Real Analysis (MA 403)
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Problem set 1

1. If $a \geq 0$ and if $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$ then prove that $|x| \leq a$ if and only if $-a \leq x \leq a$. (Here we are in real number system and $<$ and $\leq$ have their usual meaning.)

2. Assuming $\sqrt{2}$ to be irrational prove that $\sqrt{2} + \sqrt{3}$ and $\sqrt{6}$ are irrational numbers.

3. Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$. When does $\alpha = \beta$ hold ?

4. Let A be a nonempty subset or real numbers which is bounded below. Let $-A = \{-x : x \in A\}$. Prove that
$$\inf A = -\sup -A.$$

5. Assuming the fact that for every real number $x > 0$ and every natural number n there is a unique real number y such that $y^n = x$ prove that
$$(ab)^{1/n} = a^{1/n} b^{1/n} \,,$$
where a, b are positive real numbers and n is a positive integer. (When $y^n = x$, the number y is written $x^{1/n}$.)

6. With the hypothesis of Question 5 fix $b > 1$.
   (i) If m, n, p, q are integers, $n > 0, q > 0$ and $r = m/n = p/q$, prove that $(b^m)^{1/n} = (b^p)^{1/q}$. Hence it makes sense to define $b^r = (b^m)^{1/n}$. (If m is a negative integer $b^m = \frac{1}{b^{-m}}$.)
   (ii) Prove that $b^{r+s} = b^r b^s$ if r and s are rational numbers.
   (iii) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that
   $$b^r = \sup B(r) \,,$$
   when r is rational. Hence it makes sense to define
   $$b^x = \sup B(x) \,,$$
   for every real number x.
   (iv) Prove that $b^{x+y} = b^x b^y$ when x and y are real numbers.

7. Prove that in a field the axioms for addition imply the following statements.
   (i) If $x + y = x + z$ then $y = z$.
   (ii) If $x + y = x$ then $y = 0$.
   (iii) If $x + y = 0$ then $y = -x$.
   (iv) $-(-x) = x$.

8. Prove that in a field the axioms for multiplication imply the following statements.

(i) If $x \neq 0$ and $xy = xz$ then $y = z$.

(ii) If $x \neq 0$ and $xy = x$ then $y = 0$.

(iii) If $x \neq 0$ and $xy = 1$ then $y = 1/x$.

(iv) If $x \neq 0$ then $1/(1/x) = x$.

9. Prove that in a field the following statements are true.

   (i) If $0x = 0$.

   (ii) If $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.

   (iii) $(-x)y = -(xy) = x(-y)$.

   (iv) $(-x)(-y) = xy$.

10. Consider the ordered field $(\mathbb{R}, <, +, .)$. Prove that if $x, y \in \mathbb{R}$ and $x < y$ then there exists a rational number q such that $x < p < y$.

11. If S is an infinite set prove that S contains a proper countable subset.

12. Let f be a real-valued function defined on the closed interval $[0, 1]$. Suppose there is a positive number M having the following property: for every finite number of points $x_1, x_2, \ldots, x_n$ in $[0, 1]$,

$$|f(x_1) + f(x_2) + \ldots + f(x_n)| \leq M\,.$$

    Let $S = \{x \in [0, 1] \,:\, f(x) \neq 0\}$. Prove that S is at most countable.