(1) on a : $\underbrace{dS_t}_{GBM} = \mu S_t\, dt + \sigma S_t + \underbrace{dW_t}$ → brownien Martien

on pose : $Y_t = \ln S_t$

$$dY_t = d\ln S_t = \frac{dS_t}{S_t} - \frac{1}{2}\frac{dS_t^2}{S_t^2}$$

$$\Rightarrow \quad d\ln S_t = \mu\, dt + \sigma\, dW_t - \frac{1}{2}\sigma^2 dt$$

$$= \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dW_t$$

$$\int_0^T d\ln S_t = \int_0^T \left(\mu - \frac{\sigma^2}{2}\right) dt + \int_0^T \sigma\, dW_t$$

$$\exp(\ln S_t - \ln S_0) = \exp\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_t$$

$$\Rightarrow \boxed{S_t = S_0\, e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}}$$

↳ Formule du mouvement brownien géométrique

3) Maximum de vraisemblance :

On a : $dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$

On pose $Y_t = \ln S_t$ par la formule d'Itô :

$$dY_t = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma\, dW_t$$

Supposons : la suite t_i : $i = 0 \dots N$ est croissante avec : $dY_t = \Delta Y_t$:

$$\Delta Y_t = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t_n + \sigma \Delta W_n$$

$$\mathcal{L} = \sum_{n=1}^{N} \log \underbrace{f_n(\Delta Y_n)}_{\text{densité de } \Delta Y_t}$$

On a donc : $\log f_n(\Delta Y_n) = -\ln \sigma - \dfrac{\left(\Delta Y_n - (\mu - \frac{1}{2}\sigma^2)\Delta t_n\right)^2}{2\sigma^2 \Delta t_n} + c$

en μ : $\Rightarrow$ $\dfrac{\partial \log f_n(\Delta Y_n)}{\partial \mu} = \dfrac{\Delta Y_n - (\mu - \frac{1}{2}\sigma^2)\Delta t_n}{\sigma^2} \Rightarrow \dfrac{\partial \mathcal{L}}{\partial \mu} = \underbrace{\dfrac{\Delta Y - (\mu - \frac{1}{2}\sigma^2)}{\sigma^2}}_{= 0}$

$$\Rightarrow \boxed{\hat{\mu} = \frac{dY}{dt} + \frac{1}{2}\sigma^2}$$

en σ : $\Rightarrow$ $\dfrac{\partial \log f_n(\Delta X_n)}{\partial \sigma} = \dfrac{(\mu^2 - \frac{1}{4}\sigma^4)\Delta t_n - \sigma^2 - 2\mu \Delta Y_n + \Delta Y_n^2 \Delta t_n^{-1}}{\sigma^3}$

$$\Rightarrow \frac{\partial \ell}{\partial \sigma} = \frac{(\mu^2 - \frac{1}{4}\sigma^4)dt_n - N\sigma^2 - 2\mu\, dY_n + \sum \Delta Y_n^2 \Delta t_n^{-1}}{\sigma^3}$$

$$\Rightarrow \boxed{\hat{\sigma}^2 = \frac{2}{dt} \times \sqrt{dt(\mu^2 dt - 2\mu\, dY + \sum_n \Delta Y_n^2 \Delta t_n^{-1} + N^2} - N}$$