15 Bargaining games

Chapter summary

In this chapter we present bargaining games, which model situations in which two or more players bargain toward an agreed-upon outcome. The set of all possible outcomes is called the *feasible set* and each outcome can be attained only by the unanimous agreement of all players. Different players typically prefer different outcomes, which explains the bargaining aspect of the model. A default outcome, called the *disagreement point*, is realized if the players fail to reach an agreement.

A solution concept for bargaining games is a function that assigns to every bargaining game an outcome that can be looked at as the outcome that would be recommended to the players by an arbitrator or a judge. We list several desirable properties that a solution concept for two-player bargaining games could satisfy and provide the unique solution concept that satisfies all these properties, namely, the *Nash solution* for bargaining games. Variants of the Nash solution, like the Kalai–Smorodinsky solution, are obtained by imposing a different set of properties that a solution concept should satisfy. Finally, the model and some of the results are extended to bargaining games with more than two players.

It is frequently the case that two (or more) parties conduct negotiations over an issue, with the payoff to each party dependent on the outcome of the negotiation process. Examples include negotiations between employers and employees on working conditions, nations negotiating trade treaties, and company executives negotiating corporate mergers and acquisitions. In each of these cases, there is a range of outcomes available, if only the parties can come to an agreement and cooperate. Sometimes negotiations do not lead to an agreement. Employees can leave their place of work, countries can impose high tariffs, hurting mutual trade, and negotiations on mergers and acquisitions can fail, with no acquisition taking place. Such bargaining situations are typically not zero-sum: if two countries fail to agree on a trade treaty, for example, they may both suffer from decreased trade.

We will model bargaining games between two parties using a set $S \subseteq \mathbb{R}^2$, and a vector $d \in \mathbb{R}^2$. A point $x = (x_1, x_2) \in S$ represents a potential bargaining outcome expressed in units of utility, where x_i is player i's utility from the bargaining outcome, or in units of money. (Utility theory was presented in Chapter 2.) The set S thus represents the collection of possible bargaining outcomes, and the vector d represents the outcomes in the case where no agreement emerges from the bargaining process. The model presented in this chapter was introduced and first studied by Nash [1950a].

Before we continue, let us consider two simple examples of bargaining games.

Example 15.1 Suppose two players are to divide between them a potential profit of \$100. If the players come to an agreement, they divide the money based on their agreement; if they fail to agree, neither of them receives anything. This is a simple model for a situation in which the abilities of the players are complementary and both players are needed to produced a profit; examples of such situations include an investor and the holder of a patent, and an investor and a landowner.

Figure 15.1(a) depicts this game graphically. The set of possible agreements is the interval

$$S = \{(x, 100 - x) : 0 \leq x \leq 100\}, \tag{15.1}$$

and the vector of disagreement $d = (0, 0)$. The first coordinate represents the outcome for Player 1, and the second coordinate represents the outcome for Player 2. If the outcomes are interpreted in dollars, and if the two players are of similar economic status, it is reasonable for them to divide the \$100 equally.

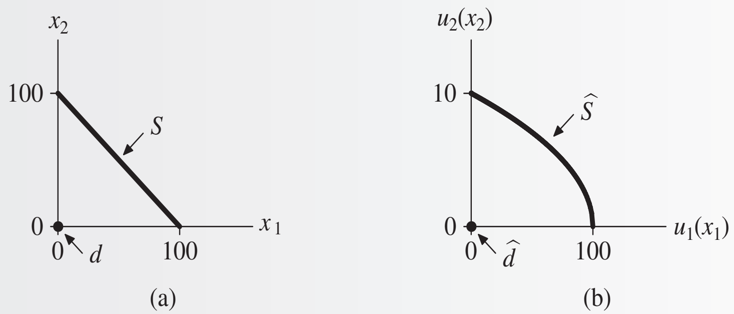

Figure 15.1 Graphic presentation of the bargaining game in Example 15.1, in dollars (a) and in utility units (b)

Suppose now that Player 1's utility from money is $u_1(x) = x$, and Player 2's utility from money is $u_1(x) = \sqrt{x}$. In utility units, the set of possible outcomes is

$$\widehat{S} = \{(x, \sqrt{100 - x}) : 0 \leq x \leq 100\}, \tag{15.2}$$

and the disagreement point is $\widehat{d} = (0, 0)$ (see Figure 15.2(b)). How will they now divide the \$100? ◀

As we can see, the set of possible agreements can be described in various ways, depending on the units in which results are measured. Which of the two depictions is preferable? We will not answer this question in this chapter, but instead only check which agreements can reasonably be expected, once the units in which outcomes are measured have been set.

Example 15.2 Larry and Sergey can, by cooperating, attain a potential profit of \$100. They need to agree on dividing this sum between them. If they cannot come to an agreement, they will not cooperate, there will be no profit, and neither will receive any payoff. If they come to an agreement, they will cooperate, and divide the money according to the agreement. However, Larry will be required

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to pay tax at a rate of 50% of his share of the profit, whereas Sergey will be taxed at a rate of only 30% of his share. In this case, the disagreement point is $d = (0, 0)$, and the set of possible agreements is the interval S between (50, 0) and (0, 70) (see Figure 15.2).

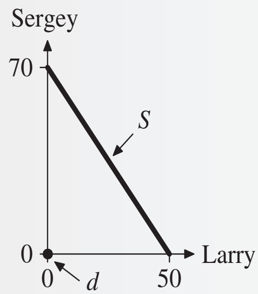

Figure 15.2 Graphic presentation of the bargaining game in Example 15.2 ◀

Even rational players may sometimes fail to reach a compromise. This is why players on occasion turn to an arbitrator, to determine a fair agreement. How should an arbitrator decide which is the fairest agreement among all the possible agreements? Under any proposed agreement, a party to the agreement who receives less than he would receive under his optimal agreement can claim that the arbitrator is unfair! For the arbitrator to be able to explain how he arrived at the proposed agreement and defend his proposal, he needs to base his method of choosing the agreement on principles agreeable to both players, and show how his proposed agreement follows from those principles. It is also desirable for the principles to determine an agreement in such a way that any other suggested agreement would fail to satisfy one or more of the principles.

The following principles, which we present in an intuitive manner, are examples of principles that can be used to guide arbitrators:

- Symmetry: If the two players are equal both in their abilities and in how disagreement will affect them (a formal definition of this concept appears later in this chapter; see Definition 15.5), then under the arbitrator's proposed agreement, both players receive the same payoff.
- Efficiency: There does not exist a possible agreement that is better for both players than the proposed agreement.

Suppose that the arbitrator lists principles according to which he plans to choose a proposed agreement. He then needs to find a function that associates every bargaining game with a proposed agreement. Such a function will be called a "solution concept." The list of principles will be expressed as a list of mathematical properties to be satisfied by this function.

If the arbitrator chooses a list of desired principles that is too long, he may discover that there is no solution concept that satisfies every principle. If the list is too short, there may be many solution concepts that satisfy all the principles. In such a case, the arbitrator needs to find a way to choose one solution concept out of the many possible solution concepts, which is tantamount to adding another principle to the list. When there is exactly one

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solution concept that satisfies all the principles, the arbitrator can propose one agreement for every bargaining game, and defend his choice.

In his mathematical model, Nash presented several desired properties, and pointed to a unique solution concept that satisfies all those properties. We will present Nash's properties, and then introduce two more properties and the solution concepts that follow from them. This approach to finding a solution concept based on a list of properties is called the *axiomatic approach*, and the properties underlying the associated solution concept are often called *axioms*.

15.1 Notation

Let $x, y \in \mathbb{R}^m$ be two vectors. Denote $x \geq y$ if $x_i \geq y_i$ for all $i \in \{1, 2, \dots, m\}$. Denote $x > y$ if $x \geq y$ and $x \neq y$. Denote $x \gg y$ if $x_i > y_i$ for all $i \in \{1, 2, \dots, m\}$.

Given vectors $x, y \in \mathbb{R}^m$, $c \in \mathbb{R}$ and sets $S, T \subseteq \mathbb{R}^m$ define

$$x + y := (x_1 + y_1, x_2 + y_2, \dots, x_m + y_m), \tag{15.3}$$

$$xy := (x_1 y_1, x_2 y_2, \dots, x_m y_m), \tag{15.4}$$

$$cx := (cx_1, cx_2, \dots, cx_m), \tag{15.5}$$

$$cS := \{cx : x \in S\}, \tag{15.6}$$

$$x + S := \{x + s : s \in S\}, \tag{15.7}$$

$$xS := \{xs : s \in S\}, \tag{15.8}$$

$$S + T := \{x + y : x \in S, y \in T\}. \tag{15.9}$$

If $x^1, x^2, \dots, x^n$ are vectors in $\mathbb{R}^m$, denote by $\operatorname{conv}\{x^1, x^2, \dots, x^n\}$ the smallest convex set in $\mathbb{R}^m$ (using the set inclusion relation) containing the points $x^1, x^2, \dots, x^n$. For example, if x, y, and z are three points in the plane and are not colinear, then $\operatorname{conv}\{x, y, z\}$ is the triangle whose vertices are x, y, and z.

Given a compact set $S \subset \mathbb{R}^2$, and a continuous function $f : S \to \mathbb{R}$, denote by

$$\operatorname{argmax}_{\{x \in S\}} f(x) := \{x \in S : f(x) \geq f(y) \quad \forall y \in S\} \tag{15.10}$$

the set of points in S at which the maximum of f is attained.

15.2 The model

Definition 15.3 *A* bargaining game *is an ordered pair* (S, d) *in which:*

- $S \subseteq \mathbb{R}^2$ *is a nonempty, compact, and convex set, called the* set of alternatives.
- $d = (d_1, d_2) \in S$ *is called the* disagreement point *(or* conflict point*).*
- *There exists an alternative* $x = (x_1, x_2) \in S$ *satisfying* $x \gg d$.

Denote the collection of all bargaining games by $\mathcal{F}$.

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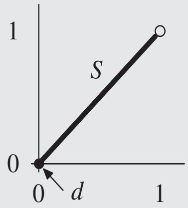

Figure 15.3 A bargaining game in which S is not closed

We interpret a bargaining game as a situation in which two players need to agree on an alternative $x = (x_1, x_2) \in S$. If they come to such an agreement, Player 1's payoff is x_1, and Player 2's payoff is x_2. If the players cannot come to an agreement, the outcome of the game is d; i.e., Player 1's payoff is d_1, and Player 2's payoff is d_2. The assumptions appearing in the definition of a bargaining games are justified as follows:

- The set of alternatives S is bounded; i.e., the maximal and minimal outcomes of each player are bounded.
- The set of alternatives S is closed, and therefore, the boundary of every sequence of possible outcomes in S is also in S. Without this assumption, it may be the case that there is no optimal solution. For example, if the set of alternatives is $S = \{(x_1, x_1) : 0 \leq x_1 < 1\}$, a half-closed and half-open interval, the players do not have a most-preferred alternative: for every proposed solution, there is a solution that is closer than it to $(1, 1)$, and is therefore more preferred by both (see Figure 15.3).
- The set of alternatives S is convex; i.e., a weighted average of possible alternatives is also an alternative. This is a reasonable assumption when we relate to outcomes as linear von Neumann–Morgenstern utilities (see Chapter 2), and the players can conduct lotteries over two (or more) alternatives. For example, a lottery that chooses one possible outcome with probability $\frac{1}{3}$, and another outcome with probability $\frac{2}{3}$, is also a possible outcome of a bargaining process.
- We assume that there exists an alternative $x \in S$ such that $x \gg d$, to avoid dealing with degenerate cases in which there is no possibility that both players can profit from an agreement. Such cases require separate proofs (Exercise 15.12).

Definition 15.4 *A* solution concept *is a function φ associating every bargaining game $(S, d) \in \mathcal{F}$ with an alternative $\varphi(S, d) \in S$.*

The interpretation we give to a solution concept φ is that if two players are playing a bargaining game (S, d), the point $\varphi(S, d)$ is the alternative that an arbitrator will propose that the players accept as an agreement.

15.3 Properties of the Nash solution

In this section, we present several properties that one can require from solution concepts of bargaining games. These properties were first proposed by John Nash in 1953, and they

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are the mathematical expression of principles that could guide an arbitrator who is called upon to propose a bargaining agreement.[1] In the next section, we will show that there exists a unique solution concept satisfying these properties. We will then critique some of these properties, and present alternative properties (which are also open to critique), leading to other solution concepts to the bargaining game.

15.3.1 Symmetry

Definition 15.5 *A bargaining game* $(S, d) \in \mathcal{F}$ *is* symmetric *if the following two properties are satisfied:*

- $d_1 = d_2$ *(the disagreement point is symmetric).*
- *If* $x = (x_1, x_2) \in S$*, then* $(x_2, x_1) \in S$.

Geometrically, symmetry implies that S is symmetric with respect to the main diagonal in $\mathbb{R}^2$, where the disagreement point is located. The symmetry property forbids the arbitrator from giving preference to one party over the other when the game is symmetric.

Definition 15.6 *A solution concept* φ *is* symmetric *(or satisfies the symmetry property) if for every symmetric bargaining game* $(S, d) \in \mathcal{F}$ *the vector* $\varphi(S, d) = (\varphi_1(S, d), \varphi_2(S, d))$ *satisfies* $\varphi_1(S, d) = \varphi_2(S, d)$.

15.3.2 Efficiency

The goal of bargaining is to improve the situations of the players. We therefore do not want to propose an alternative that can be improved upon, that is, that is strictly preferred by one player and does not harm the interests of the other player. If such an alternative exists, the arbitrator will prefer it to the proposed alternative.

Definition 15.7 *An alternative* $x \in S$ *is called an* efficient point *of* S *if there does not exist an alternative* $y \in S$*,* $y \neq x$*, such that* $y \geq x$.

Denote by $PO(S)$ the set of efficient points[2] of S.

Definition 15.8 *A solution concept* φ *is* efficient *(or satisfies the* efficiency property*) if* $\varphi(S, d) \in PO(S)$ *for each bargaining game* $(S, d) \in \mathcal{F}$.

Definition 15.9 *An alternative* $x \in S$ *is called* weakly efficient *in* S *if there is no alternative in* S *that is strictly preferred to* x *by both players; in other words, there is no alternative* $y \in S$ *satisfying* $y \gg x$.

Denote the set of weakly efficient points in S by $PO^W(S)$. It follows by definition that $PO(S) \subseteq PO^W(S)$ for each set $S \subseteq \mathbb{R}^2$; as the following example shows, this set inclusion can be a proper inclusion.

1 These properties are also called the Nash axioms in the literature.

2 PO stands for Pareto optimum, named after the Italian economist Vilfredo Pareto (1848–1923). In his 1906 book, *Manuale di Economia Politica*, Pareto developed the idea that a distribution of resources in a society is nonoptimal if it is possible to increase at least one person's welfare without decreasing the welfare of any other individual.

Example 15.10 Consider the bargaining game in Figure 15.4. The set of possible outcomes that cannot be improved from the perspective of at least one player, i.e., $PO(S)$, appears in bold in part A. The set of possible outcomes that cannot be improved from the perspective of both players, i.e., $PO^W(S)$, appears in bold in part B. For example, the outcome (30, 100) is inefficient, since the outcome (40, 100) is better from the perspective of Player 1. On the other hand, there is no outcome that is strictly better for both players than (30, 100). In other words, $(30, 100) \in PO^W(S)$, but $(30, 100) \notin PO(S)$.

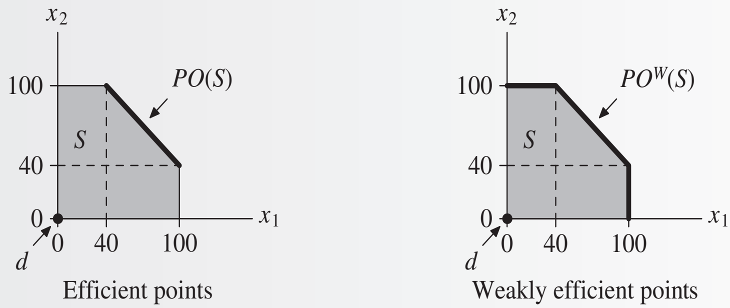

Figure 15.4 The efficient points of S in Example 15.10 ◀

Definition 15.11 *A solution concept φ is* weakly efficient *if $\varphi(S, d) \in PO^W(S)$ for each bargaining game $(S, d) \in \mathcal{F}$.*

The sets $PO(S)$ and $PO^W(S)$ are on the boundary of S, and therefore $\varphi(S, d)$ is on the boundary of S whenever φ is an efficient or a weakly efficient solution concept.

15.3.3 Covariance under positive affine transformations

When the axes of a bargaining game represent monetary payoffs, it is reasonable to require that the solution concept be *independent of the units of measurement*. In other words, if we measure the payoff to one player in cents instead of dollars, we get a different bargaining game (in which the coordinate corresponding to each point is larger by a factor of 100). In this case, we want the coordinate corresponding to the solution to change by the same ratio.

Another possible property to adopt is *covariance under translations*. This property implies that if we add a constant to each one of a certain player's payoffs, the solution will change by the same constant: the amount of money that each player has at the start of the bargaining process should not change the profit that each player gets by bargaining.

Definition 15.12 *A solution concept φ is* covariant under changes in the units of measurement *if for each bargaining game $(S, d) \in \mathcal{F}$, and every vector $a \in \mathbb{R}^2$ such that $a \gg 0$,*

$$\varphi(aS, ad) = a\varphi(S, d) = (a_1\varphi_1(S, d), a_2\varphi_2(S, d)). \tag{15.11}$$

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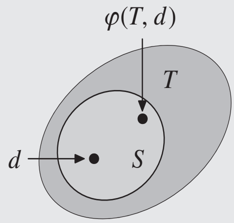

Figure 15.5 Independence of irrelevant alternatives

A solution concept φ is covariant under translations *if for each bargaining game $(S, d) \in \mathcal{F}$, and every vector $b = (b_1, b_2) \in \mathbb{R}^2$,*

$$\varphi(S + b, d + b) = \varphi(S, d) + b = (\varphi_1(S, d) + b_1, \varphi_2(S, d) + b_2). \tag{15.12}$$

We combine these two properties into one, the property of covariance under positive affine transformations. Recall that a positive affine transformation in the plane is a function $x \mapsto ax + b$, where $a, b \in \mathbb{R}^2$, and $a \gg 0$.

Definition 15.13 *A solution concept φ is* covariant under positive affine transformations *if for every bargaining game $(S, d) \in \mathcal{F}$, for every vector $a \in \mathbb{R}^2$ such that $a \gg 0$, and for every vector $b \in \mathbb{R}^2$,*

$$\varphi(aS + b, ad + b) = a\varphi(S, d) + b. \tag{15.13}$$

While covariance under multiplication by $a \gg 0$ (change of units) is reasonable when considering bargaining over money, covariance under translations is open to critique because the amount of money a player has does, in general, affect his attitude towards any extra money over which the players are bargaining. If we wish to take this into consideration, we need to consider not the amount of money a player gets, but his utility from this amount of money. In other words, covariance under positive affine transformations is a natural assumption when the alternatives are expressed as pairs of utilities: the utility of each player from the alternative. As we saw in the chapter on the theory of utility (Chapter 2), von Neumann–Morgenstern utility functions are determined only up to positive affine transformations, so that we need to impose the condition that solution concepts to bargaining games be independent of the particular representation chosen for the players' utility functions.

15.3.4 Independence of irrelevant alternatives (IIA)

Suppose that $S \subseteq T$ and $\varphi(T, d) \in S$ (see Figure 15.5); i.e., in the bargaining process of the game (T, d), the players have checked all the alternatives in T and decided that the best alternative is $\varphi(T, d)$, which happens to be located in S. What happens if the set of possible alternatives is now restricted to S? One could make the case that the players will still choose $\varphi(T, d)$, because if there were a better alternative in S, that alternative would also be available in the game (T, d), and that alternative should then have been chosen, rather than $\varphi(T, d)$.

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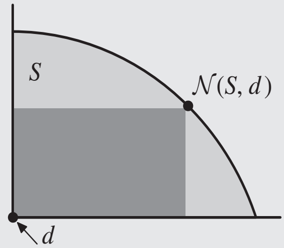

Figure 15.6 The Nash solution (the darkened rectangle is the rectangle of maximal area)

Definition 15.14 *A solution concept φ satisfies the property of* independence of irrelevant alternatives *(IIA) if for every bargaining game $(T, d) \in \mathcal{F}$, and every subset $S \subseteq T$,*

$$\varphi(T, d) \in S \quad \Longrightarrow \quad \varphi(S, d) = \varphi(T, d). \tag{15.14}$$

As we see in the intuitive "justification" presented above, this property is reasonable when a solution concept is supposed to reflect the "best" outcome: if the alternative $\varphi(T, d)$ that is the best alternative out of all the alternatives in T is in S, then if we delete the alternatives in $T \setminus S$, it will still be the best alternative. If, however, a solution concept is supposed to reflect the outcome of a compromise between the players, then it is possible to claim that even an alternative that is not chosen may influence the solution concept. In such cases, the independence of irrelevant alternatives is open to critique.

15.4 Existence and uniqueness of the Nash solution

In this section, we prove the existence of a unique solution concept that satisfies the properties of symmetry, efficiency, covariance under positive affine transformations, and independence of irrelevant alternatives. We will also present a formula for computing this solution for every bargaining game in $\mathcal{F}$.

An alternative $x \in S$ is *individually rational* in S if $x \geq d$. Since $d \in S$, the set of individually rational alternatives is not empty. If x is not individually rational in S, then at least one player strictly prefers d to x, and since each player can enforce the condition that the bargaining outcome is d, it is reasonable to suppose that such an alternative x will not be the end result of the bargaining process.

Theorem 15.15 *There exists a unique solution concept $\mathcal{N}$ for the family $\mathcal{F}$ satisfying symmetry, efficiency, covariance under positive affine transformations, and independence of irrelevant alternatives. The solution $\mathcal{N}(S, d)$ of the bargaining game (S, d) is the individually rational alternative x in S that maximizes the area of the rectangle whose bottom left vertex is d, and whose top right vertex is x (see Figure 15.6).*

The point $\mathcal{N}(S, d)$ is called the *Nash agreement point* (or the *Nash solution*) of the bargaining game (S, d). Note that for every $x \in S$ satisfying $x \geq d$, the area of the rectangle

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whose bottom left vertex is d, and whose top right vertex is x, is given by the product $(x_1 - d_1)(x_2 - d_2)$. The product $(x_1 - d_1)(x_2 - d_2)$ is called the *Nash product*. We will denote it here by $f(x)$.

We will prove the theorem in three steps. We first show that for every bargaining game in the family $\mathcal{F}$, there exists a unique alternative maximizing the area of this rectangle.

Lemma 15.16 *For every bargaining game $(S, d) \in \mathcal{F}$ there exists a unique point in the set*

$$\operatorname{argmax}_{\{x\in S, x\geq d\}}(x_1 - d_1)(x_2 - d_2). \tag{15.15}$$

Proof of Lemma 15.16: If we translate all the points in the plane by adding $-d$ to each point, we get the bargaining game $(S - d, (0, 0))$. Since the area of a rectangle is unchanged by translation, the points at which the Nash product is maximized for the bargaining game (S, d) are translated to the points at which the Nash product is maximized in the bargaining game $(S - d, (0, 0))$. We can therefore assume that, without loss of generality, $d = (0, 0)$, and then

$$f(x) = x_1 x_2. \tag{15.16}$$

The set of individually rational points in S, which we denote by $D := \{x \in S, x \geq d\}$, is the intersection of the compact and convex set S with the closed and convex set $\{x \in \mathbb{R}^2 : x \geq d\}$, so that it, too, is compact and convex. As we already noted, the set D is nonempty because it contains the disagreement point d.

Since the function f is continuous, and the set D is compact, there exists at least one point y in D at which the maximum is attained. Suppose by contradiction that there exist two distinct points y and z in D at which the maximum of f is attained. In particular,

$$y_1 y_2 = z_1 z_2. \tag{15.17}$$

Define

$$w := \tfrac{1}{2}y + \tfrac{1}{2}z. \tag{15.18}$$

Since D is convex, and $y, z \in D$, it follows that $w \in D$. We will show that

$$f(w) > f(y), \tag{15.19}$$

contradicting the fact that the Nash product is maximized at y (and at z). The assumption that $y \neq z$ therefore leads to a contradiction, hence $y = z$, and we will be able to conclude that the Nash product is maximized at a unique point.

One way to prove Equation (15.19) is to note that for every $c > 0$ the function $x_2 = \frac{c}{x_1}$ is strictly convex. For $c = y_1 y_2$, both $(y, f(y))$ and $(z, f(z))$ are on the graph of the function, and therefore $(w, f(w))$ is above the graph. In particular, $w_1 w_2 > c = y_1 y_2$.

A direct proof of the claim is as follows. In Figure 15.7, the points y and z are noted, with A, B, C, and D denoting four rectangular areas. From the figure we see that

$$y_1 z_2 + z_1 y_2 = A + 2B + C + D > A + 2B + C = y_1 y_2 + z_1 z_2. \tag{15.20}$$

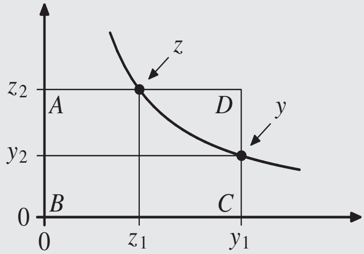

Figure 15.7 The areas of the rectangles defined by y and z are equal

Thus we have

$$\begin{aligned} f(w) = w_1 w_2 &= \left(\tfrac{1}{2}y_1 + \tfrac{1}{2}z_1\right)\left(\tfrac{1}{2}y_2 + \tfrac{1}{2}z_2\right) &(15.21)\\ &= \tfrac{1}{4}y_1y_2 + \tfrac{1}{4}y_1z_2 + \tfrac{1}{4}z_1y_2 + \tfrac{1}{4}z_1z_2 &(15.22)\\ &> \tfrac{1}{4}y_1y_2 + \tfrac{1}{4}y_1y_2 + \tfrac{1}{4}z_1z_2 + \tfrac{1}{4}z_1z_2 &(15.23)\\ &= \tfrac{1}{2}y_1y_2 + \tfrac{1}{2}z_1z_2 = f(y), &(15.24)\end{aligned}$$

where Equation (15.23) follows from Equation (15.20) and Equation (15.24) follows from Equation (15.17). In summary, $f(w) > f(y)$, which is the desired contradiction. □

From Lemma 15.16, the function $\mathcal{N}$ defined by

$$\mathcal{N}(S,d) := \operatorname{argmax}_{\{x\in S, x\geq d\}}(x_1 - d_1)(x_2 - d_2) \tag{15.25}$$

is well defined and single-valued and is therefore a solution concept. Since there exists $x \in S$ satisfying $x \gg d$, one has $\max_{\{x\in S,x\geq d\}} f(x) > 0$, and therefore $\mathcal{N}(S,d) \gg d$.

We next show that this solution concept satisfies the four properties of Theorem 15.15.

Lemma 15.17 *The solution concept $\mathcal{N}$ satisfies the properties of symmetry, efficiency, covariance under positive affine transformations, and independence of irrelevant alternatives.*

Proof:
$\mathcal{N}$ satisfies symmetry: Let (S,d) be a symmetric bargaining game, and let

$$y^* := \mathcal{N}(S,d) = \operatorname{argmax}_{\{x\in S,x\geq d\}}(x_1 - d_1)(x_2 - d_2). \tag{15.26}$$

Denote by z the point $z = (y_2^*, y_1^*)$. Since S is symmetric, and $y^* \in S$, we deduce that $z \in S$. Since $d_1 = d_2$, the area of the rectangle defined by y^* and d equals the area of the rectangle defined by z and d (see Figure 15.8):

$$f(y^*) = (y_1^* - d_1)(y_2^* - d_2) = (z_2 - d_1)(z_1 - d_2) = (z_1 - d_1)(z_2 - d_2) = f(z). \tag{15.27}$$

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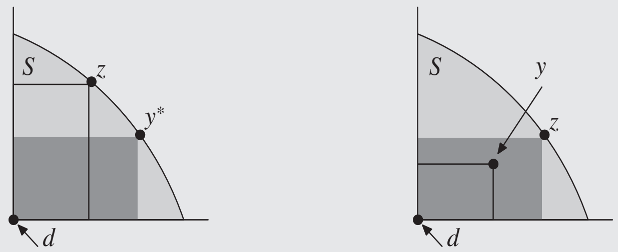

The symmetry property The efficiency property: y is not a solution

Figure 15.8

By Lemma 15.16, the maximum of f over S is attained at a unique point. Therefore $y^* = z$, leading to $y_1^* = y_2^*$, as required.

$\mathcal{N}$ *satisfies efficiency:* If y is not efficient in S then there exists $z \in S$ satisfying (a) $z \geq y$ and (b) $z \neq y$. Then the area of the rectangle defined by z and d is strictly greater than the area of the rectangle defined by y and d, and therefore $\mathcal{N}(S, d) \neq y$ (see Figure 15.8).

$\mathcal{N}$ *satisfies covariance under positive affine transformations:* The maximum of the function f over $\{x \in S, x \geq d\}$ is attained at the point $\mathcal{N}(S, d)$. Applying the positive affine transformation $x \mapsto ax + b$ to the plane combines a translation with multiplication by a positive constant at every coordinate. A translation does not change the area of a rectangle, and multiplication by $a = (a_1, a_2)$ multiplies the area of the rectangle by $a_1 a_2$. It follows that if prior to the application of the transformation the Nash product maximizes at y, then after the application of the transformation $x \mapsto ax + b$ the Nash product maximizes at $ay + b$.

$\mathcal{N}$ *satisfies independence of irrelevant alternatives:* This follows from a general fact: let $S \subseteq T$, let $g : T \to \mathbb{R}$ be a function, and let $w \in \operatorname{argmax}_{\{x \in T, x \geq d\}} g(x)$. If $w \in S$, then $w \in \operatorname{argmax}_{\{x \in S, x \geq d\}} g(x)$ (explain why the claim that $\mathcal{N}$ satisfies independence of irrelevant alternatives follows from this general fact). To see why this claim holds, note that since $w \in S$ and $S \subseteq T$,

$$\max_{\{x \in S, x \geq d\}} g(x) \geq g(w) = \max_{\{x \in T, x \geq d\}} g(x) \geq \max_{\{x \in S, x \geq d\}} g(x). \tag{15.28}$$

It follows that

$$\max_{\{x \in T, x \geq d\}} g(x) = \max_{\{x \in S, x \geq d\}} g(x), \tag{15.29}$$

and therefore $w \in \operatorname{argmax}_{\{x \in S, x \geq d\}} g(x)$. □

To complete the proof of Theorem 15.15, we need to prove the uniqueness of the solution concept.

Lemma 15.18 *Every solution concept φ satisfying symmetry, efficiency, covariance under positive affine transformations, and independence of irrelevant alternatives is identical to the solution concept $\mathcal{N}$ defined by Equation (15.25).*

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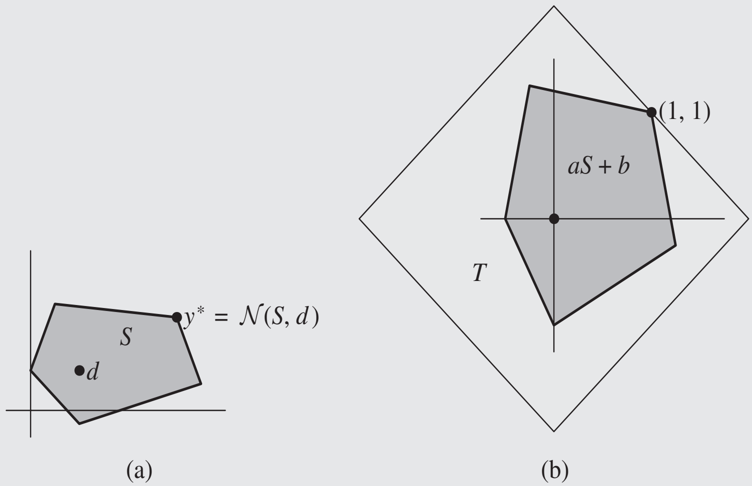

Figure 15.9 The bargaining game (S, d) (a) and the game obtained by implementation of the positive affine transformation L, along with the symmetric square T (b)

Proof: Let φ be a solution concept satisfying the four properties of the statement of the theorem. Let (S, d) be a bargaining game in $\mathcal{F}$, and denote $y^* := \mathcal{N}(S, d)$. We will show that $\varphi(S, d) = y^*$.

Step 1: Applying a positive affine transformation L.
Since there is an alternative x in S such that $x \gg d$, the point $\mathcal{N}(S, d) = y^* \in \{z \in S : z \geq d\}$ at which the Nash product is maximized satisfies $y^* \gg d$. We can therefore define a positive affine transformation L over the plane shifting d to the origin, and y^* to $(1, 1)$ (see Figure 15.9). This function is given by

$$L(x_1, x_2) = \left(\frac{x_1 - d_1}{y_1^* - d_1}, \frac{x_2 - d_2}{y_2^* - d_2} \right). \tag{15.30}$$

Since $y_1^* > d_1$ and $y_2^* > d_2$, the denominators in the definition of L are positive. The function L is of the form $L = ax + b$, where $a_1 = \frac{1}{y_1^* - d_1} > 0$, $a_2 = \frac{1}{y_2^* - d_2} > 0$, $b_1 = -\frac{d_1}{y_1^* - d_1}$, and $b_2 = -\frac{d_2}{y_2^* - d_2}$. Since the solution concept $\mathcal{N}$ satisfies covariance under positive affine transformations,

$$\mathcal{N}(aS + b, (0, 0)) = \mathcal{N}(aS + b, ad + b) = ay^* + b = (1, 1). \tag{15.31}$$

Step 2: $x_1 + x_2 \leq 2$ for every $x \in aS + b$.
Let $x \in aS + b$. Since S is convex, the set $aS + b$ is also convex (Exercise 15.7). Therefore, since both x and $(1, 1)$ are in $aS + b$, the interval connecting x and $(1, 1)$ is also in $aS + b$. In other words, for every $\varepsilon \in [0, 1]$, the point z^ε defined by

$$z^\varepsilon := (1 - \varepsilon)(1, 1) + \varepsilon x = (1 + \varepsilon(x_1 - 1), 1 + \varepsilon(x_2 - 1)) \tag{15.32}$$

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is in $aS+b$. If ε is sufficiently close to 0 then $z^\varepsilon \geq (0,0)$, and therefore z^ε is one of the points in the set $\{w \in aS+b, w \geq (0,0)\}$. It follows that for each such ε,

$$f(z^\varepsilon) \leq \max_{\{w \in aS+b, w \geq (0,0)\}} f(w) = f(\mathcal{N}(aS+b,(0,0))) = f\big((1,1)\big) = 1. \qquad (15.33)$$

Hence

$$\begin{aligned} 1 \geq f(z^\varepsilon) = z_1^\varepsilon z_2^\varepsilon &= 1 + \varepsilon(x_1 + x_2 - 2) + \varepsilon^2(x_1-1)(x_2-1) && (15.34)\\ &= 1 + \varepsilon(x_1 + x_2 - 2 + \varepsilon(x_1-1)(x_2-1)). && (15.35) \end{aligned}$$

Therefore, for every $\varepsilon > 0$ sufficiently small,

$$0 \geq \varepsilon(x_1 + x_2 - 2 + \varepsilon(x_1-1)(x_2-1)), \qquad (15.36)$$

leading to the conclusion that

$$2 + \varepsilon(x_1-1)(x_2-1) \geq x_1 + x_2. \qquad (15.37)$$

Taking the limit as ε approaches 0 yields $2 \geq x_1 + x_2$, which is what we wanted to show.

Step 3: $\varphi(S,d) = \mathcal{N}(S,d)$.
Let T be a symmetric square relative to the diagonal $x_1 = x_2$ that contains $aS+b$, with one side along the line $x_1 + x_2 = 2$ (see Figure 15.9(b)). Since $aS+b$ is compact (and thus bounded), such a square exists. By the symmetry and efficiency of φ, one has $\varphi(T,(0,0)) = (1,1)$. Since the solution concept φ satisfies independence of irrelevant alternatives, and since $aS+b$ is a subset of T containing $(1,1)$, it follows that $\varphi(aS+b,(0,0)) = (1,1)$. Since the solution concept φ satisfies covariance under positive affine transformations, one can implement the inverse transformation L^{-1} to deduce that $\varphi(S,d) = y^*$. Since $y^* = \mathcal{N}(S,d)$, we conclude that $\varphi(S,d) = \mathcal{N}(S,d)$, as required. □

15.5 Another characterization of the Nash solution

In this section, we present another geometric characterization of the solution concept $\mathcal{N}$ defined in Equation (15.25).

Definition 15.19 *Let S be a closed and convex set in $\mathbb{R}^2$, and let x be an alternative on the boundary of S. A* supporting line *of S at x is any line through x such that the set S lies in one of the closed half-planes defined by it.*

By a general theorem from the theory of Convex Sets (the Separating Hyperplane Theorem; see Section 23.2, page 943), for every convex set $S \subseteq \mathbb{R}^2$ and every point x on the boundary S there exists a supporting line of S at x.

If the boundary of S is smooth at a point x, there exists a unique supporting line of S at x, which is the tangent line to S at x. If, in contrast, x is a “corner” of S, there are several supporting lines at x (see Figure 15.10).

Recall that since the Nash solution is an efficient solution concept, $\mathcal{N}(S,d) \in PO(S)$: the Nash solution is in the boundary of S.

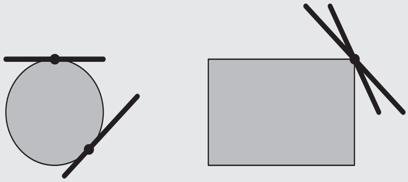

Figure 15.10 Examples of supporting lines of closed and convex sets

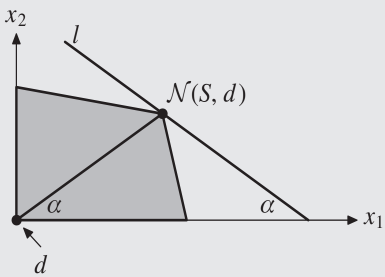

Figure 15.11 The second characterization of the Nash solution

Theorem 15.20 *For every bargaining game $(S, d) \in \mathcal{F}$, one has $\mathcal{N}(S, d) = y$ if and only if (a) $y \in PO(S)$, (b) $y \gg d$, and (c) there exists a supporting line l of S at y, such that the triangle whose vertices are d, y, and the intersection point of l with the line $x_2 = d_2$ is an equilateral triangle, whose base is the line $x_2 = d_2$ (see Figure 15.11).*

In other words, the angle between the line connecting d with $\mathcal{N}(S, d)$ and the x_1-axis equals the angle between the supporting line l and the x_1-axis.

Proof: Given a bargaining game $(S, d) \in \mathcal{F}$, denote $y^* := \mathcal{N}(S, d)$. In the proof of Lemma 15.18, we saw that there exists a positive affine transformation L such that $L(d) = (0, 0)$ and $z^* := L(y^*) = (1, 1)$. We also proved there that the set $L(S)$, the image of S under the transformation L, is a subset of the half-plane $x_1 + x_2 \leq 2$, and therefore the line $x_1 + x_2 = 2$ is a supporting line of $L(S)$ at z^* (see Figure 15.12(b)). In addition, the angle between the line connecting z^* and d, and the x_1-axis, is 45°, and this is also the angle between the line $x_1 + x_2 = 2$ and the x_1-axis. It follows that the triangle whose vertices are $L(d)$, z^*, and the intersection of the line $x_1 + x_2 = 2$ with the x_1-axis is an equilateral triangle.

The inverse transformation L^{-1} maps the supporting line $x_1 + x_2 = 2$ of $L(S)$ at $z^* = L(y^*)$ to the supporting line l of S at y^* (see Exercise 15.6), and $(1, 1)$ to $\mathcal{N}(S, d)$ (see Figure 15.12). Similarly, a positive affine transformation maps an equilateral triangle whose base is the line $x_2 = d_2$ to an equilateral triangle (whose base is parallel to the x_1-axis; Exercise 15.11). It follows that S has a supporting line at y^* defining an equilateral triangle, as required.

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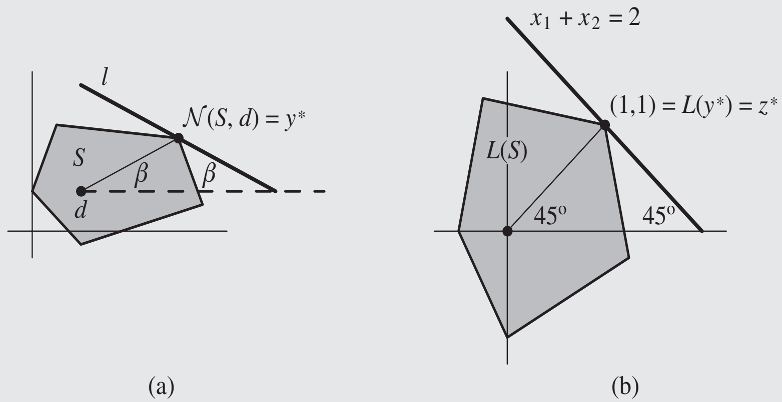

Figure 15.12 Proof of Theorem 15.20. The bargaining game prior to the implementation of L (a), and after the implementation of L (b)

We now prove the opposite direction; i.e., we will show that if there is an efficient point $y \in S$, $y \gg d$, and a supporting line l of S at y defining an equilateral triangle, then necessarily $y = \mathcal{N}(S, d)$. Since $y \gg d$, there is an affine transformation L, $L(x) = ax + b$ where $a \gg 0$, mapping d to $(0, 0)$, y to $(1, 1)$, and l to the supporting line of $aS + b$ at $ay + b$. In addition, the triangle formed by the line connecting $ad + b = (0, 0)$ with $ay + b = (1, 1)$, the image of the supporting line $al + b$, and the axis $x_1 = 0$, is an equilateral triangle.

The angle between the line connecting $(0, 0)$ to $(1, 1)$ and the x_1-axis is 45°. Since the triangle we described is an equilateral triangle, the line $al + b$ also intersects the x_1-axis at a 45° angle, and it is therefore the line $x_1 + x_2 = 2$. The point that maximizes the Nash product in the triangle whose vertices are $(0, 0)$, $(2, 0)$, and $(0, 2)$ is $(1, 1)$, and since this point is in the set $aS + b$, it follows that $\mathcal{N}(aS + b, (0, 0)) = (1, 1)$. Therefore, $y = \mathcal{N}(S, d)$, which is what we wanted to show. □

In the next two subsections we present applications of this characterization of the Nash solution.

15.5.1 Interpersonal comparison of utilities

In the previous sections, we described the Nash point as a solution concept based on several properties. This solution concept, similar to every solution concept in game theory, is convincing only insofar as the properties characterizing it are convincing. In the case of the Nash solution, the property of independence of irrelevant alternatives is open to critique (see Sections 15.3.4 (page 629) and 15.7 (page 641)). This motivates interest in the question of whether the solution concept can be characterized in a different manner. Shapley [1969] proposed such a characterization, based on the following two properties:

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- **Egalitarianism:**[3] At the solution point, the profit in utility units, relative to the disagreement point, is the same for both players.
- **Utilitarianism:**[4] The players will choose an alternative that maximizes the sum of their profits (in utility units), relative to the disagreement point.

The main difficulty in applying these properties is that within the framework of the von Neumann–Morgenstern utility theory (Chapter 2), a player's utility function is determined only up to a positive affine transformation, so there is no way to compare or add together the utilities of different players. This is called the "interpersonal comparison of utilities" problem in the literature.

Shapley's approach to this problem is that although it is impossible to compare utilities between the players over all games, in every particular game we can regard the alternative which the players finally agree upon as a reflection of the exchange rate between their utilities that emerged in the bargaining process. In particular, Shapley's suggestion is that the minus of the slope c of the supporting line[5] at the agreement point should be considered the exchange rate between the utility of the second player and the utility of the first player. The reasoning behind this suggestion is that at this point, slightly moving the agreement point within the set of efficient points $PO(S)$ yields approximately c units of profit/loss to the second player, with a unit of loss/profit to the first player (prove that this is true).

We next present Shapley's characterization of the Nash solution.

Definition 15.21 *An alternative* $x = (x_1, x_2) \in S$ *is a* solution *of a bargaining game* (S, d) *if there exists a positive number c satisfying:*

- *Egalitarianism:* $x_2 - d_2 = c(x_1 - d_1)$.
- *Utilitarianism:* $x \in \operatorname{argmax}_{y \in S}\{(y_2 - d_2) + c(y_1 - d_1)\}$.

Shapley's characterization states that the only possible candidate for a solution concept of the bargaining game, according to this definition, is the Nash solution. In that case, the constant c is minus the slope of a supporting line to S at the Nash point.

Theorem 15.22 *Let* $(S, d) \in \mathcal{F}$ *be a bargaining game such that for each efficient point satisfying* $y \gg d$ *there exists a unique tangent to* S *at* y*. Then there exists a unique alternative* $x \in S$ *that is a solution of the game according to Definition 15.21. Furthermore, this alternative is* $x = \mathcal{N}(S, d)$*.*

The proof of this theorem is accomplished using Theorem 15.20, and is left to the reader (Exercise 15.18).

15.5.2 The status quo region

Given a bargaining game (S, d), we ask what are the disagreement points that, with the same feasible set S, yield the same solution. That is, what is the collection of all the points $\widehat{d}$ such that $\mathcal{N}(S, \widehat{d}) = \mathcal{N}(S, d)$? By Theorem 15.20, if S has a unique supporting line at

3 Shapley used the term "equitability" for this concept.
4 Shapley used the term "efficiency" for this concept.
5 For simplicity, we assume here that S has a unique supporting line at the agreement point.

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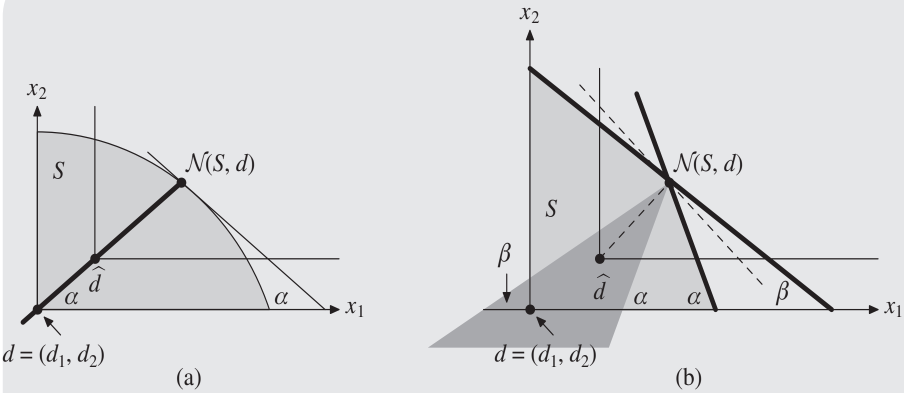

Figure 15.13 The points $\widehat{d}$ such that $\mathcal{N}(S, \widehat{d}) = \mathcal{N}(S, d)$

the point $\mathcal{N}(S, d)$, then the point $\widehat{d}$ satisfies this condition if and only if it is located on the ray emanating from $\mathcal{N}(S, d)$ and passing through d (see Figure 15.13(a)).

If there are several supporting lines for S at the point $\mathcal{N}(S, d)$, then $\widehat{d}$ may be located on any ray emanating from $\mathcal{N}(S, d)$, forming, along with one of the supporting lines, an equilateral triangle whose base is on the axis $x_2 = d_2$ (the darkened area in Figure 15.14(b)). This line (or region) is called the *status quo line* (or *status quo region*).

Suppose that in the current situation, the players are at point d. Both players are interested in signing an agreement that will improve this situation. The set of alternatives S represents the possible situations to which the players can move. Since the players believe in the Nash solution, they have an interest in following a process that will move them to $y^* = \mathcal{N}(S, d)$. For various reasons (for example, the players may not trust each other) they want to arrive at the Nash point by way of a series of interim agreements that will not change the balance of power; none of the players will have a reason to object to the common goal of reaching y^* throughout the process. Since the two parties believe in the Nash solution, we require that y^* continue to be the Nash solution in all the interim stages, where the disagreement point is the situation in the interim stage. This property is satisfied when the interim states are on the status quo line (or in the status quo region). As long as the interim state is on this line (or in this region) y^* is the Nash solution of the changing bargaining problem throughout the process.

15.6 The minimality of the properties of the Nash solution

In the previous section, we showed that there exists a unique solution concept satisfying the four properties of the Nash solution. We will show here that if one of these properties is left out, uniqueness is lost. In the following examples, for each property in turn, we will exhibit a solution concept for the family $\mathcal{F}$ with respect to which that property fails to hold, but the other three properties do hold. Since the Nash solution satisfies all four

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properties, it will follow that for every set of three of the four properties there exist at least two solution concepts satisfying these three properties.

Example 15.23 **Leaving out efficiency** Define a solution concept φ as follows:

$$\varphi(S, d) := \frac{d + \mathcal{N}(S, d)}{2}. \tag{15.38}$$

This solution is located "halfway" between the disagreement point and the Nash solution. Out of the four Nash properties, the only property that this solution concept does not satisfy is efficiency (Exercise 15.1). ◀

Example 15.24 **Leaving out symmetry: preferring one player to another** Define a solution concept φ as follows:

$$\varphi(S, d) := \mathrm{argmax}_{\{x \in PO(S):\, x \geq d\}} x_1. \tag{15.39}$$

This solution is Player 1's most-preferred alternative among the efficient and individually rational alternatives. Out of the four Nash properties, the only property that this solution concept does not satisfy is symmetry (Exercise 15.2). ◀

Example 15.25 **Leaving out covariance under positive affine transformations** For every angle $0° < \alpha < 90°$ let $\lambda_\alpha(S, d)$ be the highest point in S on the line emanating from the disagreement point d at angle α relative to the line $x_2 = d_2$ (see Figure 15.14).

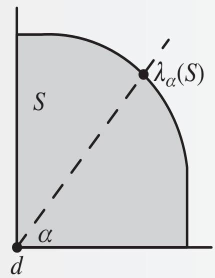

Figure 15.14 The solution concept λ_α

The solution concept $\lambda_{45°}(S, d)$ satisfies all the Nash properties except for covariance under positive affine transformations (Exercise 15.3). ◀

Example 15.26 **Leaving out independence of irrelevant alternatives** Define a solution concept φ as follows. If there exists a positive affine transformation L satisfying the property that $(L(S), L(d))$ is a symmetric bargaining game, then $\varphi(S, d) := \mathcal{N}(S, d)$. Otherwise, $\varphi(S, d) := \mathrm{argmax}_{\{x \in PO(S):\, x \geq d\}} x_1$ is the best efficient and individually rational alternative for Player 1. Out of the four Nash properties, the only one that this solution concept fails to satisfy is independence of irrelevant alternatives (Exercise 15.4). ◀

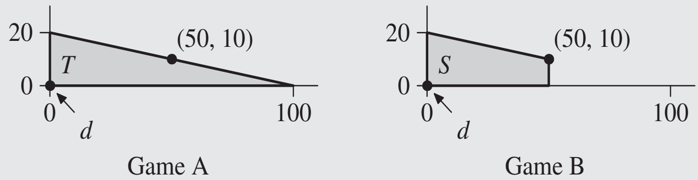

Figure 15.15 The bargaining game $(T, (0, 0))$, and the bargaining game $(S, (0, 0))$

15.7 Critiques of the properties of the Nash solution

We have seen that if one seeks a solution concept satisfying symmetry, efficiency, covariance under positive affine transformations, and independence of irrelevant alternatives, the only solution concept one obtains is the Nash solution. But are these properties reasonable? Can an arbitrator who believes in these properties persuade the contending players that these are reasonable properties upon which to base his decisions? It is easy to make a case for efficiency, because it is reasonable for the players to agree to improve one player's situation if doing so does not come at the expense of the other player.

Covariance under affine transformations, in contrast, is open to criticism, because changing the location of the disagreement point d and the set of alternatives S may give rise to new claims on the part of the players. For example, when two players of approximately equal abilities are negotiating the division of \$1,000, the symmetric solution (\$500, \$500) seems reasonable. If, in contrast, one of the players is very wealthy, there may be two opposite influences on the outcome of the bargaining process: on one hand, it may appear more just for the person who is more in need of the money to receive a greater share, with his justification being that an additional \$500 will have little effect on the wealthy player's condition, but is significant for a poor player. On the other hand, a wealthy player can exploit this fact to his advantage: he can hold out for a greater share of the money, knowing that if negotiations fail and the players walk away with nothing, this state of affairs will be harder on the poor player, who therefore has greater incentive to yield to the wealthy player's demands out of fear of being left with nothing. As we stated above, covariance under affine transformations is necessary when outcomes are stated in units of utility.

The property that has drawn the greatest share of attention is the property of independence of irrelevant alternatives. The next example is taken from Luce and Raifa [1957]. Consider the two bargaining games in Figure 15.15.

Nash's solution for both bargaining games is (50, 10). We do not dispute that this solution appears reasonable for the game $(T, (0, 0))$ (Game A), but we will present a case here that the solution appears unreasonable for the game $(S, (0, 0))$ (Game B). Suppose that both players accept (50, 10) as a fair solution for the game $(T, (0, 0))$, and then turn to playing $(S, (0, 0))$. Player 2 can now claim that alternative (50, 10) is unreasonable: in the bargaining game $(T, (0, 0))$ both players compromise to some extent to arrive at the outcome (50, 10), but in contrast, in the bargaining game $(S, (0, 0))$, the outcome

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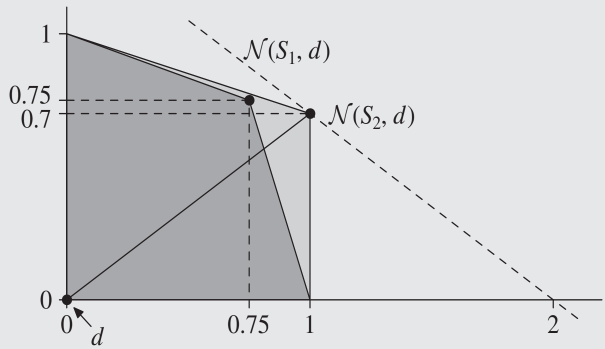

Figure 15.16 The Kalai–Smorodinsky critique of the Nash solution

(50, 10) gives Player 1 his highest possible payoff, while Player 2 does not receive his highest possible payoff. It is therefore reasonable for Player 2 to demand more than 10, by claiming that Player 1 should also compromise and receive less than he would from his best alternative.

Another critique of the Nash solution arises from the following example, given by Kalai and Smorodinsky [1975]. Let S_1 and S_2 be the two compact and convex sets in the plane defined by:

- S_1 is the convex hull of the points $(0, 0)$, $(0, 1)$, $(1, 0)$, $(0.75, 0.75)$ (the darkly shaded area in Figure 15.16).
- S_2 is the convex hull of the points $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 0.7)$ (the lightly shaded area in Figure 15.16).

The set S_2 contains the set S_1. In addition, both players are better off in the bargaining game $(S_2, (0, 0))$ than in the bargaining game $(S_1, (0, 0))$: for every point $x \in S_1$ (except for the points $(1, 0)$ and $(0, 1)$), there is a point $y \in S_2$ satisfying $y \gg x$. In this sense, by playing $(S_2, (0, 0))$, the players are both in an improved situation, and Kalai and Smorodinsky therefore claim that one should expect the solution of $(S_2, (0, 0))$ to be better for both players than the solution of $(S_1, (0, 0))$.

Since the bargaining game $(S_1, (0, 0))$ is symmetric, $\mathcal{N}(S_1, (0, 0)) = (0.75, 0.75)$. By drawing an equilateral triangle whose vertices are $(0, 0)$, $(1, 0.7)$, $(2, 0)$ (see Figure 15.16) and using Theorem 15.20, we deduce that $\mathcal{N}(S_2, (0, 0)) = (1, 0.7)$. Notice that under $\mathcal{N}$, Player 2's situation is worsened in going from $(S_1, (0, 0))$ to $(S_2, (0, 0))$ (in S_1 he receives 0.75, as opposed to 0.7 in S_2). At the same time, this critique is not unequivocal, because although both players are better off, one may argue that in a certain sense Player 1's situation is "more improved" than Player 2's, because the set of efficient points has "moved more to the right than upwards."

Additional critiques that have been applied to the Nash solution are presented in Exercises 15.22 and 15.23. Despite these critiques, the Nash solution is still regarded as the most important solution concept for bargaining games. It is more frequently applied than other proposed solution concepts, which are also open to critique.

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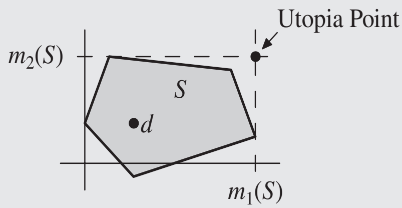

Figure 15.17 The definition of $m_1(S)$ and $m_2(S)$

15.8 Monotonicity properties

Up to now, we have dealt with solution concepts defined over the family $\mathcal{F}$ of bargaining games. In this section, we will consider solution concepts defined over other families of bargaining games. Our goal will be to replace the property of independence of irrelevant alternatives with a different property, a monotonicity property. We will present two different monotonicity properties, and study the solution concepts implied by each of them.

The first monotonicity property we will study is the following.

Definition 15.27 *A solution concept φ satisfies* full monotonicity *over a family $\mathcal{F}_0$ of bargaining games if for every pair of bargaining games (S, d) and (T, d) in $\mathcal{F}_0$ such that $S \subseteq T$,*

$$\varphi(S, d) \leq \varphi(T, d). \tag{15.40}$$

The full monotonicity property is satisfied if adding more possible outcomes does not make a player's situation worse. The second monotonicity property is more complex. For every compact set $S \subset \mathbb{R}^2$, define

$$m_1(S) := \max\{x_1 \in \mathbb{R}\colon \text{ there exists } x_2 \in \mathbb{R} \text{ such that } (x_1, x_2) \in S\}, \tag{15.41}$$

and

$$m_2(S) := \max\{x_2 \in \mathbb{R}\colon \text{ there exists } x_1 \in \mathbb{R} \text{ such that } (x_1, x_2) \in S\}. \tag{15.42}$$

The maximal payoff that Player 1 can possibly get is $m_1(S)$, and $m_2(S)$ is the maximal payoff that Player 2 can get (see Figure 15.17). The point $(m_1(S), m_2(S))$ is called the *utopia point* of the game. This point is typically not in S, as in Figure 15.17.

The second monotonicity property is, in a sense, a weaker version of full monotonicity.

Definition 15.28 *A solution concept φ satisfies* limited monotonicity *over a family $\mathcal{F}_0$ of bargaining games if for every pair of bargaining games (S, d) and (T, d) in $\mathcal{F}_0$ satisfying (a) $S \subseteq T$, (b) $m_1(S) = m_1(T)$, and (c) $m_2(S) = m_2(T)$, it is the case that*

$$\varphi(S, d) \leq \varphi(T, d). \tag{15.43}$$

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A solution concept φ which satisfies full monotonicity also satisfies limited monotonicity. As the example in Figure 15.16 shows, the Nash solution does not satisfy limited monotonicity over the family of games $\mathcal{F}$, since in that example $S_1 \subset S_2$, $m_1(S_1) = m_1(S_2) = 1$, $m_2(S_1) = m_2(S_2) = 1$, and $\mathcal{N}(S_2, d) \not\geq \mathcal{N}(S_1, d)$.

Denote by $\mathcal{F}_0$ the family of bargaining games (S, d) satisfying:

1. The set S is compact and convex.
2. The disagreement point is $d = (0, 0)$.
3. $x \geq (0, 0)$ for every $x \in S$, and there exists $x \in S$ such that $x \gg (0, 0)$.
4. **Comprehensiveness:** If $x \in S$, then the rectangle defined by $(0, 0)$ and x is also contained in S:

$$[0, x_1] \times [0, x_2] \subseteq S, \qquad \forall x \in S. \tag{15.44}$$

We already encountered the first condition in the definition of the family of bargaining games $\mathcal{F}$. The second condition states that if the players cannot arrive at a compromise, neither of them gets anything. This assumption imposes no loss of generality if we add the property of covariance under translations. The third condition states that every alternative in S is weakly preferred to the disagreement point; any other alternative would be rejected by one of the players, and therefore we can omit it from S. The fourth condition is equivalent to enabling the players to throw away (or donate to charity) some of their profits from the bargaining process.

Since the disagreement point is $(0, 0)$, it suffices to denote the bargaining games in $\mathcal{F}_0$ by S, instead of (S, d). By the definition of a bargaining game, there exists $x \in S$ satisfying $x \gg d = (0, 0)$. It follows that $m_1(S) > 0$, $m_2(S) > 0$, and $S \subseteq [0, m_1(S)] \times [0, m_2(S)]$.

In this section we will focus on bargaining games in $\mathcal{F}_0$. A solution concept over $\mathcal{F}_0$ is a function associating every bargaining game S in this family with an alternative in S.

Definition 15.29 *A solution concept φ satisfies (first-order)* homogeneity *if for every bargaining game S and every real number $c > 0$,*

$$\varphi(cS) = c\varphi(S). \tag{15.45}$$

This property is a weakening of the property of independence of units of measurement, because it requires covariance only for every linear transformation that multiplies both coordinates by the same constant.

Definition 15.30 *A solution concept φ satisfies* strict individual rationality *if for every bargaining game S,*

$$\varphi(S) \gg (0, 0). \tag{15.46}$$

This property requires that each player obtain a strictly positive profit relative to the disagreement point. Because there exists $x \in S$ satisfying $x \gg (0, 0)$, it is reasonable to require that both players profit from cooperation.

For every angle $0^\circ < \alpha < 90^\circ$, define a solution λ_α over the family $\mathcal{F}_0$ as follows: $\lambda_\alpha(S)$ is the highest point in S on the ray emanating from the disagreement point $(0, 0)$ at angle α, relative to the axis $x_2 = 0$ (see Figure 15.18).

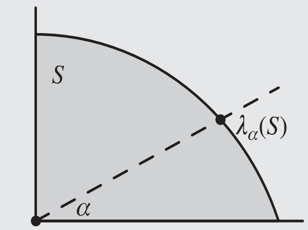

Figure 15.18 The solution λ_α

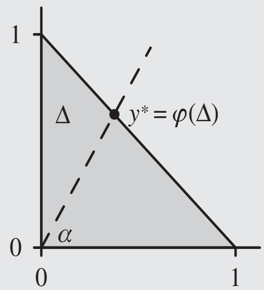

Figure 15.19 The definitions of $y^* = \varphi(\Delta)$ and the angle α

For every $0^\circ < \alpha < 90^\circ$, the solution λ_α satisfies weak efficiency, homogeneity, strict individual rationality, and full monotonicity. In addition, if $\alpha = 45^\circ$, the solution λ_α also satisfies symmetry (Exercise 15.24). The next theorem, due to Kalai [1977], shows that these are all the solution concepts satisfying these properties.

Theorem 15.31 *Let φ be a solution concept over $\mathcal{F}_0$ satisfying weak efficiency, homogeneity, strict individual rationality, and full monotonicity. Then there exists an angle $0^\circ < \alpha < 90^\circ$ such that $\varphi = \lambda_\alpha$.*

Proof: Let φ be a solution concept satisfying the four properties of the statement of the theorem.

Step 1: Defining α.
Define the set $\Delta \subset \mathbb{R}^2$ as follows:

$$\Delta = \{x \in [0,1]^2 \colon x_1 + x_2 \leq 1\}. \tag{15.47}$$

Denote $y^* := \varphi(\Delta)$ (see Figure 15.19). Since φ satisfies weak efficiency and strict individual rationality, the solution $y^* = \varphi(\Delta)$ is in the interior of the interval connecting $(0,1)$ with $(1,0)$.

Let $\alpha = \arctan(\frac{y_2^*}{y_1^*})$ be the angle of the line connecting $(0,0)$ with y^* (see Figure 15.19). Then $\varphi(\Delta) = y^* = \lambda_\alpha(\Delta)$. We will prove that $\varphi(S) = \lambda_\alpha(S)$ for every $S \in \mathcal{F}_0$. For the rest of the proof, we fix $S \in \mathcal{F}_0$.

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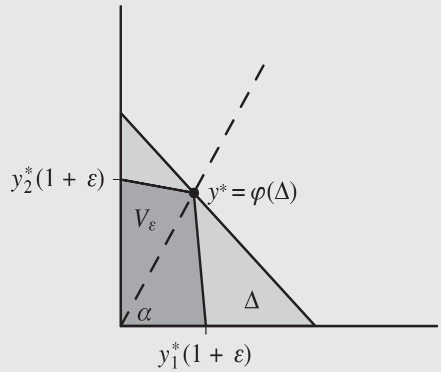

Figure 15.20 The set V_ε

Step 2: $\varphi(S) \geq \lambda_\alpha(S)$.
Recall that $\varphi(\Delta) = y^* = (y_1^*, y_2^*) \gg (0,0)$. For every $\varepsilon \in (0, \min\{\frac{1}{y_1^*} - 1, \frac{1}{y_2^*} - 1\})$, consider the convex set V_ε whose extreme points are $(0,0)$, $(0, y_2^*(1+\varepsilon))$, y^*, $(y_1^*(1+\varepsilon), 0)$ (see Figure 15.20). Since $\varepsilon < \frac{1}{y_1^*} - 1$ and $\varepsilon < \frac{1}{y_2^*} - 1$, we deduce that $V_\varepsilon \subseteq \Delta$.

We now show that $\varphi(V_\varepsilon) = y^*$. Since φ satisfies full monotonicity, $\varphi(V_\varepsilon) \leq \varphi(\Delta) = y^*$. Since φ satisfies weak efficiency, $\varphi(V_\varepsilon)$ is an efficient point of V_ε. But the only weakly efficient point x in V_ε satisfying $x \leq y^*$ is y^*, and therefore $\varphi(V_\varepsilon) = y^*$.

Denote $z^* := \lambda_\alpha(S)$. Since y^* and z^* are both points on the line emanating from $(0,0)$ and forming an angle α with the axis $x_2 = 0$,

$$\frac{y_1^*(1+\varepsilon)}{y_2^*(1+\varepsilon)} = \frac{y_1^*}{y_2^*} = \frac{z_1^*}{z_2^*}. \tag{15.48}$$

Denote $c_\varepsilon := \frac{z_1^*}{y_1^*(1+\varepsilon)} = \frac{z_2^*}{y_2^*(1+\varepsilon)}$. We deduce

$$c_\varepsilon y_1^*(1+\varepsilon) = z_1^*, \qquad c_\varepsilon y_2^*(1+\varepsilon) = z_2^*. \tag{15.49}$$

Denote

$$z_\varepsilon = c_\varepsilon y^* = \frac{z^*}{1+\varepsilon}. \tag{15.50}$$

The set $c_\varepsilon V_\varepsilon$ and the point z_ε are depicted in Figure 15.21. Note that since S is a comprehensive set, it follows that $c_\varepsilon V_\varepsilon \subseteq S$.

Since the solution concept φ satisfies homogeneity,

$$\varphi(c_\varepsilon V_\varepsilon) = c_\varepsilon \varphi(V_\varepsilon) = c_\varepsilon y^* = z_\varepsilon = \frac{z^*}{1+\varepsilon}. \tag{15.51}$$

Since the solution concept φ satisfies full monotonicity, and $c_\varepsilon V_\varepsilon \subseteq S$, we deduce that $\varphi(S) \geq \frac{z^*}{1+\varepsilon}$. This inequality holds for every $\varepsilon > 0$ that is sufficiently small. Letting ε converge to 0 yields $\varphi(S) \geq z^* = \lambda_\alpha(S)$, which is what we wanted to show.

In the example in Figure 15.21 there exists only one weakly efficient point in S that is greater than or equal to $\lambda_\alpha(S)$, and therefore in this case $\varphi(S) = \lambda_\alpha(S)$. As can be seen in

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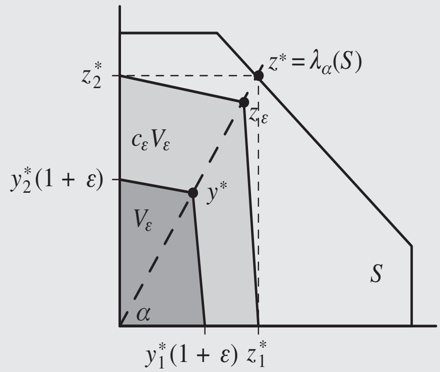

Figure 15.21 The set $c_\varepsilon V_\varepsilon$ and the point z_ε

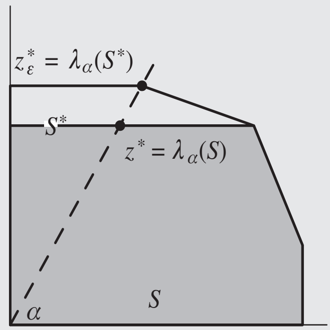

Figure 15.22 The set S (in grey) and the set S^*

Figure 15.22, when the boundary of S is parallel to one of the axes, there may be several weakly efficient points in S that are greater than or equal to $\lambda_\alpha(S)$. We will now deal with this case.

Step 3: $\varphi(S) = \lambda_\alpha(S)$.
Let $\varepsilon > 0$, and define $z_\varepsilon^* = (1+\varepsilon)z^*$. Let S^* be the smallest convex and comprehensive set containing S and z_ε^* (see Figure 15.22).

From what we showed in Step 2, $\varphi(S^*) \geq \lambda_\alpha(S^*)$. As the only weakly efficient point in S^* that is greater than or equal to $\lambda_\alpha(S^*)$ is $\lambda_\alpha(S^*)$ itself, we deduce that

$$\varphi(S^*) = \lambda_\alpha(S^*) = z_\varepsilon^* = z^*(1+\varepsilon). \tag{15.52}$$

Since $S \subset S^*$, and since φ satisfies full monotonicity,

$$\varphi(S) \leq \varphi(S^*) = z^*(1+\varepsilon), \tag{15.53}$$

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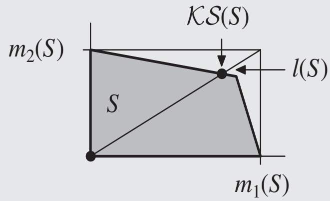

Figure 15.23 The Kalai–Smorodinsky solution

and this inequality holds for all $\varepsilon > 0$. Letting ε converge to 0 yields

$$\varphi(S) \leq z^* = \lambda_\alpha(S). \tag{15.54}$$

Together with the result of Step 2, we conclude that $\varphi(S) = \lambda_\alpha(S)$, which is what we wanted to show. □

The next theorem, which is proved in Kalai and Smorodinsky [1974], characterizes the solution concept obtained from the limited monotonicity property.

Theorem 15.32 *There exists a unique solution concept $\mathcal{KS}$ over the family $\mathcal{F}_0$ satisfying symmetry, efficiency, independence of the units of measurement, and limited monotonicity. That solution is the highest point located in S and on the line $l(S)$ connecting the disagreement point* $(0, 0)$ *with the utopia point* $(m_1(S), m_2(S))$ *(see Figure 15.23).*

The alternative $\mathcal{KS}(S)$ is called the *Kalai–Smorodinsky solution* or the *Kalai–Smorodinsky agreement point*.

Proof:
The $\mathcal{KS}$ solution is well defined: Denote by $l(S)$ the line that passes through the disagreement point $(0, 0)$ and the utopia point $(m_1(S), m_2(S))$. Since there exists $x \in S$ satisfying $x \gg (0, 0)$, necessarily $m_1(S) > 0$ and $m_2(S) > 0$. It follows that $l(S)$ is a strictly increasing line. Since $(0, 0)$ is on the line $l(S)$, there exists at least one point in S on this line. Since S is compact, the highest point in S that is on $l(S)$ is well defined, and located in S. Denote this point by $\mathcal{KS}(S)$. In particular, $\mathcal{KS}$ is a solution concept over the family $\mathcal{F}_0$.

$\mathcal{KS}$ satisfies symmetry: If S is a symmetric bargaining game, then $m_1(S) = m_2(S)$. In particular, the line $l(S)$ is the diagonal $x_1 = x_2$, and therefore the point $\mathcal{KS}(S)$, located on this line, satisfies $\mathcal{KS}_1(S) = \mathcal{KS}_2(S)$. We deduce from this that the solution $\mathcal{KS}$ satisfies symmetry.

$\mathcal{KS}$ satisfies efficiency: Let $S \in \mathcal{F}_0$. Denoting $x^* = \mathcal{KS}(S)$, we will show that x^* is an efficient point; i.e., there does not exist y in S satisfying $y \geq x^*$ and $y \neq x^*$.

If x^* is inefficient, there is $y \in S$ satisfying one of the following three properties:

1. $y \gg x^*$,
2. $y_1 > x_1^*$ and $y_2 = x_2^*$, or
3. $y_1 = x_1^*$ and $y_2 > x_2^*$.

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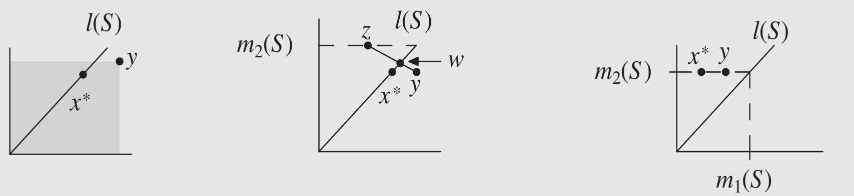

(a) Case 1 (b) Case 2 with $m_2(S) > x_2^*$ (c) Case 2 with $m_2(S) = x_2^*$

Figure 15.24 Efficiency

Case 1: There exists $y \in S$ satisfying $y \gg x^*$ (see Figure 15.24(a)).

Since S is comprehensive, the rectangle defined by $(0, 0)$ and y is contained in S, and therefore x^* is in the interior of S. In particular, x^* is not the highest point in S on the line $l(S)$, contradicting its definition. It follows that this case is impossible.

Case 2: $y_1 > x_1^*$ and $y_2 = x_2^*$.
Since $m_i(S) \geq x_i^*$ for $i = 1, 2$, we can distinguish between two alternatives: $m_2(S) > x_2^*$ (see Figure 15.24(b)) and $m_2(S) = x_2^*$ (see Figure 15.24(c)).

If $m_2(S) > x_2^*$, then there exists $z \in S$ satisfying $z_2 = m_2(S) > x_2^*$. Since S is convex, the interval connecting y with z is contained in S. But this interval contains a point w satisfying $w \gg x^*$, which is impossible, as shown in Case 1.

As we saw in Case 1, if $m_2(S) = x_2^*$, then since $y_1 > x_1^*$, necessarily $m_1(S) > x_1^*$. It follows that the line $l(S)$ is located under x^*, and does not pass through it, contradicting the definition of $\mathcal{KS}(S)$, so that this case, too, is impossible.

Case 3: $y_1 = x_1^*$ and $y_2 > x_2^*$.
This case is similar to Case 2, switching the roles of the players, and is therefore impossible.

$\mathcal{KS}$ satisfies independence of units of measurement: Let S be a bargaining game in $\mathcal{F}_0$. Denote $x^* := \mathcal{KS}(S)$. Let $a \in \mathbb{R}^2$ such that $a \gg (0, 0)$. We will show that $\mathcal{KS}(aS) = ax^*$. Since $a_1 > 0$ and $a_2 > 0$, it follows by definition that $m_i(aS) = a_i m_i(S)$ for $i = 1, 2$. Therefore, under the positive affine transformation $x \mapsto ax$ the line $l(S)$ is mapped to the line $l(aS)$. In particular, ax^* is located on the line $l(aS)$. Since $a \gg (0, 0)$, we deduce that $az \gg ax^*$ for every $z \in S$ if and only if $z \gg x^*$. Since x^* is the highest point in S on the line $l(S)$, we obtain that ax^* is the highest point in aS on the line $l(aS)$, which is what we wanted to show.

$\mathcal{KS}$ satisfies limited monotonicity: Let S and T be two bargaining games in $\mathcal{F}_0$ satisfying $m_1(S) = m_1(T)$, $m_2(S) = m_2(T)$, and $S \subseteq T$.

Since $m_1(S) = m_1(T)$ and $m_2(S) = m_2(T)$, one has $l(S) = l(T)$, and since $S \subseteq T$, the highest point in T on this line is not below the highest point in S on the line.

$\mathcal{KS}$ is the only solution satisfying these properties: Let φ be a solution concept satisfying symmetry, efficiency, independence of units of measurement, and limited monotonicity. We will prove that $\varphi = \mathcal{KS}$.

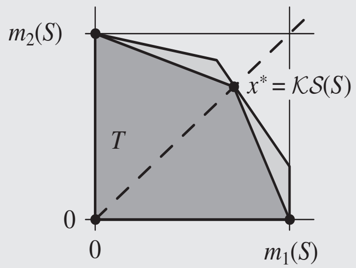

Figure 15.25 The sets T (in dark grey) and S (in light and dark grey)

Let S be a bargaining game in $\mathcal{F}_0$, and denote $x^* := \mathcal{KS}(S)$. Let T be the following set (see Figure 15.25):

$$T := \operatorname{conv}\{(0,0), (0, m_2(S)), x^*, (m_1(S), 0)\}. \tag{15.55}$$

The set T is contained in S. To see this, note that the point $(0, 0)$ is a disagreement point and is therefore in S. Since S is comprehensive, the points $(0, m_2(S))$ and $(m_1(S), 0)$ are also in S, and by the definition of the Kalai–Smorodinsky solution the point x^* is in S. Finally, the set S is convex and therefore T is contained in S. Furthermore, $\mathcal{KS}(T) = \mathcal{KS}(S) = x^*$ (why?).

Since φ and $\mathcal{KS}$ satisfy independence of units of measurement, symmetry, and efficiency, $\varphi(T) = \mathcal{KS}(T)$ (Exercise 15.10). Since φ and $\mathcal{KS}$ satisfy limited monotonicity,

$$\varphi(S) \geq \varphi(T) = \mathcal{KS}(T) = \mathcal{KS}(S) = x^*. \tag{15.56}$$

Since $\mathcal{KS}$ is efficient, the only alternative in S that is greater than or equal to x^* is x^* itself. Therefore, $\varphi(S) = x^* = \mathcal{KS}(S)$, which is what we wanted to show. □

15.9 Bargaining games with more than two players

In the previous sections we concentrated on bargaining games with two players. But there are cases in which more than two players conduct negotiations. Examples include the distribution of government ministries among governing coalitions in parliamentary democracies, and the financing of joint projects.

Definition 15.33 *A* bargaining game *is a triple* (N, S, d)*, where:*

- *N is a finite set of players.*
- *$S \subseteq \mathbb{R}^N$ is a nonempty, compact, and convex set of* alternatives.
- *$d \in \mathbb{R}^N$ is a* disagreement point.
- *There exists $x \in S$ satisfying $x \gg d$.*

Denote by $\mathcal{F}^N$ the family of bargaining games (N, S, d) with the set of players N, and by $\mathcal{F}^* = \cup_{\{N \subset \mathbb{N}\}} \mathcal{F}^N$ the family of all bargaining games with a finite number of players. As in the case of two players, a *solution concept for bargaining games over the set of players N* is a function associating each bargaining game $(N, S, d) \in \mathcal{F}^N$ with an alternative $\varphi(N, S, d) \in S$, and a *solution concept* (for the collection of all bargaining games with an arbitrary number of players) is a function associating every bargaining game $(N, S, d) \in \mathcal{F}^*$ with an alternative $\varphi(N, S, d) \in S$.

This model has the following interpretation: an alternative in S is the outcome of a bargaining process if and only if all the players agree to cooperate. In other words, in this model no proper subset of N can obtain an outcome in S except for d. A model that takes into account the possibility that various proper subcoalitions of N can on their own obtain an outcome that is preferred by their members will be presented in Chapter 16.

We now specify the properties we will use in studying bargaining games with more than two players. The definitions of efficiency, covariance under positive affine transformations, and independence of irrelevant alternatives are analogous to the definitions in Section 15.3 (page 626).

We next define the concept of symmetry for this case.

Definition 15.34 *A bargaining game $(N, S, d) \in \mathcal{F}^*$ is called a* symmetric game *if:*

- *$d_i = d_j$ for every pair of players $i, j \in N$: the disagreement point is symmetric.*
- *$(x_{\pi(i)})_{i \in N} \in S$ for every alternative $x \in S$ and every permutation π of N.*

A solution concept φ satisfies symmetry if $\varphi_i(N, S, d) = \varphi_j(N, S, d)$ for every symmetric bargaining game $(N, S, d) \in \mathcal{F}^*$, and every pair of players $i, j \in N$.

Theorem 15.15 (page 630) can be generalized to the case in which there are more than two players:

Theorem 15.35 *There exists a unique solution concept $\mathcal{N}^*$ for the family of bargaining games $\mathcal{F}^*$ satisfying the properties of symmetry, efficiency, covariance under positive affine transformations, and independence of irrelevant alternatives. For every bargaining game $(N, S, d) \in \mathcal{F}^*$, the point $\mathcal{N}^*(N, S, d)$ is the vector x in S satisfying $x \gg d$ that maximizes the product $\prod_{i \in N}(x_i - d_i)$.*

The proof is analogous to the proof in the case $|N| = 2$ (Exercise 15.29).

Another property of a solution concept, which appears in various contexts, is the property of consistency.

Definition 15.36 *A solution concept φ satisfies* consistency *if for every bargaining game $(N, S, d) \in \mathcal{F}^*$, and every nonempty set of players $I \subset N$, $(\varphi_i(I, S, d))_{i \in I}$ is the solution of the bargaining game $(I, \widehat{S}, \widehat{d})$ in which:*

- *$\widehat{d}_i = d_i$ for every $i \in I$: The disagreement point is derived from the disagreement point of the original game.*
- *$\widehat{S} = \left\{(x_i)_{i \in I} \in \mathbb{R}^I : \left((x_i)_{i \in I}, (\varphi_j(N, S, d))_{j \notin I}\right) \in S\right\}$. The set of alternatives is the set of all alternatives in S at which all the players who are not in I get the outcome offered to them under the solution concept φ.*

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A solution concept satisfies consistency if after the bargaining game has ended, every set of players who decide to go to arbitration and renegotiate only the shares of the members of that set discover that the arbitrator (who acts according to the solution concept φ) will not change the outcome they previously attained. The consistency property is the only property we have presented that involves games with different numbers of players.

A permutation π of the set of players N defines a mapping $\pi : \mathbb{R}^N \to \mathbb{R}^N$ by

$$\pi(x) := (x_{\pi(i)})_{i \in N}. \tag{15.57}$$

For a set $S \subseteq \mathbb{R}^N$ define

$$\pi(S) := \{\pi(x) : x \in S\}. \tag{15.58}$$

The bargaining game $(N, \pi(S), \pi(d))$ is the game (N, S, d), in which the names of the players have been changed according to the permutation π.

A solution concept satisfies anonymity if when the names of the players are changed, the proposed solution concept changes accordingly. Such a solution concept cannot discriminate between the players solely because of their names.

Definition 15.37 *A solution concept φ satisfies* anonymity *if for every bargaining game $(N, S, d) \in \mathcal{F}^*$, and every permutation π of the set of players N,*

$$\varphi_{\pi(i)}(\pi(S), \pi(d)) = \varphi_i(S, d), \quad \forall i \in N.$$

This property is also called independence of the names of the players *in the literature.*

Denote by $\mathcal{F}_0^*$ the family of bargaining games (N, S, d) satisfying:

1. The set of players N is a finite set.
2. The disagreement point is $d = \vec{0} := (0, 0, \cdots, 0)$.
3. The set S is a nonempty, compact, and convex set in $\mathbb{R}^N$.
4. $x \geq \vec{0}$ for every $x \in S$, and there exists $x \in S$ such that $x \gg \vec{0}$.
5. *Comprehensiveness:* If $x \in S$, then the n-dimensional rectangle defined by d and x is also contained in S:

$$\bigtimes_{i \in N} [d_i, x_i] \subseteq S, \qquad \forall x \in S. \tag{15.59}$$

Note that every solution concept that satisfies anonymity also satisfies symmetry. The following theorem is proved in Lensberg [1988].

Theorem 15.38 *The only solution concept for the family of bargaining games $\mathcal{F}_0^*$ that satisfies efficiency, anonymity, covariance under positive affine transformations, and consistency is the Nash solution $\mathcal{N}^*$.*

This characterization does not use the independence of irrelevant alternatives property. Thus, when restricted to the family of bargaining games $\mathcal{F}_0^*$, adding the consistency property, which is meaningless when $|N| = 2$, makes independence of irrelevant alternatives superfluous. Since the Nash solution $\mathcal{N}^*$ satisfies the independence of irrelevant alternatives property, one deduces that this property follows from the other properties characterizing the Nash solution $\mathcal{N}^*$.

15.10 Remarks

A discussion of the influence risk aversion has on the outcomes of bargaining games (see Exercise 15.21) can be found in Kihlstrom, Roth, and Schmeidler [1981]. The difficulty pointed out in Exercise 15.22 was first noted in Perles and Maschler [1981], which suggests an alternative solution concept that overcomes this difficulty.

15.11 Exercises

15.1 Prove that the solution concept defined in Example 15.23 (page 640) satisfies the properties of symmetry, covariance under positive affine transformations, and independence of irrelevant alternatives, but does not satisfy efficiency.

15.2 Prove that the solution concept defined in Example 15.24 (page 640) satisfies the properties of efficiency, covariance under positive affine transformations, and independence of irrelevant alternatives, but does not satisfy symmetry.

15.3 Prove that the solution concept $\lambda_{45^0}(S, d)$ (see Example 15.25 on page 640) satisfies the properties of efficiency, symmetry, and independence of irrelevant alternatives, but does not satisfy the property of covariance under affine transformations.

15.4 Prove that the solution concept defined in Example 15.26 (page 640) satisfies the properties of efficiency, symmetry, and covariance under positive affine transformations, but does not satisfy the property of independence of irrelevant alternatives.

15.5 Let (S, c) be a bargaining game, and let (T, d) be the image of (S, c) under the positive affine transformation $x \mapsto ax + b$ on $\mathbb{R}^2$; that is, $T = aS + b$ and $d = ac + b$. Prove that the set $\operatorname{argmax}_{y \in T}(y_1 - d_1)(y_2 - d_2)$ is the image of the set $\operatorname{argmax}_{x \in S}(x_1 - c_1)(x_2 - c_2)$ under this transformation.

15.6 Let $S \subset \mathbb{R}^2$, let $x \in S$ be an alternative on the boundary of S, and let l be a supporting line[6] of S at x. Let $x \mapsto ax + b$ be a positive affine transformation on $\mathbb{R}^2$. Prove that the line $al + b$ is a supporting line of the set $aS + b$ at the point $ax + b$.

15.7 Let $S \subset \mathbb{R}^2$ be a convex set, and let $a, b \in \mathbb{R}^2$. Prove that the set $aS + b$ is convex.

15.8 Let φ be a solution concept (for $\mathcal{F}$) satisfying symmetry and efficiency. Prove that for every symmetric bargaining game (S, d), the outcome $\varphi(S, d)$ is the highest point on the line $x_1 = x_2$ that is also in S (in other words, the point in S and on the line $x_1 = x_2$ for which x_1 is maximal).

15.9 Let φ be a solution concept (for $\mathcal{F}$) satisfying symmetry, efficiency, and independence of the units of measurement. Let $a, b > 0$ be positive numbers, and

6 Here we are identifying the line l with the set of points contained in it: the line l defined by the equation $\alpha x_1 + \beta x_2 = \gamma$ is identified with the collection of points $\{x \in \mathbb{R}^2 : \alpha x_1 + \beta x_2 = \gamma\}$.

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let S be the triangle whose vertices are $(0, 0)$, $(a, 0)$, and $(0, b)$. Prove that $\varphi(S, (0, 0)) = \left(\frac{a}{2}, \frac{b}{2}\right)$.

15.10 Let φ be a solution concept (for $\mathcal{F}$) satisfying symmetry, efficiency, and independence of the units of measurement. Let $a, b > 0$ be positive numbers, and let $x = (x_1, x_2)$ be a point on the ray emanating from $(0, 0)$ and passing through (a, b), satisfying $x_1 > \frac{a}{2}$. Let S be the quadrangle whose vertices are $(0, 0)$, $(a, 0)$, $(0, b)$, and x. Prove that $\varphi(S, (0, 0)) = x$.

15.11 Prove that under a positive affine transformation of the plane, an equilateral triangle whose base is the x_1-axis is transformed into an equilateral triangle whose base is parallel to the x_1-axis.

15.12 Let φ be a solution concept (for $\mathcal{F}$) satisfying the properties of efficiency and individual rationality. Let (S, d) be a bargaining game satisfying the following property: there is an alternative $x \in S$ satisfying $x \geq d$, but there is no $x \in S$ satisfying $x \gg d$. What is $\varphi(S, d)$?

15.13 A *set solution concept* for a family of bargaining games $\widetilde{\mathcal{F}}$ is a function φ associating every bargaining game (S, d) in $\widetilde{\mathcal{F}}$ with a subset of S (which may contain more than a single point). Let $f : \mathbb{R}^4 \to \mathbb{R}$ be a function. Define a set solution concept φ as follows:

$$\varphi(S, d) = \operatorname{argmax}_{x \in S} f(d, x). \tag{15.60}$$

(a) Give an example of a function f for which $\varphi(S, d)$ is not always a single point.
(b) Prove that φ satisfies independence of irrelevant alternatives. In other words, if $T \supseteq S$ and $x \in \varphi(T, d) \cap S$, then $x \in \varphi(S, d)$.

15.14 Let φ_1 and φ_2 be two solution concepts for the family of bargaining games $\mathcal{F}$. Define another solution concept φ for the family of bargaining games $\mathcal{F}$ as follows:

$$\varphi(S, d) = \tfrac{1}{2}\varphi_1(S, d) + \tfrac{1}{2}\varphi_2(S, d). \tag{15.61}$$

For each of the following properties, prove or disprove the following claim: if φ_1 and φ_2 satisfy the property, then the solution concept φ also satisfies the same property.

(a) Symmetry.
(b) Efficiency.
(c) Independence of irrelevant alternatives.
(d) Covariance under positive affine transformations.

15.15 Two players are to divide \$2,000 between them. The utility functions of the players are the amounts of money they receive: $u_1(x) = u_2(x) = x$. If they cannot come to agreement, neither of them receives anything. For the following cases, describe the bargaining game derived from the given situation, in utility units, and find its Nash solution of the game.

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(a) Given any division which the players agree upon, the first player receives his full share under the agreed division, and the second player pays a tax of 40%.
(b) Repeat the situation in the previous item, but assume that the second player pays a tax of 60%.
(c) The first player pays a tax of 20%, and the second pays a tax of 30%.

15.16 Two players are to divide \$2,000 between them. The utility function of the first player is $u_1(x) = x$. The utility function of the second player is $u_2(x) = \sqrt{x}$. For each of the following two situations, describe the bargaining game derived from the situation, in utility units, and find its Nash solution.

(a) If the two players cannot come to an agreement, neither of them receives any payoff.
(b) If the two players cannot come to an agreement, the first one receives \$16, and the second receives \$49 (note that in this case the disagreement point in the utility space is $(16, 7)$).

15.17 Find the Nash solution for the bargaining game in which

$$S = \left\{ x \in \mathbb{R}^2 \colon \frac{x_1^2}{16^2} + \frac{x_2^2}{20^2} \leq 1 \right\}, \tag{15.62}$$

and

(a) The disagreement point is $(0, 0)$.
(b) The disagreement point is $(10, 0)$.

15.18 Prove Theorem 15.22 on page 638: for every bargaining game (S, d), the only alternative that constitutes a solution concept according to Definition 15.21 (page 638) is the Nash solution $\mathcal{N}(S, d)$. Moreover, the constant c equals minus the slope of the supporting line of S at the point $\mathcal{N}(S, d)$.

15.19 Let $(S, d) \in \mathcal{F}$ be a bargaining game.

(a) Prove that there exists a unique efficient alternative in S minimizing the absolute value $|(x_1 - d_1) - (x_2 - d_2)|$. Denote this alternative by x^*.

Let Y be the collection of efficient alternatives y in S satisfying the property that the sum of their coordinates $y_1 + y_2$ is maximal.

(b) Show that the Nash solution $\mathcal{N}(S, d)$ is on the efficient boundary between x^* and the point in Y that is closest to x. In particular, if $x^* \in Y$ then $x^* = \mathcal{N}(S, d)$.

15.20 Let $(S, d) \in \mathcal{F}$ be a bargaining game. Denote by $x_2 = g(x_1)$ the equation defining the north-east boundary of S. Prove that if g is strictly concave and twice differentiable, then the point $x^* = \mathcal{N}(S, d)$ is the only efficient point x in S satisfying $-g'(x_1)(x_1 - d_1) = (x_2 - d_2)$.

15.21 Suppose two players have utility functions for money given by

$$u_1(x) = x^{\alpha_1}, \quad u_2(x) = x^{\alpha_2}, \tag{15.63}$$

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where $0 < \alpha_1 < \alpha_2 < 1$. The Arrow–Pratt risk aversion index of player i is $r_{u_i}(x) := -\frac{u_i''(x)}{u_i'(x)}$ (see Exercise 2.28 on page 37 for an explanation of this index).

(a) Are the players risk-seeking or risk-averse? In other words, are their utility functions convex or concave?
(b) Player i is more risk-averse than player j if $r_{u_i}(x) \geq r_{u_j}(x)$ for every x. Which of the two players is more risk-averse?
(c) The players are to divide between them a potential profit of A dollars, but this profit can only be realized if the players can come to agreement on how to divide it. What is the Nash solution of this bargaining game, when the outcomes are in units of utility? Which of the players receives the greater payoff?
(d) What is the effect of risk aversion on the Nash outcome of a bargaining game in this example?

15.22 Two bargaining games, $(S, (0, 0))$ and $(T, (0, 0))$, are given by

$$S = \{x \in \mathbb{R}^2_+ : 2x_1 + x_2 \leq 100\}, \tag{15.64}$$
$$T = \{x \in \mathbb{R}^2_+ : x_1 + 2x_2 \leq 100\}. \tag{15.65}$$

David and Jonathan face the following situation. With probability $\frac{1}{2}$, they will negotiate tomorrow over the bargaining game $(S, (0, 0))$, and with probability $\frac{1}{2}$, they will negotiate over the bargaining game $(T, (0, 0))$.

David and Jonathan believe in the Nash solution, but they do not know which bargaining game they will play. Jonathan proposes that they apply the Nash solution in each of the two bargaining games (when it is reached), and therefore their expected utility is

$$\tfrac{1}{2}\mathcal{N}(S, (0, 0)) + \tfrac{1}{2}\mathcal{N}(T, (0, 0)). \tag{15.66}$$

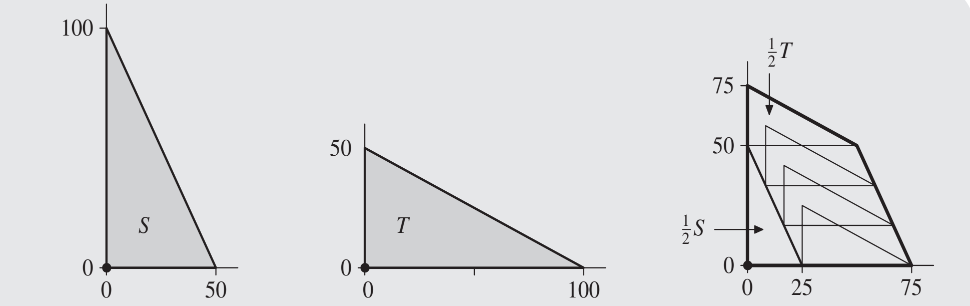
David counterproposes as follows: since the probability that they will negotiate over each of the bargaining games is $\frac{1}{2}$, the players are actually facing the bargaining game $(\frac{1}{2}S + \frac{1}{2}T, (0, 0))$, where the set $\frac{1}{2}S + \frac{1}{2}T$ is defined by

$$\tfrac{1}{2}S + \tfrac{1}{2}T = \left\{\tfrac{1}{2}x + \tfrac{1}{2}y : x \in S, y \in T\right\}. \tag{15.67}$$

They should therefore implement the Nash solution over the game $(\frac{1}{2}S + \frac{1}{2}T, (0, 0))$, which then should be

$$N\left(\tfrac{1}{2}S + \tfrac{1}{2}T, (0, 0)\right). \tag{15.68}$$

To compute the set $\frac{1}{2}S + \frac{1}{2}T$, draw the set $\frac{1}{2}S$, and "slide" the set $\frac{1}{2}T$ along its efficient points (see Figure 15.26(c)).



(a) Bargaining game $(S, (0, 0))$ (b) Bargaining game $(T, (0, 0))$ (c) Bargaining game $(\frac{1}{2}S + \frac{1}{2}T, (0, 0))$

Figure 15.26 The bargaining games $(S, (0, 0))$, $(T, (0, 0))$ and $(\frac{1}{2}S + \frac{1}{2}T, (0, 0))$

Compute $\frac{1}{2}\mathcal{N}(S, (0, 0)) + \frac{1}{2}\mathcal{N}(T, (0, 0))$, and $\mathcal{N}\left(\frac{1}{2}S + \frac{1}{2}T, (0, 0)\right)$. Did you get the same result in both cases?

15.23 Repeat Exercise 15.22 for the following bargaining games $(S, (0, 0))$ and $(T, (0, 0))$:

$$S = \{x \in \mathbb{R}^2_+ : x_1 + x_2 \leq 4\}, \tag{15.69}$$

$$T = \left\{x \in \mathbb{R}^2_+ : x_1 + x_2 \leq 5, \tfrac{3}{4}x_1 + x_2 \leq 4\right\}. \tag{15.70}$$

15.24 Prove that for every $0^\circ < \alpha < 90^\circ$, the solution λ_α defined on page 640 satisfies weak efficiency, homogeneity, strict individual rationality, and full monotonicity. In addition, if $\alpha = 45^\circ$, the solution λ_α satisfies symmetry.

15.25 Prove the minimality of the set of properties characterizing the solution concept λ_α, as listed in Theorem 15.31 (page 645). In other words, prove (by examples) that for every three properties out of the four properties mentioned in the statement of Theorem 15.31 there exists a solution concept satisfying all three properties, but not the fourth.

15.26 Solve Exercises 15.15 and 15.16, assuming the players accept the Kalai–Smorodinsky solution, not the Nash solution. To convert the bargaining game (S, d) in $\mathcal{F}$ to a bargaining game in $\mathcal{F}_0$ (over which the Kalai–Smorodinsky solution is defined), apply the positive affine transformation $x \mapsto x - d$ and remove all the points y that do not satisfy $y \geq (0, 0)$.

15.27 Solve Exercises 15.22 and 15.23, assuming the players accept the Kalai–Smorodinsky solution, rather than the Nash solution. To convert the bargaining game (S, d) in $\mathcal{F}$ to a bargaining game in $\mathcal{F}_0$ (over which the Kalai–Smorodinsky solution is defined), apply the positive affine transformation $x \mapsto x - d$ and remove all the points y that do not satisfy $y \geq (0, 0)$.

15.28 Prove the minimality of the set of properties defining the Kalai–Smorodinsky solution. In other words, prove (by examples) that for any set of three of the four

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properties characterizing the solution concept, there exists a solution concept over $\mathcal{F}_0$ that satisfies those three properties, but not the fourth property.

15.29 Prove Theorem 15.35 (page 651), which characterizes the Nash solution for bargaining games with any number of players.
Hint: Look at the function $\ln(\prod_{i=1}^{n}(x_i - d_i))$.

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