9 Games with incomplete information and common priors

Chapter summary

In this chapter we study situations in which players do not have complete information on the environment they face. Due to the interactive nature of the game, modeling such situations involves not only the *knowledge* and *beliefs* of the players, but also the whole *hierarchy of knowledge* of each player, that is, knowledge of the knowledge of the other players, knowledge of the knowledge of the other players of the knowledge of other players, and so on. When the players have beliefs (i.e. probability distributions) on the unknown parameters that define the game, we similarly run into the need to consider *infinite hierarchies of beliefs*. The challenge of the theory was to incorporate these infinite hierarchies of knowledge and beliefs in a workable model.

We start by presenting the Aumann model of incomplete information, which models the knowledge of the players regarding the payoff-relevant parameters in the situation that they face. We define the *knowledge operator*, the concept of *common knowledge*, and characterize the collection of events that are common knowledge among the players.

We then add to the model the notion of belief and prove Aumann's agreement theorem: it cannot be common knowledge among the players that they disagree about the probability of a certain event.

An equivalent model to the Aumann model of incomplete information is a *Harsanyi game with incomplete information*. After presenting the game, we define two notions of equilibrium: the Nash equilibrium corresponding to the *ex ante* stage, before players receive information on the game they face, and the Bayesian equilibrium corresponding to the *interim* stage, after the players have received information. We prove that in a Harsanyi game these two concepts are equivalent.

Finally, using games with incomplete information, we present Harsanyi's interpretation of mixed strategies.

As we have seen, a very large number of real-life situations can be modeled and analyzed using extensive-form and strategic-form games. Yet, as Example 9.1 shows, there are situations that cannot be modeled using those tools alone.

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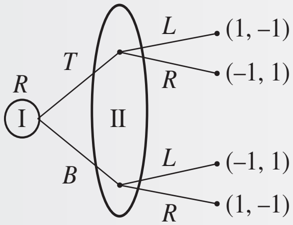
Example 9.1 Consider the Matching Pennies game, which is depicted in Figure 9.1 in both extensive form and strategic form.



		Player II	
		L	R
Player I	T	1, –1	–1, 1
	B	–1, 1	1, –1

Figure 9.1 The game of Matching Pennies, in extensive form and strategic form

Suppose that Player I knows that he is playing Matching Pennies, but believes that Player II does not know that the pure strategy R is available to her. In other words, Player I believes that Player II is convinced that she has only one pure strategy, L. Suppose further that Player II does in fact know that she (Player II) is playing Matching Pennies, with both pure strategies available. How can we model this game? Neither the extensive-form nor the strategic-form descriptions of the game enable us to model such a state of players' knowledge and beliefs. If we try to analyze this situation using only the depictions of the game appearing in Figure 9.1, we will not be able to predict how the players will play, or recommend an optimal course of action.

For example, as we showed on page 52, the optimal strategy of Player I playing Matching Pennies is the mixed strategy $[\frac{1}{2}(T), \frac{1}{2}(B)]$. But in the situation we have just described, Player I believes that Player II will play L, so that his best reply is the pure strategy T.

Note that Player I's optimal strategy depends only on how he perceives the game: what he knows about the game and what he believes Player II knows about the game. The way that Player II really perceives the game (which is not necessarily known to Player I) has no effect on the strategy chosen by Player I.

Consider next a slightly more complicated situation, in which Player I knows that he is playing Matching Pennies, he believes that Player II knows that she is playing Matching Pennies, and he believes that Player II believes that Player I does not know that the pure strategy B is available to him. Then Player I will believe that Player II believes that Player I will play strategy T, and he will therefore conclude that Player II will select strategy R, and Player I's best strategy will therefore be B.

A similar situation obtains if there is incomplete information regarding some of the payoffs. For example, suppose that Player I knows that his payoff under the strategy profile (T, L) is 5 rather than 1, but believes that Player II does not know this, and that she thinks the payoff is 1. How should Player I play in this situation? Or consider an even more complicated situation, in which both Player I and Player II know that Player I's payoff under (T, L) is 5, but Player II believes Player I does not know that she (Player II) knows this; Player II believes Player I believes Player II thinks the payoff is 1. ◄

Situations like those described in Example 9.1, in which players do not necessarily know which game is being played, or are uncertain about whether the other players know which game is being played, or are uncertain whether the other players know whether the other players know which game is being played, and so on, are called situations of

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"incomplete information." In this chapter we study such situations, and see how they can be modeled and analyzed as games.

Notice that neither of the situations described in Example 9.1 is well defined, as we have not precisely defined what the players know. For example, in the second case we did not specify what Player I knows about what Player II knows about what Player I knows about what Player II knows, and we did not touch upon what Player II knows. Consideration of hierarchies of levels of knowledge leads to the concept of common knowledge, which we touched upon in Section 4.5 (page 87). An informal definition of common knowledge is:

Definition 9.2 *A fact F is* common knowledge *among the players of a game if all the players know F, all the players know that all the players know F, all the players know that all the players know that all the players know F, and so on (for every finite number of levels).*[1]

Definition 9.2 is incomplete, because we have not yet defined what we mean by a "fact," nor have we defined the significance of the expression "knowing a fact." These concepts will be modeled formally later in this chapter, but for now we will continue with an informal exposition.

So far we have seen that in situations involving several players, incomplete knowledge of the game that is being played leads us to consider infinite hierarchies of knowledge. In decision-making situations with incomplete information, describing the information that decision makers have usually cannot be captured by labeling a given fact as "known" or "unknown." Decision makers often have assessments or beliefs about the truthfulness of various facts. For example, when a person takes out a variable-rate loan he never has precise knowledge of the future fluctuations of the interest rate (which can significantly affect the total amount of loan repayment), but he may have certain beliefs about future rates, such as "I assign probability 0.7 to the event that there will be lower interest rates over the term of the loan." To take another example, a company bidding for oil exploration rights in a certain geographical location has beliefs about the amount of oil likely to be found there and the depth of drilling required (which affects costs and therefore expected profits). A trial jury passing judgment on a defendant expresses certain collective beliefs about the question: is the defendant guilty as charged? For our purposes in this chapter, the source of such probabilistic assessments is of no importance. The assessments may be based on "objective" measurements such as geological surveys (as in the oil exploration example), on impressions (as in the case of a jury deliberating the judgment it will render in a trial), or on personal hunches and information published in the media (as in the example of the variable-rate loan). Thus, probability assessments may be objective or subjective.[2] In our models, a decision maker's beliefs will be expressed by a probability distribution function over the possible values of parameters unknown to him.

1 A simple example of a fact that is common knowledge is a *public* event: when a teacher is standing before a class, that fact is common knowledge among the students, because every student knows that every student knows . . . that the teacher is standing before the class.

2 A formal model for deriving an individual's subjective probability from his preferences was first put forward by Savage [1954], and later by Anscombe and Aumann [1963] (see also Section 2.8 on page 26).

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The most widely accepted statistical approach for dealing with decision problems in situations of incomplete information is the *Bayesian approach*.[3] In the Bayesian approach, every decision maker has a probability distribution over parameters that are unknown to him, and he chooses his actions based on his beliefs as expressed by that distribution. When several decision makers (or players) interact, knowing the probability distribution (beliefs) of each individual decision maker is insufficient: we also need to know what each one's beliefs are about the beliefs of the other decision makers, what they believe about his beliefs about the others' beliefs, and so on. This point is illustrated by the following example.

Example 9.1 (*Continued*) Returning to the Matching Pennies example, suppose that Player I attributes probability p_1 to the event: "Player II knows that R is a possible action." The action that Player I will choose clearly depends on p_1, because the entire situation hinges on the value of p_1: if $p_1 = 1$, Player I believes that Player II knows that R is an action available to her, and if $p_1 = 0$, he believes that Player II does not know that R is possible at all. If $0 < p_1 < 1$, Player I believes that it is possible that Player II knows that R is an available strategy. But the action chosen by Player I also depends on his beliefs about the beliefs of Player II: because Player I's action depends on p_1, it follows that Player II's action depends on her beliefs about p_1, namely, on her beliefs about Player I's beliefs. By the same reasoning, Player I's action depends on his beliefs about Player II's beliefs about his own beliefs, p_1. As in the case of hierarchy of knowledge, we see that determining the best course of action of a Player requires considering an infinite hierarchy of beliefs. ◀

Adding beliefs to our model is a natural step, but it leads us to an infinite hierarchy of beliefs. The concepts of knowledge and of beliefs are closely intertwined in games of incomplete information. For didactic reasons, however, we will treat the two notions separately, considering first hierarchies of knowledge and then hierarchies of beliefs.

9.1 The Aumann model of incomplete information and the concept of knowledge

In this section we will provide a formal definition of the concept of "knowledge," and then construct hierarchies of knowledge: what each player knows about what the other players know. We will start with an example to illustrate the basic elements of the model.

Example 9.3 Assume that racing cars are produced in three possible colors: gold, red, and purple. Color-blind individuals cannot distinguish between red and gold. Everyone knows that John is color-blind, but no one except Paul knows whether or not Paul is color-blind too. John and Paul are standing side by side viewing a photograph of the racing car that has just won first prize in the Grand Prix, and asking themselves what color it is. The parameter that is of interest in this example is the color of

3 The Bayesian approach is named after Thomas Bayes, 1702–1761, a British clergyman and mathematician who formulated a special case of the rule now known as Bayes' rule.

the car, which will later be called the *state of nature*, and we wish to describe the knowledge that the players possess regarding this parameter.

If the color of the car is purple, then both color-blind and non-color-blind individuals know that fact, so that both John and Paul know that the car is purple, and each of them knows that the other knows that the car is purple. If, however, the car is red or gold, then John knows that it is either red or gold. As he does not know whether or not Paul is color-blind, he does not know whether Paul knows the exact color of the car. Because Paul knows that John is color-blind, if the car is red or gold he knows that John does not know what the precise color is, and John knows that Paul knows this.

We therefore need to consider six distinct possibilities (three possibilities per car color times two possibilities regarding whether or not Paul is color-blind):

- The car is purple and Paul is not color-blind. John and Paul both know that the car is purple, they each know that the other knows that the car is purple, and so on.
- The car is purple and Paul is color-blind. Here, too, John and Paul both know that the car is purple, they each know that the other knows that the car is purple, and so on.
- The car is red and Paul is not color-blind. Paul knows the car is red; John knows that the car is red or gold; John does not know whether or not Paul knows the color of the car.
- The car is gold and Paul is not color-blind. Paul knows the car is gold; John knows that the car is red or gold; John does not know whether or not Paul knows the color of the car.
- The car is red and Paul is color-blind. Paul and John know that the car is red or gold; John does not know whether or not Paul knows the color of the car.
- The car is gold and Paul is color-blind. Paul and John know that the car is red or gold; John does not know whether or not Paul knows the color of the car.

In each of these possibilities, both John and Paul clearly know more than we have explicitly written above. For example, in the latter four situations, Paul knows that John does not know whether Paul knows the color of the car. Each of the six cases is associated with what will be defined below as a *state of the world*, which is a description of a state of nature (in this case, the color of the car) and the state of knowledge of the players. Note that the first two cases describe the same state of the world, because the difference between them (Paul's color-blindness) affects neither the color of the car, which is the parameter that is of interest to us, nor the knowledge of the players regarding the color of the car. ◀

The definition of the set of states of nature depends on the situation that we are analyzing. In Example 9.3 the color of the car was the focus of our interest – perhaps, for example, because a bet has been made regarding the color. Since the most relevant parameters in a game are the payoffs, in general we will want the states of nature to describe all the parameters that affect the payoffs of the players (these are therefore also called "payoff-relevant parameters"). For instance, if in Example 9.3 we were in a situation in which Paul's color-blindness (or lack thereof) were to affect his utility, then color-blindness would be a payoff-relevant parameter and would comprise a part of the description of the state of nature. In such a model there would be six distinct states of nature, rather than three.

Definition 9.4 *Let S be a finite set of states of nature. An* Aumann model of incomplete information *(over the set S of states of nature) consists of four components $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s})$, where:*

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- *N is a finite set of players;*
- *Y is a finite set of elements called* states of the world;[4]
- *$\mathcal{F}_i$ is a partition of Y, for each $i \in N$ (i.e., a collection of disjoint nonempty subsets of Y whose union is Y);*
- *$\mathfrak{s} : Y \to S$ is a function associating each state of the world with a state of nature.*

The interpretation is that if the "true" state of the world is ω_*, then each player $i \in N$ knows only the element of his partition $\mathcal{F}_i$ that contains ω_*. For example, if $Y = \{\omega_1, \omega_2, \omega_3\}$ and $\mathcal{F}_i = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$, then player i cannot distinguish between ω_1 and ω_2. In other words, if the state of the world is ω_1, player i knows that the state of the world is either ω_1 or ω_2, and therefore knows that the state of the world is not ω_3. For this reason, the partition $\mathcal{F}_i$ is also called the *information* of player i. The element of the partition $\mathcal{F}_i$ that contains the state of the world ω is denoted $F_i(\omega)$. For convenience, we will use the expression "the information of player i" to refer both to the partition $\mathcal{F}_i$ and to the partition element $F_i(\omega_*)$ containing the true state of the world.

Definition 9.5 *An* Aumann situation of incomplete information *over a set of states of nature S is a quintuple $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s}, \omega_*)$, where $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s})$ is an Aumann model of incomplete information and $\omega_* \in Y$.*

The state ω_* is the "true state of the world" and each player knows the partition element $F_i(\omega_*)$ in his information partition that contains the true state. A situation of incomplete information describes a knowledge structure at a particular state of the world, i.e., in a particular reality. Models of incomplete information, in contrast, enable us to analyze all possible situations.

Example 9.3 (*Continued*) An Aumann model of incomplete information for this example is as follows:

- $N = \{\text{John}, \text{Paul}\}$.
- $S = \{\text{Purple Car}, \text{Red Car}, \text{Gold Car}\}$.
- $Y = \{\omega_{g,1}, \omega_{r,1}, \omega_{g,2}, \omega_{r,2}, \omega_p\}$.
- John's partition is $\mathcal{F}_J = \{\{\omega_{g,1}, \omega_{g,2}, \omega_{r,1}, \omega_{r,2}\}, \{\omega_p\}\}$.
- Paul's partition is $\mathcal{F}_P = \{\{\omega_{g,1}, \omega_{r,1}\}, \{\omega_{g,2}\}, \{\omega_{r,2}\}, \{\omega_p\}\}$.
- The function $\mathfrak{s}$ is defined by

$$\mathfrak{s}(\omega_{g,1}) = \mathfrak{s}(\omega_{g,2}) = \text{Gold Car}, \ \mathfrak{s}(\omega_{r,1}) = \mathfrak{s}(\omega_{r,2}) = \text{Red Car}, \qquad \mathfrak{s}(\omega_p) = \text{Purple Car}.$$

The state of the world ω_p is associated with the situation in which the car is purple, in which case both John and Paul know that it is purple, and each of them knows that the other knows that the car is purple. It represents the two situations in the two first bullets on page 323, which differ only in whether Paul is color-blind or not. As we said before, these two situations are equivalent, and can be represented by the same state of the world, as long as Paul's color-blindness is not

4 We will later examine the case where Y is infinite, and show that some of the results obtained in this chapter also hold in that case.

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payoff relevant, and hence is not part of the description of the state of nature. The state of the world $\omega_{g,1}$ is associated with the situation in which the car is gold and Paul is color-blind, while the state of the world $\omega_{r,1}$ is associated with the situation in which the car is red and Paul is color-blind; in both these situations, Paul cannot distinguish which state of the world holds, because he is color-blind and cannot tell red from gold. The state of the world $\omega_{g,2}$ is associated with the situation in which the car is gold and Paul is not color-blind, while the state of the world $\omega_{r,2}$ is associated with the situation in which the car is red and Paul is not color-blind; in both these cases Paul knows the true color of the car. Therefore, $F_P(\omega_{g,2}) = \{\omega_{g,2}\}$, and $F_P(\omega_{g,1}) = \{\omega_{g,1}, \omega_{r,1}\}$.

As for John, he is both color-blind and does not know whether Paul is color-blind. He therefore cannot distinguish between the four states of the world $\{\omega_{g,1}, \omega_{r,1}, \omega_{g,2}, \omega_{r,2}\}$, so that $F_J(\omega_{g,1}) = F_J(\omega_{g,2}) = F_J(\omega_{r,1}) = F_J(\omega_{r,2}) = \{\omega_{g,1}, \omega_{r,1}, \omega_{g,2}, \omega_{r,2}\}$.

The true state of the world is one of the possible states in the set Y. The Aumann model along with the true state of the world describes the actual situation faced by John and Paul. ◀

Definition 9.6 *An* event *is a subset of Y.*

In Example 9.3 the event $\{\omega_{g,1}, \omega_{g,2}\}$ is the formal expression of the sentence "the car is gold," while the event $\{\omega_{g,1}, \omega_{g,2}, \omega_p\}$ is the formal expression of the sentence "the car is either gold or purple."

We say that an event A *obtains* in a state of the world ω if $\omega \in A$. It follows that if event A obtains in a state of the world ω and if $A \subseteq B$, then event B obtains in ω.

Definition 9.7 *Let $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s})$ be an Aumann model of incomplete information, let i be a player, let $\omega \in Y$ be a state of the world, and let $A \subseteq Y$ be an event.* Player i knows A in ω *if*

$$F_i(\omega) \subseteq A. \tag{9.1}$$

If $F_i(\omega) \subseteq A$, then in state of the world ω player i knows that event A obtains (even though he may not know that the state of the world is ω), because according to his information, all the possible states of the world, $F_i(\omega)$, are included in the event A.

Definition 9.8 *Let $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s})$ be an Aumann model of incomplete information, let i be a player, and let $A \subseteq Y$ be an event. Define an operator $K_i : 2^Y \to 2^Y$ by*[5]

$$K_i(A) := \{\omega \in Y : F_i(\omega) \subseteq A\}. \tag{9.2}$$

We will often denote $K_i(A)$, the set of all states of the world in which player i knows event A, by K_iA. Thus, player i knows event A in state of the world ω_* if and only if $\omega_* \in K_iA$. The definition implies that the set K_iA equals the union of all the elements in the partition $\mathcal{F}_i$ contained in A. The event $K_j(K_iA)$ (which we will write as K_jK_iA for short) is the event that player j knows that player i knows A:

$$K_jK_iA = \{\omega \in Y : F_j(\omega) \subseteq K_iA\}. \tag{9.3}$$

5 The collection of all subsets of Y is denoted by 2^Y.

Example 9.3 (*Continued*) Denote $A = \{\omega_p\}$, $B = \{\omega_{r,2}\}$, and $C = \{\omega_{r,1}, \omega_{r,2}\}$. Then

$$\begin{array}{lll} K_J A = \{\omega_p\} = A, & K_J B = \emptyset, & K_J C = \emptyset, \\ K_P A = \{\omega_p\} = A, & K_P B = \{\omega_{r,2}\}, & K_P C = \{\omega_{r,2}\}. \end{array}$$

The content of the expression $K_P B = \{\omega_{r,2}\}$ is that only in state of the world $\omega_{r,2}$ does Paul know that event B obtains (meaning that only in that state of the world does he know that the car is red). The content of $K_J B = \emptyset$ is that there is no state of the world in which John knows that B obtains; i.e., he never knows that the car is red and that Paul is not color-blind. From this we conclude that

$$K_J K_P C = K_J B = \emptyset. \tag{9.4}$$

This means that there is no state of the world in which John knows that Paul knows that the car is red. In contrast, $\omega_p \in K_P K_J A$, which means that in state of the world ω_p Paul knows that John knows that the state of the world is ω_p (and in particular, that the car is purple). ◀

We can now present some simple results that follow from the above definition of knowledge. The first result states that if a player knows event A in state of the world ω, then it is necessarily true that $\omega \in A$. In other words, if a player knows the event A, then A necessarily obtains (because the true state of the world is contained within it).[6]

Theorem 9.9 *$K_i A \subseteq A$ for every event $A \subseteq Y$ and every player $i \in N$.*

Proof: Let $\omega \in K_i A$. From the definition of knowledge it follows that $F_i(\omega) \subseteq A$. Since $\omega \in F_i(\omega)$ it follows that $\omega \in A$, which is what we needed to prove. □

Our second result states that if event A is contained in event B, then the states of the world in which player i knows event A form a subset of the states of the world in which the player knows event B. In other words, in every state of the world in which a player knows event A, he also knows event B.

Theorem 9.10 *For every pair of events $A, B \subseteq Y$, and every player $i \in N$,*

$$A \subseteq B \implies K_i A \subseteq K_i B. \tag{9.5}$$

Proof: We will show that $\omega \in K_i A$ implies that $\omega \in K_i B$. Suppose that $\omega \in K_i A$. By definition, $F_i(\omega) \subseteq A$, and because $A \subseteq B$, one has $F_i(\omega) \subseteq B$. Therefore, $\omega \in K_i B$, which is what we need to show. □

Our third result[7] says that if a player knows event A, then he knows that he knows event A, and conversely, if he knows that he knows event A, then he knows event A.

Theorem 9.11 *For every event $A \subseteq Y$ and every player $i \in N$, we have $K_i K_i A = K_i A$.*

Proof: Theorems 9.9 and 9.10 imply that $K_i K_i A \subseteq K_i A$. We will show that the opposite inclusion holds, namely, if $\omega \in K_i A$ then $\omega \in K_i K_i A$. If $\omega \in K_i A$ then $F_i(\omega) \subseteq A$. Therefore, for every $\omega' \in F_i(\omega)$, we have $\omega' \in F_i(\omega') = F_i(\omega) \subseteq A$. It follows that $\omega' \in K_i A$. As this is true for every $\omega' \in F_i(\omega)$, we deduce that $F_i(\omega) \subseteq K_i A$, which implies that $\omega \in K_i K_i A$. Thus, $K_i A \subseteq K_i K_i A$, which is what we wanted to prove. □

6 In the literature, this is known as the "axiom of knowledge."

7 One part of this theorem, namely, the fact that if a player knows an event, then he knows that he knows the event, is known in the literature as the "axiom of positive introspection."

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More generally, the knowledge operator K_i of player i satisfies the following five properties, which collectively are called *Kripke's S5 System*:

1. $K_i Y = Y$: the player knows that Y is the set of all states of the world.
2. $K_i A \cap K_i B = K_i(A \cap B)$: if the player knows event A and knows event B then he knows event $A \cap B$.
3. $K_i A \subseteq A$: if the player knows event A then event A obtains.
4. $K_i K_i A = K_i A$: if the player knows event A then he knows that he knows event A, and vice versa.
5. $(K_i A)^c = K_i((K_i A)^c)$: if the player does not know event A, then he knows that he does not know event A, and vice versa.[8,9]

Property 3 was proved in Theorem 9.9. Property 4 was proved in Theorem 9.11. The proof that the knowledge operator satisfies the other three properties is left to the reader (Exercise 9.1). In fact, Properties 1–5 characterize knowledge operators: for every operator $K : 2^Y \to 2^Y$ satisfying these properties there exists a partition $\mathcal{F}$ of Y that induces K via Equation (9.2) (Exercise 9.2).

Example 9.12 Anthony, Betty, and Carol are each wearing a hat. Hats may be red (r) or blue (b). Each one of the three sees the hats worn by the other two, but cannot see his or her own hat, and therefore does not know its color. This situation can be described by an Aumann model of incomplete information as follows:

- The set of players is $N = \{\text{Anthony, Betty, Carol}\}$.
- The set of states of nature is $S = \{(r,r,r), (r,r,b), (r,b,r), (r,b,b), (b,r,r), (b,r,b), (b,b,r), (b,b,b)\}$. A state of nature is described by three hat colors: that of Anthony's hat (the left letter), of Betty's hat (the middle letter), and of Carol (the right letter).
- The set of states of the world is $Y = \{\omega_{rrr}, \omega_{rrb}, \omega_{rbr}, \omega_{rbb}, \omega_{brr}, \omega_{brb}, \omega_{bbr}, \omega_{bbb}\}$.
- The function $\mathfrak{s} : Y \to S$ that maps every state of the world to a state of nature is defined by

$$\mathfrak{s}(\omega_{rrr}) = (r,r,r), \quad \mathfrak{s}(\omega_{rrb}) = (r,r,b), \quad \mathfrak{s}(\omega_{rbr}) = (r,b,r), \quad \mathfrak{s}(\omega_{rbb}) = (r,b,b),$$
$$\mathfrak{s}(\omega_{brr}) = (b,r,r), \quad \mathfrak{s}(\omega_{brb}) = (b,r,b), \quad \mathfrak{s}(\omega_{bbr}) = (b,b,r), \quad \mathfrak{s}(\omega_{bbb}) = (b,b,b).$$

The information partitions of Anthony, Betty, and Carol are as follows:

$$\mathcal{F}_{\text{A}} = \{\{\omega_{rrr}, \omega_{brr}\}, \{\omega_{rrb}, \omega_{brb}\}, \{\omega_{rbr}, \omega_{bbr}\}, \{\omega_{rbb}, \omega_{bbb}\}\}, \tag{9.6}$$

$$\mathcal{F}_{\text{B}} = \{\{\omega_{rrr}, \omega_{rbr}\}, \{\omega_{rrb}, \omega_{rbb}\}, \{\omega_{brr}, \omega_{bbr}\}, \{\omega_{brb}, \omega_{bbb}\}\}, \tag{9.7}$$

$$\mathcal{F}_{\text{C}} = \{\{\omega_{rrr}, \omega_{rrb}\}, \{\omega_{rbr}, \omega_{rbb}\}, \{\omega_{brr}, \omega_{brb}\}, \{\omega_{bbr}, \omega_{bbb}\}\}. \tag{9.8}$$

For example, when the state of the world is ω_{brb}, Anthony sees that Betty is wearing a red hat and that Carol is wearing a blue hat, but does not know whether his hat is red or blue, so that he knows that the state of the world is in the set $\{\omega_{rrb}, \omega_{brb}\}$, which is one of the elements of his

8 The first part of this property, i.e., the fact that if a player does not know an event, then he knows that he does not know it, is known in the literature as the "axiom of negative introspection."
9 For any event A, the complement of A is denoted by $A^c := Y \setminus A$.

partition $\mathcal{F}_A$. Similarly, if the state of the world is ω_{brb}, Betty knows that the state of the world is in her partition element $\{\omega_{brb}, \omega_{bbb}\}$, and Carol knows that the state of the world is in her partition element $\{\omega_{brr}, \omega_{brb}\}$.

Let R be the event "there is at least one red hat," that is,

$$R = \{\omega_{rrr}, \omega_{rrb}, \omega_{rbr}, \omega_{rbb}, \omega_{brr}, \omega_{brb}, \omega_{bbr}\}. \tag{9.9}$$

In which states of the world does Anthony know R? In which states does Betty know that Anthony knows R? In which states does Carol know that Betty knows that Anthony knows R? To begin answering the first question, note that in state of the world ω_{rrr}, Anthony knows R, because

$$F_A(\omega_{rrr}) = \{\omega_{rrr}, \omega_{brr}\} \subseteq R. \tag{9.10}$$

Anthony also knows R in each of the states of the world ω_{rrb}, ω_{rbr}, ω_{brb}, ω_{brr}, and ω_{bbr}. In contrast, in the states ω_{rbb} and ω_{bbb} he does not know R, because

$$F_A(\omega_{rbb}) = F_A(\omega_{bbb}) = \{\omega_{rbb}, \omega_{bbb}\} \not\subseteq R. \tag{9.11}$$

In summary,

$$K_A R = \{\omega \in Y : F_A(\omega) \subseteq R\} = \{\omega_{rrr}, \omega_{rbr}, \omega_{rrb}, \omega_{brb}, \omega_{brr}, \omega_{bbr}\}.$$

The analysis here is quite intuitive: Anthony knows R if either Betty or Carol (or both) is wearing a red hat, which occurs in the states of the world in the set $\{\omega_{rrr}, \omega_{rbr}, \omega_{rrb}, \omega_{brb}, \omega_{brr}, \omega_{bbr}\}$. When does Betty know that Anthony knows R? This requires calculating $K_B K_A R$.

$$\begin{aligned} K_B K_A R &= \{\omega \in Y : F_B(\omega) \subseteq K_A R\} \\ &= \{\omega \in Y : F_B(\omega) \subseteq \{\omega_{rrr}, \omega_{rbr}, \omega_{rrb}, \omega_{brb}, \omega_{brr}, \omega_{bbr}\}\} \\ &= \{\omega_{rrr}, \omega_{brr}, \omega_{rbr}, \omega_{bbr}\}. \end{aligned} \tag{9.12}$$

For example, since $F_B(\omega_{rbr}) = \{\omega_{rbr}, \omega_{rrr}\} \subseteq K_A R$ we conclude that $\omega_{rbr} \in K_B K_A R$. On the other hand, since $F_B(\omega_{brb}) = \{\omega_{brb}, \omega_{bbb}\} \not\subseteq K_A R$, it follows that $\omega_{brb} \notin K_B K_A R$. The analysis here is once again intuitively clear: Betty knows that Anthony knows R only if Carol is wearing a red hat, which only occurs in the states of the world $\{\omega_{rrr}, \omega_{brr}, \omega_{rbr}, \omega_{bbr}\}$.

Finally, we answer the third question: when does Carol know that Betty knows that Anthony knows R? This requires calculating $K_C K_B K_A R$.

$$\begin{aligned} K_C K_B K_A R &= \{\omega \in Y : F_C(\omega) \subseteq K_B K_A R\} \\ &= \{\omega \in Y : F_C(\omega) \subseteq \{\omega_{rrr}, \omega_{brr}, \omega_{rbr}, \omega_{bbr}\}\} = \emptyset. \end{aligned} \tag{9.13}$$

For example, since $F_C(\omega_{rbr}) = \{\omega_{rbr}, \omega_{rbb}\} \not\subseteq K_B K_A R$, we conclude that $\omega_{rbr} \notin K_C K_B K_A R$. In other words, there is no state of the world in which Carol knows that Betty knows that Anthony knows R. This is true intuitively, because as we saw previously, Betty knows that Anthony knows R only if Carol is wearing a red hat, but Carol does not know the color of her own hat.

This analysis enables us to conclude, for example, that in state of the world ω_{rrr} Anthony knows R, Betty knows that Anthony knows R, but Carol does not know that Betty knows that Anthony knows R. ◀

Note the distinction in Example 9.12 between states of nature and states of the world. The state of nature is the parameter with respect to which there is incomplete information: the colors of the hats worn by the three players. The state of the world includes in addition the mutual knowledge structure of the players regarding the state of nature. For example, the state of the world ω_{rrr} says a lot more than the fact that all three players are wearing red hats; for example, in this state of the world Carol knows there is at least one red hat, Carol knows that Anthony knows that there is at least one red hat, and Carol does not know that Betty knows that Anthony knows that there is at least one red hat. In Example 9.12 there is a one-to-one correspondence between the set of states of nature S and the set of states of the world Y. This is so since the mutual knowledge structure is uniquely determined by the configuration of the colors of the hats.

Example 9.13 Arthur, Harry, and Tom are in a room with two windows, one facing north and the other facing south. Two hats, one yellow and one brown, are placed on a table in the center of the room. After Harry and Tom leave the room, Arthur selects one of the hats and places it on his head. Tom and Harry peek in, each through a different window, watching Arthur (so that they both know the color of the hat Arthur is wearing). Neither Tom nor Harry knows whether or not the other player who has left the room is peeking through a window, and Arthur has no idea whether or not Tom or Harry is spying on him as he places one of the hats on his head. An Aumann model of incomplete information describing this situation is as follows:

- $N = \{\text{Arthur, Harry, Tom}\}$.
- $S = \{\text{Arthur wears the brown hat, Arthur wears the yellow hat}\}$.
- There are eight states of the world, each of which is designated by two indices: $Y = \{\omega_{\text{b},\emptyset}, \omega_{\text{b,T}}, \omega_{\text{b,H}}, \omega_{\text{b,TH}}, \omega_{\text{y},\emptyset}, \omega_{\text{y,T}}, \omega_{\text{y,H}}, \omega_{\text{y,TH}}\}$. The left index of ω indicates the color of the hat that Arthur is wearing (which is either brown or yellow), and the right index indicates which of the other players has been peeking into the room (Tom (T), Harry (H), both (TH), or neither($\emptyset$)).
- Arthur's partition contains two elements, because he knows the color of the hat on his head, but does not know who is peeking into the room: $\mathcal{F}_{\text{A}} = \{\{\omega_{\text{b},\emptyset}, \omega_{\text{b,H}}, \omega_{\text{b,T}}, \omega_{\text{b,TH}}\}, \{\omega_{\text{y},\emptyset}, \omega_{\text{y,H}}, \omega_{\text{y,I}}, \omega_{\text{y,TH}}\}\}$.
- Tom's partition contains three elements, one for each of his possible situations of information: Tom has not peeked into the room; Tom has peeked into the room and seen Arthur wearing the brown hat; Tom has peeked into the room and seen Arthur wearing the yellow hat. His partition is thus $\mathcal{F}_{\text{T}} = \{\{\omega_{\text{b},\emptyset}, \omega_{\text{b,H}}, \omega_{\text{y},\emptyset}, \omega_{\text{y,H}}\}, \{\omega_{\text{b,T}}, \omega_{\text{b,TH}}\}, \{\omega_{\text{y,T}}, \omega_{\text{y,TH}}\}\}$.

  For example, if Tom has peeked and seen the brown hat on Arthur's head, he knows that Arthur has selected the brown hat, but he does not know whether he is the only player who peeked (corresponding to the state of the world $\omega_{\text{b,T}}$) or whether Harry has also peeked (state of the world $\omega_{\text{b,TH}}$).
- Similarly, Harry's partition is $\mathcal{F}_{\text{H}} = \{\{\omega_{\text{b},\emptyset}, \omega_{\text{b,T}}, \omega_{\text{y},\emptyset}, \omega_{\text{y,T}}\}, \{\omega_{\text{b,H}}, \omega_{\text{b,TH}}\}, \{\omega_{\text{y,H}}, \omega_{\text{y,TH}}\}\}$.
- The function $\mathfrak{s}$ is defined by

$$\mathfrak{s}(\omega_{\text{b},\emptyset}) = \mathfrak{s}(\omega_{\text{b,T}}) = \mathfrak{s}(\omega_{\text{b,H}}) = \mathfrak{s}(\omega_{\text{b,TH}}) = \text{Arthur wears the brown hat;}$$

$$\mathfrak{s}(\omega_{\text{y},\emptyset}) = \mathfrak{s}(\omega_{\text{y,T}}) = \mathfrak{s}(\omega_{\text{y,H}}) = \mathfrak{s}(\omega_{\text{y,TH}}) = \text{Arthur wears the yellow hat.}$$

In this model, for example, if the true state of the world is $\omega_* = \omega_{\text{b,TH}}$, then Arthur is wearing the brown hat, and both Tom and Harry have peeked into the room. The event "Arthur is wearing the brown hat" is $B = \{\omega_{\text{b},\emptyset}, \omega_{\text{b,T}}, \omega_{\text{b,H}}, \omega_{\text{b,TH}}\}$. Tom and Harry know that Arthur's hat is brown only if they have peeked into the room. Therefore,

$$K_{\text{T}}B = \{\omega_{\text{b,T}}, \omega_{\text{b,TH}}\}, \qquad K_{\text{H}}B = \{\omega_{\text{b,H}}, \omega_{\text{b,TH}}\}. \tag{9.14}$$

Given Equation (9.14), since the set $K_{\text{H}}B$ is not included in any of the elements in Tom's partition, we conclude that $K_{\text{T}}K_{\text{H}}B = \emptyset$. In other words, in any state of the world, Tom does not know whether or not Harry knows that Arthur is wearing the brown hat, and therefore, in particular, this is the case at the given state of the world, $\omega_{\text{b,TH}}$. We similarly conclude that $K_{\text{H}}K_{\text{T}}B = \emptyset$: in any state of the world, Harry does not know that Tom knows that Arthur is wearing the brown hat (and in particular this is the case at the true state of the world, $\omega_{\mathrm{b,TH}}$). This is all quite intuitive; Tom knows that Arthur is wearing the brown hat only if he has peeked into the room, but Harry does not know whether or not Tom has peeked into the room.

Note again the distinction between a state of nature and a state of the world. The objective fact about which the players have incomplete information is the color of the hat atop Arthur's head. Each one of the four states of the world $\{\omega_{\mathrm{y},\emptyset}, \omega_{\mathrm{y,H}}, \omega_{\mathrm{y,T}}, \omega_{\mathrm{y,TH}}\}$ corresponds to the state of nature "Arthur wears the yellow hat," yet they differ in the knowledge that the players have regarding the state of nature. In the state of the world $\omega_{\mathrm{y},\emptyset}$, Arthur wears the yellow hat, but Tom and Harry do not know that, while in state of the world $\omega_{\mathrm{y,H}}$, Arthur wears the yellow hat and Harry knows that, but Tom does not know that. Note that in both of these states of the world Tom and Arthur do not know that Harry knows the color of Arthur's hat, Harry and Arthur do not know whether or not Tom knows the color of the hat, and in each state of the world there are additional statements that can be made regarding the players' mutual knowledge of Arthur's hat. ◀

The insights gleaned from these examples can be formulated and proven rigorously.

Definition 9.14 *A* knowledge hierarchy *among players in state of the world ω over the set of states of the world Y is a system of "yes" or "no" answers to each question of the form "in a state of the world ω, does player i_1 know that player i_2 knows that player i_3 knows ... that player i_l knows event A"? for any event $A \subseteq Y$ and any finite sequence $i_1, i_2, \ldots, i_l$ of players*[10] *in N.*

The answer to the question "does player i_1 know that player i_2 knows that player i_3 knows ... that player i_l knows event A?" in a state of the world ω is affirmative if $\omega \in K_{i_1} K_{i_2} \cdots K_{i_l} A$, and negative if $\omega \notin K_{i_1} K_{i_2} \cdots K_{i_l} A$. Since for every event A and every sequence of players $i_1, i_2, \ldots, i_l$ the event $K_{i_1} K_{i_2} \cdots K_{i_l} A$ is well defined and calculable in an Aumann model of incomplete information, every state of the world defines a knowledge hierarchy. We have therefore derived the following theorem.

Theorem 9.15 *Every situation of incomplete information $(N, Y, (\mathcal{F}_i)_{i \in N}, \mathfrak{s}, \omega_*)$ uniquely determines a knowledge hierarchy over the set of states of the world Y in state of the world ω_*.*

For every subset $C \subseteq S$ of the set of states of nature, we can consider the event that contains all states of the world whose state of nature is an element of C:

$$\mathfrak{s}^{-1}(C) := \{\omega \in Y : \mathfrak{s}(\omega) \in C\}. \tag{9.15}$$

For example, in Example 9.13 the set of states of nature {yellow} corresponds to the event $\{\omega_{\mathrm{y},\emptyset}, \omega_{\mathrm{y,H}}, \omega_{\mathrm{y,G}}, \omega_{\mathrm{y,TH}}\}$ in Y. Every subset of S is called an *event in S*. We define knowledge of events in S as follows: in a state of the world ω player i knows event C in S if and only if he knows the event $\mathfrak{s}^{-1}(C)$, i.e., if and only if $\omega \in K_i(\mathfrak{s}^{-1}(C))$. In the same manner, in state of the world ω player i_1 knows that player i_2 knows that player i_3 knows ... that player i_l knows event C in S if and only if in state of the world ω player i_1 knows that player i_2 knows that player i_3 knows ... that player i_l knows $\mathfrak{s}^{-1}(C)$.

10 A player may appear several times in the chain $i_1, i_2, \ldots, i_l$. For example, the chain player 2 knows that player 1 knows that player 3 knows that player 2 knows event A is a legitimate chain.

Corollary 9.16 is a consequence of Theorem 9.15 (Exercise 9.10).

Corollary 9.16 *Every situation of incomplete information* $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s}, \omega_*)$ *uniquely determines a knowledge hierarchy over the set of states of nature* S *in state of the world* ω_*.

Having defined the knowledge operators of the players, we next turn to the definition of the concept of common knowledge, which was previously defined informally (see Definition 9.2).

Definition 9.17 *Let* $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s})$ *be an Aumann model of incomplete information, let* $A \subseteq Y$ *be an event, and let* $\omega \in Y$ *be a state of the world. The event* A *is* common knowledge in ω *if for every finite sequence of players* $i_1, i_2, \ldots, i_l$,

$$\omega \in K_{i_1} K_{i_2} \ldots K_{i_{l-1}} K_{i_l} A. \tag{9.16}$$

That is, event A is common knowledge at state of the world ω if in ω every player knows event A, every player knows that every player knows event A, etc. In Examples 9.12 and 9.13 the only event that is common knowledge in any state of the world is Y (Exercise 9.12). In Example 9.3 (page 322) the event $\{\omega_{\mathrm{p}}\}$ (and every event containing it) is common knowledge in state of the world ω_{p}, and the event $\{\omega_{\mathrm{g},1}, \omega_{\mathrm{g},2}, \omega_{\mathrm{r},1}, \omega_{\mathrm{r},2}\}$ (and the event Y containing it) is common knowledge in every state of the world contained in this event.

Example 9.18 Abraham selects an integer from the set $\{5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$. He tells Jefferson whether the number he has selected is even or odd, and tells Ulysses the remainder left over from dividing that number by 4. The corresponding Aumann model of incomplete information depicting the induced situation of Jefferson and Ulysses is:

- $N = \{\text{Jefferson, Ulysses}\}$.
- $S = \{5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$: the state of nature is the number selected by Abraham.
- $Y = \{\omega_5, \omega_6, \omega_7, \omega_8, \omega_9, \omega_{10}, \omega_{11}, \omega_{12}, \omega_{13}, \omega_{14}\}$.
- The function $\mathfrak{s} : Y \to S$ is given by $\mathfrak{s}(\omega_k) = k$ for every $k \in S$.
- Since Jefferson knows whether the number is even or odd, his partition contains two elements, corresponding to the subset of even numbers and the subset of odd numbers in the set Y:

$$\mathcal{F}_{\mathrm{J}} = \{\{\omega_5, \omega_7, \omega_9, \omega_{11}, \omega_{13}\}, \{\omega_6, \omega_8, \omega_{10}, \omega_{12}, \omega_{14}\}\}. \tag{9.17}$$

- As Ulysses knows the remainder left over from dividing the number by 4, his partition contains four elements, one for each possible remainder:

$$\mathcal{F}_{\mathrm{U}} = \{\{\omega_8, \omega_{12}\}, \{\omega_5, \omega_9, \omega_{13}\}, \{\omega_6, \omega_{10}, \omega_{14}\}, \{\omega_7, \omega_{11}\}\}. \tag{9.18}$$

In the state of the world ω_6, the event that the selected number is even, i.e., $A = \{\omega_6, \omega_8, \omega_{10}, \omega_{12}, \omega_{14}\}$, is common knowledge. Indeed, $K_{\mathrm{J}} A = K_{\mathrm{U}} A = A$, and therefore it follows that $K_{i_1} K_{i_2} \ldots K_{i_{l-1}} K_{i_l} A = A$ for every finite sequence of players $i_1, i_2, \ldots, i_l$. Since $\omega_6 \in A$, it follows from Definition 9.17 that in state of the world ω_6 the event A is common knowledge among Jefferson and Ulysses. Similarly, in state of the world ω_9, the event that the selected number is odd, $B = \{\omega_5, \omega_7, \omega_9, \omega_{11}, \omega_{13}\}$, is common knowledge among Jefferson and Ulysses (verify!). ◀

Remark 9.19 *From Definition 9.17 and Theorem 9.10 we conclude that if event A is common knowledge in a state of the world ω, then every event containing A is also common knowledge in ω.* ♦

Remark 9.20 *The definition of common knowledge can be expanded to events in S: an event C in S is common knowledge in a state of the world ω if the event $\mathfrak{s}^{-1}(C)$ is common knowledge in ω. For example, in Example 9.13 in state of the world $\omega_{\mathrm{b,TH}}$ the event (in the set of states of nature) "Arthur selects the brown hat" is not common knowledge among the players (verify!).* ♦

Remark 9.21 *If event A is common knowledge in a state of the world ω, then in particular $\omega \in K_i A$ and so $F_i(\omega) \subseteq A$ for each $i \in N$. In other words, all players know A in ω.* ♦

Remark 9.22 *We can also speak of common knowledge among a subset of the players $M \subseteq N$: in a state of the world ω, event A is common knowledge among the players in M if Equation (9.16) is satisfied for any finite sequence $i_1, i_2, \ldots, i_l$ of players in M.* ♦

Theorem 9.23 states that if there is a player who cannot distinguish between ω and ω', then every event that is common knowledge in ω is also common knowledge in ω'.

Theorem 9.23 *If event A is common knowledge in state of the world ω, and if $\omega' \in F_i(\omega)$ for some player $i \in N$, then the event A is also common knowledge in state of the world ω'.*

Proof: Suppose that $\omega' \in F_i(\omega)$ for some player $i \in N$. As the event A is common knowledge in ω, for any sequence $i_1, i_2, \ldots, i_l$ of players we have

$$\omega \in K_i K_{i_1} K_{i_2} \ldots K_{i_{l-1}} K_{i_l} A. \tag{9.19}$$

Remark 9.21 implies that

$$F_i(\omega) \subseteq K_{i_1} K_{i_2} \ldots K_{i_{l-1}} K_{i_l} A. \tag{9.20}$$

Since $\omega' \in F_i(\omega') = F_i(\omega)$ it follows that $\omega' \in K_{i_1} K_{i_2} \ldots K_{i_{l-1}} K_{i_l} A$. As this is true for any sequence $i_1, i_2, \ldots, i_l$ of players, the event A is common knowledge in ω'. □

We next turn to characterizing sets that are common knowledge. Given an Aumann model of incomplete information $(N, Y, (\mathcal{F}_i)_{i \in N}, \mathfrak{s})$, define the graph $G = (Y, V)$ in which the set of vertices is the set of states of the world Y, and there is an edge between vertices ω and ω' if and only if there is a player i such that $\omega' \in F_i(\omega)$. Note that the condition defining the edges of the graph is symmetric: $\omega' \in F_i(\omega)$ if and only if $F_i(\omega) = F_i(\omega')$, if and only if $\omega \in F_i(\omega')$; hence $G = (Y, V)$ is an undirected graph.

A set of vertices C in a graph is a *connected component* if the following two conditions are satisfied:

- For every $\omega, \omega' \in C$, there exists a path connecting ω with ω', i.e., there exist $\omega = \omega_1, \omega_2, \ldots, \omega_K = \omega'$ such that for each $k = 1, 2, \ldots, K-1$ the graph contains an edge connecting ω_k and ω_{k+1}.
- There is no edge connecting a vertex in C with a vertex that is not in C.

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The *connected component of* ω in the graph, denoted by $C(\omega)$, is the (unique) connected component containing ω.

Theorem 9.24 *Let $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s})$ be an Aumann model of incomplete information and let G be the graph corresponding to this model. Let $\omega \in Y$ be a state of the world and let $A \subseteq Y$ be an event. Then event A is common knowledge in state of the world ω if and only if $A \supseteq C(\omega)$.*

Proof: First we prove that if A is common knowledge in ω, then $C(\omega) \subseteq A$. Suppose then that $\omega' \in C(\omega)$. We want to show that $\omega' \in A$. From the definition of a connected component, there is a path connecting ω with ω'; we denote that path by $\omega = \omega_1, \omega_2, \ldots, \omega_K = \omega'$. We prove by induction on k that $\omega_k \in A$, and that A is common knowledge in ω_k, for every $1 \leq k \leq K$. For $k = 1$, because the event A is common knowledge in ω, we deduce that $\omega_1 = \omega \in A$. Suppose now that $\omega_k \in A$ and A is common knowledge in ω_k. We will show that $\omega_{k+1} \in A$ and that A is common knowledge in ω_{k+1}. Because there is an edge connecting ω_k and ω_{k+1}, there is a player i such that $\omega_{k+1} \in F_i(\omega_k)$. It follows from Theorem 9.23 that the event A is common knowledge in ω_{k+1}. From Remark 9.21 we conclude that $\omega_{k+1} \in A$. This completes the inductive step, so that in particular $\omega' = \omega_K \in A$.

Consider now the other direction: if $C(\omega) \subseteq A$, then event A is common knowledge in state of the world ω. To prove this, it suffices to show that $C(\omega)$ is common knowledge in ω, because from Remark 9.19 it will then follow that any event containing $C(\omega)$, and in particular A, is also common knowledge in ω. Let i be a player in N. Because $C(\omega)$ is a connected component of G, for each $\omega' \in C(\omega)$, we have $F_i(\omega') \subseteq C(\omega)$. It follows that

$$C(\omega) \supseteq \bigcup_{\omega' \in C(\omega)} F_i(\omega') \supseteq \bigcup_{\omega' \in C(\omega)} \{\omega'\} = C(\omega). \tag{9.21}$$

In other words, for each player i the set $C(\omega)$ is the union of all the elements of $\mathcal{F}_i$ contained in it. This implies that $K_i(C(\omega)) = C(\omega)$. As this is true for every player $i \in N$, it follows that for every sequence of players $i_1, i_2, \ldots, i_l$,

$$\omega \in C(\omega) = K_{i_1} K_{i_2} \cdots K_{i_l} C(\omega), \tag{9.22}$$

and therefore $C(\omega)$ is common knowledge in ω. □

The following corollary follows from Theorem 9.24 and Remark 9.19.

Corollary 9.25 *In every state of the world $\omega \in Y$, the event $C(\omega)$ is common knowledge among the players, and it is the smallest event that is common knowledge in ω.*

For this reason, $C(\omega)$ is sometimes called the *common knowledge component* among the players in state of the world ω.

Remark 9.26 *The proof of Theorem 9.24 shows that for each player $i \in N$, the set $C(\omega)$ is the union of the elements of $\mathcal{F}_i$ contained in it, and it is the smallest event containing ω that satisfies this property. The set of all the connected components of the graph G defines a partition of Y, which is called the* meet *of $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$. This is the finest partition that satisfies the property that each partition $\mathcal{F}_i$ is a refinement of it. We can therefore formulate Theorem 9.24 equivalently as follows. Let $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s})$ be* *an Aumann model of incomplete information. Event A is common knowledge in state of the world $\omega \in Y$ if and only if A contains the element of the meet containing ω.* ♦

9.2 The Aumann model of incomplete information with beliefs

The following model extends the Aumann model of incomplete information presented in the previous section.

Definition 9.27 *An* Aumann model of incomplete information with beliefs *(over a set of states of nature S) consists of five elements $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s}, \mathbf{P})$, where:*

- *N is a finite set of players;*
- *Y is a finite set of states of the world;*
- *$\mathcal{F}_i$ is a partition of Y, for each $i \in N$;*
- *$\mathfrak{s} : Y \to S$ is a function associating a state of nature to every state of the world;*
- *$\mathbf{P}$ is a probability distribution over Y such that $\mathbf{P}(\omega) > 0$ for each $\omega \in Y$.*

Comparing this definition to that of the Aumann model of incomplete information (Definition 9.4), we have added one new element, namely, the probability distribution $\mathbf{P}$ over Y, which is called the *common prior*. In this model, a state of the world ω_* is selected by a random process in accordance with the common prior probability distribution $\mathbf{P}$. After the true state of the world has been selected by this random process, each player i learns his partition element $F_i(\omega_*)$ that contains ω_*. Prior to the stage at which private information is revealed, the players share a common prior distribution, which is interpreted as their belief about the probability that any specific state of the world in Y is the true one. After each player i has acquired his private information $F_i(\omega_*)$, he updates his beliefs. This process of belief updating is the main topic of this section.

The assumption that all the players share a common prior is a strong assumption, and in many cases there are good reasons to doubt that it obtains. We will return to this point later in the chapter. In contrast, the assumption that $\mathbf{P}(\omega) > 0$ for all $\omega \in Y$ is not a strong assumption. As we will show, a state of the world ω for which $\mathbf{P}(\omega) = 0$ is one to which all the players assign probability 0, and it can be removed from consideration in Y.

In the following examples and in the rest of this chapter, whenever the states of nature are irrelevant we will specify neither the set S nor the function $\mathfrak{s}$.

Example 9.28 Consider the following Aumann model:

- The set of players is $N = \{\mathrm{I}, \mathrm{II}\}$.
- The set of states of the world is $Y = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.
- The information partitions of the players are

$$\mathcal{F}_{\mathrm{I}} = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}, \qquad \mathcal{F}_{\mathrm{II}} = \{\{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}\}. \tag{9.23}$$

- The common prior $\mathbf{P}$ is

$$\mathbf{P}(\omega_1) = \tfrac{1}{4}, \qquad \mathbf{P}(\omega_2) = \tfrac{1}{4}, \qquad \mathbf{P}(\omega_3) = \tfrac{1}{3}, \qquad \mathbf{P}(\omega_4) = \tfrac{1}{6}. \tag{9.24}$$

A graphic representation of the players' partitions and the prior probability distribution is provided in Figure 9.2. Player I's partition elements are marked by a solid line, while Player II's partition elements are denoted by a dotted line.

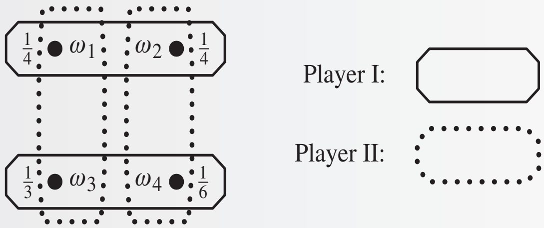

Figure 9.2 The information partitions and the prior distribution in Example 9.28

What are the beliefs of each player about the state of the world? Prior to the chance move that selects the state of the world, the players have a common prior distribution over the states of the world. When a player receives information that indicates that the true state of the world is in the partition element $F_i(\omega_*)$, he updates his beliefs about the states of the world by calculating the conditional probability given his information. For example, if the state of the world is ω_1, Player I knows that the state of the world is either ω_1 or ω_2. Player I's beliefs are therefore

$$\mathbf{P}(\omega_1 \mid \{\omega_1, \omega_2\}) = \frac{p(\omega_1)}{p(\omega_1) + p(\omega_2)} = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{4}} = \frac{1}{2}, \tag{9.25}$$

and similarly

$$\mathbf{P}(\omega_2 \mid \{\omega_1, \omega_2\}) = \frac{p(\omega_2)}{p(\omega_1) + p(\omega_2)} = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{4}} = \frac{1}{2}. \tag{9.26}$$

In words, if Player I's information is that the state of the world is in $\{\omega_1, \omega_2\}$, he attributes probability $\frac{1}{2}$ to the state of the world ω_1 and probability $\frac{1}{2}$ to the state of the world ω_2. The tables appearing in Figure 9.3 are arrived at through a similar calculation. The upper table describes Player I's beliefs, as a function of his information partition, and the lower table represents Player II's beliefs as a function of his information partition.

Player I's beliefs:

Player I's Information	ω_1	ω_2	ω_3	ω_4
$\{\omega_1, \omega_2\}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0
$\{\omega_3, \omega_4\}$	0	0	$\frac{2}{3}$	$\frac{1}{3}$

Player II's beliefs:

Player II's Information	ω_1	ω_2	ω_3	ω_4
$\{\omega_1, \omega_3\}$	$\frac{3}{7}$	0	$\frac{4}{7}$	0
$\{\omega_2, \omega_4\}$	0	$\frac{3}{5}$	0	$\frac{2}{5}$

Figure 9.3 The beliefs of the players in Example 9.28

For example, if Player II's information is $\{\omega_2, \omega_4\}$ (i.e., the state of the world is either ω_2 or ω_4), he attributes probability $\frac{3}{5}$ to the state of the world ω_2 and probability $\frac{2}{5}$ to the state of the world ω_4.

A player's beliefs will be denoted by square brackets in which states of the world appear alongside the probabilities that are ascribed to them. For example, $[\frac{3}{5}(\omega_2), \frac{2}{5}(\omega_4)]$ represents beliefs in which probability $\frac{3}{5}$ is ascribed to state of the world ω_2, and probability $\frac{2}{5}$ is ascribed to state of the world

ω_4. The calculations performed above yield the first-order beliefs of the players at all possible states of the world. These beliefs can be summarized as follows:

- In state of the world ω_1 the first-order belief of Player I is $[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)]$ and that of Player II is $[\frac{3}{7}(\omega_1), \frac{4}{7}(\omega_3)]$.
- In state of the world ω_2 the first-order belief of Player I is $[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)]$ and that of Player II is $[\frac{3}{5}(\omega_2), \frac{2}{5}(\omega_4)]$.
- In state of the world ω_3 the first-order belief of Player I is $[\frac{2}{3}(\omega_3), \frac{1}{3}(\omega_4)]$ and that of Player II is $[\frac{3}{7}(\omega_1), \frac{4}{7}(\omega_3)]$.
- In state of the world ω_4 the first-order belief of Player I is $[\frac{2}{3}(\omega_3), \frac{1}{3}(\omega_4)]$ and that of Player II is $[\frac{3}{5}(\omega_2), \frac{2}{5}(\omega_4)]$.

Given the first-order beliefs of the players over Y, we can construct the second-order beliefs, by which we mean the beliefs each player has about the state of the world and the first-order beliefs of the other player. In state of the world ω_1 (or ω_2) Player I attributes probability $\frac{1}{2}$ to the state of the world being ω_1 and probability $\frac{1}{2}$ to the state of the world being ω_2. As we noted above, when the state of the world is ω_1, the first-order belief of Player II is $[\frac{3}{7}(\omega_1), \frac{4}{7}(\omega_3)]$, and when the state of the world is ω_2, Player II's first-order belief is $[\frac{3}{5}(\omega_2), \frac{2}{5}(\omega_4)]$. Therefore:

- In state of the world ω_1 (or ω_2) Player I attributes probability $\frac{1}{2}$ to the state of the world being ω_1 and the first-order belief of Player II being $[\frac{3}{7}(\omega_1), \frac{4}{7}(\omega_3)]$, and probability $\frac{1}{2}$ to the state of the world being ω_2 and Player II's first-order belief being $[\frac{3}{5}(\omega_2), \frac{2}{5}(\omega_4)]$.

We can similarly calculate the second-order beliefs of each of the players in each state of the world:

- In state of the world ω_3 (or ω_4) Player I attributes probability $\frac{2}{3}$ to the state of the world being ω_3 and the first-order belief of Player II being $[\frac{3}{7}(\omega_1), \frac{4}{7}(\omega_3)]$, and probability $\frac{1}{3}$ to the state of the world being ω_4 and Player II's first-order belief being $[\frac{3}{5}(\omega_2), \frac{2}{5}(\omega_4)]$.
- In state of the world ω_1 (or ω_3) Player II attributes probability $\frac{3}{7}$ to the state of the world being ω_1 and the first-order belief of Player I being $[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)]$, and probability $\frac{4}{7}$ to the state of the world being ω_3 and Player I's first-order belief being $[\frac{2}{3}(\omega_3), \frac{1}{3}(\omega_4)]$.
- In state of the world ω_2 (or ω_4) Player II attributes probability $\frac{3}{5}$ to the state of the world being ω_2 and the first-order belief of Player I being $[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)]$, and probability $\frac{2}{5}$ to the state of the world being ω_4 and Player I's first-order belief being $[\frac{2}{3}(\omega_3), \frac{1}{3}(\omega_4)]$.

These calculations can be continued to arbitrarily high orders in a similar manner to yield belief hierarchies of the two players. ◀

Theorem 9.29 says that in an Aumann model, knowledge is equivalent to belief with probability 1. The theorem, however, requires assuming that $\mathbf{P}(\omega) > 0$ for each $\omega \in Y$; without that assumption the theorem's conclusion does not obtain (Exercise 9.21). In Example 9.36 we will see that the conclusion of the theorem also fails to hold when the set of states of the world is infinite.

Theorem 9.29 *Let $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s}, \mathbf{P})$ be an Aumann model of incomplete information with beliefs. Then for each $\omega \in Y$, for each player $i \in N$, and for every event $A \subseteq Y$, player i knows event A in state of the world ω if and only if he attributes probability 1 to that event:*

$$\mathbf{P}(A \mid F_i(\omega)) = 1 \iff F_i(\omega) \subseteq A. \tag{9.27}$$

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Notice that the assumption that $\mathbf{P}(\omega) > 0$ for every $\omega \in Y$, together with $\omega \in F_i(\omega)$ for every $\omega \in Y$, yields $\mathbf{P}(F_i(\omega)) > 0$ for each player $i \in N$ and every state of the world $\omega \in Y$, so that the conditional probability in Equation (9.27) is well defined.

Proof: Suppose first that $F_i(\omega) \subseteq A$. Then

$$\mathbf{P}(A \mid F_i(\omega)) \geq \mathbf{P}(F_i(\omega) \mid F_i(\omega)) = 1, \tag{9.28}$$

so that $\mathbf{P}(A \mid F_i(\omega)) = 1$. To prove the reverse implication, if $\mathbf{P}(A \mid F_i(\omega)) = 1$ then

$$\mathbf{P}(A \mid F_i(\omega)) = \frac{\mathbf{P}(A \cap F_i(\omega))}{\mathbf{P}(F_i(\omega))} = 1, \tag{9.29}$$

which yields $\mathbf{P}(A \cap F_i(\omega)) = \mathbf{P}(F_i(\omega))$. From the assumption that $\mathbf{P}(\omega') > 0$ for each $\omega' \in Y$ we conclude that $A \cap F_i(\omega) = F_i(\omega)$, that is, $F_i(\omega) \subseteq A$. □

A *situation of incomplete information with beliefs* is a vector $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s}, \mathbf{P}, \omega_*)$ composed of an Aumann model of incomplete information with beliefs $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s}, \mathbf{P})$ together with a state of the world $\omega_* \in Y$. The next theorem follows naturally from the analysis we performed in Example 9.28, and it generalizes Theorem 9.15 and Corollary 9.16 to situations of belief.

Theorem 9.30 *Every situation of incomplete information with beliefs $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s}, \mathbf{P}, \omega_*)$ uniquely determines a mutual belief hierarchy among the players over the states of the world Y, and therefore also a mutual belief hierarchy over the states of nature S.*

The above formulation is not precise, as we have not formally defined what the term "mutual belief hierarchy" means. The formal definition is presented in Chapter 11 where we will show that each state of the world is in fact a pair, consisting of a state of nature and a mutual belief hierarchy among the players over the states of nature S. The inductive description of belief hierarchies, as presented in the examples above and the examples below, will suffice for this chapter.

In Example 9.28 we calculated the belief hierarchy of the players in each state of the world. A similar calculation can be performed with respect to events.

Example 9.28 (*Continued*) Consider the situation in which $\omega_* = \omega_1$ and the event $A = \{\omega_2, \omega_3\}$. As Player I's information in state of the world ω_1 is $\{\omega_1, \omega_2\}$, the conditional probability that he ascribes to event A in state of the world ω_1 (or ω_2) is

$$\mathbf{P}(A \mid \{\omega_1, \omega_2\}) = \frac{\mathbf{P}(A \cap \{\omega_1, \omega_2\})}{\mathbf{P}(\{\omega_1, \omega_2\})} = \frac{\mathbf{P}(\{\omega_1\})}{\mathbf{P}(\{\omega_1, \omega_2\})} = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{4}} = \frac{1}{2}. \tag{9.30}$$

Because Player II's information in state of the world ω_1 is $\{\omega_1, \omega_3\}$, the conditional probability that he ascribes to event A in state of the world ω_1 (or ω_3) is

$$\mathbf{P}(A \mid \{\omega_1, \omega_3\}) = \frac{\mathbf{P}(A \cap \{\omega_1, \omega_3\})}{\mathbf{P}(\{\omega_1, \omega_3\})} = \frac{\mathbf{P}(\{\omega_3\})}{\mathbf{P}(\{\omega_1, \omega_3\})} = \frac{\frac{1}{3}}{\frac{1}{4} + \frac{1}{3}} = \frac{4}{7}. \tag{9.31}$$

Second-order beliefs can also be calculated readily. In state of the world ω_1, Player I ascribes probability $\frac{1}{2}$ to the true state being ω_1, in which case the probability that Player II ascribes to event A is $\frac{4}{7}$; he ascribes probability $\frac{1}{2}$ to the true state being ω_2, in which case the probability that Player II ascribes to event A is $(\frac{1}{4})/(\frac{1}{4}+\frac{1}{6})=\frac{2}{5}$. These are Player I's second-order beliefs about event A in state of the world ω_1. We can similarly calculate the second-order beliefs of Player II, as well as all the higher-order beliefs of the two players. ◀

Example 9.31 Consider again the Aumann model of incomplete information with beliefs presented in Example 9.28, but now with the common prior given by

$$\mathbf{P}(\omega_1)=\mathbf{P}(\omega_4)=\tfrac{1}{6},\quad \mathbf{P}(\omega_2)=\mathbf{P}(\omega_3)=\tfrac{1}{3}. \tag{9.32}$$

The partitions $\mathcal{F}_{\mathrm{I}}$ and $\mathcal{F}_{\mathrm{II}}$ are graphically depicted in Figure 9.4.

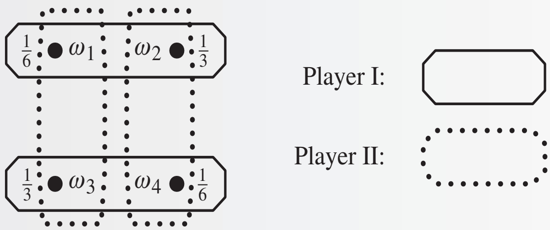

Figure 9.4 The information partitions and the prior distribution in Example 9.31

Since $\omega_1 \in F_{\mathrm{I}}(\omega_2)$, $\omega_2 \in F_{\mathrm{II}}(\omega_4)$, and $\omega_4 \in F_{\mathrm{I}}(\omega_3)$ in the graph corresponding to this Aumann model, all states in Y are connected. Hence the only connected component in the graph is Y (verify!), and therefore the only event that is common knowledge in any state of the world ω is Y (Theorem 9.24). Consider now the event $A=\{\omega_2,\omega_3\}$ and the situation in which $\omega_*=\omega_1$. What is the conditional probability that the players ascribe to A? Similarly to the calculation performed in Example 9.28,

$$\mathbf{P}(A \mid \{\omega_1,\omega_2\})=\frac{\mathbf{P}(A\cap\{\omega_1,\omega_2\})}{\mathbf{P}(\{\omega_1,\omega_2\})}=\frac{\mathbf{P}(\{\omega_2\})}{\mathbf{P}(\{\omega_1,\omega_2\})}=\frac{\frac{1}{3}}{\frac{1}{6}+\frac{1}{3}}=\frac{2}{3}, \tag{9.33}$$

and we can also readily calculate that both players ascribe probability $\frac{2}{3}$ to event A in each state of the world. Formally:

$$\left\{\omega\colon q_{\mathrm{I}}:=\mathbf{P}(A \mid F_{\mathrm{I}}(\omega))=\tfrac{2}{3}\right\}=Y,\qquad \left\{\omega\colon q_{\mathrm{II}}:=\mathbf{P}(A \mid F_{\mathrm{II}}(\omega))=\tfrac{2}{3}\right\}=Y. \tag{9.34}$$

It follows from the definition of the knowledge operator that the event "Player I ascribes probability $\frac{2}{3}$ to A" is common knowledge in each state of the world, and the event "Player II ascribes probability $\frac{2}{3}$ to A" is also common knowledge in each state of the world. In other words, in this situation the probabilities that the two players ascribe to event A are both common knowledge and equal to each other. ◀

Is it a coincidence that the probabilities q_{I} and q_{II} that the two players assign to the event A in Example 9.31 are equal (both being $\frac{2}{3}$)? Can there be a situation in which it is common knowledge that to the event A, Player I ascribes probability q_{I} and Player II

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ascribes probability q_{II}, where $q_{\mathrm{I}} \neq q_{\mathrm{II}}$? Theorem 9.32 asserts that this state of affairs is impossible.

Theorem 9.32 Aumann's Agreement Theorem (Aumann [1976]) *Let $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s}, \mathbf{P})$ be an Aumann model of incomplete information with beliefs, and suppose that $n = 2$ (i.e., there are two players). Let $A \subseteq Y$ be an event and let $\omega \in Y$ be a state of the world. If the event "Player I ascribes probability q_{I} to A" is common knowledge in ω, and the event "Player II ascribes probability q_{II} to A" is also common knowledge in ω, then $q_{\mathrm{I}} = q_{\mathrm{II}}$.*

Let us take a moment to consider the significance of this theorem before proceeding to its proof. The theorem states that if two players begin with "identical beliefs about the world" (represented by the common prior $\mathbf{P}$) but receive disparate information (represented by their respective partition elements containing ω), then "they cannot agree to disagree": if they agree that the probability that Player I ascribes to a particular event is q_{I}, then they cannot also agree that Player II ascribes a probability q_{II} to the same event, unless $q_{\mathrm{I}} = q_{\mathrm{II}}$. If they disagree regarding a particular fact (for example, Player I ascribes probability q_{I} to event A and Player II ascribes probability q_{II} to the same event), then the fact that they disagree cannot be common knowledge. Since we know that people often agree to disagree, we must conclude that either (a) different people begin with different prior distributions over the states of the world, or (b) people incorrectly calculate conditional probabilities when they receive information regarding the true state of the world.

Proof of Theorem 9.32: Let C be the connected component of ω in the graph corresponding to the given Aumann model. It follows from Theorem 9.24 that event C is common knowledge in state of the world ω. The event C can be represented as a union of partition elements in $\mathcal{F}_{\mathrm{I}}$; that is, $C = \bigcup_j F_{\mathrm{I}}^j$, where $F_{\mathrm{I}}^j \in \mathcal{F}_{\mathrm{I}}$ for each j. Since $\mathbf{P}(\omega') > 0$ for every $\omega' \in Y$, it follows that $\mathbf{P}(F_{\mathrm{I}}^j) > 0$ for every j, and therefore $\mathbf{P}(C) > 0$.

The fact that Player I ascribes probability q_{I} to the event A is common knowledge in ω. It follows that the event A contains the event C (Corollary 9.25), and therefore each one of the events $(F_{\mathrm{I}}^j)_j$. This implies that for each of the sets F_{I}^j the conditional probability of A, given that Player I's information is F_{I}^j, equals q_{I}. In other words, for each j,

$$\mathbf{P}\left(A \mid F_{\mathrm{I}}^j\right) = \frac{\mathbf{P}\left(A \cap F_{\mathrm{I}}^j\right)}{\mathbf{P}\left(F_{\mathrm{I}}^j\right)} = q_{\mathrm{I}}. \tag{9.35}$$

As this equality holds for every j, and $C = \bigcup_j F_{\mathrm{I}}^j$, it follows from Equation (9.35) that

$$\mathbf{P}(A \cap C) = \sum_j \mathbf{P}\left(A \cap F_{\mathrm{I}}^j\right) = q_{\mathrm{I}} \sum_j \mathbf{P}\left(F_{\mathrm{I}}^j\right) = q_{\mathrm{I}}\mathbf{P}(C). \tag{9.36}$$

We similarly derive that

$$\mathbf{P}(A \cap C) = q_{\mathrm{II}}\mathbf{P}(C). \tag{9.37}$$

Finally, since $\mathbf{P}(C) > 0$, Equations (9.36) and (9.37) imply that $q_{\mathrm{I}} = q_{\mathrm{II}}$, which is what we wanted to show. □

How do players arrive at a situation in which the probabilities q_{I} and q_{II} that they ascribe to a particular event A are common knowledge? In Example 9.31, each player calculates

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the conditional probability of A given a partition element of the other player, and comes to the conclusion that no matter which partition element of the other player is used for the conditioning, the conditional probability turns out to be the same. That is why q_i is common knowledge among the players for $i = \mathrm{I}, \mathrm{II}$.

In most cases the conditional probability of an event is not common knowledge, because it varies from one partition element to another. We can, however, describe a process of information transmission between the players that guarantees that these conditional probabilities will become common knowledge when the process is complete (see Exercises 9.25 and 9.26). Suppose that each player publicly announces the conditional probability he ascribes to event A given the information (i.e., the partition element) at his disposal. After each player has heard the other player's announcement, he can rule out some states of the world, because they are impossible: possible states of the world are only those in which the conditional probability that the other player ascribes to event A is the conditional probability that he publicly announced. Each player can then update the conditional probability that he ascribes to event A following the elimination of impossible states of the world, and again publicly announce the new conditional probability he has calculated. Following this announcement, the players can again rule out the states of the world in which the updated conditional probability of the other player differs from that which he announced, update their conditional probabilities, and announce them publicly. This can be repeated again and again. Using Aumann's Agreement Theorem (Theorem 9.32), it can be shown that at the end of this process the players will converge to the same conditional probability, which will be common knowledge among them (Exercise 9.28).

Example 9.33 We provide now an example of the dynamic process just described. More examples can be found in Exercises 9.25 and 9.26. Consider the following Aumann model of incomplete information:

- $N = \{\mathrm{I}, \mathrm{II}\}$.
- $Y = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.
- The information partitions of the players are

$$\mathcal{F}_{\mathrm{I}} = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}, \qquad \mathcal{F}_{\mathrm{II}} = \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4\}\}. \tag{9.38}$$

- The prior distribution is

$$\mathbf{P}_{\mathrm{II}}(\omega_1) = \mathbf{P}_{\mathrm{II}}(\omega_4) = \tfrac{1}{3}, \qquad \mathbf{P}_{\mathrm{II}}(\omega_2) = \mathbf{P}_{\mathrm{II}}(\omega_3) = \tfrac{1}{6}. \tag{9.39}$$

The partition elements $\mathcal{F}_{\mathrm{I}}$ and $\mathcal{F}_{\mathrm{II}}$ are as depicted graphically in Figure 9.5.

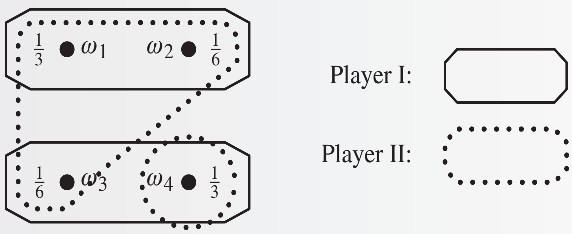

Figure 9.5 The information partitions and the prior distribution in Example 9.33

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Let $A = \{\omega_2, \omega_3\}$, and suppose that the true state of the world is ω_3. We will now trace the dynamic process described above. Player I announces the conditional probability $\mathbf{P}(A \mid \{\omega_3, \omega_4\}) = \frac{1}{3}$ that he ascribes to event A, given his information. Notice that in every state of the world Player I ascribes probability $\frac{1}{3}$ to event A, so that this announcement does not add any new information to Player II.

Next, Player II announces the conditional probability $\mathbf{P}(A \mid \{\omega_3, \omega_4\}) = \frac{1}{2}$ that he ascribes to A, given his information. This enables Player I to learn that the true state of the world is not ω_4, because if it were ω_4, Player II would have ascribed conditional probability 0 to the event A.

Player I therefore knows, after Player II's announcement, that the true state of the world is ω_3, and then announces that the conditional probability he ascribes to the event A is 1. This informs Player II that the true state of the world is ω_3, because if the true state of the world were ω_1 or ω_2 (the two other possible states, given Player II's information), Player I would have announced that he ascribed conditional probability $\frac{1}{3}$ to the event A. Player II therefore announces that the conditional probability he ascribes to the event A is 1, and this probability is now common knowledge among the two players.

It is left to the reader to verify that if the true state of the world is ω_1 or ω_2, the dynamic process described above will lead the two players to common knowledge that the conditional probability of the event A is $\frac{1}{3}$. ◀

Aumann's Agreement Theorem has important implications regarding the rationality of betting between two risk-neutral players (or two players who share the same level of risk aversion). To simplify the analysis, suppose that the two players bet that if a certain event A occurs, Player II pays Player I one dollar, and if event A fails to occur, Player I pays Player II one dollar instead. Labeling the probabilities that the players ascribe to event A as q_{I} and q_{II} respectively, Player I should be willing to take this bet if and only if $q_{\mathrm{I}} \geq \frac{1}{2}$, with Player II agreeing to the bet if and only if $q_{\mathrm{II}} \leq \frac{1}{2}$. Suppose that Player I accepts the bet. Then the fact that he has accepted the bet is common knowledge, which means that the fact that $q_{\mathrm{I}} \geq \frac{1}{2}$ is common knowledge. By the same reasoning, if Player II agrees to the bet, that fact is common knowledge, and therefore the fact that $q_{\mathrm{II}} \leq \frac{1}{2}$ is common knowledge. Using a proof very similar to that of Aumann's Agreement Theorem, we conclude that it is impossible for both facts to be common knowledge unless $q_{\mathrm{I}} = q_{\mathrm{II}} = \frac{1}{2}$, in which case the expected payoff for each player is 0, and there is no point in betting (see Exercises 9.29 and 9.30).

Note that the agreement theorem rests on two main assumptions:

- Both players share a common prior over Y.
- The probability that each of the players ascribes to event A is common knowledge among them.

Regarding the first assumption, the common prior distribution $\mathbf{P}$ is part of the Aumann model of incomplete information with beliefs and it is used to compute the players' beliefs given their partitions. As the following example shows, if each player's belief is computed from a different probability distribution, we obtain a more general model in which the agreement theorem does not hold. We will return to Aumann models with incomplete information and different prior distributions in Chapter 10.

Example 9.34 In this example we will show that if the two players have different priors, Theorem 9.32 does not hold. Consider the following Aumann model of incomplete information:

- $N = \{\mathrm{I}, \mathrm{II}\}$.
- $Y = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.
- The information that the two players have is given by

$$\mathcal{F}_{\mathrm{I}} = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}, \qquad \mathcal{F}_{\mathrm{II}} = \{\{\omega_1, \omega_4\}, \{\omega_2, \omega_3\}\}. \tag{9.40}$$

- Player I calculates his beliefs based on the following prior distribution:

$$\mathbf{P}_{\mathrm{I}}(\omega_1) = \mathbf{P}_{\mathrm{I}}(\omega_2) = \mathbf{P}_{\mathrm{I}}(\omega_3) = \mathbf{P}_{\mathrm{I}}(\omega_4) = \tfrac{1}{4}. \tag{9.41}$$

- Player II calculates his beliefs based on the following prior distribution:

$$\mathbf{P}_{\mathrm{II}}(\omega_1) = \mathbf{P}_{\mathrm{II}}(\omega_3) = \tfrac{2}{10}, \qquad \mathbf{P}_{\mathrm{II}}(\omega_2) = \mathbf{P}_{\mathrm{II}}(\omega_4) = \tfrac{3}{10}. \tag{9.42}$$

The only connected component in the graph corresponding to this Aumann model is Y (verify!), so that the only event that is common knowledge in any state of the world ω is Y. Let $A = \{\omega_1, \omega_3\}$. A quick calculation reveals that in each state $\omega \in Y$

$$\mathbf{P}_{\mathrm{I}}(A \mid F_{\mathrm{I}}(\omega)) = \tfrac{1}{2}, \qquad \mathbf{P}_{\mathrm{II}}(A \mid F_{\mathrm{II}}(\omega)) = \tfrac{2}{5}. \tag{9.43}$$

That is,

$$\left\{\omega \colon q_{\mathrm{I}} := \mathbf{P}(A \mid F_{\mathrm{I}}(\omega)) = \tfrac{1}{2}\right\} = Y, \qquad \left\{\omega \colon q_{\mathrm{II}} := \mathbf{P}(A \mid F_{\mathrm{II}}(\omega)) = \tfrac{2}{5}\right\} = Y. \tag{9.44}$$

From the definition of the knowledge operator it follows that the facts that $q_{\mathrm{I}} = \frac{1}{2}$ and $q_{\mathrm{II}} = \frac{2}{5}$ are common knowledge in every state of the world. In other words, it is common knowledge in every state of the world that the players ascribe different probabilities to the event A. This does not contradict Theorem 9.32 because the players do not share a common prior. In fact, this result is not surprising; because the players start off by "agreeing" that their initial probability distributions diverge (and that fact is common knowledge), it is no wonder that it is common knowledge among them that they ascribe different probabilities to event A (after learning which partition element they are in). ◀

Example 9.35 In this example we will show that even if the players share a common prior, if the fact that "Player II ascribes probability q_{II} to event A" is not common knowledge, Theorem 9.32 does not hold; that is, it is possible that $q_{\mathrm{I}} \neq q_{\mathrm{II}}$. Consider the following Aumann model of incomplete information:

- $N = \{\mathrm{I}, \mathrm{II}\}$.
- $Y = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.
- The players' information partitions are

$$\mathcal{F}_{\mathrm{I}} = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}, \qquad \mathcal{F}_{\mathrm{II}} = \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4\}\}. \tag{9.45}$$

- The common prior distribution is

$$\mathbf{P}(\omega_1) = \mathbf{P}(\omega_2) = \mathbf{P}(\omega_3) = \mathbf{P}(\omega_4) = \tfrac{1}{4}. \tag{9.46}$$

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The partitions $\mathcal{F}_{\mathrm{I}}$ and $\mathcal{F}_{\mathrm{II}}$ are depicted graphically in Figure 9.6.

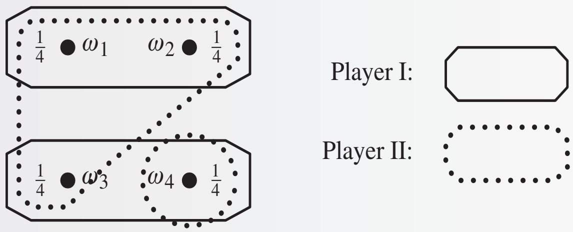

Figure 9.6 The partitions of the players in Example 9.35 and the common prior

The only connected component in the graph corresponding to this Aumann model is Y (verify!). Let $A = \{\omega_1, \omega_3\}$. In each state of the world, the probability that Player I ascribes to event A is $q_{\mathrm{I}} = \frac{1}{2}$:

$$\left\{w \in Y : q_{\mathrm{I}} = \mathbf{P}(A \mid F_{\mathrm{I}}(\omega)) = \tfrac{1}{2}\right\} = Y, \tag{9.47}$$

and therefore the fact that $q_{\mathrm{I}} = \frac{1}{2}$ is common knowledge in every state of the world.

In states of the world ω_1, ω_2, and ω_3, Player II ascribes probability $\frac{2}{3}$ to event A:

$$\left\{w \in Y : q_{\mathrm{II}} = \mathbf{P}\left(A \mid F_{\mathrm{II}}(\omega) = \tfrac{2}{3}\right)\right\} = \{\omega_1, \omega_2, \omega_3\} \not\subseteq Y, \tag{9.48}$$

and in state of the world ω_4 he ascribes probability 0 to A. Since the only event that is common knowledge in any state of the world is Y, the event "Player II ascribes probability $\frac{2}{3}$ to A" is not common knowledge in any state of the world. For that reason, the fact that $q_{\mathrm{I}} \neq q_{\mathrm{II}}$ does not contradict Theorem 9.32.

Note that in state of the world ω_1, Player I knows that the state of the world is in $\{\omega_1, \omega_2\}$, and therefore he knows that Player II's information is $\{\omega_1, \omega_2, \omega_3\}$, and thus he (Player I) knows that Player II ascribes probability $q_{\mathrm{II}} = \frac{2}{3}$ to the event A. However, the fact that Player II ascribes probability $q_{\mathrm{II}} = \frac{2}{3}$ to event A is not common knowledge among the players in the state of the world ω_1. This is so because in that state of the world Player II cannot exclude the possibility that the state of the world is ω_3 (he ascribes to this probability $\frac{1}{3}$). If the state of the world is ω_3, Player I knows that the state of the world is in $\{\omega_3, \omega_4\}$, and therefore he (Player I) cannot exclude the possibility that the state of the world is ω_4 (he ascribes to this probability $\frac{1}{2}$), in which case Player II knows that the state of the world is ω_4, and then the probability that Player II ascribes to event A is 0 ($q_{\mathrm{II}} = 0$). Therefore, in state of the world ω_1 Player II ascribes probability $\frac{1}{3}$ to the fact that Player I ascribes probability $\frac{1}{2}$ to Player II ascribing probability 0 to event A. Thus, in state of the world ω_1 Player I knows that $q_{\mathrm{II}} = \frac{2}{3}$, yet this event is not common knowledge among the players. ◀

Before we proceed, let us recall that an Aumann model consists of two elements:

- The partitions of the players, which determine the information (knowledge) they possess.
- The common prior **P** that, together with the partitions, determines the beliefs of the players.

The knowledge structure in an Aumann model is independent of the common prior **P**. Furthermore, as we saw in Example 9.34, even when there is no common prior, and instead every player has a different subjective prior distribution, the underlying knowledge structure and the set of common knowledge events are unchanged. Not surprisingly, the Agreement Theorem (Theorem 9.32), which deals with beliefs, depends on the assumption of a common prior, while the common knowledge characterization theorem (Theorem 9.24, page 333) is independent of the assumption of a common prior.

9.3 An infinite set of states of the world

Thus far in the chapter, we have assumed that the set of states of the world is finite. What if this set is infinite? With regard to set-theoretic operations, in the case of an infinite set of states of the world we can make use of the same operations that we implemented in the finite case. On the other hand, dealing with the beliefs of the players requires using tools from probability theory, which in the case of an infinite set of states of the world means that we need to ensure that this set is a measurable space.

A *measurable space* is a pair $(Y, \mathcal{F})$, with Y denoting a set, and $\mathcal{F}$ a σ-algebra over Y. This means that $\mathcal{F}$ is a family of subsets of Y that includes the empty set, is closed under complementation (i.e., if $A \in \mathcal{F}$ then $A^c = Y \setminus A \in \mathcal{F}$), and is closed under countable unions (i.e., if $(A_n)_{n=1}^\infty$ is a family of sets in $\mathcal{F}$ then $\bigcup_{n=1}^\infty A_n \in \mathcal{F}$). An event is any element of $\mathcal{F}$. In particular, the partitions of the players, $\mathcal{F}_i$, are composed solely of elements of $\mathcal{F}$.

The collection of all the subsets of Y, 2^Y, is a σ-algebra over Y, and therefore $(Y, 2^Y)$ is a measurable space. This is in fact the measurable space we used, without specifically mentioning it, in all the examples we have seen so far in which Y was a finite set. All the infinite sets of states of the world Y that we will consider in the rest of the section will be a subset of a Euclidean space, and the σ-algebra $\mathcal{F}$ will be the σ-algebra of Borel sets, that is, the smallest σ-algebra that contains all the relatively open sets[11] in Y.

The next example shows that when the set of states of the world is infinite, knowledge is not equivalent to belief with probability 1 (in contrast to the finite case; see Theorem 9.29 on page 336).

Example 9.36 Consider an Aumann model of incomplete information in which the set of players $N = \{\mathrm{I}\}$ contains only one player, the set of states of the world is $Y = [0, 1]$, the σ-algebra $\mathcal{F}$ is the σ-algebra of Borel sets,[12] and the player has no information, which means that $\mathcal{F}_{\mathrm{I}} = \{Y\}$. The common prior **P** is the uniform distribution over the interval $[0, 1]$.

Since there is only one player and his partition contains only one element, the only event that the player knows (in any state of the world ω) is Y. Let A be the set of irrational numbers in the interval $[0, 1]$, which is in $\mathcal{F}$. As the set A does not contain Y, the player does not know A. But $\mathbf{P}(A \mid F_{\mathrm{I}}(\omega)) = \mathbf{P}(A \mid Y) = \mathbf{P}(A) = 1$ for all $\omega \in Y$. ◀

11 When $Y \subseteq \mathbb{R}^d$, a set $A \subseteq Y$ is *relatively open in* Y if it is equal to the intersection of Y with an open set in $\mathbb{R}^d$.

12 In this case the σ-algebra of Borel sets is the smallest σ-algebra that contains all the open intervals in $[0, 1]$, and the intervals of the form $[0, \alpha)$ and $(\alpha, 1]$ for $\alpha \in (0, 1)$.

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Next we show that when the set of states of the world is infinite, the very notion of knowledge hierarchy can be problematic. To make use of the knowledge structure, for every event $A \in \mathcal{F}$ the event $K_i A$ must also be an element of $\mathcal{F}$: if we can talk about the event A, we should also be able to talk about the event that "player i knows A."

Is it true that for every σ-algebra, every partition $(\mathcal{F}_i)_{i \in N}$ representing the information of the players, and every event $A \in \mathcal{F}$, it is necessarily true that $K_i A \in \mathcal{F}$? When the set of states of the world is infinite, the answer to that question is no. This is illustrated in the next example, which uses the fact that there is a Borel set in the unit square whose projection onto the first coordinate is not a Borel set in the interval $[0, 1]$ (see Suslin [1917]).

Example 9.37 Consider the following Aumann model of incomplete information:

- There are two players: $N = \{\mathrm{I}, \mathrm{II}\}$.
- The space of states of the world is the unit square: $Y = [0, 1] \times [0, 1]$, and $\mathcal{F}$ is the σ-algebra of Borel sets in the unit square.
- For $i = \mathrm{I}, \mathrm{II}$, the information of player i is the i-th coordinate of ω; that is, for each $x, y \in [0, 1]$ denote

$$A_x = \{(x, y) \in Y : 0 \leq y \leq 1\}, \qquad B_y = \{(x, y) \in Y : 0 \leq x \leq 1\}. \tag{9.49}$$

A_x is the set of all points in Y whose first coordinate is x, and B_y is the set of all points in Y whose second coordinate is y. We then have

$$\mathcal{F}_{\mathrm{I}} = \{A_x : 0 \leq x \leq 1\}, \qquad \mathcal{F}_{\mathrm{II}} = \{B_y : 0 \leq y \leq 1\}. \tag{9.50}$$

In words, Player I's partition is the set of vertical sections of Y, and the partition of Player II is the set of horizontal sections of Y. Thus, for any $(x, y) \in Y$ Player I knows the x-coordinate and Player II knows the y-coordinate.

Let $E \subseteq Y$ be a Borel set whose projection onto the x-axis is not a Borel set, i.e., the set

$$F = \{x \in [0, 1] : \text{there exists } y \in [0, 1] \text{ such that } (x, y) \in E\} \tag{9.51}$$

is not a Borel set, and hence $F^c = Y \setminus F$ is also not a Borel set in $[0, 1]$. Player I knows that the event E does not obtain when the x-coordinate is not in F:

$$K_{\mathrm{I}}(E^c) = F^c \times [0, 1]. \tag{9.52}$$

This implies that despite the fact that the set E^c is a Borel set, the set of states of the world in which Player I knows the event E^c is not a Borel set. ◀

In spite of the technical difficulties indicated by Examples 9.36 and 9.37, in Chapter 10 we develop a general model of incomplete information that allows infinite sets of states of the world.

9.4 The Harsanyi model of games with incomplete information

In our treatment of the Aumann model of incomplete information, we concentrated on concepts such as mutual knowledge and mutual beliefs among players regarding the true

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state of the world. Now we will analyze games with incomplete information, which are models in which the incomplete information is about the game that the players play. In this case, a state of nature consists of all the parameters that have a bearing on the payoffs, that is, the set of actions of each player and his payoff function. This is why the state of nature in this case is also called the *payoff-relevant parameter* of the game. This model was first introduced by John Harsanyi [1967], nine years prior to the introduction of the Aumann model of incomplete information, and was the first model of incomplete information used in game theory.

The Harsanyi model consists of two elements. The first is the games in which the players may participate, which will be called "state games," and are the analog of states of nature in Aumann's model of incomplete information. The second is the beliefs that the players have about both the state games and the beliefs of the other players.

Since the information the player has of the game is incomplete, a player is characterized by his beliefs about the state of nature (namely, the state game) and the beliefs of the other players. This characterization was called by Harsanyi the *type* of the player. In fact, as we shall see, a player's type in a Harsanyi model is equivalent to his belief hierarchy in an Aumann model. Just as we did when studying the Aumann model of incomplete information, we will also assume here that the space of states of the world is finite, so that the number of types of each player is finite. We will further assume that every player knows his own type, and that the set of types is common knowledge among the players.

Example 9.38 Harry (Player I, the row player) and William (Player II, the column player) are playing a game in which the payoff functions are determined by one of the two matrices appearing in Figure 9.7. William has two possible actions (t and b), while Harry has two or three possible actions (either T and B, or T, C, and B), depending on the payoff function.

William

		t	b
Harry	T	1, 0	0, 2
	B	0, 3	1, 0

State game G_1

William

		t	b
	T	1, 1	1, 0
Harry	C	0, 2	1, 1
	B	1, 0	0, 2

State game G_1

Figure 9.7 The state games in the game in Example 9.38

Harry knows the payoff function (and therefore in particular knows whether he has two or three actions available). William only knows that the payoff functions are given by either G_1 or G_2. He ascribes probability p to the payoff function being given by G_1 and probability $1 - p$ to the payoff function being given by G_2. This description is common knowledge among Harry and William.[13]

13 In other words, both Harry and William know that this is the description of the game, each one knows that the other knows that this is the description of the game, and so on.

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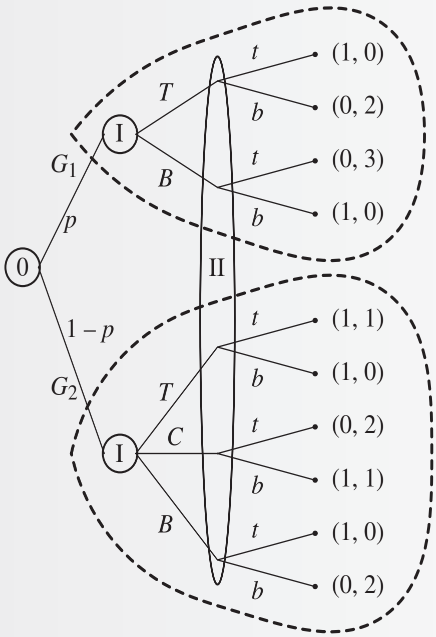

Figure 9.8 The game in Example 9.38 in extensive form

This situation can be captured by the extensive-form game appearing in Figure 9.8. In this game, Nature chooses G_1 or G_2 with probability p or $1-p$, respectively, with the choice known to Harry but not to William. In Figure 9.8 the state games are delineated by broken lines. Neither of the two state games is a subgame (Definition 3.11, page 45), because there are information sets that contain vertices from two state games. ◀

The game appearing in Figure 9.8 is the game that Harsanyi suggested as the model for the situation described in Example 9.8. Such a game is called a *Harsanyi game with incomplete information* and defined as follows.

Definition 9.39 *A* Harasanyi game with incomplete information *is a vector* $(N, (T_i)_{i\in N}, p, S, (s_t)_{t\in \times_{i\in N} T_i})$ *where:*

- *N is a finite set of players.*
- *T_i is a finite set of* types *for player i, for each $i \in N$. The set of type vectors is denoted by $T = \times_{i\in N} T_i$.*
- *$p \in \Delta(T)$ is a probability distribution over the set of type vectors*[14] *that satisfies $p(t_i) := \sum_{t_{-i}\in T_{-i}} p(t_i, t_{-i}) > 0$ for every player $i \in N$ and every type $t_i \in T_i$.*
- *S is a set of states of nature, which will be called* state games.[15] *Every state of nature $s \in S$ is a vector $s = (N, (A_i)_{i\in N}, (u_i)_{i\in N})$, where A_i is a nonempty set of actions of player i and $u_i : \times_{i\in N} A_i \to \mathbb{R}$ is the payoff function of player i.*

14 Recall that $T_{-i} = \times_{j\neq i} T_j$ and $t_{-i} = (t_j)_{j\neq i}$.

15 For the sake of notational convenience, every state game will be represented by a game in strategic form. Everything stated in this section also holds true in the case in which every state game is represented by a game in extensive form.

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- *$s_t = (N, (A_i(t_i))_{i \in N}, (u_i(t))_{i \in N}) \in S$ is the state game for the type vector t, for every $t \in T$. Thus, player i's action set in the state game s_t depends on his type t_i only, and is independent of the types of the other players.*

A game with incomplete information proceeds as follows:

- A chance move selects a type vector $t = (t_1, t_2, \ldots, t_n) \in T$ according to the probability distribution p.
- Every player i knows the type t_i that has been selected for him (i.e., his own type), but does not know the types $t_{-i} = (t_j)_{j \neq i}$ of the other players.
- The players select their actions simultaneously: each player i, knowing his type t_i, selects an action $a_i \in A_i(t_i)$.
- Every player i receives the payoff $u_i(t; a)$, where $a = (a_1, a_2, \ldots, a_n)$ is the vector of actions that have been selected by the players.

Player i, of type t_i, does not know the types of the other players; he has a belief about their types. This belief is the conditional probability distribution $p(\cdot \mid t_i)$ over the set $T_{-i} = \times_{j \neq i} T_j$ of the vectors of the types of the other players. The set of actions that player i believes that he has at his disposal is part of his type, and therefore the set $A_i(t_i)$ of actions available to player i in the state game s_t is determined by t_i only, and not by the types of other players. It is possible for a player, though, not to know the sets of actions that the other players have at their disposal, as we saw in Example 9.38. He does have beliefs about the action sets of the other players, which are derived from his beliefs about their types. The payoff of player i depends on the state game s_t and on the vector of actions a selected by the players, so that it depends on the vector of types t in two ways:

- The set of action vectors $A(t) := \times_{i \in N} A_i(t_i)$ in the state game s_t depends on t.
- The payoff function $u_i(t)$ in the state game s_t depends on t; even when the sets of action $A_j(t_j)$ do not depend on the players' types, player i's payoff depends on the types of all players.

Because a player may not know for certain the types of the other players, he may not know for certain the state of nature, which in turn implies that he may not know for certain his own payoff function. In summary, a Harsanyi game is an extensive-form game consisting of several state games related to each other through information sets, as depicted in Figure 9.8.

Remark 9.40 *The Harsanyi model, defined in Definition 9.39, provides a tool to describe the incomplete information a player may have regarding the possible action sets of the other players, their utility functions, and even the set of other players active in the game: a player j who is not active in state game s_t has type t_j such that $A_j(t_j)$ contains only one action. This interpretation makes sense because the set of equilibria in the game is independent of the payoffs to such a type t_j of player j (see Exercise 9.45).* ♦

Remark 9.41 *We note that in reality there is no chance move that selects one of the state games: the players play one, and only one, of the possible state games. The Harsanyi game is a construction we use to present the situation of interest to us, by describing the beliefs of the players about the game that they are playing. For instance, suppose that* *in in Example 9.38 the players play the game G_1. Since William does not know whether the game he is playing is G_1 or G_2, from his standpoint he plays a state game that can be either one of these games. Therefore, he constructs the extensive-form game that is described in Figure 9.8, and the situation that he faces is a Harsanyi game with incomplete information.* ♦

In the economic literature the Harsanyi model is often referred to as the *ex ante stage*[16] model as it captures the situation of the players before knowing their types. The situation obtained after the chance move has selected the types of the players is referred to as the *interim stage* model. This model corresponds to an Aumann situation of incomplete information that captures the situation in a specific state of the world (Definition 9.5, page 324).

The next theorem generalizes Example 9.38, and states that any Harsanyi game with incomplete information can be described as an extensive-form game. Its proof is left to the reader (Exercise 9.35).

Theorem 9.42 *Every (Harsanyi) game with incomplete information can be described as an extensive-form game (with moves of chance and information sets).*

9.4.1 Belief hierarchies

In any Aumann model of incomplete information we can attach a belief hierarchy to every state of the world (Theorem 9.30, page 337). Similarly, in any Harsanyi game with incomplete information we can attach a belief hierarchy to every type vector. We illustrate this point by the following example.

Example 9.43 The residents of the town of Smallville live in a closed and supportive tight-knit community. The personality characteristic that they regard as most important revolves around the question of whether a person puts his family life ahead of his career, or his career ahead of his family. Kevin, the local matchmaker, approaches two of the residents, Abe and Sarah, informing them that in his opinion they would be well suited as a couple. It is well known in the community from past experience that Kevin tends to match a man who stresses his career with a woman who emphasizes family life, and a man who puts family first with a woman who is career-oriented, but there were several instances in the past when Kevin did not stick to that rule. The distribution of past matches initiated by Kevin is presented in the following diagram (Figure 9.9).

	Family Woman	Career Woman
Family Man	$\frac{1}{10}$	$\frac{3}{10}$
Career Man	$\frac{4}{10}$	$\frac{2}{10}$

Figure 9.9 Player types with prior distribution

16 The Latin expression *ex ante* means “before.”

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The above verbal description can be presented as a Harsanyi model (without specifying the state games) in the following way:

- The set of players is $N = \{\text{Abe, Sarah}\}$.
- Abe's set of types is $T_{\text{A}} = \{\text{Careerist, Family}\}$.
- Sarah's set of types is $T_{\text{S}} = \{\text{Careerist, Family}\}$.
- Because this match is one of Kevin's matches, the probability distribution p over $T = T_{\text{A}} \times T_{\text{S}}$ is calculated from past matches of Kevin; namely, it is the probability distribution given in Figure 9.9.
- Since the state game corresponding to each pair of types is not specified, we denote the set of states of nature by $S = \{s_{CC}, s_{CF}, s_{FC}, s_{FF}\}$, without specifying the details of each state game. For each state game, the left index indicates the type of Abe ("C" for Career man, "F" for Family man) and the right index indicates the type of Sarah ("C" for Career woman, "F" for Family woman).

As each player knows his own type, armed with knowledge of the past performance of the match-maker (the prior distribution in Figure 9.9), each player can calculate the conditional probability of the type of the other player. For example, if Abe is a careerist, he can conclude that the conditional probability that Sarah is also a careerist is

$$p(\text{Sarah is a careerist} \mid \text{Abe is a careerist}) = \frac{\frac{2}{10}}{\frac{2}{10} + \frac{4}{10}} = \frac{1}{3},$$

while if Abe is a family man he can conclude that the conditional probability that Sarah is a careerist is

$$p(\text{Sarah is a careerist} \mid \text{Abe is a family man}) = \frac{\frac{3}{10}}{\frac{1}{10} + \frac{3}{10}} = \frac{3}{4}.$$

◀

Given his type, every player can calculate the infinite belief hierarchy. The continuation of our example illustrates this.

Example 9.43 (*Continued*) Suppose that Abe's type is careerist. As shown above, in that case his first-order beliefs about the state of nature is $[\frac{2}{3}(s_{CF}), \frac{1}{3}(s_{CC})]$. His second-order beliefs are as follows: he ascribes probability $\frac{2}{3}$ to the state of nature being s_{CF}, in which case Sarah's beliefs are $[\frac{1}{5}(s_{FF}), \frac{4}{5}(s_{FC})]$ (this follows from a similar calculation to the one performed above; verify!), and he ascribes probability $\frac{1}{3}$ to the state of nature being s_{CC}, in which case Sarah's beliefs are $[\frac{3}{5}(s_{CF}), \frac{2}{5}(s_{CC})]$. Abe's higher-order beliefs can similarly be calculated. ◀

When we analyze a Harsanyi game without specifying the state game corresponding to each state of nature, we will refer to it as a *Harsanyi model of incomplete information*. Such a model is equivalent to an Aumann model of incomplete information in the sense that every situation of incomplete information that can be analyzed using one model can be analyzed using the other one: a partition element $F_i(\omega)$ of player i in an Aumann model is his type in a Harsanyi model. Let $(N, (T_i)_{i \in N}, p, S, (s_t)_{t \in \times_{i \in N} T_i})$ be a Harsanyi model of incomplete information. An Aumann model of incomplete information describing the same structure of mutual information is the model in which the set of states of the world

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is the set of type vectors that have positive probability

$$Y = \{t \in T : p(t) > 0\}. \tag{9.53}$$

The partition of each player i is given by his type: for every type $t_i \in T_i$, there is a partition element $F_i(t_i) \in \mathcal{F}_i$, given as follows:

$$F_i(t_i) = \{(t_i, t_{-i}) : t_{-i} \in T_{-i}, p(t_i, t_{-i}) > 0\}. \tag{9.54}$$

The common prior is $\mathbf{P} = p$.

In the other direction, let $(N, Y, (\mathcal{F}_i)_{i \in N}, \mathfrak{s})$ be an Aumann model of incomplete information over a set S of states of nature. A corresponding Harsanyi model of incomplete information is given by the model in which the set of types of each player i is the set of his partition elements $\mathcal{F}_i$:

$$T_i = \{F_i \in \mathcal{F}_i\}, \tag{9.55}$$

and the probability distribution p is given by

$$p(F_1, F_2, \ldots, F_n) = \mathbf{P}\left(\bigcap_{i \in N} F_i\right). \tag{9.56}$$

Note that the intersection in this equation may be empty, or it may contain only one state of the world, or it may contain several states of the world. If the intersection is empty, the corresponding type vector is ascribed a probability of 0. If the intersection contains more than one state of the world, then in the Aumann model of incomplete information no player can distinguish between these states. The Harsanyi model identifies all these states as one state, and ascribes to the corresponding type vector the sum of their probabilities.

This correspondence shows that a state of the world in an Aumann model of incomplete information is a vector containing the state of nature and the type of each player. The state of nature describes the payoff-relevant parameters, and the player's type describes his beliefs. This is why we sometimes write a state of the world ω in the following form (we will expand on this in Chapter 11):

$$\omega = (\mathfrak{s}(\omega); t_1(\omega), t_2(\omega), \ldots, t_n(\omega)), \tag{9.57}$$

where $\mathfrak{s}(\omega)$ is the state of nature and $t_i(\omega)$ is player i's type in the state of the world ω.

The following conclusion is a consequence of Theorem 9.30 (page 337) and the equivalence between the Aumann model and the Harsanyi model.

Theorem 9.44 *In a Harsanyi model of incomplete information, every state of the world* $\omega = (\mathfrak{s}(\omega); t_1(\omega), t_2(\omega), \ldots, t_n(\omega))$ *uniquely determines the belief hierarchy of each player over the state of nature and the beliefs of the other players.*

9.4.2 Strategies and payoffs

In the presentation of a game with incomplete information as an extensive-form game, each type $t_i \in T_i$ corresponds to an information set of player i. It follows that a *pure* *strategy*[17] of player i is a function $s_i : T_i \to \bigcup_{t_i \in T_i} A_i(t_i)$ that satisfies

$$s_i(t_i) \in A_i(t_i), \quad \forall t_i \in T_i. \tag{9.58}$$

In words, $s_i(t_i)$ is the action specified by the strategy s_i for player i of type t_i (which is an action available to him as a player of type t_i). A *mixed strategy* of player i is, as usual, a probability distribution over his pure strategies. A *behavior strategy* σ_i of player i is a function mapping each type $t_i \in T_i$ to a probability distribution over the actions available to that type. Notationally, $\sigma_i : T_i \to \bigcup_{t_i \in T_i} \Delta(A_i(t_i))$ that satisfies

$$\sigma_i(t_i) = (\sigma_i(t_i; a_i))_{a_i \in A_i(t_i)} \in \Delta(A_i(t_i)). \tag{9.59}$$

In words, $\sigma_i(t_i; a_i)$ is the probability that player i of type t_i chooses the action a_i. A Harsanyi game is an extensive-form game with perfect recall (Definition 9.39, page 347), and therefore by Kuhn's Theorem (Theorem 4.49, page 118) every mixed strategy is equivalent to a behavior strategy. For this reason, there is no loss of generality in using only behavior strategies, which is indeed what we do in this section.

Remark 9.45 *Behavior strategy, as defined here, is a behavior strategy in a Harsanyi game in which the state game corresponding to t is a strategic-form game. If the state game is an extensive-form game, then $A_i(t_i)$ is the set of pure strategies in that game, so that $\Delta(A_i(t_i))$ is the set of mixed strategies of this state game. In this case, a strategy $\sigma_i : T_i \to \bigcup_{t_i \in T_i} \Delta(A_i(t_i))$ in which $\sigma_i(t_i) \in \Delta(A_i(t_i))$ is not a behavior strategy of the Harsanyi game. Rather, a behavior strategy is a function $\sigma_i : T_i \to \bigcup_{t_i \in T_i} \mathcal{B}_i(t_i)$ with $\sigma_i(t_i) \in \mathcal{B}_i(t_i)$ for every $t_i \in T_i$, where $\mathcal{B}_i(t_i)$ is the set of behavior strategies of player i in the state game $s_{(t_i, t_{-i})}$, which is the same for all $t_{-i} \in T_{-i}$. The distinction between these definitions is immaterial to the presentation in this section, and the results obtained here apply whether the state game is given in strategic form or in extensive form.* ♦

If the vector of the players' behavior strategies is $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, and the vector of types that is selected by the chance move is $t = (t_1, \ldots, t_n)$, then each vector of actions $(a_1, \ldots, a_n)$ is selected with probability

$$\sigma_1(t_1; a_1) \times \sigma_2(t_2; a_2) \times \cdots \times \sigma_n(t_n; a_n). \tag{9.60}$$

Player i's expected payoff, which we denote by $U_i(t; \sigma)$, is therefore

$$U_i(t; \sigma) := \mathbf{E}_\sigma[u_i(t)] = \sum_{a \in A(t)} \sigma_1(t_1; a_1) \times \cdots \times \sigma_n(t_n; a_n) \times u_i(t; a). \tag{9.61}$$

It follows that when the players implement strategy vector σ, the expected payoff in the game for player i is

$$U_i(\sigma) := \sum_{t \in T} p(t) U_i(t; \sigma). \tag{9.62}$$

This is the expected payoff for player i at the ex ante stage, that is, before he has learned what type he is. After the chance move, the vector of types has been selected, and the

17 We use the notation s_i for a pure strategy of player i, and s_t for the state game that corresponds to the type vector t.

conditional expected payoff to player i of type t_i is

$$U_i(\sigma \mid t_i) := \sum_{t_{-i} \in T_{-i}} p(t_{-i} \mid t_i) U_i((t_i, t_{-i}); \sigma), \tag{9.63}$$

where

$$p(t_{-i} \mid t_i) = \frac{p(t_i, t_{-i})}{\sum_{t'_{-i} \in T_{-i}} p(t_i, t'_{-i})} = \frac{p(t_i, t_{-i})}{p(t_i)}. \tag{9.64}$$

This is the expected payoff of player i at the interim stage. The connection between the ex ante (unconditional expected) payoff $U_i(\sigma)$ and the interim (conditional) payoff $(U_i(\sigma \mid t_i))_{t_i \in T_i}$ is given by the equation

$$U_i(\sigma) = \sum_{t_i \in T_i} p(t_i) U_i(\sigma \mid t_i). \tag{9.65}$$

Indeed,

$$\sum_{t_i \in T_i} p(t_i) U_i(\sigma \mid t_i) = \sum_{t_i \in T_i} p(t_i) \sum_{t_{-i} \in T_{-i}} p(t_{-i} \mid t_i) U_i((t_i, t_{-i}); \sigma) \tag{9.66}$$

$$= \sum_{t_{-i} \in T_{-i}} \sum_{t_i \in T_i} p(t_i) p(t_{-i} \mid t_i) U_i((t_i, t_{-i}); \sigma) \tag{9.67}$$

$$= \sum_{t \in T} p(t) U_i(t; \sigma) \tag{9.68}$$

$$= U_i(\sigma). \tag{9.69}$$

Equation (9.66) follows from Equation (9.63), Equation (9.67) is a rearrangement of sums, Equation (9.68) is a consequence of the definition of conditional probability, and Equation (9.69) follows from Equation (9.62).

9.4.3 Equilibrium in games with incomplete information

As we pointed out, Harsanyi games with incomplete information may be analyzed at two separate points in time: at the ex ante stage, before the players know their types, and at the interim stage, after they have learned what types they are. Accordingly, two different types of equilibria can be defined. The first equilibrium concept, which is Nash equilibrium in Harsanyi games, poses the requirement that no player can profit by a unilateral deviation before knowing his type. The second equilibrium concept, called Bayesian equilibrium, poses the requirement that no player i can profit by deviating at the interim stage, after learning his type t_i.

Definition 9.46 *A strategy vector σ^* is a* Nash equilibrium *if*[18] *for each player i and each strategy σ_i of player i,*

$$U_i(\sigma^*) \geq U_i(\sigma_i, \sigma^*_{-i}). \tag{9.70}$$

As every game with incomplete information can be described as an extensive-form game (Theorem 9.42), every finite extensive-form game has a Nash equilibrium in mixed

18 Recall that $\sigma^*_{-i} = (\sigma^*_j)_{j \neq i}$.

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strategies (Theorem 5.13, page 152), and every mixed strategy is equivalent to a behavior strategy, we arrive at the following conclusion:

Theorem 9.47 *Every game with incomplete information in which the set of types is finite and the set of actions of each type is finite has a Nash equilibrium in behavior strategies.*

Remark 9.48 *When the set of player types is countable, a Nash equilibrium is still guaranteed to exist (see Exercise 9.43). In contrast, when the set of player types is uncountable, it may be the case that all the equilibria involve strategies that are not measurable (see Simon [2003]).* ♦

Definition 9.49 *A strategy vector* $\sigma^* = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ *is a* Bayesian equilibrium *if for each player* $i \in N$*, each type* $t_i \in T_i$*, and each possible*[19] *action* $a_i \in A_i(t_i)$*,*

$$U_i(\sigma^* \mid t_i) \geq U_i((a_i, \sigma_{-i}^*) \mid t_i). \tag{9.71}$$

An equivalent way to define Bayesian equilibrium is by way of an auxiliary game, called the *agent-form game*.

Definition 9.50 *Let* $\Gamma = (N, (T_i)_{i\in N}, p, S, (s_t)_{t\in T})$ *be a game with incomplete information. The* agent-form game $\widehat{\Gamma}$ *corresponding to* Γ *is the following game in strategic form:*

- *The set of players is* $\cup_{i\in N} T_i$*: every type of each player in* Γ *is a player in* $\widehat{\Gamma}$.
- *The set of pure strategies of player* t_i *in* $\widehat{\Gamma}$ *is* $A_i(t_i)$*, the set of available actions of that type in the game* Γ.
- *The payoff function* $\widehat{u}_{t_i}$ *of player* t_i *in* $\widehat{\Gamma}$ *is given by*

$$\widehat{u}_{t_i}(a) := \sum_{t_{-i}\in T_{-i}} p(t_{-i} \mid t_i) u_i(t_i, t_{-i}; (a_j(t_j))_{j\in N}), \tag{9.72}$$

where $a = (a_j(t_j))_{j\in N, t_j\in T_j}$ *denotes a vector of actions of all the players in* $\widehat{\Gamma}$.

The payoff $\widehat{u}(t_i; a)$ of player t_i in $\widehat{\Gamma}$ equals the expected payoff of player i of type t_i in the game Γ when he chooses action $a_i(t_i)$, and for any $j \neq i$, player j of type t_j chooses action $a_j(t_j)$. The conditional probability in Equation (9.72) is well defined because we have assumed that $p(t_i) > 0$ for each player i and each type t_i.

Note that every behavior strategy $\sigma_i = (\sigma_i(t_i))_{t_i\in T_i}$ of player i in the game Γ naturally defines a mixed strategy for the players in T_i in the agent-form game $\widehat{\Gamma}$. Conversely, every vector of mixed strategies of the players in T_i in the agent-form game $\widehat{\Gamma}$ naturally defines a behavior strategy $\sigma_i = (\sigma_i(t_i))_{t_i\in T_i}$ of player i in the game Γ.

Theorem 9.51 relates Bayesian equilibria in a game Γ to the Nash equilibria in the corresponding agent-form game $\widehat{\Gamma}$. The proof of the theorem is left to the reader (Exercise 9.44).

Theorem 9.51 *A strategy vector* $\sigma^* = (\sigma_i^*)_{i\in N}$ *is a Bayesian equilibrium in a game* Γ *with incomplete information if and only if the strategy vector* $(\sigma_i^*(t_i))_{i\in N, t_i\in T_i}$ *is a Nash equilibrium in the corresponding agent-form game* $\widehat{\Gamma}$.

19 In Equation (9.71), $U_i((a_i, \sigma_{-i}^*) \mid t_i)$ is the payoff of player i of type t_i, when all other players use σ^*, and he plays action a_i.

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As every game in strategic form in which the set of pure strategies available to each player is finite has a Nash equilibrium (Theorem 5.13, page 152), we derive the next theorem:

Theorem 9.52 *Every game with incomplete information in which the set of types is finite and the set of actions of each type is finite has a Bayesian equilibrium (in behavior strategies).*

As already noted, the two definitions of equilibrium presented in this section (Nash equilibrium and Bayesian equilibrium) express two different perspectives on the game: does each player regard the game prior to knowing his type or after knowing it? Theorem 9.53 states that these two definitions are in fact equivalent.

Theorem 9.53 (Harsanyi [1967]) *In a game with incomplete information in which the number of types of each player is finite, every Bayesian equilibrium is also a Nash equilibrium, and conversely every Nash equilibrium is also a Bayesian equilibrium.*

In other words, no player has a profitable deviation *after* he knows which type he is if and only if he has no profitable deviation *before* knowing his type. Recall that in the definition of a game with incomplete information we required that $p(t_i) > 0$ for each player i and each type $t_i \in T_i$. This is essential for the validity of Theorem 9.53, because if there is a type that is chosen with probability 0 in a Harsanyi game, the action selected by a player of that type has no effect on the payoff. In particular, in a Nash equilibrium a player of this type can take any action. In contrast, because the conditional probabilities $p(t_{-i} \mid t_i)$ in Equation (9.64) are not defined for such a type, the payoff function of this type is undefined, and in that case we cannot define a Bayesian equilibrium.

Proof of Theorem 9.53: The idea of the proof runs as follows. Because the expected payoff of player i in a game with incomplete information is the expectation of the conditional expected payoff of all of his types t_i, and because the probability of each type is positive, it follows that every deviation that increases the expected payoff of any single type of player i also increases the overall payoff for player i in the game. In the other direction, if there is a deviation that increases the total expected payoff of player i in the game, it must necessarily increase the conditional expected payoff of at least one type t_i.

Step 1: Every Bayesian equilibrium is a Nash equilibrium.
Let σ^* be a Bayesian equilibrium. Then for each player $i \in N$, each type $t_i \in T_i$, and each action $a_i \in A_i(t_i)$,

$$U_i(\sigma^* \mid t_i) \geq U_i(a_i, \sigma^*_{-i} \mid t_i). \tag{9.73}$$

Combined with Equation (9.65) this implies that for each pure strategy s_i of player i we have

$$U_i(s_i, \sigma^*_{-i}) = \sum_{t_i \in T_i} p(t_i) U_i(s_i(t_i), \sigma^*_{-i} \mid t_i) \leq \sum_{t_i \in T_i} p(t_i) U_i(\sigma^* \mid t_i) = U_i(\sigma^*). \tag{9.74}$$

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As this inequality holds for any pure strategy s_i of player i, it also holds for any of his mixed strategies. This implies that σ_i^* is a best reply to σ_{-i}^*. Since this is true for each player $i \in N$, we conclude that σ^* is a Nash equilibrium.

Step 2: Every Nash equilibrium is a Bayesian equilibrium.
We will prove that if σ^* is not a Bayesian equilibrium, then it is also not a Nash equilibrium. As σ^* is not a Bayesian equilibrium, there is at least one player $i \in N$, type $t_i \in T_i$, and action $a_i \in A_i(t_i)$ satisfying

$$U_i(\sigma^* \mid t_i) < U_i((a_i, \sigma_{-i}^*) \mid t_i). \tag{9.75}$$

Consider a strategy $\widehat{\sigma}_i$ of player i defined by

$$\widehat{\sigma}_i(t_i') = \begin{cases} \sigma_i^*(t_i') & \text{when } t_i' \neq t_i, \\ a_i & \text{when } t_i' = t_i. \end{cases} \tag{9.76}$$

In words: strategy $\widehat{\sigma}_i$ is identical to strategy σ_i^* except in the case of type t_i, who plays a_i instead of $\sigma_i^*(t_i)$. Equations (9.65) and (9.75) then imply that

$$U_i(\widehat{\sigma}_i, \sigma_{-i}^*) = \sum_{t_i' \in T_i} p(t_i') U_i(\widehat{\sigma}_i, \sigma_{-i}^* \mid t_i') \tag{9.77}$$

$$= \sum_{t_i' \neq t_i} p(t_i') U_i(\widehat{\sigma}_i, \sigma_{-i}^* \mid t_i') + p(t_i) U_i(\widehat{\sigma}_i, \sigma_{-i}^* \mid t_i) \tag{9.78}$$

$$= \sum_{t_i' \neq t_i} p(t_i') U_i(\sigma_i^*, \sigma_{-i}^* \mid t_i') + p(t_i) U_i(a_i, \sigma_{-i}^* \mid t_i) \tag{9.79}$$

$$> \sum_{t_i' \neq t_i} p(t_i') U_i(\sigma_i^*, \sigma_{-i}^* \mid t_i') + p(t_i) U_i(\sigma_i^*, \sigma_{-i}^* \mid t_i) \tag{9.80}$$

$$= \sum_{t_i' \in T_i} p(t_i') U_i(\sigma^* \mid t_i') = U_i(\sigma^*). \tag{9.81}$$

Inequality (9.80) follows from Inequality (9.75) and the assumption that $p(t_i) > 0$ for each player i and every type $t_i \in T_i$. From the chain of equations (9.77)–(9.81) we get

$$U_i(\widehat{\sigma}_i, \sigma_{-i}^*) > U_i(\sigma^*), \tag{9.82}$$

which implies that σ^* is not a Nash equilibrium. □

We next present two examples of games with incomplete information and calculate their Bayesian equilibria.

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Example 9.54 Consider the following game with incomplete information:

- $N = \{\mathrm{I}, \mathrm{II}\}$.
- $T_{\mathrm{I}} = \{\mathrm{I}_1, \mathrm{I}_2\}$ and $T_{\mathrm{II}} = \{\mathrm{II}\}$: Player I has two types and Player II has one type.
- $p(\mathrm{I}_1, \mathrm{II}) = p(\mathrm{I}_2, \mathrm{II}) = \frac{1}{2}$: the two types of Player I have equal probabilities.
- There are two states of nature corresponding to two state games in which each player has two possible actions, and the payoff functions are given by the matrices shown in Figure 9.10.

Player II

Player I	L	R
T_1	1, 0	0, 2
B_1	0, 3	1, 0

The state game for $t = (\mathrm{I}_1, \mathrm{II})$

Player II

Player I	L	R
T_2	0, 2	1, 1
B_2	1, 0	0, 2

The state game for $t = (\mathrm{I}_2, \mathrm{II})$

Figure 9.10 The state games in Example 9.54

Because the information each player has is his own type, Player I knows the payoff matrix, while Player II does not know it (see Figure 9.11).

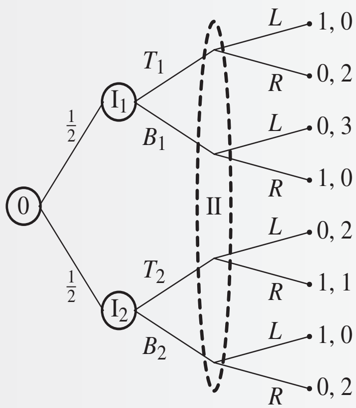

Figure 9.11 The game of Example 9.54 in extensive form

Turning to the calculation of Bayesian equilibria in this game, given such an equilibrium, denote by $[q(L), (1-q)(R)]$ the equilibrium strategy of Player II, by $[x(T_1), (1-x)(B_1)]$ the equilibrium strategy of Player I of type I_1, and by $[y(T_2), (1-y)(B_2)]$ the equilibrium strategy of Player I of type I_2 (see Figure 9.12).

Player II

Player I	q	$1-q$
x	1, 0	0, 2
$1-x$	0, 3	1, 0

Strategies in state game $t = (\mathrm{I}_1, \mathrm{II})$

Player II

Player I	q	$1-q$
y	0, 2	1, 1
$1-y$	1, 0	0, 2

Strategies in state game $t = (\mathrm{I}_2, \mathrm{II})$

Figure 9.12 The strategies of the players in the game of Example 9.54

We first show that $0 < q < 1$.

- If $q = 1$, then type I_1's best reply is T ($x = 1$) and I_2's best reply is B ($y = 0$). But Player II's best reply to this strategy is R ($q = 0$). It follows that $q = 1$ is not part of a Bayesian equilibrium.
- If $q = 0$, then type I_1's best reply is B ($x = 0$) and I_2's best reply is T ($y = 1$). But Player II's best reply to this strategy is L ($q = 1$). It follows that $q = 0$ is not part of a Bayesian equilibrium.

The conclusion is therefore that in a Bayesian equilibrium Player II's strategy must be completely mixed, so that he is necessarily indifferent between L and R. This implies that

$$\tfrac{1}{2} \cdot 3(1 - x) + \tfrac{1}{2} \cdot 2y = \tfrac{1}{2} \cdot 2x + \tfrac{1}{2}(y + 2(1 - y)), \tag{9.83}$$

giving us

$$x = \frac{1 + 3y}{5}. \tag{9.84}$$

Is every pair (x, y) satisfying Equation (9.84) part of a Bayesian equilibrium? For (x, y) to be part of a Bayesian equilibrium, it must be a best reply to q.

- If $q < \frac{1}{2}$, Player I's best reply is $x = 0$, $y = 1$, which does not satisfy Equation (9.84).
- If $q = \frac{1}{2}$, Player I's payoff is $\frac{1}{2}$ irrespective of what he plays, so that every pair (x, y) is a best reply to $q = \frac{1}{2}$.
- If $q > \frac{1}{2}$, Player I's best reply is $x = 1$, $y = 0$, which does not satisfy Equation (9.84).

This leads to the conclusion that a pair of strategies $(x, y; q)$ is a Bayesian equilibrium if and only if $q = \frac{1}{2}$ and Equation (9.84) is satisfied. Since x and y are both in the interval $[0, 1]$ we obtain

$$\tfrac{1}{5} \leq x \leq \tfrac{4}{5}, \quad 0 \leq y \leq 1, \quad x = \tfrac{1+3y}{5}. \tag{9.85}$$

We have thus obtained a continuum of Bayesian equilibria $(x, y; q)$, in all of which Player I's payoff (of either type) is $\frac{1}{2}$, and Player II's payoff is

$$\tfrac{1}{2} \cdot 3(1 - x) + \tfrac{1}{2} \cdot 2y = \tfrac{12+y}{10}. \tag{9.86}$$

◀

Example 9.55 **Cournot duopoly competition with incomplete information** Consider the duopoly competition described in Example 4.23 (page 99) when there is incomplete information regarding production costs. Two manufacturers, labeled 1 and 2, produce the same product and compete for the same market of potential customers. The manufacturers simultaneously select their production quantities, with demand determining the market price of the product, which is identical for both manufacturers. Denote by q_1 and q_2 the quantities respectively produced by manufacturers 1 and 2. The total quantity of products in the market is therefore $q_1 + q_2$. Assume that when the supply is $q_1 + q_2$ the price of each item is $2 - q_1 - q_2$. The per-item production cost for Manufacturer 1 is $c_1 = 1$, and it is common knowledge among the two manufacturers. The per-item production cost for Manufacturer 2 is known only to him, not to Manufacturer 1. All that Manufacturer 1 knows about it is that it is either $c_2^L = \frac{3}{4}$ (low cost) or $c_2^H = \frac{5}{4}$ (high cost), with equal probability. Note that the average production cost of Manufacturer 2 is 1, which is equal to Manufacturer 1's cost.

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Let us find a Bayesian equilibrium of this game. This is a game with incomplete information in which the types of each manufacturer correspond to their production costs:[20]

- $N = \{1, 2\}$.
- $T_1 = \{1\}$, $T_2 = \left\{\frac{3}{4}, \frac{5}{4}\right\}$.
- $p(1, \frac{3}{4}) = p(1, \frac{5}{4}) = \frac{1}{2}$.
- There are two states of nature, corresponding respectively to the type vectors $(1, \frac{3}{4})$ and $(1, \frac{5}{4})$. Each one of these states of nature corresponds to a state game in which the action set of each player is $[0, \infty)$ (each player can produce any nonnegative quantity of items), and the payoff functions which we provide now.

Denote by $u_i(q_1, q_2^H, q_2^L)$ the net profit of Manufacturer i as a function of the quantities of items produced by each type, where q_1 is the quantity produced by Manufacturer 1, q_2^H is the quantity produced by Manufacturer 2 if his production costs are high, and q_2^L is the quantity produced by Manufacturer 2 if his production costs are low. As Manufacturer 1 does not know the type of Manufacturer 2, his expected profit is

$$\begin{aligned} u_1\left(q_1, q_2^H, q_2^L\right) &= \tfrac{1}{2}q_1\left(2 - q_1 - q_2^H\right) + \tfrac{1}{2}q_1\left(2 - q_1 - q_2^L\right) - c_1 q_1 \\ &= q_1\left(2 - c_1 - q_1 - \tfrac{1}{2}q_2^H - \tfrac{1}{2}q_2^L\right). \end{aligned} \tag{9.87}$$

The net profit of Manufacturer 2's two possible types is

$$u_2^H\left(q_1, q_2^H, q_2^L\right) = q_2^H\left(2 - q_1 - q_2^H\right) - c_2^H q_2^H = q_2^H\left(2 - c_2^H - q_1 - q_2^H\right), \tag{9.88}$$

$$u_2^L\left(q_1, q_2^H, q_2^L\right) = q_2^L\left(2 - q_1 - q_2^L\right) - c_2^L q_2^L = q_2^L\left(2 - c_2^L - q_1 - q_2^L\right). \tag{9.89}$$

Since each manufacturer has a continuum of actions, the existence of an equilibrium is not guaranteed. Nevertheless, we will assume that an equilibrium exists, and try to calculate it. Denote by q_1^* the quantity of items produced by Manufacturer 1 at equilibrium, by q_2^{*H} the quantity produced by Manufacturer 2 at equilibrium if his production costs are high, and by q_2^{*L} the quantity he produces at equilibrium under low production costs. At equilibrium, every manufacturer maximizes his expected payoff given the strategy of the other manufacturer: q_1^* maximizes $u_1(q_1, q_2^{*H}, q_2^{*L})$, q_2^{*H} maximizes $u_2^H(q_1^*, q_2^H, q_2^{*L})$, and q_2^{*L} maximizes $u_2^L(q_1^*, q_2^{*H}, q_2^L)$. Since u_2^H is a quadratic function of q_2^H, and the coefficient of $(q_2^H)^2$ is negative, it has a maximum at the point where its derivative with respect to q_2^H vanishes. This results in

$$q_2^H = \frac{\frac{3}{4} - q_1}{2}. \tag{9.90}$$

Similarly, we differentiate u_2^L with respect to q_2^L, set the derivative to zero, and get

$$q_2^L = \frac{\frac{5}{4} - q_1}{2}. \tag{9.91}$$

20 Similar to remarks we made with respect to the Aumann model regarding the distinction between states of nature and states of the world, the type $t_2 = \frac{3}{4}$ in this Harsanyi model contains far more information than the simple fact that the per-unit production cost of Manufacturer 2 is $\frac{3}{4}$; it contains the entire belief hierarchy of Manufacturer 2 with respect to the production costs of both manufacturers. Production costs are states of nature, with respect to which there is incomplete information.

Finally, differentiate u_1 with respect to q_1 and set the derivative to zero, obtaining

$$q_1 = \frac{1 - \frac{1}{2}q_2^H - \frac{1}{2}q_2^L}{2}. \tag{9.92}$$

Insert Equations (9.90) and (9.91) in Equation (9.92) to obtain

$$q_1 = \frac{1 - \frac{1-q_1}{2}}{2}, \tag{9.93}$$

or, in other words,

$$q_1^* = \tfrac{1}{3}. \tag{9.94}$$

This leads to

$$q_2^{*H} = \frac{\frac{3}{4} - \frac{1}{3}}{2} = \tfrac{5}{24}, \tag{9.95}$$

$$q_2^{*L} = \frac{\frac{5}{4} - \frac{1}{3}}{2} = \tfrac{11}{24}. \tag{9.96}$$

The conclusion is that $(q_1^*, q_2^{*H}, q_2^{*L}) = (\frac{1}{3}, \frac{5}{24}, \frac{11}{24})$ is the unique Bayesian equilibrium of the game. Note that $q_2^{*H} < q_1^* < q_2^{*L}$: the high (inefficient) type produces less than Manufacturer 1, and the low (more efficient) type produces more than Manufacturer 1, whose production costs are the average of the production costs of the two types of Manufacturer 2.

The profits gained by the manufacturers are

$$u_1\left(\tfrac{1}{3}, \tfrac{5}{24}, \tfrac{11}{24}\right) = \tfrac{1}{3}\left(2 - 1 - \tfrac{1}{3} - \tfrac{8}{24}\right) = \tfrac{1}{9}, \tag{9.97}$$

$$u_2^H\left(\tfrac{1}{3}, \tfrac{5}{24}, \tfrac{11}{24}\right) = \tfrac{5}{24}\left(\tfrac{3}{4} - \tfrac{1}{3} - \tfrac{5}{24}\right) = \left(\tfrac{5}{24}\right)^2, \tag{9.98}$$

$$u_2^L\left(\tfrac{1}{3}, \tfrac{5}{24}, \tfrac{11}{24}\right) = \tfrac{11}{24}\left(\tfrac{5}{4} - \tfrac{1}{3} - \tfrac{11}{24}\right) = \left(\tfrac{11}{24}\right)^2. \tag{9.99}$$

Therefore Manufacturer 2's expected profit is

$$\tfrac{1}{2}\left(\tfrac{5}{24}\right)^2 + \tfrac{1}{2}\left(\tfrac{11}{24}\right)^2 \approx 0.127. \tag{9.100}$$

The case in which Manufacturer 2 also does not know his exact production cost (but knows that the cost is either $\frac{3}{4}$ or $\frac{5}{4}$ with equal probability, and thus knows that his average production cost is 1) is equivalent to the case we looked at in Example 4.23 (page 99). In that case we derived the equilibrium $q_1^* = q_2^* = \frac{1}{3}$, with the profit of each manufacturer being $\frac{1}{9}$. Comparing that figure with Equation (9.97), we see that relative to the incomplete information case, Manufacturer 1's profit is the same. Using Equations (9.98)–(9.99) and the fact that $0.127 > \frac{1}{9}$, we see that Manufacturer 2's profit when he does not know his own type is smaller than his expected profit when he knows his type; the added information is advantageous to Manufacturer 2.

We also gain insight by comparing this situation to one in which the production cost of Manufacturer 2 is common knowledge among the two manufacturers. In that case, after the selection of Manufacturer 2's type by the chance move, we arrive at a game similar to a Cournot competition with complete information, which we solved in Example 4.23 (page 99). With probability $\frac{1}{2}$ the manufacturers face a Cournot competition in which $c_1 = 1$ and $c_2 = c_2^H = \frac{5}{4}$, and with probability

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$\frac{1}{2}$ they face a Cournot competition in which $c_1 = 1$ and $c_2 = c_2^L = \frac{3}{4}$. In the first case, equilibrium is attained at $q_1^* = \frac{5}{12}$ and $q_2^* = \frac{1}{6}$, with profits of $u_1 = \left(\frac{1}{6}\right)^2$ and $u_2 = \left(\frac{5}{12}\right)^2$ (verify!). In the second case, equilibrium is attained at $q_1^* = \frac{1}{4}$ and $q_2^* = \frac{1}{2}$, corresponding to profits of $u_1 = \left(\frac{1}{4}\right)^2$ and $u_2 = \left(\frac{1}{2}\right)^2$ (verify!). The expected profits prior to the selection of the types is

$$\overline{u}_1 = \frac{1}{2}\left(\frac{1}{4}\right)^2 + \frac{1}{2}\left(\frac{5}{12}\right)^2 = \frac{1}{8}, \tag{9.101}$$

$$\overline{u}_2 = \frac{1}{2}\left(\frac{1}{6}\right)^2 + \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{5}{36}. \tag{9.102}$$

For comparison, we present in table form the profits attained by the manufacturers in each of the three cases dealt with in this example (with respect to the production costs of Manufacturer 2):

Knowledge regarding Manufacturer 2's type	Manufacturer 1's profit	Manufacturer 2's profit
Unknown to both manufacturers	$\frac{1}{9}$	$\frac{1}{9}$
Known only to Manufacturer 2	$\frac{1}{9}$	≈ 0.127
Known to both manufacturers	$\frac{1}{8}$	$\frac{5}{36}$

Note the following:

- Both manufacturers have an interest in Manufacturer 2's type being common knowledge, as opposed to the situation in which that type is unknown to both manufacturers, because $\frac{1}{8} > \frac{1}{9}$ and $\frac{5}{36} > \frac{1}{9}$.
- Both manufacturers have an interest in Manufacturer 2's type being common knowledge, as opposed to the situation in which that type is known solely to Manufacturer 2, because $\frac{1}{8} > \frac{1}{9}$ and $\frac{5}{36} > 0.127$.

This last conclusion may look surprising, because it says that Manufacturer 2 prefers that his private information regarding his production cost be exposed and made public knowledge. ◀

9.5 Incomplete information as a possible interpretation of mixed strategies

There are cases in which it is difficult to interpret or justify the use of mixed strategies in equilibria. Consider for example the following two-player game in which the payoff functions are given by the matrix in Figure 9.13.

This game has only one Nash equilibrium, with Player I playing the mixed strategy $[\frac{3}{4}(T), \frac{1}{4}(B)]$ and Player II playing the mixed strategy $[\frac{1}{2}(L), \frac{1}{2}(R)]$. The payoff at equilibrium is 0 for both players. When Player II plays strategy $[\frac{1}{2}(L), \frac{1}{2}(R)]$ Player I is indifferent between T and B. If that is the case, why should he stick to playing a mixed strategy? And even if he does play a mixed strategy, why the mixed strategy $[\frac{3}{4}(T), \frac{1}{4}(B)]$? If he plays, for example, the pure strategy T, he guarantees himself a payoff of 0 without going through the bother of randomizing strategy selection.

		Player II	
		L	R
Player I	T	$0, 0$	$0, -1$
	B	$1, 0$	$-1, 3$

Figure 9.13 The payoff matrix of a strategic-form game

		Player II	
		L	R
Player I	T	$\varepsilon\alpha, \varepsilon\beta$	$\varepsilon\alpha, -1$
	B	$1, \varepsilon\beta$	$-1, 3$

Figure 9.14 The payoff matrix of Figure 9.13 with "noise"

As we now show, this equilibrium can be interpreted as the limit of a sequence of Bayesian equilibria of games in which the players play pure strategies. The idea is to add incomplete information by injecting "noise" into the game's payoffs; each player will know his own payoffs, but will be uncertain about the payoffs of the other player. To illustrate this idea, suppose the payoff function, rather than being known with certainty, is given by the matrix of Figure 9.14.

In Figure 9.14, ε (the amplitude of the noise) is small and α and β are independently and identically distributed random variables over the interval $[-1, 1]$, with the uniform distribution. Note that for $\varepsilon = 0$ the resulting game is the original game appearing in Figure 9.13.

Suppose that Player I knows the value of α and Player II knows the value of β; i.e., each player has precise knowledge of his own payoff function. This game can be depicted as a game with incomplete information and a continuum of types, as follows:

- The set of players is $N = \{\mathrm{I}, \mathrm{II}\}$.
- The type space of Player I is $T_{\mathrm{I}} = [-1, 1]$.
- The type space of Player II is $T_{\mathrm{II}} = [-1, 1]$.
- The prior distribution over T is the uniform distribution over the square $[-1, 1]^2$.
- The state game corresponding to the pair of types $(\alpha, \beta) \in T := T_{\mathrm{I}} \times T_{\mathrm{II}}$ is given by the matrix in Figure 9.14.

The Harsanyi game that we constructed here has a continuum of types. The definition of a Harsanyi game is applicable also in this case, provided the set of type vectors is a measurable space, so that a common prior distribution can be defined. In the example presented here, the set of type vectors is $[-1, 1]^2$, which is a measurable space (with the σ-algebra of Borel sets), and the common prior distribution is the uniform distribution.

The expected payoff and the conditional expected payoff are defined analogously to the definitions in Equations (9.62) and (9.63), by replacing the summation over T (in Equation (9.62)) or on T_{-i} (in Equation (9.63)) with integration. To ensure that the expressions in these equations are meaningful, we need to require the strategies of the players to be measurable functions of their type, and the payoff functions have to be measurable as well (so that the expected payoffs are well defined). The definitions of Nash equilibrium and Bayesian equilibrium remain unchanged (Definitions 9.46 and 9.49).

Since each player has a continuum of types, the existence of a Bayesian equilibrium is not guaranteed. We will, nevertheless, assume that there exists a Bayesian equilibrium and try to identify it. In fact, we will prove that there exists an equilibrium in which the strategies are threshold strategies: the player plays one action if his type is less than or equal to a particular threshold, and he plays the other action if his type is greater than this threshold.

- Let $\alpha_0 \in [-1, 1]$, and let $s_{\mathrm{I}}^{\alpha_0}$ be the following strategy:

$$s_{\mathrm{I}}^{\alpha_0} = \begin{cases} T & \text{when } \alpha > \alpha_0, \\ B & \text{when } \alpha \leq \alpha_0. \end{cases} \tag{9.103}$$

  In words, if Player I's type is "high" ($\alpha > \alpha_0$) he plays T, and if his type is "low" ($\alpha \leq \alpha_0$) he plays B.
- Let $\beta_0 \in [-1, 1]$ and let $s_{\mathrm{II}}^{\alpha_0}$ be the following strategy:

$$s_{\mathrm{II}}^{\beta_0} = \begin{cases} L & \text{when } \beta > \beta_0, \\ R & \text{when } \beta \leq \beta_0. \end{cases} \tag{9.104}$$

  In words, if Player II's type is "high" ($\beta > \beta_0$) he plays L, and if his type is "low" ($\beta \leq \beta_0$) he plays R.

Next, we will identify two values, α_0 and β_0, for which the pair of strategies $(s_{\mathrm{I}}^{\alpha_0}, s_{\mathrm{II}}^{\beta_0})$ form a Bayesian equilibrium.

Since $\mathbf{P}(\beta > \beta_0) = \frac{1-\beta_0}{2}$ and $\mathbf{P}(\beta \leq \beta_0) = \frac{1+\beta_0}{2}$, the expected payoff of Player I of type α facing strategy $s_{\mathrm{II}}^{\beta_0}$ of Player II is

$$U_{\mathrm{I}}\big(T, s_{\mathrm{II}}^{\beta_0} \,\big|\, \alpha\big) = \varepsilon\alpha, \tag{9.105}$$

if he plays T; and it is

$$U_{\mathrm{I}}\big(B, s_{\mathrm{II}}^{\beta_0} \,\big|\, \alpha\big) = 1\frac{1-\beta_0}{2} + (-1)\frac{1+\beta_0}{2} = -\beta_0, \tag{9.106}$$

if he plays B. In order for $s_{\mathrm{I}}^{\alpha_0}$ to be a best reply to $s_{\mathrm{II}}^{\beta_0}$ the following conditions must hold:

$$\alpha > \alpha_0 \implies U_{\mathrm{I}}\big(T, s_{\mathrm{II}}^{\beta_0} \,\big|\, \alpha\big) \geq U_{\mathrm{I}}\big(B, s_{\mathrm{II}}^{\beta_0} \,\big|\, \alpha\big) \iff \varepsilon\alpha \geq -\beta_0, \tag{9.107}$$

$$\alpha \leq \alpha_0 \implies U_{\mathrm{I}}\big(T, s_{\mathrm{II}}^{\beta_0} \,\big|\, \alpha\big) \leq U_{\mathrm{I}}\big(B, s_{\mathrm{II}}^{\beta_0} \,\big|\, \alpha\big) \iff \varepsilon\alpha \leq -\beta_0. \tag{9.108}$$

From this we conclude that at equilibrium,

$$\varepsilon\alpha_0 = -\beta_0. \tag{9.109}$$

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We can similarly calculate that the expected payoff of Player II of type β facing strategy $s_{\mathrm{I}}^{\alpha_0}$ of Player I is

$$U_{\mathrm{II}}\left(s_{\mathrm{I}}^{\alpha_0}, L\big|\beta\right) = \varepsilon\beta, \tag{9.110}$$

$$U_{\mathrm{II}}\left(s_{\mathrm{I}}^{\alpha_0}, R\big|\beta\right) = (-1)\frac{1-\alpha_0}{2} + 3\frac{1+\alpha_0}{2} = 1 + 2\alpha_0. \tag{9.111}$$

In order for $s_{\mathrm{II}}^{\beta_0}$ to be a best reply against $s_{\mathrm{I}}^{\alpha_0}$, the following needs to hold:

$$\beta > \beta_0 \implies U_{\mathrm{II}}\left(s_{\mathrm{I}}^{\alpha_0}, L\big|\beta\right) \geq U_{\mathrm{II}}\left(s_{\mathrm{I}}^{\alpha_0}, R\big|\beta\right) \iff \varepsilon\beta \geq 1 + 2\alpha_0,$$
$$\beta \leq \beta_0 \implies U_{\mathrm{II}}\left(s_{\mathrm{I}}^{\alpha_0}, L\big|\beta\right) \leq U_{\mathrm{II}}\left(s_{\mathrm{I}}^{\alpha_0}, R\big|\beta\right) \iff \varepsilon\beta \leq 1 + 2\alpha_0.$$

From this we further deduce that at equilibrium

$$\varepsilon\beta_0 = 1 + 2\alpha_0 \tag{9.112}$$

must hold. The solution of Equations (9.109) and (9.112) is

$$\alpha_0 = -\frac{1}{2+\varepsilon^2}, \qquad \beta_0 = \frac{\varepsilon}{2+\varepsilon^2}. \tag{9.113}$$

The probability that Player I will play B is therefore

$$\mathbf{P}_\varepsilon(B) = \mathbf{P}\left(\alpha \leq -\frac{1}{2+\varepsilon^2}\right) = \frac{1-\frac{1}{2+\varepsilon^2}}{2} = \frac{1+\varepsilon^2}{4+2\varepsilon^2}, \tag{9.114}$$

and the probability that Player II will play R is

$$\mathbf{P}_\varepsilon(R) = \mathbf{P}\left(\beta \leq \frac{\varepsilon}{2+\varepsilon^2}\right) = \frac{1+\frac{\varepsilon}{2+\varepsilon^2}}{2} = \frac{2+\varepsilon+\varepsilon^2}{4+2\varepsilon^2}. \tag{9.115}$$

When ε approaches 0, that is, when we reduce the uncertainty regarding the payoffs down towards zero, we get

$$\lim_{\varepsilon\to 0} \mathbf{P}_\varepsilon(B) = \tfrac{1}{4}, \tag{9.116}$$

$$\lim_{\varepsilon\to 0} \mathbf{P}_\varepsilon(R) = \tfrac{1}{2}, \tag{9.117}$$

which is the mixed strategy equilibrium in the original game that began this discussion.

It follows that in the equilibrium $(s_{\mathrm{I}}^{\alpha_0}, s_{\mathrm{II}}^{\beta_0})$ each player implements a pure strategy. Moreover, for $\alpha \neq \alpha_0$, the action chosen by Player I of type α yields a strictly higher payoff than the action not chosen by him. Similarly, when $\beta \neq \beta_0$, the action chosen by Player II of type is β yields a strictly higher payoff than the action not chosen by him. Harsanyi [1973] proposed this sort of reasoning as a basis for a new interpretation of mixed strategies. According to Harsanyi, a mixed strategy can be viewed as a pure strategy of a player that can be of different types. Each type chooses a pure strategy, and different types may choose different pure strategies. From the perspective of other players, who do not know the player's type but rather have a belief (probability distribution) about the player's type, it is as if the player chooses his pure strategy randomly; that is, he is implementing a mixed strategy. It is proved in Harsanyi [1973] that this result can be applied to n-player strategic-form games in which the set of pure strategies is finite. That paper also identifies conditions guaranteeing that each equilibrium is the limit of equilibria in "games with added noise," similar to those presented in the above example, as the amplitude of the noise approaches zero.

We note that the same result obtains when the distribution of noise is not necessarily uniform over the interval $[-1, 1]$. Any probability distribution that is continuous over a compact, nonempty interval can be used (an example of such a probability distribution appears in Exercise 9.47).

9.6 The common prior assumption: inconsistent beliefs

As noted above, in both the Aumann and Harsanyi models, a situation of incomplete information can be assessed from two different perspectives: the *ex ante stage*, prior to the chance move selecting the state of the world (in the Aumann model) or the type vector (in the Harsanyi model), and the *interim stage*, after the chance move has selected the type vector and informed each player about his type, but before the players choose their actions. Prior to the selection of the state of the world, no player knows which information (the partition element in the Aumann model; the type in the Harsanyi model) he will receive; he only knows the prior distribution over the outcomes of the chance move. After the chance move, each player receives information, and updates his beliefs about the state of the world in the Aumann model (the distribution $\mathbf{P}$ conditioned on $\mathcal{F}_i(\omega)$) or about the types of the other players in the Harsanyi model (the distribution p conditioned on t_i).

The concept of interim beliefs is straightforward: a player's interim beliefs are his beliefs after they have been updated in light of new information he has privately received. In real-life situations, a player's beliefs may not be equal to his updated conditional probabilities for various reasons: errors in the calculation of conditional probability, lack of knowledge of the prior distribution, psychologically induced deviations from calculated probabilities, or in general any "subjective feeling" regarding the probability of any particular event, apart from any calculations. It therefore appears to be natural to demand that the interim beliefs be part of the fundamental data of the game, and not necessarily derived from prior distributions (whether or not those prior distributions are common). This is not the case in the Aumann and Harsanyi models: the fundamental data in these models includes a common prior distribution, with the interim beliefs derived from the common prior through the application of Bayes' rule. Assuming the existence of a common prior means adopting a very strong assumption. Can this assumption be justified? What is the exact role of the prior distribution p? Who, or what, makes that selection of the type vector (in the Harsanyi model) or the state of the world (in the Aumann model) at the beginning of the game? And how are the players supposed to "know" the prior distribution p that forms part of the game data?

When player beliefs in the interim stage are derived from one common prior by way of Bayes' rule, given the private information that the players receive, those beliefs are termed *consistent beliefs*. They are called consistent because they imply that the players' beliefs about the way the world works are identical; the only thing that distinguishes players from each other is the information each has received. In that case there is

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"no difference" between the Harsanyi depiction of the game and its depiction in the interim stage. Theorem 9.53 (page 355) states that the sets of equilibria in both depictions are identical (when the set of type vectors is finite). This means that the Aumann and Harsanyi models may be regarded as "convenient tools" for analyzing the interim stage, which is the stage in which we are really interested.

If we propose that the most relevant stage for analyzing the game is the interim stage, in which each player is equipped with his own (subjective) interim beliefs, the next question is: can every system of interim stage beliefs be described by a Harsanyi game? In other words, given a system of interim stage beliefs, can we find a prior distribution p such that the beliefs of each player's type is the conditional probability derived from p, given that the player is of that type? The next example shows that the answer to this question may be negative.

Example 9.56 Consider a model of incomplete information in which:

- there are two players: $N = \{\mathrm{I}, \mathrm{II}\}$, and
- each player has two types: $T_{\mathrm{I}} = \{\mathrm{I}_1, \mathrm{I}_2\}$, $T_{\mathrm{II}} = \{\mathrm{II}_1, \mathrm{II}_2\}$, and $T = T_{\mathrm{I}} \times T_{\mathrm{II}} = \{\mathrm{I}_1\mathrm{II}_1, \mathrm{I}_1\mathrm{II}_2, \mathrm{I}_2\mathrm{II}_1, \mathrm{I}_2\mathrm{II}_2\}$.

Suppose that in the interim stage, before actions are chosen by the players, the mutual beliefs of the players' types are given by the tables in Figure 9.15.

	II_1	II_2
I_1	3/7	4/7
I_2	2/3	1/3

Player I's beliefs

	II_1	II_2
I_1	1/2	4/5
I_2	1/2	1/5

Player II's beliefs

Figure 9.15 The mutual beliefs of the various types in the interim stage in Example 9.56

The tables in Figure 9.15 have the following interpretation. The table on the left describes the beliefs of the two possible types of Player I: Player I of type I_1 ascribes probability $\frac{3}{7}$ to the type of Player II being II_1 and probability $\frac{4}{7}$ to the type of Player II being II_2. Player I of type I_2 ascribes probability $\frac{2}{3}$ to the type of Player II being II_1 and probability $\frac{1}{3}$ to the type of Player II being II_2. The table on the right describes the beliefs of the two possible types of Player II. For example, Player II of type II_1 ascribes probability $\frac{1}{2}$ to the type of Player I being I_1 and probability $\frac{1}{2}$ to the type of Player I being I_2.

There is no common prior distribution p over $T = T_{\mathrm{I}} \times T_{\mathrm{II}}$ that leads to the beliefs described above. This can readily be seen with the assistance of Figure 9.16.

	II_1	II_2
I_1	$2x$	$4x$
I_2	$2x$	x

Figure 9.16 Conditions that must be satisfied in order for a common prior in Example 9.56 to exist

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In Figure 9.16, we have denoted $x = p(\mathrm{I}_2, \mathrm{II}_2)$. In order for the beliefs of type I_2 to correspond with the data in Figure 9.15, it must be the case that $p(\mathrm{I}_2, \mathrm{II}_1) = 2x$ (because according to Figure 9.15 type I_2 believes that the probability that Player II's type II_1 is twice the probability that his type is II_2). In order for the beliefs of type II_1 to correspond with the data in Figure 9.15, it must be the case that $p(\mathrm{I}_1, \mathrm{II}_1) = 2x$, and in order for the beliefs of type II_2 to correspond with the data in Figure 9.15, it must be the case that $p(\mathrm{I}_1, \mathrm{II}_2) = 4x$. But then the beliefs of type I_1 are $[\frac{3}{7}(\mathrm{II}_1), \frac{4}{7}(\mathrm{II}_2)]$, while according to Figure 9.16, these beliefs are $[\frac{1}{3}(\mathrm{II}_1), \frac{2}{3}(\mathrm{II}_2)]$. ◀

The incomplete information situation described in the last example is a situation of inconsistent beliefs. Such a situation cannot be described by a Harsanyi model, and it therefore cannot be described by an Aumann model. Analyzing such situations requires extending the Harsanyi model, which is what we will do in the next chapter, where we will construct a model of incomplete information in which the beliefs of the types are part of the data of the game. The question is what can be said about models of situations with inconsistent beliefs. For one thing, the concept of Bayesian equilibrium is still applicable, also when players' beliefs are inconsistent. In the definition of Bayesian equilibrium, the prior p has significance only in establishing the beliefs $p(t_{-i} \mid t_i)$ in Equation (9.63). That means that the definition is meaningful also when beliefs are not derived from a common prior. In the next chapter we will return to the topic of consistency, provide a formal definition of the concept, and define Bayesian equilibrium in general belief structures.

9.7 Remarks

Kripke's S5 system was defined in Kripke [1963] (see also Geanakoplos [1992]). The concept of common knowledge first appeared in Lewis [1969] and was independently defined in Aumann [1976]. Theorem 9.32 (page 339) is proved in Aumann [1976], in which he also proves the Characterization Theorem 9.24 (page 333) in a formulation that is equivalent to that appearing in Remark 9.26 (page 333). The same paper presents a dynamic process that leads to a posterior probability that is common knowledge. A formal description of that dynamic process is given by Geanakoplos and Polemarchakis [1982]. Further developments of this idea can be found in many papers, including Geanakoplos and Sebenius [1983], McKelvey and Page [1986], and Parikh and Krasucki [1990].

John Harsanyi proposed the Harsanyi model of incomplete information in a series of three papers titled "Games of incomplete information played by Bayesian players" (Harsanyi [1967, 1968a, 1968b]), for which he was awarded the Nobel Memorial Prize in Economics in 1994. Harsanyi also proposed the interpretation of the concept of mixed strategies and mixed equilibria as the limit of Bayesian equilibria (in Harsanyi [1973]), as explained in Section 9.5 (page 361).

For further discussions on the subject of the distinction between knowledge and probability-one belief, the reader is directed to Monderer and Samet [1989] and Vassilakis and Zamir [1993].

Exercise 9.14 is based on a question suggested by Ronen Eldan. Geanakoplos [1992] notes that the riddles on which Exercises 9.23 and 9.31 are based first appeared in Bollobás

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[1953]. Exercise 9.25 is proved in Geanakoplos and Polemarchakis [1982], from which Exercise 9.28 is also taken. Exercise 9.26 was donated to the authors by Ayala Mashiah-Yaakovi. Exercises 9.29 and 9.30 are from Geanakoplos and Sebenius [1983]. Exercise 9.33 is taken from Geanakoplos [1992]. Exercise 9.34 is the famous "coordinated attack problem," studied in the field of distributed computing. The formulation of the exercise is from Halpern [1986]. Exercise 9.39 is from Harsanyi [1968a]. Exercise 9.40 is based on Spence [1974]. Exercise 9.41 is based on Akerlof [1970]. Exercise 9.46 is the "Electronic Mail game" of Rubinstein [1989]. Exercise 9.53 is based on Aumann [1987].

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9.8 Exercises

In the exercises in this chapter, all announcements made by the players are considered common knowledge, and the game of each exercise is also considered common knowledge among the players.

9.1 Prove that the knowledge operator K_i (Definition 9.8, page 325) of each player i satisfies the following properties:

(a) $K_i Y = Y$: player i knows that Y is the set of all states.
(b) $K_i A \cap K_i B = K_i(A \cap B)$: player i knows event A and knows event B if and only if he knows event $A \cap B$.
(c) $(K_i A)^c = K_i((K_i A)^c)$: player i does not know event A if and only if he knows that he does not know event A.

9.2 This exercise shows that the Kripke S5 system characterizes the knowledge operator.

Let Y be a finite set, and let $K : 2^Y \to 2^Y$ be an operator that associates with each subset A of Y a subset $K(A)$ of Y. Suppose that the operator K satisfies the following properties:

(i) $K(Y) = Y$.
(ii) $K(A) \cap K(B) = K(A \cap B)$ for every pair of subsets $A, B \subseteq Y$.
(iii) $K(A) \subseteq A$ for every subset $A \subseteq Y$.
(iv) $K(K(A)) = K(A)$ for every subset $A \subseteq Y$.
(v) $(K(A))^c = K((K(A))^c)$ for every subset $A \subseteq Y$.

Associate with each $\omega \in Y$ a set $F(\omega)$ as follows:

$$F(\omega) := \bigcap \{A \subseteq Y, \omega \in K(A)\}. \tag{9.118}$$

(a) Prove that $\omega \in F(\omega)$ for each $\omega \in Y$.
(b) Prove that if $\omega' \in F(\omega)$, then $F(\omega) = F(\omega')$. Conclude from this that the family of sets $\mathcal{F} := \{F(\omega), \omega \in Y\}$ is a partition of Y.

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(c) Let K' be the knowledge operator defined by the partition $\mathcal{F}$:

$$K'(A) = \{\omega \in Y : F(\omega) \subseteq A\}. \tag{9.119}$$

Prove that $K' = K$.

(d) Which of the five properties listed above did you use in order to prove that $K' = K$?

9.3 Prove that in Kripke's S5 system (see page 327), the fourth property, $K_i K_i A = K_i A$, is a consequence of the other four properties.

9.4 Consider an Aumann model of incomplete information in which

$N = \{1, 2\}$,
$Y = \{1, 2, 3, 4, 5, 6, 7\}$,
$\mathcal{F}_1 = \{\{1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}\}$,
$\mathcal{F}_2 = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7\}\}$.

Let $A = \{1\}$ and $B = \{1, 2, 3, 4, 5, 6\}$. Identify the events $K_1 A$, $K_2 A$, $K_2 K_1 A$, $K_1 K_2 A$, $K_1 B$, $K_2 B$, $K_2 K_1 B$, $K_1 K_2 B$, $K_1 K_2 K_1 B$, $K_2 K_1 K_2 B$.

9.5 Emily, Marc, and Thomas meet at a party to which novelists and poets have been invited. Every attendee at the party is either a novelist or a poet (but not both). Every poet knows all the other poets, but every novelist does not know any of the other attendees, whether they are poets or novelists. What do Emily, Marc, and Thomas know about each other's professions? Provide an Aumann model of incomplete information that describes this situation (there are several ways to do so).

9.6 I love Juliet, and I know that Juliet loves me, but I do not know if Juliet knows that I love her. Provide an Aumann model of incomplete information that describes this situation, and specify a state of the world in that model that corresponds to this situation (there are several possible ways of including higher-order beliefs in this model).

9.7 Construct an Aumann model of incomplete information for each of the following situations, and specify a state of the world in that model which corresponds to the situation (there are several possible ways of including higher-order beliefs in each model):

(a) Mary gave birth to a baby, and Herod knows it.
(b) Mary gave birth to a baby, and Herod does not know it.
(c) Mary gave birth to a baby, Herod knows it, and Mary knows that Herod knows it.
(d) Mary gave birth to a baby, Herod knows it, but Mary does not know that Herod knows it.
(e) Mary gave birth to a baby, Herod does not know it, and Mary does not know whether Herod knows it or not.

9.8 Romeo composes a letter to Juliet, and gives it to Tybalt to deliver to Juliet. While on the way, Tybalt peeks at the letter's contents. Tybalt gives Juliet the letter, and

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Juliet reads it immediately, in Tybalt's presence. Neither Romeo nor Juliet knows that Tybalt has read the letter.

Answer the following questions relating to this story:

(a) Construct an Aumann model of incomplete information in which all the elements of the story above regarding the knowledge possessed by Romeo, Tybalt, and Juliet regarding the content of the letter hold true (there are several possible ways to do this). Specify a state of the world in the model that corresponds to the situation described above.
(b) In the state of the world you specified above, does Romeo know that Juliet has read the letter? Justify your answer.
(c) In the state of the world you specified above, does Tybalt know that Romeo knows that Juliet has read the letter? Justify your answer.
(d) Construct an Aumann model of incomplete information in which, in addition to the particulars of the story presented above, the following also holds: "Tybalt does not know that Juliet does not know that Tybalt read the letter," and specify a state of the world in your model that corresponds to this situation.

9.9 George, John, and Thomas are standing first, second, and third in a line, respectively. Each one sees the persons standing in front of him. James announces: "I have three red hats and two white hats. I will place a hat on the head of each one of you." After James places the hats, he asks Thomas (who can see the hats worn by John and George) if he knows the color of the hat on his own head. Thomas replies "no." He then asks John (who sees only George's hat) whether he knows the color of the hat on his own head, and he also replies "no." Finally, he asks George (who cannot see any of the hats) if he knows the color of the hat on his own head.

(a) Construct an Aumann model of incomplete information that contains 7 states of the world and describes this situation.
(b) What are the partitions of George, John, and Thomas after James's announcement and before he asked Thomas whether he knows the color of his hat?
(c) What are the partitions of George and John after Thomas's response and before John responded to James's question?
(d) What is George's partition after hearing John's response?
(e) What is George's answer to James's question? Does this answer depend on the the state of the world, that is, on the colors of the hats that the three wear?

9.10 Prove Corollary 9.16 (page 331): every situation of incomplete information $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s}, \omega_*)$ over a set of states of nature S uniquely determines a knowledge hierarchy among the players over the set of states of nature S in state of the world ω_*.

9.11 Consider an Aumann model of incomplete information in which $N = \{\mathrm{I}, \mathrm{II}\}$, $Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $\mathcal{F}_{\mathrm{I}} = \{\{1\}, \{2, 3\}, \{4, 5\}, \{6\}, \{7\}, \{8, 9\}\}$, and $\mathcal{F}_{\mathrm{II}} = \{\{1\}, \{2, 5\}, \{3\}, \{4, 7\}, \{6, 9\}, \{8\}\}$. What are the connected components in the graph corresponding to this Aumann model? Which events are common knowledge in state

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of the world $\omega = 1$? Which events are common knowledge in state of the world $\omega = 9$? Which events are common knowledge in state of the world $\omega = 5$?

9.12 Show that in Examples 9.12 (page 327) and 9.13 (page 329), in each state of the world, the only event that is common knowledge is Y.

9.13 Consider an Aumann model of incomplete information in which $N = \{\text{I}, \text{II}\}$, $Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $\mathcal{F}_{\text{I}} = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}\}$, and $\mathcal{F}_{\text{II}} = \{\{1, 5\}, \{2, 6\}, \{3, 4\}, \{7\}, \{8, 9\}\}$. Answer the following questions:

(a) What are the connected components in the graph corresponding to the Aumann model?
(b) Which events are common knowledge in state of the world $\omega = 1$? In $\omega = 7$? In $\omega = 8$?
(c) Denote by A the event $\{1, 2, 3, 4, 5\}$. Find the shortest sequence of players $i_1, i_2, \ldots, i_k$ such that in the state of the world $\omega = 1$ it is not the case that i_1 knows that i_2 knows that $\ldots i_{k-1}$ knows that i_k knows event A.

9.14 A digital clock showing the hours between 00:00 to 23:59 hangs on a wall; the digits on the clock are displayed using straight lines, as depicted in the accompanying figure.

William and Dan are both looking at the clock. William sees only the top half of the clock (including the midline) while Dan sees only the bottom half of the clock (including the midline). Answer the following questions:

(a) At which times does William know the correct time?
(b) At which times does Dan know the correct time?
(c) At which times does William know that Dan knows the correct time?
(d) At which times does Dan know that William knows the correct time?
(e) At which times is the correct time common knowledge among William and Dan?
(f) Construct an Aumann model of incomplete information describing this situation. How many states of nature, and how many states of the world, are there in your model?

9.15 Prove that if in an Aumann model of incomplete information the events A and B are common knowledge among the players in state of the world ω, then the event $A \cap B$ is also common knowledge among the players in ω.

9.16 Given an Aumann model of incomplete information, prove that event A is common knowledge in every state of the world in A if and only if $K_1 K_2 \cdots K_n A = A$, where $N = \{1, 2, \ldots, n\}$ is the set of players.

9.17 Prove that in an Aumann model of incomplete information with n players, every event that is common knowledge among the players in state of the world ω is also common knowledge among any subset of the set of players (Remark 9.22, page 332).

9.18 Give an example of an Aumann model of incomplete information with a set of players $N = \{1, 2, 3\}$ and an event A that is not common knowledge among all the players N, but is common knowledge among players $\{2, 3\}$.

9.19 (a) In state of the world ω, Andrew knows that Sally knows the state of nature. Does this imply that Andrew knows the state of nature in ω? Is the fact that Sally knows the state of nature common knowledge among Andrew and Sally in ω?
(b) In every state of the world, Andrew knows that Sally knows the state of nature. Does this imply that Andrew knows the state of nature in every state of the world? Is the fact that Sally knows the state of nature common knowledge among Andrew and Sally in every state of the world?
(c) In state of the world ω, Andrew knows that Sally knows the state of the world. Does this imply that Andrew knows the state of the world in ω? Is the fact that Sally knows the state of the world common knowledge among Andrew and Sally in ω?

9.20 Let $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s}, \mathbf{P})$ be an Aumann model of incomplete information with beliefs, and let $W \subseteq Y$ be an event. Prove that $(N, W, (\mathcal{F}_i \cap W)_{i\in N}, \mathbf{P}(\cdot \mid W))$ is also an Aumann model of incomplete information with beliefs, where for each player $i \in N$

$$\mathcal{F}_i \cap W = \{F \cap W : F \in \mathcal{F}_i\} \tag{9.120}$$

is the partition $\mathcal{F}_i$ restricted to W, and $\mathbf{P}(\cdot \mid W)$ is the conditional distribution of $\mathbf{P}$ over W.

9.21 Prove that without the assumption that $\mathbf{P}(\omega) > 0$ for all $\omega \in Y$, Theorem 9.29 (page 336) does not obtain.

9.22 This exercise generalizes Aumann's Agreement Theorem to a set of players of arbitrary finite size. Given an Aumann model of incomplete information with beliefs $(N, Y, (\mathcal{F}_i)_{i\in N}, \mathfrak{s}, \mathbf{P})$ with n players, suppose that for each $i \in N$, the fact that player i ascribes probability q_i to an event A is common knowledge among the players. Prove that $q_1 = q_2 = \cdots = q_n$.
Hint: Use Theorem 9.32 on page 339 and Exercise 9.17.

9.23 Three individuals are seated in a room. Each one of them is wearing a hat, which may be either red or white. Each of them sees the hats worn by the others, but cannot see his own hat (and in particular does not know its color). The true situation is that every person in the room is wearing a red hat.

(a) Depict this situation as a Harsanyi model of incomplete information, where a player's type is the color of his hat, and specify the vector of types corresponding to the true situation.
(b) Depict this situation as an Aumann model of incomplete information, and specify the state of the world corresponding to the true situation.

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(c) A stranger enters the room, holding a bell. Once a minute, he rings the bell while saying "If you know that the color of the hat on your head is red, leave this room immediately." Does anyone leave the room after a few rings? Why?

(d) At a certain point in time, the announcer says, "At least one of you is wearing a red hat." He continues to ring the bell once a minute and requesting that those who know their hat to be red to leave. Use the Aumann model of incomplete information to prove that after the third ring, all three hat-wearers will leave the room.

(e) What information did the announcer add by saying that at least one person in the room was wearing a red hat, when this was known to everyone before the announcement was made?
Hint: See Example 9.12 on page 327.

(f) Generalize this result to n individuals (instead of 3).

9.24 Prove that in an Aumann model of incomplete information with a common prior $\mathbf{P}$, if in a state of the world ω Player 1 knows that Player 2 knows A, then $\mathbf{P}(A \mid F_1(\omega)) = 1$.

9.25 Consider an Aumann model of incomplete information with beliefs in which

$$N = \{\mathrm{I}, \mathrm{II}\},$$
$$Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9\},$$
$$\mathcal{F}_{\mathrm{I}} = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}\},$$
$$\mathcal{F}_{\mathrm{II}} = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{9\}\},$$
$$\mathbf{P}(\omega) = \tfrac{1}{9}, \quad \forall \omega \in Y.$$

Let $A = \{1, 5, 9\}$, and suppose that the true state of the world is $\omega_* = 9$. Answer the following questions:

(a) What is the probability that Player I (given his information) ascribes to the event A?

(b) What is the probability that Player II ascribes to the event A?

(c) Suppose that Player I announces the probability you calculated in item (a) above. How will that affect the probability that Player II now ascribes to the event A?

(d) Suppose that Player II announces the probability you calculated in item (c). How will that affect the probability that Player I ascribes to the event A, after hearing Player II's announcement?

(e) Repeat the previous two questions, with each player updating his conditional probability following the announcement of the other player. What is the sequence of conditional probabilities the players calculate? Does the sequence converge, or oscillate periodically (or neither)?

(f) Repeat the above, with $\omega_* = 8$.

(g) Repeat the above, with $\omega_* = 6$.

(h) Repeat the above, with $\omega_* = 4$.

(i) Repeat the above, with $\omega_* = 1$.

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9.26 Repeat Exercise 9.25, using the following Aumann model of incomplete information with beliefs:

$N = \{\mathrm{I}, \mathrm{II}\},$
$Y = \{1, 2, 3, 4, 5\},$
$\mathcal{F}_{\mathrm{I}} = \{\{1, 2\}, \{3, 4\}, \{5\}\},$
$\mathcal{F}_{\mathrm{II}} = \{\{1, 3, 5\}, \{2\}, \{4\}\},$
$\mathbf{P}(\omega) = \frac{1}{5}, \quad \forall \omega \in Y.$

for $A = \{1, 4\}$ and $\omega_* = 3$.

9.27 Repeat Exercise 9.25 when the two players have different priors over Y:

$$\mathbf{P}_{\mathrm{I}}(\omega) = \frac{\omega}{45}, \quad \forall \omega \in Y, \tag{9.121}$$

$$\mathbf{P}_{\mathrm{II}}(\omega) = \frac{10 - \omega}{45}, \quad \forall \omega \in Y. \tag{9.122}$$

9.28 This exercise generalizes Exercise 9.25. Let $(N, Y, \mathcal{F}_{\mathrm{I}}, \mathcal{F}_{\mathrm{II}}, \mathfrak{s}, \mathbf{P})$ be an Aumann model of incomplete information with beliefs in which $N = \{\mathrm{I}, \mathrm{II}\}$ and let $A \subseteq Y$ be an event. Consider the following process:

- Player I informs Player II of the conditional probability $\mathbf{P}(A \mid F_{\mathrm{I}}(\omega))$.
- Player II informs Player I of the conditional probability that he ascribes to event A given the partition element $F_{\mathrm{II}}(\omega)$ and Player I's announcement.
- Player I informs Player II of the conditional probability that he ascribes to event A given the partition element $F_{\mathrm{I}}(\omega)$ and all the announcements so far.
- Repeat indefinitely.

Answer the following questions:

(a) Prove that the sequence of conditional probabilities that Player I announces converges; that the sequence of conditional probabilities that Player II announces also converges; and that both sequences converge to the same limit.
(b) Prove that after at most $2|Y|$ announcements the sequence of announcements made by the players becomes constant.

9.29 The "No Trade Theorem" mentioned on page 341 is proved in this exercise. Let $(N, Y, \mathcal{F}_{\mathrm{I}}, \mathcal{F}_{\mathrm{II}}, \mathfrak{s}, \mathbf{P})$ be an Aumann model of incomplete information with beliefs where $N = \{\mathrm{I}, \mathrm{II}\}$, let $f : Y \to \mathbb{R}$ be a function, and let $\omega_* \in Y$ be a state of the world in Y. Suppose[21] that the fact that $\mathbf{E}[f \mid \mathcal{F}_{\mathrm{I}}](\omega) \geq 0$ is common knowledge in ω_*, and that the fact that $\mathbf{E}[f \mid \mathcal{F}_{\mathrm{II}}](\omega) \leq 0$ is also common knowledge in ω_*. In other words, the events $A_{\mathrm{I}} := \{\omega \colon \mathbf{E}[f \mid \mathcal{F}_{\mathrm{I}}](\omega) \geq 0\}$ and $A_{\mathrm{II}} := \{\omega \colon \mathbf{E}[f \mid \mathcal{F}_{\mathrm{II}}](\omega) \leq 0\}$ are common knowledge in ω_*. Prove that the event $D := \{\omega \in Y \colon \mathbf{E}[f \mid \mathcal{F}_{\mathrm{I}}](\omega) = \mathbf{E}[f \mid \mathcal{F}_{\mathrm{II}}](\omega) = 0\}$ is common knowledge in the state of the world ω_*.

21 Recall that the conditional expectation $\mathbf{E}[f \mid \mathcal{F}_{\mathrm{I}}]$ is the function on Y that is defined by $\mathbf{E}[f \mid \mathcal{F}_{\mathrm{I}}](\omega) := \mathbf{E}[f \mid F_{\mathrm{I}}(\omega)]$ for each $\omega \in Y$.

9.30 This exercise is similar to Exercise 9.25, but instead of announcing the probability of a particular event given their private information, the players announce whether or not the expectation of a particular random variable is positive or not, given their private information. This is meant to model trade between two parties to an agreement, as follows. Suppose that Ralph (Player 2) owns an oil field. He expects the profit from the oil field to be negative, and therefore intends to sell it. Jack is of the opinion that the oil field can yield positive profits, and is therefore willing to purchase it (for the price of \$0). Jack and Ralph arrive at different determinations regarding the oil field because they have different information. We will show that no trade can occur under these conditions, because of the following exchange between the parties:

- Jack: I am interested in purchasing the oil field; are you interested in selling?
- Ralph: Yes, I am interested in selling; are you interested in purchasing?
- Jack: Yes, I am interested in purchasing; are you still interested in selling?
- And so on, until one of the two parties announces that he has no interest in a deal.

The formal description of this process is as follows. Let $(N, Y, \mathcal{F}_1, \mathcal{F}_2, \mathfrak{s}, \mathbf{P})$ be an Aumann model of incomplete information with beliefs where $N = \{\mathrm{I}, \mathrm{II}\}$, let $f : Y \to \mathbb{R}$ be a function, and let $\omega \in Y$ be a state of the world. $f(\omega)$ represents the profit yielded by the oil field at the state of the world ω. At each stage, Jack will be interested in the deal only if the conditional expectation of f given his information is positive, and Ralph will be interested in the deal only if the conditional expectation of f given his information is negative. The process therefore looks like this:

- Player I states whether or not $\mathbf{E}[f \mid \mathcal{F}_{\mathrm{I}}](\omega) > 0$ (implicitly doing so by expressing or not expressing interest in purchasing the oil field). If he says "no" (i.e., his expectation is less than or equal to 0), the process ends here.
- If the process gets to the second stage, Player II states whether his expectation of f, given the information he has received so far, is negative or not. The information he has includes $F_{\mathrm{II}}(\omega)$ and the affirmative interest of Player I in the first stage. If Player II now says "no" (i.e., his expectation is greater than or equal to 0), the process ends here.
- If the process has not yet ended, Player I states whether his expectation of f, given the information he has received so far, is positive or not. The information he has includes $F_{\mathrm{I}}(\omega)$ and the affirmative interest of Player II in the second stage. If Player I now says "no" (i.e., his expectation is less than or equal to 0), the process ends here.
- And so on. The process ends the first time either Player I's expectation of f, given his information, is not positive, or Player II's expectation of f, given his information, is not negative.

Show that this process ends after a finite number of stages. In fact, show that the number of stages prior to the end of the process is at most $\max\{2|\mathcal{F}_{\mathrm{I}}| - 1, 2|\mathcal{F}_{\mathrm{II}}| - 1\}$.

9.31 Peter has two envelopes. He puts 10^k euros in one and 10^{k+1} euros in the other, where k is the outcome of the toss of a fair die. Peter gives one of the envelopes to

Mark and one to Luke (neither Mark nor Luke knows the outcome of the toss). Mark and Luke both go to their respective rooms, open the envelopes they have received, and observe the amounts in them.

(a) Depict the situation as a model with incomplete information, where the state of nature is the amounts in Mark and Luke's envelopes.
(b) Mark finds 1,000 euros in his envelope, and Luke finds 10,000 euros in his envelope. What is the true state of the world in your model?
(c) According to the information Mark has, what is the expected amount of money in Luke's envelope?
(d) According to the information Luke has, what is the expected amount of money in Mark's envelope?
(e) Peter enters Mark's room and asks him whether he would like to switch envelopes with Luke. If the answer is positive, he goes to Luke's room and informs him: "Mark wants to switch envelopes with you. Would you like to switch envelopes with him?" If the answer is positive, he goes to Mark's room and tells him: "Luke wants to switch envelopes with you. Would you like to switch envelopes with him?" This process repeats itself as long as the answer received by Peter from Mark and Luke is positive.

Use your model of incomplete information to show that the answers of Mark and Luke will be positive at first, and then one of them will refuse to switch envelopes. Who will be the first to refuse? Assume that each of the two would like to change envelopes if the conditional expectation of the amount of money in the other's envelope is higher than the amount in his envelope.

9.32 The setup is just as in the previous exercise, but now Peter tells Mark and Luke that they can switch the envelopes if and only if both of them have an interest in switching envelopes: each one gives Peter a sealed envelope with "yes" or "no" written in it, and the switch is effected only if both envelopes read "yes". What will be Mark and Luke's answers after having properly analyzed the situation? Justify your answer.

9.33 The setup is again as in Exercise 9.31, but this time Peter chooses the integer k randomly according to a geometric distribution with parameter $\frac{1}{2}$, that is, $\mathbf{P}(k = n) = \frac{1}{2^n}$ for each $n \in \mathbb{N}$. How does this affect your answers to the questions in Exercise 9.31?

9.34 Two divisions of Napoleon's army are camped on opposite hillsides, both overlooking the valley in which enemy forces have massed. If both divisions attack their enemy simultaneously, victory is assured, but if only one division attacks alone, it will suffer a crushing defeat. The division commanders have not yet coordinated a joint attack time. The commander of Division A wishes to coordinate a joint attack time of 6 am the following day with the commander of Division B. Given the stakes involved, neither commander will give an order to his troops to attack until he is absolutely certain that the other commander is also attacking simultaneously. The only way the commanders can communicate with each other is by courier. The travel

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time between the two camps is an hour's trek through enemy-held territory, exposing the courier to possible capture by enemy patrols. It turns out that on that night no enemy patrols were scouting in the area. How much time will pass before the two commanders coordinate the attack? Justify your answer.

9.35 Prove Theorem 9.42 on page 349: every game with incomplete information can be described as an extensive-form game.

9.36 Describe the following game with incomplete information as an extensive-form game. There are two players $N = \{\mathrm{I}, \mathrm{II}\}$. Each player has three types, $T_{\mathrm{I}} = \{\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3\}$ and $T_{\mathrm{II}} = \{\mathrm{II}_1, \mathrm{II}_2, \mathrm{II}_3\}$, with common prior:

$$p(\mathrm{I}_k, \mathrm{II}_l) = \frac{k(k+l)}{78}, \qquad 1 \le k, l \le 3. \tag{9.123}$$

The number of possible actions available to each type is given by the index of that type: the set of actions of Player I of type I_k contains k actions $\{1, 2, \ldots, k\}$; the set of actions of Player II of type II_l contains l actions $\{1, 2, \ldots, l\}$. When the type vector is $(\mathrm{I}_k, \mathrm{II}_l)$, and the vector of actions chosen is $(a_{\mathrm{I}}, a_{\mathrm{II}})$, the payoffs to the players are given by

$$\begin{aligned} u_{\mathrm{I}}(\mathrm{I}_k, \mathrm{II}_l; a_{\mathrm{I}}, a_{\mathrm{II}}) &= (k+l)(a_{\mathrm{I}} - a_{\mathrm{II}}), \\ u_{\mathrm{II}}(\mathrm{I}_k, \mathrm{II}_l; a_{\mathrm{I}}, a_{\mathrm{II}}) &= (k-l)a_{\mathrm{I}}a_{\mathrm{II}}. \end{aligned} \tag{9.124}$$

For each player, and each of his types, write down the conditional probability that the player ascribes to each of the types of the other player, given his own type.

9.37 Find a Bayesian equilibrium in the game described in Example 9.38 (page 346). *Hint:* To find a Bayesian equilibrium, you may remove weakly dominated strategies.

9.38 Find a Bayesian equilibrium in the following game with incomplete information:

- $N = \{\mathrm{I}, \mathrm{II}\}$.
- $T_{\mathrm{I}} = \{\mathrm{I}_1, \mathrm{I}_2\}$ and $T_{\mathrm{II}} = \{\mathrm{II}_1\}$: Player I has two types, and Player II has one type.
- $p(\mathrm{I}_1, \mathrm{II}_1) = \frac{1}{3}, \quad p(\mathrm{I}_2, \mathrm{II}_1) = \frac{2}{3}$.
- Every player has two possible actions, and state games are given by the following matrices:

Player II

Player I	L	R
T	2, 0	0, 3
B	0, 4	1, 0

The state game for $t = (\mathrm{I}_1, \mathrm{II}_1)$

Player II

Player I	L	R
T	0, 3	3, 1
B	2, 0	0, 1

The state game for $t = (\mathrm{I}_2, \mathrm{II}_1)$

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9.39 Answer the following questions for the zero-sum game with incomplete information with two players I and II, in which each player has two types, $T_{\mathrm{I}} = \{\mathrm{I}_1, \mathrm{I}_2\}$ and $T_{\mathrm{II}} = \{\mathrm{II}_1, \mathrm{II}_2\}$, the common prior over the type vectors is

$$p(\mathrm{I}_1, \mathrm{II}_1) = 0.4,\ \ p(\mathrm{I}_1, \mathrm{II}_2) = 0.1,\ \ p(\mathrm{I}_2, \mathrm{II}_1) = 0.2,\ \ p(\mathrm{I}_2, \mathrm{II}_2) = 0.3,$$

and the state games are given by

Player II

Player I		*L*	*R*
	T	2	5
	B	–1	20

The state game for $t = (\mathrm{I}_1, \mathrm{II}_1)$

Player II

Player I		*L*	*R*
	T	–24	–36
	B	0	24

The state game for $t = (\mathrm{I}_1, \mathrm{II}_2)$

Player II

Player I		*L*	*R*
	T	28	15
	B	40	4

The state game for $t = (\mathrm{I}_2, \mathrm{II}_1)$

Player II

Player I		*L*	*R*
	T	12	20
	B	2	13

The state game for $t = (\mathrm{I}_2, \mathrm{II}_2)$

(a) List the set of pure strategies of each player.
(b) Depict the game in strategic form.
(c) Calculate the value of the game and find optimal strategies for the two players.

9.40 Signaling games This exercise illustrates that a college education serves as a form of signaling to potential employers, in addition to expanding the knowledge of students. A young person entering the job market may be talented or untalented. Suppose that one-quarter of high school graduates are talented, and the rest untalented. A recent high school graduate, who knows whether or not he is talented, has the option of spending a year traveling overseas or enrolling at college (we will assume that he or she cannot do both) before applying for a job. An employer seeking to fill a job opening cannot know whether or not a job applicant is talented; all he knows is that the applicant either went to college or traveled overseas. The payoff an employer gets from hiring a worker depends solely on the talents of the hired worker (and not on his educational level), while the payoff to the youth depends on what he chose to do after high school, on his talents (because talented students enjoy their studies at college more than untalented students), and on whether or not he gets a job. These payoffs are described in the following tables (where the employer is the row player and the youth is the column player, so that a payoff vector of (x, y) represents a payoff of x to the employer and y to the youth).

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Payoff matrix if youth is untalented (Employer rows, Youth columns):

	Travel	Study
Hire	0, 6	0, 2
Don't Hire	3, 3	3, −3

Payoff matrix if youth is talented (Employer rows, Youth columns):

	Travel	Study
Hire	8, 6	8, 4
Don't Hire	3, 3	3, 1

(a) Depict this situation as a Harsanyi game with incomplete information.
(b) List the pure strategies of the two players.
(c) Find two Bayesian equilibria in pure strategies.

9.41 **Lemon Market** This exercise illustrates that in situations in which a seller has more information than a buyer, transactions might not be possible. Consider a used car market in which a fraction q of the cars ($0 \leq q \leq 1$) are in good condition and $1 - q$ are in bad condition (lemons). The seller (Player 2) knows the quality of the car he is offering to sell while the buyer (Player 1) does not know the quality of the car that he is being offered to buy. Each used car is offered for sale at the price of $\$p$ (in units of thousands of dollars). The payoffs to the seller and the buyer, depending on whether or not the transaction is completed, are described in the following tables:

State game if car in good condition:

	Sell	Don't Sell
Buy	$6 - p, p$	0, 5
Don't Buy	0, 5	0, 5

State game if car in bad condition:

	Sell	Don't Sell
Buy	$4 - p, p$	0, 0
Don't Buy	0, 0	0, 0

Depict this situation as a Harsanyi game with incomplete information, and for each pair of parameters p and q, find all the Bayesian equilibria.

9.42 Nicolas would like to sell a company that he owns to Marc. The company's true value is an integer between 10 and 12 (including 10 and 12), in millions of dollars. Marc has to make a take-it-or-leave-it offer, and Nicolas has to decide whether to accept the offer or reject it. If Nicolas accepts the offer, the company is sold, Nicolas's payoff is the amount that he got, and Marc's payoff is the difference between the company's true value and the amount that he paid. If Nicolas rejects the offer, the company is not sold, Nicolas's payoff is the value of the company, and Marc's payoff is 0. For each one of the following three information structures, describe the situation as a game with incomplete information, and find all the Bayesian equilibria in the corresponding game. In each case, the description of the situation is common knowledge among the players. In determining Nicolas's action set, note that Nicolas knows what Marc's offer is when he decides whether or not to accept the offer.

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(a) Neither Nicolas nor Marc knows the company's true value; both ascribe probability $\frac{1}{3}$ to each possible value.
(b) Nicolas knows the company's true value, whereas Marc does not know it, and ascribes probability $\frac{1}{3}$ to each possible value.
(c) Marc does not know the company's worth and ascribes probability $\frac{1}{3}$ to each possible value. Marc further ascribes probability p to the event that Nicolas knows the value of the company, and probability $1 - p$ to the event that Nicolas does not know the value of the company, and instead ascribes probability $\frac{1}{3}$ to each possible value.

9.43 Prove that in each game with incomplete information with a finite set of players, where the set of types of each player is a countable set, and the set of possible actions of each type is finite, there exists a Bayesian equilibrium (in behavior strategies).

Guidance: Suppose that the set of types of player i, T_i, is the set of natural numbers $\mathbb{N}$. Denote $T_i^k := \{1, 2, \ldots, k\}$ and $T^k = \times_{i \in N} T_i^k$. Let p^k be the probability distribution p conditioned on the set T^k:

$$p^k(t) = \begin{cases} \frac{p(t)}{p(T^k)} & t \in T^k, \\ 0 & t \notin T^k. \end{cases} \tag{9.125}$$

Prove that for a sufficiently large k, the denominator $p(T^k)$ is positive and therefore the probability distribution p^k is well defined. Show that for each k, the game in which the probability distribution over the types is p^k has an equilibrium, and any accumulation point of such equilibria, as k goes to infinity, is an equilibrium of the original game.

9.44 Prove Theorem 9.51 on page 354: a strategy vector $\sigma^* = (\sigma_i^*)_{i \in N}$ is a Bayesian equilibrium in a game Γ with incomplete information if and only if the strategy vector $(\sigma_i^*(t_i))_{i \in N, t_i \in T_i}$ is a Nash equilibrium in the agent-form game $\widehat{\Gamma}$. (For the definition of an agent-form game, see Definition 9.50 on page 354.)

9.45 This exercise shows that in a game with incomplete information, the payoff function of an inactive type has no effect on the set of equilibria. Let $\Gamma = (N, (T_i)_{i \in N}, p, S, (s_t)_{t \in \times_{i \in N} T_i})$, where $s_t = (N, (A_i(t_i), u_i(t))_{i \in N})$ for each $t \in \times_{i \in N} T_i$, be a game with incomplete information in which there exists a player j and a type t_j^* of player j such that $|A_j(t_j^*)| = 1$. Let $\widehat{\Gamma}$ be a game with incomplete information that is identical to Γ, except that the payoff function $\widehat{u}_j(t_j^*)$ of player j of type t_j^* may be different from $u_j(t_j^*)$, that is, $\widehat{u}_i(t; a) = u_i(t; a)$ if $t_j \neq t_j^*$ or $i \neq j$. Show that the two games Γ and $\widehat{\Gamma}$ have the same set of Bayesian equilibria.

9.46 **Electronic Mail game** Let $L > M > 0$ be two positive real numbers. Two players play a game in which the payoff function is one of the following two, depending on the value of the state of nature s, which may be 1 or 2:

The state game for $s = 1$:

		Player II: A	Player II: B
Player I	A	M, M	$1, -L$
	B	$-L, 0$	$0, 0$

The state game for $s = 2$:

		Player II: A	Player II: B
Player I	A	$0, 0$	$0, -L$
	B	$-L, 1$	M, M

The probability that the state of nature is $s = 2$ is $p < \frac{1}{2}$. Player I knows the true state of nature, and Player II does not know it. The players would clearly prefer to coordinate their actions and play (A, A) if the state of nature is $s = 1$ and (B, B) if the state is $s = 2$, which requires that both of them know what the true state is. Suppose the players are on opposite sides of the globe, and the sole method of communication available to them is e-mail. Due to possible technical communications disruptions, there is a probability of $\varepsilon > 0$ that any e-mail message will fail to arrive at its destination. In order to transfer information regarding the state of nature from Player I to Player II, the two players have constructed an automated system that sends e-mail from Player I to Player II if the state of nature is $s = 2$, and does not send any e-mail if the state is $s = 1$. To ensure that Player I knows that Player II received the message, the system also sends an automated confirmation of receipt of the message (by e-mail, of course) from Player II to Player I the instant Player I's message arrives at Player II's e-mail inbox. To ensure that Player II knows that Player I received the confirmation message, the system also sends an automated confirmation of receipt of the confirmation message from Player I to Player II the instant Player II's confirmation arrives at Player I's e-mail inbox. The system then proceeds to send an automated confirmation of the receipt of the confirmation of the receipt of the confirmation, and so forth. If any of these e-mail messages fail to arrive at their destinations, the automated system stops sending new messages. After communication between the players is completed, each player is called upon to choose an action, A or B.

Answer the following questions:

(a) Depict the situation as a game with incomplete information, in which each type of each player is indexed by the number of e-mail messages he has received.
(b) Prove that the unique Bayesian equilibrium where Player I plays A when $s = 1$ is for both players to play A under all conditions.
(c) How would you play if you received 100 e-mail confirmation messages? Explain your answer.

9.47 In the example described in Section 9.5 (page 361), for each $\varepsilon \in [0, 1]$ find Bayesian equilibria in threshold strategies, where α has uniform distribution over the interval $\left[\frac{1}{4}, \frac{1}{2}\right]$ and β has uniform distribution over the interval $\left[-\frac{1}{3}, \frac{2}{3}\right]$.

9.48 In each of the two strategic-form games whose matrices appear below, find all the equilibria. For each equilibrium, describe a sequence of games with incomplete

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information in which the amplitude of the noise converges to 0, and find Bayesian equilibria in pure strategies in each of these games, such that when the amplitude of the noise converges to 0, the probability that each of the players will choose a particular action converges to the corresponding probability in the equilibrium of the original game (see Section 9.5 on page 361).

Game A (Player I chooses rows, Player II chooses columns):

	L	R
T	1, 5	4, 1
B	2, 1	0, 3

Game B (Player I chooses rows, Player II chooses columns):

	L	R
T	3, 4	2, 2
B	1, 1	2, 1

9.49 Consider a Harsanyi game with incomplete information in which $N = \{\mathrm{I}, \mathrm{II}\}$, $T_{\mathrm{I}} = \{\mathrm{I}_1, \mathrm{I}_2\}$, and $T_{\mathrm{II}} = \{\mathrm{II}_1, \mathrm{II}_2\}$. The mutual beliefs of the types in this game in the interim stage, before actions are chosen, are

	II_1	II_2
I_1	1/4	3/4
I_2	2/3	1/3

Player I's beliefs

	II_1	II_2
I_1	3/11	9/13
I_2	8/11	4/13

Player II's beliefs

and the state games are given by

The state game for $t = (\mathrm{I}_1, \mathrm{II}_1)$ (Player I chooses rows, Player II chooses columns):

	L	R
T	1	0
B	0	0

The state game for $t = (\mathrm{I}_1, \mathrm{II}_2)$ (Player I chooses rows, Player II chooses columns):

	L	R
T	0	1
B	0	0

The state game for $t = (\mathrm{I}_2, \mathrm{II}_1)$ (Player I chooses rows, Player II chooses columns):

	L	R
T	0	0
B	1	0

The state game for $t = (\mathrm{I}_2, \mathrm{II}_2)$ (Player I chooses rows, Player II chooses columns):

	L	R
T	0	0
B	0	1

Are the beliefs of the players consistent? In other words, can they be derived from common prior beliefs? If you answer no, justify your answer. If you answer yes, find the common prior, and find a Bayesian equilibrium in the game.

9.50 Repeat Exercise 9.49, with the following mutual beliefs:

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	II_1	II_2
I_1	1/3	2/3
I_2	3/4	1/4

Player I's beliefs

	II_1	II_2
I_1	3/5	1/6
I_2	2/5	5/6

Player II's beliefs

9.51 Two or three players are about to play a game: with probability $\frac{1}{2}$ the game involves Players 1 and 2 and with probability $\frac{1}{2}$ the game involves Players 1, 2, and 3. Players 2 and 3 know which game is being played. In contrast, Player 1, who participates in the game under all conditions, does not know whether he is playing against Player 2 alone, or against both Players 2 and 3. If the game involves Players 1 and 2 the game is given by the following matrix, where Player 1 chooses the row, and Player 2 chooses the column:

	L	R
T	0, 0	2, 1
B	2, 1	0, 0

with Player 3 receiving no payoff. If the game involves all three players, the game is given by the following two matrices, where Player 1 chooses the row, Player 2 chooses the column, and Player 3 chooses the matrix:

W

	L	R
T	1, 2, 4	0, 0, 0
B	0, 0, 0	2, 1, 3

E

	L	R
T	2, 1, 3	0, 0, 0
B	0, 0, 0	1, 2, 4

(a) What are the states of nature in this game?
(b) How many pure strategies does each player have in this game?
(c) Depict this game as a game with incomplete information.
(d) Describe the game in extensive form.
(e) Find two Bayesian equilibria in pure strategies.
(f) Find an additional Bayesian equilibrium by identifying a strategy vector in which all the players of all types are indifferent between their two possible actions.

9.52 This exercise generalizes Theorems 9.47 (page 354) and 9.53 (page 355) to the case where the prior distributions of the players differ.

Let $(N, (T_i)_{i\in N}, (p_i)_{i\in N}, S, (s_t)_{t\in \times_{i\in N} T_i})$ be a game with incomplete information where each player has a different prior distribution: for each $i \in N$, player i's prior distribution is p_i. For each strategy vector σ, define the payoff function U_i as

$$U_i(\sigma) := \sum_{t\in T} p_i(t) U_i(t;\sigma), \qquad (9.126)$$

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and the payoff of player i of type t_i by

$$U_i(\sigma \mid t_i) := \sum_{t_{-i} \in T_{-i}} p_i(t_{-i} \mid t_i) U_i((t_i, t_{-i}); \sigma). \tag{9.127}$$

A strategy vector σ^* is a *Nash equilibrium* if for every player $i \in N$ and every strategy σ_i of player i,

$$U_i(\sigma^*) \geq U_i(\sigma_i, \sigma^*_{-i}), \tag{9.128}$$

and it is a *Bayesian equilibrium* if for every player $i \in N$, every type $t_i \in T_i$, and every strategy σ_i of player i,

$$U_i(\sigma^* \mid t_i) \geq U_i(\sigma_i, \sigma^*_{-i} \mid t_i). \tag{9.129}$$

(a) Prove that a Nash equilibrium exists when the number of players is finite and each player has finitely many types and actions.

(b) Prove that if each player assigns positive probability to every type of every player, i.e., if $p_i(t_j) := \sum_{t_{-j} \in T_{-j}} p_i(t_j, t_{-j}) > 0$ for every $i, j \in N$ and every $t_j \in T_j$, then every Nash equilibrium is a Bayesian equilibrium, and every Bayesian equilibrium is a Nash equilibrium.

9.53 In this exercise, we explore the connection between correlated equilibrium (see Chapter 8) and games with incomplete information.

(a) Let $\Gamma = (N, (T_i)_{i \in N}, p, S, (s_t)_{t \in \times_{i \in N} T_i})$ be a game with incomplete information, where the set of states of nature S contains only one state, which is a game in strategic form $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$; that is, $s_t = G$ for every $t \in \times_{i \in N} T_i$. The game G is called "the base game" of Γ. Denote the set of action vectors by $A = \times_{i \in N} A_i$. Every strategy vector σ in Γ naturally induces a distribution μ_σ over the vectors in A:

$$\mu_\sigma(a) = \sum_{\omega \in \Omega} p(\omega) \times \sigma_1(t_1; a_1) \times \sigma_2(t_2; a_2) \times \cdots \times \sigma_n(t_n; a_n). \tag{9.130}$$

Prove that if a strategy vector σ^* is a Bayesian equilibrium of Γ, then the distribution μ_{σ^*} defined in Equation (9.130) is a correlated equilibrium in the base game G.

(b) Prove that for every strategic-form game $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$, and every correlated equilibrium μ in this game there exists a game with incomplete information $\Gamma = (N, (T_i)_{i \in N}, p, S, (s_t)_{t \in \times_{i \in N} T_i})$ in which the set of states of nature S contains only one state, and that state corresponds to the base game, $s_t = G$ for every $t \in \times_{i \in N} T_i$, and there exists a Bayesian equilibrium σ^* in the game Γ, such that $\mu(a) = \sum_{\omega \in \Omega} p(\omega) \times \sigma^*_1(t_1; a_1) \times \sigma^*_2(t_2; a_2) \times \cdots \times \sigma^*_n(t_n; a_n)$ for every $a \in A$.

9.54 Carolyn and Maurice are playing the game "Chicken" (see Example 8.3 on page 303). Both Carolyn and Maurice know that Maurice knows who won the Wimbledon tennis tournament yesterday (out of three possible tennis players, Jim, John, and Arthur, who each had a probability of one-third of winning the tournament), but Carolyn does not know who won the tournament.

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(a) Describe this situation as a game with incomplete information, and find the set of Bayesian equilibria of this game.
(b) Answer the first two questions of this exercise, under the assumption that both Carolyn and Maurice only know whether or not Jim has won the tournament.
(c) Answer the first two questions of this exercise, under the assumption that Maurice only knows whether or not Jim has won the tournament, while Carolyn only knows whether or not John has won the tournament.

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