

SE328:Topology

Hyosang Kang¹

¹Division of Mathematics
School of Interdisciplinary Studies
DGIST

Week 03

Definition

Let X, Y be two topological spaces. The **product topology** on the product $X \times Y$ is the topology generated by the subset of the form

$$U \times V$$

where U, V are open subsets in X, Y respectively.

Theorem

If \mathcal{B} is a basis for the topology of X and \mathcal{C} is a basis for the topology of Y , then the collection

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B}, C \in \mathcal{C}\}$$

is a basis for the topology of $X \times Y$.

Definition

Let $\pi_1 : X \times Y \rightarrow X$ and $\pi_2 : X \times Y \rightarrow Y$ be defined by

$$\pi_1(x, y) = x, \quad \pi_2(x, y) = y.$$

The maps π_1, π_2 are called the projections of $X \times Y$ onto its factors.

Theorem

The collection

$$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ is open in } X\}$$

is a subbasis for the topology $X \times Y$.

Definition

Let X be a topological space with the topology \mathcal{T} . Let $Y \subset X$ be a subset. The collection

$$\mathcal{T}' = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology of Y , called the **subspace topology**. With this topology, Y is called a subspace of X .

Lemma

If \mathcal{B} is a basis for the topology of X , then the collection

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology of Y .

Lemma

Let Y be a subspace of X . If U is open in Y and Y is open in X , then U is open in X .

Theorem

If A is a subspace of X and B is a subspace of Y , then the product topology on $A \times B$ is the same topology $A \times B$ inherits as a subspace of $X \times Y$.

Example

Let X be a set with order relation. Assume $|X| > 1$. Let \mathcal{B} a collection of subsets of the following types:

1. all open intervals (a, b) in X ;
2. all half-intervals $[a_0, b)$ where a_0 is the smallest element (if any) of X ;
3. all half-interval $(a, b_0]$ where b_0 is the largest element (if any) of X .

Then the collection \mathcal{B} is a basis for a topology of X , called the order topology.

Suppose that $Y = [0, 1) \cup \{2\}$. Show that the one-point set $\{2\}$ is open in the subspace topology $Y \subset \mathbb{R}$, while it is not open in the order topology, $\{2\}$ is not open.

Example

Let $I = [0, 1]$. The dictionary order on $I \times I$ is the restriction of dictionary order on $\mathbb{R} \times \mathbb{R}$. Show that dictionary order topology on $I \times I$ is not the same as the subspace topology obtained from dictionary order topology on $\mathbb{R} \times \mathbb{R}$.

Definition

Given an ordered set X , a subset $Y \subset X$ is called convex if for each pair of points $a < b$, the interval (a, b) lies in Y .

Theorem

Let X be an ordered set in the order topology. Let Y be a subset of X that is convex in X . Then the order topology on Y is the same as the topology Y as the subspace of X .

Example

A map $f : X \rightarrow Y$ is an **open map** if for every open set $U \subset X$ the image $f(U)$ is open in Y . Show that $\pi_1 : X \times Y \rightarrow X$ and $\pi_2 : X \times Y \rightarrow Y$ are open maps.

Definition

A subset A of a topological space X is **close** if $X - A$ is open.

Example

1. A subset $[a, b] \subset \mathbb{R}$ is closed in the standard topology.
2. A subset $\{(x, y) \mid x, y \geq 0\} \subset \mathbb{R}^2$ is closed in the standard topology.
3. In the finite complement topology of X , every closed set is either X itself or finite subset.

Theorem

Let X be a topological space. Then

- 1. \emptyset and X are closed.*
- 2. Arbitrary intersection of closed sets is closed.*
- 3. Finite union of closed sets is closed.*

Theorem

Let Y be a subspace of X . A subset $A \subset Y$ is closed (in Y) if and only if $A = A \cap C$ for some closed subset $C \subset X$.

Theorem

Let Y be a subspace of X . If $A \subset Y$ is closed in Y and Y is closed in X , then A is closed in X .

Definition

Let A be a subset of topological space X . The **interior** of A is the union of all open sets contained in A . The **closure** of A is the intersection of all closed sets containing A . The interior of A is denoted by $\text{Int}A$. The closure of A is denoted by \overline{A} .

Theorem

Let Y be a subspace of X . Let A be a subset of Y . Let \overline{A} is the closure of A in X . Then the closure of A in Y equals $\overline{A} \cap Y$.

Theorem

Let A be a subset of a topological space X .

- 1. $x \in \overline{A}$ if and only if every open set U containing x intersects A .*
- 2. Let \mathcal{B} be a basis of X . $x \in \overline{A}$ if and only if $x \in B \in \mathcal{B}$ implies $B \cap A$.*

Remark

The phrase “ U is open set containing x ” equal to the statement “ U is (open) neighborhood of x .”

Definition

Let A be a subset of a topological space X . An element $x \in X$ is a **limit point** of A if every neighborhood of x intersects A in some point other than x itself.

Theorem

Let A be a subset of a topological space X . Let A' be the set of all limit points of A . Then

$$\overline{A} = A \cup A'$$

Definition

Given a topological space X , a sequence x_1, x_2, \dots **converges to** $x_\infty \in X$ if for each neighborhood U of x_∞ , there exists an integer $N > 0$ such that $x_i \in U$ for all $i > N$.

Definition

A topological space X is **Hausdorff** if for each pair x_1, x_2 of distinct points of X , there exist neighborhoods U_1, U_2 of x_1, x_2 respectively which are disjoint.

Theorem

Every finite set in a Hausdorff space is closed.

Definition

A topological space X has the T_1 **axiom** if finite point set is closed.

Example

The finite complement topology on \mathbb{R} has the T_1 axiom but it is not Hausdorff.

Theorem

Every sequence in a Hausdorff space converges at most one point.

Example

1. Every ordered set is Hausdorff in the order topology.
2. The product of two Hausdorff space is Hausdorff.
3. A subspace of Hausdorff space is Hausdorff.

Example

X is Hausdorff if and only if the **diagonal** $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X \times X$.

Example

The **boundary** of a subset $A \subset X$, denoted by $\text{Bd}A$ or ∂A , is defined by

$$\partial A = \overline{A} \cap \overline{X - A}$$

1. Show that $\overline{A} \cap \partial A = \emptyset$.
2. Show that $\overline{A} = \partial A \cup \text{Int}A$
3. Show that $\partial A = \emptyset$ if and only if A is both open and closed.
4. Show that U is open if and only if $\partial U = \overline{U} - U$.
5. Prove or disprove: $U = \text{Int}(\overline{U})$.