INDIAN INSTITUTE OF TECHNOLOGY BOMBAY

Department of Mathematics

SI 427 (Probability Theory)

Tutorial Sheet-IV

All random objects are in $(\Omega, \mathcal{F}, P)$ unless told otherwise.

1. Let $A, B, C \in \mathcal{F}$ be such that A, C are independent and B, C are independent. Is $I_A + I_B$ independent of I_C? Justify your answer.

2. Let X, Y be independent random variables. Also let $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Show that $f \circ X$ and $f \circ Y$ are independent.

3. Find all random variables which are independent of itself.

4. Let X, Y be random variables such that
$$P\{X \leq x, Y \leq y\} \ = \ P\{X \leq x\} \, P\{Y \leq y\}, \text{ for all } x, y \in \mathbb{R}\,.$$
show that
$$P\{X < x, Y < y\} \ = \ P\{X < x\} \, P\{Y < y\}, \text{ for all } x, y \in \mathbb{R}\,.$$

5. Give an example of random variables X, Y, Z which are pairwise independent but not independent.

6. Give an example of random variables X_1, X_2, X_3 and X_4 which are independent.

7. Give an example of $\{A_n\}$ such that
$$\liminf_{n \to \infty} A_n \neq \limsup_{n \to \infty} A_n\,.$$

8. Let $\{A_n\}$, $\{B_n\}$ be sequences of subsets of a non empty set Ω. Show that
$$\limsup_{n \to \infty} A_n \bigcup \limsup_{n \to \infty} B_n \ = \ \limsup_{n \to \infty} (A_n \cup B_n)\,.$$

9. Let $\{A_n\}$, $\{B_n\}$ be sequences of subsets of a non empty set Ω. Show that
$$\limsup_{n \to \infty} (A_n \cap B_n) \subseteq \limsup_{n \to \infty} A_n \bigcap \limsup_{n \to \infty} B_n\,.$$

10. Give an example of sequences of sets $\{A_n\}$, $\{B_n\}$ such that
$$\limsup_{n\to\infty}(A_n \cap B_n) \neq \limsup_{n\to\infty} A_n \bigcap \limsup_{n\to\infty} B_n \,.$$

11. Let $\{A_n\}$ be a sequence of subsets of a non empty set Ω. Show that
$$\limsup_{n\to\infty}(A_n \cap A_{n+1}^c) \;=\; \limsup_{n\to\infty} A_n \setminus \liminf_{n\to\infty} A_n \,.$$

12. Let $\{A_n\}$, $\{B_n\}$ be sequences of subsets of a non empty set Ω such that
$$\limsup_{n\to\infty} A_n \;=\; \liminf_{n\to\infty} A_n$$
and
$$\limsup_{n\to\infty} B_n \;=\; \liminf_{n\to\infty} B_n \,.$$
Show that
$$\limsup_{n\to\infty}(A_n \cup B_n) \;=\; \liminf_{n\to\infty}(A_n \cup B_n) \;=\; \limsup_{n\to\infty} A_n \cup \limsup_{n\to\infty} B_n \,.$$

13. Let $\{A_n\}$, $\{B_n\}$ be sequences of subsets from $\mathcal{F}$ such that
$$P\Big(\limsup_{n\to\infty} A_n\Big) \;=\; P\Big(\limsup_{n\to\infty} B_n\Big) \;=\; 1\,.$$
Prove or disprove that
$$P\Big(\limsup_{n\to\infty} A_n \cap B_n\Big) \;=\; 1\,.$$

14. Let $\{A_n\}$ be a sequence of events from $\mathcal{F}$ satisfying the following.
(i)The events $A_{i_1}, A_{i_2}, \ldots, A_{i_n}$ are independent if $|i_k - i_j| \geq 2$, $n \geq 2$.
(ii)
$$\sum_{n=1}^{\infty} P(A_n) \;=\infty\,.$$
Show that
$$P\Big(\limsup_{n\to\infty} A_n\Big) \;=\; 1\,.$$

15. Let $\{A_n\}$ be a sequence of events from $\mathcal{F}$ such that
$$\sum_{n=1}^{\infty} P(A \cap A_n) \;=\; \infty \text{ for all } A \in \mathcal{F} \text{ with } P(A) > 0\,.$$
Show that
$$P\Big(\limsup_{n\to\infty} A_n\Big) \;=\; 1\,.$$