2024-01-SI423: Linear Algebra & its Applications

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Basic Notations and Definitions

- Let $\mathbb{F}$ be a field, (e.g., think $\mathbb{F} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$).
- For $n \in \mathbb{N}$, we have $\mathbb{F}^n = \{(a_1, \ldots, a_n) \mid a_i \in \mathbb{F}\}$.
  By e_i, we mean the point in $\mathbb{F}^n$ with ith coordinate 1, and all other coordinates 0.
  Note: We will write the points in $\mathbb{F}^n$ as column verctors.
- The set of all $m \times n$ matrices with entries in $\mathbb{F}$ is denoted $M_{m\times n}(\mathbb{F})$. If $m = n$, it is denoted $M_n(\mathbb{F})$.
- Let x be an indeterminate over $\mathbb{F}$. Then the set of all polynomials (in x with coefficients in $\mathbb{F}$) is $\mathcal{P}(\mathbb{F}) = \{f \mid \exists\, n \in \mathbb{N}, a_0, \ldots, a_n \in \mathbb{F}$ such that $f = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \mid a_i \in \mathbb{F}\}$.
  This is also denoted $\mathbb{F}[x]$.
- If X and Y are non-empty sets, then $\mathcal{F}(X, Y) = \{f : X \longrightarrow Y \mid f$ is a function$\}$, i.e., $\mathcal{F}(X, Y)$ is the set of all functions from X to Y.

1. Linear Equations

Notation and Definitions:

- Consider the system of m linear equations (S) in n unknowns given by:

$$\begin{gathered} a_{11}x_1 + \cdots + a_{1j}x_j + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{i1}x_1 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n = b_i \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mj}x_j + \cdots + a_{mn}x_n = b_m. \end{gathered}$$

  The short-hand notation for (S) is: $\sum_{j=1}^{n} a_{ij}x_j = b_i$ for $1 \leq i \leq m$.

  The matrix form of (S) is

$$\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{pmatrix} \qquad \text{or simply } A\mathbf{x} = \mathbf{b},$$

  where $A \in M_{m\times n}(\mathbb{F})$, with $A_{ij} = a_{ij}$, $\mathbf{x} = (x_1, \ldots, x_n)^t$ and $\mathbf{b} = (b_1, \ldots, b_m)^t \in \mathbb{F}^m$.

- $\mathbf{u} = (u_1, \ldots, u_n)^t \in \mathbb{F}^n$ is a *solution* of (S) if $A\mathbf{u} = \mathbf{b}$. The set of all solutions of (S), called the *solution set* of (S), is $K = \{\mathbf{u} \in \mathbb{F}^n \mid \mathbf{u}$ is a solution of (S)$\}$. The system (S) is *consistent* if K is non-empty, and *inconsistent* otherwise.
- The *associated homogeneous system* of (S), denoted (S_h) is $A\mathbf{x} = 0$. Its solution set is K_h.

PROBLEMS:

1. I have 10, 3, 6, 7, 5, 1 and 2 currency notes respectively of the following denominations: Rs. 10, Rs. 20, Rs. 50, Rs. 100, Rs. 500, Rs. 1,000 and Rs. 2,000. How many notes of each denomination will I need to pay for a purchase worth Rs. 2,760 from a shop that accepts only cash?

2. A cake shop offers three sizes of snack boxes containing chips packets, samosas, and cake slices. Each small box contains 1 chips packet, 3 samosas, and 3 cake slices. Each medium box contains 2 chips packets, 4 samosas, and 6 cake slices. Each large box contains 4 chips packets, 8 samosas, and 6 cake slices. You need 24 chips packets, 50 samosas, and 48 cake slices for your new year party. How many of the three type of snack boxes you should order?

3. It is known that sodium hydroxide and carbonic acid react to give sodium carbonate and water. The chemical equation can be written as:

$$wNaOH + xH_2CO_3 \to yH_2O + zNa_2CO_3.$$

Find the values of w, x, y and z to balance this equation.

4. The values of k such that the linear system

$$(S) \quad \begin{cases} x+y+z &= 2 \\ x+4y-z &= k \\ 2x+y+4z &= k^2 \end{cases}$$

i) has no solutions is ______.
ii) has a unique solution is ______.
iii) has infinitely many solutions is ______.

Solve the system in the cases when it is consistent.

5. In the following linear system, find all values for a and b for which the resulting system has

$$(S) \quad \begin{cases} x + y + z &= 2 \\ 2x + 3y + 2z &= 5 \\ 2x + 3y + (a^2+1)z &= b \end{cases}$$

i) no solutions.
ii) a unique solution.
iii) infinitely many solutions.

6. Record the age, height and weight of each member of your group as a vector $\overline{x}$ in $\mathbb{R}^3$. Find the average as a vector $\overline{\mu} \in \mathbb{R}^3$.
What does the data set $\overline{x} - \overline{\mu}$ represent? What is its average?

7. A solution of the equation $x+y=1$ is (__, __). What is the solution set of the equation $x+y=1$ in $\mathbb{R}^2$? How do you write it as a subset of $\mathbb{R}^2$? What geometric object does it represent? How is it related to the solution of the equation $x+y=0$?
Answer the same questions for the equation $2x+3y=5$.

8. A solution of the system $x-5y+z=8$ is (__, __, __). What is the complete solution set K of the system $x-5y+z=8$ in $\mathbb{R}^3$? Show that the solution set K_h of the system $x-5y+z=0$ in $\mathbb{R}^3$ can be written as $\{s(5,1,0)+t(-1,0,1) \mid s,t \in \mathbb{R}\}$. (This is called a PARAMETRIC REPRESENTATION).
What geometric object do K and K_h represent? How are they related?
Answer the same questions for the equation $x+2y+3z=6$.

9. If (S) is the system of equations $y+5z=0$; $x-z=0$; then its solution set K is ______. The geometric object it represents is ______. What is its parametric representation?

10. Let $f = a_0 + a_1x + a_2x^2$ be a polynomial with a_0, a_1, $a_2 \in \mathbb{R}$. Find a_i such that $f(1)=1$, $f(2)=2$ and $f(3)=5$. Is this choice unique?
Find all cubic polynomials satisfying the same conditions.

11. The equation of the plane passing through $(2, -5, -1)$, $(0, 4, 6)$ and $(-3, 7, 1)$ is ______.
How would you write it parametrically?

12. The equations of the line passing through $(3, -2, 4)$ and $(-5, 7, 1)$ are ______.
What would be its parametric form?

13. Let $e_1 = (1, 0, 0)^t$, $e_2 = (0, 1, 0)^t$ and $e_3 = (0, 0, 1)^t$. Give a system of linear equations having
(i) e_1 as its only solution. (ii) e_1, e_2 and e_3 as solutions. (iii) e_1 and e_2 as solutions, but not e_3.

14. Let $u = (-1, 2, 2)^t$, $v = (2, a, -5)^t \in \mathbb{R}^3$. Find a such that u and v perpendicular to each other. For what values of a, if any, are u and v parallel to each other?

15. Let $b = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, A be a 3×3 invertible matrix such that $A^{-1} = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$. Find the solution set of $Ax = b$. (Try to answer this question without computing A).

16. Consider the following vectors in $\mathbb{R}^4$:
$v_1 = (-2, 0, 1, 3)^t$, $v_2 = (1, 1, 1, 1)^t$, $v_3 = (2, 0, 0, 0)^t$, $w_1 = (2, 3, 5, 9)^t$ and $w_2 = (3, 3, 5, 9)^t$.
Do there exist $c_1, c_2, c_3 \in \mathbb{R}$ such that (a) $w_1 = c_1v_1 + c_2v_2 + c_3v_3$? (b) $w_2 = c_1v_1 + c_2v_2 + c_3v_3$?

17. Let $v_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, w_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ be vectors in $\mathbb{R}^3$. A solution to the equation $c_1v_1 + c_2v_2 + c_3w_1 + c_4w_2 = 0$ is $(c_1, c_2, c_3, c_4) =$ ______. Can you find all possible solutions?
Identify a matrix A, such that $(c_1, c_2, c_3, c_4)^t$ is a solution to the system $A\mathbf{x} = 0$.

18. Let $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \\ 2 & 5 \end{pmatrix}$. Is there a matrix B such that $BA = I_2$? Is there matrix C such that $AC = I_3$?
Find all solutions to $Ax = 0$.

19. Let (S) be a system of linear equations with solution set K, (S_h) be the associated homogeneous system, with solution set K_h. If u_1, $u_2 \in K$, v_1, $v_2 \in K_h$, $c \in \mathbb{R}$, what all can you conclude about $u_1 + u_2$, $v_1 + v_2$, $u_1 - u_2$, $v_1 - v_2$, cu_1 and cv_1?

20. Let $a, b \in \mathbb{R}$. Is the solution set $K \subset \mathbb{R}$ of the system $ax = b$ always a singleton set? If yes, prove it. If not, write down (with justification) what are the possibilities for K, and identify conditions when it is a singelton set.

21. Let $a, b, c \in \mathbb{R}$. Is the solution set $K \subset \mathbb{R}^2$ of the system $ax + by = c$ always a line? Always infinite? If yes, prove the statements. If not, write down (with proof) what are the possibilities for K, and identify conditions when it is a line.

22. When does the equation $a_1x_1 + \cdots + a_nx_n = b$ have a solution (i.e., when is its solution set K non-empty)?
Assuming that there is a solution, identify exactly when there is a unique solution, i.e., if K is non-empty, when is it a singleton?
Identify exactly when K is an infinite set.