SE328:Topology

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Definition

Let X, Y be two topological spaces. A function $f : X \to Y$ is called a **continuous** function if $f^{-1}(V)$ is open for every open subset $V \subset Y$.

Proposition

For a function $f : X \to Y$ between topological spaces X, Y, the followings are equivalent:

1. *f is continuous;*
2. *for every subset $U \subset X$, we have $f(\overline{U}) \subset \overline{f(U)}$;*
3. *for every closed subset $V \subset Y$, the set $f^{-1}(V)$ is closed;*
4. *for every point $x \in X$ and a neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.*

Definition

A one-to-one correspondence $f : X \to Y$ is called a **homeomorphism** if f and its inverse f^{-1} are continuous.

Example

Let The map

$$\begin{aligned} f : [0,1) &\to \mathbf{S}^1 \\ x &\mapsto (\cos 2\pi x, \sin 2\pi x) \end{aligned}$$

is bijective and continuous, but its inverse is not continuous.

Proposition

The following methods constructs a continuous functions:

1. *A constant function is continuous.*
2. *For a subset* $A \subset X$*, the inclusion* $\iota : A \hookrightarrow X$ *is continuous.*
3. *For two continuous functiosn* $f : X \to Y$ *and* $g : Y \to Z$*, the composite* $g \circ f : X \to Z$ *is continuous.*
4. *Given a continuous function* $f : X \to Y$ *and a subspace* $A \subset X$*, the restriction* $f|_A : A \to Y$ *is continuous.*

5. Let $f : X \to Y$ be a continuous function. For a subspace $Y_1 \subset Y$ containing $f(X)$, the *restriction of range*

$$f : X \to Y_1$$

is continuous. Similarly, if Y admits a subspace topology induced from $Y_2 \supset Y$, the *expansion of range* $f : X \to Y_2$ is continuous.

6. Let $\{U_\alpha\}$ be a collection of open subsets in X such that $X = \bigcup_\alpha U_\alpha$. (Such collection is called the *open covering* of X.) A map $f : X \to Y$ is continuous if $f|_{U_\alpha}$ is continuous for all α.

Theorem

Let $X = A \cup B$ where $A, B \subset X$ are closed. If $f : A \to Y$ and $g : B \to Y$ are continuous and $f(x) = g(x)$ for all $x \in A \cap B$, then the function $h : X \to Y$ defined by $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$ is continuous.

Theorem

Let $f_1 : A \to X$, $f_2 : A \to Y$ be functions which define $f : A \to X \times Y$ as $f(x) = (f_1(x), f_2(x))$. Then f is continuous if and only if f_1, f_2 are continuous.

Example

A family of subsets $\{A_\alpha\}$ of X is called **locally finite** if every point $x \in X$ admits a neighborhood which intersects only finitely many A_α. Suppose that $\{A_\alpha\}$ is a locally finite *closed* covering of X and $f : X \to Y$ is a function whose restrictions $f|_{A_\alpha}$ are continuous. Show that f is also continuous.

Example

Let $F : X \times Y \to Z$ be a continuous function. Show that for each $x_0 \in X$ and $y_0 \in Y$, the functions $f(x) = F(x, y_0)$ and $g(y) = F(x_0, y)$ are continuous. Show that the converse does not hold. (Use the function below as an counter example.)

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Definition

Let $\{A_\alpha\}$ be a collection of sets indexed by $\alpha \in J$. The **cartesian product** denoted by $\prod_\alpha A_\alpha$ is the set of all J-tuples

$$\mathbf{x} : J \to \bigcup_\alpha A_\alpha; \quad (\alpha) \in A_\alpha$$

Definition

The topology on $\prod_\alpha A_\alpha$ generated by the basis

$$\mathcal{B} = \{\prod_\alpha U_\alpha \mid U_\alpha \subset A_\alpha \text{ is open}\}$$

is called the **box topology**. The topology generated by the subbasis

$$\mathcal{S} = \{\pi_\alpha^{-1}(U_\alpha) \mid U_\alpha \subset A_\alpha \text{ is open}\}$$

is called the **product topology**.

Remark

The basis of product topology consists of the set $\prod_\alpha U_\alpha$ where U_α is open subset of A_α such that $U_\alpha = A_\alpha$ for all but finitely many α. Therefore, the box topology is finer than the product topology. For the finite product space, these topologies are the same.

Theorem

In either topologies on $\prod_\alpha A_\alpha$*,*

- $\prod_\alpha A_\alpha$ *is Hausdorff if each* A_α *is Hausdorff;*
- $\prod_\alpha \overline{A_\alpha} = \overline{\prod_\alpha A_\alpha}$.

Theorem

Let $f : A \to \prod_{\alpha \in J} X_\alpha$ *be defined by* $f(a) = (f_\alpha(a))_{\alpha \in J}$. *Let* $\prod_\alpha X_\alpha$ *has the product topology. Then* f *is continuous if and only if all* $f_\alpha : A \to X_\alpha$ *is continuous.*

Example

Let us consider the box topology on $\mathbb{R}^\omega$. Then $f(t) = (t, t, \cdots)$ is not continuous.

Definition

A **metric** on a set X is a function

$$d : X \times X \to \mathbf{R}$$

such that

1. $d(x, y) \geq 0$ for all $x \in X$ where the equality holds when $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

The **ball** of radius ε (or simply ε-**ball**) centered at $x \in X$ is the subset of X defined by

$$B^d_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$$

If there is no ambiguity on which metric we refer to, we simply write $B_\varepsilon(x)$. The topology on X generated by the basis consists of all ε-balls, $\varepsilon \in \mathbf{R}_+$, is called the **metric topology** on X with respect to the metric d.

Proposition

Given a metric space (X, d), *the bounded metric* $\overline{d}(x, y) = \min\{d(x, y), 1\}$ *defines the same topology.*

Definition

On $\mathbb{R}^n$, there are two metrics.

- For $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$, define

  $$\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2}$$

  The **euclidean metric** is the metric defined by

  $$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$$

- The square metric is defined by

  $$\rho(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, \cdots, |x_n - y_n|\}$$

Theorem

On $\mathbb{R}^n$, the metric topologies induced by d and ρ are the same.

Definition

The metric $\overline{\rho}$ on $\mathbb{R}^J$ defined by

$$\overline{\rho}(\mathbf{x}, \mathbf{y}) = \sup\{\overline{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}$$

is called the **uniform metric**. The topology generated by $\overline{\rho}$ is called the **uniform topology** on $\mathbb{R}^J$.

Theorem

The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology. If J is infinite, all three topologies are different.

Theorem

The metric

$$\overline{D}(\mathbf{x}, \mathbf{y}) = \sup\left\{\frac{\overline{d}(x_i, y_i)}{i}\right\}$$

is a metric that induces the product topology on $\mathbb{R}^\omega$.

Example

Let $X \subset \mathbb{R}^\omega$ be a subset consists of $\mathbf{x} = (x_1, \cdots)$ where $\sum_{i=1}^{\infty} x_i^2$ converges. Define

$$d(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^{\infty} (x_i - y_i)^2 \right)^2$$

The topology generated by this metric is called the l^2**-topology**.

1. Show that l^2-topology is finer than the uniform topology and coarser than the box topology on X.
2. Let $\mathbf{R}^\infty \subset X$ be a subset of X consist of all points $\mathbf{x} = (x_1, \cdots)$ such that $\lim x_i = 0$. Show that uniform, l^2, box, product topologies are all distinct on $\mathbb{R}^\infty$.
3. Compare four topologies on $H = \prod_{n \in \mathbb{Z}_+} [0, 1/n]$.