2 Utility theory

Chapter summary

The objective of this chapter is to provide a quantitative representation of players' preference relations over the possible outcomes of the game, by what is called a *utility function*. This is a fundamental element of game theory, economic theory, and decision theory in general, since it facilitates the application of mathematical tools in analyzing game situations whose outcomes may vary in their nature, and often be uncertain.

The utility function representation of preference relations over uncertain outcomes was developed and named after John von Neumann and Oskar Morgenstern. The main feature of the von Neumann–Morgenstern utility is that it is linear in the probabilities of the outcomes. This implies that a player evaluates an uncertain outcome by its *expected utility*.

We present some properties (also known as axioms) that players' preference relations can satisfy. We then prove that any preference relation having these properties can be represented by a von Neumann–Morgenstern utility and that this representation is determined up to a positive affine transformation. Finally we note how a player's attitude toward risk is expressed in his von Neumann–Morgenstern utility function.

2.1 Preference relations and their representation

A game is a mathematical model of a situation of interactive decision making, in which every decision maker (or *player*) strives to attain his "best possible" outcome, knowing that each of the other players is striving to do the same thing.

But what does a player's "best possible" outcome mean? The outcomes of a game need not be restricted to "Win," "Loss," or "Draw." They may well be monetary payoffs or non-monetary payoffs, such as "your team has won the competition," "congratulations, you're a father," "you have a headache," or "you have granted much-needed assistance to a friend in distress."

To analyze the behavior of players in a game, we first need to ascertain the set of outcomes of a game and then we need to know the preferences of each player with respect to the set of outcomes. This means that for every pair of outcomes x and y, we need to know for each player whether he prefers x to y, whether he prefers y to x, or whether he is indifferent between them. We denote by O the set of outcomes of the game. The preferences of each player over the set O are captured by the mathematical concept that is termed *preference relation*.

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Definition 2.1 *A* preference relation *of player i over a set of outcomes O is a binary relation denoted by $\succsim_i$.*

A binary relation is formally a subset of $O \times O$, but instead of writing $(x, y) \in \succsim_i$ we write $x \succsim_i y$, and read that as saying "player i either prefers x to y or is indifferent between the two outcomes"; sometimes we will also say in this case that the player "weakly prefers" x to y. Given the preference relation $\succsim_i$ we can define the corresponding *strict preference relation* $\succ_i$, which describes when player i strictly prefers one outcome to another:

$$x \succ_i y \iff x \succsim_i y \text{ and } y \not\succsim_i x. \tag{2.1}$$

We can similarly define the *indifference* relation $\approx_i$, which expresses the fact that a player is indifferent between two possible outcomes:

$$x \approx_i y \iff x \succsim_i y \text{ and } y \succsim_i x. \tag{2.2}$$

We will assume that every player's preference relation satisfies the following three properties.

Assumption 2.2 *The preference relation $\succsim_i$ over O is* complete*; that is, for any pair of outcomes x and y in O either $x \succsim_i y$, or $y \succsim_i x$, or both.*

Assumption 2.3 *The preference relation $\succsim_i$ over O is* reflexive*; that is, $x \succsim_i x$ for every $x \in O$.*

Assumption 2.4 *The preference relation $\succsim_i$ over O is* transitive*; that is, for any triple of outcomes x, y, and z in O, if $x \succsim_i y$ and $y \succsim_i z$ then $x \succsim_i z$.*

The assumption of completeness says that a player should be able to compare any two possible outcomes and state whether he is indifferent between the two, or has a definite preference for one of them, in which case he should be able to state which is the preferred outcome. One can imagine real-life situations in which this assumption does not obtain, where a player is unable to rank his preferences between two or more outcomes (or is uninterested in doing so). The assumption of completeness is necessary for the mathematical analysis conducted in this chapter.

The assumption of reflexivity is quite natural: every outcome is weakly preferred to itself.

The assumption of transitivity is needed under any reasonable interpretation of what a preference relation means. If this assumption does not obtain, then there exist three outcomes x, y, z such that $x \succsim_i y$ and $y \succsim_i z$, but $z \succ_i x$. That would mean that if a player were asked to choose directly between x and z he would choose z, but if he were first asked to choose between z and y and then between the outcome he just preferred (y) and x, he would choose x, so that his choices would depend on the order in which alternatives are offered to him. Without the assumption of transitivity, it is unclear what a player means when he says that he prefers z to x.

The greater than or equal to relation over the real numbers $\geq$ is a familiar preference relation. It is complete and transitive. If a game's outcomes for player i are sums of dollars, it is reasonable to suppose that the player will compare different outcomes using this preference relation. Since using real numbers and the $\geq$ ordering relation is very convenient for the purposes of conducting analysis, it would be an advantage to be able

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in general to represent game outcomes by real numbers, and player preferences by the familiar $\geq$ relation. Such a representation of a preference relation is called a *utility function*, and is defined as follows.

Definition 2.5 *Let O be a set of outcomes and $\succsim$ be a complete, reflexive, and transitive preference relation over O. A function $u : O \to \mathbb{R}$ is called a* utility function representing *$\succsim$ if for all $x, y \in O$,*

$$x \succsim y \iff u(x) \geq u(y). \tag{2.3}$$

In other words, a utility function u is a function associating each outcome x with a real number $u(x)$ in such a way that the more an outcome is preferred, the larger is the real number associated with it.

If the set of outcomes is finite, any complete, reflexive, and transitive preference relation can easily be represented by a utility function.

Example 2.6 Suppose that $O = \{a, b, c, d\}$ and the preference relation $\succsim$ is given by

$$a \succ b \approx c \succ d. \tag{2.4}$$

Note that although the relation is defined only on part of the set of all pairs of outcomes, the assumptions of reflexivity and transitivity enable us to extend the relation to every pair of outcomes. For example, from the above we can immediately conclude that $a \succ c$.

The utility function u defined by

$$u(a) = 22,\ u(b) = 13,\ u(c) = 13,\ u(d) = 0, \tag{2.5}$$

which represents $\succsim$. There are, in fact, a continuum of utility functions that represent this relation, because the only condition that a utility function needs to meet in order to represent $\succsim$ is

$$u(a) > u(b) = u(c) > u(d). \tag{2.6}$$

◀

The following theorem, whose proof is left to the reader (Exercise 2.2), generalizes the conclusion of the example.

Theorem 2.7 *Let O be a set of outcomes and let $\succsim$ be a complete, reflexive, and transitive preference relation over O. Suppose that u is a utility function representing $\succsim$. Then for every monotonically strictly increasing function $v : \mathbb{R} \to \mathbb{R}$, the composition $v \circ u$ defined by*

$$(v \circ u)(x) = v(u(x)) \tag{2.7}$$

is also a utility function representing $\succsim$.

Given the result of this theorem, a utility function is often called an *ordinal* function, because it represents only the order of preferences between outcomes. The numerical values that a utility function associates with outcomes have no significance, and do not in any way represent the "intensity" of a player's preferences.

2.2 Preference relations over uncertain outcomes: the model

Once we have represented a player's preferences by a utility function, we need to deal with another problem: the outcome of a game may well be uncertain and determined by a lottery. This can occur for two reasons:

- The game may include moves of chance. Examples of such games include backgammon and Monopoly (where dice are tossed) and bridge and poker (where the shuffling of the deck introduces chance into the game). In many economic situations, an outcome may depend on uncertain factors such as changes in currency conversion rates or the valuation of stocks in the stock market, and the outcome itself may therefore be uncertain. The most convenient way to model such situations is to describe some of the determining factors as lottery outcomes.
- One or more of the players may play in a non-deterministic manner, choosing moves by lottery. For example, in a chess match, a player may choose his opening move by tossing a coin. The formal analysis of strategies that depend on lotteries will be presented in Chapter 5.

Example 2.8 Consider the following situation involving one player who has two possible moves, T and B. The outcome is the amount of dollars that the player receives. If she chooses B, she receives \$7,000. If she chooses T, she receives the result of a lottery that grants a payoff of \$0 or \$20,000 with equal probability. The lottery is denoted by $[\frac{1}{2}(\$20{,}000), \frac{1}{2}(\$0)]$. What move can we expect the player to prefer? The answer depends on the player's attitude to risk. There are many people who would rather receive \$7,000 with certainty than take their chances with a toss of a coin determining whether they receive \$20,000 or \$0, while others would take a chance on the large sum of \$20,000. Risk attitude is a personal characteristic that varies from one individual to another, and therefore affects a player's preference relation. ◀

To analyze situations in which the outcome of a game may depend on a lottery over several possible outcomes, the preference relations of players need to be extended to cover preferences over lotteries involving the outcomes.

Given an extended preference relation of a player, which includes preferences over both individual outcomes and lotteries, we can again ask whether such a relation can be represented by a utility function. In other words, can we assign a real number to each lottery in such a way that one lottery is preferred by the player to another lottery if and only if the number assigned to the more-preferred lottery is greater than the number assigned to the less-preferred lottery?

A convenient property that such a utility function can satisfy is linearity, meaning that the number assigned to a lottery is equal to the expected value of the numbers assigned to the individual outcomes over which the lottery is being conducted. For example, if $L = [px, (1-p)y)]$ is a lottery assigning probability p to outcome x, and probability $1-p$ to outcome y, then the linearity requirement would imply that

$$u(L) = pu(x) + (1-p)u(y). \tag{2.8}$$

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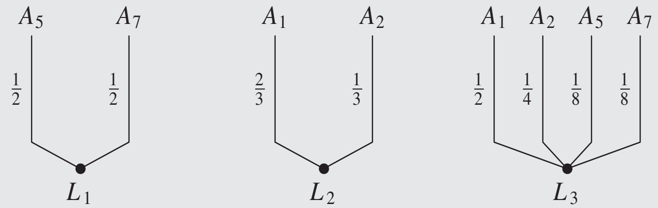

Figure 2.1 Lotteries over outcomes

Such a utility function is linear in the probabilities p and $1 - p$; hence the name. The use of linear utility functions is very convenient for analyzing games in which the outcomes are uncertain (a topic studied in depth in Section 5.5 on page 172). But we still need to answer the question which preference relation of a player (over lotteries of outcomes) can be represented by a linear utility function, as expressed in Equation (2.8)?

The subject of linear utility functions was first explored by the mathematician John von Neumann and the economist Oskar Morgenstern [1944], and it is the subject matter of this chapter.

Suppose that a decision maker is faced with a decision determining which of a finite number of possible outcomes, sometimes designated "prizes," he will receive. (The terms "outcome" and "prize" will be used interchangeably in this section.) Denote the set of possible outcomes by $O = \{A_1, A_2, \ldots, A_K\}$.

In Example 2.8 there are three outcomes $O = \{A_1, A_2, A_3\}$, where $A_1 = \$0$, $A_2 = \$7{,}000$, and $A_3 = \$20{,}000$.

Given the set of outcomes O, the relevant space for conducting analysis is the set of lotteries over the outcomes in O. Figure 2.1 depicts three possible lotteries over outcomes.

The three lotteries in Figure 2.1 are: L_1, a lottery granting A_5 and A_7 with equal probability; L_2, a lottery granting A_1 with probability $\frac{2}{3}$ and A_2 with probability $\frac{1}{3}$; and L_3 granting A_1, A_2, A_5, and A_7 with respective probabilities $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, and $\frac{1}{8}$.

A lottery L in which outcome A_k has probability p_k (where $p_1, \ldots, p_K$ are nonnegative real numbers summing to 1) is denoted by

$$L = [p_1(A_1), p_2(A_2), \ldots, p_K(A_K)], \tag{2.9}$$

and the set of all lotteries over O is denoted by $\mathcal{L}$.

The three lotteries in Figure 2.1 can thus be written as

$$L_1 = \left[\tfrac{1}{2}(A_5), \tfrac{1}{2}(A_7)\right], \quad L_2 = \left[\tfrac{2}{3}(A_1), \tfrac{1}{3}(A_2)\right],$$
$$L_3 = \left[\tfrac{1}{2}(A_1), \tfrac{1}{4}(A_2), \tfrac{1}{8}(A_5), \tfrac{1}{8}(A_7)\right].$$

The set of outcomes O may be regarded as a subset of the set of lotteries $\mathcal{L}$ by identifying each outcome A_k with the lottery yielding A_k with probability 1. In other words, receiving outcome A_k with certainty is equivalent to conducting a lottery that yields A_k with probability 1 and yields all the other outcomes with probability 0,

$$[0(A_1), 0(A_2), \ldots, 0(A_{k-1}), 1(A_k), 0(A_{k+1}), \ldots, 0(A_K)]. \tag{2.10}$$

We will denote a preference relation for player i over the set of all lotteries by $\succsim_i$, so that $L_1 \succsim_i L_2$ indicates that player i either prefers lottery L_1 to lottery L_2 or is indifferent between the two lotteries.

Definition 2.9 *Let $\succsim_i$ be a preference relation for player i over the set of lotteries $\mathcal{L}$. A* utility function u_i *representing the preferences of player i is a real-valued function defined over $\mathcal{L}$ satisfying*

$$u_i(L_1) \geq u_i(L_2) \iff L_1 \succsim_i L_2 \quad \forall L_1, L_2 \in \mathcal{L}. \tag{2.11}$$

In words, a utility function is a function whose values reflect the preferences of a player over lotteries.

Definition 2.10 *A utility function u_i is called* linear *if for every lottery $L = [p_1(A_1), p_2(A_2), \ldots, p_K(A_K)]$, it satisfies*[1]

$$u_i(L) = p_1 u_i(A_1) + p_2 u_i(A_2) + \cdots + p_K u_i(A_K). \tag{2.12}$$

As noted above, the term "linear" expresses the fact that the function u_i is a linear function in the probabilities $(p_k)_{k=1}^K$. If the utility function is linear, the utility of a lottery is the expected value of the utilities of the outcomes. A linear utility function is also called a *von Neumann–Morgenstern utility function*.

Which preference relation of a player can be represented by a linear utility function? First of all, since $\geq$ is a transitive relation, it cannot possibly represent a preference relation $\succsim_i$ that is not transitive. The transitivity assumption that we imposed on the preferences over the outcomes O must therefore be extended to preference relations over lotteries. This alone, however, is still insufficient for the existence of a linear utility function over lotteries: there are complete, reflexive, and transitive preference relations over the set of simple lotteries that cannot be represented by linear utility functions (see Exercise 2.18).

The next section presents four requirements on preference relations that ensure that a preference relation $\succsim_i$ over O can be represented by a linear utility function. These requirements are also termed *the von Neumann–Morgenstern axioms*.

2.3 The axioms of utility theory

Given the observations of the previous section, we would like to identify which preference relations $\succeq_i$ over lotteries can be represented by linear utility functions u_i. The first requirement that must be imposed is that the preference relation be extended beyond the set of simple lotteries to a larger set: the set of compound lotteries.

Definition 2.11 *A* compound lottery *is a lottery of lotteries.*

A compound lottery is therefore given by

$$\widehat{L} = [q_1(L_1), q_2(L_1), \ldots, q_J(L_J)], \tag{2.13}$$

1 Given the identification of outcomes with lotteries, we use the notation $u_i(A_k)$ to denote the utility of the lottery in Equation (2.10), in which the probability of receiving outcome A_k is one.

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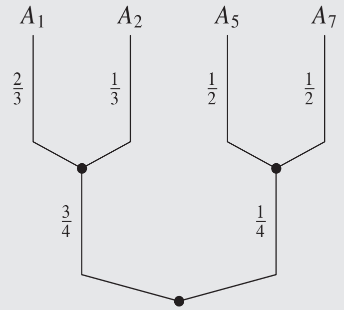

Figure 2.2 An example of a compound lottery

where $q_1, \ldots, q_J$ are nonnegative numbers summing to 1, and $L_1, \ldots, L_J$ are lotteries in $\mathcal{L}$. This means that for each $1 \leq j \leq J$ there are nonnegative numbers $(p_k^j)_{k=1}^K$ summing to 1 such that

$$L_j = \left[p_1^j(A_1), p_2^j(A_1), \ldots, p_K^j(A_K)\right]. \tag{2.14}$$

Compound lotteries naturally arise in many situations. Consider, for example, an individual who chooses his route to work based on the weather: on rainy days he travels by Route 1, and on sunny days he travels by Route 2. Travel time along each route is inconstant, because it depends on many factors (beyond the weather). We are therefore dealing with a "travel time to work" random variable, whose value depends on a lottery of a lottery: there is some probability that tomorrow morning will be rainy, in which case travel time will be determined by a probability distribution depending on the factors affecting travel along Route 1, and there is a complementary probability that tomorrow will be sunny, so that travel time will be determined by a probability distribution depending on the factors affecting travel along Route 2.

We will show in the sequel that under proper assumptions there is no need to consider lotteries that are more compound than compound lotteries, namely, lotteries of compound lotteries. All our analysis can be conducted by limiting consideration to only one level of compounding.

To distinguish between the two types of lotteries with which we will be working, we will call the lotteries in $L \in \mathcal{L}$ *simple lotteries*. The set of compound lotteries is denoted by $\widehat{\mathcal{L}}$.

A graphic depiction of a compound lottery appears in Figure 2.2. Denoting $L_1 = [\frac{2}{3}(A_1), \frac{1}{3}(A_2)]$ and $L_2 = [\frac{1}{2}(A_5), \frac{1}{2}(A_7)]$, the compound lottery in Figure 2.2 is

$$\widehat{L} = \left[\tfrac{3}{4}(L_1), \tfrac{1}{4}(L_2)\right]. \tag{2.15}$$

Every simple lottery L can be identified with the compound lottery $\widehat{L}$ that yields the simple lottery L with probability 1:

$$\widehat{L} = [1(L)]. \tag{2.16}$$

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As every outcome A_k is identified with the simple lottery

$$L = [0(A_1), \dots, 0(A_{k-1}), 1(A_k), 0(A_{k+1}), \dots, 0(A_K)], \tag{2.17}$$

it follows that an outcome A_k is also identified with the compound lottery $[1(L)]$, in which L is the simple lottery defined in Equation (2.17).

Given these identifications, the space we will work with will be the set of compound lotteries,[2] which includes within it the set of simple lotteries $\mathcal{L}$, and the set of outcomes O.

We will assume from now on that the preference relation $\succsim_i$ is defined over the set of compound lotteries. Player i's utility function, representing his preference relation $\succsim_i$, is therefore a function $u_i : \widehat{\mathcal{L}} \to \mathbb{R}$ satisfying

$$u_i(\widehat{L}_1) \geq u_i(\widehat{L}_2) \quad \Longleftrightarrow \quad \widehat{L}_1 \succsim_i \widehat{L}_2, \quad \forall \widehat{L}_1, \widehat{L}_2 \in \widehat{\mathcal{L}}. \tag{2.18}$$

Given the identification of outcomes with simple lotteries, $u_i(A_k)$ and $u_i(L)$ denote the utility of compound lotteries corresponding to the outcome A_k and the simple lottery L, respectively.

Because the preference relation is complete, it determines the preference between any two outcomes A_i and A_j. Since it is transitive, the outcomes can be ordered, from most preferred to least preferred. We will number the outcomes (recall that the set of outcomes is finite) in such a way that

$$A_K \succsim_i \cdots \succsim_i A_2 \succsim_i A_1. \tag{2.19}$$

2.3.1 Continuity

Every reasonable decision maker will prefer receiving \$300 to \$100, and prefer receiving \$100 to \$0, that is,

$$\$300 \succ_i \$100 \succ_i \$0. \tag{2.20}$$

It is also a reasonable assumption that a decision maker will prefer receiving \$300 with probability 0.9999 (and \$0 with probability 0.0001) to receiving \$100 with probability 1. It is reasonable to assume he would prefer receiving \$100 with probability 1 to receiving \$300 with probability 0.0001 (and \$0 with probability 0.9999). Formally,

$$[0.9999(\$300), 0.0001(\$0)] \succ_i 100 \succ_i [0.0001(\$300), 0.9999(\$0)].$$

The higher the probability of receiving \$300 (and correspondingly, the lower the probability of receiving \$0), the more the lottery will be preferred. By continuity, it is reasonable to suppose that there will be a particular probability p at which the decision maker will be indifferent between receiving \$100 and a lottery granting \$300 with probability p and \$0 with probability $1 - p$:

$$100 \approx_i [p(\$300), (1 - p)(\$0)]. \tag{2.21}$$

2 The set of lotteries, as well as the set of compound lotteries, depends on the set of outcomes O, so that in fact we should denote the set of lotteries by $L(O)$, and the set of compound lotteries by $\widehat{\mathcal{L}}(O)$. For the sake of readability, we take the underlying set of outcomes O to be fixed, and we will not specify this dependence in our formal presentation.

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The exact value of p will vary depending on the decision maker: a pension fund making many investments is interested in maximizing expected profits, and its p will likely be close to $\frac{1}{3}$. The p of a risk-averse individual will be higher than $\frac{1}{3}$, whereas for the risk lovers among us p will be less than $\frac{1}{3}$. Furthermore, the size of p, even for one individual, may be situation-dependent: for example, a person may generally be risk averse, and have p higher than $\frac{1}{3}$. However, if this person has a pressing need to return a debt of \$200, then \$100 will not help him, and his p may be temporarily lower than $\frac{1}{3}$, despite his risk aversion.

The next axiom encapsulates the idea behind this example.

Axiom 2.12 (Continuity) *For every triplet of outcomes $A \succsim_i B \succsim_i C$, there exists a number $\theta_i \in [0, 1]$ such that*

$$B \approx_i [\theta_i(A), (1 - \theta_i)(C)]. \tag{2.22}$$

2.3.2 Monotonicity

Every reasonable decision maker will prefer to increase his probability of receiving a more-preferred outcome and lower the probability of receiving a less-preferred outcome. This natural property is captured in the next axiom.

Axiom 2.13 (Monotonicity) *Let α, β be numbers in $[0, 1]$, and suppose that $A \succ_i B$. Then*

$$[\alpha(A), (1 - \alpha)(B)] \succsim_i [\beta(A), (1 - \beta)(B)] \tag{2.23}$$

if and only if $\alpha \geq \beta$.

Assuming the Axioms of Continuity and Monotonicity yields the next theorem, whose proof is left to the reader (Exercise 2.4).

Theorem 2.14 *If a preference relation satisfies the Axioms of Continuity and Monotonicity, and if $A \succsim_i B \succsim_i C$, and $A \succ_i C$, then the value of θ_i defined in the Axiom of Continuity is unique.*

Corollary 2.15 *If a preference relation $\succsim_i$ over $\widehat{\mathcal{L}}$ satisfies the Axioms of Continuity and Monotonicity, and if $A_K \succ_i A_1$, then for each $k = 1, 2, \ldots, K$ there exists a unique $\theta_i^k \in [0, 1]$ such that*

$$A_k \approx_i \left[\theta_i^k(A_K), \left(1 - \theta_i^k\right)(A_1)\right]. \tag{2.24}$$

The corollary and the fact that $A_1 \approx_i [0(A_K), 1(A_1)]$ and $A_K \approx_i [1(A_K), 0(A_1)]$ imply that

$$\theta_i^1 = 0, \quad \theta_i^K = 1. \tag{2.25}$$

2.3.3 Simplification of lotteries

The next axiom states that the only considerations that determine the preference between lotteries are the probabilities attached to each outcome, and not the way that the lottery is conducted. For example, if we consider the lottery in Figure 2.2, with respect to the probabilities attached to each outcome that lottery is equivalent to lottery L_3 in Figure 2.1:

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in both lotteries the probability of receiving outcome A_1 is $\frac{1}{2}$, the probability of receiving outcome A_2 is $\frac{1}{4}$, the probability of receiving outcome A_5 is $\frac{1}{8}$, and the probability of receiving outcome A_7 is $\frac{1}{8}$. The next axiom captures the intuition that it is reasonable to suppose that a player will be indifferent between these two lotteries.

Axiom 2.16 (Axiom of Simplification of Compound Lotteries) *For each* $j = 1, \ldots, J$, *let* L_j *be the following simple lottery:*

$$L_j = \left[p_1^j(A_1), p_2^j(A_2), \ldots, p_K^j(A_K)\right], \tag{2.26}$$

and let $\widehat{L}$ *be the following compound lottery:*

$$\widehat{L} = [q_1(L_1), q_2(L_2), \ldots, q_J(L_J)]. \tag{2.27}$$

For each $k = 1, \ldots, K$ *define*

$$r_k = q_1 p_k^1 + q_2 p_k^2 + \cdots + q_J p_k^J; \tag{2.28}$$

this is the overall probability that the outcome of the compound lottery $\widehat{L}$ *will be* A_k. *Consider the simple lottery*

$$L = [r_1(A_1), r_2(A_2), \ldots, r_K(A_K)]. \tag{2.29}$$

Then

$$\widehat{L} \approx_i L. \tag{2.30}$$

As noted above, the motivation for the axiom is that it should not matter whether a lottery is conducted in a single stage or in several stages, provided the probability of receiving the various outcomes is identical in the two lotteries. The axiom ignores all aspects of the lottery except for the overall probability attached to each outcome, so that, for example, it takes no account of the possibility that conducting a lottery in several stages might make participants feel tense, which could alter their preferences, or their readiness to accept risk.

2.3.4 Independence

Our last requirement regarding the preference relation $\succsim_i$ relates to the following scenario. Suppose that we create a new compound lottery out of a given compound lottery by replacing one of the simple lotteries involved in the compound lottery with a different simple lottery. The axiom then requires a player who is indifferent between the original simple lottery and its replacement to be indifferent between the two corresponding compound lotteries.

Axiom 2.17 (Independence) *Let* $\widehat{L} = [q_1(L_1), \ldots, q_J(L_J)]$ *be a compound lottery, and let* M *be a simple lottery. If* $L_j \approx_i M$ *then*

$$\widehat{L} \approx_i [q_1(L_1), \ldots, q_{j-1}(L_j), q_j(M), q_{j+1}(L_{j+1}), \ldots, q_J(L_J)]. \tag{2.31}$$

One can extend the Axioms of Simplification and Independence to compound lotteries of any order (i.e., lotteries over lotteries over lotteries ... over lotteries over outcomes) in a natural way. By induction over the levels of compounding, it follows that the player's

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preference relation over all compound lotteries (of any order) is determined by the player's preference relation over simple lotteries (why?).

2.4 The characterization theorem for utility functions

The next theorem characterizes when a player has a linear utility function.

Theorem 2.18 *If player i's preference relation $\succsim_i$ over $\widehat{\mathcal{L}}$ is complete and transitive, and satisfies the four von Neumann–Morgenstern axioms (Axioms 2.12, 2.13, 2.16, and 2.17), then this preference relation can be represented by a linear utility function.*

The next example shows how a player whose preference relation satisfies the von Neumann–Morgenstern axioms compares two lotteries based on his utility from the outcomes of the lottery.

Example 2.19 Suppose that Joshua is choosing which of the following two lotteries he prefers:

- $[\frac{1}{2}(\text{New car}), \frac{1}{2}(\text{New computer})]$ – a lottery in which his probability of receiving a new car is $\frac{1}{2}$, and his probability of receiving a new computer is $\frac{1}{2}$.
- $[\frac{1}{3}(\text{New motorcycle}), \frac{2}{3}(\text{Trip around the world})]$ – a lottery in which his probability of receiving a new motorcycle is $\frac{1}{3}$, and his probability of receiving a trip around the world is $\frac{2}{3}$.

Suppose that Joshua's preference relation over the set of lotteries satisfies the von Neumann–Morgenstern axioms. Then Theorem 2.18 implies that there is a linear utility function u representing his preference relation. Suppose that according to this function u:

$$\begin{aligned} u(\text{New Car}) &= 25, \\ u(\text{Trip around the world}) &= 14, \\ u(\text{New motorcycle}) &= 3, \\ u(\text{New computer}) &= 1. \end{aligned}$$

Then Joshua's utility from the first lottery is

$$u\left(\left[\tfrac{1}{2}(\text{New Car}), \tfrac{1}{2}(\text{New computer})\right]\right) = \tfrac{1}{2} \times 25 + \tfrac{1}{2} \times 1 = 13, \tag{2.32}$$

and his utility from the second lottery is

$$u\left(\left[\tfrac{1}{3}(\text{New motorcycle}), \tfrac{2}{3}(\text{Trip around the world})\right]\right) = \tfrac{1}{3} \times 3 + \tfrac{2}{3} \times 14 = \tfrac{31}{3} = 10\tfrac{1}{3}. \tag{2.33}$$

It follows that he prefers the first lottery (whose outcomes are a new car and a new computer) to the second lottery (whose outcomes are a new motorcycle and a trip around the world). ◀

Proof of Theorem 2.18: We first assume that $A_K \succ_i A_1$, i.e., the most-desired outcome A_K is strictly preferred to the least-desired outcome A_1. If $A_1 \approx_i A_K$, then by transitivity, the player is indifferent between all the outcomes. That case is simple to handle, and we will deal with it at the end of the proof.

Step 1: Definition of a function u_i over the set of lotteries.
By Corollary 2.15, for each $1 \leq k \leq K$ there exists a unique real number $0 \leq \theta_i^k \leq 1$ satisfying

$$A_k \approx_i \left[\theta_i^k(A_K), \left(1-\theta_i^k\right)(A_1)\right]. \tag{2.34}$$

We now define a function u_i over the set of compound lotteries $\widehat{\mathcal{L}}$. Suppose $\widehat{L} = [q_1(L_1), \ldots, q_J(L_J)]$ is a compound lottery, in which $q_1, \ldots, q_J$ are nonnegative numbers summing to 1, and $L_1, \ldots, L_J$ are simple lotteries given by $L_j = [p_1^j(A_1), \ldots, p_K^j(A_K)]$.

For each $1 \leq k \leq K$ define

$$r_k = q_1 p_k^1 + q_2 p_k^2 + \cdots + q_J p_k^J. \tag{2.35}$$

This is the probability that the outcome of the lottery is A_k. Define a function u_i on the set of compound lotteries $\widehat{\mathcal{L}}$:

$$u_i(\widehat{L}) = r_1\theta_i^1 + r_2\theta_i^2 + \cdots + r_K\theta_i^k. \tag{2.36}$$

It follows from (2.36) that, in particular, every simple lottery $L = [p_1(A_1), \ldots, p_K(A_K)]$ satisfies

$$u_i(L) = \sum_{k=1}^{K} p_k\theta_i^k. \tag{2.37}$$

Step 2: $u_i(A_k) = \theta_i^k$ for all $1 \leq k \leq K$.
Outcome A_k is equivalent to the lottery $L = [1(A_k)]$, which in turn is equivalent to the compound lottery $\widehat{L} = [1(L)]$. The outcome of this lottery $\widehat{L}$ is A_k with probability 1, so that in this case

$$r_l = \begin{cases} 1 & \text{if } l = k, \\ 0 & \text{if } l \neq k. \end{cases} \tag{2.38}$$

We deduce that

$$u_i(A_k) = \theta_i^k, \qquad \forall k \in \{1, 2, \ldots, K\}. \tag{2.39}$$

Since $\theta_i^1 = 0$ and $\theta_i^K = 1$, we deduce that in particular $u_i(A_1) = 0$ and $u_i(A_K) = 1$.

Step 3: The function u_i is linear.
To show that u_i is linear, it suffices to show that for each simple lottery $L = [p_1(A_1), \ldots, p_K(A_K)]$,

$$u_i(L) = \sum_{k=1}^{K} p_k u_i(A_k). \tag{2.40}$$

This equation holds, because Equation (2.37) implies that the left-hand side of this equation equals $\sum_{i=1}^{K} p_k\theta_i^k$, and Equation (2.39) implies that the right-hand side also equals $\sum_{i=1}^{K} p_k\theta_i^k$.

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Step 4: $\widehat{L} \approx_i [u_i(\widehat{L})(A_K), (1 - u_i(\widehat{L}))(A_1)]$ for every compound lottery $\widehat{L}$.
Let $\widehat{L} = [q_1(L_1), \ldots, q_J(L_J)]$ be a compound lottery, where

$$L_j = \left[p_1^j(A_1), \ldots, p_K^j(A_K)\right], \quad \forall j = 1, 2, \ldots, J. \tag{2.41}$$

Denote, as before,

$$r_k = \sum_{j=1}^{J} q_j p_j^k, \quad \forall k = 1, 2, \ldots, K. \tag{2.42}$$

By the Simplification Axiom,

$$\widehat{L} \approx_i [r_1(A_1), r_2(A_2), \ldots, r_K(A_K)]. \tag{2.43}$$

Denote $M_k = [\theta_i^k(A_K), (1 - \theta_i^k)(A_1)]$ for every $1 \le k \le K$. By definition, $A_k \approx_i M_k$ for every $1 \le k \le K$. Therefore, K applications of the Independence Axiom yield Equation (2.43).

$$\widehat{L} \approx_i [r_1(M_1), r_2(M_2), \ldots, r_K(M_K)]. \tag{2.44}$$

Since all the lotteries $(M_k)_{k=1}^K$ are lotteries over outcomes A_1 and A_K, the lottery on the right-hand side of Equation (2.44) is also a lottery over these two outcomes. Therefore, if we denote by r_* the total probability of A_K in the lottery on the right-hand side of Equation (2.44), then

$$r_* = \sum_{k=1}^{K} r_k \theta_i^k = u_i(\widehat{L}), \tag{2.45}$$

and the Simplification Axiom implies that

$$\widehat{L} \approx_i [r_*(A_K), (1 - r_*)(A_1)] = [u_i(\widehat{L})(A_K), (1 - u_i(\widehat{L}))(A_1)]. \tag{2.46}$$

Step 5: The function u_i is a utility function.
To prove that u_i is a utility function, we need to show that for any pair of compound lotteries $\widehat{L}$ and $\widehat{L}'$

$$\widehat{L} \succsim_i \widehat{L}' \quad \Longleftrightarrow \quad u_i(\widehat{L}) \ge u_i(\widehat{L}') \;\; \forall \widehat{L}_1, \widehat{L}_2 \in \widehat{\mathcal{L}}, \tag{2.47}$$

and this follows from Step 4, and the Monotonicity Axiom. This concludes the proof, under the assumption that $A_K \succ_i A_1$.

We next turn to deal with the degenerate case in which the player is indifferent between all the outcomes:

$$A_1 \approx_i A_2 \approx_i \cdots \approx_i A_K. \tag{2.48}$$

By the Axioms of Independence and Simplification, the player is indifferent between *any* two simple lotteries. To see why, consider the simple lottery $L = [p_1(A_1), \ldots, p_K(A_K)]$. By repeated use of the Axiom of Independence,

$$L \approx_i [p_1(A_1), p_2(A_1), \ldots, p_K(A_1)]. \tag{2.49}$$

The Axiom of Simplification implies that $L \approx_i [1(A_1)]$, so every compound lottery $\widehat{L}$ satisfies $\widehat{L} \approx_i [1(A_1)]$. It follows that the player is indifferent between any two compound lotteries, so that any constant function u_i, represents his preference relation. □

Theorem 2.18 implies that if a player's preference relation satisfies the von Neumann–Morgenstern axioms, then in order to know the player's preferences over lotteries it suffices to know the utility he attaches to each individual outcome, because the utility of any lottery can then be calculated from these utilities (see Equation (2.37) and Example 2.19).

Note that the linearity of utility functions in the probabilities of the individual outcomes, together with the Axiom of Simplification, implies the linearity of utility functions in the probabilities of simple lotteries. In words, if L_1 and L_2 are simple lotteries and $\widehat{L} = [q(L_1), (1-q)(L_2)]$, then $u_i(\widehat{L}) = [qu_i(L_1) + (1-q)u_i(L_2)]$ (see Exercise 2.11).

2.5 Utility functions and affine transformations

Definition 2.20 *Let $u : X \to \mathbb{R}$ be a function. A function $v : X \to \mathbb{R}$ is a* positive affine transformation *of u if there exists a positive real number $\alpha > 0$ and a real number β such that*

$$v(x) = \alpha u(x) + \beta, \quad \forall x \in X. \tag{2.50}$$

The definition implies that if v is a positive affine transformation of u, then u is a positive affine transformation of v (Exercise 2.19).

The next theorem states that every affine transformation of a utility function is also a utility function.

Theorem 2.21 *If u_i is a linear utility function representing player i's preference relation $\succsim_i$, then every positive affine transformation of u_i is also a linear utility function representing $\succsim_i$.*

Proof: Let $\succsim_i$ be player i's preference relation, and let $v_i = \alpha u_i + \beta$ be a positive affine transformation of u_i. In particular, $\alpha > 0$. The first step is to show that v_i is a utility function representing $\succsim_i$. Let $\widehat{L}_1$ and $\widehat{L}_2$ be compound lotteries. We will show that $\widehat{L}_1 \succsim_i \widehat{L}_2$ if and only if $v_i(\widehat{L}_1) \geq v_i(\widehat{L}_2)$.

Note that since u_i is a utility function representing $\succsim_i$,

$$\begin{aligned} \widehat{L}_1 \succsim_i \widehat{L}_2 &\Longleftrightarrow u_i(\widehat{L}_1) \geq u_i(\widehat{L}_2) && (2.51)\\ &\Longleftrightarrow \alpha u_i(\widehat{L}_1) + \beta \geq \alpha u_i(\widehat{L}_2) + \beta && (2.52)\\ &\Longleftrightarrow v_i(\widehat{L}_1) \geq v_i(\widehat{L}_2), && (2.53)\end{aligned}$$

which is what we needed to show.

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Next, we need to show that v_i is linear. Let $L = [p_1(A_1), p_2(A_2), \dots, p_K(A_K)]$ be a simple lottery. Since $p_1 + p_2 + \cdots + p_K = 1$, and u_i is linear, we get

$$v_i(L) = \alpha u_i(L) + \beta \tag{2.54}$$
$$= \alpha(p_1 u_i(A_1) + p_2 u_i(A_2) + \cdots + p_K u_i(A_K)) + (p_1 + p_2 + \cdots + p_K)\beta \tag{2.55}$$
$$= p_1 v_i(A_1) + p_2 v_i(A_2) + \cdots + p_K v_i(A_K), \tag{2.56}$$

which shows that v_i is linear. □

The next theorem states the opposite direction of the previous theorem. Its proof is left to the reader (Exercise 2.21).

Theorem 2.22 *If u_i and v_i are two linear utility functions representing player i's preference relation, where that preference relation satisfies the von Neumann–Morgenstern axioms, then v_i is a positive affine transformation of u_i.*

Corollary 2.23 *A preference relation of a player that satisfies the von Neumann–Morgenstern axioms is representable by a linear utility function that is uniquely determined up to a positive affine transformation.*

2.6 Infinite outcome set

We have so far assumed that the set of outcomes O is finite. A careful review of the proofs reveals that all the results above continue to hold if the following conditions are satisfied:

- The set of outcomes O is any set, finite or infinite.
- The set of simple lotteries $\mathcal{L}$ contains every lottery over a finite number of outcomes.
- The set of compound lotteries $\widehat{\mathcal{L}}$ contains every lottery over a finite number of simple lotteries.
- The player has a complete, reflexive, and transitive preference relation over the set of compound lotteries $\widehat{\mathcal{L}}$.
- There exists a (weakly) most-preferred outcome $A_K \in O$: the player (weakly) prefers A_K to any other outcome in O.
- There exists a (weakly) least-preferred outcome $A_1 \in O$: the player (weakly) prefers any other outcome in O to A_1.

In Exercise 2.22, the reader is asked to check that Theorems 2.18 and 2.22, and Corollary 2.23, hold in this general model.

2.7 Attitude towards risk

There are people who are risk averse, people who are risk neutral, and people who are risk seeking. The risk attitude of an individual can change over time; it may depend, for

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example, on the individual's family status or financial holdings. How does risk attitude affect a player's utility function?

In this section, we will assume that the set of outcomes is given by the interval $O = [-R, R]$: the real number $x \in [-R, R]$ represents the monetary outcome that the player receives. We will assume that every player prefers receiving more, in dollars, to receiving less, so that $x \succ_i y$ if and only if $x > y$. We will similarly assume that the player has a complete, reflexive, and transitive preference relation over the set of compound lotteries that satisfies the von Neumann–Morgenstern axioms.

Denote by u_i player i's utility function. As previously noted, the function u_i is determined by player i's utility from every outcome of a lottery. These utilities are given by a real-valued function $U_i : \mathbb{R} \to \mathbb{R}$. In words, for every $x \in O$,

$$U_i(x) := u_i([1(x)]). \tag{2.57}$$

Since players are assumed to prefer getting as large a monetary amount as possible, U_i is a monotonically increasing function.

By the assumption that each player's preference relation satisfies the von Neumann–Morgenstern axioms, it follows that for every simple lottery $L = [p_1(x_1), p_2(x_2), \ldots, p_K(x_k)]$,

$$u_i(L) = \sum_{k=1}^{K} p_k U_i(x_k) = \sum_{k=1}^{K} p_k u_i([1(x_k)]). \tag{2.58}$$

The significance of this equation is that the utility $u_i(L)$ of a lottery L is the expected utility of the resulting payoff.

Given a lottery $L = [p_1(x_1), p_2(x_2), \ldots, p_K(x_k)]$ with a finite number of possible outcomes, we will denote by μ_L the expected value of L, given by

$$\mu_L = \sum_{i=1}^{K} p_k x_k. \tag{2.59}$$

Definition 2.24 *A player i is termed* risk neutral *if for every lottery L with a finite number of possible outcomes,*

$$u_i(L) = u_i([1(\mu_L)]). \tag{2.60}$$

A player i is termed risk averse *if for every lottery L with a finite number of possible outcomes,*

$$u_i(L) \le u_i([1(\mu_L)]). \tag{2.61}$$

A player i is termed risk seeking *(or risk loving) if for every lottery L with a finite number of possible outcomes,*

$$u_i(L) \ge u_i([1(\mu_L)]). \tag{2.62}$$

Using Definition 2.24, to establish a player's risk attitude, we need to compare the utility he ascribes to every lottery with the utility he ascribes to the expected value of that lottery. Conducting such a comparison can be exhausting, because it involves checking the condition with respect to every possible lottery. The next theorem, whose proof is

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left to the reader (Exercise 2.23), shows that it suffices to conduct the comparisons only between lotteries involving pairs of outcomes.

Theorem 2.25 *A player i is risk neutral if and only if for each $p \in [0, 1]$ and every pair of outcomes $x, y \in \mathbb{R}$,*

$$u_i([p(x), (1-p)(y)]) = u_i([1(px + (1-p)y)]). \tag{2.63}$$

A player i is risk averse if and only if for each $p \in [0, 1]$ and every pair of outcomes $x, y \in \mathbb{R}$,

$$u_i([p(x), (1-p)(y)]) \leq u_i([1(px + (1-p)y)]). \tag{2.64}$$

A player i is risk seeking if and only if for each $p \in [0, 1]$ and every pair of outcomes $x, y \in \mathbb{R}$,

$$u_i([p(x), (1-p)(y)]) \geq u_i([1(px + (1-p)y)]). \tag{2.65}$$

Example 2.26 Consider a player whose preference relation is represented by the utility function $U_i(x)$ that is depicted in Figure 2.3, which is concave.

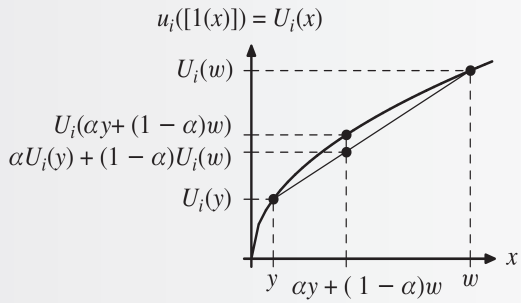

Figure 2.3 The utility function of a risk-averse player

The figure depicts the graph of the function U_i, which associates each x with the utility of the player from definitely receiving outcome x (see Equation (2.57)). We will show that the concavity of the function U_i is an expression of the fact that player i is risk averse. Since the function U_i is concave, the chord connecting the two points on the graph of the function passes underneath the graph. Hence for every $y, w \in \mathbb{R}$ and every $\alpha \in (0, 1)$,

$$\begin{aligned} u_i([1(\alpha y + (1-\alpha)w))]) &= U_i(\alpha y + (1-\alpha)w) && (2.66)\\ &> \alpha U_i(y) + (1-\alpha)U_i(w) && (2.67)\\ &= \alpha u_i([1(y)]) + (1-\alpha)u_i([1(w)]) && (2.68)\\ &= u_i([\alpha(y), (1-\alpha)(w)]). && (2.69)\end{aligned}$$

In words, player i prefers receiving with certainty the expectation $\alpha y + (1-\alpha)w$ to receiving y with probability α and w with probability $1-\alpha$, which is precisely what risk aversion means. ◀

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As Example 2.26 suggests, one's attitude to risk can be described in simple geometrical terms, using the utility function.

Theorem 2.27 *A player i, whose preference relation satisfies the von Neumann–Morgenstern axioms, is risk neutral if and only if U_i is a linear function, he is risk averse if and only if U_i is a concave function, and he is risk seeking if and only if U_i is a convex function.*

Proof: Since by assumption the player's preference relation satisfies the von Neumann–Morgenstern axioms, the utility of every simple lottery $L = [p_1(x_1), p_2(x_2), \ldots, p_K(x_K)]$ is given by

$$u_i(L) = \sum_{k=1}^{K} p_k u_i([1(x_k)]) = \sum_{k=1}^{K} p_k U_i(x_k). \tag{2.70}$$

A player is risk averse if and only if $u_i(L) \leq u_i([1(\mu_L)]) = U_i(\mu_L)$, or, in other words, if and only if

$$\sum_{k=1}^{K} p_k U_i(x_k) = u_i(L) \leq U_i(\mu_L) = U_i\left(\sum_{k=1}^{K} p_k x_k\right). \tag{2.71}$$

In summary, a player is risk averse if and only if

$$\sum_{k=1}^{K} p_k U_i(x_k) \leq U_i\left(\sum_{k=1}^{K} p_k x_k\right). \tag{2.72}$$

This inequality holds for every $(x_1, x_2, \ldots, x_K)$ and for every vector of nonnegative numbers $(p_1, p_2, \ldots, p_K)$ summing to 1, if and only if U_i is concave. Similarly, player i is risk seeking if and only if

$$\sum_{k=1}^{K} p_k U_i(x_k) \geq U_i\left(\sum_{k=1}^{K} p_k x_k\right). \tag{2.73}$$

This inequality holds for every $(x_1, x_2, \ldots, x_K)$ and for every vector of nonnegative numbers $(p_1, p_2, \ldots, p_K)$ summing to 1, if and only if U_i is convex.

A player is risk neutral if and only if he is both risk seeking and risk neutral. Since a function is both concave and convex if and only if it is linear, player i is risk neutral if and only if U_i is linear. □

2.8 Subjective probability

A major milestone in the study of utility theory was attained in 1954, with Leonard Savage's publication of *The Foundations of Statistics*. Savage generalized von Neumann and Morgenstern's model, in which the probability of each outcome in every lottery is "objective" and known to the participants. That model is reasonable when the outcome is determined by a flip of a coin or a toss of dice, but in most of the lotteries we face in real life, probabilities are often unknown. Consider, for example, the probability of a major earthquake occurring over the next year in the San Fernando Valley, or the probability that a particular candidate will win the next presidential election. The exact probabilities of these occurrences are unknown. Different people will differ in their assessments of these probabilities, which are subjective. In addition, as noted above, people often fail to perceive probability correctly, so that their perceptions contradict the laws of probability.

Savage supposed that there is an infinite set of states of the world, Ω; each state of the world is a complete description of all the variables characterizing the players, including the information they have. Players are asked to choose between "gambles," which formally are functions $f : \Omega \to O$. What this means is that if a player chooses gamble f, and the state of the world (i.e., the true reality) is ω, then the outcome the player receives is $f(\omega)$. Players are assumed to have complete, reflexive, and transitive preference relations over the set of all gambles. For example, if $E, F \subset \Omega$ are two events, and A_1, A_2, A_3, and A_4 are outcomes, a player can compare a gamble in which he receives A_1 if the true state is in E and A_2 if the true state is not in E, with a gamble in which he receives A_3 if the true state is in F and A_4 if the true state is not in F.

Savage proved that if the preference relation of player i satisfies certain axioms, then there exists a probability distribution q_i over Ω and a function $u_i : O \to \mathbb{R}$ representing player i's preference relation. In other words, the player, by preferring one gamble to another, behaves as if he is maximizing expected utility, where the expected utility is calculated using the probability distribution q_i:

$$u_i(f) = \int_\Omega u_i(f(\omega))\mathrm{d}q_i(\omega). \tag{2.74}$$

Similarly to von Neumann–Morgenstern utility, the utility of f is the expected value of the utility of the outcomes, with q_i representing player i's subjective probability, and utility u_i representing the player's preferences (whether or not he is conscious of using a probability distribution and a utility function at all).

A further development in subjective probability theory, slightly different from Savage's, was published by Anscombe and Aumann [1963].

2.9 Discussion

Theoretically, a simple interview is all that is needed to ascertain a player's utility function, assuming his preference relation satisfies the von Neumann–Morgenstern axioms. One can set the utility of A_1, the least-preferred outcome, to be 0, the utility of A_k, the most-preferred outcome, to be 1, and then find, for every $k \in \{2, 3, \ldots, K-1\}$, the values of θ_i^k such that the player is indifferent between A_K and the lottery $[\theta_i^k(A_K), (1-\theta_i^k)(A_1)]$.

Experimental evidence shows that in interviews, people often give responses that indicate their preferences do not always satisfy the von Neumann–Morgenstern axioms. Here are some examples.

2.9.1 The assumption of completeness

The assumption of completeness appears to be very reasonable, but it should not be regarded as self-evident. There are cases in which people find it difficult to express clear

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preferences between outcomes. For example, imagine a child whose parents are divorced, who is asked whether he prefers a day with his mother or his father. Many children find the choice too difficult, and refuse to answer the question.

2.9.2 The assumption of transitivity

Give a person a sufficiently large set of choices between outcomes, and you are likely to discover that his declared preferences contradict the assumption of transitivity. Some of these "errors" can be corrected by presenting the player with evidence of inconsistencies, careful analysis of the answers, and attempts to correct the player's valuations.

Violations of transitivity are not always due to inconsistencies on the part of an individual player. If a "player" is actually composed of a group of individuals, each of whom has a transitive preference relation, it is possible for the group's collective preferences to be non-transitive. The next example illustrates this phenomenon.

Example 2.28 **The Condorcet Paradox** Three alternative political policies, A, B, and C, are being debated. It is suggested that a referendum be conducted to choose between them. The voters, however, have divided opinions on the relative preferences between the policies, as follows:

Democrats:	$A \succ_D B \succ_D C$
Republicans:	$B \succ_R C \succ_R A$
Independents:	$C \succ_I A \succ_I B$

Suppose that the population is roughly equally divided between Democrats, Republicans, and Independents. It is possible to fashion a referendum that will result in a nearly two-thirds majority approving any one of the alternative policies. For example, if the referendum asks the electorate to choose between A and B, a majority will vote $A \succ B$. If, instead, the referendum presents a choice between B and C, a majority will vote $B \succ C$; and a similar result can be fashioned for $C \succ A$. Which of these three policies, then, can we say the electorate prefers?

The lack of transitivity in preferences resulting from the use of the majority rule is an important subject in "social choice theory" (see Chapter 21). This was first studied by Condorcet[3] (see Example 21.1 on page 854). ◀

2.9.3 Perceptions of probability

If a person's preference relation over three possible outcomes A, B, and C satisfies $A \succ B \succ C$, we may by trial and error present him with various different probability values p, until we eventually identify a value p_0 such that

$$B \approx [p_0(A), (1 - p_0)(C)]. \tag{2.75}$$

Let's say, for example, that the player reports that he is indifferent between the following:

$$\$7{,}000 \approx \left[\tfrac{2}{3}(\$20{,}000), \tfrac{1}{3}(\$0)\right]. \tag{2.76}$$

3 Marie Jean Antoine Nicolas Caritat, Marquis de Condorcet, 1743–94, was a French philosopher and mathematician who wrote about political science.

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Empirically, however, if the same person is asked how large x must be in order for him to be indifferent between the following:

$$\$7{,}000 \approx \left[\tfrac{2}{3}(\$x), \tfrac{1}{3}(\$0)\right], \tag{2.77}$$

the answer often[4] differs from $20,000.

This shows that the perceptions of probability that often occur naturally to decision makers may diverge from the mathematical formulations. People are not born with internal calculators, and we must accept the fact that what people perceive may not always follow the laws of probability.

2.9.4 The Axiom of Simplification

The Axiom of Simplification states that the utility of a compound lottery depends solely on the probability it eventually assigns to each outcome. We have already noted that this ignores other aspects of compound lotteries; for example, it ignores the pleasure (or lack of pleasure) a participant gains from the very act of participating in a lottery. It is therefore entirely possible that a person may prefer a compound lottery to a simple lottery with exactly the same outcome probabilities, or vice versa.

2.9.5 Other aspects that can influence preferences

People's preferences change over time and with changing circumstances. A person may prefer steak to roast beef today, and roast beef to steak tomorrow.

One also needs to guard against drawing conclusions regarding preferences to quickly given answers to interview questions, because the answers are liable to depend on the information available to the player. Take, for example, the following story, based on a similar story appearing in Luce and Raifa [1957]. A man at a restaurant asks a waiter to list the available items on the menu. The waiter replies "steak and roast beef." The man orders the roast beef. A few minutes later, the waiter returns and informs him that he forgot to note an additional item on the menu, filet mignon. "In that case," says the restaurant guest, "I'll have the steak, please."

Does this behavior reveal inconsistency in preferences? Not necessarily. The man may love steak, but may also be concerned that in most restaurants, the steak is not served sufficiently tender to his taste. He therefore orders the roast beef, confident that most chefs know how to cook a decent roast. When he is informed that the restaurant serves filet mignon, he concludes that there is a high-quality chef in the kitchen, and feels more confident in the chef's ability to prepare a tender steak.

In other words, the fact that given a choice between steak and roast beef, a player chooses roast beef, does not necessarily mean that he prefers roast beef to steak. It may only indicate that the quality of the steak is unknown, in which case choosing "steak" may translate into a lottery between quality steak and intolerable steak. Before receiving additional information, the player ascribes low probability to receiving quality steak. After the additional information has been given, the probability of quality steak increases in the player's estimation, thus affecting his choice. The player's preference of steak to roast

4 The authors wish to thank Reinhard Selten for providing them with this example.

beef has not changed at all over time, but rather his perception of the lottery with which he is presented.

This story illustrates how additional information can bring about changes in choices without contradicting the assumptions of utility theory.

Another story, this one a true event that occurred during the Second World War on the Pacific front,[5] seems to contradict utility theory. A United States bomber squadron, charged with bombing Tokyo, was based on the island of Saipan, 3000 kilometers from the bombers' targets. Given the vast distance the bombers had to cover, they flew without fighter-plane accompaniment and carried few bombs, in order to cut down on fuel consumption. Each pilot was scheduled to rotate back to the United States after 30 successful bombing runs, but Japanese air defenses were so efficient that only half the pilots sent on the missions managed to survive 30 bombing runs.

Experts in operations research calculated a way to raise the odds of overall pilot survival by increasing the bomb load carried by each plane – at the cost of placing only enough fuel in each plane to travel in one direction. The calculations indicated that increasing the number of bombs per plane would significantly reduce the number of required bombing runs, enabling three-quarters of the pilots to be rotated back to the United States immediately, without requiring them to undertake any more missions. The remaining pilots, however, would face certain death, since they would have no way of returning to base after dropping their bombs over Tokyo.

If the pilots who are sent home are chosen randomly, then the pilots were, in fact, being offered the lottery

$$\left[\tfrac{3}{4}(\text{Life}), \tfrac{1}{4}(\text{Death})\right],$$

in place of their existing situation, which was equivalent to the lottery

$$\left[\tfrac{1}{2}(\text{Life}), \tfrac{1}{2}(\text{Death})\right].$$

Every single pilot rejected the suggested lottery outright. They all preferred their existing situation.

Were the pilots lacking a basic understanding of probability? Were they contradicting the von Neumann–Morgenstern axioms? One possible explanation for why they failed to act in accordance with the axioms is that they were optimists by nature, believing that "it will not happen to me." But there are other explanations, that do not necessarily lead to a rejection of standard utility theory. The choice between life and death may not have been the only factor that the pilots took into account. There may also have been moral issues, such as taboos against sending some comrades on certain suicide missions while others got to return home safely. In addition, survival rates are not fixed in war situations. There was always the chance that the war would take a dramatic turn, rendering the suicide missions unnecessary, or that another ingenious solution would be found. And indeed, a short time after the suicide mission suggestion was raised, American forces captured the island of Iwo Jima. The air base in Iwo Jima was sufficiently close to Tokyo, only

5 The story was related to the authors by Kenneth Arrow, who heard of it from Merrill F. Flood.

600 kilometers away, to enable fighter planes to accompany the bombers, significantly raising the survival rates of American bombers, and the suicide mission suggestion was rapidly consigned to oblivion.[6]

2.10 Remarks

The authors wish to thank Tzachi Gilboa and Peter Wakker for answering several questions that arose during the composition of this chapter.

The Sure-Thing Principle, which appears in Exercise 2.12, first appeared in Savage [1954]. first presented in Marschak [1950] and Nash [1950a]. The property described in Exercise 2.14 is called "Betweenness." Exercise 2.15 is based on a column written by John Branch in *The New York Times* on August 30, 2010. Exercise 2.25 is based on Rothschild and Stiglitz [1970], which also contains an example of the phenomenon appearing in Exercise 2.27. The Arrow–Pratt measure of absolute risk aversion, which appears in Exercise 2.28, was first defined by Arrow [1965] and Pratt [1964].

2.11 Exercises

2.1 Prove the following claims:

(a) A strict preference relation $\succ$ is anti-symmetric and transitive.[7]
(b) An indifference relation $\approx$ is symmetric and transitive.[8]

2.2 Prove Theorem 2.7 (page 11): let O be a set of outcomes, and let $\succsim$ be a complete, reflexive, and transitive relation over O. Suppose that u is a utility function representing $\succsim$. Prove that for every monotonically increasing function $v : \mathbb{R} \to \mathbb{R}$, the composition $v \circ u$ defined by

$$(v \circ u)(x) = v(u(x)) \tag{2.78}$$

is also a utility function representing $\succsim$.

2.3 Give an example of a countable set of outcomes O and a preference relation $\succsim$ over O, such that every utility function representing $\succsim$ must include values that are not integers.

2.4 Prove Theorem 2.14 (page 17): if a preference relation $\succsim_i$ satisfies the axioms of continuity and monotonicity, and if $A \succsim_i B \succsim_i C$ and $A \succ_i C$, then there exists a unique number $\theta_i \in [0, 1]$ that satisfies

$$B \approx_i [\theta_i(A), (1 - \theta_i)(B)]. \tag{2.79}$$

6 Bombing missions emanating from Iwo Jima also proved to be largely inefficient – only ten such missions were attempted – but American military advances in the Spring of 1945 rapidly made those unnecessary as well.
7 A relation $\succ$ is *anti-symmetric* if for each x, y, if $x \succ y$, then it is not the case that $y \succ x$.
8 A relation $\approx$ is *symmetric* if for each x, y, if $x \approx y$, then $y \approx x$.

2.5 Prove that the von Neumann–Morgenstern axioms are independent. In other words, for every axiom there exists a set of outcomes and a preference relation that does not satisfy that axiom but does satisfy the other three axioms.

2.6 Prove the converse of Theorem 2.18 (page 19): if there exists a linear utility function representing a preference relation $\succsim_i$ of player i, then $\succsim_i$ satisfies the von Neumann–Morgenstern axioms.

2.7 Suppose that a person whose preferences satisfy the von Neumann–Morgenstern axioms, and who always prefers more money to less money, says that:

- he is indifferent between receiving \$500 and participating in a lottery in which he receives \$1,000 with probability $\frac{2}{3}$ and receives \$0 with probability $\frac{1}{3}$;
- he is indifferent between receiving \$100 and participating in a lottery in which he receives \$500 with probability $\frac{3}{8}$ and receives \$0 with probability $\frac{5}{8}$.

(a) Find a linear utility function representing this person's preferences, and in addition satisfying $u(\$1{,}000) = 1$ and $u(\$0) = 0$.

(b) Determine which of the following two lotteries will be preferred by this person:
- A lottery in which he receives \$1,000 with probability $\frac{3}{10}$, \$500 with probability $\frac{1}{10}$, \$100 with probability $\frac{1}{2}$, and \$0 with probability $\frac{1}{10}$, or
- A lottery in which he receives \$1,000 with probability $\frac{2}{10}$, \$500 with probability $\frac{3}{10}$, \$100 with probability $\frac{2}{10}$, and \$0 with probability $\frac{3}{10}$.

(c) Is it possible to ascertain which of the following two lotteries he will prefer? Justify your answer.
- A lottery in which he receives \$1,000 with probability $\frac{3}{10}$, \$500 with probability $\frac{1}{10}$, \$100 with probability $\frac{1}{2}$, and \$0 with probability $\frac{1}{10}$.
- Receiving \$400 with probability 1.

(d) Is it possible to ascertain which of the following two lotteries he will prefer? Justify your answer.
- A lottery in which he receives \$1,000 with probability $\frac{3}{10}$, \$500 with probability $\frac{1}{10}$, \$100 with probability $\frac{1}{2}$, and \$0 with probability $\frac{1}{10}$.
- Receiving \$600 with probability 1.

2.8 How would the preferences between the two lotteries in Exercise 2.7(b) change if $u(\$1{,}000) = 8$ and $u(\$0) = 3$? Justify your answer.

2.9 Suppose that a person whose preferences satisfy the von Neumann–Morgenstern axioms says that his preferences regarding outcomes A, B, C, and D satisfy

$$C \approx_i \left[\tfrac{3}{5}(A), \tfrac{2}{5}(D)\right], \quad B \approx_i \left[\tfrac{3}{4}(A), \tfrac{1}{4}(C)\right], \quad A \succ_i D. \tag{2.80}$$

Determine which of the following two lotteries will be preferred by this person:

$$L_1 = \left[\tfrac{2}{5}(A), \tfrac{1}{5}(B), \tfrac{1}{5}(C), \tfrac{1}{5}(D)\right] \text{ or } L_2 = \left[\tfrac{2}{5}(B), \tfrac{3}{5}(C)\right]. \tag{2.81}$$

2.10 What would be your answer to Exercise 2.9 if $D \succ_i A$ instead of $A \succ_i D$? Relate your answer to this exercise with your answer to Exercise 2.9.

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2.11 Prove that if u_i is a linear utility function, then

$$u_i(\widehat{L}) = \sum_{j=1}^{J} q_j u_i(L_j) \tag{2.82}$$

is satisfied for every compound lottery $\widehat{L} = [q_1(L_1), q_2(L_2), \dots, q_J(L_J)]$.

2.12 The Sure-Thing Principle Prove that a preference relation that satisfies the von Neumann–Morgenstern axioms also satisfies

$$[\alpha(L_1), (1-\alpha)(L_3)] \succ [\alpha(L_2), (1-\alpha)(L_3)] \tag{2.83}$$

if and only if

$$[\alpha(L_1), (1-\alpha)(L_4)] \succ [\alpha(L_2), (1-\alpha)(L_4)]. \tag{2.84}$$

for any four lotteries L_1, L_2, L_3, L_4, and any $\alpha \in [0, 1]$.

2.13 Suppose a person whose preferences satisfy the von Neumann–Morgenstern axioms says that with respect to lotteries L_1, L_2, L_3, L_4, his preferences are $L_1 \succ L_2$ and $L_3 \succ L_4$. Prove that for all $0 \le \alpha \le 1$,

$$[\alpha(L_1), (1-\alpha)(L_3)] \succ [\alpha(L_2), (1-\alpha)(L_4)]. \tag{2.85}$$

2.14 Suppose a person whose preferences satisfy the von Neumann–Morgenstern axioms says that with respect to lotteries L_1 and L_2, his preference is $L_1 \succ L_2$. Prove that for all $0 < \alpha \le 1$,

$$[\alpha(L_1), (1-\alpha)(L_2)] \succ L_2. \tag{2.86}$$

2.15 A tennis player who is serving at the beginning of a point has two attempts to serve; if the ball does not land within the white lines of the opponent's court on his first attempt, he receives a second attempt. If the second attempt also fails to land in the opponent's court, the serving player loses the point. If the ball lands in the opponent's court during either attempt, the players volley the ball over the net until one or the other player wins the point.

While serving, a player has two alternatives. He may strike the ball with great force, or with medium force. Statistics gathered from a large number of tennis matches indicate that if the server strikes the ball with great force, the ball lands in the opponent's court with probability 0.65, with the server subsequently winning the point with probability 0.75. If, however, the server strikes the ball with medium force, the ball lands in the opponent's court with probability 0.9, with the server subsequently winning the point with probability 0.5.

In most cases, servers strike the ball with great force on their first-serve attempts, and with medium force on their second attempts.

(a) Assume that there are two possible outcomes: winning a point or losing a point, and that the server's preference relation over compound lotteries satisfies the von Neumann–Morgenstern axioms. Find a linear utility function representing the server's preference relation.

(b) Write down the compound lottery that takes place when the server strikes the ball with great force, and when he strikes the ball with medium force.

(c) The server has four alternatives: two alternatives in her first-serve attempt (striking the ball with great force or with medium force), and similarly two alternatives in his second serve attempt if the first attempt failed. Write down the compound lotteries corresponding to each of these four alternatives. Note that in this case the compound lotteries are of order 3: lotteries over lotteries over lotteries.

(d) Which compound lottery is most preferred by the server, out of the four compound lotteries you identified in item (c) above? Is this alternative the one chosen by most tennis players?

2.16 Ron eats yogurt every morning. Ron especially loves yogurt that comes with a small attached container containing white and dark chocolate balls, which he mixes into his yogurt prior to eating it. Because Ron prefers white chocolate to dark chocolate, he counts the number of white chocolate balls in the container, his excitement climbing higher the greater the number of white chocolate balls. One day, Ron's brother Tom has an idea for increasing his brother's happiness: he will write to the company producing the yogurt and ask them to place only white chocolate balls in the containers attached to the yogurt! To Tom's surprise, Ron opposes this idea: he prefers the current situation, in which he does not know how many white chocolate balls are in the container, to the situation his brother is proposing, in which he knows that each container has only white chocolate balls. Answer the following questions.

(a) Write down the set of outcomes in this situation, and Ron's preference relation over those outcomes.

(b) Does Ron's preference relation over lotteries satisfy the von Neumann–Morgenstern axioms? Justify your answer.

2.17 A farmer wishes to dig a well in a square field whose coordinates are (0, 0), (0, 1000), (1000, 0), and (1000, 1000). The well must be located at a point whose coordinates (x, y) are integers. The farmer's preferences are lexicographic: if $x_1 > x_2$, he prefers that the well be dug at the point (x_1, y_1) to the point (x_2, y_2), for all y_1, y_2. If $x_1 = x_2$, he prefers the first point only if $y_1 > y_2$.

Does there exist a preference relation over compound lotteries over pairs of integers (x, y), $0 \leq x, y \leq 1000$, that satisfies the von Neumann–Morgenstern axioms and extends the lexicographic preference relation? If so, give an example of a linear utility function representing such a preference relation, and if not, explain why such a preference relation does not exist.

2.18 In this exercise, we will show that in the situation described in Exercise 2.17, when the coordinates (x, y) can be any real numbers in the square $[0, 1000]^2$, there does not exist a utility function that represents the lexicographic preference relation. Suppose, by contradiction, that there does exist a preference relation over $[0, 1000]^2$ that represents the lexicographic preference relation.

(a) Prove that for each $(x, y) \in [0, 1000]^2$ there exists a unique $\theta_{x,y} \in [0, 1]$ such that the farmer is indifferent between locating the well at point (x, y) and a lottery in which the well is located at point $(0, 0)$ with probability $1 - \theta_{x,y}$ and located at point $(1000, 1000)$ with probability $\theta_{x,y}$.

(b) Prove that the function $(x, y) \mapsto \theta_{x,y}$ is injective, that is, $\theta_{x',y'} \neq \theta_{x,y}$ whenever $(x', y') \neq (x, y)$.

(c) For each x, define $A_x := \{\theta_{x,y} \colon y \in [0, 1000]\}$. Prove that for each x the set A_x contains at least two elements, and that the sets $\{A_x, x \in [0, 1]\}$ are pairwise disjoint.

(d) Prove that if $x_1 < x_2$ then $\theta_1 < \theta_2$ for all $\theta_1 \in A_{x_1}$ and for all $\theta_2 \in A_{x_2}$.

(e) Prove that there does not exist a set $\{A_x : x \in [0; 1]\}$ satisfying (c) and (d).

(f) Deduce that there does not exist a utility function over $[0, 1000]^2$ that represents the lexicographic preference relation.

(g) Which of the von Neumann–Morgenstern axioms is not satisfied by the preference relation in this exercise?

2.19 Prove that if v is a positive affine transformation of u, then u is a positive affine transformation of v.

2.20 Prove that if v is a positive affine transformation of u, and if w is a positive affine transformation of v, then w is a positive affine transformation of u.

2.21 Prove Theorem 2.22 (page 23): suppose a person's preferences, which satisfy the von Neumann–Morgenstern axioms, are representable by two linear utility functions u and v. Prove that v is a positive affine transformation of u.

2.22 Let O be an infinite set of outcomes. Let $\mathcal{L}$ be the set of all lotteries over a finite number of outcomes in O, and let $\widehat{\mathcal{L}}$ be the set of all compound lotteries over a finite number of simple lotteries in $\mathcal{L}$. Suppose that a player has a complete, reflexive, and transitive preference relation $\succeq$ over the set of compound lotteries $\widehat{\mathcal{L}}$ that satisfies the von Neumann–Morgenstern axioms, and also satisfies the property that O contains a most-preferred outcome A_K, and a least-preferred outcome A_1, that is, $A_K \succeq A \succeq A_1$ holds for every outcome A in O. Answer the following questions:

(a) Prove Theorem 2.18 (page 19): there exists a linear utility function that represents the player's preference relation.

(b) Prove Theorem 2.22 (page 23): if u and v are two linear utility functions of the player that represent $\succsim$, then v is a positive affine transformation of u.

(c) Prove Corollary 2.23 (page 23): there exists a unique linear utility function (up to a positive affine transformation) representing the player's preference relation.

2.23 Prove Theorem 2.25 on page 25.

2.24 Recall that a linear utility function u_i over lotteries with outcomes in the interval $[-R, R]$ defines a utility function U_i over payoffs in the interval $[-R, R]$ by setting $U_i(x) := u_i([1(x)])$. In the other direction, every function $U_i : [-R, R] \to \mathbb{R}$ defines a linear utility function u_i over lotteries with outcomes in the interval $[-R, R]$ by $u_i([p_1(x_1), p_2(x_2), \ldots, p_K(x_K)]) := \sum_{k=1}^{K} p_k U_i(x_k)$.

For each of the following functions U_i defined on $[-R, R]$, determine whether it defines a linear utility function of a risk-neutral, risk-averse, or risk-seeking player,

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or none of the above: (a) $2x+5$, (b) $-7x+5$, (c) $7x-5$, (d) x^2, (e) x^3, (f) e^x, (g) $\ln(x)$, (h) x for $x \geq 0$, and $6x$ for $x < 0$, (i) $6x$ for $x \geq 0$, and x for $x < 0$, (j) $x^{3/2}$ for $x \geq 0$, x for $x < 0$, (k) $x/\ln(2+x)$, for $x \geq 0$, and x for $x < 0$. Justify your answers.

2.25 In this exercise, we show that a risk-averse player dislikes the addition of noise to a lottery.

Let $U : \mathbb{R} \to \mathbb{R}$ be a concave function, let X be a random variable with a finite expected value, and let Y be a random variable that is independent of X and has an expected value 0. Define $Z = X + Y$. Prove that $\mathbf{E}[U(X)] \geq \mathbf{E}[U(Z)]$.

2.26 In this exercise, we show that in choosing between two random variables with the same expected value, each with a normal distribution, a risk-averse player will prefer the random variable that has a smaller variance.

Let $U : \mathbb{R} \to \mathbb{R}$ be a concave function, and let X be a random variable with a normal distribution, expected value μ, and standard deviation σ. Let $\lambda > 1$, and let Y be a random variable with a normal distribution, expected value μ, and standard deviation $\lambda\sigma$.

(a) Prove that $U(\mu + c) + U(\mu - c) \geq U(\mu + c\sqrt{\lambda}) + U(\mu - c\sqrt{\lambda})$ for all $c > 0$.
(b) By a proper change of variable, and using item (a) above, prove that

$$\int_{-\infty}^{\infty} u(x) \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma}} \mathrm{d}x \geq \int_{-\infty}^{\infty} u(y) \frac{1}{\sqrt{2\pi}\lambda\sigma} \mathrm{e}^{-\frac{(y-\mu)^2}{2\lambda\sigma}} \mathrm{d}y. \qquad (2.87)$$

(c) Conclude that $\mathbf{E}[U(X)] \geq \mathbf{E}[U(Y)]$.

2.27 In Exercises 2.25 and 2.26, a risk-averse player, in choosing between two random variables with the same expected value, prefers the random variable with smaller variance. This exercise shows that this does not always hold: sometimes a risk-averse player called upon to choose between two random variables with the same expected value will actually prefer the random variable with greater variance.

Let $U(x) = 1 - \mathrm{e}^{-x}$ be a player's utility function.

(a) Is the player risk averse, risk neutral, or risk seeking? Justify your answer.

For each $a \in (0, 1)$ and each $p \in (0, 1)$, let $X_{a,p}$ be a random variable whose distribution is

$$\mathbf{P}(X_{a,p} = 1 - a) = \frac{1-p}{2}, \quad \mathbf{P}(X_{a,p} = 1) = p, \quad \mathbf{P}(X_{a,p} = 1 + a) = \frac{1-p}{2}.$$

(b) Calculate the expected value $\mathbf{E}[X_{a,p}]$ and the variance $\mathrm{Var}(X_{a,p})$ for each $a \in (0, 1)$ and each $p \in (0, 1)$.
(c) Let $c^2 = a^2(1-p)$. Show that the expected value of the lottery $X_{a,p}$ is given by

$$\mathbf{E}[U(X_{a,p})] = 1 - \frac{1}{2e}\left((\mathrm{e}^a + \mathrm{e}^{-a} + 2)\frac{c^2}{a^2} - 2\right), \qquad (2.88)$$

which is not a constant function in a and p.

(d) Show that there exist $a_1, a_2, p_1, p_2 \in (0,1)$ such that

$$\mathbf{E}[X_{a_1,p_1}] = \mathbf{E}[X_{a_2,p_2}], \text{ and } \mathrm{Var}(X_{a_1,p_1}) = \mathrm{Var}(X_{a_2,p_2}), \tag{2.89}$$

but $\mathbf{E}[U(X_{a_1,p_1})] < \mathbf{E}[U(X_{a_2,p_2})]$.

(e) Conclude that there exist $a_1, a_2, p_1, p_2 \in (0,1)$ such that

$$\mathbf{E}[X_{a_1,p_1}] = \mathbf{E}[X_{a_2,p_2}], \quad \mathrm{Var}(X_{a_1,p_1}) < \mathrm{Var}(X_{a_2,p_2}), \quad \text{and} \quad \mathbf{E}[U(X_{a_1,p_1})] < \mathbf{E}[U(X_{a_2,p_2})]. \tag{2.90}$$

2.28 The Arrow–Pratt measure of absolute risk aversion Let U_i be a monotonically increasing, strictly concave, and twice continuously differentiable function over $\mathbb{R}$, and let i be a player for which U_i is his utility function for money. The Arrow–Pratt measure of absolute risk-aversion for player i is

$$r_{U_i}(x) := -\frac{U_i''(x)}{U_i'(x)}. \tag{2.91}$$

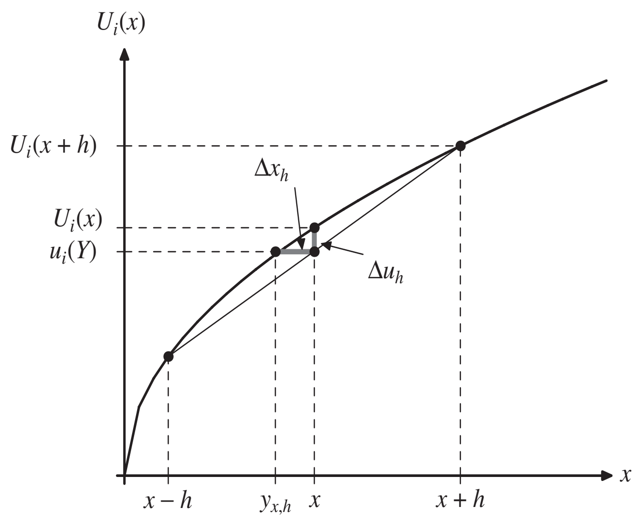
The purpose of this exercise is to understand the meaning of this measure.

(a) Suppose the player has \$$x$, and is required to participate in a lottery in which he stands to gain or lose a small amount \$$h$, with equal probability. Denote by Y the amount of money the player will have after the lottery is conducted. Calculate the expected value of Y, $\mathbf{E}[Y]$, and the variance of Y, $\mathrm{Var}(Y)$.

(b) What is the utility of the lottery, $u_i(Y)$, for this player? What is the player's utility loss due to the fact that he is required to participate in the lottery; in other words, what is $\Delta u_h := U_i(x) - u_i(Y)$?

(c) Prove that $\lim_{h \to 0} \frac{\Delta u_h}{h^2} = -\frac{U_i''(x)}{2}$.

(d) Denote by $y_{x,h}$ the amount of money that satisfies $u_i(y_{x,h}) = u_i(Y)$, and by Δx_h the difference $\Delta x_h := x - y_{x,h}$. Explain why $\Delta x_h \geq 0$. Make use of the following figure in order to understand the significance of the various sizes.



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(e) Using the fact that $\lim_{h\to 0} \frac{\Delta u_h}{\Delta x_h} = U_i'(x)$, and your answers to the items (b) and (d) above, prove that

$$\lim_{h\to 0} \frac{\Delta x_h}{\mathrm{Var}(Y)} = -\frac{U_i''(x)}{2U_i'(x)} = \tfrac{1}{2} r_{U_i}(x). \qquad (2.92)$$

We can now understand the meaning of the Arrow–Pratt measure of absolute risk aversion $r_{U_i}(x)$: it is the sum of money, multiplied by the constant $\frac{1}{2}$, that a player starting out with \$$x$ is willing to pay in order to avoid participating in a fair lottery over an infinitesimal amount \$$h$ with expected value 0, measured in units of lottery variance.

(f) Calculate the Arrow–Pratt measure of absolute risk aversion for the following utility functions: (a) $U_i(x) = x^\alpha$ for $0 < \alpha < 1$, (b) $U_i(x) = 1 - e^{-\alpha x}$ for $\alpha > 0$.

(g) A function U_i exhibits *constant absolute risk aversion* if r_{U_i} is a constant function (i.e., does not depend on x). It exhibits *increasing absolute risk aversion* if r_{U_i} is an increasing function in x, and exhibits *decreasing absolute risk aversion* if r_{U_i} is a decreasing function in x. Check which functions in part (g) exhibit constant, increasing, or decreasing absolute risk aversion.

2.29 Which of the von Neumann–Morgenstern axioms were violated by the preferences expressed by the Second World War pilots in the story described on page 30?

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