}_{\mathcal{B}}$ (i.e. $\forall x \in A$, $\exists B_x \in \mathcal{B}$ s.t. $x \in B_x \subset A$).

Suff. to show $A = \bigcup_{x \in A} B_x$

Obviously, $\bigcup_{x \in A} B_x \subset A$.

If $x' \in A$, then $x' \in B_{x'}$, thus $x \in \bigcup_{x \in A} B_x$ ▨

Lem $\mathcal{T} = \mathcal{T}_{\mathcal{C}}$.

pf) "$\subset$" $A \in \mathcal{T} \Rightarrow \forall x \in A$ $\exists C \in \mathcal{C}$ s.t. $x \in C \subset A \Rightarrow A \in \mathcal{T}_{\mathcal{C}}$.

"$\supset$" $A \in \mathcal{T}_{\mathcal{C}} \Rightarrow \forall x \in A$ ———— " ———— $\Rightarrow A \in \mathcal{T}$.

Why is $\mathcal{C}$ a basis? Need to check conditions.

① $\forall x \in X$. $\exists ?\ C \in \mathcal{C}$ s.t. $x \in C$

Take $\mathcal{U} = X$ in the definition of $\mathcal{C}$

(★) ② $\forall C_1, C_2 \in \mathcal{C}$, $\forall x \in C_1 \cap C_2$ $\exists ?\ C_3 \in \mathcal{C}$ s.t. $x \in C_3 \subset C_1 \cap C_2$.

~~Want to claim that $C_3 = C_1 \cap C_2 \in \mathcal{C}$.~~

~~(i.e. $\forall x \in C_3$, $\exists C \in \mathcal{C}$ s.t. $x \in C \subset C_3$).~~

~~$C_1 \cap C_2$: open in $X \Rightarrow C_3 = C_1 \cap C_2$ is open.~~

~~$\mathcal{U} = C_3 \Rightarrow \forall x \in \mathcal{U}$ $\exists C \in \mathcal{C}$ s.t. $x \in C \subset \mathcal{U}$.~~

=

By def. of $\mathcal{C}$ ~~$\Leftrightarrow \forall x \in C_1 \cap C_2$ $\exists C \in \mathcal{C}$ s.t. $x \in C \subset C_1 \cap C_2$~~

~~$\Leftrightarrow$ The def. of $C_1 \cap C_2 \in \mathcal{C}$.~~

Need more work!

$\left(\begin{array}{l} C_1, C_2 \in \mathcal{C} \Rightarrow C_1, C_2 : \text{open.} \Rightarrow C_1 \cap C_2 : \text{open.} \\ \forall x \in \underbrace{C_1 \cap C_2}_{\text{open}}, \ \underline{\exists} C_3 \in \mathcal{C} \text{ s.t. } x \in C_3 \subset C_1 \cap C_2 \\ \qquad \text{exists by def. of } \mathcal{C}_3 \end{array} \right)$

Lemma

Let $\mathcal{B}, \mathcal{B}'$ be bases for the topology $\underset{\mathcal{T}_{\mathcal{B}}}{\overset{}{\mathcal{T}}}$, $\mathcal{T}'$. The followings are equivalent:

$\mathcal{T}_{\mathcal{B}} = \mathcal{T}$, $\mathcal{T}_{\mathcal{B}'} = \mathcal{T}'$

1. $\mathcal{T}'$ is finer than $\mathcal{T}$.
2. For each $x \in X$ and a basis element $B \in \mathcal{B}$ containing x, there is $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Lem $\mathcal{T}_{\mathcal{B}} \subset \mathcal{T}_{\mathcal{B}'} \iff \forall B \in \mathcal{B}\ \forall x \in B\ \exists B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$.

pf) "$\Rightarrow$" $B \in \mathcal{B}$ i.e. $B \in \mathcal{T}_{\mathcal{B}} \Rightarrow B \in \mathcal{T}_{\mathcal{B}'}$

$\Rightarrow \forall x \in B\ \exists B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$

"$\Leftarrow$" $A \in \mathcal{T}_{\mathcal{B}} \iff A = \bigcup_{\alpha} B_\alpha \quad \underline{B_\alpha} \in \mathcal{B}$

$\Rightarrow \forall B_\alpha \in \mathcal{B}\ \forall x \in B_\alpha\ \exists B_\alpha' \in \mathcal{B}'$ s.t. $x \in B_\alpha' \subset B_\alpha$

$\Rightarrow \forall x \in A\ \exists \alpha$ s.t. $x \in B_\alpha\ \exists B_\alpha'$ s.t.

$x \in \underset{\underset{\mathcal{B}'}{\Uparrow}}{B_\alpha'} \subset B_\alpha \subset A$.

$\Rightarrow A \in \mathcal{T}_{\mathcal{B}'}$. ∎

Example

1. Let $\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{R}\}$. The topology generated by $\mathcal{B}$ is called the **standard topology** on $\mathbb{R}$.
2. Let $\mathcal{B}' = \{[a, b) \mid a, b \in \mathbb{R}\}$. The topology generated by $\mathcal{B}'$ is called the **lower limit topology** on $\mathbb{R}$.
3. Let $K = \{1/n \mid n \in \mathbb{Z}_+\}$, and $\mathcal{B}'' = \{\boxed{(a, b) - K} \mid a, b \in \mathbb{R}\} \cup \mathcal{B}$. The toplogy generated by $\mathcal{B}''$ is called the **K- toplogy** on $\mathbb{R}$.

Lemma

The lower limit topology and K-topology are strictly finer than the standard topology, but not comparable with each other.

- $(-1,1) \setminus K$ is NOT open in the lower limit top.
- $[0,1)$ is NOT open in K-top.

Ex $\mathcal{B}'' = \{(a,b) \setminus K\}$ is a basis.

pf. ① $x \in \mathbb{R}$. $\exists (a,b) \in \mathcal{B}$ s.t. $x \in (a,b)$. if $a = x-\varepsilon$ $\varepsilon > 0$
$b = x+\varepsilon$

$\mathcal{B} \subset \mathcal{B}''$

② $\underset{B_1}{\underset{\shortparallel}{(a_1, b_1)}}$, $\underset{B_2}{\underset{\shortparallel}{(a_2, b_2)}} \in \mathcal{B}'' \Rightarrow \underset{x}{\underset{\ni}{(a_1,b_1) \cap (a_2, b_2)}}$: open interval as B_3.

$\underset{B_1}{\underset{\shortparallel}{(a_1, b_1) \setminus K}}$. $\underset{B_2}{\underset{\shortparallel}{(a_2, b_2)}} \in \mathcal{B}'' \Rightarrow \forall x \in B_1 \cap B_2$. $\exists B_3 \in \mathcal{B}''$ s.t. $x \in B_3 \subset B_1 \cap B_2$

Take $B_3 = ((a_1, b_1) \cap (a_2, b_2)) \setminus K$. $\boxtimes$

Lem ① $\mathcal{B}'' \supset \mathcal{B} \Rightarrow \mathcal{T}_{\mathcal{B}''} \supset \mathcal{T}_{\mathcal{B}}$

② $(a,b) \in \mathcal{B}$, $\forall x \in (a,b)$, $x \in \underset{\mathcal{B}'}{\underset{\cap}{[x,b)}} \subset (a,b)$

$\Rightarrow \mathcal{T}_{\mathcal{B}'} \supset \mathcal{T}_{\mathcal{B}}$.

"strict finer"

$(-1,1) \setminus K \in \mathcal{B}''$. is open K-top.

Is $(-1,1) \setminus K$ open in standard top.?

If so, $\forall x \in (0,1) \setminus K$, $\exists (a,b) \in \mathcal{B}$ s.t. $x \in (a,b) \subset (-1,1) \setminus K$.

Pick $x = 0$. and $(a,b) \ni 0$. $b > 0$

Then $\exists$ large n s.t. $\frac{1}{n} < b$. $\frac{1}{n} \notin (-1,1) \setminus K$.
$\frac{1}{n} \in (a,b)$

$(a,b) \not\subset (-1,1) \setminus K$. cont.

$[0,1) \in \mathcal{B}'$ open in lower limit top

$x = 0 \in [0,1)$. If $[0,1)$ is open in the stand. top.

$\exists (a,b) \in \mathcal{B}$ s.t. $0 \in (a,b) \subset [0,1)$.

$a < 0$. $\exists a' < 0$, $a < a'$ so that

$a' \in (a,b)$ but $a' \notin [0,1)$. cont.

Definition

A subbasis $\mathcal{S}$ for a topology of X is a collection of subsets of X whose union equals X. The topology generated by the subbasis $\mathcal{S}$ is the collection of all union of finite intersections of elements of $\mathcal{S}$.

generated by $\mathcal{S}$

$$\mathcal{T}_{\mathcal{S}} = \{ \bigcup_{\alpha} (\bigcap_{i=1}^{n_\alpha} A_i^\alpha) \mid A_i^\alpha \in \mathcal{S} \}.$$

Example

Let $\mathcal{A}$ be a (sub)$\boxed{\text{basis}}$ for a topology on X. Show that the topology generated by $\mathcal{A}$ is the intersection of all topologies on X that contains $\mathcal{A}$.

$$\mathcal{T}_{\mathcal{A}} = \bigcap_{\mathcal{A} \subset \mathcal{T}} \mathcal{T}$$

Rmk An subset $\mathcal{U} \subset X$ is open in $\mathcal{T}$ if and only if $\forall x \in \mathcal{U}\ \exists\, \mathcal{U}' \in \mathcal{T}$ s.t. $x \in \mathcal{U}' \subset \mathcal{U}$.

pf). "$\subset$" $\mathcal{U} \in \mathcal{T}_{\mathcal{A}}$, $\forall x \in \mathcal{U}\ \exists A \in \mathcal{A}$ s.t. $\underline{x \in A \subset \mathcal{U}} \Rightarrow \mathcal{U} \in \mathcal{T}$.

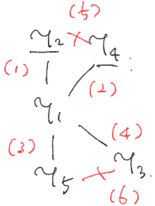
"$\supset$". $\mathcal{T}_{\mathcal{A}} \supset \mathcal{A}$. $\mathcal{T}_{\mathcal{A}} \supset \bigcap_{\mathcal{A} \subset \mathcal{T}} \mathcal{T}$ obvious

Example

Consider the following topologies on $\mathbb{R}$.

1. $\mathcal{T}_1$: the standard topology.
2. $\mathcal{T}_2$: the K-topology
3. $\mathcal{T}_3$: the finite complement topology. $\mathcal{T}_3$: generated by $\mathcal{U}$ where $X-\mathcal{U}=X$ or finite.
4. $\mathcal{T}_4$: the upper limit topology, having all sets $(a, b]$ as basis.
5. $\mathcal{T}_5$: the topology having all sets $\boxed{(-\infty, a)}$] as a basis.

Determine which topolgy contains the other.



$\mathcal{U} = (-\infty, a_1) \cup (a_1, a_2) \cup \cdots \cup (a_n, \infty)$.

(6). $(-\infty, 0) \cup (0, \infty) \in \mathcal{T}_3$.

$1 \in (-\infty, 0) \cup (0, \infty) = A$

If A is open in $\mathcal{T}_5$, $\exists (-\infty, \mu)$ s.t. $1 \in (-\infty, \mu) \subset A$

$\Rightarrow \mu > 1$, thus $0 \in (-\infty, \mu)$ but $0 \notin A$. cont.

$(-\infty, a) \in \mathcal{T}_5$.

If $(-\infty, a) \in \mathcal{T}_3$, then $\forall x \in (-\infty, a)$ $\exists \mathcal{U} \in \mathcal{T}_3$

s.t. $x \in \mathcal{U} \subset \underbrace{(-\infty, a)}_{B}$

$|\mathbb{R} \setminus \mathcal{U}| < +\infty$. $\Rightarrow$ $\mathcal{U} = (-\infty, a_0) \cup (a_0, a_1) \cup \cdots \cup \underbrace{(a_n, \infty)}_{= A'}$.

$A' \not\subset B$. cont. $\boxed{}$