**INDIAN INSTITUTE OF TECHNOLOGY, BOMBAY**

*Department of Mathematics*

*SI 427 (Probability Theory)*

**Tutorial Sheet-II**

1. Let $X$ be a constant function and $\mathcal{F}$ is a $\sigma$-field. Show that $X$ is a random variable with respect to $\mathcal{F}$.

2. Let $X$ be a constant function. Find $\sigma(X)$.

3. Let $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $\mathcal{F} = \{\emptyset, \{1, 2, 3, 4\}, \{5, 6, 7, 8, 9, 10\}, \Omega\}$. Define $X(\omega) = \omega^2, \omega \in \Omega$. Show that $X$ is not a random variable with respect to $\mathcal{F}$.

4. Let $\Omega$ be a sample space and $\mathcal{F}$ be a $\sigma$-field. Show that $I_A$ is a random variable with respect to $\mathcal{F}$ iff $A \in \mathcal{F}$.

5. Let $\Omega$ be a sample space and $\mathcal{F}$ be a $\sigma$-field. Then $X : \Omega \to \mathbb{R}$ is a random variable with respect to $\mathcal{F}$ iff $X^{-1}(O) \in \mathcal{F}$ for all open set $O$ of $\mathbb{R}$.

6. Let $X$ and $Y$ be random variables. Show that
   (i) $e^{XY}$ (ii) $X^+ := \max\{X, 0\}$ (iii) $|X + Y|$ are random variables.

7. Let $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $X$ is a random variable. Show that $f \circ X$ is a random variable.
   Hint: Under $f$, inverse images of open sets are open.

8. Let $X$ be random variable such that $X > 0$. Show that $\ln X$ is a random variable.

9. Let $X : \Omega \to \mathbb{R}$. Show that $\sigma(X)$ is the smallest $\sigma$-field with respect to which $X$ is a random variable.

10. Let $A$ and $B$ be mutually exclusive events in $(\Omega,\, \mathcal{F},\, P)$. Find $\sigma(I_A + I_B)$.

11. Let $\Omega$ be a sample space and $\mathcal{F}$ be a $\sigma$-field and $X, Y$ be random variables such that $\sigma(X) = \sigma(Y)$. Show that $\sigma(X + Y) \subseteq \sigma(X)$.

12. Let $\Omega$ be a sample space and $\mathcal{F}$ be a $\sigma$-field and $X$ be a random variable. Show that $\sigma(X^2) \subseteq \sigma(X)$.

13. Give an example of a random variable such that $\sigma(X^2) \neq \sigma(X)$ and also a random variable with $\sigma(X^2) = \sigma(X)$.

14. Let $\Omega$ and $X$ as in Q3. Is there exists a $\sigma$-field with respect to it, $X$ is a random variable? If so, find the smallest such $\sigma$-field.

15. Let $X_n, n \geq 1$ be a sequence of random variables on $(\Omega, \mathcal{F})$ such that there exists a constant $M$ such that $|X_n| \leq M$ for all $n \geq 1$. Show that $\limsup_{n\to\infty} X_n$ is a random variable with respect to $\mathcal{F}$.