Name: Solver 1

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IIT Bombay
CS 6001: GT&AMD
Midsem Exam, 2022-23-I
Date: September 19, 2022

CS6001: Game Theory and Algorithmic Mechanism Design

Total: **$10 \times 4 = 40$ marks,** *Duration:* **2 hours, ATTEMPT ALL QUESTIONS**

Instructions:

1. This question paper and answersheet contains a total of 11 pages (11 sides of paper, page 2 is blank). Please verify.
2. Write your name, roll number, department, section on **every side of every sheet** of this booklet. Use only **black/blue ball-point pen**.
3. Write final answers neatly with a pen **only in the given boxes**.
4. Use the rough sheets for scratch works / attempts to solution. **Write only the final solution (which may be a sequence of logical arguments) in a precise and succinct manner in the boxes provided**. Do not provide unnecessarily elaborate steps. The space within the boxes are sufficient for the correct and precise answers.
5. Submit your answerscripts to the teaching staff when you leave the exam hall or the time runs out (whichever is earlier). **Your exam will not be graded if you fail to return the paper**.
6. **This is a closed book, notes, internet exam. No communication device, e.g., cellphones, iPad, etc., is allowed**. Keep it switched off in your bag and keep the bag away from you. If anyone is found in possession of such devices during the exam, that answerscript may be disqualified for evaluation and DADAC may be invoked.

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Problem 1 (2 + 4 + 4 points). Consider the game with payoffs as depicted in Figure 1. Player 1 is the row player and her payoff is written first in every cell, and Player 2 is the column player.

		Player 2		
		L	C	R
Player 1	T	$0,4$	$1,2$	$0,0$
	M	$3,7$	$2,1$	$1,5$
	B	$0,3$	$6,2$	$8,4$

Figure 1: A two player normal form game with each having three strategies.

(a) Show the residual game after eliminating the strictly dominated strategies for each player. Explain which strategies are eliminated for each player and in which sequence.

T is strictly dominated by M, hence eliminated. After that, C is strictly dominated by both L and R. Hence the residual game becomes

	L	R
M	3,7	1,5
B	0,3	8,4

(b) Find all the pure strategy Nash equilibria (PSNEs) of this game.

By inspection, we find that both (M, L) and (B, R) are PSNEs of the residual game. Since eliminated strategies were strictly dominated, these two strategy profiles are PSNEs in the original game as well.

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(c) Find all the mixed strategy Nash equilibria (MSNEs) of this game that are not PSNEs.

Show that smaller supports in this game either falls back to the PSNEs or leads to impossibilities such that it can't be a valid support.

		y	$1-y$
		L	R
x	M	3, 7	1, 5
$1-x$	B	0, 3	8, 4

The full support turns out to be the only possibility.

From condn. ① of MSNE characterization:

(a) $3y + 1\cdot(1-y) = 8(1-y) \Rightarrow \frac{y}{1-y} = \frac{7}{3} \Rightarrow y = \frac{7}{10}$

(b) $7x + 3(1-x) = 5x + 4(1-x) \Rightarrow \frac{x}{1-x} = \frac{1}{2} \Rightarrow x = \frac{1}{3}$

The only MSNE that is not a PSNE:

$\left(\left(0, \frac{1}{3}, \frac{2}{3}\right), \left(\frac{7}{10}, 0, \frac{3}{10}\right)\right)$ [in the original game. however correct answers for the residual game will also be given full credit]

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Problem 2 (1 + 3 + 2 + 4 points). Consider a game with two players each having two strategies. Player 1's strategies are T and B, and that of Player 2 are L and R. Suppose for each strategy profile, there is a function $\phi : S_1 \times S_2 \to \mathbb{R}$, that maps the profile into the set of real numbers. The function is given as follows:

$$\phi(T, L) = 5, \quad \phi(B, L) = 4, \quad \phi(B, R) = 2, \quad \phi(T, R) = 7.$$

Suppose, the Players 1 and 2 picked strategies B and R respectively to begin with. Now, starting with Player 2, the players alternatingly pick strategies. The players are rational and aim to maximize their payoff, which is given in the form of the following difference equation.

$$u_i(s_i, s_{-i}) - u_i(s'_i, s_{-i}) = \phi(s_i, s_{-i}) - \phi(s'_i, s_{-i}).$$

The equation above implies that for every agent i, the difference in her utility when she changes her strategy from s'_i to s_i is identical to the corresponding change in the *common* function ϕ in the way described above, when all other players' strategies remain fixed.

(a) Consider the following policy of the players. While taking their turns, each player unilaterally (keeping the other player's strategy fixed at its current choice) maximizes her payoff by choosing her strategy. Let us call this a **greedy** policy. Does this greedy policy adopted by all the players converge in the given game? This means that, does this iterative alternating method starting with Player 2 reach a strategy profile from where no agent has any rational reason to update her strategy.

Yes, it converges.

(b) Explain your answer above and show the steps to convergence (if it does).

The iterative greedy policy of the players change the strategy profile as follows:

$$(B,R) \xrightarrow{\text{player 2's utility increases by 2}} (B,L) \xrightarrow{\text{player 1's utility increases by 1}} (T,L) \xrightarrow{\text{player 2's utility increases by 2}} (T,R)$$

Initial state

(T, R) is the converged strategy profile since player 1 or 2 does not gain any further by changing their strategies.

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(c) If it converges, which profile does it converge to? Write "NA" if it does not converge.

Converges to (T, R)

(d) If it converges, show that this profile is a pure strategy Nash equilibrium (PSNE). Write "NA" if the answer to the previous question was NA too.

By definition (s_i^*, s_{-i}^*) is a PSNE if

$$u_i(s_i^*, s_{-i}^*) - u_i(s_i, s_{-i}^*) \geq 0 \quad \forall s_i \in S_i \quad \forall i \in N.$$

Here, $u_1(T, R) - u_1(B, R) = 7 - 2 = 5 > 0$

and $u_2(T, R) - u_2(T, L) = 7 - 5 = 2 > 0$

hence (T, R) is a PSNE of this game.

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Problem 3 (2 + 4 + 4 points). Answer the following sub-questions.

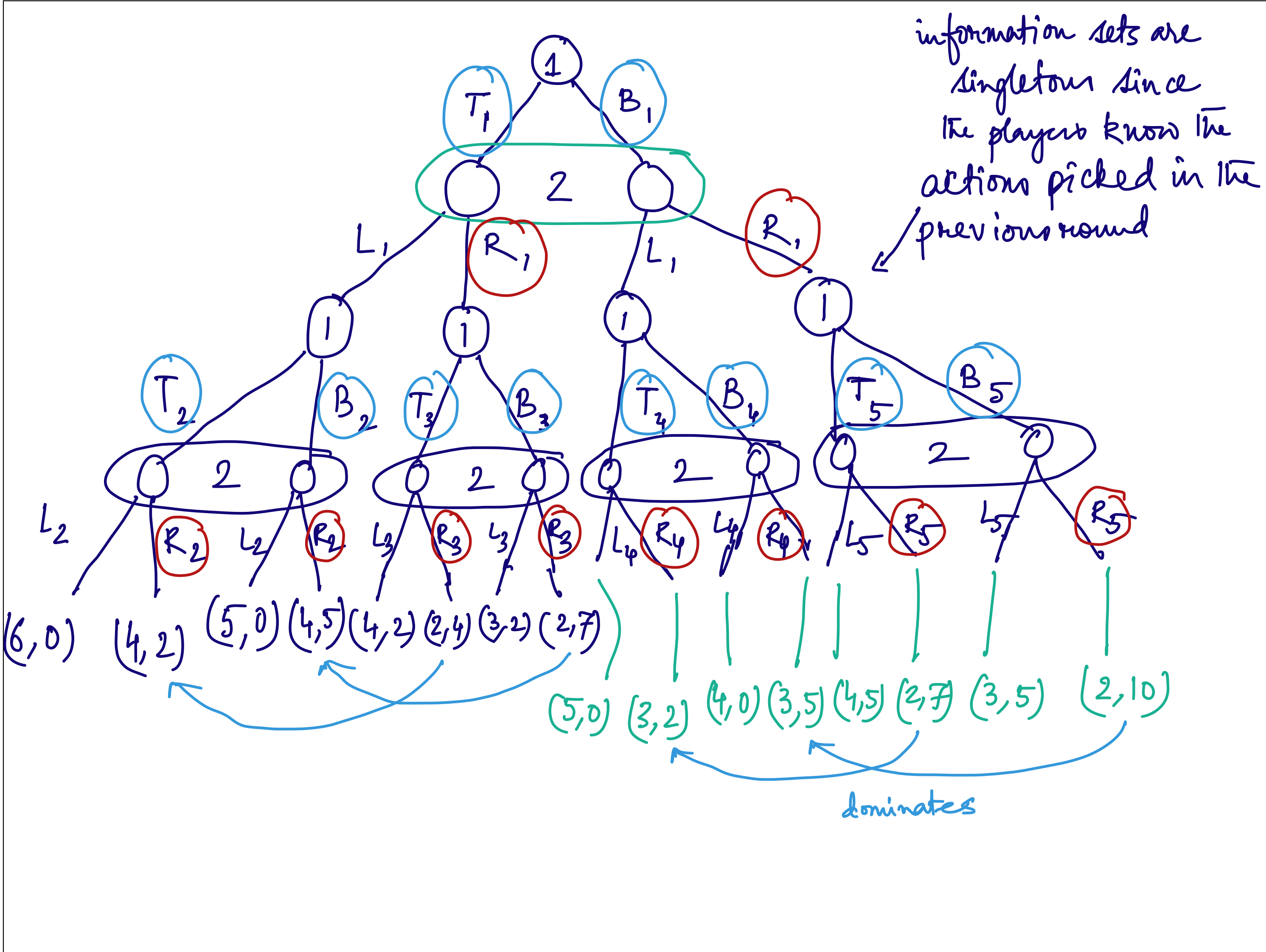
(a) Find all the MSNEs of the following two-player game.

Player 2

Player 1		L	R
	T	$3,0$	$1,2$
	B	$2,0$	$1,5$

Note that L is strictly dominated by R and can be eliminated.
Since utility of player 1 is same for both T and B in the residual game, they both can remain on the support and can be mixed in any possible way. Hence MSNE
$((x, 1-x), (0,1))$ $x \in [0,1]$

(b) Suppose the players play the game twice; after the first time they have played the game, they know the actions chosen by both of them, and hence each player may condition his action in the second stage on the actions that were chosen in the first stage. Players utilities are additive in the two stages. Describe this two-stage game as an extensive-form game along with appropriate utilities.



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(c) Find all the subgame perfect Nash equilibria of this two-stage game.

To find the subgame perfect NE, we apply backward induction. Note that at the leaf nodes playing R is a strictly dominant strategy for player 2. Given that, player 1 is indifferent between T and B in the second stage from below. Given that, strategy R strictly dominates L at the third level from below for player 2. This makes player 1 indifferent between T and B.

The SPNE strategies are shown circled in the previous diagram.

Blue denotes the strategies of player 1, while red denotes the strategies of player 2.

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Problem 4 (1 + 1 + (1 + 1 + 1) + (1 + 1 + 1) + 1 + 1 points). Consider the following game with incomplete information.

- $N = \{1, 2\}$.
- $\Theta_1 = \{\theta_1^1, \theta_1^2\}, \Theta_2 = \{\theta_2\}$, i.e., Player 1 has two types while Player 2 has only one type.
- Common prior, $p(\theta_1^1, \theta_2) = \frac{1}{3}, \quad p(\theta_1^2, \theta_2) = \frac{2}{3}$.
- Every player has two possible actions, and state games are given by the following matrices.

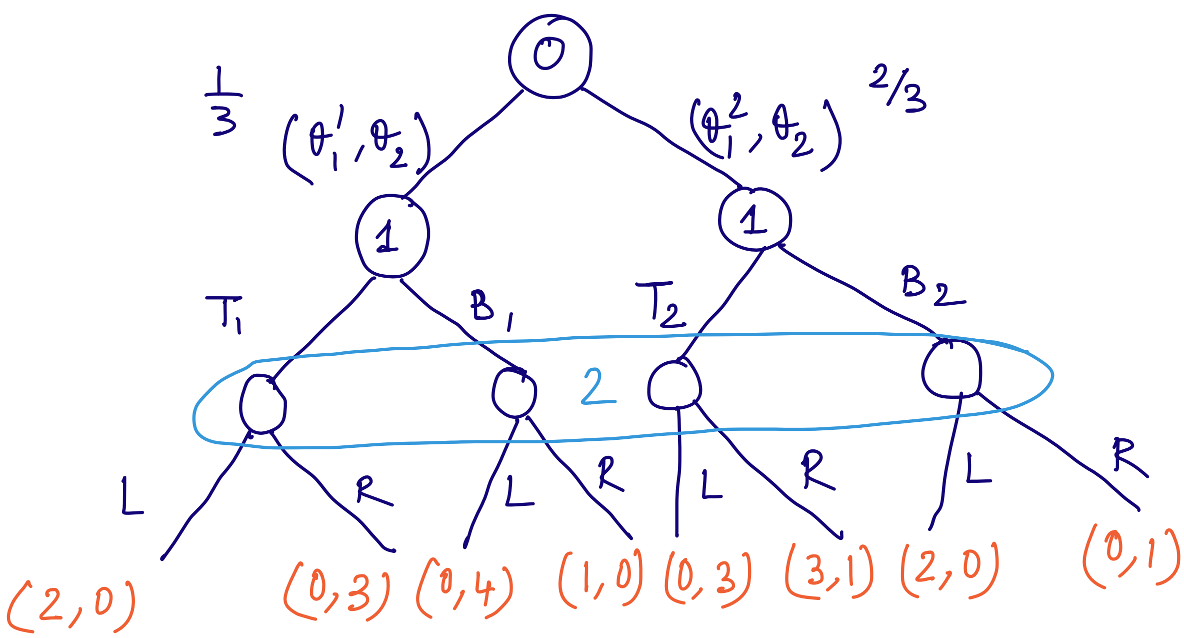
Player 1 \ Player 2	L	R
T	$2, 0$	$0, 3$
B	$0, 4$	$1, 0$

Game for $\theta = (\theta_1^1, \theta_2)$

Player 1 \ Player 2	L	R
T	$0, 3$	$3, 1$
B	$2, 0$	$0, 1$

Game for $\theta = (\theta_1^2, \theta_2)$

(a) Draw the imperfect information extensive form game (IIEFG) representation of this game with nature (Player 0) making the first move and then Player 1 making her move.



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(b) To find a Bayesian equilibrium of this game, we will consider the Harsanyi representation of the game by treating each type as a player. Hence, consider the expanded player set $\tilde{N} = \{\theta_1^1, \theta_1^2, \theta_2\}$, and the support profile of $(\{T\}, \{T, B\}, \{L, R\})$. Apply the MSNE characterization theorem on this setup.

Recall the characterization theorem (stated here to refresh your memory, and not in the most formal way).

- Part 1 of the theorem says that for each player, the expected utility of the player must be the same for all strategies in the support.
- Part 2 of the theorem says that for each player, the expected utility of the player in its support must be at least as much as that of the strategies outside of its support.

Let $\sigma_{\theta_1^2} = (x, 1-x)$ and $\sigma_{\theta_2} = (q, 1-q)$ denote the mixed strategies of the last two players in this support of the expanded game.

What does part 2 of the characterization theorem give when applied for player θ_1^1?

$u_{\theta_1^1}(T, \sigma_{-\theta_1^1}) \geqslant u_{\theta_1^1}(B, \sigma_{-\theta_1^1}) \Rightarrow 2xq + 2(1-x)q \geqslant x(1-q) + (1-x)(1-q)$

$\Rightarrow 2q \geqslant 1-q \Rightarrow q \geqslant \frac{1}{3}$

(c) What are the values of the utilities:

- $u_{\theta_1^2}(T, T, \sigma_{\theta_2}) =$ $3(1-q)$
- $u_{\theta_1^2}(T, \cancel{T}^{B}, \sigma_{\theta_2}) =$ $2q$

What do you get by applying part 1 of the characterization theorem for player θ_1^2?

Equate the two terms above and get

$q = 3/5$

(d) What are the values of the utilities:

- $u_{\theta_2}(T, \sigma_{\theta_1^2}, L) =$ $2x$
- $u_{\theta_2}(T, \sigma_{\theta_1^2}, R) =$ $5/3$

What do you get by applying part 1 of the characterization theorem for player θ_2?

equate them and get $x = 5/6$

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(e) Is this a valid MSNE of the expanded game, i.e., does both the conditions of the characterization theorem hold good? Explain why.

Yes, because for players θ_1^2 and θ_2, we verified condn. ① of the characterization theorem and condn.-② is vacuously satisfied.
For player θ_1^1, Condition ① is thrivially satisfied and condn. ② requires $q \geq \frac{1}{3}$ (as shown in part (b))
Hence $q = 3/5$ satisfies this condition.

(f) What is the Bayesian equilibrium, (σ_1, σ_2), emerging from the calculations above? Note that, a typical strategy for player i in a Bayesian game is $\sigma_i : \Theta_i \to \Delta A_i$. Hence, write the answer such that it gives the complete description of σ_i's.

The Bayesian equilibrium in the original game is therefore

$\sigma_1(\theta_1^1) = (1, 0)$ [pure strategy T]

$\sigma_1(\theta_1^2) = \left(\frac{5}{6}, \frac{1}{6}\right)$

$\sigma_2(\theta_2) = \left(\frac{3}{5}, \frac{2}{5}\right)$

END OF QUESTION PAPER. GOOD LUCK!