Tutorial 1: CS 215, Fall 2024

Prof. Sunita Sarawagi

12 August 2024

1. Let X be a random variable. If Var(X) = 0, then prove that (*Hint:* Use Chebyshev's Inequality)
$$P\{X = E[X]\} = 1$$

2. Let X and Y be two Random Variables.
   (a) If they are independent, show that they are uncorrelated (i.e. Corr(X, Y) = 0).
   (b) If they are uncorrelated (i.e. Corr(X, Y) = 0), does it imply they are independent? Prove or find a counterexample.

3. Let (X_i, Y_i), $i = 1, ..., n$ be a sequence of independent and identically distributed random vectors. That is, X_1, Y_1 is independent of, and has the same distribution as, X_2, Y_2, and so on. Although X_i and Y_i can be dependent, X_i and Y_j are independent when $i \neq j$. Let
$$\mu_x = E[X_i],\ \mu_y = E[Y_i]$$
$$\sigma_x^2 = Var(X_i),\ \sigma_y^2 = Var(Y_i)$$
$$\rho = Corr(X_i, Y_i)$$
Find Corr($\sum_{i=1}^n X_i, \sum_{i=1}^n Y_i$) (You might need to prove $Var(X + Y) = Var(X) + Var(Y)$ for independent R.V. X and Y)

4. A deck of 52 cards is shuffled and a bridge hand of 13 cards is dealt out. Let X and Y denote, respectively, the number of aces and the number of spades in the hand.
   (a) Show that X and Y are uncorrelated.
   (b) Are they independent?

5. (Gambler's fallacy) Mr. Jones has devised a gambling system for winning at roulette. When he bets, he bets on red and places a bet only when the 10 previous spins of the roulette have landed on a black number. He reasons that his chance of winning is quite large because the probability of 11 consecutive spins resulting in black is quite small. What do you think of this system?

6. Show that for any events E and F,
$$P(E|E \cup F) \geq P(E|F)$$
In words, probability that E occurs given E or F occurs must be larger than if we just know that only F occurs. (Hint: Compute $P(E|E \cup F)$ by conditioning on whether F occurs)

7. Consider the following game played with an ordinary deck of 52 playing cards: The cards are shuffled and then turned over one at a time. At any time, the player can guess that the next card to be turned over will be the ace of spades; if it is, then the player wins. In addition, the player is said to win if the ace of spades has not yet appeared when only one card remains and no guess has yet been made. What is a good strategy? What is a bad strategy?

8. (a) A gambler has a fair coin and a two-headed coin in his pocket. He selects one of the coins at random; when he flips it, it shows heads. What is the probability that it is the fair coin?
   (b) Suppose that he flips the same coin a second time and, again, it shows heads. Now what is the probability that it is the fair coin?
   (c) Suppose that he flips the same coin a third time and it shows tails. Now what is the probability that it is the fair coin?

9. There are k + 1 coins in a box. When flipped, the i^{th} coin will turn up heads with probability i/k, i = 0, 1, … , k. A coin is randomly selected from the box and is then repeatedly flipped. If the first n flips all result in heads, what is the conditional probability that the (n + 1) flip will do likewise? Calculate for large k. (*Hint:* For large n, use approximation $\frac{1}{n}\sum_{i=0}^{n}\left(\frac{i}{n}\right)^m \approx \int_0^1 x^m dx$)

10. A person tried by a 3-judge panel is declared guilty if at least 2 judges cast votes of guilty. Suppose that when the defendant is in fact guilty, each judge will independently vote guilty with probability 0.7, whereas when the defendant is in fact innocent, this probability drops to 0.2. If 70 percent of defendants are guilty, compute the conditional probability that judge number 3 votes guilty given that

    (a) judges 1 and 2 vote guilty

    (b) judges 1 and 2 cast 1 guilty and 1 not guilty vote

    (c) judges 1 and 2 both cast not guilty votes