20 The nucleolus

Chapter summary

This chapter is devoted to the study of the *nucleolus*, which is, like the Shapley value, a single-point solution concept for coalitional games. The notion that underlies the nucleolus is that of *excess*: the excess of a coalition at a vector x in $\mathbb{R}^N$ is the difference between the worth of the coalition and the total amount that the members of the coalition receive according to x. When the excess is positive, the members of the coalition are not content with the total amount that they together receive at x, which is less than the worth of the coalition. Each vector x in $\mathbb{R}^N$ corresponds to a vector of 2^N excesses of all coalitions. The nucleolus of a coalitional game relative to a set of vectors in $\mathbb{R}^N$ consists of the vectors in that set whose vector of excesses are minimal in the *lexicographic order*. It is proved that the nucleolus relative to any compact set is nonempty and if the set is also convex, then the nucleolus relative to that set consists of a single vector.

The nucleolus of the game is the nucleolus relative to the set of imputations, that is, the set of efficient and individually rational vectors. The *prenucleolus* of a coalitional game is its nucleolus relative to the set of preimputations, that is, the set of all efficient vectors. Both the nucleolus and the prenucleolus are defined for any coalition structure.

The prenucleolus of a coalitional game is characterized in Section 20.5 in terms of balanced collections of coalitions. This characterization is used to prove that the prenucleolus is a consistent solution concept; that is, it satisfies the Davis–Maschler reduced game property. In Section 20.7 we show that for a weighted majority game, the nucleolus is the unique representation of the game that satisfies some desirable properties.

In Section 20.8 the nucleolus is applied to bankruptcy problems. The Babylonian Talmud (a Jewish text that records rabbinic discussions held between the second and fifth centuries AD) presents a solution concept for bankruptcy problems suggested by Rabbi Nathan. We prove that for each bankruptcy problem one can define a coalitional game whose nucleolus coincides with the Rabbi Nathan solution of the bankruptcy problem.

In this chapter we present the nucleolus, a solution concept for coalitional games that, like the Shapley value, is a single-valued solution that exists for every coalitional game. The nucleolus was first defined in Schmeidler [1969]. We consider coalitional games $(N; v)$

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with a set of players $N = \{1, 2, \ldots, n\}$. As before, for every vector $x \in \mathbb{R}^N$ we denote

$$\begin{cases} x(S) := \sum_{i \in S} x_i, & \emptyset \subset S \subseteq N, \\ x(\emptyset) := 0. \end{cases} \tag{20.1}$$

20.1 Definition of the nucleolus

Definition 20.1 *For every vector $x \in \mathbb{R}^N$, and every coalition $S \subseteq N$,*

$$e(S, x) := v(S) - x(S) \tag{20.2}$$

is called the excess *of coalition S at x.*

When x_i is a payoff to player i, the excess $e(S, x)$ measures how dissatisfied the members of S are with the vector x. If the excess is positive, the members of S are not satisfied with x, because they could band together to form S, obtain $v(S)$, and then divide that sum in such a way that each member of S receives more than he receives under x. The smaller the excess, the less the members of S are dissatisfied. When the excess is negative, the members of S, as a coalition, are satisfied with x, and the more negative the excess is, the more satisfied they are, because collectively they are receiving at x more than they could receive working together as a coalition.

Recall that the set of imputations of the coalitional game $(N; v)$ is the set $X(N; v)$ defined by

$$X(N; v) := \{x \in \mathbb{R}^N : x(N) = v(N), \quad x_i \geq v(i) \quad \forall i \in N\}. \tag{20.3}$$

This is the set of vectors that are *efficient*, that is, satisfy $x(N) = v(N)$, and *individually rational*, that is, satisfy $x_i \geq v(i)$ for every player $i \in N$. The core is the set of imputations satisfying in addition *coalitional rationality*, that is, $x(S) \geq v(S)$ for every coalition $S \subseteq N$. This can be expressed using the notion of excess, as follows:

$$\mathcal{C}(N; v) = \{x \in \mathbb{R}^N : x(N) = v(N), \quad e(S, x) \leq 0 \quad \forall S \subseteq N\}. \tag{20.4}$$

When the core is empty, then given any imputation, there will be at least one coalition with positive excess. In that case, we may wish to minimize the excesses as much as possible. The nucleolus proposes a way of achieving this end. We proceed to define it.

Given a vector $x \in \mathbb{R}^N$, we compute the excess of all the coalitions at x, and we write them in decreasing order from left to right,

$$\theta(x) = (e(S_1, x), e(S_2, x), \ldots, e(S_{2^n}, x)), \tag{20.5}$$

where $\{S_1, S_2, \ldots, S_{2^n}\}$ are all the coalitions, indexed such that

$$e(S_1, x) \geq e(S_2, x) \geq \ldots \geq e(S_{2^n}, x). \tag{20.6}$$

Two elements in this sequence are $e(\emptyset, x)$ and $e(N, x)$. By definition $v(\emptyset) = 0$ for every coalitional game $(N; v)$, and $x(\emptyset) = 0$ for every $x \in \mathbb{R}^N$. It follows that $e(\emptyset, x) = 0$ for every coalitional game $(N; v)$ and every vector $x \in \mathbb{R}^N$. Also, $e(N, x) = 0$ for every efficient vector $x \in \mathbb{R}^N$.

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Note that this indexing of the coalitions is determined only up to equality of excesses. If for example $e(S_k, x) = e(S_l, x)$, then swapping S_k and S_l will not change the vector $\theta(x)$. When passing to another vector $\theta(y)$, a different letter must be used to denote the coalitions,

$$\theta(y) = (e(R_1, y), e(R_2, y), \dots, e(R_{2^n}, y)), \tag{20.7}$$

where the coalitions $(R_k)_{k=1}^{2^n}$ are ordered in decreasing excess order, and the ordering is a permutation of the previous ordering. To avoid relabeling the coalitions every time we consider a different imputation, denote the k-th coordinate of $\theta(x)$ by $\theta_k(x)$, without writing explicitly the corresponding coalition at the k-th coordinate, and write the vector $\theta(x)$ as

$$\theta(x) = (\theta_1(x), \theta_2(x), \dots, \theta_{2^n}(x)), \tag{20.8}$$

where

$$\theta_1(x) \geq \theta_2(x) \geq \dots \geq \theta_{2^n}(x). \tag{20.9}$$

We have thus defined 2^n functions $\theta_k : \mathbb{R}^N \to \mathbb{R}$ for $k \in \{1, 2, \dots, 2^n\}$, where $\theta_k(x)$ is the k-th coordinate of $\theta(x)$. Note that there may be cases in which $\theta(x) = \theta(y)$, even though $x \neq y$ (explain why).

To minimize the excesses we compare any two vectors $\theta(x)$ and $\theta(y)$ lexicographically.

Definition 20.2 *Let $a = (a_1, a_2, \dots, a_d)$ and $b = (b_1, b_2, \dots, b_d)$ be two vectors in $\mathbb{R}^d$. Then $a \succsim_L b$ if either $a = b$, or there exists an integer k, $1 \leq k \leq d$, such that $a_k > b_k$, and $a_i = b_i$ for every $1 \leq i < k$. This order relation is termed the* lexicographic order.

As usual, the strong order $\succ_L$ derived from $\succsim_L$ is $a \succ_L b$ if $a \succsim_L b$ and $b \not\succsim_L a$. It follows that $a \succ_L b$ if $a \neq b$ and the first coordinate at which a differs from b is greater in a than in b. Moreover, $a \approx_L b$ if and only if $a = b$.

The lexicographic relation is reflexive, transitive, and complete, but not continuous (Exercise 20.4): there exists a sequence $(a^n)_{n \in \mathbb{N}}$ of vectors in $\mathbb{R}^d$ converging to a (in the Euclidean metric) and a vector b such that $a^n \prec_L b$ for all $n \in \mathbb{N}$, but $a \succ_L b$.

Example 20.3 **The gloves game** (see Example 17.5, page 690) Consider the following three-player coalitional game, with the coalitional function given by

$$v(1) = v(2) = v(3) = v(1, 2) = 0, \quad v(1, 3) = v(2, 3) = v(1, 2, 3) = 1. \tag{20.10}$$

Computing the excesses with respect to the vectors $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $y = (0, 0, 1)$, $z = (\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$, and $w = (-\frac{1}{3}, \frac{1}{3}, 1)$ yields the table shown in Figure 20.1

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S	$e(S,x)$	$e(S,y)$	$e(S,z)$	$e(S,w)$
$\emptyset$	0	0	0	0
$\{1\}$	$-\frac{1}{3}$	0	$-\frac{1}{6}$	$\frac{1}{3}$
$\{2\}$	$-\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{1}{3}$
$\{3\}$	$-\frac{1}{3}$	-1	$-\frac{1}{2}$	-1
$\{1,2\}$	$-\frac{2}{3}$	0	$-\frac{1}{2}$	0
$\{1,3\}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$
$\{2,3\}$	$\frac{1}{3}$	0	$\frac{1}{6}$	$-\frac{1}{3}$
$\{1,2,3\}$	0	0	0	0

Figure 20.1 The excesses of all coalitions at x, y, z, and w

Writing the excesses in decreasing order gives

$$\theta(x) = \left(\tfrac{1}{3}, \tfrac{1}{3}, 0, 0, -\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{2}{3}\right), \tag{20.11}$$
$$\theta(y) = (0, 0, 0, 0, 0, 0, 0, -1), \tag{20.12}$$
$$\theta(z) = \left(\tfrac{1}{3}, \tfrac{1}{6}, 0, 0, -\tfrac{1}{6}, -\tfrac{1}{3}, -\tfrac{1}{2}, -\tfrac{1}{2}\right), \tag{20.13}$$
$$\theta(w) = \left(\tfrac{1}{3}, \tfrac{1}{3}, 0, 0, 0, -\tfrac{1}{3}, -\tfrac{1}{3}, -1\right). \tag{20.14}$$

Therefore,

$$\theta(y) \prec_L \theta(z) \prec_L \theta(x) \prec_L \theta(w). \tag{20.15}$$

It can be proved that the vector y satisfies $\theta(y) \prec_L \theta(u)$ for every imputation $u \in X(N; v)$ (Exercise 20.7). ◀

Definition 20.4 *Let $(N; v)$ be a coalitional game and let $K \subseteq \mathbb{R}^N$. The* nucleolus *of the game $(N; v)$ relative to K is the set*

$$\mathcal{N}(N; v; K) := \{x \in K : \theta(x) \precsim_L \theta(y), \quad \forall y \in K\}. \tag{20.16}$$

The nucleolus emerges as the solution that an arbitrator would recommend for dividing the quantity $v(N)$ among the players if he uses the following procedure: he first seeks the imputations x such that $\theta_1(x)$ is minimal; since $\theta_1(x)$ measures the magnitude of the maximal complaint against x, the arbitrator wishes to minimize it. After accomplishing this, from among the vectors minimizing the maximal complaint, the arbitrator turns to seeking those vectors that minimize the second-highest complaint, $\theta_2(x)$, and so on.

If the set K is not compact, there may not be a vector x in K that is minimal in the lexicographic order, and therefore the nucleolus may be empty (Exercise 20.8). We will later prove the converse direction; if K is compact, the nucleolus relative to K is not empty (Corollary 20.10). In this section we study the nucleolus relative to some sets K that are not necessarily compact, such as closed sets that may not be bounded. The first set we will consider is the set of imputations $X(N; v)$, defined in Equation (20.3). If we drop the requirement of individual rationality in Equation (20.3), we get the set $X^0(N; v)$

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of *preimputations*:

$$X^0(N;v) := \{x \in \mathbb{R}^N : x(N) = v(N)\}. \tag{20.17}$$

This is an unbounded set that contains the set of imputations: $X(N;v) \subset X^0(N;v)$. We can similarly define the sets of imputations and preimputations for any coalitional structure $\mathcal{B}$:

$$X(\mathcal{B};v) := \{x \in \mathbb{R}^N : x(B) = v(B) \quad \forall B \in \mathcal{B}, \quad x_i \geq v(i) \quad \forall i \in N\}, \tag{20.18}$$
$$X^0(\mathcal{B};v) := \{x \in \mathbb{R}^N : x(B) = v(B) \quad \forall B \in \mathcal{B}\}. \tag{20.19}$$

Remark 20.5 *Note that every vector x in either $X(\mathcal{B};v)$ or $X^0(\mathcal{B};v)$ satisfies $x(N) = \sum_{B\in\mathcal{B}} v(B)$: the sum of the coordinates is the same for all vectors in both sets.* ♦

Definition 20.6 *The* nucleolus *of a coalitional game $(N;v)$ is the nucleolus relative to the set of imputations $X(N;v)$, that is, $\mathcal{N}(N;v;X(N;v))$. The* prenucleolus *of a game is the nucleolus relative to the set of preimputations $X^0(N;v)$; i.e., it is the set $\mathcal{N}(N;v;X^0(N;v))$. For every coalitional structure $\mathcal{B}$, the* nucleolus *for $\mathcal{B}$ is the nucleolus relative to the set of imputations $X(\mathcal{B};v)$, i.e., $\mathcal{N}(N;v;X(\mathcal{B};v))$, and the* prenucleolus *for $\mathcal{B}$ is the nucleolus relative to the set of preimputations $X^0(\mathcal{B};v)$, i.e., $\mathcal{N}(N;v;X^0(\mathcal{B};v))$.*

For the sake of simplifying the notation, we will henceforth write $\mathcal{N}(N;v)$ in place of $\mathcal{N}(N;v;X(N;v))$, and call that the nucleolus of the game $(N;v)$. Similarly, we will write $\mathcal{PN}(N;v)$ in place of $\mathcal{N}(N;v;X^0(N;v))$ and call that the prenucleolus of the game $(N;v)$. For every coalitional structure $\mathcal{B}$, we will write $\mathcal{N}(N;v;\mathcal{B})$ and $\mathcal{PN}(N;v;\mathcal{B})$ in place of $\mathcal{N}(N;v;X(\mathcal{B};v))$ and $\mathcal{N}(N;v;X^0(\mathcal{B};v))$, and call them the nucleolus and the prenucleolus, respectively, of the game $(N;v)$ for the coalitional structure $\mathcal{B}$.

20.2 Nonemptiness and uniqueness of the nucleolus

We start by showing that if the set K is compact, then the nucleolus is nonempty. To this end we express $\theta_k(x)$ as the maximum of the minimum of the excesses.

Theorem 20.7 *For every k, $1 \leq k \leq 2^n$,*

$$\theta_k(x) = \max_{\text{different } S_1,\ldots,S_k} \min\{e(S_1,x),\ldots,e(S_k,x)\}. \tag{20.20}$$

For $k = 1$, Equation (20.20) takes the following form,

$$\theta_1(x) = \max_{S\subseteq N} e(S,x), \tag{20.21}$$

which is an expression of the fact that $\theta_1(x)$ is the maximal excess at x. The interpretation of the right-hand side of Equation (20.20) is that for every k different coalitions $S_1, S_2, \ldots, S_k$, we compute the minimum among all their excesses at x. This yields a list of $\binom{2^n}{k}$ numbers. The maximum among these numbers is $\theta_k(x)$ (the k-th element of $\theta(x)$).

We will prove the theorem for the special case of $k = 2$. For $k > 2$ the proof is left to the reader (Exercise 20.11).

Proof: For $k = 2$, the statement of the theorem can be formulated more generally. Let A be a finite set of real numbers. Then the second greatest element[1] in A is $\max_{\{x,y\in A,x\neq y\}} \min\{x, y\}$.

To see this, denote the elements of A as $a_1, a_2, \dots, a_K$, and assume without loss of generality that

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_K. \tag{20.22}$$

The second greatest element in A is a_2. We will show that $\max_{\{x,y\in A,x\neq y\}} \min\{x, y\} = a_2$. Note that $\min\{a_i, a_j\} \leq a_2$ for every pair of distinct elements $a_i, a_j \in A$, with equality if and only if $\{a_i, a_j\} = \{a_1, a_2\}$. Thus, $\max_{\{x,y\in A,x\neq y\}} \min\{x, y\} = a_2$, as claimed. □

Corollary 20.8 *For every $k = 1, 2, \dots, 2^n$, the function θ_k is continuous.*

Proof: For every coalition $S \subseteq N$ the function $e(S, x) = v(S) - x(S)$ is linear in x, and therefore in particular it is a continuous function. Since the minimum of a finite number of continuous functions is a continuous function, we deduce that the function $x \mapsto \min\{e(S_1, x), \dots, e(S_k, x)\}$ is continuous for every set of k coalitions $\{S_1, \dots, S_k\}$. Since the maximum of a finite number of continuous functions is also a continuous function, by Equation (20.20) θ_k is a continuous function. □

Theorem 20.9 *For every coalitional game $(N; v)$ and any nonempty and compact set $K \subseteq \mathbb{R}^N$, the nucleolus of the game $(N; v)$ relative to K is a nonempty compact set.*

Proof: Since θ_1 is a continuous function, the set

$$X_1 := \left\{ x \in K : \theta_1(x) = \min_{y\in K} \theta_1(y) \right\} \tag{20.23}$$

is compact and nonempty (Exercise 20.3). Define X_k for $2 \leq k \leq 2^n$ inductively by

$$X_k := \left\{ x \in X_{k-1} : \theta_k(x) = \min_{y\in X_{k-1}} \theta_k(y) \right\}. \tag{20.24}$$

We prove by induction over k that the sets $(X_k)_{k=2}^{2^n}$ are compact and nonempty. Suppose that the set X_{k-1} is compact and nonempty. Since θ_k is a continuous function, by applying the inductive hypothesis, we deduce that the set X_k is also compact and nonempty. To conclude the proof, note that X_{2^n} is the nucleolus of the game. □

The set $X(\mathcal{B}; v)$ of imputations for the coalitional structure $\mathcal{B}$ is compact. It is nonempty if $v(B) \geq \sum_{i\in B} v(i)$ for every coalition $B \in \mathcal{B}$. We therefore have the following corollary.

Corollary 20.10 *The nucleolus $\mathcal{N}(N; v; \mathcal{B})$ of a coalitional game $(N; v)$ for any coalitional structure $\mathcal{B}$ is a compact set. If $v(B) \geq \sum_{i\in B} v(i)$ for every coalition $B \in \mathcal{B}$, then $\mathcal{N}(N; v; \mathcal{B})$ is also nonempty.*

1 Here we mean the weakly second-greatest element, i.e., when arranging the elements in decreasing order. It may happen that the greatest element is equal to the second-greatest element.

Theorem 20.11 *Let $(N; v)$ be a coalitional game, and let $K \subseteq \mathbb{R}^N$ be a nonempty closed set satisfying the following property: there exists a real number c such that*

$$\sum_{i \in N} x_i = c, \quad \forall x \in K. \tag{20.25}$$

Then $\mathcal{N}(N; v; K)$ is a nonempty compact set.

A set K satisfying the property described by Equation (20.25) may not be bounded, and therefore may not be compact. In particular, Equation (20.25) holds for the closed set $X^0(\mathcal{B}; v)$ of preimputations for any coalitional structure $\mathcal{B}$ (Remark 20.5), and we deduce the following corollary.

Corollary 20.12 *The prenucleolus of a coalitional game, for any coalitional structure, is a nonempty compact set.*

Proof of Theorem 20.11: The outline of the proof proceeds as follows. We will choose a point $y \in K$, and use it to define a compact set $\widetilde{K}$ that is contained in K, contains y, and satisfies $\theta(y) \prec_L \theta(z)$ for every $z \in K \setminus \widetilde{K}$. Since $\widetilde{K}$ is nonempty and compact, the nucleolus relative to $\widetilde{K}$ is a nonempty and compact set (Theorem 20.9). Since for every x in the nucleolus relative to $\widetilde{K}$, the vector $\theta(x)$ is less than or equal to $\theta(y)$ in the lexicographic order, which is less than $\theta(z)$, for all $z \in K \setminus \widetilde{K}$, it follows that the nucleolus relative to $\widetilde{K}$ equals the nucleolus relative to K, and therefore the latter is also nonempty and compact.

We now turn to the construction of the set $\widetilde{K}$. Let $y \in K$, and denote

$$\mu = \theta_1(y) = \max_{S \subseteq N} e(S, y). \tag{20.26}$$

Define

$$\widetilde{K} = \left\{ x \in K : \max_{S \subseteq N} e(S, x) \leq \mu \right\}. \tag{20.27}$$

By definition, $\widetilde{K} \subseteq K$, and $y \in \widetilde{K}$, and hence the set $\widetilde{K}$ is nonempty. If $z \in K \setminus \widetilde{K}$ then $\theta_1(z) = \max_{T \subseteq N} e(T, z) > \mu = \theta_1(y)$, and hence $\theta(y) \prec_L \theta(z)$.

Finally, we show that the set $\widetilde{K}$ is compact. Since $\widetilde{K}$ is defined by weak linear inequalities, it is a closed set. We will show that $\widetilde{K}$ is also bounded. Let $x \in \widetilde{K}$. By the definition of μ, one has $e(S, x) \leq \mu$ for every coalition $S \subseteq N$. Setting $S = \{i\}$ yields

$$\mu \geq e(\{i\}, x) = v(i) - x_i, \tag{20.28}$$

and therefore $x_i \geq v(i) - \mu$ for every $i \in N$. On the other hand, since $\sum_{i \in N} x_i = c$,

$$x_i = c - \sum_{\{j : j \neq i\}} x_j \leq c - \sum_{\{j : j \neq i\}} (v(j) - \mu) = c + |N - 1|\mu - \sum_{\{j : j \neq i\}} v(j). \tag{20.29}$$

To summarize, the set $\widetilde{K}$ is contained in the following product of intervals:

$$\widetilde{K} \subseteq \bigtimes_{i \in N} \left[v(i) - \mu, c + |N - 1|\mu - \sum_{\{j : j \neq i\}} v(j) \right]; \tag{20.30}$$

hence $\widetilde{K}$ is a bounded set. $\square$

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Theorem 20.13 *Let $(N; v)$ be a coalitional game and let $K \subseteq \mathbb{R}^N$ be a convex set. Then $\mathcal{N}(N; v; K)$ contains at most one point.*

Proof: Let x and y be two points in the nucleolus. We will prove that $x = y$. Denote

$$\theta(x) = (e(S_1, x), e(S_2, x), \ldots, e(S_{2^n}, x)), \tag{20.31}$$

$$\theta(y) = (e(R_1, y), e(R_2, y), \ldots, e(R_{2^n}, y)), \tag{20.32}$$

and

$$\theta\left(\frac{x+y}{2}\right) = \left(e\left(T_1, \frac{x+y}{2}\right), e\left(T_2, \frac{x+y}{2}\right), \ldots, e\left(T_{2^n}, \frac{x+y}{2}\right)\right). \tag{20.33}$$

Since x and y are in the nucleolus, by definition $\theta(x) \approx_L \theta(y)$, and therefore $\theta(x) = \theta(y)$, i.e.,

$$\theta_k(x) = \theta_k(y), \quad 1 \leq k \leq 2^n. \tag{20.34}$$

Since K is a convex set, $\frac{x+y}{2}$ is also in K. For every coalition $T \subseteq N$

$$2e\left(T, \frac{x+y}{2}\right) = 2v(T) - (x+y)(T) = e(T, x) + e(T, y), \tag{20.35}$$

and therefore

$$2\theta\left(\frac{x+y}{2}\right) = (e(T_1, x) + e(T_1, y), e(T_2, x) + e(T_2, y), \ldots, e(T_{2^n}, x) + e(T_{2^n}, y)). \tag{20.36}$$

Since S_1 maximizes the excess at x we deduce that

$$e(T_1, x) \leq e(S_1, x), \tag{20.37}$$

and since R_1 maximizes the excess at y,

$$e(T_1, y) \leq e(R_1, y). \tag{20.38}$$

Since $e(S_1, x) = e(R_1, y)$ (Equation (20.34)), Equations (20.36), (20.37), and (20.38) imply that

$$e\left(T_1, \frac{x+y}{2}\right) = \frac{e(T_1, x) + e(T_1, y)}{2} \leq e(S_1, x). \tag{20.39}$$

If $e(T_1, \frac{x+y}{2}) < e(S_1, x)$, then $\theta(\frac{x+y}{2}) \prec_L \theta(x)$, contradicting the assumption that x is in the nucleolus. It follows that $e(T_1, \frac{x+y}{2}) = e(S_1, x)$, and therefore

$$e(T_1, x) + e(T_1, y) = e(S_1, x) + e(R_1, y). \tag{20.40}$$

Using Equations (20.37) and (20.38) we deduce that $e(T_1, x) = e(S_1, x)$ and $e(T_1, y) = e(R_1, y)$, and therefore T_1 maximizes the excess at x and at y; i.e, by changing the order of the coalitions, one can write

$$\theta(x) = (e(T_1, x), e(S'_2, x), \ldots, e(S'_{2^n}, x)), \tag{20.41}$$

and

$$\theta(y) = (e(T_1, y), e(R'_2, y), \ldots, e(R'_{2^n}, y)), \tag{20.42}$$

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where $T_1, S'_2, \ldots, S'_{2^n}$ are obtained from $S_1, S_2, \ldots, S_{2^n}$, by swapping S_1 with T_1. A similar statement holds for $T_1, R'_2, \ldots, R'_{2^n}$. Continuing by induction, for every k such that $1 \leq k \leq 2^n$, we show that $e(T_k, x) = e(S_k, x)$ and $e(T_k, y) = e(R_k, y)$. In other words,

$$e(T, x) = e(T, y), \quad \forall T \subseteq N. \tag{20.43}$$

In particular setting $T = \{i\}$, we deduce that for every player $i \in N$,

$$v(i) - x_i = e(\{i\}, x) = e(\{i\}, y) = v(i) - y_i, \tag{20.44}$$

and therefore $x_i = y_i$. This means that $x = y$, which is what we needed to show. □

When K is not a convex set, the nucleolus may contain more than one point (Exercise 20.14).

Corollary 20.14 *For every coalitional structure, the prenucleolus of a coalitional game $(N; v)$ for that coalitional structure consists of a single preimputation. If the set of imputations for the coalitional structure is nonempty, then the nucleolus for this coalitional structure consists of a single imputation as well.*

Proof: Since the set $X^0(\mathcal{B}; v)$ is convex for every coalitional structure $\mathcal{B}$, the claim for the prenucleolus follows from Corollary 20.12 and Theorem 20.13. Since the set $X(\mathcal{B}; v)$ is convex for every coalitional structure $\mathcal{B}$, the claim for the nucleolus follows from Corollary 20.10 and Theorem 20.13. □

As the nucleolus contains a single vector if it is not empty, and the prenucleolus always contains a single vector, we call these two vectors respectively "the nucleolus" and "the prenucleolus" of the game, and view them as vectors in $\mathbb{R}^N$. The i-th coordinates of these vectors are denoted by $\mathcal{N}_i(N; v)$ and $\mathcal{PN}_i(N; v)$ respectively. Similarly, for every coalitional structure $\mathcal{B}$ we call the single vector contained in the nucleolus (if it is nonempty) and in the prenucleolus for $\mathcal{B}$ "the nucleolus for $\mathcal{B}$" and "the prenucleolus for $\mathcal{B}$," and we view them as vectors in $\mathbb{R}^N$.

The nucleolus and the prenucleolus may not coincide (Exercise 20.15). As the next theorem states, when the prenucleolus of a game is an imputation, then the nucleolus and the prenucleolus coincide.

Theorem 20.15 *For every coalitional structure $\mathcal{B}$, if the prenucleolus x^* of a coalitional game $(N; v)$ for $\mathcal{B}$ is individually rational, i.e., $x^*_i \geq v(i)$ for all $i \in N$, then x^* is also the nucleolus of $(N; v)$ for $\mathcal{B}$.*

Proof: Let x^* be the prenucleolus of the game $(N; v)$ for $\mathcal{B}$. Since by assumption this vector is individually rational, it is in $X(\mathcal{B}; v)$. Because every imputation is also in particular a preimputation, $\theta(x^*) \precsim_L \theta(x)$ for every $x \in X(N; v)$. It follows that the vector x^* is also the nucleolus of the game $(N; v)$ for $\mathcal{B}$. □

20.3 Properties of the nucleolus

We have seen two single-valued solution concepts for coalitional games, the Shapley value and the nucleolus (or prenucleolus). We will now study what these two solution concepts

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have in common, and what properties distinguish them. This analysis may prove useful in determining which solution concept is more appropriate for each specific application.

Theorem 20.16 *The nucleolus is covariant under strategic equivalence. That is, for every coalitional game $(N; v)$, every set $K \subseteq \mathbb{R}^N$, every $a > 0$, and every $b \in \mathbb{R}^N$,*

$$\mathcal{N}(N; v; aK + b) = a\mathcal{N}(N; v; K) + b. \tag{20.45}$$

This theorem, whose proof is left to the reader as an exercise (Exercise 20.17), parallels the covariance under strategic equivalence of the Shapley value (Claim 18.18, page 755).

The next theorem states that for every coalitional structure $\mathcal{B}$, symmetric players who are members of the same coalition in $\mathcal{B}$ receive equal payoffs in the nucleolus and in the prenucleolus for $\mathcal{B}$. Recall that players i and j are symmetric players if $v(S \cup \{i\}) = v(S \cup \{j\})$ for every coalition S that contains neither player i nor player j. This theorem parallels the symmetry property of the Shapley value (Claim 18.18, page 755).

Theorem 20.17 *Let $(N; v)$ be a coalitional game and let $\mathcal{B}$ be a coalitional structure, and let i and j be symmetric players who are members of the same coalition in $\mathcal{B}$. Then $\mathcal{N}_i(N; v; \mathcal{B}) = \mathcal{N}_j(N; v; \mathcal{B})$ and $\mathcal{PN}_i(N; v; \mathcal{B}) = \mathcal{PN}_j(N; v; \mathcal{B})$.*

Proof: Let i and j be symmetric players in the same coalition in $\mathcal{B}$; i.e., there exists a coalition $B \in \mathcal{B}$ such that $i, j \in B$. Denote $x^* = \mathcal{N}(N; v; \mathcal{B})$, and let y be the vector obtained from x^* by swapping the payoffs of players i by j:

$$y_k = \begin{cases} x_i^* & \text{if } k = j, \\ x_j^* & \text{if } k = i, \\ x_k^* & \text{if } k \notin \{i, j\}. \end{cases} \tag{20.46}$$

Since players i and j are in the same coalition in $\mathcal{B}$, and since $x^* \in X(\mathcal{B}; v)$, it follows that $y \in X(\mathcal{B}; v)$. We will show that $\theta(x^*) \approx_L \theta(y)$, and therefore y is also minimal in the lexicographic order. In particular, y is also in the nucleolus. Since the nucleolus (or the prenucleolus) contains a single vector, it must be that $x^* = y$, and in particular $x_i^* = x_j^*$.

To show that $\theta(x^*) \approx_L \theta(y)$, define a bijection φ from the set of coalitions to itself:

$$\varphi(S) = \begin{cases} S, & \text{if } i \in S, j \in S, \\ S, & \text{if } i \notin S, j \notin S, \\ (S \setminus \{i\}) \cup \{j\}, & \text{if } i \in S, j \notin S, \\ (S \setminus \{j\}) \cup \{i\}, & \text{if } i \notin S, j \in S. \end{cases} \tag{20.47}$$

Since players i and j are symmetric, $v(\varphi(S)) = v(S)$ for every coalition $S \subseteq N$. The definitions of φ and y imply that $e(S, x^*) = e(\varphi(S), y)$ for every coalition $S \subseteq N$. Therefore the sets of excesses with respect to x^* and y are equal:

$$\{e(S, x^*) \colon S \subseteq N\} = \{e(S, y) \colon S \subseteq N\}. \tag{20.48}$$

Thus, arranging the collections of excesses at x^* and at y in decreasing order yields the same vector, and hence $\theta(x^*) \approx_L \theta(y)$. □

Theorems 20.16 and 20.17 enable one to write explicitly a formula for the nucleolus of two-player games.

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Theorem 20.18 *Let $(N; v)$ be a two-player coalitional game. If $v(1,2) \geq v(1) + v(2)$, then the nucleolus is*

$$\left(\frac{v(1,2) + v(1) - v(2)}{2}, \frac{v(1,2) - v(1) + v(2)}{2}\right). \tag{20.49}$$

This imputation is called the *standard solution* of the game. To prove Theorem 20.18 one uses the fact that the nucleolus is symmetric and covariant under strategic equivalence. The condition $v(1,2) \geq v(1) + v(2)$ guarantees that the set of imputations $X(N; v)$ is nonempty, thereby ensuring that the nucleolus is nonempty. Since the prenucleolus and the Shapley value are also symmetric solution concepts that are covariant under strategic equivalence, Equation (20.49) characterizes these two solution concepts in two-player coalitional games. The condition $v(1,2) \geq v(1) + v(2)$ is, however, not needed for their characterizations, since they are defined for all coalitional games.

Theorem 20.19 *Let $i \in N$ be a null player in a coalitional game $(N; v)$. Then under both the nucleolus and the prenucleolus, player i's payoff is 0; i.e., $\mathcal{N}_i(N; v) = \mathcal{PN}_i(N; v) = 0$.*

This theorem does not hold for general coalitional structures (Exercise 20.16).

Proof: Let i be a null player in the coalitional game $(N; v)$. We will first prove the claim for the nucleolus x^*. Suppose by contradiction that $x_i^* \neq 0$. Since $x^* \in X(N; v)$, one has $x_i^* \geq v(i) = 0$, and since $x_i^* \neq 0$, it follows that $x_i^* > 0$. We will show that by transferring a small amount from player i to all the other players, it is possible to create an imputation y satisfying $\theta(y) \prec_L \theta(x^*)$, thereby contradicting the assumption that x^* is the nucleolus. Since i is a null player, it follows that for every coalition T that does not contain i,

$$e(T \cup \{i\}, x^*) = v(T \cup \{i\}) - x^*(T \cup \{i\}) \tag{20.50}$$

$$= v(T) - x^*(T) - x_i^* \tag{20.51}$$

$$= e(T, x^*) - x_i^*. \tag{20.52}$$

Since $e(N; x^*) = 0$, Equations (20.50)–(20.52) for $T = N \setminus \{i\}$ imply that

$$\theta_1(x^*) \geq e(N \setminus \{i\}, x^*) = e(N, x^*) + x_i^* = x_i^* > 0. \tag{20.53}$$

Let y be the vector derived from x^* by having player i transfer an amount $\frac{x_i^*}{n}$ to every other player:

$$y_j = \begin{cases} \frac{x_i^*}{n} & \text{if } j = i, \\ x_j^* + \frac{x_i^*}{n} & \text{if } j \neq i. \end{cases} \tag{20.54}$$

Since x^* is an imputation, it follows that y is also an imputation (why?). To prove that $\theta(y) \prec_L \theta(x^*)$ we will show that $\theta_1(y) < \theta_1(x^*)$. To this end we will prove that $e(T, y) < \theta_1(x^*)$ for every coalition $T \subseteq N$.

For the coalition $T = \emptyset$ we have

$$e(\emptyset, y) = 0 < \theta_1(x^*). \tag{20.55}$$

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For every nonempty coalition T that does not contain player i we have

$$e(T, y) = e(T, x^*) - \frac{|T|}{n} x_i^* < e(T, x^*) \leq \theta_1(x^*). \tag{20.56}$$

From Equations (20.50)–(20.52) we deduce that $e(T, y) = e(T \setminus \{i\}) - x_i^*$ for every coalition T that contains player i, and, therefore,

$$e(T, y) = e(T, x^*) + \frac{n - |T|}{n} x_i^* = e(T \setminus \{i\}, x^*) - \frac{|T|}{n} x_i^* \tag{20.57}$$

$$\leq \theta_1(x^*) - \frac{|T|}{n} x_i^* < \theta_1(x^*). \tag{20.58}$$

We thus proved that $\theta_1(y) = \max_{T \subseteq N} e(T, y) < \theta_1(x^*)$, thereby contradicting the fact that x^* is the nucleolus.

Suppose now that x^* is the prenucleolus, and assume by way of contradiction that $x_i^* \neq 0$. If $x_i^* > 0$ we obtain a contradiction as in the case of the nucleolus. Suppose then that $x_i^* < 0$. We will prove that in this case as well the vector y defined in Equation (20.54) satisfies $\theta(y) \prec_L \theta(x^*)$. Since $x_i^* < 0$, the vector y is derived from x^* by having every other player transfer to player i the amount $\frac{|x_i^*|}{n}$. Since x^* is a preimputation, y is a preimputation as well. Since player i is a null player,

$$\theta_1(x^*) \geq e(\{i\}, x_i^*) = -x_i^* > 0. \tag{20.59}$$

For the coalition $T = N$ we have $e(N, y) = 0 < \theta_1(x^*)$. For every coalition T that does not contain player i we have $|T| < n$, and by Equations (20.50)–(20.52), we also have $e(T, x^*) = e(T \cup \{i\}, x^*) + x_i^*$. Therefore,

$$e(T, y) = e(T, x^*) - \frac{|T|}{n} x_i^* = e(T \cup \{i\}, x^*) + \frac{n - |T|}{n} x_i^* < e(T \cup \{i\}, x^*) \leq \theta_1(x^*). \tag{20.60}$$

For every coalition T that contains player i and is not N we have

$$e(T, y) = e(T, x^*) + \frac{n - |T|}{n} x_i^* < e(T, x^*) \leq \theta_1(x^*). \tag{20.61}$$

Therefore also in this case $\theta_1(y) = \max_{T \subseteq N} e(T, y) < \theta_1(x^*)$, contradicting the fact that x^* is the prenucleolus. □

The Shapley value is the only single-valued solution concept satisfying the properties of efficiency, symmetry, null player, and additivity. Both the nucleolus and the prenucleolus satisfy the properties of efficiency, symmetry, and null player. If we show that the Shapley value differs from the nucleolus and the prenucleolus, then it will follow that the nucleolus and prenucleolus do not satisfy additivity. Indeed, there are examples of coalitional games in which the nucleolus and the prenucleolus differ from the Shapley value, such as the gloves game (Examples 17.5 and 18.24), where the nucleolus and the prenucleolus are $(0, 0, 1)$, while the Shapley value is $(\frac{1}{6}, \frac{1}{6}, \frac{2}{3})$.

An important property shared by the nucleolus and the prenucleolus is that both of them are in the core, if the core is nonempty. This property does not hold for the Shapley value

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(Example 18.24, page 759), another proof that the Shapley value does not coincide with the nucleolus and the prenucleolus.

Theorem 20.20 *If the core of a coalitional game $(N; v)$ for the coalitional structure $\mathcal{B}$ is nonempty, then the nucleolus for $\mathcal{B}$ is in the core, and it coincides with the prenucleolus for $\mathcal{B}$.*

Proof: Let x be an imputation in the core of a coalitional game $(N; v)$ for the coalitional structure $\mathcal{B}$, and let x^* be the prenucleolus for $\mathcal{B}$. Since x is in the core, $x(S) \geq v(S)$ for every coalition S, and therefore

$$e(S, x) = v(S) - x(S) \leq 0, \quad \forall S \subseteq N. \tag{20.62}$$

This implies that $\theta_1(x) \leq 0$. Since x^* is the prenucleolus, $\theta(x^*) \precsim_L \theta(x)$. Hence $\theta_1(x^*) \leq \theta_1(x) \leq 0$. By definition, $\theta_1(x^*) = \max_{S \subseteq N} e(S, x^*)$, and therefore for every coalition $S \subseteq N$ we have $e(S, x^*) \leq 0$; i.e., $x^*(S) \geq v(S)$, and therefore x^* is in the core.

Since x^* is in the core, $x_i^* \geq v(i)$, and therefore $x^* \in X(N; v)$. By Theorem 20.15, x^* is also the nucleolus. □

As the next theorem shows, the nucleolus is also in the bargaining set (the bargaining set is studied in Chapter 19).

Theorem 20.21 *The nucleolus of a coalitional game $(N; v)$ for any coalitional structure $\mathcal{B}$ is in the bargaining set for $\mathcal{B}$:*

$$\mathcal{N}(N; v; \mathcal{B}) \in \mathcal{M}(N; v; \mathcal{B}). \tag{20.63}$$

Proof: If the core is nonempty, then Theorems 20.20 and 19.12 (page 786) imply that

$$\mathcal{N}(N; v; \mathcal{B}) \in \mathcal{C}(N; v; \mathcal{B}) \subseteq \mathcal{M}(N; v; \mathcal{B}). \tag{20.64}$$

Suppose that the core $\mathcal{C}(N; v; \mathcal{B})$ is empty. Let $x^* = \mathcal{N}(N; v; \mathcal{B})$ be the nucleolus for the coalitional structure $\mathcal{B}$. In particular $x^* \notin \mathcal{C}(N; v; \mathcal{B})$, and therefore $\theta_1(x^*) > 0$ (Exercise 20.26). If $x^* \notin \mathcal{M}(N; v; \mathcal{B})$, then there is a player k who has a justified objection at x^* against player l, where players k and l are in the same coalition of $\mathcal{B}$.

Since player k has an objection at x^* against player l, there is a coalition S containing player k and not player l with positive excess. Since S is a justified objection against player l, all the coalitions containing player l but not player k have less excess (Theorem 19.14, page 787).

Order the coalitions by decreasing excess at x^*. Denote by a the maximal excess of the coalitions containing player k and not player l. Denote by b the maximal excess of the coalitions containing player l and not player k. All the coalitions whose excesses are greater than a either contain both player k and player l, or contain neither k nor l (Figure 20.2). Denote $\delta := \min\{a - b, x_l^* - v(l)\}$. As we showed previously, $a > b$. Since player k has a justified objection against player l, it must be the case that $x_l^* > v(l)$; otherwise player l would have a counterobjection using the coalition $\{l\}$. We deduce that $\delta > 0$.

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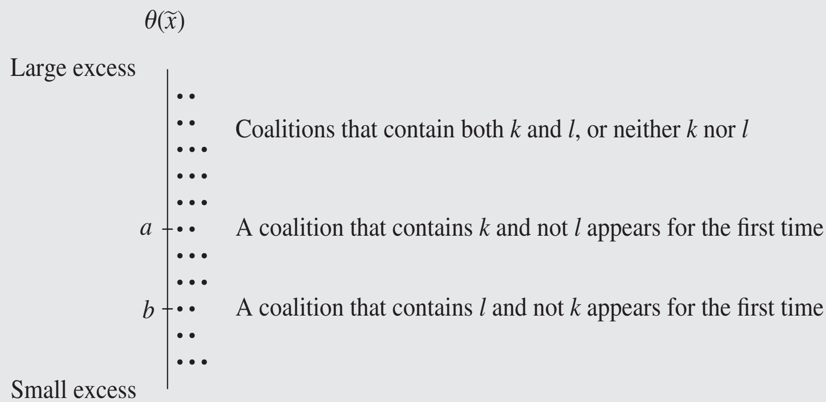

Figure 20.2 The coalitions and ordered excesses at x^*

Define a vector y derived from x^* by transferring from player l to player k the amount of $\frac{\delta}{2}$:

$$y_i = \begin{cases} x_i^* & \text{if } i \notin \{l, k\}, \\ x_i^* - \frac{\delta}{2} & \text{if } i = l, \\ x_i^* + \frac{\delta}{2} & \text{if } i = k. \end{cases} \tag{20.65}$$

We first prove that y is an imputation. Since $y(N) = x^*(N) = v(N)$ it follows that y is an efficient vector. Since the imputation x^* is individually rational, $y_i \geq x_i^* \geq v(i)$ for every player $i \neq l$. As for player $i = l$, since by the definition of δ we have $x_l^* \geq v(l) + \delta$,

$$y_l = x_l^* - \frac{\delta}{2} \geq v(l) + \frac{\delta}{2} > v(l). \tag{20.66}$$

Hence y is also an individually rational vector, and therefore an imputation.

What is the relationship between $\theta(x^*)$ and $\theta(y)$?

- The excess of the coalitions containing both k and l, and of the coalitions that contain neither player k nor player l, do not change in passing from x^* to y.
- The excesses of coalitions containing player k but not player l are reduced by $\frac{\delta}{2}$.
- The excesses of coalitions containing player l but not player k are increased by $\frac{\delta}{2}$.

The excesses of coalitions whose excess is above a in Figure 20.2 do not change. The excess of at least one coalition at height a is reduced by a positive amount. The only coalitions whose excesses can increase are those with excesses b or less. By the definition of δ, these excesses do not exceed a. We conclude that this transfer has decreased the vector of the excesses in lexicographic order: $\theta(y) \prec_L \theta(x^*)$. This contradicts the fact that x^* is the nucleolus. The contradiction follows from the assumption that $x^* \notin \mathcal{M}(N; v; \mathcal{B})$, and hence the claim that $x^* \in \mathcal{M}(N; v; \mathcal{B})$ is proved. $\square$

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20.4 Computing the nucleolus

In this section, we present a procedure for computing the nucleolus $\mathcal{N}(N; v)$ by solving a sequence of linear programs. The complexity of the algorithm is exponential in the number n of players, and therefore it is useful only for small games. A brief review of linear programming appears in Section 23.3 (page 945).

The procedure is based on the conditions an imputation must satisfy to be the nucleolus; it finds an imputation x whose vector of excesses $\theta(x)$ is minimal in the lexicographic order. In the first step, the procedure finds all the vectors whose maximal excess $\theta_1(x)$ is as small as possible. From among these the procedure finds all the vectors whose second-largest excess is as small as possible, and so on. As the length of the vector of excesses is finite, the procedure eventually halts and yields an imputation, which is the nucleolus.

Step 1: Minimizing the maximal excess

Solve the following linear program with unknowns $x_1, x_2, \ldots, x_n, t$.

$$\begin{array}{ll} \text{Compute:} & \min t, \\ \text{subject to:} & e(S,x) \leq t, \quad \forall S \subseteq N, \\ & x(N) = v(N), \\ & x_i \geq v(i), \quad \forall i \in N. \end{array} \tag{20.67}$$

Denote by θ_1 the value of this program, and by X_1 the set of vectors at which the minimum of the excesses is attained:

$$X_1 := \{x \in X(N; v) \colon e(S,x) \leq \theta_1, \quad \forall S \subseteq N\}. \tag{20.68}$$

Denote by Σ_1 the set of all coalitions at which the maximal excess θ_1 is attained at all $x \in X_1$:

$$\Sigma_1 := \{S \subseteq N \colon e(S,x) = \theta_1, \quad \forall x \in X_1\}. \tag{20.69}$$

The set Σ_1 is nonempty. To see this, note that by the definition of θ_1, we have $e(S,x) \leq \theta_1$ for every $x \in X_1$, and for every coalition S. If Σ_1 were empty, then for every coalition $S \subseteq N$ there would exist a vector $x^S \in X_1$ satisfying $e(S, x^S) < \theta_1$. But then the average $y := \frac{1}{2^n} \sum_{S \subseteq N} x^S$ would satisfy $e(S, y) < \theta_1$ for every coalition S (Exercise 20.32), contradicting the definition of θ_1.

Step 2: Minimizing the second-largest excess

Solve the following linear program with unknowns $x_1, x_2, \ldots, x_n, t$:

$$\begin{array}{ll} \text{Compute:} & \min t, \\ \text{subject to:} & e(S,x) = \theta_1, \quad \forall S \in \Sigma_1, \\ & e(S,x) \leq t, \quad \forall S \notin \Sigma_1, \\ & x(N) = v(N), \\ & x_i \geq v(i), \quad \forall i \in N. \end{array} \tag{20.70}$$

Denote the value of this program by θ_2, by X_2 the set of all vectors at which the minimum is attained, and by Σ_2 the set of all coalitions (a) that are not in Σ_1, and (b) at which the value θ_2 is attained for all $x \in X_2$. As before, Σ_2 is not empty.

Continue implementing this procedure iteratively in a similar manner to define disjoint collections of coalitions $\Sigma_3, \Sigma_4, \ldots$. Since these collections are disjoint, there exists $L > 0$ such that Σ_L is nonempty, and the union $\cup_{l=1}^{L} \Sigma_l$ contains all the coalitions. The set of imputations X_L contains a single vector, the nucleolus of the game. The reader is asked to prove that the algorithm indeed calculates the nucleolus in Exercise 20.31.

20.5 Characterizing the prenucleolus

In this section, we prove a theorem that can be used to check whether a preimputation x is the prenucleolus of a coalitional game $(N; v)$ by considering only the vector of excesses $\theta(x)$, without comparing it to $\theta(y)$ for $y \neq x$.

Definition 20.22 *A system of equalities and inequalities is* tight *if it has at least one solution, and at every solution of the system every inequality obtains as an equality.*

Example 20.23 The following system of inequalities is tight:

$$x + y \leq 7, \tag{20.71}$$
$$x + 2y \geq 7, \tag{20.72}$$
$$x \geq 7, \tag{20.73}$$

because the only solution of this system is $x = 7$, $y = 0$, and in this solution all inequalities hold as equalities. In contrast, the following system is not tight:

$$x + y \leq 7, \tag{20.74}$$
$$x + 2y \geq 7, \tag{20.75}$$
$$x \geq 6, \tag{20.76}$$

because $x = 7$, $y = 0$ is a solution of the system at which the third inequality does not obtain as an equality. ◀

The next theorem characterizes balanced collections of coalitions using tight systems of equations. We first recall the definition of a balanced collection of coalitions (Definition 17.11, page 693).

Definition 20.24 *A collection $\mathcal{D}$ of coalitions is* balanced *if there exist positive numbers $(\delta_S)_{S\in\mathcal{D}}$ satisfying*

$$\sum_{\{S\in\mathcal{D}:\, i\in S\}} \delta_S = 1, \quad \forall i \in N. \tag{20.77}$$

The vector $(\delta_S)_{S\in\mathcal{D}}$ is called a vector of balancing weights *of $\mathcal{D}$.*

Theorem 20.25 *A collection $\mathcal{D}$ of subsets of N is a balanced collection if and only if the following system of equations, with $|N|$ unknowns $(y_i)_{i\in N}$, is tight:*

$$\begin{cases} y(N) = 0, & \\ y(S) \geq 0, & \forall S \in \mathcal{D}. \end{cases} \tag{20.78}$$

Example 20.26 Suppose that $N = \{1, 2, 3\}$. The system of equations (20.78) corresponding to the balanced collection $\mathcal{D} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ is

$$y_1 + y_2 + y_3 = 0, \tag{20.79}$$

$$y_1 + y_2 \geq 0, \tag{20.80}$$

$$y_1 + y_3 \geq 0, \tag{20.81}$$

$$y_2 + y_3 \geq 0. \tag{20.82}$$

The only solution of this system is $y_1 = y_2 = y_3 = 0$ (verify!), in which the inequalities obtain as equalities, and hence this system is tight.

The system of equations corresponding to the balanced collection $\mathcal{D} = \{\{1\}, \{2, 3\}\}$ is

$$y_1 + y_2 + y_3 = 0, \tag{20.83}$$

$$y_1 \geq 0, \tag{20.84}$$

$$y_2 + y_3 \geq 0. \tag{20.85}$$

All the solutions of this system are of the form $y_1 = 0$, $y_3 = -y_2$ (verify!), in which the inequalities obtain as equalities, and hence this system is tight as well.

The system of equations corresponding to the collection $\mathcal{D} = \{\{1, 2\}, \{1, 2, 3\}\}$ is

$$y_1 + y_2 + y_3 = 0, \tag{20.86}$$

$$y_1 + y_2 \geq 0, \tag{20.87}$$

$$y_1 + y_2 + y_3 \geq 0. \tag{20.88}$$

One solution to this system is $y_1 = y_2 = 1$, $y_3 = -2$ (verify!). In this solution not all the inequalities hold as equalities (since $y_1 + y_2 > 0$), and hence the system is not tight. Indeed, the collection $\mathcal{D}$ is not balanced (verify), in accordance with Theorem 20.25. ◀

We now present the proof of Theorem 20.25, which uses the Duality Theorem from the theory of linear programming (Theorem 23.46, page 950).

Proof of Theorem 20.25: Suppose first that $\mathcal{D}$ is a balanced collection, and that $(\delta_S)_{S\in\mathcal{D}}$ is a vector of balancing weights of $\mathcal{D}$. We will show that the system of equations (20.78) is tight.

The zero vector ($y_i = 0$ for all i) is a solution of the system of equations (20.78), and the system therefore has at least one solution. We will show that at every solution y of this system, all inequalities obtain as equalities. Let y be a vector in $\mathbb{R}^N$ satisfying the system of equations (20.78). By Lemma 17.16 (page 696),

$$\sum_{S\in\mathcal{D}} \delta_S y(S) = y(N) = 0. \tag{20.89}$$

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Since $\delta_S > 0$ and $y(S) \geq 0$ for every coalition $S \in \mathcal{D}$, it follows that $y(S) = 0$ for every $S \in \mathcal{D}$. Thus, the system of equations is tight.

Next, suppose that the system of equations (20.78) is tight. We will show that the collection $\mathcal{D}$ is a balanced collection of coalitions. Consider the following linear program with unknowns $(\beta_S)_{S\in\mathcal{D}}, \gamma, \delta$:

$$\begin{array}{ll} \text{Compute:} & Z_P := \max 0, \\ \text{subject to:} & \sum_{\{S\in\mathcal{D}:\, i\in S\}} \beta_S + \gamma - \delta = \sum_{\{S\in\mathcal{D}:\, i\in S\}}(-1), \forall i \in N, \\ & \beta_S \geq 0, \quad \forall S \in \mathcal{D}, \\ & \gamma \geq 0, \\ & \delta \geq 0. \end{array} \tag{20.90}$$

To construct a vector of balancing weights for $\mathcal{D}$ we will use the optimal solution to program (20.90). Since the objective function is the zero function, showing the existence of an optimal solution only requires showing that there exists a solution to this linear program. This is achieved by checking the dual linear program. The dual linear program is the following program with unknowns $(y_i)_{i\in N}$ (verify!):

$$\begin{array}{ll} \text{Compute:} & Z_D := \min \sum_{S\in\mathcal{D}}(-y(S)), \\ \text{subject to:} & y(S) \geq 0, \quad \forall S \in \mathcal{D}, \\ & y(N) \geq 0, \\ & -y(N) \geq 0. \end{array} \tag{20.91}$$

Since $y = \vec{0}$ is a solution of the dual linear program (20.91), the set of possible solutions of that linear program is nonempty. The system of constraints of program (20.91) is system (20.78). Since system (20.78) is tight, at every solution of the dual linear program the constraints obtain with equality, and therefore $Z_D = 0$. By the Duality Theorem (Theorem 23.46, page 950) the set of possible solutions of program (20.90) is nonempty (and $Z_P = Z_D = 0$). Let $((\beta_S)_{S\in\mathcal{D}}, \gamma, \delta)$ be a solution of the primal linear program (20.90). Write the first constraint in this program as

$$\sum_{\{S\in\mathcal{D}:\, i\in S\}} (1 + \beta_S) = \delta - \gamma, \quad \forall i \in N. \tag{20.92}$$

Since $\beta_S \geq 0$ for every $S \in \mathcal{D}$, the left-hand side of this equation is positive (for all $i \in N$), and therefore the right-hand side is also positive, that is, $\delta - \gamma > 0$. Define

$$\lambda_S := \frac{1 + \beta_S}{\delta - \gamma}, \quad \forall S \in \mathcal{D}. \tag{20.93}$$

Then $(\lambda_S)_{S\in\mathcal{D}}$ is a vector of positive numbers satisfying

$$\sum_{\{S\in\mathcal{D}:\, i\in S\}} \lambda_S = 1, \quad \forall i \in N, \tag{20.94}$$

which implies that the collection $\mathcal{D}$ is balanced with the vector of balancing weights $(\lambda_S)_{S\in\mathcal{D}}$. □

Definition 20.27 *Let $(N; v)$ be a coalitional game, and let $x \in X^0(N; v)$ be a preimputation. For every $\alpha \in \mathbb{R}$, denote by $\mathcal{D}(\alpha, x)$ the collection of nonempty coalitions S satisfying*

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$e(S,x) \geq \alpha$:

$$\mathcal{D}(\alpha, x) := \{S \subset N \colon S \neq \emptyset, e(S,x) \geq \alpha\}. \tag{20.95}$$

The collection of coalition $\mathcal{D}(\alpha, x)$ is related to the vector $\theta(x)$ in the following way. Denote the different values of the excesses at x by $a_1, a_2, \ldots, a_p$, where $a_1 > a_2 > \cdots > a_p$. Then,

$$\theta(x) = (a_1, a_1, \ldots, a_1, a_2, \ldots, a_2, \ldots, a_p, \ldots, a_p). \tag{20.96}$$

Note that since $e(N,x) = e(\emptyset, x) = 0$ we necessarily have $a_p \leq 0$. In this notation,

$$\mathcal{D}(a_1, x) \subset \mathcal{D}(a_2, x) \subset \cdots \subset \mathcal{D}(a_p, x) = \{S \colon S \subset N, S \neq \emptyset\}, \tag{20.97}$$

the collection $\mathcal{D}(a_1, x)$ contains all coalitions with maximal excess at x (that is, with excess a_1), the collection $\mathcal{D}(a_2, x)$ contains all coalitions whose excess at x is either a_1 or a_2, etc. For every α for which this collection is nonempty, this collection is one of the p collections $(\mathcal{D}(a_k, x))_{k=1}^p$, since

$$\mathcal{D}(\alpha, x) = \begin{cases} \emptyset & \alpha > a_1, \\ \mathcal{D}(a_k, x) & a_{k+1} < \alpha \leq a_k, \\ \{S \colon S \subset N, S \neq \emptyset\} & \alpha \leq a_p. \end{cases} \tag{20.98}$$

Theorem 20.28 *If x^* is the prenucleolus of a coalitional game $(N; v)$, then for every $\alpha \in \mathbb{R}$ such that the collection $\mathcal{D}(\alpha, x^*)$ is nonempty, the following system of equations is tight:*

$$y(N) = 0, \tag{20.99}$$

$$y(S) \geq 0, \quad \forall S \in \mathcal{D}(\alpha, x^*). \tag{20.100}$$

Conversely, if for the vector $x \in X^0(N; v)$, and for every $\alpha \in \mathbb{R}$ for which the collection $\mathcal{D}(\alpha, x)$ is nonempty, the following system of equations is tight:

$$y(N) = 0, \tag{20.101}$$

$$y(S) \geq 0, \quad \forall S \in \mathcal{D}(\alpha, x), \tag{20.102}$$

then x is the prenucleolus of the game.

Proof:
Step 1: If x^* is the prenucleolus, then the system (20.99)–(20.100) is tight, for every α such that $\mathcal{D}(\alpha, x^*) \neq \emptyset$.
Suppose that x^* is the prenucleolus, and let $\alpha \in \mathbb{R}$ satisfy $\mathcal{D}(\alpha, x^*) \neq \emptyset$. To show that the system (20.99)–(20.100) is tight, define for every $\varepsilon > 0$ and every solution y of the system (20.99)–(20.100) a vector $z_\varepsilon \in \mathbb{R}^N$ by

$$z_\varepsilon := x^* + \varepsilon y. \tag{20.103}$$

Since $y(N) = 0$,

$$z_\varepsilon(N) = x^*(N) + \varepsilon y(N) = x^*(N) = v(N), \tag{20.104}$$

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and therefore $z_\varepsilon \in X^0(N; v)$. Since $\lim_{\varepsilon\to 0} z_\varepsilon = x^*$ it follows that for every coalition $S \subseteq N$ we have $\lim_{\varepsilon\to 0} e(S, z_\varepsilon) = e(S, x^*)$. If $S \in \mathcal{D}(\alpha, x^*)$ and $T \notin \mathcal{D}(\alpha, x^*)$, then $e(S, x^*) \geq \alpha > e(T, x^*)$. It follows that for $\varepsilon > 0$ sufficiently small, $e(S, z_\varepsilon) > e(T, z_\varepsilon)$. Choose ε to be small enough for the inequality $e(S, z_\varepsilon) > e(T, z_\varepsilon)$ to hold for all $S \in \mathcal{D}(\alpha, x^*)$ and all $T \notin \mathcal{D}(\alpha, x^*)$. Since $y(S) \geq 0$ for every coalition $S \in \mathcal{D}(\alpha, x^*)$, the excess of every such coalition S satisfies

$$e(S, z_\varepsilon) = v(S) - (x^*(S) + \varepsilon y(S)) = e(S, x^*) - \varepsilon y(S) \leq e(S, x^*). \quad (20.105)$$

We have therefore shown that for $\varepsilon > 0$ sufficiently small,

$$e(S, z_\varepsilon) \leq e(S, x^*), \forall S \in \mathcal{D}(\alpha, x^*), \quad (20.106)$$

$$e(T, z_\varepsilon) < e(S, z_\varepsilon), \forall S \in \mathcal{D}(\alpha, x^*), T \notin \mathcal{D}(\alpha, x^*). \quad (20.107)$$

If one of the inequalities in Equation (20.106) were a strict inequality, then $\theta(z_\varepsilon) \prec_L \theta(x^*)$ would hold, which is impossible since x^* is the prenucleolus. This leads to the conclusion that every inequality in (20.106) holds as an equality, and this and Equation (20.105) imply that $y(S) = 0$ for every coalition $S \in \mathcal{D}(\alpha, x^*)$. In particular, the system of equations (20.99)–(20.100) is tight.

Step 2: If the system of equations (20.101)–(20.102) is tight for every α such that $\mathcal{D}(\alpha, x) \neq \emptyset$, then x is the prenucleolus.
Let $x \in X^0(N; v)$ satisfy the property that the system of equations (20.101)–(20.102) is tight for every α such that $\mathcal{D}(\alpha, x) \neq \emptyset$. Let x^* be the prenucleolus. We need to prove that $x = x^*$. We will show that $\theta(x^*) = \theta(x)$, which, by the uniqueness of the prenucleolus, implies that $x = x^*$.

Denote the excesses of the coalitions at the preimputation x by $a_1, a_2, \ldots, a_p$, where $a_1 > a_2 > \cdots > a_p$. In other words,

$$\theta(x) = (a_1, a_1, \ldots, a_1, a_2, \ldots, a_2, \ldots, a_p, \ldots, a_p). \quad (20.108)$$

In this notation, $\mathcal{D}(a_1, x) \subset \mathcal{D}(a_2, x) \subset \cdots \subset \mathcal{D}(a_p, x) = \{S \subset N, S \neq \emptyset\}$. Let $a_0 > a_1$. Then $\mathcal{D}(a_0, x) = \emptyset$. We will prove by induction over t that $\mathcal{D}(a_t, x) = \mathcal{D}(a_t, x^*)$ for every $t = 0, 1, \ldots, p$, and $e(S, x) = e(S, x^*)$ for every $S \in \mathcal{D}(a_t, x)$. Since $\mathcal{D}(a_p, x) = \{S \subset N, S \neq \emptyset\}$, $e(\emptyset, x) = 0 = e(\emptyset, x^*)$, and $e(N, x) = 0 = e(N, x^*)$, this implies that $\theta(x) = \theta(x^*)$.

The case $t = 0$:
$\mathcal{D}(a_0, x) = \emptyset$, because $a_0 > a_1$. Since x^* is the prenucleolus, it follows that for every coalition S,

$$e(S, x^*) \leq \theta_1(x^*) \leq \theta_1(x) = a_1 < a_0, \quad (20.109)$$

and therefore $\mathcal{D}(a_0, x^*) = \emptyset = \mathcal{D}(a_0, x)$.

The case $t \geq 1$:
Assume as the induction hypothesis that $\mathcal{D}(a_{t-1}, x) = \mathcal{D}(a_{t-1}, x^*)$, and that $e(S, x) = e(S, x^*)$ for every coalition $S \in \mathcal{D}(a_{t-1}, x)$.

Denote by l_{t-1} the number of coalitions in $\mathcal{D}(a_{t-1}, x)$, and by $\widehat{l_t}$ the number of coalitions S satisfying $e(S, x) = a_t$. By the induction hypothesis, the first l_{t-1} coordinates of $\theta(x)$

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equal the first l_{t-1} coordinates of $\theta(x^*)$, and the next $\widehat{l_t}$ coordinates in the vector $\theta(x)$ equal a_t. Since $\theta(x^*) \precsim_L \theta(x)$, it follows that $e(S, x^*) \leq a_t$ for every coalition S that is not in $\mathcal{D}(a_{t-1}, x^*)$.

Define $y := x^* - x \in \mathbb{R}^N$. Then for every coalition $S \subseteq N$,

$$e(S, x) - e(S, x^*) = (v(S) - x(S)) - (v(S) - x^*(S)) = x^*(S) - x(S) = y(S). \tag{20.110}$$

Moreover, since both x^* and x are preimputations,

$$y(N) = x^*(N) - x(N) = v(N) - v(N) = 0. \tag{20.111}$$

Consider the following system of equations:

$$y(N) = 0, \tag{20.112}$$

$$y(S) \geq 0, \quad \forall S \in \mathcal{D}(a_t, x). \tag{20.113}$$

This system of equations is the system (20.101)–(20.102) for $\alpha = a_t$, and therefore it is tight. The vector $y = x^* - x$ is a solution of this system of equations. To see this, note that by the induction hypothesis, $e(S, x) = e(S, x^*)$ for every coalition $S \in \mathcal{D}(a_{t-1}, x)$, and by Equation (20.110), $y(S) = 0$ for every such coalition. As we saw earlier, $e(S, x) = a_t \geq e(S, x^*)$ for every $S \in \mathcal{D}(a_t, x) \setminus \mathcal{D}(a_{t-1}, x)$. Finally, $y(N) = 0$ by Equation (20.111). Because system (20.112)–(20.113) is tight, $y(S) = 0$ for every coalition $S \in \mathcal{D}(a_t, x)$, and therefore using Equation (20.110) one has

$$0 = y(S) = e(S, x) - e(S, x^*), \quad \forall S \in \mathcal{D}(a_t, x). \tag{20.114}$$

This implies that $\mathcal{D}(a_t, x) \subseteq \mathcal{D}(a_t, x^*)$, and that

$$\theta_{l_{t-1}+1}(x^*) = \theta_{l_{t-1}+2}(x^*) = \cdots = \theta_{l_{t-1}+\widehat{l_t}}(x^*) = a_t. \tag{20.115}$$

It remains to prove that $\mathcal{D}(a_t, x) = \mathcal{D}(a_t, x^*)$. Since $\theta_{l_{t-1}+\widehat{l_t}+1}(x) = a_{t+1}$, and since $\theta(x^*) \precsim_L \theta(x)$, we deduce that

$$\theta_{l_{t-1}+\widehat{l_t}+1}(x^*) \leq \theta_{l_{t-1}+\widehat{l_t}+1}(x) = a_{t+1} < a_t, \tag{20.116}$$

which implies that the set of coalitions whose excesses at x equal a_t equals the set of coalitions whose excesses at x^* equal a_t. We conclude that $\mathcal{D}(a_t, x) = \mathcal{D}(a_t, x^*)$, completing the proof of the inductive step, and the proof of Theorem 20.28. □

The next theorem is an immediate corollary of Theorems 20.25 and 20.28.

Theorem 20.29 (Kohlberg) *A necessary and sufficient condition for x^* to be the prenucleolus of a coalitional game $(N; v)$ is for $\mathcal{D}(\alpha, x)$ to be a balanced collection for every $\alpha \in \mathbb{R}$ for which this collection is nonempty.*

With regard to the nucleolus, Kohlberg's Theorem is more complicated. We present the theorem here without proof.

Theorem 20.30 *Denote by $\mathcal{D}_0 = \{\{i\} : i \in N\}$ the collection of all coalitions containing only one player. A necessary and sufficient condition for x^* to be the nucleolus of a coalitional game $(N; v)$ is that, for every α such that $\mathcal{D}(\alpha, x^*)$ is nonempty, the collection*

$\mathcal{D}(\alpha, x^*) \cup \mathcal{D}_0$ is a weakly balanced collection of coalitions[2] *with positive coefficients for the coalitions in $\mathcal{D}(\alpha, x^*)$.*

The nucleolus and the prenucleolus coincide in a large class of games that includes most games studied in applications.

Definition 20.31 *A coalitional game $(N; v)$ is* 0-monotonic *if its 0-normalization is a monotonic game, or equivalently (Exercise 16.27, page 683), if*

$$v(S \cup \{i\}) \geq v(S) + v(i), \quad \forall S \subset N, \forall i \notin S. \tag{20.117}$$

Every superadditive game (Definition 16.8, page 671), as well as every convex game (Definition 17.51, page 717), is a 0-monotonic game.

Theorem 20.32 *In 0-monotonic games, the nucleolus and the prenucleolus coincide.*

To prove this last theorem, we need the following theorem.

Theorem 20.33 *Let $i \in N$ be a player, and let $\mathcal{D}$ be a balanced collection of coalitions such that player i is a member of every coalition in $\mathcal{D}$. Then $\mathcal{D}$ contains only one coalition, namely, $\mathcal{D} = \{N\}$.*

Proof: Let $(\delta_D)_{D \in \mathcal{D}}$ be a vector of balancing weights of $\mathcal{D}$. Then $\sum_{\{S \in \mathcal{D}:\, j \in S\}} \delta_D = 1$ for each player $j \in N$. Since player i is a member of every coalition in $\mathcal{D}$, by setting $j = i$ we deduce that $\sum_{S \in \mathcal{D}} \delta_D = 1$. Let j be any player in N. Since all the weights $(\delta_D)_{D \in \mathcal{D}}$ are positive,

$$1 = \sum_{\{S \in \mathcal{D}:\, j \in D\}} \delta_D \leq \sum_{S \in \mathcal{D}} \delta_D = 1. \tag{20.118}$$

It follows that $\sum_{\{S \in \mathcal{D}:\, j \in D\}} \delta_D = \sum_{S \in \mathcal{D}} \delta_D$, and since all the weights are positive, $\{S \in \mathcal{D} :\, j \in D\} = \mathcal{D}$; that is, every player j is a member of every coalition in $\mathcal{D}$. Since every coalition in the collection may appear only once, $\mathcal{D} = \{N\}$, as claimed. □

Proof of Theorem 20.32: Let $(N; v)$ be a 0-monotonic game, and let x^* be its prenucleolus. We will show that the prenucleolus is individually rational: $x^*_i \geq v(i)$ for every $i \in N$, and it therefore coincides with the nucleolus (Theorem 20.15, page 809).

Suppose by contradiction that there exists a player i for whom $x^*_i < v(i)$. Then for every $S \subseteq N \setminus \{i\}$,

$$\begin{aligned}
e\left(S \cup \{i\}, x^*\right) &= v\left(S \cup \{i\}\right) - x^*(S \cup \{i\}) &\quad (20.119)\\
&= v\left(S \cup \{i\}\right) - x^*(S) - x^*_i &\quad (20.120)\\
&\geq v(S) + v(i) - x^*(S) - x^*_i &\quad (20.121)\\
&= e(S, x^*) + (v(i) - x^*_i) &\quad (20.122)\\
&> e(S, x^*). &\quad (20.123)
\end{aligned}$$

2 Recall that a collection $\mathcal{D}$ of coalitions is weakly balanced if there are nonnegative numbers $(\alpha_S)_{S \in \mathcal{D}}$ such that $\sum_{S \in \mathcal{D}} \alpha_S \chi^S = \chi^N$.

Equation (20.121) holds because the game is 0-monotonic, and Equation (20.123) holds because by assumption $x_i^* < v(i)$. Thus, player i is a member of every coalition with maximal excess. Let $\mathcal{D}$ be the set of all such coalitions. By Kohlberg's Theorem (Theorem 20.29), this is a balanced collection, and player i is a member of every coalition in it. By Theorem 20.33, $\mathcal{D} = \{N\}$. However, since x^* is the prenucleolus, it satisfies $v(N) = x^*(N)$, and therefore $e(N, x^*) = 0$. Since $\mathcal{D}$ contains the coalitions with maximal excess, the excesses of all the other coalitions must be strictly less than 0. In particular,

$$0 > e(\{i\}, x^*) = v(i) - x_i^*, \tag{20.124}$$

which contradicts the assumption that $x_i^* < v(i)$. This contradiction proves that $x_i^* \geq v(i)$ for every player i, which is what we wanted to show. □

20.6 The consistency of the nucleolus

In Chapter 18, we showed that the Shapley value satisfies the property of consistency with respect to the Hart–Mas-Colell reduced game. In this section, we prove that the prenucleolus satisfies the property of consistency, with respect to the other notion of reduced game, the Davis–Maschler reduced game, which we discussed in Chapter 17. We begin by recalling the definition of a Davis–Maschler reduced game, and the definition of consistency.

Definition 20.34 *Let $(N; v)$ be a coalitional game, let S be a nonempty coalition, and let $x \in \mathbb{R}^N$ be a preimputation. The* Davis–Maschler reduced game *to coalition S at x, denoted by $(S; w_S^x)$, is the coalitional game with the set of players S, and a coalitional function w_S^x defined by*

$$w_S^x(R) = \begin{cases} \max_{Q \subseteq S^c}(v(R \cup Q) - x(Q)) & \emptyset \neq R \subset S, \\ x(S) & R = S, \\ 0 & R = \emptyset. \end{cases} \tag{20.125}$$

Definition 20.35 *A solution concept φ satisfies the* Davis–Maschler reduced game property *if for every game coalitional $(N; v)$, for every nonempty coalition $S \subseteq N$, and for every vector $x \in \varphi(N; v)$,*

$$(x_i)_{i \in S} \in \varphi\left(S; w_S^x\right). \tag{20.126}$$

Theorem 20.36 *The prenucleolus satisfies the Davis–Maschler reduced game property.*

Proof: Let $(N; v)$ be a coalitional game, let x^* be the prenucleolus of the game $(N; v)$, and let S be a nonempty coalition. Denote the restriction of x^* to the coalition S by

$$x_S^* := (x_i^*)_{i \in S}. \tag{20.127}$$

These are the coordinates of the members of S in the prenucleolus. For each coalition $R \subseteq S$, denote by $e(R, x^*; v)$ the excess of coalition R at x^* in the game $(N; v)$, and by $e(R, x_S^*; w_S^{x^*})$ the excess of this coalition at x_S^* in the reduced game $(S; w_S^{x^*})$.

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To check that x^*_S is the prenucleolus of the reduced game $(S; w^{x^*}_S)$, we must first ascertain that x^*_S is a preimputation of this game. By the definition of the reduced game $w^{x^*}_S(S) = x^*(S)$, and therefore $x^*_S \in X^0(S; w^{x^*}_S)$.

For every $\alpha \in \mathbb{R}$, define

$$\mathcal{D}^S_w(\alpha, x^*_S) := \left\{R \subset S\colon R \neq \emptyset, \quad e\left(R, x^*_S; w^{x^*}_S\right) \geq \alpha\right\}. \tag{20.128}$$

The collection of coalitions $\mathcal{D}^S_w$ is the analogue, in the reduced game, to the collection of coalitions whose excess at x^* in the original game is greater than or equal to α, which is

$$\mathcal{D}_v(\alpha, x^*) = \{R \subset N\colon R \neq \emptyset, \quad e(R, x^*; v) \geq \alpha\}. \tag{20.129}$$

We now compare $\mathcal{D}^S_w(\alpha, x^*_S)$ and $\mathcal{D}_v(\alpha, x^*_S)$. For every coalition R in $\mathcal{D}^S_w\left(\alpha, x^*_S\right)$, $R \notin \{\emptyset, S\}$,

$$e\left(R, x^*_S; w^{x^*}_S\right) = w^{x^*}_S(R) - x^*(R) \tag{20.130}$$

$$= \max_{Q \subseteq S^c}(v(R \cup Q) - x^*(Q)) - x^*(R) \tag{20.131}$$

$$= e(R \cup Q_R, x^*; v). \tag{20.132}$$

Here, $Q_R \subseteq S^c$ is the coalition at which the maximum of Equation (20.131) is attained. Therefore, if $\mathcal{D}^S_w(\alpha, x^*_S)$ is nonempty, then $\mathcal{D}_v(\alpha, x^*)$ is also nonempty, because if $R \in \mathcal{D}^S_w(\alpha, x^*_S)$, then $R \cup Q_R \in \mathcal{D}_v(\alpha, x^*)$.

In the other direction, suppose that $T \in D_v(\alpha, x^*)$. Denote $R = T \cap S$. Then $T = R \cup (T \setminus S)$. If R is neither the empty set nor the set S, then

$$e\left(R, x^*_S; w^{x^*}_S\right) = w^{x^*}_S(R) - x^*(R) \tag{20.133}$$

$$= \max_{Q \subseteq S^c}\left(v(R \cup Q) - x^*(Q)\right) - x^*(R) \tag{20.134}$$

$$\geq v(R \cup (T \setminus S)) - x^*(T \setminus S) - x^*(R) \tag{20.135}$$

$$= e(R \cup (T \setminus S), x^*; v) \tag{20.136}$$

$$= e(T, x^*; v) \geq \alpha, \tag{20.137}$$

where Equation (20.135) holds because $Q = T \setminus S$ is one of the coalitions in the maximization in Equation (20.134). In particular, $R \in \mathcal{D}^S_w(\alpha, x^*_S)$. Summarizing what we have proved so far: for every $\alpha \in \mathbb{R}$,

- if $R \in \mathcal{D}^S_w(\alpha, x^*_S)$, there exists a coalition T containing R satisfying $T \in D_v(\alpha, x^*)$ and $T \cap S = R$;
- if $T \in \mathcal{D}_v(\alpha, x^*)$, and if $\emptyset \neq T \cap S \subset S$, then $T \cap S \in \mathcal{D}^S_w(\alpha, x^*_S)$ (and if $T \cap S$ is empty or equal to S, then $e(T \cap S, x^*_S; w^{x^*}_S) = 0$).

By the above, the collection $\mathcal{D}^S_w(\alpha, x^*_S)$ is derivable from the collection $\mathcal{D}_v(\alpha, x^*)$ by removing players who are not members of S from every coalition. This act of removing players may leave only the coalition S, or the empty coalition, and in those cases these coalitions must be removed, because by definition $\mathcal{D}^S_w(\alpha, x^*_S)$ does not contain these coalitions. It may happen, of course, that a coalition R emerges several times during this process. In that case, only one copy of that coalition is included in $\mathcal{D}^S_w(\alpha, x^*_S)$.

By Kohlberg's Theorem (Theorem 20.29), since x^* is the prenucleolus, $\mathcal{D}_v(\alpha, x^*)$ is a balanced collection for every $\alpha \in \mathbb{R}$ for which the collection $\mathcal{D}_v(\alpha, x^*)$ is not empty.

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Removing players who are not in S from the balanced collection $\mathcal{D}_v(\alpha, x^*)$ leaves either an empty collection or a nonempty collection that is also balanced (as follows from Definition 20.24). It follows that $\mathcal{D}_w^S(\alpha, x_S^*)$ is a balanced collection, for every $\alpha \in \mathbb{R}$, if it is not empty, and then Kohlberg's Theorem implies that x_S^* is the prenucleolus of the reduced game. □

The consistency of the nucleolus is a more complex issue than the consistency of the prenucleolus. First of all, in a game reduced to a coalition S with respect to the nucleolus, the set of imputations may be empty, in which case the nucleolus is also empty.

Example 20.37 Consider the three-player simple majority game $(N; v)$; i.e, $N = \{1, 2, 3\}$ and the coalitional function is

$$v(S) = \begin{cases} 1 & |S| \geq 2, \\ 0 & |S| \leq 1. \end{cases} \tag{20.138}$$

Since every pair of players is symmetric, the nucleolus is $x^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The Davis–Maschler reduced game to the coalition $S = \{1, 2\}$ at the nucleolus x^* is the game $(S; w_S^{x^*})$ where

$$w_S^{x^*}(1) = w_S^{x^*}(2) = w_S^{x^*}(1, 2) = \tfrac{2}{3} \tag{20.139}$$

(check that this is true). Since $w_S^{x^*}(1) + w_S^{x^*}(2) > w_S^{x^*}(1, 2)$, the set of imputations $X(S; w_S^{x^*})$ is empty, and the nucleolus of the game $(S; w_S^{x^*})$ is therefore empty. ◀

Example 20.37 shows that the nucleolus does not satisfy the Davis–Maschler reduced game property over the family of all games. For the reduced game property to be meaningful in this context, we need to restrict our attention to coalitional games $(N; v)$ that satisfy $X(S; w_S^{x^*}) \neq \emptyset$ for every nonempty coalition S, where x^* is the nucleolus of the game.

By Theorem 20.32, in 0-monotonic games the prenucleolus and the nucleolus coincide. The following theorem states that the nucleolus satisfies the Davis–Maschler reduced game property over the family of 0-monotonic games that satisfy the condition that for every nonempty coalition S, the game $(S; w_S^{x^*})$ is also 0-monotonic.

Theorem 20.38 *Let $(N; v)$ be a 0-monotonic game, with nucleolus x^*. If for each coalition $S \subseteq N$ the Davis–Maschler reduced game $(S; w_S^{x^*})$ is 0-monotonic, then x_S^* is the nucleolus of the game $(S; w_S^{x^*})$.*

Proof: By Theorem 20.32, x^* is the prenucleolus of the game $(N; v)$. By Theorem 20.36, x_S^* is the prenucleolus of the reduced game $(S; w_S^{x^*})$. Since this game is 0-monotonic, by Theorem 20.32 x_S^* is also the nucleolus of the reduced game. □

20.7 Weighted majority games

In this section we study the nucleolus in a large class of simple games, the class of weighted majority games, which are pervasive in the study of elections and committee decision-making. We start by recalling the definition of a simple game.

Definition 20.39 *A coalitional game* $(N; v)$ *is a* simple game *if the worth of every coalition is* 0 *or* 1*; i.e.,* $v(S) \in \{0, 1\}$ *for every coalition* $S \subseteq N$.

Definition 20.40 *In a simple game* $(N; v)$*:*

- *A coalition* S *is a* winning coalition *if* $v(S) = 1$.
- *A coalition* S *is a* losing coalition *if* $v(S) = 0$.
- *A coalition* S *is a* minimal winning coalition *if it is a winning coalition, and every one of its proper subcoalitions is a losing coalition.*

A simple game is *monotonic* if $v(S) = 1$ and $T \supseteq S$ implies that $v(T) = 1$. When a game is simple and monotonic, it is determined by the set of its minimal winning coalitions, which we denote by $\mathcal{W}^m$.

Definition 20.41 *A coalitional game* $(N; v)$ *is a* constant-sum game *if it satisfies*

$$v(S) + v(S^c) = v(N), \quad \forall S \subseteq N. \tag{20.140}$$

In this section, we concentrate on a subclass of the class of simple games: the class of simple strong games.

Definition 20.42 *A simple game* $(N; v)$ *is* strong *if it is monotonic, constant sum, and satisfies* $v(N) = 1$.

By Equation (20.140), in a simple strong game the complement of a winning coalition is a losing coalition, and vice versa.

The class of weighted majority games is a proper subclass of the class of simple games.

Definition 20.43 *A simple game* $(N; v)$ *is a* weighted majority game *if there exist non-negative numbers* $q, w_1, w_2, \ldots, w_n$ *such that*

$$v(S) = 1 \quad \Longleftrightarrow \quad w(S) \geq q, \tag{20.141}$$

$$v(S) = 0 \quad \Longleftrightarrow \quad w(S) < q, \tag{20.142}$$

where $w(S) = \sum_{i \in S} w_i$. *The quantity* q *is the* quota*, and* w_i *is the* weight of player i*, for every* $i \in N$. *We denote the weighted majority game by* $[q; w]$*, where* $w = (w_1, w_2, \ldots, w_n)$*, and call* $[q; w]$ *a* representation *of the game* $(N; v)$.

By definition, a weighted majority game is simple and monotonic, but not every simple and monotonic game is a weighted majority game, even if it is constant sum (see the example of a projective game, Exercise 16.10 on page 680). As mentioned in Chapter 16, weighted majority games are generally the appropriate type of games to use for modeling elections or majority decision-making. The players in these models may be political parties, where the weight of each player is the fraction of the seats in the parliament occupied by that party. The quota is determined by the specific rules used for adopting a decision. For example, if a majority of $\frac{2}{3}$ is required to pass a decision, then q is $\frac{2}{3}$ of the members of parliament. If only a simple majority is required, then q is the smallest integer that is greater than half the number of members of parliament.

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By multiplying the quota and all the weights by the same positive constant, we obtain another representation of the same game; such a multiplication does not affect the coalitional function v of the game. A weighted majority game therefore has many possible representations. The next example shows that a game may also have different representations that cannot be derived from each other by multiplication by a positive constant.

Example 20.44 Consider the following simple majority game $(N; v)$ in which $N = \{1, 2, 3\}$ and the coalitional function is given by

$$v(1) = v(2) = v(3) = 0, \quad v(1,2) = v(1,3) = v(2,3) = v(1,2,3) = 1. \qquad (20.143)$$

This game has many possible representations, such as, for example, $[3; 2, 2, 2]$, $[5; 4, 4, 1]$, and $[6; 3, 4, 5]$ (check that each of these is indeed a representation of the game). Which of these representations seems the most "natural"? Since the game is symmetric, one may regard the representation $[3; 2, 2, 2]$ as the most natural. This representation has the property that the total weight of the members of every minimal winning coalition is the same (in this case, this total weight is 4). ◀

This property of a natural representation can be generalized as follows.

Definition 20.45 *A representation* $[q; w_1, w_2, \ldots, w_n]$ *of a weighted majority game* $(N; v)$ *is a* homogeneous representation *if the sum* $\sum_{i \in S} w_i$ *is the same for every minimal winning coalition* $S \in \mathcal{W}^m$.

In Example 20.44, the representation $[3; 2, 2, 2]$ is homogeneous, while the representations $[5; 4, 4, 1]$ and $[6; 3, 4, 5]$ are not homogeneous. The representation $[4; 2, 2, 2]$ is yet another homogeneous representation of the same game.

Not every weighted majority game has a homogeneous representation. For example, the game $[5; 2, 2, 2, 1, 1, 1]$ is a game without a homogeneous representation (Exercise 20.41).

Definition 20.46 *A weighted majority game* $(N; v)$ *is* homogeneous *if it has a homogeneous representation.*

The goal of this section is to prove that in every constant-sum weighted majority game, the nucleolus is a set of weights for a representation of the game, and in fact that representation is homogeneous if the game is homogeneous.

Definition 20.47 *A representation* $[q; w]$ *of a weighted majority game* $(N; v)$ *is* normalized *if* $\sum_{i=1}^{n} w_i = 1$.

If a game has a representation, it has a normalized representation. Normalizing the three representations of the game in Example 20.44 shows that a game may have several normalized representations. Note that if $[q; w]$ is a homogeneous representation of a game, then its normalization is also a homogeneous representation of the same game. It follows, in particular, that every homogeneous game has a normalized homogeneous representation. A homogeneous game can have several normalized representations. For example, for every $q \in (\frac{1}{3}, \frac{2}{3}]$, $[q; \frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$ is a normalized representation of the coalitional

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game in Example 20.44. As the following example shows, a game may have different normalized homogeneous representations that differ from each other by their weights.

Example 20.48 **Dictator game** Let $(N; v)$ be a simple strong game, and suppose that there is a player i such that $v(i) = 1$. Since the game is both monotonic and constant sum,

$$v(S) = 1 \quad \Longleftrightarrow \quad i \in S. \tag{20.144}$$

The only minimal winning coalition of the game is $\{i\}$: $\mathcal{W}^m = \{\{i\}\}$.

The game has several normalized homogeneous representations: for every $q \in (\frac{1}{2}, 1]$, every representation $[q; w]$ satisfying (a) $w_i = q$ and (b) $\sum_{j \in N} w_j = 1$ is a normalized homogeneous representation of the game. In such a game, all the players in the set $N \setminus \{i\}$ are null players. The only normalized vector of weights in which the weight of each null player is zero is

$$w_j = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases} \tag{20.145}$$

Therefore, all the normalized homogeneous representations of the game have the form $[q; w]$, where $q \in (0, 1]$. The vector of weights w is both the Shapley value and the nucleolus of the game, since under both solution concepts null players receive 0. This is also the only vector in the core. ◀

Let $[q; w_1, w_2, \ldots, w_n]$ be a constant-sum weighted majority game. If we increase the quota to $\min_{S \in \mathcal{W}^m} w(S)$, we get another representation of the same game, different from the original one when $q \neq \min_{S \in \mathcal{W}^m} w(S)$ (Exercise 20.43). This is the maximal quota that does not change the set of winning coalitions. This motivates the following definition.

Definition 20.49 *Let $(N; v)$ be a simple game, and let $w \in \mathbb{R}^N$ be a vector. Define*

$$q(w) := \min_{S \in \mathcal{W}^m} w(S). \tag{20.146}$$

Note that $q(w)$ depends on the coalitional function v, since v determines the set of minimal winning coalitions. Since the quota $q(w)$ is determined by the weights, when we say that w is a representation of the game, we will mean that $[q(w); w]$ is a representation. When $[q(w); w]$ is a homogeneous representation, then $w(S) = q(w)$ for every minimal winning coalition S.

Given a simple strong game $(N; v)$, every imputation $x \in X(N; v)$ defines a weighted majority game $[q(x); x]$. As a weighted majority game, this game is simple and monotonic but not necessarily a constant-sum game (Exercises 20.44 and 20.45). In particular, the weighted majority game $[q(x); x]$ is not necessarily a representation of the game $(N; v)$ with which we started. We would like to know for which imputations the weighted majority game $[q(x); x]$ is a representation of the original game $(N; v)$.

Theorem 20.50 *Let $(N; v)$ be a simple strong game, and let $x \in X(N; v)$. Then $[q(x); x]$ is a representation of $(N; v)$ if and only if $q(x) > \frac{1}{2}$.*

Proof: Since $x \in X(N; v)$, it follows in particular that $x(N) = \sum_{i \in N} x_i = v(N) = 1$. Suppose first that $[q(x); x]$ is a representation of $(N; v)$. In other words, a coalition is a winning coalition if and only if the total weight of its members is greater than or equal to $q(x)$. We will show that $q(x) > \frac{1}{2}$. Let S_* be a minimal winning coalition in which $q(x)$

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is attained, i.e., $x(S_*) = q(x)$. Since S_* is a winning coalition, and since a simple strong game is a constant-sum game, the complementary coalition S_*^c is a losing coalition and hence $x(S_*^c) < q(x)$. One deduces from this that

$$q(x) > x(S_*^c) = 1 - x(S_*) = 1 - q(x), \tag{20.147}$$

and therefore $q(x) > \frac{1}{2}$, as we wanted to show.

Suppose next that $q(x) > \frac{1}{2}$. We will show that $[q(x); x]$ is a representation of $(N; v)$. For this, we need to show that $v(S) = 1$ if and only if $x(S) \geq q(x)$. If S is a winning coalition, then it must contain a minimal winning coalition, and hence $x(S) \geq q(x)$. On the other hand, if S is a losing coalition, since the game is constant sum, S^c is a winning coalition, hence $x(S^c) \geq q(x)$, and then

$$x(S) = x(N) - x(S^c) = 1 - x(S^c) \leq 1 - q(x) < \tfrac{1}{2} < q(x). \tag{20.148}$$

We proved that if S is a losing coalition, then $x(S) < q(x)$, and therefore if $x(S) \geq q(x)$ then $v(S) = 1$, as required. □

Theorem 20.51 *Let $(N; v)$ be a simple strong game and let $x^* = \mathcal{N}(N; v)$ be the nucleolus of the game. Then $q(x^*) \geq q(x)$ for all $x \in X(N; v)$.*

Proof: Let $x \in X(N; v)$. Since x^* is the nucleolus, $\theta_1(x^*) \leq \theta_1(x)$, i.e,

$$\max_{S \subseteq N} e(S, x^*) \leq \max_{S \subseteq N} e(S, x). \tag{20.149}$$

Now,

$$\max_{S \subseteq N} e(S, x) = \max \left\{ \max_{\{S:\, v(S)=1\}} e(S, x), \max_{\{S:\, v(S)=0\}} e(S, x) \right\} \tag{20.150}$$
$$= \max_{\{S:\, v(S)=1\}} e(S, x) \tag{20.151}$$
$$= \max_{\{S:\, v(S)=1\}} (1 - x(S)) \tag{20.152}$$
$$= \max_{S \in \mathcal{W}^m} (1 - x(S)) \tag{20.153}$$
$$= 1 - \min_{S \in \mathcal{W}^m} x(S) \tag{20.154}$$
$$= 1 - q(x). \tag{20.155}$$

Equation (20.151) holds because $\max_{\{S:\, v(S)=1\}} (v(S) - x(S)) \geq 0$ (since N is a winning coalition with excess 0), and on the other hand $\max_{\{S:\, v(S)=0\}} e(S, x) \leq 0$ (since for a losing coalition $v(S) = 0$ and $x(S) \geq 0$, because $x \in X(N; v)$). Equation (20.153) holds because $x_i \geq 0$ for all players (since $x \in X(N; v)$), and therefore maximizing $1 - x(S)$ over all winning coalitions is equivalent to maximizing this quantity over all minimal winning coalitions. By setting $x = x^*$ in Equations (20.150)–(20.155) we get $\max_{S \subseteq N} e(S, x^*) = 1 - q(x^*)$. By Equation (20.149) we obtain

$$q(x^*) = 1 - \max_{S \subseteq N} e(S, x^*) \geq 1 - \max_{S \subseteq N} e(S, x) = q(x), \tag{20.156}$$

as claimed. □

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The next theorem states that the nucleolus is a normalized representation of strong weighted majority games.

Theorem 20.52 *Let* $(N; v)$ *be a strong weighted majority game, and let* x^* *be its nucleolus. Then* $[q(x^*); x^*]$ *is a normalized representation of the game* $(N; v)$.

Proof: Let $[q; w]$ be a normalized representation of $(N; v)$. Then $q(w) > \frac{1}{2}$ by Theorem 20.50, and $q(x^*) \geq q(w)$ by Theorem 20.51; hence $q(x^*) > \frac{1}{2}$. Since $x^*(N) = v(N) = 1$, we may again use Theorem 20.50, to conclude that $[q(x^*); x^*]$ is a normalized representation of the game. □

Theorem 20.53 *Let* $(N; v)$ *be a homogeneous weighted majority game satisfying* $v(N) = 1$. *Then it has a homogeneous representation in which the weight of each null player is* 0.

Proof: Let $[q; w]$ be a homogeneous representation of the game. Then $q(w) = w(S)$ for every minimal winning coalition S. Let D be the set of null players. Define a quota $\widehat{q}$ and a vector of weights $\widehat{w}$ in which the weight of null players is 0 as follows:

$$\widehat{q} := q(w), \tag{20.157}$$

$$\widehat{w}_i := \begin{cases} w_i & i \notin D, \\ 0 & i \in D. \end{cases} \tag{20.158}$$

We will ascertain that $[\widehat{q}; \widehat{w}]$ is a homogeneous representation of the game. To this end, we will check that $\widehat{w}(S) = \widehat{q}$ for every minimal winning coalition S. Indeed, since a minimal winning coalition cannot contain a null player, it follows that for every minimal winning coalition S,

$$\widehat{w}(S) = \sum_{i \in S} w_i = q(w) = \widehat{q}. \tag{20.159}$$

□

Theorem 20.54 *Let* $(N; v)$ *be a homogeneous and constant-sum weighted majority game satisfying* $v(N) = 1$. *Then the nucleolus is the only homogeneous normalized representation in which the weight of every null player is* 0.

Proof: If there is a player i such that $v(i) = 1$, then i is a dictator (explain why). As we saw in Example 20.48, the nucleolus in that case is the only normalized representation in which the weight of every null player is 0. We will therefore consider the case in which $v(i) = 0$ for every player i.

Since $v(i) = 0$ for each player i, and since $v(N) = 1$, the set of imputations $X(N; v)$ is nonempty, and hence, by Corollary 20.14, the nucleolus is a nonempty set that contains a single vector x^*.

Denote by D the set of all the null players in the game. Since the game is homogeneous, it has a homogeneous representation. By Theorem 20.53, the game has a homogeneous representation in which the weight of every null player is 0. Denote that representation by $[q(y); y]$. Then $y(S) = q(y)$ for every minimal winning coalition $S \in \mathcal{W}^m$. Define a

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polytope P as follows:[3]

$$P := \{x \in \mathbb{R}^n : x_i \geq v(i),\ \forall i \in N, x(N) = v(N),$$
$$x(S) \geq q(y)\ \forall S \in \mathcal{W}^m,\ x_i = 0\ \forall i \in D\}. \tag{20.160}$$

The first two conditions in the definition of P guarantee that P is contained in $X(N; v)$. Since the game is a weighted majority game, if $x \in P$ then $x(S) \geq q(y)$ for every winning coalition $S \subseteq N$. Since y is a homogeneous representation in which the weight of every null player is 0, $y \in P$, and in particular $y \in X(N; v)$.

We next show that the nucleolus, x^*, is also in P. Since $x^* \in X(N; v)$, the vector x^* satisfies the first two conditions in the definition of P. By Theorem 20.52, the nucleolus is a representation of the game. By Theorem 20.51 one has $x^*(S) \geq q(x^*) \geq q(y)$ for every $S \in \mathcal{W}^m$. Finally, since the nucleolus satisfies the null player property (Theorem 20.19, page 811), $x^*_i = 0$ for every player $i \in D$, thus proving that indeed $x^* \in P$.

We will now show that the set P contains only one vector, and that in particular $y = x^*$. This will imply that the nucleolus is the unique homogeneous representation in which the weight of every null player is 0. If the set P contains at least two imputations, then in particular it has at least two extreme points, and one of them, z, must be distinct from y. This then implies that some of the inequalities defining P that did not obtain as equalities for y obtain as equalities for z. Since for y, all inequalities of the form $x^*(S) \geq q(y)$ obtain as equalities, at least one of the inequalities $x_i \geq v(i)$ obtains as an equality for z and as a strict inequality for y. This necessarily occurs for $i \notin D$, since $y_i = 0 = v(i)$ for every $i \in D$.

Let j be a player who is not in D, and for whom $z_j = 0$. Since j is not a null player, there is a coalition S such that $v(S) = 0$ and $v(S \cup \{j\}) = 1$. Let S_0 be a minimal coalition (with respect to set inclusion) such that this property holds. Then $S_0 \cup \{j\} \in \mathcal{W}^m$ (verify this). Since the coalition S_0 is a losing coalition, and since the game is constant sum, the complementary coalition S_0^c, containing j, is a winning coalition. Hence

$$q(y) \leq z\left(S_0^c\right) = 1 - z(S_0) = 1 - z(S_0 \cup \{j\}) \leq 1 - q(y). \tag{20.161}$$

The first inequality holds because $z(T) \geq q(y)$ for every winning coalition T, and S_0^c is a winning coalition. The last inequality holds for the same reason, and because $S_0 \cup \{j\}$ is a winning coalition. The second equality holds because $z_j = 0$. Equation (20.161) implies that $q(y) \leq \frac{1}{2}$, contradicting Theorem 20.50. The contradiction proves that P contains a single imputation, as we wanted to show. □

20.8 The bankruptcy problem

Many legislators have contended with the issue of the best way to divide the assets of a bankrupt entity among the creditors, given that the total sum of the debts owed by the bankrupt entity is greater than the available assets. To date, various bankruptcy rules apply

3 Recall that a *polytope* in $\mathbb{R}^n$ is the convex hull of a finite number of points in $\mathbb{R}^n$.

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in different societies. In the Babylonian Talmud,[4] in Chapter Ten of Tractate Kethubot, the following item appears, attributed to Rabbi Nathan:

If a man who was married to three wives died, and the *kethubah*[5] of one was a *maneh*,[6] of the other two hundred *zuz*, and of the third three hundred *zuz*, and the estate [was worth] only one *maneh*, [the sum] is divided equally. If the estate [was worth] two hundred *zuz* [the claimant] of the *maneh* receives fifty *zuz* [and the claimants respectively] of the two hundred and three hundred *zuz* [receive each] three gold *denarii*.[7] If the estate [was worth] three hundred *zuz*, [the claimant] of the *maneh* receives fifty *zuz* and [the claimant] of the two hundred *zuz* [receives] a *maneh*, while [the claimant] of the three hundred *zuz* [receives] six gold *denarii*.

The Talmud's prescription of division of the estate (in units of *zuz*) is summarized in the following table. To simplify the analysis, we will call the first wife Anne, the second wife Betty, and the third wife Carol.

		Anne	Betty	Carol
	Debt:	100	200	300
The Estate	100	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
	200	50	75	75
	300	50	100	150

The idea behind this seemingly strange division of the estate appears to be unclear. When the size of the estate is small, the creditors divide the assets equally among them; when the size of the estate is large, the creditors divide the assets proportionally to their claims; and when the size of the estate is intermediate, the division seems inexplicable.

In this section, we relate the above Talmudic bankruptcy division recommendation to the nucleolus of an appropriate game in coalitional form. We begin by considering yet another passage from the Babylonian Talmud, appearing in Tractate Baba Metzia, Chapter 1, Mishnah 1:

Two [persons appearing before a court] hold a garment. One of them says, "I found it," and the other says "I found it"; one of them says "it is all mine" and the other says, "it is all mine." Then the one shall swear that his share in it is not less than half, and the other shall swear that his share in it is not less than half, and [the value of the garment] shall then be divided between them. If one says "it is all mine" and the other says "half of it is mine," he who says "it is all mine" shall swear that his share in it is not less than three quarters, and he who says "half of it is mine" shall swear that his share in it is not less than a quarter. The former then receives three quarters.

In other words, the Talmud here is recommending that if two parties each claim one hundred percent ownership of an asset, the asset should be divided among them equally.

4 The Babylonian Talmud is a Jewish text that records rabbinic discussions on Jewish law, ethics, customs, and philosophy held between the second and fifth centuries AD. The quotations below are taken from The Babylonian Talmud, translated and edited by Rabbi Dr. Isidore Epstein, Soncino Press, London. Available at: http://www.halakhah.com.

5 The *kethubah* is the traditional Jewish wedding document, and it specifies promises the groom makes to the bride, including the amount of money to be given to her in the event of the annulment of the marriage, either by the death of the husband or divorce.

6 The basic monetary unit in this passage is *zuz*, which was an ancient Near Eastern silver coin. One *maneh* is equivalent to 100 *zuz*.

7 The *denarium* (plural *denarii*) was an ancient Roman coin. One gold *denarium* was equivalent to 25 *zuz*. Therefore, 3 gold *denarii* were equivalent to 75 *zuz*, and 6 gold *denarii* were equivalent to 150 *zuz*.

If one party claims one hundred percent ownership of an asset, and the other party claims only fifty percent ownership, then the division should be $\frac{3}{4} : \frac{1}{4}$.

The logic behind this Talmudic passage is clear. If both parties have an equal claim to the asset, by symmetry they should divide it equally. If, in contrast, one party claims all the asset, and the second party claims only half of the asset, then in effect both parties agree that at least one half of the asset belongs to the first party, and therefore there can be no dispute that the first party should get at least half of the asset. With respect to the second half of the asset, the claimants are symmetric, and therefore that half should be divided equally.

In the general case, if a person claims that his share in an asset is p, while another person claims that his share of the same asset is q (where $0 \leq p \leq 1$, $0 \leq q \leq 1$ and $p + q > 1$), the first person receives, by this reasoning, $(1 - q) + \frac{p+q-1}{2}$ of the asset, and the second person[8] receives $(1 - p) + \frac{p+q-1}{2}$. This solution for two-creditor bankruptcy problems is called the "contested garment" solution.

20.8.1 The model

Definition 20.55 *A* bankruptcy problem *is given by:*

1. *a set of players $N = \{1, 2, \ldots, n\}$,*
2. *for each player $i \in N$, a nonnegative real number $d_i \in \mathbb{R}_+$,*
3. *and a nonnegative real number $E \in \mathbb{R}_+$ such that $E < \sum_{i \in N} d_i$.*

We interpret N to be a set of creditors, E to be the total worth of the assets of a bankrupt entity, and d_i to be the amount of money that the bankrupt entity owes creditor i. Under this interpretation, the condition $E < \sum_{i \in N} d_i$ states that the debtor lacks sufficient capital to repay his debts. If this inequality does not hold, then the debtor can pay all that he owes, and he cannot justifiably be declared bankrupt. A bankruptcy problem will be denoted by $[E; d_1, \ldots, d_n]$, or by $[E; d]$ for short, where $d = (d_1, d_2, \ldots, d_n)$.

Definition 20.56 *An* allocation *for a bankruptcy problem $[E; d_1, \ldots, d_n]$ is a vector $x \in \mathbb{R}^N_+$ satisfying $x(N) = E$.*

An allocation is thus a suggested way of dividing the assets of the bankrupt entity among the creditors.

Definition 20.57 *A* solution concept for bankruptcy problems *is a function φ associating every bankruptcy problem $[E; d_1, \ldots, d_n]$ with an allocation.*

20.8.2 The case $n = 2$

The Babylonian Talmud, in its passage on the "contested garment," suggests a solution for every two-creditor bankruptcy problem $[E; d_1, d_2]$. Suppose, without loss of generality, that $d_2 \geq d_1$.

8 There is another passage in the Babylonian Talmud (Tractate Yevamoth, page 38) in which this reasoning is applied. There, one party claims 50% of an asset, and a second party claims $66\frac{2}{3}\%$ of the same asset. The Talmud then instructs the first party to yield 50% of the asset to the second party, and instructs the second party to yield $33\frac{1}{3}\%$ of the asset to the first party, with the two parties then dividing the remainder equally between them.

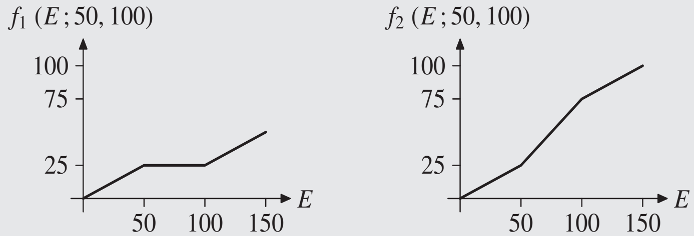

Figure 20.3 The functions $E \mapsto f_1(E;50,100)$ and $E \mapsto f_2(E;50,100)$

If $d_1 \geq E$, then each creditor is in effect claiming that the entire asset belongs to him, and the asset should therefore be divided equally between them. This yields the allocation

$$\left(\frac{E}{2}, \frac{E}{2}\right). \tag{20.162}$$

If $d_1 \leq E \leq d_2$, then the first creditor agrees that $E - d_1$ of the estate belongs to the second creditor, with both creditors claiming ownership over the remainder. This yields the allocation

$$\left(\frac{d_1}{2}, E - \frac{d_1}{2}\right). \tag{20.163}$$

Finally, we analyze the case where $d_1 \leq d_2 \leq E$ (and $d_1 + d_2 > E$). In this case, the first creditor agrees that $E - d_1$ belongs to the second creditor, and the second creditor agrees that $E - d_2$ belongs to the first creditor. That leaves the remainder under dispute, i.e.,

$$E - (E - d_1) - (E - d_2) = d_1 + d_2 - E, \tag{20.164}$$

which is then divided equally between the creditors. The resulting allocation is thus

$$\left(\frac{E + d_1 - d_2}{2}, \frac{E + d_2 - d_1}{2}\right). \tag{20.165}$$

This discussion can be summarized as follows. Define a function f associating a vector in $\mathbb{R}^2$ to every two-creditor bankruptcy problem $[E; d_1, d_2]$ as follows:

$$f(E; d_1, d_2) = \begin{cases} \left(\frac{E}{2}, \frac{E}{2}\right) & \text{if } E \leq d_2, d_1, \\ \left(\frac{d_1}{2}, E - \frac{d_1}{2}\right) & \text{if } d_1 \leq E \leq d_2, \\ \left(E - \frac{d_2}{2}, \frac{d_2}{2}\right) & \text{if } d_2 \leq E \leq d_1, \\ \left(\frac{E+d_1-d_2}{2}, \frac{E+d_2-d_1}{2}\right) & \text{if } d_2, d_1 \leq E. \end{cases} \tag{20.166}$$

The Babylonian Talmud, in Tractate Baba Metzia, states that if the debt owed to the two creditors is d_1 and d_2 respectively, and the assets of the bankrupt entity sum to E, the first creditor receives $f_1(E; d_1, d_2)$, and the second creditor receives $f_2(E; d_1, d_2)$. Note that $f_1(E; d_1, d_2) + f_2(E; d_1, d_2) = E$ for every d_1 and d_2. The function f is continuous in E, d_1, d_2. Moreover, for fixed d_1 and d_2 the functions $E \mapsto f_1(E; d_1, d_2)$ and $E \mapsto f_2(E; d_1, d_2)$ are monotonic and nondecreasing (see the example in Figure 20.3).

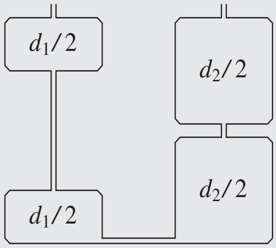

Figure 20.4 A physical manifestation of the "contested garment" solution

Figure 20.4 depicts an implementation of the function f using "communicating vessels" composed of containers and pipes. The claims of the creditors (i.e., the debts owned by the bankrupt entity) are depicted by containers of unit base area, where the claim of d_i is represented by two containers of height $\frac{d_i}{2}$, one "on the ground," and the other "hanging from the ceiling"; the two containers are connected by a dimensionless pipe, with the two containers touching the ground also connected by a dimensionless pipe.

To compute $f_1(E; d_1, d_2)$ and $f_2(E; d_1, d_2)$, we let a volume of liquid equal to E flow into the system. By the communicating vessels law, the height of the liquid on both sides of the system must be equal. The volume of liquid on the left side represents the payment given to the first creditor, and the volume of liquid on the right side represents the payment given to the second creditor. When the volume of liquid flowing is at most d_1, the amount of liquid is divided equally between the two sides. After a volume of d_1 has entered the system, every additional volume of liquid flowing in, up to the amount $d_2 - d_1$, will be added only to the right side. After a volume of d_2 has entered the system, every additional volume of liquid flowing in will be divided equally between the two sides of the system.

20.8.3 The case $n > 2$

When there are more than two creditors, we will find a solution concept based on the "contested garment" solution, using the *principle of consistency*. This principle relates to the following situation. Suppose that n creditors, faced with a bankruptcy problem $[E; d_1, \ldots, d_n]$, divide the estate between them using the solution concept φ. Creditor i and creditor j can now unite their shares $\varphi_i(E; d_1, \ldots, d_n) + \varphi_j(E; d_1, \ldots, d_n)$, and divide this sum between them using the "contested garment" solution. A solution concept φ is consistent with the "contested garment" solution if for every two creditors i and j, after uniting their shares and applying the "contested garment" solution to the two-creditor problem that involves the two of them, they end up with the same shares allocated to them originally, namely, $\varphi_i(E; d_1, \ldots, d_n)$ and $\varphi_j(E; d_1, \ldots, d_n)$.

Consider the example of the three widows mentioned in the Babylonian Talmud's Tractate Kethubot. When the estate is 200, Anne, Betty, and Carol receive 50, 75, and 75 respectively. Together, Anne and Betty receive 125. If they divide this sum between them according to the "contested garment" solution (Equation (20.166)), then Anne gets

$$f_1(125; 100, 200) = 50, \tag{20.167}$$

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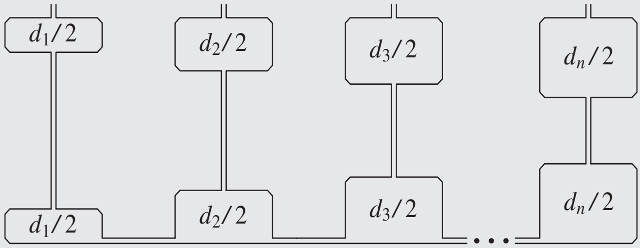

Figure 20.5 A physical implementation of the consistent solution

and Betty gets

$$f_2(125; 100, 200) = 75. \tag{20.168}$$

These sums are identical to the sums that the women receive according to the Talmud. If Betty and Carol were to divide the 150 that they have together received according to the "contested garment" solution Betty gets

$$f_1(150; 200, 300) = 75, \tag{20.169}$$

and Carol gets

$$f_2(150; 200, 300) = 75. \tag{20.170}$$

A similar result is obtained if we change the value of the estate to 300 or decrease it to 100. We therefore see that the Talmud's recommended division of the estate of the man with three wives is consistent with the "contested garment" solution.

We are now ready to give a formal definition of the principle of consistency with the "contested garment" solution.

Definition 20.58 *A solution concept φ is consistent with the "contested garment" solution if for every bankruptcy problem $[E; d_1, \ldots, d_n]$ and every two creditors i and j, $i \neq j$,*

$$f_1(x; d_i, d_j) = \varphi_i(E; d_1, \ldots, d_n), \tag{20.171}$$

where $x = \varphi_i(E; d_1, \ldots, d_n) + \varphi_j(E; d_1, \ldots, d_n)$.

Theorem 20.59 *There exists a unique solution concept that is consistent with the "contested garment" solution.*

The unique solution concept satisfying Theorem 20.59 is called the *Rabbi Nathan solution*.

Proof:

Step 1: There exists at least one solution concept consistent with the "contested garment" solution.

Suppose, without loss of generality, that $d_1 \leq d_2 \leq \cdots \leq d_n$. We will construct a system of containers similar to the one constructed in Figure 20.4, with the containers ordered by increasing size from left to right (see Figure 20.5).

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Pour liquid of volume E into this system of containers. Denote by x_i the amount of liquid in container i (in both its parts), $i \in \{1, 2, \dots, n\}$. Then $x_1 + x_2 + \cdots + x_n = E$, and this system therefore defines a solution $(x_1, x_2, \dots, x_n)$ to the bankruptcy problem, which we will call the "containers solution." If we restrict ourselves to two of the containers, i and j, the amount $x_i + x_j$ is divided between the two containers in accordance with the case $n = 2$, which is the physical implementation of the "contested garment" solution. The containers solution therefore defines a solution concept consistent with the "contested garment" solution, which is what we needed to show.

Step 2: There is at most one solution concept consistent with the "contested garment" solution.

Let φ be a solution concept to the bankruptcy problem consistent with the "contested garment" solution. We will show that φ coincides with the above-described containers solution. Let $[E; d_1, \dots, d_n]$ be a bankruptcy problem, and consider the corresponding system of containers (see Figure 20.5). Disconnect all horizontal pipes between containers and pour into container i a volume of liquid equal to $\varphi_i(E; d_1, \dots, d_n)$. Since the solution concept φ is consistent with the "contested garment" solution, it follows that for every pair of containers, the liquid attains the same height. To see this, note that consistency with the "contested garment" solution implies that for every pair of containers i and j, if we construct a system containing only the containers i and j connected by a horizontal pipe (as in Figure 20.4), into which we pour liquid of volume $\varphi_i(E; d_1, \dots, d_n) + \varphi_j(E; d_1, \dots, d_n)$, the height of the liquid in both containers will be identical, the amount of liquid in container i being $\varphi_i(E; d_1, \dots, d_n)$, and the amount of liquid in container j being $\varphi_j(E; d_1, \dots, d_n)$.

It follows that the height of the liquid is the same in all n containers; hence if we open the horizontal pipes the height of the liquid in the containers will remain the same. After doing so, we arrive at the system depicted in Figure 20.5. In other words, $\varphi_i(E; d_1, \dots, d_n)$ equals the amount creditor i receives in the containers solution, and therefore φ coincides with the containers solution. In particular, the containers solution is the unique solution concept that is consistent with the "contested garment" solution. $\square$

Using the physical system described above in the proof of the theorem, and considering money to be a continuous variable, we can describe the Rabbi Nathan solution to a bankruptcy problem $[E; d]$ as follows:

- Divide (in a continuous manner) the money equally only among the creditors who have not yet received half of their claims; a creditor who has already received half of his claim stops receiving any more money until all creditors receive half of their claims.
- Once each creditor has received half of his claim, we continue to give money (in a continuous manner) to the creditor with the maximal remaining claim, until there are two creditors left with maximal remaining claims.
- We continue to give money (in a continuous manner) to the two creditors with the maximal remaining claims, until there are three creditors left with maximal remaining claims, and so on.

Exercise 20.53 provides an explicit formula for the Rabbi Nathan solution.

20.8.4 The nucleolus of a bankruptcy problem

For every nonempty set of creditors S, denote by $d(S) := \sum_{i\in S} d_i$ the sum of their claims according to d, and set $d(\emptyset) := 0$. For every bankruptcy problem $[E; d_1, d_2, \ldots, d_n]$ define a game $(N; v)$ where the set of players is the set of creditors, and the coalitional function is defined as follows:

$$v(S) := \max\left\{E - d(S^c), 0\right\}, \qquad \forall S \subseteq N. \tag{20.172}$$

The quantity $v(S)$ is the part of E that is not in dispute, i.e, the part of E that remains to the members of S if every member of $S^c = N \setminus S$ gets his whole claim. If $d(S^c) \geq E$, then nothing remains for the members of S that is under dispute, and then $v(S) = 0$. Note that $v(N) = E$ and $v(\emptyset) = 0$.

Example 20.60 **The case of two creditors** Let $[E; d_1, d_2]$ be a bankruptcy problem with two creditors. The game corresponding to this problem is

$$\begin{array}{llll} v(1) = 0, & v(2) = 0, & v(1,2) = E & \text{if } E \leq d_1, d_2, \\ v(1) = E - d_2, & v(2) = 0, & v(1,2) = E & \text{if } d_2 < E \leq d_1, \\ v(1) = 0, & v(2) = E - d_1, & v(1,2) = E & \text{if } d_1 < E \leq d_2, \\ v(1) = E - d_2, & v(2) = E - d_1, & v(1,2) = E & \text{if } d_1, d_2 < E. \end{array} \tag{20.173}$$

The nucleolus of this game (as given in Theorem 20.18, page 811) coincides with the "contested garment" solution, which we computed in Section 20.8.2 on page 833 (verify!). ◀

The conclusion of Example 20.60 is summarized in the following theorem.

Theorem 20.61 *When $n = 2$, the "contested garment" solution coincides with the nucleolus of the coalitional game $(N; v)$ defined in Equation (20.172).*

The rest of this section is devoted to proving that when the number of creditors is greater than two, the nucleolus coincides with the Rabbi Nathan solution. The proof is conducted in several steps. We first prove that the game defined in Equation (20.172) is 0-monotonic (Theorem 20.62). Then we prove that for every imputation x in the game $(N; v)$, and for every nonempty coalition T, the game corresponding to the bankruptcy problem $[x(T); (d_i)_{i\in T}]$ in which the set of creditors is T and the estate is $x(T)$ is the Davis–Maschler reduced game of the game $(N; v)$ (Theorem 20.63). By Theorem 20.38 (page 825), it follows that the nucleolus is a consistent solution concept in the set of coalitional games corresponding to a bankruptcy problem. Since in two-player games the nucleolus and the Rabbi Nathan solution coincide, it follows that these two solution concepts coincide in every coalitional game corresponding to a bankruptcy problem.

Theorem 20.62 *For every bankruptcy game $[E; d]$, the coalitional game $(N; v)$ defined by Equation (20.172) is 0-monotonic.*

Proof: To prove that the game $(N; v)$ is 0-monotonic, we need to prove that for every coalition $S \subset N$ and every player $i \notin S$,

$$v(S \cup \{i\}) \geq v(S) + v(i). \tag{20.174}$$

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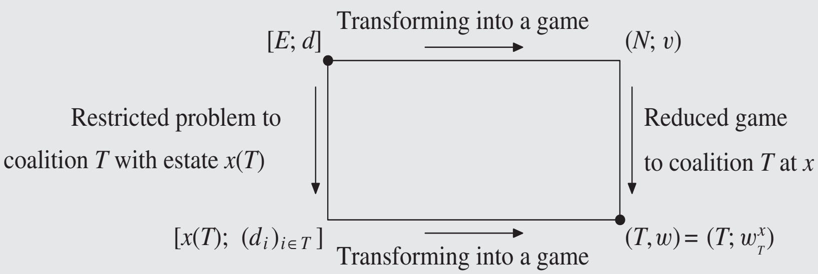

Figure 20.6 The commutative diagram according to Theorem 20.63

By the definition of the coalitional function v, the game $(N; v)$ is monotonic (verify!). If $v(i) = 0$ or $v(S) = 0$, then by the monotonicity of the game $(N; v)$ it follows that Equation (20.174) holds. Suppose therefore that $v(i) > 0$ and $v(S) > 0$. In particular,

$$0 < v(i) = E - d(N \setminus \{i\}) < d_i, \tag{20.175}$$

where the right-hand side inequality follows from the assumption that $d(N) > E$, and

$$0 < v(S) = E - d(N \setminus \{S\}). \tag{20.176}$$

Therefore

$$v(S \cup \{i\}) \geq E - d((S \cup \{i\})^c) \tag{20.177}$$
$$= E - d(S^c) + d_i \tag{20.178}$$
$$> v(S) + v(i), \tag{20.179}$$

where Equation (20.177) holds[9] by the definition of v, and Equation (20.179) holds by Equations (20.175) and (20.176). □

Theorem 20.63 *Let $(N; v)$ be the coalitional game corresponding to a bankruptcy problem $[E; (d_i)_{i \in N}]$. Let x be a vector satisfying $x(N) = E$ and $0 \leq x_i \leq d_i$ for every $i \in N$, and let $T \subseteq N$ be a nonempty coalition. Let $[x(T), (d_i)_{i \in T}]$ be the bankruptcy problem restricted to T with estate $x(T)$, and let $(T; w)$ be the corresponding coalitional game. Then $(T; w) = (T; w_T^x)$, where $(T; w_T^x)$ is the Davis–Maschler reduced game of $(N; v)$ to coalition T at x.*

The theorem implies that the diagram in Figure 20.6 is commutative.

In words, the same game $(T; w)$ is derived under both of the following procedures: (a) converting the bankruptcy problem $[E; d]$ to a game, and then reducing it to the coalition T at x; (b) restricting the bankruptcy problem to the coalition T, with estate $\widehat{E} = x(T)$, and then converting that bankruptcy problem to a game.

Proof of Theorem 20.63: We want to prove that $w_T^x = w$ holds for the coalition T, i.e., $w_T^x(R) = w(R)$ for every coalition $R \subseteq T$.

9 Actually, since $0 < v(S) \leq v(S \cup \{i\})$, Equation (20.177) holds with equality.

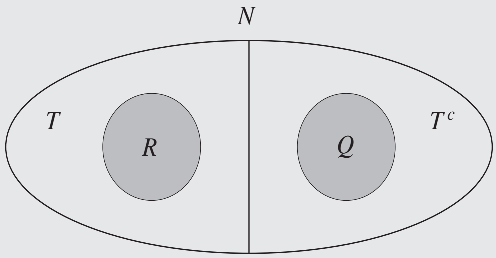

Figure 20.7 A depiction of the equality $N \setminus Q = R \cup (N \setminus (R \cup Q))$

Step 1: $w_T^x(T) = w(T)$ and $w_T^x(\emptyset) = w(\emptyset)$.
For every $c \in \mathbb{R}$, denote $c_+ = \max\{c, 0\}$, and note that

$$a_+ - b_+ \leq (a - b)_+. \tag{20.180}$$

In this notation, $v(S) = \max\{E - d(S^c), 0\} = (E - d(S^c))_+$ for every coalition $S \subseteq N$, and $w(R) = (x(T) - d(T \setminus R))_+$, for every coalition $R \subseteq T$.

Since $(T; w)$ is the game corresponding to a reduced bankruptcy problem,

$$w(T) = x(T), \qquad w(\emptyset) = 0. \tag{20.181}$$

The equality $w(\emptyset) = 0$ holds because $x_i \leq d_i$ for every i, and therefore $w(\emptyset) = (x(T) - d(T))_+ = 0$. Since $(T; w_T^x)$ is a Davis–Maschler reduced game to T at x,

$$w_T^x(T) = x(T), \qquad w_T^x(\emptyset) = 0. \tag{20.182}$$

From Equations (20.181) and (20.182) one obtains

$$w(T) = w_T^x(T), \qquad w(\emptyset) = w_T^x(\emptyset), \tag{20.183}$$

which is what we wanted to show.

Step 2: $w_T^x(R) \leq w(R)$ for every coalition $R \subset T$.
Let Q be a coalition contained in T^c at which the maximum in the definition of $w_T^x(R)$ is attained,

$$w_T^x(R) = v(R \cup Q) - x(Q) = (E - d(N \setminus (R \cup Q)))_+ - x(Q). \tag{20.184}$$

Since $R \subset T$ and $Q \subseteq T^c$ one has that $N \setminus Q = R \cup (N \setminus (R \cup Q))$ (see Figure 20.7).

$x(Q) \geq 0$ implies that $x(Q) = (x(Q))_+$, and therefore by Equations (20.184) and (20.180),

$$\begin{aligned} w_T^x(R) &= (E - d(N \setminus (R \cup Q)))_+ - x(Q)_+ && (20.185)\\ &\leq (E - d(N \setminus (R \cup Q)) - x(Q))_+ && (20.186)\\ &= (x(N) - d(N \setminus (R \cup Q)) - x(Q))_+. && (20.187) \end{aligned}$$

Because $x(N) - x(Q) = x(N \setminus Q) = x(T) + x(T^c \setminus Q)$, along with $d(N \setminus (R \cup Q)) = d(T \setminus R) + d(T^c \setminus Q)$, one has

$$w_T^x(R) \leq (x(T) + x(T^c \setminus Q) - d(T \setminus R) - d(T^c \setminus Q))_+. \tag{20.188}$$

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Since $x_i \leq d_i$ for every player i, the difference $x(T^c \setminus Q) - d(T^c \setminus Q)$ is nonpositive; therefore,

$$w_T^x(R) \leq (x(T) - d(T \setminus R))_+ = w(R). \tag{20.189}$$

Step 3: $w_T^x(R) \geq w(R)$ for every coalition $R \subset T$.
Let $R \subset T$ be a coalition. By the definition of the reduced game, for $Q = N \setminus T$,

$$\begin{aligned} w_T^x(R) &\geq v(R \cup (N \setminus T)) - x(N \setminus T) && (20.190)\\ &= (E - d(T \setminus R))_+ - x(N \setminus T) && (20.191)\\ &= (E - d(T \setminus R))_+ - x(N) + x(T) && (20.192)\\ &= (E - d(T \setminus R))_+ - E + x(T) && (20.193)\\ &\geq E - d(T \setminus R) - E + x(T) && (20.194)\\ &= x(T) - d(T \setminus R). && (20.195) \end{aligned}$$

By the definition of the reduced game, for $Q = \emptyset$,

$$w_T^x(R) \geq v(R) = (E - d(N \setminus R))_+ \geq 0. \tag{20.196}$$

By Equations (20.195) and (20.196),

$$w_T^x(R) \geq (x(T) - d(T \setminus R))_+ = w(R). \tag{20.197}$$

We deduce that $w_T^x(R) = w(R)$ for every coalition $R \subseteq T$, thus concluding the proof. □

Theorem 20.64 *Let $(N; v)$ be the coalitional game corresponding to a bankruptcy problem $[E; d]$, and let x^* be its nucleolus. Then for every coalition $T \subseteq N$ the Davis–Maschler reduced game to the coalition T relative to x^*, i.e., the game $(T; w_T^{x^*})$, is 0-monotonic.*

Proof: We first show that $0 \leq x_i^* \leq d_i$ for all $i \in N$. The game $(N; v)$ is a convex game (Exercise 20.55), and in particular its core is nonempty (Theorem 17.55, page 719). It follows that the nucleolus x^* lies in the core of the game (Theorem 20.20, page 813). Therefore,

$$x_i^* \geq v(i) = (E - d(N \setminus \{i\}))_+ \geq 0. \tag{20.198}$$

Since $x^*(N) = E$, and since x^* is in the core, it follows that for every player i,

$$E - x_i^* = x^*(N \setminus \{i\}) \geq v(N \setminus \{i\}) = (E - d_i)_+ \geq E - d_i, \tag{20.199}$$

yielding the conclusion $x_i^* \leq d_i$, as claimed.

By Theorem 20.63 it follows that $(T; w_T^{x^*})$ is the game corresponding to the bankruptcy problem $[x^*(T); (d_i)_{i \in T}]$ and therefore by Theorem 20.62 this is a 0-monotonic game. □

By Theorem 20.38 (page 825), the previous results deliver the following corollary, which states that the nucleolus satisfies the reduced game property over the family of games corresponding to the bankruptcy problem.

Corollary 20.65 *Let $(N; v)$ be the coalitional game corresponding to the bankruptcy problem $[E; d]$, let x^* be its nucleolus, and let $T \subseteq N$ be a nonempty coalition. Then $(x^*)_{i\in T}$ is the nucleolus of the game $(T; w_T^{x^*})$.*

We are ready to formulate and prove the central theorem of this section:

Theorem 20.66 *Let $[E; d_1, d_2, \ldots, d_n]$ be a bankruptcy problem, and let $(N; v)$ be the coalitional game defined by Equation (20.172). Then the Rabbi Nathan solution to the bankruptcy problem coincides with the nucleolus of the game $(N; v)$.*

Proof: Define a solution concept φ to the bankruptcy problem as follows. For every bankruptcy problem $[E; d]$, $\varphi(E; d)$ is the nucleolus of the game $(N; v)$ corresponding to $[E; d]$. We have to prove that φ coincides with the Rabbi Nathan solution.

By Corollary 20.65, for every coalition $T = \{i, j\}$ where i and j are distinct players, the nucleolus of the game $(T; w_T^{x^*})$ is (x_i^*, x_j^*). By Theorem 20.61, (x_i^*, x_j^*) is the "contested garment" solution of the problem $[x(T); d_i, d_j]$. It follows that the solution concept φ is consistent with the "contested garment" solution. Since the Rabbi Nathan solution is the only solution concept for the family of bankruptcy problems consistent with the "contested garment" solution, the nucleolus of the game $(N; v)$ coincides with the Rabbi Nathan solution. This concludes the proof of the theorem. □

20.9 Discussion

In the chapter on the Shapley value, we proved that the Shapley value is the only single-valued solution concept satisfying the properties of efficiency, symmetry, covariance under strategic equivalence, and consistency relative to the Hart–Mas-Colell reduced game (Theorem 18.39, page 771). Sobolev [1975] shows that the same principles characterize the prenucleolus, if we replace Hart and Mas-Colell's notion of the reduced game by that of Davis and Maschler.[10] This shows that an arbitrator who is thinking of recommending the Shapley value or the prenucleolus as a solution concept should first consider both the Davis–Maschler reduced game and the Hart–Mas-Colell reduced game, and see which one is more appropriate for the given situation.

For defining the coalitional game corresponding to a bankruptcy problem, it is possible to consider other coalitional functions than that given in Equation (20.172), as for example

$$w(S) := E - \sum_{i\in S^c} d_i, \quad \forall S \subseteq N. \tag{20.200}$$

This coalitional function is appropriate for situations in which the debt must be paid, and creditors might pay out of their own pocket other creditors who have higher claims. It can be shown (Exercise 20.56) that in this case, if x is the Shapley value of the game, then the Hart–Mas-Colell reduced game $(S; w_S^x)$ coincides with the game corresponding to the restricted bankruptcy problem $[x(S); (d_i)_{i\in S}]$. Therefore, in such situations, it may be more appropriate to use the Shapley value. In summary, the system of properties characterizing

10 Sobolev used the property of anonymity (independence from changing the names of the players), which is a stronger assumption than symmetry. Orshan [1993] showed that assuming symmetry is sufficient for the result.

single-valued solution concepts can be of use in deciding which solution is appropriate in a given situation.

A solution for the bankruptcy problem is *consistent* if, after some of the creditors leave the game, taking with them the amounts allocated to them under the solution, and the rest of the creditors then contend with the resulting restricted bankruptcy problem where the amount to be divided is what remains after the other creditors have left, they discover that the solution to the restricted problem gives each of them exactly what he received under the original problem.

Definition 20.67 *A solution concept for a bankruptcy problem* φ *is* consistent *if for every bankruptcy problem* $[E; d]$*, and every set of creditors* $T \subseteq N$*,*

$$\varphi_i(E; d) = \varphi_i(x; (d_j)_{j \in T}), \quad \forall i \in T, \tag{20.201}$$

where $x = \sum_{j \in T} \varphi_j(E; d)$.

The Rabbi Nathan solution is consistent (Exercise 20.57), and several other consistent solutions exist (such as the proportional division solution). Kaminski [2000] proves that every consistent solution for bankruptcy problems can be described by a system of containers of various sizes, and that the converse also holds; i.e., every system of containers defines a consistent solution.

20.10 Remarks

The procedure presented in this chapter for computing the nucleolus was suggested by Bezalel Peleg, and first appeared in Kopelowitz [1967]. The results in Section 20.5 (page 816) are from Kohlberg [1971]. The proof of Theorem 20.25 (page 817) presented here is from Peleg and Sudhölter [2003].

Weighted majority games were first defined in von Neumann and Morgenstern [1944]. That book also presents an example of a game without a homogeneous representation. Isbell [1959] posed the question of whether it is possible to choose, among all the representations of a weighted majority game, one normalized representation that is, in a certain sense, the most "natural" representation. Isbell did not formally define the term "natural representation," but required the following property that such a representation should satisfy: if a game is homogeneous, then the "natural" representation thus chosen must be homogeneous. The answer to this question, appearing in Theorem 20.54 (page 830), was given by Peleg [1968], and extended to a larger class of games by Sudhölter [1996].

The explanation of the Talmudic passages presented in this chapter is based on Aumann and Maschler [1985]. The implementation of the solution using a system of containers was first suggested by Kaminski [2000], although the use of systems of containers for finding equilibria goes back as far as Fisher [1891].

Exercise 20.16 is Example 5.6.3 in Peleg and Sudhölter [2003]. The game "My Aunt and I," as presented in Exercise 20.21, first appeared in Davis and Maschler [1965]. That article includes a report on correspondence that Davis and Maschler conducted with several researchers on the question "how will my aunt and one of my brothers divide the profit that will accrue to them if they form the coalition {aunt, brother}?" Various answers were given to this question. The game was included in an empirical study of twelve games, with the results of that study appearing in Selten and Schuster [1968]. Exercise 20.28 is based on Maschler, Peleg, and Shapley [1979].

20.11 Exercises

20.1 Is the function $x \mapsto \sum_{S\subseteq N} e(S,x)$ constant on the set of preimputations $X^0(N;v)$? Prove this claim, or show that it is incorrect.

20.2 Prove that the lexicographic relation is transitive.

20.3 Let $K \subseteq \mathbb{R}^n$, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Denote the set of points in K at which the minimum of f is attained by

$$\operatorname{argmin}_{x\in K} f(x) = \left\{ y \in K : f(y) = \min_{z\in K} f(z) \right\}. \tag{20.202}$$

Prove that if K is a compact set and f is a continuous function, then the set $\operatorname{argmin}_{x\in K} f(x)$ is compact and nonempty.

20.4 Find a sequence of vectors $(x^n)_{n=1}^\infty$ in $\mathbb{R}^2$ converging to x, and a vector $y \in \mathbb{R}^2$ such that (a) $x^n \succ_L y$ for all $n \in \mathbb{N}$, but (b) $x \prec_L y$.

20.5 Consider the three-player coalitional game with the following coalitional function:

$$\begin{aligned} &v(1)=0, \quad v(2)=1, \quad v(3)=4, \quad v(1,2)=2, \quad v(1,3)=6,\\ &v(2,3)=-1, \quad v(1,2,3)=5. \end{aligned} \tag{20.203}$$

(a) Compute $\theta((1,1,3))$, $\theta((1,3,1))$ and $\theta((3,1,1))$.
(b) Arrange the three vectors in decreasing lexicographic order.

20.6 Let $x, y \in X(N;v)$ be imputations, and suppose that $\theta(x) \succsim_L \theta(y)$. Denote $z = \frac{1}{2}x + \frac{1}{2}y$. Is it necessarily true that $\theta(x) \succsim_L \theta(z) \succsim_L \theta(y)$? Either prove this statement, or provide a counterexample.

20.7 (a) Prove that in the gloves game (Example 20.3) the imputation $y = (0,0,1)$ satisfies $\theta(y) \prec_L \theta(u)$ for every $u \in X(N;v)$, $u \neq y$.
(b) Prove that this imputation also satisfies $\theta(y) \prec_L \theta(u)$ for every $u \in X^0(N;v)$, $u \neq y$.

20.8 (a) Find a two-player coalitional game $(N;v)$, and a bounded set K that is not closed, such that the nucleolus $\mathcal{N}(N;v;K)$ is the empty set.
(b) Find a two-player coalitional game $(N;v)$, and a closed and unbounded set K, such that the nucleolus $\mathcal{N}(N;v;K)$ is the empty set.

20.9 Let $(N;v)$ be a coalitional game satisfying the following property: there is an imputation $x \in X(N;v)$ such that all the excesses at x are nonnegative.

(a) Prove that x is the only imputation in the game, and in particular it is the nucleolus.

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(b) Is x necessarily also the prenucleolus? Either prove this statement, or provide a counterexample.

20.10 Player i in a coalitional game $(N; v)$ is a *dummy player* if

$$v(S \cup \{i\}) = v(S) + v(i), \quad \forall S \subseteq N \setminus \{i\}. \tag{20.204}$$

Prove that if player i is a dummy player in a game $(N; v)$ then under both the nucleolus and the prenucleolus, player i's payoff is $v(i)$, that is, $\mathcal{N}_i(N; v) = v(i)$ and $\mathcal{PN}_i(N; v) = v(i)$.

20.11 Complete the proof of Theorem 20.7 (page 805): prove that for every k, $2 \leq k \leq 2^n$,

$$\theta_k(x) = \max_{\text{different } S_1,\ldots,S_k} \min\{e(S_1, x), \ldots, e(S_k, x)\}. \tag{20.205}$$

20.12 Compute the nucleolus of the three-player coalitional game with the following coalitional function:

$$v(1) = v(2) = v(3) = v(2,3) = 0, \quad v(1,2) = v(1,3) = v(1,2,3) = 1. \tag{20.206}$$

20.13 Compute the prenucleolus of the three-player coalitional game with the following coalitional function:

$$v(1) = v(2) = v(3) = v(2,3) = 0, \quad v(1,2) = v(1,3) = v(1,2,3) = -1. \tag{20.207}$$

20.14 Let $(N; v)$ be the three-player coalitional game with the following coalitional function:

$$v(S) = 0 \Longleftrightarrow |S| \leq 1, \tag{20.208}$$

$$v(S) = 1 \Longleftrightarrow |S| \geq 2. \tag{20.209}$$

Let K be the triangle in $\mathbb{R}^3$ whose vertices are $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, and let K_0 be its boundary. Compute the nucleolus of the game $(N; v)$ relative to K and relative to K_0.

20.15 Compute the nucleolus and the prenucleolus of the three-player coalitional game $(N; v)$ in which $v(1, 2) = 1$ and $v(S) = 0$ for every other coalition S. Does the nucleolus coincide with the prenucleolus?

20.16 Let $(N; v)$ be a five-player coalitional game with the following coalitional function:

$$v(S) = \begin{cases} 1 & \text{if } S = \{1,2\} \text{ or } S = \{1,2,5\}, \\ 2 & \text{if } S = \{3,4\} \text{ or } S = \{3,4,5\}, \\ 0 & \text{for any other coalition } S. \end{cases} \tag{20.210}$$

Answer the following questions:

(a) Are there null players in this game? If so, which players are null players?
(b) Prove that the nucleolus and the prenucleolus for the coalitional structure $\mathcal{B} = \{\{1, 2, 5\}, \{3\}, \{4\}\}$ are

$$\mathcal{N}(N; v) = \mathcal{PN}(N; v) = (0, 0, 0, 0, 1). \tag{20.211}$$

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20.17 Prove that the nucleolus is covariant under strategic equivalence: for every coalitional game $(N; v)$, for every set $K \subseteq \mathbb{R}^N$, for every $a > 0$, and every set $b \in \mathbb{R}^N$,

$$\mathcal{N}(N; av + b; aK + b) = a\mathcal{N}(N; v; K) + b. \tag{20.212}$$

20.18 Find a two-player coalitional game $(N; v)$, and a V-shaped set K, i.e., a set that is the union of two line segments sharing an edge point, such that the nucleolus of $(N; v)$ relative to K is the two edge points of K.

20.19 Compute the nucleolus of the weighted majority game $[q; 2, 2, 3, 3]$ for every quota $q > 0$.

20.20 Compute the nucleolus of the coalitional game $(N; v)$ where $N = \{1, 2, 3, 4\}$ and the coalitional function v is given by

$$v(S) = \begin{cases} i & \text{if } S = \{i\}, \\ 0 & \text{otherwise.} \end{cases} \tag{20.213}$$

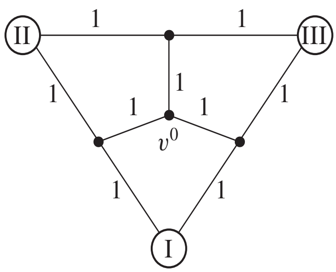
20.21 **My Aunt and I** Auntie Betty can complete a certain job together with me or with any of my three brothers, with the payment for the work being \$1,000, but she must choose one of the four of us. All four of us brothers together (without Auntie Betty) can also complete the same job.

(a) Describe this situation as a coalitional game.
(b) Compute the nucleolus of the game for the coalitional structure $\mathcal{B} = \{N\}$.
(c) Compute the nucleolus of the game for the coalitional structure

$$\mathcal{B} = \{\{\text{Auntie Betty, Me}\}, \{\text{Brother } A\}, \{\text{Brother } B\}, \{\text{Brother } C\}\}. \tag{20.214}$$

20.22 Define the nucleolus of a cost game $(N; c)$.

20.23 Compute the core and the nucleolus of the following spanning tree game (see Section 16.1.7, page 666). v^0 is the central point to which Players I, II, and III, who are physically located at the vertices of a triangle (as depicted in the next figure), wish to connect. The cost associated with every edge in the figure is one unit.



20.24 Let $(N; v)$ be a coalitional game, and let $\mathcal{B}$ be a coalitional structure. For every $\alpha > 0$, and every imputation $x \in \mathbb{R}^N$, define

$$f(x, \alpha) = \sum_{k=1}^{2^n} \alpha^k \theta_k(x). \tag{20.215}$$

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Prove that for $\alpha > 0$ sufficiently small, the minimum of the function $x \mapsto f(x, \alpha)$ in the set $X(\mathcal{B}; v)$ is attained at the nucleolus. Formally, there exists $\alpha_0 > 0$ such that for every $\alpha \in (0, \alpha_0)$, the minimum of the function $x \mapsto f(x, \alpha)$ is attained at the nucleolus.

20.25 Let $K \subseteq \mathbb{R}^N$ be a closed set satisfying the following property: there exists a real number c such that $\sum_{i \in N} x_i \le c$ for all $x \in K$. Prove that the nucleolus relative to K is nonempty.

20.26 Prove that the vector $x \in X(\mathcal{B}; v)$ is in the core $\mathcal{C}(N; v; \mathcal{B})$ if and only if $\theta_1(x) \le 0$.

20.27 Prove that if the ε-core $\mathcal{C}_\varepsilon(N; v)$ is nonempty, then $\mathcal{N}(N; v) \subseteq \mathcal{C}_\varepsilon(N; v)$. Deduce that the nucleolus is contained in the minimal core, and that if the core is nonempty, then the nucleolus is contained in the core. For the definition of $\mathcal{C}_\varepsilon(N; v)$ and the minimal core, see Exercise 17.33 on page 740.

20.28 The nucleolus of a three-player coalitional game can be computed as follows.[11] First, find the minimal core of the game (see Exercise 17.33 on page 740). The minimal core is defined by linear inequalities corresponding to the various coalitions. The inequality corresponding to the coalition S corresponds to the line H_S. Changing the value of ε then corresponds to parallel displacement of these lines. In the process of reducing the value of ε, some of the lines H_S cannot be displaced without causing the ε-core to become empty. These lines are then left unmoved in the process, while other lines are moved as the value of ε is reduced. Every time a line is encountered such that continuing to move it will cause the ε-core to be empty, that line is no longer moved. This process is continued until only one point remains, which is the nucleolus.

For example, in the game in Exercise 17.33 (page 740) the minimal core is on the line defined by $x_1 = 1$, and the boundary points of the minimal core are located at $(1, 5, 3)$ and $(1, 2, 6)$. Move the constraints $x_1 + x_3$ and $x_1 + x_2$ (and x_1 and x_2) another unit and a half, until they meet, yielding the nucleolus $(1, 3\frac{1}{2}, 4\frac{1}{2})$.

(a) Prove that this process indeed finds the nucleolus.
(b) Compute, using this process, the nucleolus in the games in Exercise 17.33 (page 740).

20.29 Suppose that $(N; v)$ is a coalitional game such that the set of imputations $X(N; v)$ is nonempty, and such that the nucleolus x^* differs from the prenucleolus $\widehat{x}$. Prove that the nucleolus is located on the boundary of the set $X(N; v)$.
Hint: Show that if the nucleolus is not on the boundary of $X(N; v)$, then there is a point $z = \alpha x^* + (1 - \alpha)\widehat{x} \in X(N; v)$, where $\alpha \in (0, 1)$, and use the ideas presented in the proof of Theorem 20.13 (page 808).

20.30 Prove that the nucleolus is independent of the names given to the players; i.e., if $\pi : N \to N$ is a permutation of N, and the coalitional games $(N; v)$ and $(N; w)$

11 The procedure presented here can be generalized to an arbitrary number of players, but the generalization is beyond the scope of this book.

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satisfy $w(S) = v(\pi(S))$ for every $S \subseteq N$, and if x^{*v} and x^{*w} are the nucleoli of these games, then $x_i^w = x_{\pi(i)}^v$ for every player $i \in N$.

20.31 Prove that the algorithm described in Section 20.4 (page 815) is well defined (i.e., show that there exists L for which the sets $\Sigma_1, \Sigma_2, \ldots, \Sigma_L$ are nonempty and the union $\cup_{l=1}^{L} \Sigma_l$ contains all the coalitions), and show that X_L contains only one point, which is the nucleolus.
Guidance: show that the nucleolus is contained in each of the sets $X_1, X_2, \ldots, X_L$.

20.32 Let $x_1, x_2, \ldots, x_L$ be imputations in the game $(N; v)$. Suppose that $\theta_1(x_l) = \theta_1(x_{l'})$ for every l, l' satisfying $1 \leq l, l' \leq L$, and denote $\Sigma_1 := \{S \subseteq N : e(S, x_l) = \theta_1(x_l) \ \ \forall l \in \{1, 2, \ldots, L\}\}$. Prove that the collection of coalitions Σ_1 is nonempty.

20.33 Give an example of a monotonic game that is not 0-monotonic.

20.34 Show that a simple, constant-sum game need not be 0-monotonic.

20.35 Prove that every convex game is 0-monotonic.

20.36 Show that every null player is a dummy player, but there are dummy players that are not null players.

20.37 Using Theorem 20.29 (page 821), prove that the imputation $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is not the prenucleolus of the gloves game that appears in Example 20.3 and that the prenucleolus of this game is $y = (0, 0, 1)$.

20.38 Eilon contacts Michael, claiming that the second direction in the proof of Theorem 20.28 on page 819 (if the system of equations given by Equations (20.99)–(20.100) is tight for x, then x is the nucleolus) does not hold: “Denote the nucleolus by x^*, and let x be an imputation,” explains Eilon. “Denote $y = x^* - x$. Then $y(N) = 0$. Let α_1 be the greatest value of the excesses at $\theta(x^*)$. Then for every coalition S,

$$y(S) = (x^* - x)(S) = e(S, x) - e(S, x^*). \tag{20.216}$$

Since x^* is the nucleolus, and since α_1 is the greatest value of the excesses, it must be the case that $e(S, x) \geq 0$ for every coalition $S \in \mathcal{D}(\alpha_1, x^*)$. In other words, $y(S) \geq 0$ for every such coalition. Since in the first part of the theorem, it is proved that relative to the nucleolus, the system of equations is tight, one deduces that $y(S) = 0$ for every $S \in \mathcal{D}(\alpha_1, x^*)$. Continuing by induction to the next level, one concludes that $y(S) = 0$ for every coalition S, which implies that $x = x^*$. But this cannot hold since x was chosen arbitrarily.”

Is Eilon correct? If not, where is the flaw in his argument?

20.39 Find a weighted majority game that is not constant sum, and satisfies the property that if S is a winning coalition, then S^c is a losing coalition.

20.40 Find a weighted majority game that is not constant sum, satisfying the property that if S is a losing coalition, then S^c is a winning coalition.

20.41 Prove that the game $[5; 2, 2, 2, 1, 1, 1]$ does not have a homogeneous representation.

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20.42 Prove that if $[q; w_1, w_2, \ldots, w_n]$ and $[\widehat{q}; \widehat{w}_1, \widehat{w}_2, \ldots, \widehat{w}_n]$ are two representations of the same simple game, then $[q + \widehat{q}; w_1 + \widehat{w}_1, w_2 + \widehat{w}_2, \ldots, w_n + \widehat{w}_n]$ is also a representation of that game.

20.43 Prove that if $[q; w]$ is a representation of a weighted majority game $(N; v)$, then $[q(w); w]$ is also a representation of that game.

20.44 Let $(N; v)$ be a simple strong game, and let $x \in X(N; v)$ be an imputation. Recall that $q(x) := \min_{S \in \mathcal{W}^m} x(S)$. Answer the following questions:

(a) Show by example that the game $[q(x); x]$ is not necessarily a simple strong game. Which property in the definition of a simple strong game may not hold?
(b) Denote by $\mathcal{W}^m_x$ the set of minimal winning coalitions in the game $[q(x); x]$. Prove that $\mathcal{W}^m \subseteq \mathcal{W}^m_x$.
(c) Give an example showing that the inclusion $\mathcal{W}^m \subseteq \mathcal{W}^m_x$ can be strict.

20.45 Let $(N; v)$ be a coalitional game with $N = \{1, 2, 3\}$ and a coalitional function given by

$$v(S) = \begin{cases} 1 & |S| \geq 2, \\ 0 & |S| \leq 1. \end{cases} \tag{20.217}$$

Answer the following questions:

(a) Is the game $(N; v)$ a simple strong game?
(b) For which imputations $x \in X(N; v)$ is $[q(x); x]$ a simple strong game?
(c) For which imputations $x \in X(N; v)$ does the game $[q(x); x]$ represent the game $(N; v)$?

20.46 Find the Rabbi Nathan solution of the following bankruptcy problems:

(a) There are four wives, who are owed 100, 200, 300, and 400, respectively, out of an estate of 350.
(b) There are four wives, who are owed 100, 200, 300, and 400, respectively, out of an estate of 720.
(c) There are five wives, who are owed 60, 120, 180, 240, and 300, respectively, out of an estate of 600.

20.47 Prove directly that the function f defined in Equation (20.166) is continuous, its first coordinate is monotonically nondecreasing in d_1 for every fixed d_2, and its second coordinate is monotonically nondecreasing for in d_2 for every fixed d_1.

20.48 Which of the following properties are satisfied by the Rabbi Nathan solution to bankruptcy problems? (For each property, provide either a proof or a counterexample.)

(a) Symmetry (creditors with identical claims receive identical payments).
(b) Null player (a creditor with a claim of 0 receives nothing).

(c) Covariance under strategic equivalence: for every $a > 0$ and every $b \in \mathbb{R}^N$ that satisfy $ad_i + b_i > 0$ for every $i \in N$ we have

$$\varphi\left(aE + \sum_{i \in N} b_i; ad_1 + b_1, \ldots, ad_n + b_n\right) = a\varphi(E; d_1, \ldots, d_n) + b. \tag{20.218}$$

(d) Additivity:

$$\varphi(E; d_1, \ldots, d_n) + \varphi(\widetilde{E}; \widetilde{d}_1, \ldots, \widetilde{d}_n) = \varphi(E + \widetilde{E}; d_1 + \widetilde{d}_1, \ldots, d_n + \widetilde{d}_n). \tag{20.219}$$

20.49 Jeff owes Sam \$140, and owes Harry \$80. Jeff declares bankruptcy, because he has only \$100, and the decision of how to divide his \$100 between Harry and Sam comes before a court.

The court is composed of a three-judge panel, John, Clarence, and Ruth. John is convinced that the most proper division is [40 : 60], using the following reasoning. "There is no dispute over the first \$20," he claims, "because both agree that they go to Sam. Regarding the remaining 80, both parties have legitimate claims, and they must therefore divide that sum equally between them."

Clarence, in contrast, claims that the most proper division is [20 : 80], explaining: "Suppose each of them could take the amount owed to him, \$80 to Harry and \$140 to Sam. That would lead to a deficit of \$120. This deficit should be divided equally between the creditors; i.e., each of them should yield \$60 of what he claims."[12]

Ruth claims that the most proper division is [44.44 : 55.55], since "The total sum should be divided proportionally to the debts owed: Harry should get $\frac{80}{80+140} \times 100$, not a penny more or less."

For each judge, describe a solution concept to bankruptcy problems with n creditors that is consistent with his or her solution for the bankruptcy problem with two creditors.

20.50 Define the following solution concept φ to bankruptcy problems by

$$\varphi_i(E; d_1, d_2, \ldots, d_n) := \frac{d_1}{d_1 + d_2 + \cdots + d_n} \times E. \tag{20.220}$$

This is the proportional division between the creditors.

(a) Construct a system of containers that implements this solution.
(b) Prove that this is a consistent solution concept.
(c) Which of the properties listed in Exercise 20.48 are satisfied by this solution?

20.51 Repeat Exercise 20.50, using the following solution concept: divide the estate in a continuous manner equally between the creditors, until one (or more) of the

12 Note that under this procedure, a situation may develop in which a creditor may need to pay. For example, if $E = 20$, the debt to Sam is \$10, and the debt to Harry is \$100, under Clarence's procedure Sam needs to pay \$35, and Harry receives \$55.

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creditors has received his or her entire claim. Continue to divide the remaining amount of money between the remaining creditors, until one (or more) of the creditors has received his or her entire claim, and so on, until all the money in the estate runs out.

20.52 Let φ be a solution concept to the bankruptcy problem. The dual solution of φ is the solution concept φ^* defined as follows:

$$\varphi_i^*(E;d) = d_i - \varphi_i(D-E,d), \quad \forall i \in N, \tag{20.221}$$

where $D := \sum_{i=1}^n d_i$ is the sum of the claims. Since $D - E$ is the sum that the debtor cannot pay the creditors, φ^* divides the loss between the players in the same way that φ divides the profits.

(a) Prove that φ^* is a solution concept.
(b) Prove that the Rabbi Nathan solution is self-dual, i.e., satisfies $\varphi^* = \varphi$.
(c) Find the dual solution to proportional division defined in Exercise 20.50.
(d) Find the dual solution to the solution proposed by Clarence in Exercise 20.49.

20.53 Let $[E; d_1, d_2, \ldots, d_n]$ be a bankruptcy problem, where $d_1 \le d_2 \le \cdots \le d_n$. Denote $D := \sum_{i=1}^n d_i$, and for every $i \in N$, denote $D^i := \sum_{j=1}^i d_j$. In particular, $D = D^n$. Show that the Rabbi Nathan solution to the problem $[E; d]$ is given by the following function φ:

$$\varphi(E;d) := \begin{cases} \left(\frac{E}{n}, \frac{E}{n}, \ldots, \frac{E}{n}\right), & \text{if } E \le n\frac{D^1}{2}, \\ \left(\frac{d_1}{2}, \ldots, \frac{d_i}{2}, \frac{E-D^i}{n-i}, \ldots, \frac{E-D^i}{n-i}\right), & \text{if } 1 \le i < n \text{ and } \frac{D^i}{2} + (n-i)\frac{d_i}{2} \\ & \quad < E \le \frac{D^i}{2} + (n-i-1)\frac{d_{i+1}}{2}, \\ d - \varphi(D-E;d), & \text{if } \frac{D}{2} < E \le D. \end{cases} \tag{20.222}$$

Remark: For the case $\frac{D}{2} < E \le D$, use Exercise 20.52.

20.54 Consider the following situation. A corporation has regular shareholders, and preferred share holders. If the corporation goes bankrupt, every shareholder is supposed to get the value of the shares he holds. During the division of the assets of the bankrupt corporation, preferred shareholders take precedence: they first receive the value of the shares they hold, and only afterwards do the regular shareholders receive what is left of the money.

Formally, a *corporate bankruptcy problem* is given by:

- E: the total worth of the assets of the corporation.
- n: the number of preferred shareholders.
- m: the number of regular shareholders.
- $d_1, \ldots, d_n$: the values of the shares of the preferred shareholders.
- $d_{n+1}, \ldots, d_{n+m}$: the values of the shares of the regular shareholders.

A *solution concept* for the corporation bankruptcy problem is a function associating every corporate bankruptcy problem $[E; d_1, \ldots, d_n; d_{n+1}, \ldots, d_{n+m}]$ with a vector of nonnegative numbers $(x_1, \ldots, x_{n+m})$ satisfying:

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- $\sum_{i=1}^{n+m} x_i = E$.
- If $d_1 + \cdots + d_n \leq E$, then $x_{n+1} + \cdots + x_{n+m} = 0$.
- If $d_1 + \cdots + d_n \geq E$, then $x_i = d_i$ for every $1 \leq i \leq n$.

A solution concept for the corporate bankruptcy problem is *consistent with the "contested garment" solution* if for every corporate bankruptcy problem $[E; d_1, \ldots, d_n; d_{n+1}, \ldots, d_{n+m}]$, and every pair of players i and j, if i and j hold the same class of shares, then

$$\varphi_i(E; d_1, \ldots, d_n; d_{n+1}, \ldots, d_{n+m}) = f_1(x; d_i, d_j), \tag{20.223}$$

where

$$x = \varphi_i(E; d_1, \ldots, d_n; d_{n+1}, \ldots, d_{n+m}) + \varphi_j(E; d_1, \ldots, d_n; d_{n+1}, \ldots, d_{n+m}), \tag{20.224}$$

and f_1 is the first coordinate of the function defined in Equation (20.166) on page 834.

(a) Construct a system of containers implementing a solution to the corporate bankruptcy problem that is consistent with the "contested garment" solution.
(b) Prove that there exists at most one solution concept to the corporate bankruptcy problem that is consistent with the "contested garment" solution.

20.55 Let $[E; d_1, d_2, \ldots, d_n]$ be a bankruptcy problem satisfying $\sum_{i=1}^{n} d_i > E$. Prove that the game $(N; v)$ defined by

$$v(S) := \max\left\{E - \sum_{i \notin S} d_i, 0\right\} \tag{20.225}$$

is convex.

20.56 Prove that if one associates a bankruptcy problem $[E; d_1, d_2, \ldots, d_n]$ with the coalitional game $(N; w)$ where $w(S) = E - \sum_{i \in S^c} d_i$, and if $\widetilde{x}$ is the Shapley value of this game, then the Hart–Mas-Colell reduced game $\left(S; w_S^{\widetilde{x}}\right)$ is the coalitional game corresponding to the bankruptcy problem $[\widetilde{x}(S); (d_i)_{i \in S}]$.

For the definition of the Hart–Mas-Colell reduced game, see Definition 18.33 on page 768.

20.57 Prove that the Rabbi Nathan solution is consistent according to Definition 20.67 (page 843).

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