Real Analysis (MA 403)
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Problem set 2

1. Let $\overline{E}$ and E° be respectively the closure and interior of a set E.
   (i) Do E and $\overline{E}$ have the same interiors ?
   (ii) Do E and E° have the same closures ?
2. Construct a compact set of real numbers whose limit points form a countable set.
3. Consider the set of rational numbers $\mathbb{Q}$ with usual (modulus) metric. If p, q are prime numbers with $p < q$ prove that $[\sqrt{p}, \sqrt{q}] \cap \mathbb{Q}$ is closed and bounded in $\mathbb{Q}$ but not compact in $\mathbb{Q}$. Is $[\sqrt{p}, \sqrt{q}] \cap \mathbb{Q}$ an open set in $\mathbb{Q}$?
4. Let (X, d) be a metric space. Prove that a set $K \subseteq X$ is compact if and only if every infinite subset of K has a limit point in K.
5. Let (X, d) be a metric space in which every infinite subset has a limit point. Prove that X is separable.
6. Prove that every nonempty open set in $\mathbb{R}$ can be expressed as a union of at most countable open intervals.
7. A set E in a metric space (X, d) is said to be *perfect* if E is closed and every point in E is a limit point of E. Prove that every nonempty perfect set in $\mathbb{R}^n$ is uncountable.
8. Are closures and interiors of connected sets always connected ?
9. If A and B are disjoint closed sets in a metric space (X, d), prove that they are separated. What happens if A and B are disjoint open sets ?
10. Prove that a set in $\mathbb{R}$ is connected if and only if it is an interval. Does an analogous statement hold for $\mathbb{R}^n$, $n > 1$, that is, is it true that k-cells or open k-cells are the only connected proper subsets of $\mathbb{R}^n$?