Introduction

What is game theory?

Game theory is the name given to the methodology of using mathematical tools to model and analyze situations of interactive decision making. These are situations involving several decision makers (called *players*) with different goals, in which the decision of each affects the outcome for all the decision makers. This interactivity distinguishes game theory from standard decision theory, which involves a single decision maker, and it is its main focus. Game theory tries to predict the behavior of the players and sometimes also provides decision makers with suggestions regarding ways in which they can achieve their goals.

The foundations of game theory were laid down in the book *The Theory of Games and Economic Behavior*, published in 1944 by the mathematician John von Neumann and the economist Oskar Morgenstern. The theory has been developed extensively since then and today it has applications in a wide range of fields. The applicability of game theory is due to the fact that it is a context-free mathematical toolbox that can be used in any situation of interactive decision making. A partial list of fields where the theory is applied, along with examples of some questions that are studied within each field using game theory, includes:

- **Theoretical economics.** A market in which vendors sell items to buyers is an example of a game. Each vendor sets the price of the items that he or she wishes to sell, and each buyer decides from which vendor he or she will buy items and in what quantities. In models of markets, game theory attempts to predict the prices that will be set for the items along with the demand for each item, and to study the relationships between prices and demand. Another example of a game is an auction. Each participant in an auction determines the price that he or she will bid, with the item being sold to the highest bidder. In models of auctions, game theory is used to predict the bids submitted by the participants, the expected revenue of the seller, and how the expected revenue will change if a different auction method is used.
- **Networks.** The contemporary world is full of networks; the Internet and mobile telephone networks are two prominent examples. Each network user wishes to obtain the best possible service (for example, to send and receive the maximal amount of information in the shortest span of time over the Internet, or to conduct the highest-quality calls using a mobile telephone) at the lowest possible cost. A user has to choose an Internet service provider or a mobile telephone provider, where those providers are also players in the game, since they set the prices of the service they provide. Game theory tries to predict the behavior of all the participants in these markets. This game is more complicated from the perspective of the service providers than from the perspective

of the buyers, because the service providers can cooperate with each other (for example, mobile telephone providers can use each other's network infrastructure to carry communications in order to reduce costs), and game theory is used to predict which cooperative coalitions will be formed and suggests ways to determine a "fair" division of the profit of such cooperation among the participants.

- **Political science.** Political parties forming a governing coalition after parliamentary elections are playing a game whose outcome is the formation of a coalition that includes some of the parties. This coalition then divides government ministries and other elected offices, such as parliamentary speaker and committee chairmanships, among the members of the coalition. Game theory has developed indices measuring the power of each political party. These indices can predict or explain the division of government ministries and other elected offices given the results of the elections. Another branch of game theory suggests various voting methods and studies their properties.
- **Military applications.** A classical military application of game theory models a missile pursuing a fighter plane. What is the best missile pursuit strategy? What is the best strategy that the pilot of the plane can use to avoid being struck by the missile? Game theory has contributed to the field of defense the insight that the study of such situations requires strategic thinking: when coming to decide what you should do, put yourself in the place of your rival and think about what he/she would do and why, while taking into account that he/she is doing the same and knows that you are thinking strategically and that you are putting yourself in his/her place.
- **Inspection.** A broad family of problems from different fields can be described as two-player games in which one player is an entity that can profit by breaking the law and the other player is an "inspector" who monitors the behavior of the first player. One example of such a game is the activities of the International Atomic Energy Agency, in its role of enforcing the Treaty on the Non-Proliferation of Nuclear Weapons by inspecting the nuclear facilities of signatory countries. Additional examples include the enforcement of laws prohibiting drug smuggling, auditing of tax declarations by the tax authorities, and ticket inspections on public trains and buses.
- **Biology.** Plants and animals also play games. Evolution "determines" strategies that flowers use to attract insects for pollination and it "determines" strategies that the insects use to choose which flowers they will visit. Darwin's principle of the "survival of the fittest" states that only those organisms with the inherited properties that are best adapted to the environmental conditions in which they are located will survive. This principle can be explained by the notion of *Evolutionarily Stable Strategy*, which is a variant of the notion of *Nash equilibrium*, the most prominent game-theoretic concept. The introduction of game theory to biology in general and to evolutionary biology in particular explains, sometimes surprisingly well, various biological phenomena.

Game theory has applications to other fields as well. For example, to philosophy it contributes some insights into concepts related to morality and social justice, and it raises questions regarding human behavior in various situations that are of interest to psychology. Methodologically, game theory is intimately tied to mathematics: the study of game-theoretic models makes use of a variety of mathematical tools, from probability and combinatorics to differential equations and algebraic topology. Analyzing game-theoretic models sometimes requires developing new mathematical tools.

Traditionally, game theory is divided into two major subfields: strategic games, also called noncooperative games, and coalitional games, also called cooperative games. Broadly speaking, in strategic games the players act independently of each other, with each player trying to obtain the most desirable outcome given his or her preferences, while in coalitional games the same holds true with the stipulation that the players can agree on and sign binding contracts that enforce coordinated actions. Mechanisms enforcing such contracts include law courts and behavioral norms. Game theory does not deal with the quality or justification of these enforcement mechanisms; the cooperative game model simply assumes that such mechanisms exist and studies their consequences for the outcomes of the game.

The categories of strategic games and coalitional games are not well defined. In many cases interactive decision problems include aspects of both coalitional games and strategic games, and a complete theory of games should contain an amalgam of the elements of both types of models. Nevertheless, in a clear and focused introductory presentation of the main ideas of game theory it is convenient to stick to the traditional categorization. We will therefore present each of the two models, strategic games and coalitional games, separately. Chapters 1–14 are devoted to strategic games, and Chapters 15–20 are devoted to coalitional games. Chapters 21 and 22 are devoted to social choice and stable matching, which include aspects of both noncooperative and cooperative games.

How to use this book

The main objective of this book is to serve as an introductory textbook for the study of game theory at both the undergraduate and the graduate levels. A secondary goal is to serve as a reference book for students and scholars who are interested in an acquaintance with some basic or advanced topics of game theory. The number of introductory topics is large and different teachers may choose to teach different topics in introductory courses. We have therefore composed the book as a collection of chapters that are, to a large extent, independent of each other, enabling teachers to use any combination of the chapters as the basis for a course tailored to their individual taste. To help teachers plan a course, we have included an abstract at the beginning of each chapter that presents its content in a short and concise manner.

Each chapter begins with the basic concepts and eventually goes farther than what may be termed the "necessary minimum" in the subject that it covers. Most chapters include, in addition to introductory concepts, material that is appropriate for advanced courses. This gives teachers the option of teaching only the necessary minimum, presenting deeper material, or asking students to complement classroom lectures with independent readings or guided seminar presentations. We could not, of course, include all known results of game theory in one textbook, and therefore the end of each chapter contains references to other books and journal articles in which the interested reader can find more material for a deeper understanding of the subject. Each chapter also contains exercises, many of which are relatively easy, while some are more advanced and challenging.

This book was composed by mathematicians; the writing is therefore mathematically oriented, and every theorem in the book is presented with a proof. Nevertheless, an effort has been made to make the material clear and transparent, and every concept is illustrated with examples intended to impart as much intuition and motivation as possible. The book is appropriate for teaching undergraduate and graduate students in mathematics, computer science and exact sciences, economics and social sciences, engineering, and life sciences. It can be used as a textbook for teaching different courses in game theory, depending on the level of the students, the time available to the teacher, and the specific subject of the course. For example, it could be used in introductory level or advanced level semester courses on coalitional games, strategic games, a general course in game theory, or a course on applications of game theory. It could also be used for advanced mini-courses on, e.g., incomplete information (Chapters 9, 10, and 11), auctions (Chapter 12), or repeated games (Chapters 13 and 14). As mentioned previously, the material in the chapters of the book will in many cases encompass more than a teacher would choose to teach in a single course. This requires teachers to choose carefully which chapters to teach and which parts to cover in each chapter. For example, the material on strategic games (Chapters 4 and 5) can be taught without covering extensive-form games (Chapter 3) or utility theory (Chapter 2). Similarly, the material on games with incomplete information (Chapter 9) can be taught without teaching the other two chapters on models of incomplete information (Chapters 10 and 11).

For the sake of completeness, we have included an appendix containing the proofs of some theorems used throughout the book, including Brouwer's Fixed Point Theorem, Kakutani's Fixed Point Theorem, the Knaster–Kuratowski–Mazurkiewicz (KKM) Theorem, and the separating hyperplane theorem. The appendix also contains a brief survey of linear programming. A teacher can choose to prove each of these theorems in class, assign the proofs of the theorems as independent reading to the students, or state any of the theorems without proof based on the assumption that students will see the proofs in other courses.