8 Correlated equilibria

Chapter summary

This chapter introduces the concept of *correlated equilibrium* in strategic-form games. The motivation for this concept is that players' choices of pure strategies may be correlated due to the fact that they use the same random events in deciding which pure strategy to play. Consider an extended game that includes an observer who recommends to each player a pure strategy that he should play. The vector of recommended strategies is chosen by the observer according to a probability distribution over the set of pure strategy vectors, which is commonly known among the players. This probability distribution is called a correlated equilibrium if the strategy vector in which all players follow the observer's recommendations is a Nash equilibrium of the extended game.

The probability distribution over the set of strategy vectors induced by any Nash equilibrium is a correlated equilibrium. The set of correlated equilibria is a polytope that can be calculated as a solution of a set of linear equations.

In Chapters 4, 5, and 7 we considered strategic-form games and studied the concept of equilibrium. One of the underlying assumptions of those chapters was that the choices made by the players were independent. In practice, however, the choices of players may well depend on factors outside the game, and therefore these choices may be correlated. Players can even coordinate their actions among themselves.

A good example of such correlation is the invention of the traffic light: when a motorist arrives at an intersection, he needs to decide whether to cross it, or alternatively to give right of way to motorists approaching the intersection from different directions. If the motorist were to use a mixed strategy in this situation, that would be tantamount to tossing a coin and entering the intersection based on the outcome of the coin toss. If two motorists approaching an intersection simultaneously use this mixed strategy, there is a positive probability that both of them will try to cross the intersection at the same time – which means that there is a positive probability that a traffic accident will ensue. In some states in the United States, there is an equilibrium rule that requires motorists to stop before entering an intersection, and to give right of way to whoever arrived at the intersection earlier. The invention of the traffic light provided a different solution: the traffic light informs each motorist which pure strategy to play, at any given time. The traffic light thus correlates the pure strategies of the players. Note that the traffic light does not, strictly speaking, choose a pure strategy for the motorist; it recommends a pure strategy. It is in the interest of each motorist to follow that recommendation, even if we suppose there are no traffic police watching, no cameras, and no possible court summons awaiting a motorist who disregards the traffic light's recommendation.

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The concept of correlated equilibrium, which is an equilibrium in a game where players' strategies may be correlated, is the subject of this chapter. As we will show, correlation can be beneficial to the players.

8.1 Examples

Example 8.1 **Battle of the Sexes** Consider the Battle of the Sexes game, as depicted in Figure 8.1 (see also Example 4.21 on page 98). The game has three equilibria (verify that this is true):

1. (F, F): the payoff is $(2, 1)$.
2. (C, C): the payoff is $(1, 2)$.
3. $([\frac{2}{3}(F), \frac{1}{3}(C)], [\frac{1}{3}(F), \frac{2}{3}(C)])$: in this equilibrium, every player uses mixed strategies. The row player plays $[\frac{2}{3}(F), \frac{1}{3}(C)]$ – he chooses F with probability two-thirds, and T with probability one-third. The column player plays $[\frac{1}{3}(F), \frac{2}{3}(C)]$. The expected payoff in this case is $(\frac{2}{3}, \frac{2}{3})$.

		Player II F	Player II C
Player I	F	2, 1	0, 0
Player I	C	0, 0	1, 2

Figure 8.1 The Battle of the Sexes

The first two equilibria are not symmetric; in each one, one of the players yields to the preference of the other player. The third equilibrium, in contrast, is symmetric and gives the same payoff to both players, but that payoff is less than 1, the lower payoff in each of the two pure equilibria.

The players can correlate their actions in the following way. They can toss a fair coin. If the coin comes up heads, they play (F, F), and if it comes up tails, they play (C, C). The expected payoff is then $(1\frac{1}{2}, 1\frac{1}{2})$. Since (F, F) and (C, C) are equilibria, the process we have just described is an equilibrium in an extended game, in which the players can toss a coin and choose their strategies in accordance with the result of the coin toss: after the coin toss, neither player can profit by unilaterally deviating from the strategy recommended by the result of the coin toss. ◀

The reasoning behind this example is as follows: if we enable the players to conduct a joint (public) lottery, prior to playing the game, they can receive as an equilibrium payoff every convex combination of the equilibrium payoffs of the original game. That is, if we denote by V the set of equilibrium payoffs in the original game, every payoff in the convex hull of V is an equilibrium payoff in the extended game in which the players can conduct a joint lottery prior to playing the game.

The question naturally arises whether it is possible to create a correlation mechanism, such that the set of equilibrium payoffs in the game that corresponds to this mechanism includes payoffs that are not in the convex hull of V. The following examples show that the answer to this question is affirmative.

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Example 8.2 Consider the three-player game depicted in Figure 8.2, in which Player I chooses the row (T or B), Player II chooses the column (L or R), and Player III chooses the matrix (l, c, or r).

l

	L	R
T	0, 1, 3	0, 0, 0
B	1, 1, 1	1, 0, 0

c

	L	R
T	2, 2, 2	0, 0, 0
B	2, 2, 0	2, 2, 2

r

	L	R
T	0, 1, 0	0, 0, 0
B	1, 1, 1	1, 0, 3

Figure 8.2 The payoff matrix of Example 8.2

We will show that the only equilibrium payoff of this game is (1, 1, 1), but there exists a correlation mechanism that induces an equilibrium payoff of (2, 2, 2). In other words, every player gains by using the correlation mechanism. Since (1, 1, 1) is the only equilibrium payoff of the original game, the vector (2, 2, 2) is clearly outside the convex hull of the original game's set of equilibrium payoffs.

Step 1: The only equilibrium payoff is (1, 1, 1).
We will show that every equilibrium is of the form $(B, L, [\alpha(l), (1-\alpha)(r)])$, for some $0 \le \alpha \le 1$. (Check that the payoff given by any strategy vector of this form is (1, 1, 1), and that each of these strategy vectors is indeed an equilibrium.) To this end we eliminate strictly dominated strategies (see definition 4.6 on page 86). We first establish that at every equilibrium there is a positive probability that the pair of pure strategies chosen by Players II and III will not be (L, c). To see this, when Player II plays L, strategy l strictly dominates strategy c for Player III, so it cannot be the case that at equilibrium Player II plays L with probability 1 and Player III plays c with probability 1.

We next show that at every equilibrium, Player I plays strategy B. To see this, note that the pure strategy B weakly dominates T (for Player I). In addition, if the probability of (L, c) is not 1, strategy B yields a strictly higher payoff to Player I than strategy T. It follows that the pure strategy T cannot be played at equilibrium.

Finally, we show that at every equilibrium Player II plays strategy L and Player III plays either l or r. To see this, note that after eliminating strategy T, strategy r strictly dominates c for Player III, hence Player III does not play c at equilibrium, and after eliminating strategy c, strategy L strictly dominates R for Player II. We are left with only two entries in the matrix: (B, L, l) and (B, L, r), both of which yield the same payoff, (1, 1, 1). Thus any convex combination of these two matrix entries is an equilibrium, and there are no other equilibria.

Step 2: The construction of a correlation mechanism leading to the payoff (2, 2, 2).
Consider the following mechanism that the players can implement:

- Players I and II toss a fair coin, but do not reveal the result of the coin toss to Player III.
- Players I and II play either (T, L) or (B, R), depending on the result of the coin toss.
- Player III chooses strategy c.

Under the implementation of this mechanism, the action vectors that are chosen (with equal probability) are (T, L, c) and (B, R, c), hence the payoff is (2, 2, 2).

Finally, we check that no player has a unilateral deviation that improves his payoff. Recall that because the payoff function is multilinear, it suffices to check whether or not this is true for a deviation to a pure strategy. If Player III deviates and chooses l or r, his expected payoff is $\frac{1}{2} \times 3 + \frac{1}{2} \times 0 = 1\frac{1}{2}$, and hence he cannot gain from deviating. Players I and II cannot profit from deviating, because whatever the outcome of the coin toss is, the payoff to each of them is 2, the maximal payoff in the game. ◄

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For the mechanism described in Figure 8.2 to be an equilibrium, it is necessary that Players I and II know that Player III does not know the result of the coin toss. In other words, while every payoff in the convex hull of the set of equilibrium payoffs can be attained by a *public* lottery, to attain a payoff outside the convex hull of V it is necessary to conduct a lottery that is not public, in which case different players receive different partial information regarding the result of the lottery.

Example 8.3 **The game of "Chicken"** Consider the two-player non-zero-sum game depicted in Figure 8.3.

		Player II	
		L	R
Player I	T	6, 6	2, 7
	B	7, 2	0, 0

Figure 8.3 The game of "Chicken"

The following background story usually accompanies this game. Two drivers are racing directly towards each other down a single-lane road. The first to lose his nerve and swerve off the road before the cars collide is the loser of the game, the "chicken." In this case, the utility of the loser is 2, and the utility of the winner is 7. If neither player drives off the road, the cars collide, both players are injured, and they each have a utility of 0. If they both swerve off the road simultaneously, the utility of each of them is 6.

The game has three equilibria (check that this is true):

1. The players play (T, R). The payoff is $(2, 7)$.
2. The players play (B, L). The payoff is $(7, 2)$.
3. The players play $\left(\left[\frac{2}{3}(T), \frac{1}{3}(B)\right], \left[\frac{2}{3}(L), \frac{1}{3}(R)\right]\right)$. The payoff is $(4\frac{2}{3}, 4\frac{2}{3})$.

Consider the following mechanism, in which an outside observer gives each player a recommendation regarding which action to take, but the observer does not reveal to either player what recommendation the other player has received. The observer chooses between three action vectors, (T, L), (T, R), and (B, L), with equal probability (see Figure 8.4).

	L	R
T	$\frac{1}{3}$	$\frac{1}{3}$
B	$\frac{1}{3}$	0

Figure 8.4 The distribution that the observer uses to choose the action vector

After conducting a lottery to choose one of the three action vectors, the observer provides Player I with a recommendation to play the first coordinate of the vector that was chosen, and he provides Player II with a recommendation to play the second coordinate of that vector. For example, if the action vector (T, L) has been chosen, the observer recommends T to Player I and L to Player II. If Player I receives a recommendation to play T, the conditional probability that Player II has received a recommendation to play L is $\frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{3}} = \frac{1}{2}$, which is also the conditional probability that he has received a recommendation to play R. In contrast, if Player I receives a recommendation to play B, he knows that Player II has received L as his recommended action.

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We now show that neither player can profit by a unilateral deviation from the recommendation received from the observer. As we stated above, if the recommendation to Player I is to play T, Player II has received a recommendation to play L with probability $\frac{1}{2}$, and a recommendation to play R with probability $\frac{1}{2}$. Player I's expected payoff if he follows the recommended strategy of T is therefore $\frac{1}{2} \times 6 + \frac{1}{2} \times 2 = 4$, while his expected payoff if he deviates and plays B is $\frac{1}{2} \times 7 + \frac{1}{2} \times 0 = 3\frac{1}{2}$. In this case, Player I cannot profit by unilaterally deviating from the recommended strategy. If the recommendation to Player I is to play B, then with certainty Player II has received a recommendation to play L. The payoff to Player I in this case is then 7 if he plays the recommended strategy B, and only 6 if he deviates to T. Again, in this case, Player I cannot profit by deviating from the recommended strategy. By symmetry, Player II similarly cannot profit by not following his recommended strategy. It follows that this mechanism induces an equilibrium in the extended game with an outside observer. The expected equilibrium payoff is

$$\tfrac{1}{3}(6, 6) + \tfrac{1}{3}(7, 2) + \tfrac{1}{3}(2, 7) = (5, 5), \tag{8.1}$$

which lies outside the convex hull of the three equilibrium payoffs of the original game, $(2, 7)$, $(7, 2)$, and $(4\frac{2}{3}, 4\frac{2}{3})$. (A quick way to become convinced of this is to notice that the sum of the payoffs in the vector $(5, 5)$ is 10, while the sum of the payoffs in the three equilbrium payoffs is either 9 or $9\frac{1}{3}$, both of which are less than 10.) ◀

Examples 8.1 and 8.3 show that the way to attain a high payoffs for both players is to avoid the "worst" payoff (0, 0). This cannot be accomplished if the players implement independent mixed strategies; it requires correlating the players' actions. We have made the following assumptions regarding the extended game:

- The game includes an observer, who recommends strategies to the players.
- The observer chooses his recommendations probabilistically, based on a probability distribution that is commonly known to the players.
- The recommendations are private, with each player knowing only the recommendation addressed to him or her.
- The mechanism is common knowledge[1] among the players: each player knows that this mechanism is being used, each player knows that the other players know that this mechanism is being used, each player knows that the other players know that the other players know that this mechanism is being used, and so forth.

As we will see in the formal definition of correlated equilibria in the next section, the fact that the recommendations are privately provided to each player does not exclude the possibility that the recommendations may be public (in which case the recommendations to each player are identical), or that a player can deduce which recommendations the other players have received given the recommendation he has received, as we saw in Example 8.3: in the correlated equilibrium of the game of "Chicken," if Player I receives the recommendation to play B, he can deduce that Player II's recommended strategy is L.

1 See Definition 4.9 (page 87). The formal definition of common knowledge is Definition 9.2 on page 321.

8.2 Definition and properties of correlated equilibrium

The concept of correlated equilibrium formally captures the sort of correlation that we saw in Example 8.3. In that example, we added an outside observer to the strategic game G who chooses a pure strategy vector, and recommends that each player play his part in this vector. We will now present the formal definition of this concept. To distinguish between the strategies in the strategic-form game G and the strategies in the game that includes the observer we will call pure strategies in G *actions*.

Let $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a strategic-form game, where N is the set of players, S_i is the set of actions of player $i \in N$, and $u_i : S \to \mathbb{R}$ is player i's payoff function, where $S = \times_{i\in N} S_i$ is the set of strategy vectors. For every probability distribution p over the set S, define a game $\Gamma^*(p)$ as follows:

- An outside observer probabilistically chooses an action vector from S, according to the probability distribution p.
- To each player $i \in N$ the observer reveals s_i, but not s_{-i}. In other words, the observer reveals to player i his coordinate in the action vector that was chosen; to be interpreted as the recommended action to play.
- Each player i chooses an action $s'_i \in S_i$ (s'_i may be different from the action revealed by the observer).
- The payoff of each player i is $u_i(s'_1, \dots, s'_n)$.

This describes an extensive-form game with information sets.

A presentation of the extensive-form game corresponding to the game of "Chicken," with the addition of the correlation mechanism described above, is shown in Figure 8.5. Near every chance move in the figure, we have noted the respective recommendation of the observer for that choice. The actions T_1 and T_2 in the figure correspond to the action T in the strategic-form game: T_1 represents the possible action T when the observer's recommendation is T; T_2 represents the possible action T when the observer's recommendation is B. Actions B_1 and B_2 similarly correspond to action B, and so forth.

The information revealed by the observer to player i will be termed a *recommendation*: the observer recommends that player i play the action s_i in the original game. The player is not obligated to follow the recommendation he receives, and is free to play a different action (or to use a mixed action, i.e., to conduct a lottery in order to choose between several actions). A player's pure strategy in an extensive-form game with information sets is a function that maps each of that player's information sets to a possible action. Since every information set in the game $\Gamma^*(p)$ is associated with a recommendation of the observer, and the set of possible actions at each information set of player i is S_i, we obtain the following definition of a pure strategy in $\Gamma^*(p)$.

Definition 8.4 *A (pure)* strategy *of player i in the game $\Gamma^*(p)$ is a function $\tau_i : S_i \to S_i$ mapping every recommendation s_i of the observer to an action $\tau_i(s_i) \in S_i$.*

Suppose the observer has recommended that player i play the action s_i. This fact enables player i to deduce the following regarding the recommendations that the other players

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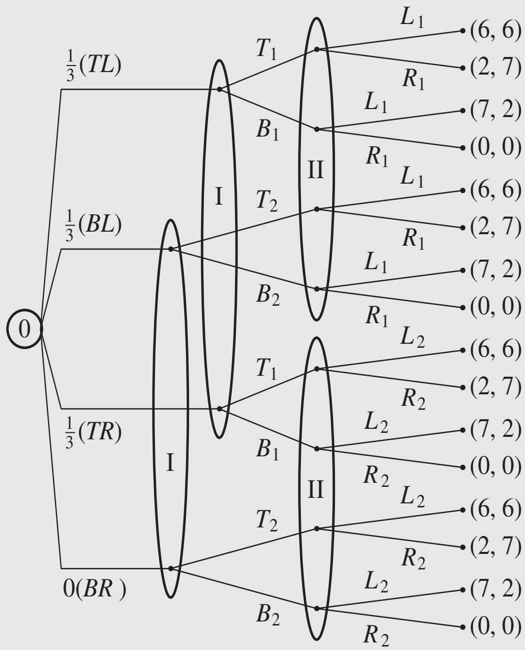

Figure 8.5 The game of "Chicken," for the probability distribution p given in Figure 8.1, in extensive form

have received: since the probability that player i receives recommendation s_i is

$$\sum_{t_{-i}\in S_{-i}} p(s_i, t_{-i}), \tag{8.2}$$

the conditional probability that the observer has chosen the action vector $s = (s_i, s_{-i})$ is

$$p(s_{-i} \mid s_i) = \frac{p(s_i, s_{-i})}{\sum_{t_{-i}\in S_{-i}} p(s_i, t_{-i})}. \tag{8.3}$$

The conditional probability in Equation (8.3) is defined when the denominator is positive, i.e., when the probability that player i receives recommendation s_i is positive. When $\sum_{t_{-i}\in S_{-i}} p(s_i, t_{-i}) = 0$, the probability that player i receives recommendation s_i is zero, and in this case the conditional probability $p(s_{-i} \mid s_i)$ is undefined.

One strategy available to player i is to follow the observer's recommendation. For each player $i \in N$, define a strategy τ_i^* by:

$$\tau_i^*(s_i) = s_i, \quad \forall s_i \in S_i. \tag{8.4}$$

Is the pure strategy vector $\tau^* = (\tau_1^*, \ldots, \tau_n^*)$, in which each player i follows the observer's recommendation, an equilibrium? As might be expected, the answer to that question depends on the probability distribution p, as specified in the following theorem.

Theorem 8.5 *The strategy vector τ^* is an equilibrium of the game $\Gamma^*(p)$ if and only if*

$$\sum_{s_{-i}\in S_{-i}} p(s_i, s_{-i})u_i(s_i, s_{-i}) \geq \sum_{s_{-i}\in S_{-i}} p(s_i, s_{-i})u_i(s_i', s_{-i}), \quad \forall i, \forall s_i, s_i' \in S_i. \tag{8.5}$$

Proof: The strategy vector τ^*, in which each player follows the recommendation he receives, is an equilibrium if and only if no player i can profit by deviating to a strategy that differs from his recommendation. Equation (8.3) implies that the payoff that player i has under the action vector τ^*, when his recommended action is s_i, is

$$\sum_{s_{-i}\in S_{-i}} \left(\frac{p(s_i, s_{-i})}{\sum_{t_{-i}\in S_{-i}} p(s_i, t_{-i})} \times u_i(s_i, s_{-i}) \right). \tag{8.6}$$

Suppose player i decides to deviate and play action s_i' instead of s_i, while the other players follow the recommendations (i.e., play τ^*_{-i}). The distribution of the actions of the other players is given by the conditional probability in Equation (8.3), and therefore player i's expected payoff if he deviates to action s_i' is

$$\sum_{s_{-i}\in S_{-i}} \left(\frac{p(s_i, s_{-i})}{\sum_{t_{-i}\in S_{-i}} p(s_i, t_{-i})} \times u_i(s_i', s_{-i}) \right). \tag{8.7}$$

This means that the strategy vector τ^* is an equilibrium if and only if for each player $i \in N$, for each action $s_i \in S_i$ for which $\sum_{s_{-i}\in S_{-i}} p(s_i, s_{-i}) > 0$, and for each action $s_i' \in S_i$:

$$\begin{aligned}
&\sum_{s_{-i}\in S_{-i}} \left(\frac{p(s_i, s_{-i})}{\sum_{t_{-i}\in S_{-i}} p(s_i, t_{-i})} \times u_i(s_i, s_{-i}) \right) \\
&\quad \geq \sum_{s_{-i}\in S_{-i}} \left(\frac{p(s_i, s_{-i})}{\sum_{t_{-i}\in S_{-i}} p(s_i, t_{-i})} \times u_i(s_i', s_{-i}) \right).
\end{aligned} \tag{8.8}$$

When the denominator of this equation is positive, we can reduce both sides of the inequality to obtain Equation (8.5). When $\sum_{t_{-i}\in S_{-i}} p(s_i, t_{-i}) = 0$, Equation (8.5) holds true with equality: since $(p(s_i, t_{-i}))_{t_{-i}\in S_{-i}}$ are nonnegative numbers, it is necessarily the case that $p(s_i, t_{-i}) = 0$ for each $t_{-i} \in S_{-i}$, and hence both sides of the inequality in Equation (8.5) are identically zero. □

We can now define the concept of correlated equilibrium.

Definition 8.6 *A probability distribution p over the set of action vectors S is called a* correlated equilibrium *if the strategy vector τ^* is a Nash equilibrium of the game $\Gamma^*(p)$. In other words, for every player $i \in N$:*

$$\sum_{s_{-i}\in S_{-i}} p(s_i, s_{-i})u_i(s_i, s_{-i}) \geq \sum_{s_{-i}\in S_{-i}} p(s_i, s_{-i})u_i(s_i', s_{-i}), \quad \forall s_i, s_i' \in S_i. \tag{8.9}$$

Every strategy vector σ induces a probability distribution p_σ over the set of action vectors S,

$$p_\sigma(s_1, \ldots, s_n) := \sigma_1(s_1) \times \sigma_2(s_2) \times \cdots \times \sigma_n(s_n). \tag{8.10}$$

Under a Nash equilibrium σ^* the actions that each player chooses with positive probability are only those that give him maximal payoffs given that the other players implement the strategy vector σ^*_{-i},

$$u_i(s_i, \sigma^*_{-i}) \geq u_i(s'_i, \sigma^*_{-i}), \quad \forall s_i \in \mathrm{supp}(\sigma^*_i), \forall s'_i \in S_i. \tag{8.11}$$

This leads to the following theorem (whose proof is left to the reader in Exercise 8.2).

Theorem 8.7 *For every Nash equilibrium σ^*, the probability distribution p_{σ^*} is a correlated equilibrium.*

As Theorem 8.7 indicates, correlated equilibrium is in a sense an extension of the Nash equilibrium concept. When we relate to a Nash equilibrium σ^* as a correlated equilibrium we mean the probability distribution p_{σ^*} given by Equation (8.10). For example, the convex hull of the set of Nash equilibria is the set

$$\mathrm{conv}\{p_{\sigma^*} : \sigma^* \text{ is a Nash equilibrium}\} \subseteq \Delta(S). \tag{8.12}$$

Since every finite normal-form game has a Nash equilibrium, we deduce the following corollary.

Corollary 8.8 *Every finite strategic-form game has a correlated equilibrium.*

Theorem 8.9 *The set of correlated equilibria of a finite game is convex and compact.*

Proof: Recall that a half-space in $\mathbb{R}^m$ is defined by a vector $\alpha \in \mathbb{R}^m$ and a real number $\beta \in \mathbb{R}$, by the following equation:

$$H^+(\alpha, \beta) := \left\{ x \in \mathbb{R}^m : \sum_{i=1}^{m} \alpha_i x_i \geq \beta \right\}. \tag{8.13}$$

A half-space is a convex and closed set. Equation (8.9) implies that the set of correlated equilibria of a game is given by the intersection of a finite number of half-spaces. Since an intersection of convex and closed spaces is convex and closed, the set of correlated equilibria is convex and closed. Since the set of correlated equilibria is a subset of the set of probability distributions S, it is a bounded set, and so we conclude that it is a convex and compact set. □

Remark 8.10 *A* polytope *in $\mathbb{R}^d$ is the convex hull of a finite number of points in $\mathbb{R}^d$. The minimal set of points satisfying the condition that the polytope is its convex hull is called the set of* extreme points *of the polytope. (For the definition of the extreme points of a general set see Definition 23.2 on page 917.) Every bounded set defined by the intersection of a finite number of half-spaces is a polytope, from which it follows that the set of correlated equilibria of a game is a polytope. Since there exist efficient algorithms for finding the extreme points of a polytope (such as the simplex algorithm), it is relatively* *easy to compute correlated equilibria, in contrast to computing Nash equilibria, which is computationally hard. (See, for example, Gilboa and Zemel [1989].)* ♦

Example 8.1 (*Continued*) Consider again the Battle of the Sexes, which is the two-player game shown in Figure 8.6.

Player II

		F	C
Player I	F	1, 2	0, 0
	C	0, 0	2, 1

Figure 8.6 Battle of the Sexes

We will compute the correlated equilibria of this game. Denote a probability distribution over the action vectors by $p = [\alpha(F, F), \beta(F, C), \gamma(C, F), \delta(C, C)]$. Figure 8.7 depicts this distribution graphically.

Player II

		F	C
Player I	F	α	β
	C	γ	δ

Figure 8.7 Graphic representation of the probability distribution p

For a probability distribution $p = [\alpha(F, F), \beta(F, C), \gamma(C, F), \delta(C, C)]$ to be a correlated equilibrium, the following inequalities must be satisfied (see Equation (8.9)):

$$\alpha u_1(F, F) + \beta u_1(F, C) \geq \alpha u_1(C, F) + \beta u_1(C, C), \tag{8.14}$$
$$\gamma u_1(C, F) + \delta u_1(C, C) \geq \gamma u_1(F, F) + \delta u_1(F, C), \tag{8.15}$$
$$\alpha u_2(F, F) + \gamma u_2(C, F) \geq \alpha u_2(F, C) + \gamma u_2(C, C), \tag{8.16}$$
$$\beta u_2(F, C) + \delta u_2(C, C) \geq \beta u_2(F, F) + \delta u_2(C, F), \tag{8.17}$$
$$\alpha + \beta + \gamma + \delta = 1, \tag{8.18}$$
$$\alpha, \beta, \gamma, \delta \geq 0. \tag{8.19}$$

Entering the values of the game matrix into these equations, we get

$$2\alpha \geq \beta, \qquad \delta \geq 2\gamma, \qquad 2\delta \geq \beta, \qquad \alpha \geq 2\gamma. \tag{8.20}$$

In other words, both α and δ must be greater than 2γ and $\frac{\beta}{2}$. The set of possible payoffs of the game (the triangle formed by the coordinates (0, 0), (1, 2), and (2, 1)) is shown in Figure 8.8, with the game's three Nash equilibrium payoffs ((1, 2), (2, 1), $(\frac{2}{3}, \frac{2}{3})$) along with the set of correlated equilibrium payoffs (the dark triangle formed by (1, 2), (2, 1), and $(\frac{2}{3}, \frac{2}{3})$). In this case, the set of correlated equilibrium payoffs is the convex hull of the Nash equilibrium payoffs.

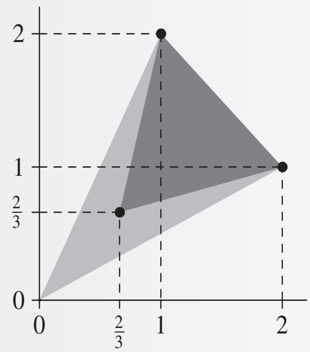

Figure 8.8 The set of possible payoffs, the set of correlated equilibrium payoffs, and the Nash equilibrium payoffs of the game in Figure 8.1

Example 8.3 (*Continued*) The payoff matrix of the game in this example is shown in Figure 8.9.

Player II

Player I		L	R
	T	6, 6	2, 7
	B	7, 2	0, 0

Figure 8.9 The game of "Chicken"

A probability distribution over the set of action vectors is again denoted by $p = [\alpha(T, L), \beta(T, R), \gamma(B, L), \delta(B, R)]$ (see Figure 8.10).

Player II

Player I		L	R
	T	α	β
	B	γ	δ

Figure 8.10 Graphic depiction of the probability distribution p

For the probability distribution p to be a correlated equilibrium (see Equation (8.9)), the following inequalities must be satisfied:

$$6\alpha + 2\beta \geq 7\alpha, \quad 7\gamma \geq 6\gamma + 2\delta, \quad 6\alpha + 2\gamma \geq 7\alpha, \quad 7\beta \geq 6\beta + 2\delta. \tag{8.21}$$

The equations imply that both β and γ must be greater than 2δ and $\frac{\alpha}{2}$. The set of possible payoffs of the game (the rhombus formed by the coordinates (0, 0), (7, 2), (2, 7), and (6, 6)) is shown

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in Figure 8.11, along with the game's three Nash equilibrium payoffs ((7, 2), (2, 7), and $(4\frac{2}{3}, 4\frac{2}{3})$), with their convex hull (the dark triangle) and the set of correlated equilibrium payoffs (the dark-grey rhombus formed by $(3\frac{2}{5}, 3\frac{2}{5})$, (7, 2), (2, 7), and $(5\frac{1}{4}, 5\frac{1}{4})$).

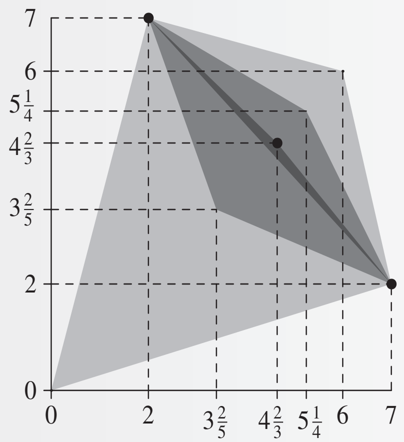

Figure 8.11 The set of possible payoffs (light rhombus), the Nash equilibrium payoffs, the convex hull of the Nash equilibrium payoffs (dark triangle), and the correlated equilibrium payoffs (dark rhombus) of the game in Figure 8.3 ◀

Example 8.11 Consider the two-player game depicted in Figure 8.12, which resembles the Battle of the Sexes, but is not symmetric between the players. The game has three equilibria: (T, L), (B, R), and $[\frac{3}{5}(T), \frac{2}{5}(B)], [\frac{2}{3}(L), \frac{1}{3}(R)]$.

Player II

Player I		L	R
	T	1, 2	0, 0
	B	0, 0	2, 3

Figure 8.12 The payoff matrix of the game in Example 8.11

We will compute the correlated equilibria of the game. For a probability distribution over the set of action vectors $p = [\alpha(T, L), \beta(T, R), \gamma(B, L), \delta(B, R)]$ to be a correlated equilibrium, the following inequalities must be satisfied (see Equation (8.9)):

$$\alpha \geq 2\beta, \tag{8.22}$$

$$2\delta \geq \gamma, \tag{8.23}$$

$$2\alpha \geq 3\gamma, \tag{8.24}$$

$$3\delta \geq 2\beta, \tag{8.25}$$

$$\alpha + \beta + \gamma + \delta = 1, \tag{8.26}$$

$$\alpha, \beta, \gamma, \delta \geq 0. \tag{8.27}$$

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Note that the constraint $\alpha + \beta + \gamma + \delta = 1$ implies that:

$$2\delta \geq \gamma \quad \Longleftrightarrow \quad \alpha + \beta + \tfrac{3}{2}\gamma \leq 1, \tag{8.28}$$

$$3\delta \geq 2\beta \quad \Longleftrightarrow \quad \alpha + \tfrac{5}{3}\beta + \gamma \leq 1. \tag{8.29}$$

Figure 8.13 shows the sets defined by each of the four inequalities in Equations (8.22)–(8.25), along with the constraints that α, β, and γ be nonnegative, and that $\delta = 1 - \alpha - \beta - \gamma \geq 0$. The intersection of these four sets is the set of correlated equilibria. To find this set, we will seek out its extreme points. The set of all the correlated equilibria is the subset of $\mathbb{R}^3$ defined by the intersection of eight half-spaces (Equations (8.22)–(8.25), along with the constraints that $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$, and $\alpha + \beta + \gamma \leq 1$). Note that in this case, if $\alpha + \frac{5}{3}\beta + \gamma \leq 1$ then $\alpha + \beta + \gamma \leq 1$, and hence there is no need explicitly to require that $\alpha + \beta + \gamma \leq 1$. In addition, if we look at the hyperplanes defining these half-spaces, we notice that three of them intersect at one point (there are $\binom{7}{3} = 35$ such intersection points, some of them identical to each other). Each such intersection point satisfying all the constraints is an extreme point.

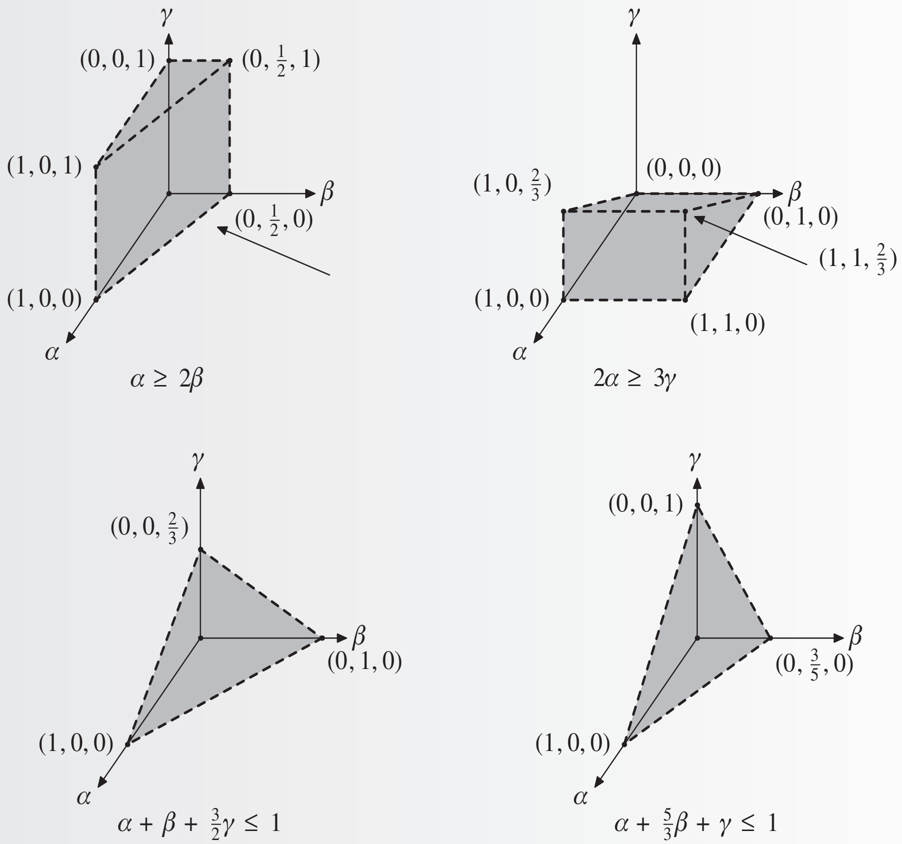

Figure 8.13 The sets defined by the inequalities in Equations (8.22)–(8.25)

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A simple, yet tedious, calculation reveals that the set of all the correlated equilibria has five extreme points (recall that $\delta = 1 - \alpha - \beta - \gamma$):

$$(\alpha, \beta, \gamma) = (0, 0, 0), \tag{8.30}$$
$$(\alpha, \beta, \gamma) = (1, 0, 0), \tag{8.31}$$
$$(\alpha, \beta, \gamma) = \left(\tfrac{6}{11}, \tfrac{3}{11}, 0\right), \tag{8.32}$$
$$(\alpha, \beta, \gamma) = \left(\tfrac{1}{2}, 0, \tfrac{1}{3}\right), \tag{8.33}$$
$$(\alpha, \beta, \gamma) = \left(\tfrac{2}{5}, \tfrac{1}{5}, \tfrac{4}{15}\right). \tag{8.34}$$

It follows that the set of all the correlated equilibria is the smallest convex set containing these five points (see Figure 8.14). The three equilibrium points are: (T, L) corresponding to the point $(1, 0, 0)$, (B, R) corresponding to the point $(0, 0, 0)$, and $([\frac{3}{5}(T), \frac{2}{5}(B)], [\frac{2}{3}(L), \frac{1}{3}(R)])$ corresponding to the point $(\frac{2}{5}, \frac{1}{5}, \frac{4}{15})$. In general, the Nash equilibria need not correspond to extreme points of the set of correlated equilibria.

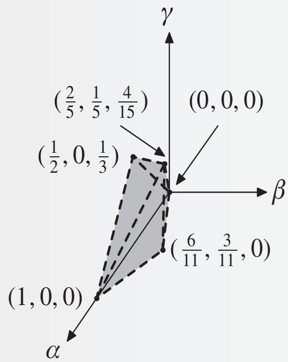

Figure 8.14 The set of correlated equilibria of the game in Example 8.11 ◀

8.3 Remarks

This chapter is based on Aumann [1974], a major work in which the concept of correlated equilibrium was developed. The game in Exercise 8.21 was suggested by Yannick Viossat, in response to a question posed by Ehud Lehrer.

8.4 Exercises

8.1 What is the set of possible payoffs of the following game (the Battle of the Sexes game; see Example 8.1 on page 301) if:

(a) the players are permitted to decide, and commit to, the mixed strategies that each player will use;

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(b) the players are permitted to make use of a public lottery that chooses a strategy vector and instructs each player which pure strategy to choose.

		Player II	
		F	C
Player I	F	1, 2	0, 0
	C	0, 0	2, 1

8.2 Prove Theorem 8.7 on page 308: for every Nash equilibrium σ^* in a strategic-form game, the probability distribution p_{σ^*} that σ^* induces on the set of action vectors S is a correlated equilibrium.

8.3 The set of all probability distributions p_σ over the set of action vectors S that are induced by Nash equilibria σ is

$$W := \{p_\sigma : \sigma \text{ is a Nash equilibrium}\} \subseteq \Delta(S). \tag{8.35}$$

Prove that any point in the convex hull of W is a correlated equilibrium.

8.4 Prove that in every correlated equilibrium, the payoff to each player i is at least his maxmin value in mixed strategies.

$$\underline{v}_i = \max_{\sigma_i \in \Sigma_i} \min_{\sigma_{-i} \in \Sigma_{-i}} U_i(\sigma_i, \sigma_{-i}). \tag{8.36}$$

8.5 Given a strategic-form game $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$, write out a linear program whose set of solution vectors is the set of correlated equilibria of the game.

8.6 Let $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ and $\widehat{G} = (N, (S_i)_{i\in N}, (\widehat{u}_i)_{i\in N})$ be strategically equivalent games (see Definition 5.34 on page 174). What is the relation between the set of correlated equilibria of G and the set of correlated equilibria of $\widehat{G}$? What is the relation between the set of correlated equilibrium payoffs of G and the set of correlated equilibrium payoffs of $\widehat{G}$? Justify your answers.

8.7 Let $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a game in strategic form, and let $\widehat{G}$ be the game derived from G by a process of iterated elimination of strictly dominated strategies. What is the relation between the set of correlated equilibria of G and the set of correlated equilibria of $\widehat{G}$? Justify your answer.

8.8 Find the correlated equilibrium that maximizes the sum of the players' payoffs in Example 8.1 (page 301), and in Example 8.3 (page 303).

8.9 Find a correlated equilibrium whose expected payoff is $(\frac{40}{9}, \frac{36}{9})$ in the game of "Chicken" (Example 8.3 on page 303).

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8.10 In the following game, compute all the Nash equilibria, and find a correlated equilibrium that is not in the convex hull of the Nash equilibria.

Player II

Player I		L	C	R
	T	0, 0	2, 4	4, 2
	M	4, 2	0, 0	2, 4
	B	2, 4	4, 2	0, 0

8.11 Repeat Exercise 8.10 for the following game.

Player II

Player I		L	R
	T	8, 8	4, 9
	B	9, 4	1, 1

8.12 In this exercise, we present an extension of the correlated equilibrium concept. Let $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a strategic-form game, and $(M_i)_{i\in N}$ be finite sets of messages. For each probability distribution q over the product set $M := \times_{i\in N} M_i$ define a game $\Gamma^*_M(q)$ as follows:

- An outside observer chooses a vector of messages $m = (m_i)_{i\in N} \in M$ probabilistically, using the probability distribution q.
- The observer reveals m_i to player $i \in N$, but not m_{-i}. In other words, the observer reveals to player i his coordinate in the vector of messages that has been chosen.
- Each player i chooses an action $s_i \in S_i$.
- Each player i has payoff $u_i(s_1, \ldots, s_n)$.

This is a generalization of the game $\Gamma^*(p)$, which is $\Gamma^*_M(q)$ for the case $M_i = S_i$ for every player i and $q = p$. Answer the following questions:

(a) What is the set of behavior strategies of player i in the game $\Gamma^*_M(q)$?
(b) Show that every vector of behavior strategies induces a probability distribution over the set of action vectors $S = \times_{i\in N} S_i$.
(c) Prove that at every Nash equilibrium of $\Gamma^*_M(q)$, the probability distribution induced on the set of pure strategy vectors S is a correlated equilibrium.

8.13 Show that there exists a unique correlated equilibrium in the following game, in which $a, b, c, d \in (-\frac{1}{4}, \frac{1}{4})$. Find this correlated equilibrium. What is the limit of the correlated equilibrium payoff as a, b, c, and d approach 0?

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		Player II	
		L	R
Player I	T	$1, 0$	$c, 1+d$
	B	$0, 1$	$1+a, b$

8.14 Let s_i be a strictly dominated action of player i. Is there a correlated equilibrium under which s_i is chosen with positive probability, i.e., $\sum_{s_{-i}\in S_{-i}} p(s_i, s_{-i}) > 0$? Justify your answer.

8.15 Prove that in a two-player zero-sum game, every correlated equilibrium payoff to Player I is the value of the game in mixed strategies.

8.16 In this and the following exercise, we will show that the result of Exercise 8.15 partially obtains for equilibrium strategies. Prove that if p is a correlated equilibrium of a two-player zero-sum game, then for every recommendation s_{I} that Player I receives with positive probability, the conditional probability $(p(s_{\mathrm{II}} \mid s_{\mathrm{I}}))_{s_{\mathrm{II}}\in S_{\mathrm{II}}}$ is an optimal strategy for Player II. Deduce from this that the marginal distribution of p over the set of actions of each of the players is an optimal strategy for that player.

8.17 In the following two-player zero-sum game, find the value of the game, the optimal strategies of the two players, and the set of correlated equilibria. Does every correlated equilibrium lie in the convex hull of the product distributions that correspond to pairs of optimal strategies?

		Player II		
		L	C	R
	T	0	0	1
Player I	M	1	1	0
	B	1	1	0

8.18 Prove that the set-valued function that assigns to every game its set of correlated equilibria is an upper semi-continuous mapping.[2] In other words, let $(G^k)_{k\in\mathbb{N}}$ be a sequence of games $(G^k) = (N, (S_i)_{i\in N}, (u_i^k)_{k\in\mathbb{N}})$, all of which share the same set of players N and the same sets of actions $(S_i)_{i\in N}$. Further suppose that for each player i, the sequence of payoff functions $(u_i^k)_{k\in\mathbb{N}}$ converges to a limit u_i,

$$\lim_{k\to\infty} u_i^k(s) = u_i(s), \quad \forall s \in S. \tag{8.37}$$

2 A set-valued function $F : X \to Y$ between two topological spaces is called *upper semi-continuous* if its graph $\mathrm{Graph}(F) = \{(x, y) : y \in F(x)\}$ is a closed set in the product space $X \times Y$.

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Suppose that for each $k \in \mathbb{N}$ the probability distribution p^k is a correlated equilibrium of G^k, and the sequence $(p^k)_{k\in\mathbb{N}}$ converges to a limit p,

$$\lim_{k\to\infty} p^k(s) = p(s), \quad \forall s \in S. \tag{8.38}$$

Prove that p is a correlated equilibrium of the game $(N, (S_i)_{i\in N}, (u_i)_{i\in N})$.

8.19 A Nash equilibrium $\sigma^* = (\sigma_i^*)_{i\in N}$ is called a *strict equilibrium* if for every player i and every action $s_i \in S_i$ satisfying $\sigma_i^*(s_i) = 0$,

$$u_i(\sigma^*) > u_i(s_i, \sigma_{-i}^*). \tag{8.39}$$

In words, if player i deviates by playing an action that is not in the support of σ_i^* then he loses. A correlated equilibrium p is called a *strict correlated equilibrium* if the strategy vector τ^* is a strict equilibrium in the game $\Gamma^*(p)$.

Answer the following questions:

(a) Does every game in strategic form have a strict correlated equilibrium? If your answer is yes, provide a proof. If your answer is no, provide a counterexample.
(b) Find all the strict correlated equilibria of the following two-player game.

Player II

Player I		L	R
	T	4, 2	3, 4
	B	5, 1	0, 0

8.20 Harry (Player I) is to choose between the payoff vector (2, 1) and playing the following game, as a row player, against Harriet (Player II), the column player:

Player II

Player I		L	R
	T	0, 0	1, 3
	B	4, 2	0, 0

(a) What are Harry's pure strategies in this game? What are Harriet's?
(b) What are the Nash equilibria of the game?
(c) What is the set of correlated equilibria of the game?

8.21 Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be positive numbers. Consider the two-player strategic game with the following payoff matrix.

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Player II

Player I

x_1, y_1	$0, 0$	$0, 0$	...	$0, y_1$
$0, 0$	x_2, y_2	$0, 0$	...	$0, y_2$
$0, 0$	$0, 0$	x_3, y_3	...	$0, y_3$
...	...	...	...	...
$x_1, 0$	$x_2, 0$	$x_3, 0$	...	x_n, y_n

(a) Find the set of Nash equilibria of this game.
(b) Prove that the set of correlated equilibria of this game is the convex hull of the set of Nash equilibria.

8.22 Let A and B be two sets in $\mathbb{R}^2$ satisfying:

- $A \subseteq B$;
- A is a union of a finite number of rectangles;
- B is the convex hull of a finite number of points.

Prove that there is a two-player strategic-form game satisfying the property that its set of Nash equilibrium payoffs is A, and its set of correlated equilibrium payoffs is B.
Hint: Make use of the game in Exercise 8.21, along with Exercise 5.44 in Chapter 5.

8.23 Let x, y, a, b be positive numbers. Consider the two-player strategic-form game with the following payoff matrix, in which Player I chooses a row, and Player II chooses a column.

$x-1, y+1$	$0, 0$	$0, 0$	$x+1,\ y-1$	$0, y$
$x+1, y-1$	$x-1, y+1$	$0, 0$	$0, 0$	$0, y$
$0, 0$	$x+1,\ y-1$	$x-1,\ y+1$	$0, 0$	$0, y$
$0, 0$	$0, 0$	$x+1,\ y-1$	$x-1, y+1$	$0, y$
$x, 0$	$x, 0$	$x, 0$	$x, 0$	a, b

(a) Find the set of Nash equilibria of this game.
(b) Find the set of correlated equilibria of this game.

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