CS 228 : Logic in Computer Science

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Normal Forms : CNF Validity

Let $\varphi = C_1 \wedge C_2 \wedge \cdots \wedge C_n$ be in CNF.

- Checking if φ is satisfiable is NP-complete.
- Checking if φ is valid is polynomial time. Why?
- Question raised in class : If validity check is polynomial time, so should be satisfiability. Is this true?
- If φ is valid, it is indeed satisfiable
- If φ is not valid, then...?

Normal Forms : DNF Satisfiability

Let $\varphi = D_1 \vee D_2 \vee \cdots \vee D_n$ be in DNF.

- Checking if φ is valid is NP-complete. Why?
- Checking if φ is satisfiable is polynomial time. Why?

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- Truth table of φ : φ is false when $p = T, q = F$ and $p = F, q = T$.
- CNF equivalent is $(\neg p \vee q) \wedge (p \vee \neg q)$.

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- Prove that any equivalent DNF formula has 2^n clauses
- Call an assignment *minimal* if it maps exactly one of p_i, q_i to 1
- There are 2^n *minimal* assignments, satisfying clauses in φ'
- Show that no two *minimal* assignments satisfy the same clause of φ' (hence there must be 2^n clauses in φ')

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- $\mathsf{min}(\alpha, \beta)(p)$=$\mathsf{min}(\alpha(p), \beta(p))$ for each variable p with the assumption that $0 < 1$, 0 represents false and 1 represents true

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- $\min(\alpha, \beta) \not\models p_i \vee q_i$, $\min(\alpha, \beta) \not\models \varphi'$
- However, if $\alpha \models D_j$ and $\beta \models D_j$ for some clause D_j of φ', then $\min(\alpha, \beta) \models D_j$ and hence $\min(\alpha, \beta) \models \varphi'$, a contradiction.

Think of an example where DNF to CNF explodes.