19 The bargaining set

Chapter summary

In this chapter we present the bargaining set, which is a set solution concept for coalitional games. The idea behind the bargaining set is that when the players consider how to divide the worth of a coalition among themselves, a player who is unsatisfied with the suggested imputation can object to it. An *objection*, which is directed against another player, roughly claims: "I deserve more than my suggested share and you should transfer part of your suggested share to me because . . . " The player against whom the objection is made may or may not have a *counterobjection*. An objection that meets with no counterobjection is a *justified objection*. The bargaining set consists of all imputations in which no player has a justified objection against any other player.

It follows from the definition of an objection that in any imputation in the core no player has an objection, and therefore the core is always a subset of the bargaining set. It is proved that contrary to the core, the bargaining set is never empty. In convex games the bargaining set coincides with the core.

In Chapter 17 we noted that the core, as a solution concept for coalitional games, suffers from a significant drawback: in many cases, the conditions that the core must satisfy are too strong, and as a result, there is no imputation that satisfies all of them. Consequently, in many games the core is the empty set. In this chapter, we present another solution concept, termed "the bargaining set," which imposes weaker conditions, and yields a recommended solution for every coalitional structure, provided that there exists at least one feasible imputation for that structure.

Example 19.1 An advertising agency seeks two celebrities to star in an advertising campaign. Three celebrities, Anna, Ben, and Carl, are approached, with the intention that two of them will be chosen for the advertising campaign. The advertising agency is persuaded that an advertisement depicting a man and a woman is generally more effective than one depicting two men, and it therefore offers a pair of celebrities comprised of a man and a woman \$1,000,000, and offers a pair comprised of two men only \$500,000. This situation may be depicted as a game in coalitional form (all payoffs are in thousands of dollars).

$$\begin{aligned}
&v(\emptyset) = 0,\\
&v(\text{Anna}) = v(\text{Ben}) = v(\text{Carl}) = 0,\\
&v(\text{Ben, Carl}) = 500,\\
&v(\text{Anna, Ben}) = v(\text{Anna, Carl}) = 1{,}000,\\
&v(\text{Anna, Ben, Carl}) = 0.
\end{aligned}$$

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Remark 19.2 *We define $v(\text{Anna, Ben, Carl}) = 0$ because the advertising campaign cannot include three celebrities. We could alternatively define $v(\text{Anna, Ben, Carl}) = 1{,}000$, with all three celebrities forming a coalition that sends only two of them to be photographed for the advertisements.* ♦

Which coalition will be formed? Suppose that Anna and Ben form a coalition, without Carl's participation. How will they divide the $1,000,000 they are paid?

It is readily verified that this game has an empty core for any coalitional structure (the core is empty also if we set $v(\text{Anna, Ben, Carl} = 1{,}000)$). Experiments conducted using games similar to this game indicate that the players usually raise offers and counteroffers, in the hope of being included in the coalition that eventually forms. A typical bargaining process looks something like this (see Maschler [1978] and Kahan and Rapoport [1984]):

Stage	Offer	Coalitional Structure	*Imputation* Anna	Ben	Carl
1	Ben	{Carl}, {Anna, Ben}	500	500	0
2	Carl	{Ben}, {Anna, Carl}	600	0	400
3	Ben	{Carl}, {Anna, Ben}	700	300	0
4	Carl	{Ben}, {Anna, Carl}	800	0	200
5	Ben	{Anna}, {Carl, Ben}	0	250	250
6	Anna	{Carl}, {Anna, Ben}	740	260	0
7	Carl	{Ben}, {Anna, Carl}	750	0	250

After this bargaining process is completed, Anna and Carl form a coalition, dividing the $1,000,000 they are paid among them as ($750,000, $250,000).

Experimental evidence indicates that the results of bargaining processes in this game are usually very close to one of the following imputations (again, in thousands of dollars in the order (Anna, Ben, Carl)):[1]

$$(750, 250, 0), \qquad (750, 0, 250), \qquad (0, 250, 250),$$

with slight variations in various directions, since a person who sees he is about to be left out is usually willing to yield a bit in the bargaining process to increase his chance of being included in a two-player coalition. Some experimental evidence indicates that each one of the above imputations is the average of payoffs obtained when the appropriate coalitional structure is formed. There are other empirical results pointing to more equitable outcomes, perhaps because Anna stands to lose more than the other players if she is not included in a two-player coalition, resulting in greater willingness on her part to yield in the bargaining process.

What characterizes these outcomes, and how can they be generalized to a solution concept? To answer this question, we first look at an imputation that is not included in the above set. For example, suppose that Anna proposes that she and Ben form a coalition, and offers to divide the money according to the imputation

$$x = (800, 200, 0).$$

Ben can be expected to be dissatisfied with this offer. He tells Anna that she should give him some of the 800 that she suggests for herself, because otherwise he will approach Carl with an offer

1 Other possible foreseeable outcomes are presented later in this chapter.

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to form a coalition {Ben, Carl}, with the imputation:

$$y = (0, 250, 250).$$

Carl will certainly agree to such an offer, and both Ben and Carl profit more from y than from x. We term the pair ({Ben, Carl}, y) an *objection of Ben against Anna at* x.

Anna may respond by saying that she also has an objection to x: Ben needs to give her an even share greater than 800 since she can approach Carl and offer to form the coalition {Anna, Carl}, with the imputation:

$$z = (810, 0, 190).$$

The pair ({Anna, Carl}, z) is an objection of Anna against Ben at x. But there is a difference between these objections: Ben's objection is *justified*, while Anna's objection is unjustified.

Why is Anna's objection unjustified? Suppose that Anna approaches Carl with the suggestion that they form a coalition and divide the money according to imputation z. Ben can prevent this from happening by offering to form a coalition with Carl and dividing the money according to y. Both Ben and Carl stand to gain from this: Ben receives 250 (instead of the 200 he receives according to x), and Carl receives 250 (instead of the 190 Anna promises him under z). Anna has no chance of realizing her objection ({Anna, Carl}, z), and we therefore regard it as unrealistic. In this case, we say that ({Ben, Carl}, y) is a *counterobjection of Ben to Anna's objection* z.

Why is Ben's objection justified? Because Anna has no counterobjection to it. Suppose that Ben indeed offers to form a coalition with Carl and divide the money according to y. To prevent this from happening, Anna must suggest forming a coalition with Carl, with an imputation giving Carl at least 250. But then all she will have left for herself is at most 750 – less than the 800 she received under x. In other words, Anna cannot defend the 800 she receives under x, and therefore Ben's objection against Anna is justified.

If the coalition {Anna, Ben} is formed, we claim that neither Anna nor Ben has a justified objection to the imputation $\widehat{x} = (750, 250, 0)$. For example, if Anna objects to $\widehat{x}$ by proposing to form the coalition {Anna, Carl} with an imputation $(750 + \varepsilon, 0, 250 - \varepsilon)$, where $\varepsilon > 0$, then as a counterobjection Ben can suggest the coalition {Ben, Carl} with the imputation (0, 250, 250). If Ben objects to $\widehat{x}$ by proposing the coalition {Ben, Carl} with an imputation $(0, 250 + \varepsilon, 250 - \varepsilon)$, Anna can counterobject by proposing to form the coalition {Anna, Carl} with the imputation (750, 0, 250).

For the coalitional structure {{Anna, Ben}, {Carl}} the imputation $\widehat{x} = (750, 250, 0)$ is the only imputation at which every objection of one player against another player in the same coalition can be met with a counterobjection. (Exercise 19.7). ◀

19.1 Definition of the bargaining set

We are now ready for the formal presentation of the bargaining set. Let $(N; v)$ be a coalitional game. Recall that for every vector $x \in \mathbb{R}^N$ we set $x(\emptyset) = 0$, and for every nonempty coalition $S \subseteq N$ we have denoted $x(S) = \sum_{i \in S} x_i$. Recall also that a coalitional structure $\mathcal{B}$ is a partition of the set of players into disjoint, nonempty coalitions whose union is N, and $X(\mathcal{B}; v)$ is the set of imputations relative to $\mathcal{B}$:

$$X(\mathcal{B}; v) := \{x \in \mathbb{R}^N : x(B) = v(B) \ \ \forall B \in \mathcal{B}, \qquad x_i \geq v(i) \ \ \forall i \in N\}. \qquad (19.1)$$

An imputation in $X(\mathcal{B}; v)$ is therefore a way to divide the worth of each coalition in $\mathcal{B}$ among its members, where each player receives at least what he can get by himself.

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Definition 19.3 *Let $x \in X(\mathcal{B}; v)$ be an imputation, and let $k \neq l$ be two players belonging to the same coalition in $\mathcal{B}$. An* objection *of player k against player l at x is a pair (C, y) such that:*

1. *$C \subseteq N$ is a coalition containing k but not l: $k \in C$, $l \notin C$.*
2. *$y \in \mathbb{R}^C$ is a vector of real numbers satisfying $y(C) = v(C)$, and $y_i > x_i$ for each player $i \in C$.*

The interpretation of an objection of player k against player l is that player k demands that player l give some of his payoff to player k, because otherwise player k can approach the members of coalition C (which contains player k but not player l, and therefore does not require player l's approval) and suggest dividing the worth of the coalition C among its members in such a way that each of them receives more than he receives under x.

Remark 19.4 *An objection can be raised by a player only against another player in the same coalition in the coalitional structure. This implies, among other things, that there are no objections in a one-player coalition. The point of an objection is not for the objection to be realized, but rather to bring about a situation in which the objecting player receives a greater payoff, at the expense of the player against whom the objection is raised. However, the coalition C with which player k raises an objection against player l may be any coalition, not necessarily a subcoalition of the coalition in $\mathcal{B}$ that contains players k and l.* ♦

The core can be characterized using the concept of an objection (Exercise 19.1).

Theorem 19.5 *The core of the coalitional game $(N; v)$ for the coalitional structure $\mathcal{B} = \{N\}$ is the set of all imputations at which no player has an objection against another player.*

For other coalitional structures, the statement of the above theorem holds only if the core is nonempty (Exercises 19.2 and 19.3).

Theorem 19.6 *If the core of the coalitional game $(N; v)$ for a coalitional structure $\mathcal{B}$ is nonempty, then it is the set of all imputations in $X(\mathcal{B}; v)$ at which no player has an objection against another player in the same coalition in $\mathcal{B}$.*

Definition 19.7 *Let (C, y) be an objection of player k against player l at x. A* counter-objection *of player l against player k is a pair (D, z) satisfying:*

1. *D is a coalition containing l but not k: $l \in D$, $k \notin D$.*
2. *$z \in \mathbb{R}^D$, and $z(D) = v(D)$.*
3. *$z_i \geq x_i$ for every player $i \in D \setminus C$.*
4. *$z_i \geq y_i$ for every player $i \in D \cap C$.*

Player l has a counterobjection to player k's objection against him if he can find a coalition D containing himself, but not player k, and a way to divide the worth of coalition D among its members in such a way that each member of $D \setminus C$ receives at least what he receives

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under x, and each member of $D \cap C$ receives at least what he receives under y (which is what player k promises to give him in his objection to x.)

Remark 19.8 *It is possible for a counterobjection to satisfy the property that $D \cap C = \emptyset$. In that case, the fourth condition in Definition 19.7 is satisfied vacuously. When the objection (C, y) is relative to the coalitional structure $\mathcal{B}$, the coalition D used for the counterobjection, just like the coalition C used for the objection, can be any coalition in N, and not necessarily a subcoalition of a coalition in $\mathcal{B}$.* ♦

Definition 19.9 *An objection (C, y) of player k against player l at x is a* justified objection *if player l has no counterobjection to it.*

We are now ready to define the bargaining set of a coalitional game $(N; v)$ for any coalition structure $\mathcal{B}$.

Definition 19.10 *Let $(N; v)$ be a coalitional game, and $\mathcal{B}$ a coalitional structure. The* bargaining set *relative to the coalitional structure $\mathcal{B}$ is the set $\mathcal{M}(N; v; \mathcal{B})$ of imputations in $X(\mathcal{B}; v)$ at which no player has a justified objection against any other player in his coalition.*

In other words, the bargaining set is the set of all imputations in $X(\mathcal{B}; v)$ at which every objection of one player against another player in his coalition in the coalitional structure $\mathcal{B}$, is met by a counterobjection.

We next present a few simple properties satisfied by the bargaining set.

Theorem 19.11 *If $\mathcal{B} = \{\{1\}, \{2\}, \dots, \{n\}\}$, then*

$$\mathcal{M}(N; v; \mathcal{B}) = \{(v(1), v(2), \dots, v(n))\}. \tag{19.2}$$

Proof: The vector $(v(1), v(2), \dots, v(n))$ is the only imputation in $X(\mathcal{B}; v)$. It is therefore the only vector that could possibly be in the bargaining set. Since there are no objections in one-player coalitions, there are in particular no justified objections, and therefore the bargaining set for this coalitional structure contains a single vector, $(v(1), v(2), \dots, v(n))$. □

Since there can be no objection raised at an imputation that is in the core, and therefore certainly no justified objection, we obtain the following result.

Theorem 19.12 *For every coalitional game $(N; v)$ and every coalitional structure $\mathcal{B}$, the bargaining set relative to $\mathcal{B}$ contains the core relative to $\mathcal{B}$:*

$$\mathcal{M}(N; v; \mathcal{B}) \supseteq \mathcal{C}(N; v; \mathcal{B}). \tag{19.3}$$

Thus, the bargaining set contains the core, but it may contain imputations that are not in the core. In Exercise 19.16, we will see an example that shows that there are situations in which imputations that are not in the core are more intuitive solutions than imputations in the core (Exercise 19.16). There are cases in which the core is empty, and then the bargaining set, which is never empty, can provide a solution to the game.

Example 19.1 (*Continued*) It is left to the reader to ascertain that the following vectors are in the bargaining set, for various coalitional structures:

Imputation	Coalitional structure
(750, 250, 0)	{{Anna, Ben}, Carl}
(750, 0, 250)	{{Anna, Carl}, Ben}
(0, 250, 250)	{Anna}, {Carl, Ben}
(0, 0, 0)	{Anna}, {Ben},{Carl}
(0, 0, 0)	{Anna, Ben, Carl} if $v(N) = 0$
$(666\frac{2}{3}, 166\frac{2}{3}, 166\frac{2}{3})$	{Anna, Ben, Carl} if $v(N) = 1{,}000$

In contrast, the core of the game is empty, relative to every coalitional structure. ◀

Definition 19.13 *Let $(N; v)$ be a coalitional game, let $x \in \mathbb{R}^N$, and let $S \subseteq N$ be a coalition. The* excess *of the coalition S at x is*

$$e(S, x) := v(S) - x(S). \tag{19.4}$$

The excess $e(S, x)$ measures the extent to which the members of S are dissatisfied when receiving payments according to x: if the excess is positive, the members of S are dissatisfied with x, because they could band together to form a coalition, receive $v(S)$, and divide that sum among them in such a way that every member of S gets more than he gets under x. The smaller the excess, the less the members of S are dissatisfied. When the excess is negative, the members of S are satisfied with x, and the more negative the excess, the luckier they consider themselves.

The following theorem illustrates the importance of considering the excess in studying the bargaining set.

Theorem 19.14 *Let $(N; v)$ be a coalitional game, let $\mathcal{B}$ be a coalitional structure, and let k and l be two players in the same coalition in $\mathcal{B}$. If (C, y) is a justified objection of player k against player l, and if D is a coalition containing player l but not player k, then $e(D, x) < e(C, x)$.*

Proof: The intuition behind this theorem is as follows. The excess $e(C, x)$ is the total sum that player $k \in C$ can divide among the members of C under an objection (C, y). If $e(D, x)$ is greater than or equal to $e(C, x)$, player l can divide among the members of $D \cap C$ at least what player k gives them under objection (C, y), thereby creating a counterobjection. Thus, if (C, y) has no counterobjection, then $e(D, x) < e(C, x)$ must hold.

Formally, define a vector $z \in \mathbb{R}^D$ as follows:

$$z_i = \begin{cases} x_i & \text{if } i \in D \setminus (C \cup \{l\}), \\ y_i & \text{if } i \in D \cap C, \\ v(D) - x(D \setminus (C \cup \{l\})) - y(D \cap C) & \text{if } i = l. \end{cases} \tag{19.5}$$

Note that $z(D) = v(D)$. The pair (D, z) satisfies Conditions (1), (2), and (4) of the definition of a counterobjection, and satisfies Condition (3) for all $i \neq l$. Since (C, y)

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is a justified objection, (D, z) is not a counterobjection of player l against player k, and therefore necessarily Condition (3) in the definition of a counterobjection for $i = l$ does not hold, that is, $z_l < x_l$. Now,

$$\begin{aligned}
e(D, x) &= v(D) - x(D) && (19.6)\\
&= v(D) - x(D \setminus (C \cup \{l\})) - x(D \cap C) - x_l - y(D \cap C) + y(D \cap C) && (19.7)\\
&= z_l - x(D \cap C) - x_l + y(D \cap C) && (19.8)\\
&< y(C \cap D) - x(D \cap C) && (19.9)\\
&\leq y(C) - x(C) && (19.10)\\
&= v(C) - x(C) = e(C, x), && (19.11)
\end{aligned}$$

where Equation (19.8) holds by the definition of z_l, Equation (19.9) holds because $z_l < x_l$, and Equation (19.10) holds because $y_i > x_i$ for all $i \in C \setminus D$. In other words, $e(D, x) < e(C, x)$, which is what we needed to show. □

19.2 The bargaining set in two-player games

Suppose that $N = \{1, 2\}$. If $v(1, 2) < v(1) + v(2)$, then the set $X(\{N\}; v)$ is empty, and therefore $\mathcal{M}(N; v; \mathcal{B})$ is empty.

If $v(1, 2) \geq v(1) + v(2)$, there are no objections at any vector $x \in X(\{N\}; v)$. To see this, note that the only coalition that player k can form for use in an objection is the coalition containing himself alone, but if $x \in X(\{N\}; v)$, then $x_i \geq v(i)$, and therefore player i cannot use this coalition for an objection at x. It follows that the bargaining set coincides with the core and with the set of imputations, namely, it is the line segment

$$\mathcal{M}(N; v; \{N\}) = \{(x, v(1, 2) - x) \colon v(1) \leq x \leq v(1, 2) - v(2)\}. \qquad (19.12)$$

Recall that according to Theorem 19.11, for the coalitional structure $\mathcal{B} = \{\{1\}, \{2\}\}$,

$$\mathcal{M}(N; v; \mathcal{B}) = \mathcal{C}(N; v; \mathcal{B}) = X(\mathcal{B}; v) = \{(v(1), v(2))\}. \qquad (19.13)$$

19.3 The bargaining set in three-player games

Suppose that $N = \{1, 2, 3\}$. Relative to the coalitional structure $\mathcal{B} = \{\{1\}, \{2\}, \{3\}\}$,

$$\mathcal{M}(N; v; \mathcal{B}) = \mathcal{C}(N; v; \mathcal{B}) = X(\mathcal{B}; v) = \{(v(1), v(2), v(3))\}. \qquad (19.14)$$

We next turn to studying the bargaining set for the coalitional structure $\mathcal{B} = \{\{1, 2\}, \{3\}\}$: Players 1 and 2 form a coalition, and Player 3 forms a one-player coalition. In this case,

$$X(\mathcal{B}; v) = \left\{x \in \mathbb{R}^3 \colon x_1 \geq v(1), x_2 \geq v(2), x_3 = v(3), x_1 + x_2 = v(1, 2)\right\}. \qquad (19.15)$$

A necessary condition for Player 1 to fail to have a justified objection against Player 2 at $x \in X(\mathcal{B}; v)$, is that at least one of the following expressions holds:

$$\boxed{x_1 + x_3 \geq v(1, 3)}\boxed{x_2 = v(2)}\boxed{v(1, 3) - x_1 \leq v(2, 3) - x_2} \qquad (19.16)$$

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The reason for this is as follows. At every objection, Player 1 must join a coalition with Player 3. If the left-hand inequality in (19.16) holds, then Player 2 cannot join an objection of Player 1, since Player 1 cannot offer him more than he gets under x. If the equality in Equation (19.16) holds, then for every objection of Player 1 against Player 2, Player 2 has a counterobjection $(\{2\}, v(2))$. If the right-hand inequality in Equation (19.16) holds, then for every objection of Player 1 that includes Player 3, Player 2 can offer Player 3 more in a counterobjection. For every $x \in X(\mathcal{B}; v)$ at which at least one of these expressions holds, Player 1 has no justified objection against Player 2. The opposite also holds: if neither of these inequalities hold, Player 1 has a justified objection against Player 2 (Exercise 19.5).

Theorem 19.15 *In a three-player game $(N; v)$, the bargaining set relative to the coalitional structure $\mathcal{B} = \{\{1, 2\}, \{3\}\}$ is the set of vector payoffs in $X(\mathcal{B}; v)$ satisfying at least one expression in each line of the following table. (each line has two numbers in the form $k \to l$ on the left-hand side, which express that that line contains the conditions that must be met for player k to fail to have a justified objection against player l).*

$1 \to 2$	$x_1 + x_3 \geq v(\{1, 3\})$	$x_2 = v(2)$	$v(\{1, 3\}) - x_1 \leq v(\{2, 3\}) - x_2$
$2 \to 1$	$x_2 + x_3 \geq v(\{2, 3\})$	$x_1 = v(1)$	$v(\{2, 3\}) - x_2 \leq v(\{1, 3\}) - x_1$

(19.17)

A larger system is needed for the coalitional structure $\{N\}$. A payoff vector $x \in X(\{N\}; v)$ is in the bargaining set relative to the coalitional structure $\{N\}$ if and only if it satisfies at least one expression in each line in the following system (as before, each line is specified by $l \leftarrow k$ and contains the conditions necessary for player k to fail to have a justified objection against player l):

$1 \to 2$	$x_1 + x_3 \geq v(1, 3)$	$x_2 = v(2)$	$v(1, 3) - x_1 \leq v(2, 3) - x_2$
$2 \to 1$	$x_2 + x_3 \geq v(2, 3)$	$x_1 = v(1)$	$v(2, 3) - x_2 \leq v(1, 3) - x_1$
$1 \to 3$	$x_1 + x_2 \geq v(1, 2)$	$x_3 = v(3)$	$v(1, 2) - x_1 \leq v(2, 3) - x_3$
$3 \to 1$	$x_2 + x_3 \geq v(2, 3)$	$x_1 = v(1)$	$v(2, 3) - x_3 \leq v(1, 2) - x_1$
$2 \to 3$	$x_1 + x_2 \geq v(1, 2)$	$x_3 = v(3)$	$v(1, 2) - x_2 \leq v(1, 3) - x_3$
$3 \to 2$	$x_1 + x_3 \geq v(1, 3)$	$x_2 = v(2)$	$v(1, 3) - x_3 \leq v(1, 2) - x_2$

(19.18)

Checking whether a particular vector x is in the bargaining set is relatively easy to do; it involves checking that at most 22 conditions are met (explain why). Computing the entire bargaining set requires solving 3^6 systems each with 10 linear equations (equalities and inequalities; explain why). The following theorem, whose proof is left to the reader (Exercise 19.11), shows that when a three-player game is 0-monotonic[2] computing the bargaining set is simplified.

Theorem 19.16 *In a 0-monotonic, three-player game, the bargaining set for the coalitional structure $\{N\}$ coincides with the core, if the core is nonempty, and it contains only one imputation, if the core is empty.*

Computing the bargaining set for a three-player game that is not 0-monotonic is a problem with high computational complexity. This complexity is vastly increased when

2 A coalitional game is 0-*mononotonic* if its 0-normalization is monotonic, or, equivalently, if $v(S \cup \{i\}) \geq v(S) + v(i)$ for every coalition S and every player $i \notin S$ (Exercise 16.27 on page 683).

there are four or more players, because then for every objection of player k against player l we need to list the conditions negating the possibility of the formation of a counterobjection for each subset $N \setminus \{k, l\}$. For example, to determine if a particular point x is in the bargaining set for the coalitional structure $\{N\}$ in a four-player game, it suffices to check 197 simple inequalities. But computing all the points in the bargaining set directly requires solving 150^{12} systems, each one containing 41 inequalities. To date there is no known practical method for computing the bargaining set in games with a large number of players, which is why indirect methods are often used to compute the bargaining set for specific families of games.

The set of solutions for a system of weak linear inequalities is an intersection of a finite number of half-spaces and is therefore a closed and convex set. The bargaining set is contained in the set of vector payoffs, which is a compact set. This leads to the following theorem, whose proof is left to the reader (Exercise 19.21).

Theorem 19.17 *For every coalitional game $(N; v)$ and every coalitional structure $\mathcal{B}$, the bargaining set $\mathcal{M}(N; v; \mathcal{B})$ is a finite union of polytopes.*[3]

Another property of the bargaining set that is a corollary of our discussion so far is:

Theorem 19.18 *When the values of the coalitional functions are all in a subfield of the real numbers (such as the field of the rational numbers), then all the vertices of the polytopes determining the bargaining set have coordinates in that subfield.*

The reason this theorem holds is that the operations required to solve a finite system of linear equations are multiplying and dividing the coefficients in the equations, and application of those operators always yields results in the same field in which the coefficients are located.

The main significance of the bargaining set is that it may be regarded as a set of suggested ways of dividing the worth of the coalition among its members, which is applicable also in cases in which the core is empty, because, as the next theorem states, the bargaining set is not empty provided that the set of imputation is not empty.

Theorem 19.19 *For every game in coalitional form $(N; v)$ and every coalitional*[4] *structure $\mathcal{B}$, if $X(\mathcal{B}; v) \neq \emptyset$, then $\mathcal{M}(N; v; \mathcal{B}) \neq \emptyset$.*

We will prove this theorem for the coalitional structure $\{N\}$. The proof of the general case appears in the chapter on the nucleolus (Theorem 20.21 on page 813), where we show that the nucleolus is always contained in the bargaining set, and the nonemptiness of the nucleolus follows from simple considerations involving continuous functions defined on compact sets.

The proof presented here is important both because of its contribution to the understanding of the structure of the bargaining set, and because it can be extended to a proof of the nonemptiness of the bargaining set in games without transferable utility, where the proof using the nucleolus is not applicable. The proof makes use of the following definition and two theorems.

3 Recall that a polytope is a compact set that is the intersection of a finite number of half-spaces.

4 Since by definition $\mathcal{M}(N; v; \mathcal{B})$ is contained in $X(\mathcal{B}; v)$, if $X(\mathcal{B}; v) = \emptyset$, then $\mathcal{M}(N; v; \mathcal{B}) = \emptyset$.

Definition 19.20 *Let $(N; v)$ be a coalitional game, let k and l be two players, and let $x \in X(N; v)$. We say that player k is* stronger than player l at x*, denoted by $k \succ_x l$, if player k has a justified objection against player l at x (relative to the coalitional structure $\{N\}$).*

The following example shows that the "stronger than" relation is not necessarily transitive.

Example 19.21 Let $(N; v)$ be a coalitional game where $N = \{1, 2, 3, 4, 5\}$, and the coalitional function is

$$v(1, 2, 3, 4, 5) = v(1, 2, 3) = v(2, 4, 5) = 30, \tag{19.19}$$

$$v(1, 4) = 40, \tag{19.20}$$

$$v(3, 5) = 20, \tag{19.21}$$

$$v(S) = 0, \quad \text{for every other coalition } S. \tag{19.22}$$

Consider the imputation $x = (10, 10, 10, 0, 0) \in X(N; v)$. At x:

- $(\{1, 4\}, (11, 0, 0, 29, 0))$ is a justified objection of Player 1 against Player 2 (check!). It follows that $1 \succ_x 2$.
- $(\{2, 4, 5\}, (0, 11, 0, 1, 18))$ is a justified objection of Player 2 against Player 3 (check!). It follows that $2 \succ_x 3$.
- We now show that $1 \not\succ_x 3$, leading to the conclusion that the "stronger than" relation is not transitive. The only way that Player 1 can raise an objection against Player 3 involves the coalition $\{1, 4\}$. But for every such objection, Player 3 can respond with a counterobjection $(0, 0, 10, 0, 10)$, with coalition $\{3, 5\}$.

The last item also shows that $3 \not\succ_x 1$. This indicates that although the "stronger than" relation is not transitive, it is likely to be acyclic, and in fact this is true, as stated in the next theorem. ◀

Theorem 19.22 *The relation $\succ_x$ is acyclic, i.e., if $1 \succ_x 2 \succ_x 3 \succ_x \ldots \succ_x t-1 \succ_x t$, then it $t \succ_x 1$ does not hold.*

Proof: Suppose by contradiction that $1 \succ_x 2 \succ_x 3 \succ_x \ldots \succ_x t-1 \succ_x t$ and $t \succ_x 1$. Suppose that the justified objections used in the above sequence involve coalitions $S_1, S_2, \ldots, S_t$, respectively; i.e., S_i is the coalition that player i uses for his justified objection against player $i+1$, for $i = 1, 2, \ldots, t-1$, and S_t is the coalition that player t uses for his objection against Player 1.

Consider the excesses $e(S_1, x), e(S_2, x), \ldots, e(S_t, x)$. Since the coalitions $S_1, S_2, \ldots, S_t$ are used in justified objections, these excesses must be positive. Relabel the names of the players in such a way that $e(S_t, x)$ is the maximal excess among these excesses (there may be several indices i at which this maximum is attained). Then coalition S_t cannot include Player 1 as a member, since it is used in an objection against him.

We show by induction that the coalition S_t includes players $\{1, 2, \ldots, t\}$. This contradicts the fact, just mentioned, that it cannot include Player 1. The contradiction establishes what we wanted to show, namely, that $\succ_x$ is an acyclic relation.

Since S_t is used in an objection of player t, it must include that player as a member. Suppose by induction that S_t contains players $\{i+1, i+2, \ldots, t\}$. We show that it must

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include player i as well. To do so, we make use of Theorem 19.14 on page 787: the coalition S_i is used in a justified objection of player i against player $i+1$ and by the inductive hypothesis the coalition S_t contains player $i+1$. If $i \notin S_t$ were to hold, it would follow from Theorem 19.14 that $e(S_t, x) < e(S_i, x)$ in contradiction to the assumption that $e(S_t, x)$ is the maximal excess among the excesses $(e(S_j, x))_{j=1}^t$. Hence $i \in S_t$, completing the inductive step of the proof and the proof of the theorem. □

The next theorem is a corollary of an important theorem in topology known as the KKM Theorem (after Knaster–Kuratowski–Mazurkiewicz). A proof of this theorem appears in Section 23.1.4 (page 941). A different proof, using Brouwer's Fixed Point Theorem, is given as a guided exercise (Exercise 23.31 page 953).

Theorem 19.23 (KKM) *Consider the* $(n-1)$*-dimensional simplex*

$$X(n) = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1, x_i \geq 0 \quad i = 1, \ldots, n \right\}. \tag{19.23}$$

Let $X_1, X_2, \ldots, X_n$ *be compact subsets of* $X(n)$ *satisfying*

$$\{x \in X(n) : x_i = 0\} \subseteq X_i, \quad i = 1, \ldots, n, \tag{19.24}$$

and whose union is $X(n)$*:*

$$\bigcup_{i=1}^n X_i = X(n). \tag{19.25}$$

Then their intersection is nonempty:

$$\bigcap_{i=1}^n X_i \neq \emptyset. \tag{19.26}$$

Recall that we are assuming in this section that the coalitional structure is $\{N\}$. For every pair of players k and l, denote by Y_{kl} the set of vector payoffs at which player k has a justified objection against player l:

$$Y_{kl} := \{x \in X(N; v) : k \succ_x l\}. \tag{19.27}$$

Theorem 19.24 *The set* Y_{kl} *is relatively open*[5] *in* $X(N; v)$*.*

Proof: Let $x \in Y_{kl}$. We will show that every imputation located in a small neighborhood of x is also in Y_{kl}. Towards that goal we will prove that if (C, y) is a justified objection of player k against player l at x then (C, y) is also a justified objection of player k against player l at every payoff vector in a sufficiently small neighborhood of x.

We will first prove that (C, y) is an objection of player k against player l at each payoff vector in a sufficiently small neighborhood of x. Denote $\delta := \min_{i \in C}(y_i - x_i)$. Since (C, y) is a justified objection of player k against player l at x, it is an objection and therefore $\delta > 0$. By Definition 19.3, (C, y) is an objection of player k against player l at every payoff vector $\widehat{x}$ satisfying $|\widehat{x}_i - x_i| < \delta$ for all $i \in C$.

5 That is, Y_{kl} is the intersection of an open set in $\mathbb{R}^N$ with $X(N; v)$.

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We now prove that there exists a sufficiently small neighborhood of x satisfying the property that at every vector in that neighborhood (C, y) is a justified objection of player k against player l. Suppose by contradiction that this is not true. Then there exists a sequence $(x^m)_{m\in\mathbb{N}}$ of payoff vectors converging to x such that (C, y) is not a justified objection of player k against player l at x^m, for all $m \in \mathbb{N}$. In other words, for every $m \in \mathbb{N}$ there exists a counterobjection (D^m, z^m) of player l against player k at x^m to the objection (C, y):

- $l \in D^m$, $k \notin D^m$ and $z^m(D^m) = v(D^m)$.
- $z^m_i \geq x^m_i$ for all $i \in D^m \setminus C$.
- $z^m_i \geq y_i$ for all $i \in D^m \cap C$.

Since the number of coalitions is finite, there is a coalition D^* appearing an infinite number of times in the sequence $(D^m)_{m\in\mathbb{N}}$. The set of payoff vectors is compact and therefore every subsequence of the sequence (z^m) has a subsequence converging to the limit z^*. By taking the limit in the subsequence we deduce that:

- $l \in D^*$, $k \notin D^*$ and $z^*(D^*) = v(D^*)$.
- $z^*_i \geq x^*_i$ for all $i \in D^* \setminus C$.
- $z^*_i \geq y_i$ for all $i \in D^* \cap C$.

It follows that (D^*, z^*) is a counterobjection of player l against player k at x to the objection (C, y), and therefore (C, y) is not a justified objection, contradicting our assumption. This contradiction proves that the set Y_{kl} is a relatively open set in $X(N; v)$. □

We turn now to the proof of Theorem 19.19, which states that for the coalitional structure $\{N\}$ the bargaining set is not empty provided that the set of imputations is not empty.

Proof of Theorem 19.19 in the case that $\mathcal{B} = \{N\}$: The bargaining set is covariant under strategic equivalence (Exercise 19.6). By Theorem 16.7 (page 670) we may assume without loss of generality that the game is $0-0$, $0-1$, or $0-(-1)$ normalized.

We will first deal with the interesting case in which the game is $0-1$ normalized, and then treat the other two cases, where the proof is rather simple.

When the game is $0-1$ normalized, the set of imputations for the coalitional structure $\{N\}$ is

$$X(N; v) = \left\{ x \in \mathbb{R}^N : \sum_{i=1}^n x_i = 1, x_i \geq 0 \quad i = 1, \ldots, n \right\}. \tag{19.28}$$

This is an $(n-1)$-dimensional simplex. Define n subsets of the simplex by:

$$X_i := \{x \in X(N; v) : \text{there is no justified objection against player } i \text{ at } x\}. \tag{19.29}$$

Claim 19.25 *X_i contains the set $\{x \in X(N; v) : x_i = 0\}$, for all $i = 1, 2, \ldots, n$.*

Proof: If $x_i = 0$, the pair $(\{i\}, 0)$ constitutes a counterobjection of player i to any objection raised against him. □

Claim 19.26 $\cup_{i=1}^n X_i = X(N; v)$.

Proof: Suppose by contradiction that there exists a point x in $X(N; v)$ that is not in $\cup_{i=1}^n X_i$. Let j be any player. Since $x \notin X_j$, at x there is a player k_j who has a justified objection against j, that is, $k_j \succ_x j$. This holds for every player j, and in particular, there is a player j_1 such that $j_1 \succ_x 1$, there is a player j_2 such that $j_2 \succ_x j_1$, there is a player j_3 such that $j_3 \succ_x j_2$, and so on. Since the number of players is finite, the sequence $(j_m)_{m=1}^{\infty}$ must contain a player who appears at least twice; i.e., there exist m and l satisfying $j_m = j_{m+l}$. In particular,

$$j_m \succ_x j_{m+1} \succ_x j_{m+2} \succ_x \cdots \succ_x j_{m+l-1} \succ_x j_{m+l} = j_m.$$

The existence of such a sequence contradicts the fact that the relation $\succ_x$ is acyclic (Theorem 19.22). The contradiction proves that our starting assumption was false, and hence $\cup_{i=1}^{n} X_i = X(N; v)$. □

Claim 19.27 *For every i, the set X_i is closed.*

Proof: X_i is a set of imputations at which there are no justified objections against player i. The complement $X(N; v) \setminus X_i$ is therefore the set of imputations at which at least one player has a justified objection against player i; i.e.,

$$X(N; v) \setminus X_i = \bigcup_{\{k:\, k \neq i\}} Y_{ki}. \tag{19.30}$$

By Theorem 19.24, the sets $(X_{ki})_{i \in N}$ are all relatively open in $X(N; v)$. Therefore $X(N; v) \setminus X_i$, as the union of relatively open sets, is itself relatively open in $X(N; v)$, and thus its complement $X_i$is relatively closed in $X(N; v)$. Since the set $X(N; v)$ is closed in $\mathbb{R}^N$ it follows that X_i is also a closed set in $\mathbb{R}^N$, as claimed. □

These three claims show that the sets $\{X_1, X_2, \ldots, X_n\}$ satisfy the conditions of the KKM Theorem (Theorem 19.23), and their intersection is therefore nonempty. This intersection is the bargaining set,

$$\bigcap_{i=1}^{n} X_i = \mathcal{M}(N; v; \{N\}), \tag{19.31}$$

since $x \in \bigcap_{i=1}^{n} X_i$ if and only if no player has a justified objection at x against any other player. This completes the proof in the case that the game is $0-1$ normalized.

If the game is $0-0$ normalized, the only imputation in $X(\{N\}; v)$ is $(0, 0, \ldots, 0)$. This imputation is in the bargaining set, because at it there are no objections that can be raised by any player against any other player (explain why).

If the game is $0–(-1)$ normalized, the set $X(\{N\}; v)$ is empty, and the statement of the theorem holds vacuously. This completes the proof of Theorem 19.19 for the coalitional structure $\mathcal{B} = \{N\}$. □

19.4 The bargaining set in convex games

There are several classes of games in which it is known that the bargaining set for the coalitional structure $\{N\}$ coincides with the core. This property, when it holds, constitutes a strong recommendation for choosing a point in the core as the solution to such a game, because at every imputation that is not in the core, there is a player who has a justified

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objection against another player. We prove in this section that in convex games the core and the bargaining set coincide for the coalitional structure $\mathcal{B} = \{N\}$. Recall that for this coalition structure we use the short notation $\mathcal{C}(N; v)$ for $\mathcal{C}(N; v; \{N\})$. For the definition of a convex game see Definition 17.51 (page 717).

Theorem 19.28 *For every convex game* $(N; v)$

$$\mathcal{C}(N; v) = \mathcal{M}(N; v; \{N\}). \tag{19.32}$$

Proof: By Theorem 19.12 on page 786, we know that $\mathcal{C}(N; v) \subseteq \mathcal{M}(N; v; \{N\})$, and we therefore need to show that $\mathcal{C}(N; v) \supseteq \mathcal{M}(N; v; \{N\})$. Let $x \notin \mathcal{C}(N; v)$ be an imputation that is not in the core. We need to show that it is not in the bargaining set, i.e., there is a justified objection at x.

Since the vector x is fixed throughout the proof, we denote by $e(S) = e(S, x) = v(S) - x(S)$ the excess of the coalition S at x. The function e associates each coalition with a real number, the excess, and we can therefore view $(N; e)$ as a coalitional game, namely, the game in which the worth of each coalition is its excess at x. We first show that $(N; e)$ is a convex game. To see this, note that

$$e(A) + e(B) = v(A) + v(B) - x(A) - x(B) \tag{19.33}$$
$$\leq v(A \cup B) + v(A \cap B) - x(A \cup B) - x(A \cap B) \tag{19.34}$$
$$= e(A \cup B) + e(A \cap B). \tag{19.35}$$

The inequality in Equation (19.34) holds because $(N; v)$ is a convex game, and $x(A) + x(B) = x(A \cup B) + x(A \cap B)$. Next, define the game $(N; \widehat{e})$ to be the monotonic cover of the game $(N; e)$, that is,

$$\widehat{e}(S) = \max_{R \subseteq S} e(R). \tag{19.36}$$

We will show that $(N; \widehat{e})$ is also a convex game. In fact, the proof we provide is valid for any convex game and thus proves that the monotonic cover of any convex game is a convex game. Let S and T be two coalitions. Denote by R and R' the coalitions at which the maximum is attained in the definition of $\widehat{e}(S)$ and $\widehat{e}(T)$ respectively:

$$\widehat{e}(S) = \max_{P \subseteq S} e(P) : = e(R) \tag{19.37}$$
$$\widehat{e}(T) = \max_{P' \subseteq T} e(P') : = e(R'). \tag{19.38}$$

Then

$$\widehat{e}(S) + \widehat{e}(T) = e(R) + e(R'). \tag{19.39}$$

Since $(N; e)$ is a convex game,

$$e(R) + e(R') \leq e(R \cup R') + e(R \cap R'). \tag{19.40}$$

Since $R \cup R' \subseteq S \cup T$ and $R \cap R' \subseteq S \cap T$,

$$e(R \cup R') + e(R \cap R') \leq \max_{P \subseteq S \cup T} e(P) + \max_{P \subseteq S \cap T} e(P) = \widehat{e}(S \cup T) + \widehat{e}(S \cap T). \tag{19.41}$$

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Using Equations (19.39)–(19.41) we deduce that

$$\widehat{e}(S)+\widehat{e}(T)\leq\widehat{e}(S\cup T)+\widehat{e}(S\cap T). \tag{19.42}$$

Since this inequality holds for every S and T, the game $(N;\widehat{e})$ is indeed convex.

From among all the coalitions C that have maximal excess at x choose one, C^*, that is maximal with respect to set inclusion:

$$e(S)\leq e(C^*),\qquad\forall S\subseteq N, \tag{19.43}$$

$$e(S)< e(C^*),\qquad\forall S\text{ such that }C^*\subset S\subseteq N. \tag{19.44}$$

Since the imputation x is not in the core, there exists a coalition S such that $v(S)>x(S)$, i.e., $e(S)>0$. Since $e(C^*)$ is maximal,

$$\widehat{e}(C^*)=\max_{P\subseteq C^*}e(P)=e(C^*)>0. \tag{19.45}$$

Consider the game $(C^*;\widehat{e})$, which is the restriction of the game $(N;\widehat{e})$ to the players in C^*. This is a convex game, because it is a subgame of a convex game (Exercise 17.46 on page 743). In particular, its core is not empty (Theorem 17.55 on page 719). Let $y\in\mathcal{C}(C^*;\widehat{e})$ be an imputation in the core of this game. This imputation satisfies

$$y(C^*)=\widehat{e}(C^*)=e(C^*), \tag{19.46}$$

$$y(R)\geq\widehat{e}(R)\geq e(R),\quad\forall R\subset C^*. \tag{19.47}$$

In particular, for $R=\{i\}$,

$$y_i\geq\widehat{e}(i)=\max\{e(i),e(\emptyset)\}\geq e(\emptyset)=0. \tag{19.48}$$

In words, every imputation in the core of $(C^*;\widehat{e})$ is nonnegative (in all its coordinates).

We will now show that there is a player $k\in C^*$ for whom (C^*,y) is a justified objection against every player $l\notin C^*$. Since $y(C^*)=\widehat{e}(C^*)>0$, there is a player $k\in C^*$ such that $y_k>0$. By Equation (19.45) one has $e(C^*)>0=e(N)$, and therefore $C^*\neq N$. In particular, there exists a player $l\notin C^*$. We will show that player k has a justified objection against player l at x.

Let $\varepsilon>0$ be sufficiently small such that

$$y_k>(|C^*|-1)\varepsilon. \tag{19.49}$$

Define:

$$z_i=x_i+y_i+\varepsilon,\qquad\forall i\in C^*\setminus\{k\}, \tag{19.50}$$

$$z_k=x_k+y_k-(|C^*|-1)\varepsilon. \tag{19.51}$$

Then $z_i>x_i$ for every $i\in C^*$, and by Equation (19.46)

$$z(C^*)=x(C^*)+y(C^*)=x(C^*)+e(C^*)=v(C^*). \tag{19.52}$$

It follows that $(C^*;z)$ is an objection of player k against player l. We will show that this objection is justified. To do so, choose an arbitrary coalition D containing player l but not player k. For a counterobjection, player l must give each player i in $D\cap C^*$ at least z_i,

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and every player i in $D \setminus C^*$ at least x_i. But this is impossible, since

$$
\begin{aligned}
z(D \cap C^*) + x(D \setminus C^*) &= z(D \cap C^*) + x(D) - x(D \cap C^*) && (19.53)\\
&\geq y(D \cap C^*) + x(D) && (19.54)\\
&\geq \widehat{e}(D \cap C) + x(D) && (19.55)\\
&\geq e(D \cap C^*) + x(D) && (19.56)\\
&\geq e(D) + e(C^*) - e(D \cup C^*) + x(D) && (19.57)\\
&> e(D) + x(D) && (19.58)\\
&= v(D). && (19.59)
\end{aligned}
$$

Equation (19.54) holds by Equation (19.50) and since $k \notin D \cap C^*$, Equation (19.55) holds because $y \in \mathcal{C}(C^*; \widehat{e})$ and $\widehat{e}(S) \geq e(S)$ for every $S \subseteq C^*$, Equation (19.57) holds because $(C^*; e)$ is a convex game, and the inequality in Equation (19.58) holds because the choice of C^* necessarily implies that $e(C^*) > e(D \cup C^*)$ (since $D \cup C^* \supset C^*$, which follows from player l being contained in $D \cup C^*$ but not in C^*).

It follows that D cannot be used for a counterobjection. Since this is true for any coalition D containing player l but not player k, the objection (C^*, y) is justified. We have shown that the imputation x, which is not in the core, is also not in the bargaining set, and therefore $\mathcal{C}(N; v) \supseteq \mathcal{M}(N; v; \{N\})$, which is what we wanted to show. □

19.5 Discussion

The following iterative procedure, which identifies an imputation in the bargaining set, is due to Stearns [1968]. Let $(N; v)$ be a coalitional game such that $X(N; v) \neq \emptyset$. Start at an arbitrary imputation $x^0 \in X(N; v)$. If it is in the bargaining set, the procedure terminates successfully. Otherwise, at x^0 there exists a player who has a justified objection against another player. It can be readily checked that if player k_0 has a justified objection against player l_0, then there exists a minimal positive number $\delta_{k_0,l_0}(x^0)$ such that the transfer of $\delta_{k_0,l_0}(x^0)$ from the payoff of player l_0 to that of player k_0 yields an imputation in which player k_0 no longer has a justified objection against player l_0. Choose one of the justified objections at x^0, and implement such a transfer of payoffs. This leads to a new imputation x^1. Repeat the process on x^1: if there are justified objections at this imputation, choose one of them, say a justified objection of player k_1 against player l_1, and create a new imputation x^2 by transferring the sum $\delta_{k_1,l_1}(x^1)$ from the payoff of player l_1 to that of player k_1, and so on.

If the process terminates successfully after a finite number of steps, we have found an imputation in the bargaining set. It is possible, however, that the resulting sequence is infinite. In such a case it can be shown that the sequence converges, but not necessarily to an imputation in the bargaining set: for example, there may be Players 1, 2, and 3 who, throughout the iterative procedure, transfer smaller and smaller amounts of payoff between each other, but Player 4 has a justified objection against Player 5 at every step of the procedure that is never canceled by a transfer between them. If, however, we ensure that in the above-described procedure there are an infinite number of times at which the

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transfers that are implemented are those where $\delta_{k_m,l_m}(x^m)$ is maximal, then the sequence $(x^m)_{m=1}^\infty$ converges to an imputation in the bargaining set.

This can be viewed as a dynamic justification for our interpretation of a justified objection as a demand by one player to receive a "transfer payment" from another player.

19.6 Remarks

The first variant of the bargaining set was presented in Aumann and Maschler [1964]. The variant of the bargaining set presented in this chapter first appeared in Davis and Maschler [1967], who also proved that the bargaining set is nonempty relative to the coalitional structure $\{N\}$. Peleg [1967] generalized this result to any coalitional structure.

Exercise 19.19 is from Peleg and Sudhölter [2003], page 74, Example 4.1.19. Exercise 19.20 is a special case of a more general theorem proved in Solymosi [1999].

19.7 Exercises

19.1 Prove Theorem 19.5 on page 785: for the coalitional structure $\{N\}$ the core is the set of all imputations in $X(N; v)$ at which no player has an objection against any other player.

19.2 Prove Theorem 19.6 on page 785: for every coalitional structure $\mathcal{B}$, when the core relative to this coalitional structure is not empty, the core is the set of all imputations in $X(\mathcal{B}; v)$ at which no player has an objection against any other player.

19.3 In this exercise, we show that Theorem 19.6 does not hold without the condition that the core is nonempty. Let $(N; v)$ be a two-player coalitional game with payoff function

$$v(1) = v(2) = 0, \qquad v(1,2) = 1, \tag{19.60}$$

and let $\mathcal{B} = \{\{1\}, \{2\}\}$.

(a) Show that the core relative to the coalitional structure $\mathcal{B}$ is empty.
(b) Find an imputation in $X(\mathcal{B}; v)$ that is in the bargaining set $\mathcal{M}(N; v; \mathcal{B})$.

19.4 Prove that if player k has a justified objection against player l at x , then player l does not have a justified objection against player k at x.

19.5 Prove that in a three-player coalitional game $(N; v)$, if for an imputation x none of the three equations in (19.16) holds, then Player 1 has a justified objection against Player 2.

19.6 Prove that the bargaining set is covariant under strategic equivalence. In other words, if $(N; v)$ and $(N; w)$ are two coalitional games with the same set of players, and if there exist $a > 0$ and $b \in \mathbb{R}^N$ such that

$$w(S) = av(S) + b(S), \quad \forall S \subseteq N, \tag{19.61}$$

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then for every coalitional structure $\mathcal{B}$,

$$x \in \mathcal{M}(N; v; \mathcal{B}) \iff ax + b \in \mathcal{M}(N; w; \mathcal{B}). \tag{19.62}$$

19.7 Show that in Example 19.1 (page 782) for every coalitional structure the bargaining set contains a single payoff vector, which is the corresponding payoff vector in the table on page 787.

19.8 For $N = \{1, 2, 3\}$, compute the bargaining set of the coalitional game $(N; v)$ relative to the coalitional structure $\{\{1, 2\}, \{3\}\}$, for each of the following coalitional functions:

(a) $v(1) = v(2) = v(3) = 0, \quad v(1, 2) = 40, \quad v(1, 3) = 50, \quad v(2, 3) = 60,$
$v(1, 2, 3) = 100.$
(b) $v(1) = v(2) = v(3) = 0, \quad v(1, 2) = 80, \quad v(1, 3) = 20, \quad v(2, 3) = 30,$
$v(1, 2, 3) = 100.$

19.9 Repeat Exercise 19.8 for the coalitional structure $\{N\}$.

19.10 Compute the bargaining set of the game "My Aunt and I" presented in Exercise 20.21 on page 846 for the coalitional structure $\{\{1, 2\}, \{3\}, \{4\}, \{5\}\}$, where the Players 1 and 2 are my aunt and I, respectively.

19.11 Prove Theorem 19.16 on page 789: in a 0-monotonic three-player game, for the coalitional structure $\{N\}$, the bargaining set coincides with the core if the core is nonempty, and is a single point if the core is empty.

19.12 Prove that in Definition 19.7 (page 785), the inequality in Condition 4 can be replaced by a strict inequality. In other words, if at x player k has a justified objection against player l, then he has a justified objection against player l also if in the definition of a justified objection the inequalities in Condition 4 are replaced by strict inequalities. Note that the justified objection may be different, under the different definitions.

19.13 Give an example in which the bargaining set is empty if strict inequalities are required in Condition 3 of Definition 19.7 (page 785).

19.14 Let $N = \{1, 2, 3, 4\}$. Write down a list of conditions guaranteeing that in the coalitional structure $\{N\}$ Player 1 has no justified objection against Player 2.

Comment: The 12 permutations of this list of conditions, along with the requirement that the imputation be in $X(\{N\}; v)$, determine the bargaining set relative to this coalitional structure.

19.15 **The gloves game** Two sellers go to the market. Each has a left-hand glove. At the same time, three other sellers come to the market, each of whom has a right-hand glove. Only pairs of gloves, each pair containing a right-hand glove and a left-hand glove, can be sold to customers. The net profit from selling one pair of gloves is \$10.

(a) Write down this game's coalitional function.
(b) Compute the bargaining set of this game relative to the coalitional structure $\{N\}$.

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19.16 Consider the market game (as described in Exercise 16.12 on page 681) in which the initial endowment of each member of $N_1 = \{1, 2\}$ is $(1, 0)$, the initial endowment of each member of $N_2 = \{3, 4, 5\}$ is $(0, \frac{1}{2})$ and for each coalition S,

$$v(S) = \min\left\{|S \cap N_1|, \tfrac{1}{2}|S \cap N_2|\right\}. \tag{19.63}$$

(a) Prove that $(0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is the only vector in the core of this game relative to the coalitional structure $\mathcal{B} = \{N\}$.

(b) Prove that the bargaining set relative to the coalitional structure $\{N\}$ is

$$\left\{(\alpha, \alpha, \beta, \beta, \beta) \colon 0 \leq \alpha \leq \tfrac{3}{4}, \beta = \tfrac{1}{2} - \tfrac{2}{3}\alpha\right\}. \tag{19.64}$$

(c) Explain why points in the bargaining set that are not in the core may be more reasonable than points in the core, in this example.

19.17 Compute the core and the bargaining set relative to the coalitional structure $\{N\}$ in a game similar to the market game of Exercise 19.16, but where the initial endowment of the members of N_2 is $(0, 1)$. What is the relationship between the results obtained for market games in this exercise and Exercise 19.16, and the results obtained for the gloves game in Exercise 19.15.

19.18 Find a game $(N; v)$, a coalitional structure, and an imputation x such that: (a) $1 \succ_x 3$, (b) $1 \sim_x 2$, and (c) $2 \sim_x 3$, where the notation $k \sim_x l$ means that the players k and l are members of the same coalition in the coalitional structure, and neither of them has a justified objection against the other.

19.19 In the weighted majority game $[3; 1, 1, 1, 1, 1, 0]$ with six players, Player 6 is a null player. Prove that despite this, the vector $(\frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{2}{7})$ is in the bargaining set relative to the coalitional structure $\{N\}$. What is the intuitive explanation for the fact that a null player can obtain a positive payoff in the bargaining set?

19.20 A simple coalitional game $(N; v)$ is called a *veto control game* if there is a player i such that $v(S) = 0$ for every coalition that does not contain i. Prove that given a monotonic, veto control game, the core relative to the coalitional structure $\{N\}$ coincides with the bargaining set relative to the same coalitional structure (see also Exercise 17.12 on page 737).

19.21 Prove Theorem 19.17 (page 790): in a coalitional game, for every coalitional structure, the bargaining set is a finite union of polytopes.

19.22 The statement of Theorem 19.28 is formulated for the coalitional structure $\mathcal{B} = \{N\}$. Where was this used in the proof of the theorem?

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