SE328:Topology

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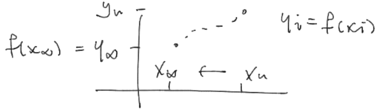
Week 04

Definition

Let X, Y be two topological spaces. A function $f : X \to Y$ is called a **continuous** function if $f^{-1}(V)$ is open for every open subset $V \subset Y$.

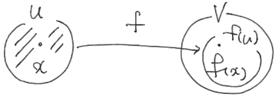


Proposition

For a function $f : X \to Y$ between topological spaces X, Y, the followings are equivalent:

"f preserves the convergent sequence in u".

1. *f is continuous;*
2. *for every subset $U \subset X$, we have $f(\overline{U}) \subset \overline{f(U)}$;*
3. *for every closed subset $V \subset Y$, the set $f^{-1}(V)$ is closed;*
4. *for every point $x \in X$ and a neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.*



PF ① ⇒ ②

SUPPOSE $\exists$ SOME SET $\mathcal{U} \subset X$ s.t. $f(\bar{\mathcal{U}}) \not\subset \overline{f(\mathcal{U})}$.

THAT IS, $\exists x \in \bar{\mathcal{U}}$ S.T. $f(x) \notin \overline{f(\mathcal{U})}$.

IF $x \in \mathcal{U}$. THEN $f(x) \in f(\mathcal{U}) \subset \overline{f(\mathcal{U})}$ CONTRADICTION.

LET $x \in \bar{\mathcal{U}} \setminus \mathcal{U} = \mathcal{U}' \setminus \mathcal{U}$.

$\forall$ NBHD W OF x. INTERSECTS $\mathcal{U}$ AT THE POINT OTHER THAN x.

IF $f(x) \notin \overline{f(\mathcal{U})}$, $f(x) \notin f(\mathcal{U})$ AND $f(x)$ IS NOT a LIMIT POINT OF $f(\mathcal{U})$

THAT IS, $\exists$ NBHD V OF $f(x)$ SUCH THAT V DOES NOT INTERSECTS $f(\mathcal{U})$.

LET $W = f^{-1}(V) \underset{\text{OPEN}}{\subset} X$. THEN W INTERSECTS $\mathcal{U}$ AT SOME POINT. CONTRADICTION.

② ⇒ ③

LET $V \underset{\text{CLOSED}}{\subset} Y$, $f^{-1}(V) = C \subset X$

THEN $f(C) \subset V$.

BY THE ASSUMPTION $f(\bar{C}) \subset \overline{f(C)} \subset \bar{V} = V$.

THUS $\bar{C} \subset f^{-1}(V) = C$

THUS $C = \bar{C}$ AND C IS CLOSED.

③ ⇒ ①

$\mathcal{U} \underset{\text{OPEN}}{\subset} Y$.

$C = \mathcal{U}^c \underset{\text{CLOSED}}{\subset} Y \Rightarrow f^{-1}(C) = X - f^{-1}(\mathcal{U}) \underset{\text{closed}}{\subset} X. \Rightarrow f^{-1}(\mathcal{U}) \underset{\text{OPEN}}{\subset} X.$

① ⇒ ④

$f(x) \in V \underset{\text{OPEN}}{\subset} Y. \Rightarrow x \in f^{-1}(V) \underset{\text{OPEN}}{\subset} X.$

$\exists$ NBHD $x \in \mathcal{U} \subset f^{-1}(V)$.

$f(\mathcal{U}) \subset V$. ▨

④ ⇒ ①.

$V \underset{\text{OPEN}}{\subset} Y$.

$x \in f^{-1}(V)$ (i.e. $f(x) \in V$).

$\exists x \in \mathcal{U} \underset{\text{OPEN}}{\subset} X$ s.t. $f(\mathcal{U}) \subset V$.

THAT IS $x \in \mathcal{U} \subset f^{-1}(V) \underset{\text{OPEN}}{\subset} X$. ▨

Definition

A one-to-one correspondence $f : X \to Y$ is called a **homeomorphism** if f and its inverse f^{-1} are continuous.

Example

"homeo tells us that two top. are THE SAME."

Let The map

subspace top in $\mathbb{R}$ — subspace of stand $\mathbb{R}^2$.

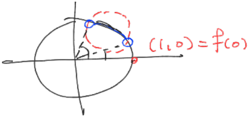
I = OPEN ARC

If $I \ni (1,0)$, THEN

$f^{-1}(I) = \underbrace{[0,\alpha) \cup (\beta,1)}_{} \subset [0,1)$

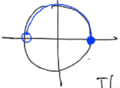
$((-\alpha,\alpha) \cup (\beta,1)) \cap [0,1)$

$$f : [0,1) \to \mathbf{S}^1 \subset \mathbb{R}^2.$$
$$x \mapsto (\cos 2\pi x, \sin 2\pi x)$$



is bijective and continuous, but its inverse is not continuous.

THE BASIS OF THE SUBSPACE TOP. OF S^1 IS $\{\underbrace{\{(\cos\theta, \sin\theta) \mid \theta \in (a,b)\}}_{\text{OPEN INTERVAL IN } S^1}\}$.

$f([0,\frac{1}{2}))$ =  (NOT AN OPEN ARC)

$\cap$ OPEN $[0,1)$

If f^{-1} CONTI, THEN $f(U)$ IS OPEN $\forall\, U \underset{\text{OPEN}}{\subset} [0,1)$.

Proposition

The following methods constructs a continuous functions:

1. *A constant function is continuous.* subspace top.
2. *For a subset $A \subset X$, the inclusion $\iota : A \hookrightarrow X$ is continuous.*
3. *For two continuous functiosn $f : X \to Y$ and $g : Y \to Z$, the composite $g \circ f : X \to Z$ is continuous.*
4. *Given a continuous function $f : X \to Y$ and a subspace $A \subset X$, the restriction $f|_A : A \to Y$ is continuous.*

Subspace top.

PROP. ② $\underset{\text{OPEN}}{U \subset X}$. $\quad \iota^{-1}(U) = U \cap A \underset{\text{OPEN}}{\subset} A$.

③ $\underset{\text{OPEN}}{W \subset Z} \Rightarrow g^{-1}(W) \underset{\text{OPEN}}{\subset} Y \Rightarrow f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W) \underset{\text{OPEN}}{\subset} X.$

④ $\underset{\text{OPEN}}{U \subset Y} \Rightarrow f^{-1}(U) \underset{\text{OPEN}}{\subset} X \Rightarrow f^{-1}(U) \cap A = (f|_A)^{-1}(U) \underset{\text{OPEN}}{\subset} A.$

⑤ $\underset{\text{OPEN}}{V \subset Y_1} \Rightarrow V = W \cap Y_1,\ \underset{\text{OPEN}}{W \subset Y}.$

$$f^{-1}(V) = f^{-1}(W \cap Y_1) = \underbrace{f^{-1}(W)}_{\text{OPEN IN}} \cap \underbrace{f^{-1}(Y)}_{=X} \underset{\text{OPEN}}{\subset} X.$$

⑥ $\underset{\text{OPEN}}{U \subset Y} \Rightarrow f^{-1}(U) = \bigcup_\alpha f^{-1}(U) \cap U_\alpha = \bigcup \underbrace{(f|_{U_\alpha})^{-1}(U)}_{\substack{\text{OPEN IN } U_\alpha \\ \Rightarrow \text{ OPEN IN } X.}} \underset{\text{OPEN}}{\subset} X.$

5. Let $f : X \to Y$ be a continuous function. For a subspace $Y_1 \subset Y$ containing $f(X)$, the *restriction of range*

$$f : X \to Y_1$$

is continuous. Similarly, if Y admits a subspace topology induced from $Y_2 \supset Y$, the *expansion of range* $f : X \to Y_2$ is continuous.

6. Let $\{U_\alpha\}$ be a collection of open subsets in X such that $X = \bigcup_\alpha U_\alpha$. (Such collection is called the *open covering* of X.) A map $f : X \to Y$ is continuous if $f|_{U_\alpha}$ is continuous for all α.

$A' \not\subset A$ Ex. $A = [0,1) \cup \{2\}$. $\Rightarrow$ $A' = [0,1]$

$\cap$

$\mathbb{R}$.

Ex $A = \{ \frac{1}{n} \mid n \in \mathbb{Z}_+ \}$. $\Rightarrow$ $A' = \{0\}$.

Thm $h : X \to Y$ is conti.

pf) $C \subset X$ NTS : $h(\overline{C}) \subset \overline{h(C)}$ $\overline{A} = A, \overline{B} = B$

$$C = (C \cap A) \cup (C \cap B) \Rightarrow \overline{C} = \overline{C \cap A} \cup \overline{C \cap B} = (\overline{C} \cap A) \cup (\overline{C} \cap B)$$

$$h(\overline{C}) = h(\overline{C} \cap A) \cup h(\overline{C} \cap B).$$

$$= f(\underbrace{\overline{C} \cap A}_{\substack{\shortparallel \\ \text{closure of} \\ C \text{ on } A}}) \cup g(\underbrace{\overline{C} \cap B}_{\substack{\shortparallel \\ \text{closure} \\ \text{of } C \text{ on } B.}}) \underset{\substack{\uparrow \\ \text{conti. of } f, g}}{\subset} \underbrace{\overline{f(C \cap A)} \cup \overline{g(C \cap B)}}_{\substack{\shortparallel \\ \overline{h(C)}}}.$$

∎

Thm $f = (f_1, f_2)$ conti $\Leftrightarrow$ f_1, f_2 conti.

pf) "$\Rightarrow$". NTS : $\pi_1 : X \times Y \to X$, $\pi_2 : X \times Y \to Y$. conti.

$$\underset{\text{open}}{U \subset X} \Rightarrow \pi_1^{-1}(U) = U \times Y \underset{\text{open}}{\subset X \times Y}.$$

$\Rightarrow$ $f_1 = \pi_1 \circ f$, $f_2 = \pi_2 \circ f$: conti.

"$\Leftarrow$". $\underset{\text{open}}{U \times V \subset X \times Y}$: open basis $(\underset{\text{open}}{U \subset X}, \underset{\text{open}}{V \subset Y})$

$$f^{-1}(U \times V) = \{ a \in A \mid f_1(a) \in U, f_2(a) \in V \}$$

$$= \underbrace{f_1^{-1}(U) \cap f_2^{-1}(V)}_{\text{open in } A} \underset{\text{open}}{\subset A}.$$

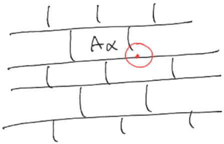
∎

Theorem

Let $X = A \cup B$ where $A, B \subset X$ are closed. If $f : A \to Y$ and $g : B \to Y$ are continuous and $f(x) = g(x)$ for all $x \in A \cap B$, then the function $h : X \to Y$ defined by $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$ is continuous.

Theorem

Let $f_1 : A \to X$, $f_2 : A \to Y$ be functions which define $f : A \to X \times Y$ as $f(x) = (f_1(x), f_2(x))$. Then f is continuous if and only if f_1, f_2 are continuous.



tiling $\mathbb{R}^2$ with "bricks".

Example

A family of subsets $\{A_\alpha\}$ of X is called **locally finite** if every point $x \in X$ admits a neighborhood which intersects only finitely many A_α. Suppose that $\{A_\alpha\}$ is a locally finite *closed* covering of X and $f : X \to Y$ is a function whose restrictions $f|_{A_\alpha}$ are continuous. Show that f is also continuous.

$\mathcal{U} \underset{\text{open}}{\subset} Y$. NTS: $f^{-1}(\mathcal{U}) \underset{\text{open}}{\subset} X$.

$x \in f^{-1}(\mathcal{U})$. $\exists$ nbhd $x \in V$ intersects only $A_1, \cdots, A_n$.

Let $A = \bigcup_{i=1}^{n} A_i$. consider $f|_A$: Since $f|_{A_i}$ is conti, $f|_A$ is conti.

That is, $(f|_A)^{-1}(\mathcal{U} \cap A) \ni x$ admits a nbhd V' s.t. $x \in V' \subset (f|_A)^{-1}(\mathcal{U} \cap A)$.

Therefore $x \in V' \subset f^{-1}(\mathcal{U})$ and thus $f^{-1}(\mathcal{U})$ is open.

Example

Let $F : X \times Y \to Z$ be a continuous function. Show that for each $x_0 \in X$ and $y_0 \in Y$, the functions $f(x) = F(x, y_0)$ and $g(y) = F(x_0, y)$ are continuous. Show that the converse does not hold. (Use the function below as an counter example.)

$X = \mathbb{R}$, $Y = \mathbb{R}$.

$f(x) = F(x, 0) = 0$.

$g(y) = F(0, y) = 0$. → conti.

$$F(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

f conti.

pf) $\underset{\text{closed}}{C \subset X}$. $f^{-1}(C) = \{x \in X \mid F(x, y_0) \subset C\}$

$= \pi_1(\underbrace{F^{-1}(C) \cap X \times \{y_0\}}_{\text{closed}})$

$\underset{\text{closed}}{\subset X}$.

$\boxed{\text{Fact}}$: $\pi_1 : X \times Y \to X$ closed map

$(\underset{\text{closed}}{\pi_1(C)} \subset \underset{\text{closed}}{X}.)$

$X \setminus \pi_1(C) = \{x \in X \mid x \notin \pi_1(C)\}$

$= \{x \in X \mid \nexists y \ (x, y) \in C\}$.

$$\left(\begin{array}{l} F\big|_{X\times\{y_0\}} = \text{restriction of the domain of } F \\ \quad\quad\quad\quad \ni \text{contt.} \\ X\times\{y_0\} = X \ \ \text{(the same top. sp.)}. \end{array} \right)$$

Suppose that $x \in X \setminus \pi_1(C)$.

and $\underline{\forall \text{ nbhd } x\in U \text{ intersects } \pi_1(C)}$ ☆

Since $C \underset{\text{closed}}{\subset} X\times Y$. for any $y\in Y$.

$(x,y)\notin C$, i.e. $(x,y)\in (X\times Y)\setminus C$

thus $\exists U_1\times V_1$ of (x,y) s.t.

open basis nbhd. $(x,y)\in U_1\times V_1 \subset (X\times Y)\setminus C$

$\underline{x'} \in U_1 \cap \underline{\pi_1(C)} \ni \exists y'$ s.t. $(x',y')\in C$.

$\underline{\mathbb{R}^\omega = \{ \aleph \}.}$ order top.

Definition

Let $\{A_\alpha\}$ be a collection of sets indexed by $\alpha \in J$. The **cartesian product** denoted by $\prod_\alpha A_\alpha$ is the set of all J-tuples

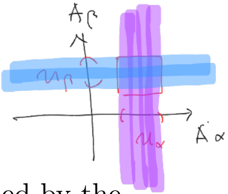
$$\mathbf{x} : J \to \bigcup_\alpha A_\alpha; \ x(\alpha) \in A_\alpha.$$

Definition

top. space

The topology on $\prod_\alpha A_\alpha$ generated by the basis

$$\mathcal{B} = \{\prod_\alpha U_\alpha \mid U_\alpha \subset A_\alpha \text{ is open}\}$$



is called the **box topology**. The topology generated by the subbasis

$$\mathcal{S} = \{\pi_\alpha^{-1}(U_\alpha) \mid U_\alpha \subset A_\alpha \text{ is open}\}$$

is called the **product topology**.

$\bigcup_\alpha \bigcap_{i=1}^{N} U_i^\alpha \qquad U_i^\alpha = \pi_{\alpha_i}^{-1}(U_{\alpha_i})$

The basis elt of prod top are of the form

$\underbrace{U_{\alpha_1} \times U_{\alpha_2} \times \cdots \times U_{\alpha_n}} \times \prod_{\substack{\alpha \neq \alpha_i \\ i=1,\cdots,n}} A_\alpha$

$\prod U_\alpha$: basis of prod top $\ni U_\alpha = A_\alpha$ except for finitely many α.

Remark

The basis of product topology consists of the set $\prod_\alpha U_\alpha$ where U_α is open subset of A_α such that $U_\alpha = A_\alpha$ for all but finitely many α. Therefore, the box topology is finer than the product topology. For the finite product space, these topologies are the same.

Theorem

In either topologies on $\prod_\alpha A_\alpha$,

- $\prod_\alpha A_\alpha$ *is Hausdorff if each* A_α *is Hausdorff;*
- $\underline{\prod_\alpha \overline{\cancel{A_\alpha}}} = \overline{\prod_\alpha \cancel{A_\alpha}}$. B_α B_α $\quad B_\alpha \subset A_\alpha$ $\quad$ ☆.

pf) $x \neq y \in \prod_\alpha A_\alpha$. NTS $U \ni x$. $V \ni y$ $U \cap V = \emptyset$.

"box" For each α, $U_\alpha, V_\alpha \underset{\text{open}}{\subset} A_\alpha$ s.t. $x_\alpha \in U_\alpha$, $y_\alpha \in V_\alpha$

$U_\alpha \cap V_\alpha = \emptyset$.

$U = \prod U_\alpha$, $V = \prod V_\alpha$

"prod". Pick α s.t. $x_\alpha \neq y_\alpha$

$\exists\ \underset{A_\alpha}{\underset{\cap\ \text{open}}{U_\alpha}} \ni x_\alpha$, $\underset{A_\alpha}{\underset{\cap\ \text{open}}{V_\alpha}} \ni y_\alpha$, $U_\alpha \cap V_\alpha = \emptyset$.

$U = U_\alpha \times \prod_{\beta \neq \alpha} A_\beta$, $V = V_\alpha \times \prod_{\beta \neq \alpha} A_\beta$

Thm $f: A \to \prod X_\alpha$ conti. iff $f_\alpha : A \to X_\alpha$ conti.

"$\Rightarrow$". NTS: $\pi_\beta : \prod X_\alpha \to X_\beta$ conti.

$U \underset{\text{open}}{\subset} X_\beta$ $\pi_\beta^{-1}(U) = \underset{\beta}{\underset{\uparrow}{U}} \times \prod_{\alpha \neq \beta} X_\alpha$: open basis

$f_\alpha = \pi_\alpha \circ f$ conti.

"$\Leftarrow$". $U = U_{\alpha_1} \times \cdots \times U_{\alpha_n} \times \prod_{\alpha \neq \alpha_i} X_\alpha \underset{\text{open}}{\subset} \prod X_\alpha$

$f^{-1}(U) = \underbrace{f_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap f_{\alpha_n}^{-1}(U_{\alpha_n})}_{\text{open}} \underset{\text{open}}{\subset} A$. $\square$

Theorem

Let $f : A \to \boxed{\prod_{\alpha \in J} X_\alpha}$ be defined by $f(a) = (f_\alpha(a))_{\alpha \in J}$. Let $\prod_\alpha X_\alpha$ has the product topology. Then f is continuous if and only if all $f_\alpha : A \to X_\alpha$ is continuous.

Example

Let us consider the box topology on $\mathbb{R}^\omega$. Then $f(t) = (t, t, \cdots)$ is not continuous.

$f_\alpha(t) = t$

$\mathcal{U} = \prod_{n=1}^{\infty} (\cancel{0}\ {-\frac{1}{n}}, \frac{1}{n}) \subset \mathbb{R}^\omega$ in box top
open

$f^{-1}(\mathcal{U}) = \cancel{\ } \ \{0\}$. NOT open.
$\cap$
$\mathbb{R}^\omega$.

If $t \in f^{-1}(\mathcal{U})$ (for simplicity, assume $t > 0$)
then $\exists\, N > 0$ s.t. $\frac{1}{N} < t$ so then $(-\frac{1}{N}, \frac{1}{N}) \not\ni t$.

Definition

A **metric** on a set X is a function

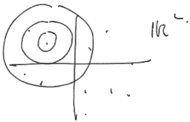
$$d : X \times X \to \mathbf{R}$$

such that

1. $d(x, y) \geq 0$ for all $x \in X$ where the equality holds when $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

The **ball** of radius ε (or simply ε-**ball**) centered at $x \in X$ is the subset of X defined by

$$B_\varepsilon^d(x) = \{y \in X \mid d(x, y) < \varepsilon\}$$



If there is no ambiguity on which metric we refer to, we simply write $B_\varepsilon(x)$. The topology on X generated by the basis consists of all ε-balls, $\varepsilon \in \mathbf{R}_+$, is called the **metric topology** on X with respect to the metric d.

Rmk $d(x,y) = (\Sigma|x_i - y_i|^2)^{1/2}$ on $\mathbb{R}^2$

metric top on $\mathbb{R}^2$ w.r.t. d = stand. top.

Proposition

Given a metric space (X, d), the bounded metric $\overline{d}(x,y) = \min\{d(x,y), 1\}$ *defines the same topology.*

Definition

On $\mathbb{R}^n$, there are two metrics.

- For $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$, define

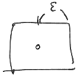
$$\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2}$$

The **euclidean metric** is the metric defined by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$$

$B^d_\varepsilon =$ 

- The square metric is defined by

$B^\rho_\varepsilon =$ 

$$\rho(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, \cdots, |x_n - y_n|\}$$

Theorem

On $\mathbb{R}^n$, the metric topologies induced by $\bar{d}$ d *and* ρ *are the same.*

Definition

The metric $\overline{\rho}$ on $\mathbb{R}^J$ defined by least upper bound

$$\overline{\rho}(\mathbf{x}, \mathbf{y}) = \sup\{\overline{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}$$

is called the **uniform metric**. The topology generated by $\overline{\rho}$ is called the **uniform topology** on $\mathbb{R}^J$.

Theorem

The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology. If J is infinite, all three topologies are different.

prod $\subset$ uniform $\subset$ box.

$\underline{B_1^{\bar\rho}(0)} = \{ y \in \mathbb{R}^\omega \mid \sup\{|y_i|\} < 1 \}$. NOT open in prod.

$0 = (0, 0, \dots)$

$\underline{y} = (0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots) \Rightarrow \sup\{|y_i|\} = \sup_{n \geq 2}\{\frac{1}{n}\} = \frac{1}{2}$

$= \bar\rho(0, y)$

CANNOT FIND $(-\varepsilon_1, \varepsilon_1) \times \dots \times (-\varepsilon_n, \varepsilon_n) \times \mathbb{R} \times \dots \mathbb{R} \times \dots \subset B_1^{\bar\rho}(0)$

PROBLEM!

$\prod_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}) \subset \mathbb{R}^\omega$ in box top.

open

NOT open in unif. top.

If $\underbrace{B_\varepsilon^{\bar\rho}(0)}$ $\subset$ $\underbrace{\prod (-\frac{1}{n}, \frac{1}{n})}$, then

choose $N > 0$ s.t. $\frac{1}{N} < \varepsilon$

then the ith comp. $(i > N)$ does not lie in $(-\frac{1}{i}, \frac{1}{i})$

Theorem

The metric

$$\overline{D}(\mathbf{x}, \mathbf{y}) = \sup\left\{\frac{\overline{d}(x_i, y_i)}{i}\right\}$$

is a metric that induces the product topology on $\mathbb{R}^\omega$.

Example

Let $X \subset \mathbb{R}^\omega$ be a subset consists of $\mathbf{x} = (x_1, \cdots)$ where $\sum_{i=1}^{\infty} x_i^2$ converges. Define

$$d(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^{\infty} (x_i - y_i)^2 \right)^2$$

The topology generated by this metric is called the l^2-**topology**.

prod $\subset$ unif $\subset$ l $\subset$ box.

1. Show that l^2-topology is finer than the uniform topology and coarser than the box topology on X.
2. ~~Let $\mathbf{R}^\infty \subset X$ be a subset of X consist of all points $\mathbf{x} = (x_1, \cdots)$ such that $\lim x_i = 0$. Show that uniform, l^2, box, product topologies are all distinct on $\mathbb{R}^\infty$.~~
3. Compare four topologies on $H = \prod_{n \in \mathbb{Z}_+} \boxed{[0, 1/n]}$. $\subset X$.

$\boxed{B_2^\rho(0)}$ if $\frac{1}{n} < \varepsilon$, then $[0, \frac{1}{n}] \ni x, y \Rightarrow |x-y| < \varepsilon$. prod = unif.