IIT Bombay
CS 6001: GT&AMD
Quiz 1, 2023-24-I
Date: September 5, 2023

Roll No : Anonymous
e.g., 190040001

Dept.: CSE
e.g., CSE

CS6001: Game Theory and Algorithmic Mechanism Design

Total: **10 × 3 = 30 marks,** *Duration:* **1 hour, ATTEMPT ALL QUESTIONS**

Instructions:

1. This question-and-answersheet booklet contains a total of 6 sheets of paper (11 pages, pages 2 and 12 are blank). Please verify.
2. Write your roll number and department on **every side of every sheet** (except the blank sheet) of this booklet. Use only **black/blue ball-point pen**. The first 5 minutes of additional time is given exclusively for this activity.
3. Write final answers neatly with a pen **only in the given boxes**.
4. Use the rough sheets for scratch works / attempts to solution. **Write only the final solution (which may be a sequence of logical arguments) in a precise and succinct manner in the boxes provided**. Do not provide unnecessarily elaborate steps. The space within the boxes are sufficient for the correct and precise answers.
5. Submit your answerscripts to the teaching staff when you leave the exam hall or the time runs out (whichever is earlier). **Your exam will not be graded if you fail to return the paper**.
6. **This is a closed book, notes, internet exam. No communication device, e.g., cellphones, iPad, etc., is allowed**. Keep it switched off in your bag and keep the bag away from you. If anyone is found in possession of such devices during the exam, that answerscript may be disqualified for evaluation and DADAC may be invoked.
7. One A4 assistance sheet (text **only on one side**) is allowed for the exam.

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Problem 1 (10 points). Café Coffee Day and Domino's Pizza are contemplating providing stalls during Mood Indigo 2023 on the street in front of the hostel blocks. Naturally, each of them is targeting to get as many students as possible to visit their stalls and hence wants to position their stalls accordingly. Assume the following for simplicity.

- The street is an interval $[0, 1]$. Stalls can be placed anywhere along this street. This includes stalls being co-located as well.
- The student population is uniform over this interval, and every student prefers to go to the stall that is closest to him/her – in case of a tie, the stalls get equal proportion of the population.
- The student proportion can also be any real number in $[0, 1]$. The stalls' utilities are the fraction of the population they get in their stall.

(a) Say, player 1 is the left player and player 2 is the right player. Left player's position is l and right player's is r, $l \leqslant r$. Write down the utilities of both the players in terms of l and r. **3 points.**

$$u_1(l, r) = \begin{cases} \frac{l+r}{2} & , \text{ if } l < r \\ \frac{1}{2} & , \text{ if } l = r \end{cases}$$

$$u_2(l, r) = \begin{cases} 1 - \frac{l+r}{2} & , \text{ if } l < r \\ \frac{1}{2} & , \text{ if } l = r \end{cases}$$

(b) Find a pure strategy Nash equilibrium of this game. **3 points.**

If $l < r$, Then each player has an incentive to move closer to the other player which strictly increases her payoff. Hence, none of these cases where $l < r$ can be a PSNE. At $l = r \neq \frac{1}{2}$, each agent gains strictly by moving towards the center of the interval $[0,1]$. This rules out PSNE at those positions as well. At $l = r = \frac{1}{2}$, no agent gains by moving in any direction – the utilities are strictly larger $(= \frac{1}{2})$ at this point than any other point on the left or right. Hence

$l = r = \frac{1}{2}$ is a PSNE of this game.

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(c) Is the PSNE unique? Why? **1 point.**

Yes. As argued in the above answer, all other possible choices of l and r has strict improvement possible by at least one player. Hence, the choice $l = r = \frac{1}{2}$ is the only PSNE of this game.

(d) Suppose, now Pizza Hut decides to enter this game and provide a stall. What will be a PSNE now? The rules remain the same – every student prefers the closest stall, and in case of ties, the stalls split the population equally. **3 points.**

Consider the position of the third player as x. WLOG (the labels of the players do not matter since the game is symmetric) assume the positions chosen are $l < x < r$.

Clearly, this is not a PSNE since both l and r improves by moving closer to x.

If $l = x < r$, again r benefits by moving towards left.

If $l < x = r$, l benefits by moving towards right.

So, none of these cases can be a PSNE.

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The only remaining option is $l = x = r$, in which case each of these players receive an utility of $\frac{1}{3}$. However, each player gains strictly by moving right (or left) by a small amount if $l = x = r \leqslant \frac{1}{2}$ (or $l = x = r > \frac{1}{2}$), where the utility is close to $\frac{1}{2}$ or higher.

Hence, no PSNE exists in this case.

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Problem 2 (10 points). Prove that a finite two-player zero sum game has a **mixed strategy Nash equilibrium** (MSNE) iff

$$\max_{\sigma_1\in\Delta S_1}\min_{\sigma_2\in\Delta S_2} u(\sigma_1,\sigma_2) = \min_{\sigma_2\in\Delta S_2}\max_{\sigma_1\in\Delta S_1} u(\sigma_1,\sigma_2) = u(\sigma_1^*,\sigma_2^*), \tag{1}$$

where σ_1^* and σ_2^* are the maxmin and minmax **mixed** strategies of players 1 and 2 respectively.

Note: the result we proved in the class was for pure strategy Nash equilibrium. Do the proof *from first principles* (assuming no known inequalities between the terms above) in two directions as mentioned below. Clearly show every step of the derivation and the required arguments.

(a) **Necessary direction:** the game has an MSNE ⇒ Equation (1). **1 point.**

Define $\overline{v} = \min_{\sigma_2\in\Delta S_2}\max_{\sigma_1\in\Delta S_1} u(\sigma_1,\sigma_2)$, $\underline{v} = \max_{\sigma_1\in\Delta S_1}\min_{\sigma_2\in\Delta S_2} u(\sigma_1,\sigma_2)$

First, we show the relationship between $\overline{v}$ and $\underline{v}$

$$u(\sigma_1,\sigma_2) \geqslant \min_{\sigma_2\in\Delta S_2} u(\sigma_1,\sigma_2) \quad \text{[By defn of "min"]} \quad \forall \sigma_1,\sigma_2$$

$$\Rightarrow \max_{\sigma_1\in\Delta S_1} u(\sigma_1,\sigma_2) \geqslant u(\sigma_1^*,\sigma_2) \geqslant \max_{\sigma_1\in\Delta S_1}\min_{\sigma_2\in\Delta S_2} u(\sigma_1,\sigma_2) \quad \forall \sigma_2 \text{ where } \sigma_1^*\in\arg\max_{\sigma_1}\min_{\sigma_2} u(\sigma_1,\sigma_2)$$

$$\Rightarrow \min_{\sigma_2}\max_{\sigma_1} u(\sigma_1,\sigma_2) \geqslant \max_{\sigma_1}\min_{\sigma_2} u(\sigma_1,\sigma_2)$$

$$\Rightarrow \overline{v} \geqslant \underline{v} \quad \text{----} \ ①$$

Since MSNE exists, say (σ_1^*,σ_2^*)

$$u(\sigma_1^*,\sigma_2^*) \geqslant u(\sigma_1,\sigma_2^*) \quad \forall \sigma_1\in\Delta S_1 \ \text{[by defn]}$$
$$\geqslant \max_{\sigma_1} u(\sigma_1,\sigma_2^*)$$
$$\geqslant \min_{\sigma_2}\max_{\sigma_1} u(\sigma_1,\sigma_2) \quad \text{[since } \sigma_2^* \text{ is a specific strategy]}$$
$$= \overline{v} \quad \text{---} \ ②$$

Similarly, $u(\sigma_1^*,\sigma_2^*) \leqslant u(\sigma_1^*,\sigma_2) \quad \forall \sigma_2\in\Delta S_2$ [by defn]

$$\Rightarrow u(\sigma_1^*,\sigma_2^*) \leqslant \min_{\sigma_2} u(\sigma_1^*,\sigma_2)$$
$$\leqslant \max_{\sigma_1}\min_{\sigma_2} u(\sigma_1,\sigma_2) = \underline{v}$$

Since $u(\sigma_1^*,\sigma_2^*) \leqslant \underline{v} \leqslant \overline{v} \leqslant u(\sigma_1^*,\sigma_2^*)$

$\Rightarrow u(\sigma_1^*,\sigma_2^*) = \overline{v} = \underline{v}$ [also shows that σ_1^* and σ_2^* are maxmin and minmax strategies resp.]

(b) **Sufficient direction:** Equation (1) ⇒ the game has an MSNE. **1 point.**

Given, $u(\sigma_1^*, \sigma_2^*) = \overline{v} = \underline{v}$ and σ_1^*, σ_2^* are maxmin and minmax strategies respectively.

$$u(\sigma_1, \sigma_2^*) \leqslant \max_{\sigma_1} u(\sigma_1, \sigma_2^*) \quad \text{[by defn. of max]} \quad \forall \sigma_1$$

$$= \min_{\sigma_2} \max_{\sigma_1} u(\sigma_1, \sigma_2) \quad [\text{since } \sigma_2^* \text{ is a minmax strategy}]$$

$$= \overline{v} = u(\sigma_1^*, \sigma_2^*) \quad \text{(given)}$$

Similarly

$$u(\sigma_1^*, \sigma_2) \geqslant \min_{\sigma_2} u(\sigma_1^*, \sigma_2)$$

$$= \max_{\sigma_1} \min_{\sigma_2} u(\sigma_1, \sigma_2)$$

$$= \underline{v} = u(\sigma_1^*, \sigma_2^*)$$

Hence, (σ_1^*, σ_2^*) is an MSNE

This completes the proof.

(c) Is Equation (1) always true for a finite two-player zero sum game? (Yes/No) **1 point.**

Yes.

(d) Justify your answer above, i.e., if no, provide a counterexample, if yes, prove it (you may use results from the class and in this question, but clearly state the results you are using). **2 points.**

We know from (a) and (b) having an MSNE is equivalent to Eq (1)
By Nash's theorem, we know an MSNE always exists in a finite game – hence also exists for finite 2-player zero sum games.
Therefore, we conclude that Eq (1) always holds for such games.

[Aside: the value $\bar{v} = \underline{v} = v$ is called the value of a matrix game]
This is just an information point, not part of the answer

(e) In the following payoff matrix of a two-person zero-sum game, no pure strategy Nash equilibrium exists.

		Player 2	
		L	R
Player 1	T	a	b
	B	c	d

What inequalities must the numbers a, b, c, d satisfy? **2 points.**

None of the pure strategy profiles are PSNEs, hence one of the following two must be true

Case 1: $a > c$ and $a > b$, so that TL and BL are not PSNEs

i.e., $a > \max\{b,c\}$

and $d > b$ and $d > c$, so that BR and TR are not PSNEs

i.e. $d > \max\{b,c\}$. Together $\min\{a,d\} > \max\{b,c\}$ - - ①

Case 2: Similarly, $\min\{b,c\} > \max\{a,d\}$.

(f) Find the MSNE(s) of this game. **3 points.**

Under the above conditions, the support of MSNE for both players has to be the full strategy set.

Let T is played with prob. p (hence B with $(1-p)$)

L is - - - - - q (- - - R with $(1-q)$)

Using the characterization theorem of MSNE

$$p = \frac{d-c}{a-b+d-c}, \quad q = \frac{d-b}{a-c+d-b}.$$

Problem 3 (10 points). For each of the following two games represented in the normal form, determine whether or not it can represent a perfect information extensive-form game (PIEFG). If so, describe a corresponding PIEFG; if not, explain why it is not possible.

(a) **Game A** **4 points.**

Player 2

		a	b
Player 1	A	$1, 1$	$5, 3$
	B	$3, 0$	$5, 3$
	C	$1, 1$	$0, 4$
	D	$3, 0$	$5, 3$

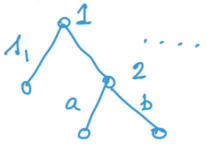
This game cannot be represented as a PIEFG.

Suppose not, i.e., it can be represented. The PIEFG must have one vertex where player 2 plays and has two actions: a and b.

Claim: this vertex has to be the root of the PIEFG.

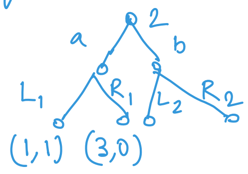
In every other case, where this vertex is NOT the root, there must be one strategy s_1 of player 1 where

$$\left(u_1(s_1,a),\ u_2(s_1,a)\right) = \left(u_1(s_1,b),\ u_2(s_1,b)\right)$$



Since, no such s_1 exists, player 2 must play at the root. ∎

Now,



The rest of the PIEFG has to be filled where A, B, C, D are broken into $\underset{A}{\underline{L_1L_2}}$, $\underset{C}{\underline{L_1R_2}}$, $\underset{B}{\underline{R_1L_2}}$, $\underset{D}{\underline{R_1R_2}}$

If one fills these strategies in any way (WLOG the one shown on the figure), there is not feasible way to fill the other two terminal vertex utilities such that they match the utilities given by the matrix.

Hence, this NFG does not come from any PIEFG.

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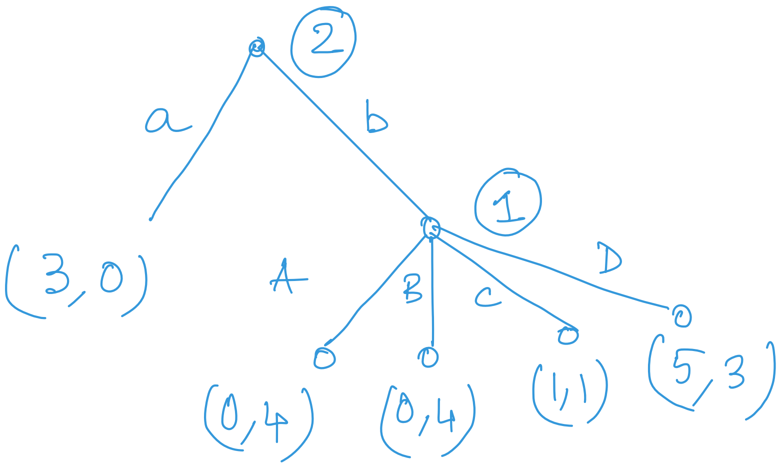
(b) **Game B** **4 points.**

Player 2 (columns), Player 1 (rows)

	a	b
A	$3, 0$	$0, 4$
B	$3, 0$	$0, 4$
C	$3, 0$	$1, 1$
D	$3, 0$	$5, 3$

This game does have a PIEFG :



(c) Find the subgame-perfect Nash equilibrium(a) of these games (write 'NA' if PIEFG representation is not possible). **2 points.**

Game A: NA

Game B: (D, b)

END OF QUESTION PAPER. GOOD LUCK!