CS 228 : Logic in Computer Science

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- Q4: Can you write an algorithm to answer Q1 and Q2?
- Q5: Can you “prove” any factually correct statement using the chosen logic *L*?
- Q6: How is logic *L* used in computer science?
- Q7: What are the techniques needed to go about these questions?

Some Members of the mini-zoo

- Propositional Logic
- First Order Logic
- Monadic Second Order Logic
- Propositional Dynamic Logic
- Linear Temporal Logic
- Computational Tree Logic

More if time permits!

References

- To start with, the text book of Huth and Ryan : Logic for CS.
- As we go ahead, lecture notes/monographs/other text books.
- Classes : Slot 4. Tutorial: To discuss.

Propositional Logic

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- Parantheses as required
- Example : $[p \wedge (q \vee r)] \rightarrow [\neg r \wedge p]$
- $\neg$ binds tighter than $\vee, \wedge$, which bind tighter than $\rightarrow$. In the absence of parantheses, $p \rightarrow q \rightarrow r$ is read as $p \rightarrow (q \rightarrow r)$

Natural Deduction

- If it rains, Alice is outside and does not have any raingear with her, she will get wet. $\varphi = (R \land \mathit{AliceOut} \land \neg \mathit{RG}) \rightarrow \mathit{AliceWet}$

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- It is raining, and Alice is outside, and is not wet. $\psi = (R \wedge \mathit{AliceOut} \wedge \neg \mathit{AliceWet})$

Natural Deduction

- If it rains, Alice is outside and does not have any raingear with her, she will get wet. $\varphi = (R \wedge \mathit{AliceOut} \wedge \neg \mathit{RG}) \rightarrow \mathit{AliceWet}$
- It is raining, and Alice is outside, and is not wet. $\psi = (R \wedge \mathit{AliceOut} \wedge \neg \mathit{AliceWet})$
- So, Alice has her rain gear with her. RG
- Thus, $\chi = \varphi \wedge \psi \rightarrow \mathit{RG}$. You can deduce RG from $\varphi \wedge \psi$.
- Is χ valid? Is χ satisfiable?

Two Examples of Natural Deduction

Solve Sudoku

Consider the following kid's version of Sudoku.

	2	4	
1			3
4			2
	1	3	

Rules:

- Each row must contain all numbers 1-4
- Each column must contain all numbers 1-4
- Each 2×2 block must contain all numbers 1-4
- No cell contains 2 or more numbers

Encoding as Propositional Satisfiability

- Proposition $P(i, j, n)$ is true when cell (i, j) has number n

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- Each row must contain all 4 numbers
  - Row 1: $[P(1,1,1) \vee P(1,2,1) \vee P(1,3,1) \vee P(1,4,1)] \wedge$
    $[P(1,1,2) \vee P(1,2,2) \vee P(1,3,2) \vee P(1,4,2)] \wedge$
    $[P(1,1,3) \vee P(1,2,3) \vee P(1,3,3) \vee P(1,4,3)] \wedge$
    $[P(1,1,4) \vee P(1,2,4) \vee P(1,3,4) \vee P(1,4,4)]$

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    $[P(1,1,4) \vee P(1,2,4) \vee P(1,3,4) \vee P(1,4,4)]$
  - Row 2: $[P(2,1,1) \vee \ldots$
  - Row 3: $[P(3,1,1) \vee \ldots$
  - Row 4: $[P(4,1,1) \vee \ldots$

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  $[P(1,1,4) \vee P(2,1,4) \vee P(3,1,4) \vee P(4,1,4)]$
- Column 2: $[P(1,2,1) \vee \ldots$
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- Column 4: $[P(1,4,1) \vee \ldots$

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$$[P(1,1,1) \vee P(1,2,1) \vee P(2,1,1) \vee P(2,2,1)] \wedge$$
$$[P(1,1,2) \vee P(1,2,2) \vee P(2,1,2) \vee P(2,2,2)] \wedge$$
$$[P(1,1,3) \vee P(1,2,3) \vee P(2,1,3) \vee P(2,2,3)] \wedge$$
$$[P(1,1,4) \vee P(1,2,4) \vee P(2,1,4) \vee P(2,2,4)]$$

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- Upper right block contains all numbers 1-4:

$$[P(1,3,1) \vee P(1,4,1) \vee P(2,3,1) \vee P(2,4,1)] \wedge \dots$$

- Lower left block contains all numbers 1-4:

$$[P(3,1,1) \vee P(3,2,1) \vee P(4,1,1) \vee P(4,2,1)] \wedge \dots$$

- Lower right block contains all numbers 1-4:

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Encoding as Propositional Satisfiability

No cell contains 2 or more numbers

- For cell(1,1):

$$P(1,1,1) \rightarrow [\neg P(1,1,2) \wedge \neg P(1,1,3) \wedge \neg P(1,1,4)] \wedge$$

$$P(1,1,2) \rightarrow [\neg P(1,1,1) \wedge \neg P(1,1,3) \wedge \neg P(1,1,4)] \wedge$$

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- Similar for other cells

Encoding as Propositional Satisfiability

Encoding Initial Configuration:

$$P(1,2,2) \wedge P(1,3,4) \wedge P(2,1,1) \wedge P(2,4,3) \wedge$$

$$P(3,1,4) \wedge P(3,4,2) \wedge P(4,2,1) \wedge P(4,3,3)$$

Solving Sodoku

To solve the puzzle, just conjunct all the above formulae and find a satisfiable truth assignment!

Gold Rush

(**Box1**) *The gold is not here*

(**Box2**) *The gold is not here*

(**Box3**) *The gold is in Box 2*

Only one message is true; the other two are false. Which box has the gold?

Solve Gold Rush

- Propositions $M1, M2, M3$ representing messages in boxes 1,2,3
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  - $\neg(M1 \wedge M2 \wedge M3)$, $M1 \vee M2 \vee M3$,
  - $(\neg M1 \wedge \neg M2) \vee (\neg M1 \wedge \neg M3) \vee (\neg M2 \wedge \neg M3)$

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  - Conjunct all these, and call the formula φ.
  - Is there a unique satisfiable assignment for φ?
  - For example, is $M1 = \mathit{true}$ a part of the satisfiable assignment?

A Proof Engine for Natural Deduction

- If it rains, Alice is outside and does not have any raingear with her, she will get wet. $\varphi = (\mathit{R} \wedge \mathit{AliceOut} \wedge \neg \mathit{RG}) \rightarrow \mathit{AliceWet}$
- It is raining, and Alice is outside, and is not wet. $\psi = (\mathit{R} \wedge \mathit{AliceOut} \wedge \neg \mathit{AliceWet})$
- So, Alice has her rain gear with her. *RG*
- Thus, $\chi = \varphi \wedge \psi \rightarrow \mathit{RG}$.
- Given φ, ψ, can we “prove” *RG*?

A Proof Engine

- Given a formula φ in propositional logic, how to "prove" φ if φ is valid?
- What is a proof engine?
- Show that this proof engine is sound and complete
  - Completeness: Any fact that can be captured using propositional logic can be proved by the proof engine
  - Soundness: Any formula that is proved to be valid by the proof engine is indeed valid

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- $\varphi_1, \ldots, \varphi_n \vdash \psi$: This is called a sequent. $\varphi_1, \ldots, \varphi_n$ are premises, and ψ, the conclusion.
- Given $\varphi_1, \ldots, \varphi_n$, we can deduce or prove ψ. What was the sequent in the Alice example?

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- For example, $\neg p \to q, q \to r, \neg r \vdash p$ is a sequent. How do you prove this?
- Proof rules to be carefully chosen, for instance you should not end up proving something like $p \wedge q \vdash \neg q$

The Rules of the Proof Engine

Rules for Natural Deduction

The and introduction rule denoted $\wedge i$

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi}$$

Rules for Natural Deduction

The and elimination rule denoted $\wedge e_1$

$$\frac{\varphi \wedge \psi}{\varphi}$$

The and elimination rule denoted $\wedge e_2$

$$\frac{\varphi \wedge \psi}{\psi}$$

A first proof using $\wedge i$, $\wedge e_1$, $\wedge e_2$

- Show that $p \wedge q, r \vdash q \wedge r$

1.	$p \wedge q$	premise
2.		

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2.	r	premise
3.	q	$\wedge e_2$ 1
4.		

A first proof using $\wedge i, \wedge e_1, \wedge e_2$

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1.	$p \wedge q$	premise
2.	r	premise
3.	q	$\wedge e_2$ 1
4.	$q \wedge r$	$\wedge i$ 3,2

Rules for Natural Deduction

The rule of double negation elimination $\neg\neg e$

$$\frac{\neg\neg\varphi}{\varphi}$$

The rule of double negation introduction $\neg\neg i$

$$\frac{\varphi}{\neg\neg\varphi}$$

Rules for Natural Deduction

The implies elimination rule or Modus Ponens MP

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}$$

Another Proof

- Show that $p, p \to q, p \to (q \to \neg\neg r) \vdash r$

1.	$p \to (q \to \neg\neg r)$	premise
2.		

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2.	$p \to q$	premise
3.	p	premise
4.	$q \to \neg\neg r$	MP 1,3
5.		

Another Proof

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1.	$p \rightarrow (q \rightarrow \neg\neg r)$	premise
2.	$p \rightarrow q$	premise
3.	p	premise
4.	$q \rightarrow \neg\neg r$	MP 1,3
5.	q	MP 2,3
6.		

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1.	$p \rightarrow (q \rightarrow \neg\neg r)$	premise
2.	$p \rightarrow q$	premise
3.	p	premise
4.	$q \rightarrow \neg\neg r$	MP 1,3
5.	q	MP 2,3
6.	$\neg\neg r$	MP 4,5
7.		

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- Show that $p, p \to q, p \to (q \to \neg\neg r) \vdash r$

1.	$p \to (q \to \neg\neg r)$	premise
2.	$p \to q$	premise
3.	p	premise
4.	$q \to \neg\neg r$	MP 1,3
5.	q	MP 2,3
6.	$\neg\neg r$	MP 4,5
7.	r	$\neg\neg e$ 6