11 The universal belief space

Chapter summary

In this chapter we construct the *universal belief space*, which is a belief space that contains all possible situations of incomplete information of a given set of players over a certain set of states of nature. The construction is carried out in a straightforward way. Starting from a given set of states of nature S and a set of players N we construct, step by step, the space of all possible hierarchies of beliefs of the players in N. The space of all possible hierarchies of beliefs of each player is proved to be a well-defined compact set T, called the *universal type space*. It is then proved that a type of a player is a joint probability distribution over the set S and the types of the other players. Finally, the universal belief space Ω is defined as the Cartesian product of S with n copies of T; that is, an element of Ω, called *state of the world*, consists of a state of nature and a list of types, one for each player.

Chapters 9 and 10 focused on models of incomplete information and their properties. A belief space Π with a set of players N on a set of states of nature S, is given by a set of states of the world Y, and, for each state of the world $\omega \in Y$, a corresponding state of nature $\mathfrak{s}(\omega) \in S$ and a belief $\pi_i(\omega) \in \Delta(Y)$ for each player $i \in N$. As we saw, the players' beliefs determine hierarchies of beliefs over the states of nature, that is, beliefs about the state of nature, beliefs about beliefs about the state of nature, beliefs about beliefs about beliefs about the state of nature, and so on (see Example 9.28 on page 334 for an Aumann model of incomplete information, Example 9.43 on page 350 for a Harsanyi model of incomplete information, and page 390 for a hierarchy of beliefs in a more general belief space). The players' hierarchies of beliefs are thus derived from the model of incomplete information, and they are not an element of the model.

In reality, when individuals analyze a situation with incomplete information they do not write down a belief space. They do, however, have hierarchies of beliefs over the state of nature: an investor ascribes a certain probability to the event "the interest rate next year will be 3%," he ascribes a possibly different probability to the event "the interest rate next year will be 3% and the other investor ascribes probability at least 0.7 to the interest rate next year being 3%," and he similarly ascribes probabilities to events that involve higher levels of beliefs. It therefore seems more natural to have the belief hierarchies as part of the data of the situation. In other words, we wish to describe a situation of incomplete information by the set of states of nature S and the players' belief hierarchies on S. Does such a description correspond to a belief space as defined in Section 10? This chapter is devoted to the affirmative answer of this question: starting from belief hierarchies we will construct the belief space that yields these belief hierarchies.

We will first define the concept of a belief hierarchy of a player, and construct a space $\Omega = \Omega(N, S)$ containing all possible hierarchies of beliefs of the set of players N about the set of states of nature S. We will then prove that this space is a belief space. This will imply that every belief space of the set of players N on the set of states of nature S is a belief subspace of the space Ω. That is why the space Ω will be called the *universal belief space*.

In constructing the universal belief space Ω, we will assume that the space of states of nature S is a compact set in a metric space. This assumption is satisfied, in particular, when S is a finite set, or when it is a closed and bounded set in a finite-dimensional Euclidean space.

Remark 11.1 *As we will now show, the assumption that the space of states of nature is compact is not a strong assumption. Suppose that the players in N are facing a strategic-form game in which the set of actions of player i is A_i, which may be a finite or an infinite set. We argue that under mild assumptions, such a game can be presented as a point in a compact subset of a metric space. Denote by $A = \times_{i \in N} A_i$ the set of action vectors. Assume that the preference relation of each player satisfies the von Neumann–Morgenstern axioms, and that each player has a most-preferred and a least-preferred outcome (see Section 2.6 for a generalization of the von Neumann–Morgenstern axioms to infinite sets of outcomes). It follows that the preference relation of each player i can be presented by a bounded linear utility function u_i. Since the utility function of every player is determined up to a positive affine transformation, we may suppose that the utility function of each player takes values in the range* $[0, 1]$.

As we saw in Chapters 9 and 10, a state of nature is a state game in strategic form that the players face. Suppose for now that the set of actions A_i of player i is common knowledge among the players. Then a state of nature is described by a vector of utility functions $(u_i)_{i \in N}$, i.e., by an element in $S := [0, 1]^A$: a list of payoff vectors for each action vector.

When the sets of actions are finite, the set S is compact, i.e., the set of states of nature is a compact set. When the sets of actions are compact (not necessarily finite), the set of states of nature is a compact set if we consider only state games in which the utility functions of all players are Lipschitz functions with a given constant. ♦

Recall that for every set X in a topological space, $\Delta(X)$ is the space of all probability distributions over X. We endow $\Delta(X)$ with the weak-$*$ topology. In the weak-$*$ topology, a sequence of distributions $(\mu^j)_{j \in \mathbb{N}}$ converges to a probability distribution μ if and only if for every continuous function $f : X \to \mathbb{R}$,

$$\lim_{j \to \infty} \int_X f(x) \mathrm{d}\mu^j(x) = \int_X f(x) \mathrm{d}\mu(x). \tag{11.1}$$

This topology is a metric topology: there exists a metric over the set $\Delta(X)$ satisfying the property that the collection of open sets in the weak-$*$ topology is identical with the collection of open sets generated by open balls in this metric.

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A fundamental property of this topology, which we will often make use of, is the following theorem, which follows from Riestz's representation theorem.

Theorem 11.2 *If X is a compact set in a metric space, then $\Delta(X)$ is a compact metric space (in the weak-$*$ topology).*

For the proof of this theorem, see Conway [1990], Theorem V.3.1, and Claim III.5.4. Further properties of this topology will be presented, as needed, in the course of the chapter.

11.1 Belief hierarchies

We begin by constructing all the belief spaces in the most direct and general possible way. A player's belief is described by a distribution over the parameter about which he is uncertain, i.e., over the state of nature. Denote by X_k the space of all belief hierarchies of order k. In particular, X_1 includes all the possible beliefs of a player about the states of nature; X_2 includes all possible beliefs of a player about the state of nature, and on the beliefs of the other players about the state of nature; X_3 includes all the beliefs of order 1 and 2 and the beliefs about the second-order beliefs of all the other players, etc.

Definition 11.3 *The space of* belief hierarchies of order k *of a set of players N on the set of states of nature S is the space X_k defined inductively as follows:*

$$X_1 := \Delta(S), \tag{11.2}$$

and for every $k \geq 2$,

$$\begin{aligned} X_k : &= X_{k-1} \times \Delta(S \times (X_{k-1})^{n-1}) \\ &= X_{k-1} \times \Delta(S \times \underbrace{X_{k-1} \times X_{k-1} \times \cdots \times X_{k-1}}_{n-1 \text{ times}}). \end{aligned} \tag{11.3}$$

An element $\mu_k \in X_k$ is called a belief of order k, *or a* belief hierarchy of order k.

Every probability distribution over S can be a first-order belief of a player in a game. A second-order belief (or hierarchy) is a first-order belief and a joint distribution over the set of states of nature and the first-order beliefs of the other players. In general, a $(k + 1)$-order belief includes a belief of order k and a joint distribution over the vectors of length n composed of a state of nature and the $n - 1$ beliefs of order k of the other players. Note that the joint distribution over $S \times (X_k)^{n-1}$ is not necessarily a product distribution. This means that a player can believe that there is a correlation between the beliefs of order k of the other players, and between those beliefs and the state of nature. This can happen, for example, if the player believes that one or more of the other players knows the state of nature, or if some other players have common information on the state of nature.

Since the first component of a $(k + 1)$-order belief is a k-order belief, and the first component of a k-order belief is a $(k - 1)$-order belief, and so on, a $(k + 1)$-order belief defines the player's beliefs of order $1, 2, \ldots, k$. This is the reason that a $(k + 1)$-order belief is also called a "belief hierarchy of order $k + 1$."

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Example 11.4 Suppose that there are two players $N = \{\text{Benjamin, George}\}$, and two states of nature $S = \{s_1, s_2\}$. The space of belief hierarchies of order 1 of every player is $X_1 = \Delta(S)$, and a first-order belief is of the form $[p_1(s_1), (1 - p_1)(s_2)]$: "I ascribe probability p_1 to the state of nature being s_1, and probability $1 - p_1$ to the state of nature being s_2."A second-order belief is an element in $X_2 = X_1 \times \Delta(S \times X_1)$, for example, "I ascribe probability p_2 to the state of nature being s_1, probability $1 - p_2$ to the state of nature being s_2 (an element of X_1), probability α_1 to the state of nature being s_1 and the belief of the second player on the state of nature being $[q_1(s_1), (1 - q_1)(s_2)]$, probability α_2 to the state of nature being s_1 and the belief of the second player on the state of nature being $[q_2(s_1), (1 - q_2)(s_2)]$, and probability $1 - \alpha_1 - \alpha_2$ to the state of nature being s_2 and the belief of the second player on the state of nature being $[q_3(s_1), (1 - q_3)(s_2)]$ (this is an element in $\Delta(S \times X_1)$).

Note that each of these beliefs can be those of either Benjamin or George: the belief spaces, at any order, of the players are identical." ◀

While the concept of a belief hierarchy is an intuitive one, the detailed mathematical description of a belief hierarchy may be extremely cumbersome. Despite this, we can prove mathematical properties of belief hierarchies that will eventually enable us to construct the universal belief space, which is the space of all possible belief hierarchies.

Theorem 11.5 *For each $k \in \mathbb{N}$, the set X_k is a compact set in a metric space.*

Proof: The theorem is proved by induction on k. Because S is a compact set in a metric space, Theorem 11.2 implies that $X_1 = \Delta(S)$ is also a compact set in a metric space.

Let $k \geq 1$, and suppose by induction that X_k is a compact set. It follows that the set $S \times (X_k)^{n-1}$ is also compact, as the Cartesian product of compact sets in metric spaces. By Theorem 11.2 again, the set $\Delta(S \times (X_k)^{n-1})$ is a compact subset of a metric space in the weak-$*$ topology. We deduce that the set $X_{k+1} = X_k \times \Delta(S \times (X_k)^{n-1})$ as the Cartesian product of two compact sets in metric spaces is also a compact set in a metric space. □

A k-order belief of a player is an element of X_k. Can every element in X_k be an "acceptable" belief of a player? The answer to this question is negative.

Example 11.4 (*Continued*) In this example, where $N = \{\text{Benjamin, George}\}$ and $S = \{s_1, s_2\}$, an element in X_1 is of the form $[p_1(s_1), (1 - p_1)(s_2)]$, and every such element is an "acceptable" first-order belief of a player. We will show, however, that not every second-order belief is "acceptable."

A second-order belief of a player is a pair $\mu_2 = (\mu_1, \nu_1)$, where μ_1 is a first-order belief of the player, and ν_1 is a probability distribution over $S \times X_1$. In other words, ν_1 is a probability distribution over vectors of the form (s, ρ), where s is a state of nature, and ρ is a first-order belief of the other player.

On page 443, we gave an example of a second-order belief of a player where:

- The first-order belief is $\mu_1 = [p_2(s_1), (1 - p_2)(s_2)]$.
- The distribution ν_1 ascribes probability α_1 to the state of nature being s_1 and the first-order belief of the second player being $[q_1(s_1), (1 - q_1)(s_2)]$.
- The distribution ν_1 ascribes probability α_2 to the state of nature being s_1 and the first-order belief of the second player being $[q_2(s_1), (1 - q_2)(s_2)]$.
- The distribution ν_1 ascribes probability $1 - \alpha_1 - \alpha_2$ to the state of nature being s_2 and the first-order belief of the second player being $[q_3(s_1), (1 - q_3)(s_2)]$.

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If a player's belief is "acceptable," we expect the player to be able to answer the question "what is the probability you ascribe to the state of nature being s_1?" When the second-order belief of the player is $\mu_2 = (\mu_1, \nu_1)$, he can answer this question in two different ways. On the one hand, μ_1 is a first-order belief; i.e., it is a probability distribution over S, so that the answer to our question is the probability that μ_1 ascribes to s_1. In the example above, that answer is p_2. On the other hand, ν_1 is a probability distribution over $S \times X_1$, so that the answer to our question is the probability that the marginal distribution of ν_1 over S ascribes to s_1. In the above example, that answer is $\alpha_1 + \alpha_2$. For the probability that the player ascribes to the state of nature s_1 to be well defined, we must require that $p_2 = \alpha_1 + \alpha_2$. In general, for a second-order belief of a player to be "acceptable," the marginal distribution of ν_1 over S must coincide with μ_1, which is also a probability distribution over S.

Note that according to the player's second-order belief, the probability that the other player ascribes to the state of nature being s_1 is $\alpha_1 q_1 + \alpha_2 q_2 + (1 - \alpha_1 - \alpha_2) q_3$. It follows that even if the player's belief is "acceptable," i.e., if $p_2 = \alpha_1 + \alpha_2$, if $\alpha_1 q_1 + \alpha_2 q_2 + (1 - \alpha_1 - \alpha_2) q_3 \neq p_2$, then the player believes that the other player ascribes a probability to the state of nature being s_1 that is different from the probability that he himself ascribes to that event. Thus, the inequality $\alpha_1 q_1 + \alpha_2 q_2 + (1 - \alpha_1 - \alpha_2) q_3 \neq p_2$ does not mean that the player's belief is unacceptable, because the player may believe that the other player does not agree with him. ◀

The condition $p_2 = \alpha_1 + \alpha_2$, which emerged in the above discussion, is a mathematical condition constraining the distributions that comprise a belief hierarchy. Its purpose is to ensure that the beliefs in a belief hierarchy do not contradict each other. This condition is called the *coherency condition*. To define the coherency condition precisely, denote by μ_{k+1} a belief hierarchy of order $k+1$, i.e., an element in X_{k+1}, for every $k \geq 0$. We will present conditions that ensure that such a hierarchy is coherent.

Since by the inductive definition (Definition 11.3) an element of X_{k+1} is $\mu_{k+1} \in X_k \times \Delta(S \times (X_k)^{n-1})$, we write $\mu_{k+1} = (\mu_k, \nu_k)$, where $\mu_k \in X_k$ and $\nu_k \in \Delta(S \times X_k^{n-1})$. We similarly write $\mu_k = (\mu_{k-1}, \nu_{k-1})$, where $\mu_{k-1} \in X_{k-1}$ and $\nu_{k-1} \in \Delta(S \times (X_{k-1})^{n-1})$. Note that[1]

$$\begin{aligned} \nu_k \in \Delta(S \times (X_k)^{n-1}) &= \Delta(S \times (X_{k-1} \times \Delta(S \times (X_{k-1})^{n-1}))^{n-1}) \\ &= \Delta(S \times (X_{k-1})^{n-1} \times (\Delta(S \times (X_{k-1})^{n-1}))^{n-1}). \end{aligned} \tag{11.4}$$

The marginal distribution of ν_k over $S \times (X_{k-1})^{n-1}$ is the player's belief about the $(k-1)$-order beliefs of the other players. For μ_{k+1} to be coherent, we require that the marginal distribution of ν_k over $S \times (X_{k-1})^{n-1}$ to be equal to the probability distribution ν_{k-1} over $S \times (X_{k-1})^{n-1}$, which comprises part of μ_k. We also require that the players believe that the beliefs of the other players be coherent: ν_k must ascribe probability 1 to the event that the lower-order beliefs of the other players are also coherent. These conditions together lead to the following inductive definition of Z_k, the set of all coherent belief hierarchies of order k (for each $k \in \mathbb{N}$).

1 The spaces $(S \times (X_{k-1})^{n-1})^{n-1}$ and $S \times (X_{k-1})^{n-1} \times \left(\Delta(S \times (X_{k-1})^{n-1})\right)^{n-1}$ in Equation (11.4) differ from each other in the order of the coordinates. Here and in the sequel we will relate to these spaces as if they were identical, identifying the corresponding coordinates in the two spaces.

Definition 11.6 *For each $k \in \mathbb{N}$, the* space of coherent belief hierarchies of order k *is the space Z_k defined inductively as follows:*

$$Z_1 := X_1 = \Delta(S), \tag{11.5}$$

$$Z_2 := \{\mu_2 = (\mu_1, \nu_1) \in Z_1 \times \Delta(S \times (Z_1)^{n-1}) : \tag{11.6}$$
$$\text{the marginal distribution of } \nu_1 \text{ over } S \text{ equals } \mu_1\}.$$

For each $k \geq 2$,

$$Z_{k+1} := \{\mu_{k+1} = (\mu_k, \nu_k) \in Z_k \times \Delta(S \times (Z_k)^{n-1}) :$$
$$\text{the marginal distribution of } \nu_k \text{ over } S \times (Z_{k-1})^{n-1} \text{ equals } \nu_{k-1} \text{ where } \mu_k = (\mu_{k-1}, \nu_{k-1})\}. \tag{11.7}$$

An element in the set Z_k is called a coherent belief hierarchy of order k.

In words, every belief of order 1 of a player is coherent; a second-order belief hierarchy $\mu_2 = (\mu_1, \nu_1)$ is a coherent belief hierarchy if the marginal distribution of ν_1 over S equals μ_1; for $k \geq 2$, a $(k+1)$-order belief hierarchy $\mu_{k+1} = (\mu_k, \nu_k)$ is a coherent belief hierarchy of order $k+1$ if:

- $\mu_k = (\mu_{k-1}, \nu_{k-1})$ is a coherent belief hierarchy of order k.
- ν_k is a probability distribution over $S \times (Z_k)^{n-1}$.
- The marginal distribution of ν_k over $S \times (X_{k-1})^{n-1}$ equals ν_{k-1}.

One can prove by induction that $Z_k \subseteq X_k$: every coherent belief hierarchy is a belief hierarchy (Exercise 11.3). As mentioned before, the coherency condition requires the beliefs of a player to be well defined. If the coherency condition is not met, then, for example, the probability that the player ascribes to event A may be $\frac{1}{3}$ according to his k-order belief, and $\frac{2}{5}$ according to his l-order belief. This is, of course, meaningless: the mathematical structure must reflect the intuition that the question "What is the probability that the player ascribes to an event A?" has an unequivocal answer. To understand the content of the coherency condition, note that a belief hierarchy of order k of any player defines a belief hierarchy for all orders l less than k for that player. Indeed, $\mu_k = (\mu_{k-1}, \nu_{k-1})$, where $\mu_{k-1} \in Z_{k-1}$ is the player's belief hierarchy of order $k-1$ and $\nu_{k-1} \in \Delta(S \times (Z_{k-1})^{n-1})$ is that player's belief on the states of nature and on the belief hierarchies of order $k-1$ of the other players. Similarly, $\mu_{k-1} = (\mu_{k-2}, \nu_{k-2})$, where $\mu_{k-2} \in Z_{k-2}$ is the players' belief hierarchy of order $k-2$ and $\nu_{k-2} \in \Delta(S \times (Z_{k-2})^{n-1})$ is that player's belief about the states of nature and about the belief hierarchies of order $k-2$ of the other players. Continuing in this way, we arrive at the conclusion that in effect a belief hierarchy of order k is equivalent to a vector

$$\mu_k = (\mu_1; \nu_1, \nu_2, \ldots, \nu_{k-1}), \qquad k \geq 2, \tag{11.8}$$

where μ_1 is the player's belief about the state of nature, and $\nu_l \in \Delta(S \times (Z_l)^{n-1})$ is a probability distribution over the states of nature and the belief hierarchies of order l of the other players, for all $2 \leq l < k$. As the next theorem states, the coherency condition guarantees that all of these distributions "agree" with each other. The proof of the theorem is left to the reader (Exercise 11.6).

Theorem 11.7 *Let $\mu_k = (\mu_1; \nu_1, \nu_2, \ldots, \nu_{k-1}) \in Z_k$ be a coherent belief hierarchy of order k, and let l_1, l_2 be integers satisfying $1 \leq l_1 \leq l_2 \leq k$. Then:*

1. *The marginal distribution of ν_1 over S equals μ_1.*
2. *The marginal distribution of ν_{l_2} over $S \times (Z_{l_1})^{n-1}$ is ν_{l_1}.*

The following theorem is a reformulation of Definition 11.6, and it details which pairs (μ_k, ν_k) form coherent beliefs of order $k+1$.

Theorem 11.8 *Let $\mu_k = (\mu_1; \nu_1, \nu_2, \ldots, \nu_{k-1}) \in Z_k$ be a coherent belief hierarchy of order k, and let $\nu_k \in \Delta(S \times (X_k)^{n-1})$. The pair (μ_k, ν_k) is a coherent belief hierarchy of order $k+1$ if and only if the following conditions are met:*

- *ν_k ascribes probability 1 to $S \times (Z_k)^{n-1}$.*
- *For $k = 1$, the marginal distribution of ν_1 over S is μ_1.*
- *For $k > 1$, the marginal distribution of ν_k over $S \times (X_{k-1})^{n-1}$ equals ν_{k-1}, where $\mu_k = (\mu_{k-1}, \nu_{k-1})$.*

From Theorem 11.8 it follows that if the belief of player i is coherent, then for every finite sequence of players $i_1, i_2, \ldots, i_l$, player i believes (ascribes probability 1) to i_1 believing that player i_2 believes ... that the belief hierarchy of order $k - l$ of player i_l is coherent.

Example 11.9 In this example we present a situation of incomplete information and write down the belief hierarchy of one of the players. Phil wonders what the color of the famous Shwezigon Pagoda in Burma is, and whether his brother Don knows what it is. The states of nature are the possible colors of the pagoda: s_{b} (blue), s_{g} (gold), s_{p} (purple), s_{r} (red), s_{w} (white), and so on. Phil does not know the color of the pagoda; he ascribes probability $\frac{1}{3}$ to the pagoda being red and probability $\frac{2}{3}$ to its being gold. Phil's first-order belief is therefore

$$\mu_1 = \left[\tfrac{1}{3}(s_{\mathrm{r}}), \tfrac{2}{3}(s_{\mathrm{g}})\right] \in \Delta(S). \tag{11.9}$$

Phil also believes that if the pagoda is red, then Don ascribes probability $\frac{1}{2}$ to the pagoda being red and probability $\frac{1}{2}$ to its being blue. He also believes that if the pagoda is gold then Don ascribes probability 1 to its being gold. Phil's second-order belief is $\mu_2 = (\mu_1, \nu_1)$ where μ_1 is given by Equation (11.9) and

$$\nu_1 = \left[\tfrac{1}{3}\left(s_{\mathrm{r}}, \left[\tfrac{1}{2}(s_{\mathrm{r}}), \tfrac{1}{2}(s_{\mathrm{b}})\right]\right), \tfrac{2}{3}\left(s_{\mathrm{g}}, 1[s_{\mathrm{g}}]\right)\right] \in \Delta(S \times Z_1). \tag{11.10}$$

In addition, Phil believes that if the pagoda is red, then the following conditions are met:

- Don ascribes probability $\frac{1}{2}$ to "the pagoda is red, and Phil believes that the pagoda is purple."
- Don ascribes probability $\frac{1}{2}$ to "the pagoda is blue, and Phil believes that the pagoda is red."

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Phil also believes that if the pagoda is gold, then Don ascribes probability 1 to "the pagoda is gold, and Phil ascribes probability 1 to the pagoda being white." Phil's third-order belief is $\mu_3 = (\mu_2, \nu_2)$ with μ_2 as defined above and

$$\nu_2 = \left[\tfrac{1}{3}\left(s_r, \left[\tfrac{1}{2}(s_r), \tfrac{1}{2}(s_b)\right], \left[\tfrac{1}{2}(s_r, [1(s_p)]), \tfrac{1}{2}(s_b, [1(s_r)])\right]\right), \tfrac{2}{3}(s_g, [1(s_g)], [1(s_g, [1(s_w)])])\right]. \tag{11.11}$$

ν_2 is Phil's belief about Don's belief. In Equation (11.11), we see that Phil believes that if the state of nature is s_r, then Don's first-order belief is $[\frac{1}{2}(s_r), \frac{1}{2}(s_b)]$ and Don's second-order belief is $([\frac{1}{2}(s_r), \frac{1}{2}(s_b)], [\frac{1}{2}(s_r, [1(s_p)]), \frac{1}{2}(s_b, [1(s_r)])])$. Phil also believes that if the state of nature is s_g, then Don's first-order belief is $[1(s_g)]$ and Don's second-order belief is $([1(s_g)], [1(s_g, [1(s_w)])])$.

We can now check the meaning of the coherence condition in this example. First, Phil's belief is coherent:

- The marginal distribution of ν_1 over S is μ_1.
- The projection of ν_2 on $S \times Z_1$ is ν_1.

Second, Phil believes that Don's belief is coherent. Indeed, the second-order belief that Phil ascribes to Don is coherent in both the states of nature s_r and s_g. Note that Phil indeed has beliefs about Don's beliefs, but Don's true beliefs are not expressed in Phil's belief about Don's beliefs; the latter may in fact differ from Phil's beliefs about Don's beliefs. ◄

Does there exist a coherent belief hierarchy of order k for every k? Can every coherent belief hierarchy of order k be extended to a coherent belief hierarchy of order $k+1$; in other words, given a coherent belief hierarchy μ_k of order k, can we find a coherent belief hierarchy μ_{k+1} of order $k+1$ such that $\mu_{k+1} = (\mu_k, \nu_k)$? The answer to these questions is yes. With respect to the first question, if $s_0 \in S$ is a given state of nature, then the following sentence defines a coherent belief hierarchy order k (Exercise 11.7): "I ascribe probability 1 to the state of nature being s_0, I ascribe probability 1 to the other players ascribing probability 1 to the state of nature being s_0, I ascribe probability 1 to each of the other players ascribing probability 1 to each of the other players ascribing probability 1 to the state of nature being s_0, and so on, up to level k."

The proof that every coherent belief hierarchy of order k can be extended to a coherent belief hierarchy of order $k+1$ is more complicated, and we will present it next. We start by showing that for every k, the set Z_k is compact.

Theorem 11.10 *For each $k \in \mathbb{N}$, the set Z_k is compact in X_k.*

Proof: The theorem is proved by induction on k. Start with $k = 1$. By definition, $Z_1 = \Delta(S)$, which is a compact set in a metric space (see the proof of Theorem 11.5).

Let $k \geq 1$, and suppose by induction that Z_k is a compact set in X_k. Since the set X_{k+1} is compact in a metric space (Theorem 11.5), to prove that Z_{k+1} is a compact set in X_{k+1} it suffices to prove that the set Z_{k+1} is a closed set. To this end we need to show that the limit of every convergent sequence of points in Z_{k+1} is also in Z_{k+1}. This follows from the following two well-known facts regarding the weak-$*$ topology:

- Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of probability distributions over a space X, which converges in the weak-$*$ topology to a probability distribution μ, and satisfies, for a compact

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set $T \subseteq X$, the condition $\mu_n(T) = 1$ for all $n \in \mathbb{N}$. Then $\mu(T) = 1$ (this follows from Theorem 2.1 in Billingsley [1999]).

- Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of probability distributions over a product space $X \times Y$ converging in the weak-$*$ topology to a probability distribution μ. Denote by ν^j the marginal distribution of μ^j over X, and by ν the marginal distribution of μ over X. Then the sequence $(\nu_n)_{n\in\mathbb{N}}$ converges in the weak-$*$ topology to ν (see Theorem 2.8 in Billingsley [1999]).

Indeed, let $(\mu^j_{k+1})_{j\in\mathbb{N}}$ be a sequence of points in Z_{k+1} converging to the limit μ_{k+1} in X_{k+1}. Denote $\mu^j_{k+1} = (\mu^j_k, \nu^j_k)$ and $\mu_{k+1} = (\mu_k, \nu_k)$. By Equation (11.8), μ^j_{k+1} ascribes probability 1 to $S \times (Z_k)^{n-1}$, and the marginal probability distribution of ν^j_k over $S \times (Z_k)^{n-1}$ is ν^j_{k-1}. By the two above-mentioned facts, these two properties also hold for the limits μ_k and ν_k. By Theorem 11.8, we deduce that $\mu_{k+1} \in Z_{k+1}$, which is what we needed to show. □

We are now ready to prove that every coherent belief hierarchy $\mu_k \in Z_k$ of order k can be extended to a coherent belief hierarchy $\mu_{k+1} = (\mu_k, \nu_k) \in Z_{k+1}$. Since the set Z_1 is nonempty, it will follow from this in particular that for any $k \in \mathbb{N}$ the set Z_k is nonempty.

Theorem 11.11 *For any $k \in \mathbb{N}$ and every coherent belief hierarchy μ_k of order k there exists $\nu_k \in \Delta(S \times (Z_k)^{n-1})$ such that the pair (μ_k, ν_k) is a coherent belief hierarchy of order $k + 1$.*

We will in effect be proving that there exists a continuous function $h_k : Z_k \to \Delta(S \times (Z_k)^{n-1})$ such that $(\mu_k, h_k(\mu_k)) \in Z_{k+1}$ for every $\mu_k \in Z_k$. If we define a function $f_k : Z_k \to Z_{k+1}$ by

$$f_k(\mu_k) = (\mu_k, h_k(\mu_k)), \tag{11.12}$$

the function f_k will be a continuous function associating every coherent belief hierarchy of order k with a coherent belief hierarchy of order $k + 1$, such that the projection of f_k to the first coordinate is the identity function.

Proof: We prove the existence of the continuous function h_k by induction on k. We start with the case $k = 1$. Let $s_1 \in S$ be a state of nature. The distribution $[1(s_1)] \in \Delta(S)$ is a first-order belief hierarchy in which the player ascribes probability 1 to s_1. For each $\mu_1 \in Z_1 = \Delta(S)$, consider the product[2] distribution $\nu_1 := \mu_1 \otimes [1(s_1)]^{n-1}$ over $S \times (Z_1)^{n-1}$. The pair $\mu_2 := (\mu_1, \nu_1)$ is a second-order belief hierarchy: the player believes that the probability distribution of the state of nature is μ_1, and that each of the other players ascribes probability 1 to the state of nature being s_1.

Define a function $h_1 : Z_1 \to \Delta(S \times (Z_1)^{n-1})$ as follows:

$$h_1(\mu_1) := \mu_1 \otimes [1(s_1)]^{n-1}. \tag{11.13}$$

2 When $\mu_1 \in \Delta(X_1)$ and $\mu_2 \in \Delta(X_2)$ are two probability distributions, the product distribution $\mu_1 \otimes \mu_2 \in \Delta(X_1 \times X_2)$ is the unique probability distribution over $X_1 \times X_2$ that satisfies $(\mu_1 \otimes \mu_2)(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2)$ for every pair of measurable sets A_1 in X_1 and A_2 in X_2.

As we saw earlier, the pair $(\mu_1, h_1(\mu_1))$ is a coherent second-order belief hierarchy. Moreover, the function h_1 is continuous (why?). We have thus completed the proof for the case $k = 1$.

Suppose by induction that there exists a continuous function $h_k : Z_k \to \Delta(S \times (Z_k)^{n-1})$ satisfying $(\mu_k, h_k(\mu_k)) \in Z_{k+1}$ for all $\mu_k \in Z_k$. By Equation (11.12) this function defines a function $f_k : Z_k \to Z_{k+1}$. We now proceed to construct the function h_{k+1}.

For every $k \in \mathbb{N}$ set $Y_k := S \times (Z_k)^{n-1}$. For every coherent belief hierarchy of order $k+1$, $\mu_{k+1} = (\mu_k, \nu_k)$, the component ν_k is a probability distribution over $S \times (Z_k)^{n-1} = Y_k$. Note that

$$\begin{aligned} Y_{k+1} &= S \times (Z_{k+1})^{n-1} \subseteq S \times (Z_k \times \Delta(S \times (Z_k)^{n-1}))^{n-1} \\ &= S \times (Z_k)^{n-1} \times (\Delta(S \times (Z_k)^{n-1}))^{n-1} = Y_k \times (\Delta(Y_k))^{n-1}. \end{aligned} \tag{11.14}$$

We will denote an element of Y_k by $(s, (\mu_{k,j})_{j=1}^{n-1})$, where $\mu_{k,j} \in Z_k$ for all $j = 1, 2, \ldots, n-1$. Using Equation (11.12), changing the order of the coordinates yields

$$\left(s, (f_k(\mu_{k,j}))_{j=1}^{n-1}\right) = \left(s, (\mu_{k,j}, h_k(\mu_{k,j}))_{j=1}^{n-1}\right) = \left(s, (\mu_{k,j})_{j=1}^{n-1}, (h_k(\mu_{k,j}))_{j=1}^{n-1}\right); \tag{11.15}$$

i.e., the projection of $(s, (f_k(\mu_{k,j}))_{j=1}^{n-1})$ on Y_k is $(s, (\mu_{k,j})_{j=1}^{n-1})$.

For every measurable set $A \subseteq Y_k$ define a set $F_k(A) \subseteq Y_{k+1}$ as follows:

$$F_k(A) := \left\{\left(s, (f_k(\mu_{k,j}))_{j=1}^{n-1}\right) : \left(s, (\mu_{k,j})_{j=1}^{n-1}\right) \in A\right\} \subseteq Y_{k+1}. \tag{11.16}$$

This set includes all the coherent belief hierarchy vectors of order $k+1$ of the other players derived by expanding the coherent belief hierarchy vectors of order k contained in A by using f_k. By the induction assumption, $F_k(A)$ is not empty when $A \neq \emptyset$ (because $(s, (f_k(\mu_{k,j})_{j=1}^{n-1})) \in F_k(A)$ for every $\mu_k \in A$) and is contained in $S \times (Z_{k+1})^{n-1}$.

Consider next the inverse function of F_k: for every measurable set $B \subseteq Y_{k+1}$ define

$$F_k^{-1}(B) := \left\{\left(s, (\mu_{k,j})_{j=1}^{n-1}\right) : \left(s, \left(f_k(\mu_{k,j})_{j=1}^{n-1}\right)\right) \in B\right\} \subseteq Y_k. \tag{11.17}$$

This is the set of all elements of Y_k that are mapped by f_k to the elements of B. Since the function f_k is continuous, it is in particular a measurable function, and therefore the set $F_k^{-1}(B)$ is also measurable.[3] We next define an element $\nu_{k+1} \in \Delta(Y_{k+1})$ as follows: for every measurable set $B \subseteq Y_{k+1}$,

$$\nu_{k+1}(B) := \nu_k(F_k^{-1}(B)). \tag{11.18}$$

Define the function $h_{k+1} : Z_{k+1} \to \Delta(Y_{k+1})$ by

$$h_{k+1}(\mu_{k+1}) := \nu_{k+1}. \tag{11.19}$$

The probability distribution ν_{k+1} is a distribution over Y_{k+1}. By Equation (11.14), ν_{k+1} is also a probability distribution over the set $Y_k \times (\Delta(Y_k))^{n-1}$ whose support is Y_{k+1}. We need to check that the marginal distribution of ν_{k+1} over Y_k is ν_k. To do so, we consider a measurable set $A \subseteq Y_k$ and check that

$$\nu_{k+1}(A \times (\Delta(Y_k))^{n-1}) = \nu_k(A). \tag{11.20}$$

3 To show that the set $F_k^{-1}(B)$ is measurable, it suffices to show that the function f_k is a measurable function. We choose to show that this function is continuous because it is easier to do so than to show directly that it is measurable.

Now,

$$\nu_{k+1}(A \times (\Delta(Y_k))^{n-1}) = \nu_k(F_k^{-1}(A \times (\Delta(Y_k))^{n-1}) \tag{11.21}$$
$$= \nu_k\Big(\Big\{\big(s, (\mu_{k,j})_{j=1}^{n-1}\big) : \big(s, f_k(\mu_{k,j})_{j=1}^{n-1}\big) \in A \times (\Delta(Y_k))^{n-1}\Big\}\Big) \tag{11.22}$$
$$= \nu_k(A). \tag{11.23}$$

Finally, we show that h_{k+1} is a continuous function. Let $(\mu_{k+1}^l)_{l\in\mathbb{N}}$ be a sequence of probability distributions over Y_k converging to the limit μ_{k+1} in the weak-$*$ topology. Denote $\nu_{k+1}^l := h_{k+1}(\mu_{k+1}^l)$ and $\nu_{k+1} = h_{k+1}(\mu_{k+1})$. We need to show that for every continuous function $g : Y_{k+1} \to \mathbb{R}$,

$$\lim_{l\to\infty} \int_{S\times(Z_{k+1})^{n-1}} g\big(s, (\widetilde{\mu}_{k+1,j})_{j=1}^{n-1}\big)\mathrm{d}\nu_{k+1}^l\big(s, (\widetilde{\mu}_{k+1,j})_{j=1}^{n-1}\big)$$
$$= \int_{S\times(Z_{k+1})^{n-1}} g\big(s, (\widetilde{\mu}_{k+1,j})_{j=1}^{n-1}\big)\mathrm{d}\nu_{k+1}\big(s, (\widetilde{\mu}_{k+1,j})_{j=1}^{n-1}\big). \tag{11.24}$$

This follows directly from $\mu_{k+1} = f_k(\mu_k)$, along with the fact that if g and f_k are continuous functions then the composition $g(s, (f(\mu_{k,j}))_{j=1}^{n-1})$ is a continuous function, where $\mu_{k+1,j} = (\mu_{k,j}, \nu_{k,j})$. □

11.2 Types

The sequence $(Z_k)_{k=1}^{\infty}$ of the spaces of coherent belief hierarchies has a special structure. Define a projection $\rho : Z_{k+1} \to Z_k$ as follows: if $\mu_{k+1} = (\mu_k, \nu_k) \in Z_{k+1}$, then $\rho(\mu_{k+1}) := \mu_k$. Theorem 11.11 implies that $\rho(Z_{k+1}) = Z_k$. Such a structure is called a *projective structure*, and it enables us to define the projective limit as follows.

Definition 11.12 The projective limit[4] *of the sequence of the spaces* $(Z_k)_{k=1}^{\infty}$ *is the space* T *of all the sequences* $(\mu_1, \mu_2, \ldots) \in \times_{k=1}^{\infty} Z_k$, *where for every* $k \in \mathbb{N}$ *the belief hierarchy* $\mu_k \in Z_k$ *is the projection of the belief hierarchy* $\mu_{k+1} \in Z_{k+1}$ *on* Z_k. *In other words, there exists a distribution* $\nu_k \in \Delta(S \times Z_k^{n-1})$ *such that* $\mu_{k+1} = (\mu_k, \nu_k)$. *The projective limit* T *is called the* universal type space.

An element in the universal type space is a sequence of finite belief hierarchies, satisfying the condition that for each k, the belief hierarchy of order $k+1$ is an extension of the belief hierarchy of order k. Such an element is called a *type*, a term due to Harsanyi.

Definition 11.13 *An element* $t = (\mu_1, \mu_2, \ldots) \in T$ *is called a* type.

A player's type is sometimes called his "state of mind," since it contains answers to all questions regarding the player's beliefs (of any order) about the state of nature. A player's belief hierarchy defines his beliefs to all orders: his beliefs about the state of nature, his beliefs about the beliefs of the other players, his beliefs about their second-order beliefs,

4 The projective limit is also called the *inverse limit*. This definition is a special case of the more general definition of the projective limit of an infinite sequence of spaces on which a projective operator is defined, from which the name "projective limit" is derived.

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and so on. We assume that a player's type is all the relevant information that the player has about the situation, and in what follows we will relate to a type as all the information in a player's possession.

Let $t = (\mu_1, \mu_2, \ldots)$ be a player's type. Since the distribution μ_k is a coherent belief hierarchy of order k, as previously noted, it follows that for every list of players $i_1, i_2, \ldots, i_l$, the player believes that player i_1 believes that player i_2 believes that $\ldots$ believes that player i_l's belief hierarchy of order $k - l$ is coherent. Since for every $k \in \mathbb{N}$, the first component of μ_{k+1} is μ_k, a player of type t believes that the fact that "the players' beliefs are coherent" is common belief among the players (Definition 10.9 on page 393).

As the following example shows, when the set of players contains only one player, and there are two states of nature, the universal type space can be simply described. This observation is extended to any finite set of states of nature in Exercise 11.9.

Example 11.14 In this example, we will construct the universal type space when there is one player, $N = \{\mathrm{I}\}$, and two states of nature, $S = \{s_1, s_2\}$. By definition,

$$X_1 = \Delta(S), \tag{11.25}$$

$$X_k = \Delta(S)^{k-1} \times \Delta(S) = (\Delta(S))^k, \quad \forall k \geq 2. \tag{11.26}$$

The coherency condition implies that the player's type (there is only one player here) is entirely determined by his first-order beliefs. The universal type space in this case is homeomorphic to the set $[0, 1]$: for every $p \in [0, 1]$, the element t_p corresponds to the type ascribing probability p to the state of nature s_1, and probability $1 - p$ to the state of nature s_2. ◀

When the set of players contains two or more players, the mathematical structure of the universal type space is far more complicated, because in that case a second-order belief hierarchy is a distribution over distributions, a third-order belief hierarchy is a distribution over distributions over distributions, and so on. The only way to analyze universal type spaces tractably requires simplifying their mathematical description. We will therefore consider several mathematical properties of the universal type space T that will be useful towards that end. Since a type is an element of the product space $\times_{k=1}^{\infty} Z_k$, a natural topology over the universal type space, which we will use, is the topology induced by the product topology on this space.

Theorem 11.15 *The universal type space T is a compact space.*

Proof: As previously stated, every coherent belief hierarchy μ_k of order k uniquely defines an element

$$(\mu_1, \mu_2, \ldots, \mu_k) \in Z_1 \times Z_2 \times \cdots \times Z_k, \tag{11.27}$$

where μ_l is the projection of μ_k on Z_l for every l, $1 \leq l \leq k$. Denote by $T_k \subseteq Z_1 \times Z_2 \times \cdots Z_k$ the set containing all k-order coherent belief hierarchies and their projections. Z_k is a compact space for every $k \in \mathbb{N}$ (Theorem 11.10), and therefore the Cartesian product $Z_1 \times Z_2 \times \cdots Z_k$ is also compact. We will now show that $T_k \subseteq Z_1 \times Z_2 \times \cdots \times Z_k$ is a compact set. To see this, note that for every $l = 1, 2, \ldots, k$, the projection $\rho_{k,l}$:

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$Z_k \to Z_l$ is a continuous function; hence T_k, which is the image of the compact set Z_k under the continuous mapping $(\rho_{k,1}, \rho_{k,2}, \dots, \rho_{k,k})$, is a compact set.

Tychonoff's Theorem (see, for example, Theorem I.8.5 in Dunford and Schwartz [1988]) states that the (finite or infinite) Cartesian product of compact spaces is a compact space in the product topology. It follows that

$$\widehat{T}_k := T_k \times Z_{k+1} \times Z_{k+2} \times \cdots \tag{11.28}$$

is also a compact set for every $k \in \mathbb{N}$. Since $T = \bigcap_{k \in \mathbb{N}} \widehat{T}_k$ we conclude that T, as the intersection of compact sets, is a compact set in $Z_1 \times Z_2 \times \cdots$. □

The topology over T is the collection of open sets in T. In order to study the probability distributions over T, it is necessary first to define a σ-algebra over T. A natural σ-algebra is the σ-algebra of the Borel sets: this is the minimal σ-algebra over T that contains all the open sets in T. The next theorem provides us with another way of defining the type of a player. It says that the type of a player is a probability distribution over the states of nature and the types of the other players.

Theorem 11.16 *The universal type space T satisfies*[5]

$$T = \Delta(S \times T^{n-1}). \tag{11.29}$$

To be more precise, we will prove that there exists a natural homeomorphism[6] $\varphi : \Delta(S \times T^{n-1}) \to T$.

Proof: An element in T is a vector of the form $(\mu_1, \mu_2, \dots)$, satisfying $\mu_k = \rho(\mu_{k+1})$ for all $k \in \mathbb{N}$.

In the proof we will use the following:

$$\begin{aligned} S \times T^{n-1} &\subseteq S \times \underbrace{(Z_1 \times Z_2 \times \cdots) \times \cdots \times (Z_1 \times Z_2 \times \cdots)}_{n-1 \text{ times}} \\ &= S \times (Z_1)^{n-1} \times (Z_2)^{n-1} \times \cdots . \end{aligned} \tag{11.30}$$

Step 1: Definition of the function $\varphi : \Delta(S \times T^{n-1}) \to T$.
We will show that every distribution $\lambda \in \Delta(S \times T^{n-1})$ uniquely determines an element $(\mu_1, \mu_2, \dots) \in T$. The belief hierarchies of all finite orders are defined as follows. Let μ_1 be the marginal distribution of λ over S. For each $k \geq 1$, let ν_k be the marginal distribution of λ over $S \times (Z_k)^{n-1}$ (see Equation (11.30)). Inductively define $\mu_{k+1} = (\mu_k, \nu_k)$ for each $k \geq 1$. To show that the resulting sequence $(\mu_1, \mu_2, \dots)$ is a type in T, we need to show that for each $k \in \mathbb{N}$, the projection of μ_{k+1} on Z_k is μ_k, which follows from the definitions of ν_{k+1} and ν_k and from the fact that T contains only coherent types.

Step 2: The function φ is continuous.
The claim obtains because if $(\lambda^l)_{l \in \mathbb{N}}$ is a sequence of probability distributions defined over the probability space $X \times Y$ converging to λ in the weak-$*$ topology, and if μ^l is the marginal distribution of λ^l over X, then the sequence of distributions $(\mu^l)_{l \in \mathbb{N}}$ converges

5 The σ-algebra over $S \times T^{n-1}$ is the product σ-algebra.
6 A *homeomorphism* between two spaces X and Y is a continuous bijection $f : X \to Y$, whose inverse $f^{-1} : Y \to X$ is also continuous.

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in the weak-$*$ topology to the marginal distribution μ of λ over X (see Theorem 2.8 in Billingsley [1999]).

Step 3: The function φ is injective.
We will show that λ can be reconstructed from $\varphi(\lambda)$, for each $\lambda \in \Delta(S \times T^{n-1})$. Denote $\varphi(\lambda) = (\mu_1, \mu_2, \ldots)$. Recall that $\mu_{k+1} = (\mu_k, \nu_k)$, where ν_k is a probability distribution over the space $S \times (Z_k)^{n-1}$. Because Z_k contains all the hierarchies of all orders $1, 2, \ldots, k$, it follows that ν_k is a probability distribution over the space $S \times (Z_1)^{n-1} \times (Z_2)^{n-1} \times \cdots \times (Z_{k-1})^{n-1}$. Since $\mu_k = (\mu_{k-1}, \nu_{k-1})$, the marginal distribution of ν_k over $S \times (Z_1)^{n-1} \times (Z_2)^{n-1} \times \cdots \times (Z_{k-2})^{n-1}$ is the probability distribution ν_{k-1}. From the Kolmogorov Extension Theorem (see, for example, Theorem II.3.4 in Shiryaev [1995]) it follows that there exists a unique distribution λ^* over the space $S \times (Z_1)^{n-1} \times (Z_2)^{n-1} \times \cdots$ satisfying the condition that for every $k \in \mathbb{N}$, the marginal distribution of λ^* over $S \times (Z_1)^{n-1} \times (Z_2)^{n-1} \times \cdots \times (Z_{k-1})^{n-1}$ is ν_k. Since the marginal distribution of λ^* over $S \times (Z_1)^{n-1} \times (Z_2)^{n-1} \times \cdots \times (Z_{k-1})^{n-1}$ equals the marginal distribution of λ over these spaces, the uniqueness of the extension implies that $\lambda = \lambda^*$.

Step 4: The function φ is surjective.
Let $(\mu_1, \mu_2, \ldots) \in T$ be a type. As we saw in Step 3, a type in T defines a unique distribution $\lambda \in \Delta(S \times T^{n-1})$. The reader is asked to ascertain that $\varphi(\lambda)$ equals $(\mu_1, \mu_2, \ldots)$.

Step 5: The function φ is a homeomorphism.
Every continuous, injective, and surjective function φ from a compact space to a Hausdorff space[7] is a homeomorphism (see Claim I.5.8 in Dunford and Schwartz [1988]). $\square$

11.3 Definition of the universal belief space

Definition 11.17 *The* universal belief space *is*

$$\Omega = \Omega(N, S) = S \times T^n. \tag{11.31}$$

By definition, the universal belief space is determined by the set of states of nature and by the number of players. To understand the meaning of Definition 11.17, write Equation (11.31) in the following form:

$$\Omega = S \times \left(\bigtimes_{i \in N} T_i \right), \tag{11.32}$$

where $T_i = T$ for all $i \in N$. The space T_i is called *player i's type space*. It is the same space for all the players, and is the universal type space. An element of Ω is a *state of the world*, and denoted by ω to distinguish it from the states of nature, which are elements

7 A topological space is a *Hausdorff* space if (a) every set containing a single point is closed, and (b) for every pair of distinct points there exist two disjoint and open sets, each of which contains one point but not the other. The space T is a Hausdorff space (Exercise 11.11).

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of S. A state of the world is therefore a vector

$$\omega = (\mathfrak{s}(\omega), t_1(\omega), t_2(\omega), \ldots, t_n(\omega)). \tag{11.33}$$

The first coordinate $\mathfrak{s}(\omega)$ is the state of nature at the state of the world ω, and $t_i(\omega)$ is player i's type at this state of the world. In other words, a state of the world is characterized by the state of nature $\mathfrak{s}(\omega)$, and the vector of types of the players, $(t_i(\omega))_{i\in N}$, at that state of the world. We will assume that all a player knows is his own type. While in the Aumann and Harsanyi models the belief hierarchies of the players can be computed at every state of the world, in the universal belief space these hierarchies are part of the data defining the state of the world: a state of the world consists of a state of nature and the players' belief hierarchies.

Example 11.14 (*Continued*) We have seen that when $N = \{\mathrm{I}\}$ and $S = \{s_1, s_2\}$, the universal type space T is homeomorphic to the interval $[0, 1]$. In this case, the universal belief space is $\Omega = \Omega(\{\mathrm{I}\}, S) = S \times [0, 1]$. For every $p \in [0, 1]$, at the state of the world $\omega = (s_1, p)$, the state of nature is s_1, and the player ascribes probability p to the state of nature being s_1, and probability $1 - p$ to the state of nature being s_2. ◀

Remark 11.18 *As we saw in Theorem 11.16, a type $t_i(\omega) \in T = \Delta(S \times T^{n-1})$ is a probability distribution over the vectors of the state of nature and the list of the $n-1$ types of the other players. Since $\Omega = (S \times (\times_{j\neq i} T_j)) \times T_i$, and because at every state of the world ω every player i knows his own type $t_i(\omega)$, we can regard $t_i(\omega)$ also as a probability distribution over Ω, where the marginal distribution over T_i is the degenerate distribution at the point $\{t_i(\omega)\}$. From here on, we will assume that $t_i(\omega)$ is indeed a probability distribution over Ω.* ◆

Recall that a belief space of the set of players N on the set of states of nature S is an ordered vector $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$, where $(Y, \mathcal{Y})$ is a measure space of states of the world, $\mathfrak{s} : Y \to S$ is a measurable function associating a state of nature with every state of the world, and $\pi_i : Y \to \Delta(Y)$ is a function associating a probability distribution over Y with every state of the world and every player $i \in N$, which satisfies the conditions of coherency and measurability. The next two theorems justify the name "universal belief space" that we gave to Ω: Theorem 11.19 states that Ω naturally defines a belief space, and Theorem 11.20 states that every belief space is a belief subspace of Ω. It follows that the space $\Omega(N, S)$ contains all the possible situations of incomplete information of a set of players N on a set of states of nature S.

Denote by $\mathcal{Y}^*$ the product σ-algebra over the set Ω defined by Equation (11.32).

Theorem 11.19 *The ordered vector $\Pi^* = (\Omega, \mathcal{Y}^*, \mathfrak{s}, (t_i)_{i\in N})$ is a belief space, where Ω is the universal belief space and $\mathfrak{s}$ and $(t_i)_{i\in N}$ are projections on the $n + 1$ coordinates of the state of the world (see Equation (11.33)).*

Proof: We will show that all the conditions defining a belief space are satisfied. The space $(\Omega, \mathcal{Y}^*)$ is a measurable space. Since $\mathcal{Y}^*$ is a σ-algebra, the functions $\mathfrak{s}$ and t_i are measurable functions.

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We will next show that the functions $(t_i)_{i\in N}$ satisfy the coherency condition (see Definition 10.1 on page 387). As stated in Remark 11.18, for every player $i \in N$ and every $\omega \in \Omega$, the type $t_i(\omega)$ is a probability distribution over Ω. Since t_i is a measurable function, the set $\{\omega' \in \Omega\colon t_i(\omega') = t_i(\omega)\}$ is measurable, and by definition, the probability distribution $t_i(\omega)$ ascribes probability 1 to this set, showing that the coherency condition is satisfied.

Finally, we check that for every $i \in N$, the function t_i satisfies the measurability condition (see Definition 10.1 on page 387). To do so, we need to show that for every measurable set E in Ω, the function $t_i(E \mid \cdot) : \Omega \to [0, 1]$ is measurable. We prove this by showing that for every $x \in [0, 1]$, the set $G_x = \{\omega \in \Omega\colon t_i(E \mid \omega) > x\}$ is measurable. By the definition of the weak-$*$ topology, for every continuous function $f : \Omega \to \mathbb{R}$ and every $x \in [0, 1]$, the set

$$A_{f,x} := \left\{\omega \in \Omega\colon \int_\Omega f(\omega')\mathrm{d}t_i(\omega' \mid \omega) > x\right\} \tag{11.34}$$

is measurable. Let $\mathcal{F}$ be the family of continuous functions $f : \Omega \to (0, \infty)$ satisfying the condition $f(\omega) > 1$ for all $\omega \in E$. Let $\mathcal{F}_0$ be a countable dense subset of $\mathcal{F}$ (why does such a set exist?). The intersection $\cap_{f\in\mathcal{F}_0} A_{f,x}$, as the intersection of a countable number of measurable sets, is measurable, and is equal to G_x (why?). □

Theorem 11.20 *Every belief space Π of a set of players N on a set of states of nature S is a belief subspace (see Definition 10.21 on page 400) of the universal belief space $\Omega(N, S)$ defined in Theorem 11.19.*

To be precise, every belief space $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$ is homomorphic to a belief subspace of the belief space Π^*, in the following sense: the belief hierarchy of every player i at every state of the world $\omega \in Y$ equals his belief hierarchy at the state of the world in Π^* corresponding to ω, under the homomorphism.

Proof: Let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$ be a belief space. As we stated on page 390, for every state of the world $\omega \in Y$ and every player $i \in N$, we can associate an infinite belief hierarchy that describes the beliefs of player i at the state of the world ω. Denote this belief hierarchy by $t_i(\omega)$. For each $\omega \in Y$, the vector $\varphi(\omega) := (\mathfrak{s}(\omega), t_1(\omega), t_2(\omega), \ldots, t_n(\omega))$ is a state of the world in the universal belief space. Note that if there are two states of the world $\omega, \omega' \in Y$ satisfying the conditions that the belief hierarchy of every player i in ω equals his belief hierarchy in ω', and if these two states of the world are associated with the same state of nature, then $\varphi(\omega) = \varphi(\omega')$ (this happens, for example, in the second belief space in Example 10.13 on page 394). The definition implies that the belief hierarchy of every player i at every state of the world $\omega \in Y$ equals his belief hierarchy in $\varphi(\omega)$. Consider the set

$$\widehat{Y} := \{\varphi(\omega)\colon \omega \in Y\} \subseteq \Omega. \tag{11.35}$$

It is left to the reader to check that the set $\widehat{Y}$ is a belief subspace of Π^* (Exercise 11.10). □

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Theorem 11.20 implies, for example, that in each of the examples in Section 10.3 (page 394), the belief space is a belief subspace of an appropriate universal belief space. For example, each of the belief spaces described in Examples 10.17 (page 396), 10.18 (page 398) and 10.19 (page 399), is a subspace of the universal belief space $\Omega(N, S)$, where $N = \{\mathrm{I}, \mathrm{II}\}$ and $S = \{s_{11}, s_{12}, s_{21}, s_{22}\}$.

11.4 Remarks

The universal belief space was first constructed and studied by Mertens and Zamir [1985]. Heifetz and Samet [1998] discuss a construction of the universal belief space using measure-theoretical tools, without any use of topological structures. Aumann [1999] constructs the universal belief space using a semantic approach. The reader interested in the weak-$*$ topology is directed to Dunford and Schwartz [1988] (Chapter V.12), Conway [1990], or Billingsley [1999].

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11.5 Exercises

11.1 Joshua and his army are planning to circle Jericho seven times. Will the wall come tumbling down? The king of Jericho reports that:

- I ascribe probability 0.8 to "the wall of the city will fall, and Joshua ascribes probability 0.6 to the wall falling."
- I ascribe probability 0.2 to "the wall of the city will not fall, and Joshua ascribes probability 0.5 to the wall falling."

Answer the following questions:

(a) What is the set of states of nature corresponding to the above description.
(b) What is the king's first-order belief? What is his second-order belief?
(c) Can Joshua's first-order belief be ascertained from the above description? Justify your answer.

11.2 Construct a belief space of the set of players $N = \{\text{Don}, \text{Phil}\}$ on the set of states of nature $S = \{s_{\mathrm{b}}, s_{\mathrm{g}}, s_{\mathrm{p}}, s_{\mathrm{r}}, s_{\mathrm{w}}\}$ describing the situation in Example 11.9 (page 446), and indicate at which state of the world Phil's belief hierarchy of order 3 is the hierarchy described in the example. There may be more than one correct answer.

11.3 Prove that $Z_k \subseteq X_k$ for each $k \geq 1$: every coherent belief hierarchy of order k (Definition 11.6 on page 445) is a belief hierarchy of order k (Definition 11.3 on page 442).

11.4 Consider the following belief space, where the set of players is $N = \{\mathrm{I}, \mathrm{II}\}$, and the set of states of nature is $S = \{s_1, s_2\}$:

State of the world	$\mathfrak{s}(\cdot)$	$\pi_{\mathrm{I}}(\cdot)$	$\pi_{\mathrm{II}}(\cdot)$
ω_1	s_1	$[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)]$	$[1(\omega_1)]$
ω_2	s_2	$[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)]$	$[\frac{3}{4}(\omega_2), \frac{1}{4}(\omega_3)]$
ω_3	s_1	$[1(\omega_3)]$	$[\frac{3}{4}(\omega_2), \frac{1}{4}(\omega_3)]$

Write out the belief hierarchies of orders 1, 2, and 3 of Player I, at each state of the world.

11.5 Roger reports:

- I ascribe probability 0.3 to "the Philadelphia Phillies will win the World Series next year, Chris ascribes probability 0.4 to their winning the World Series and to me ascribing probability 0.4 that they will win the World Series, and Chris ascribes probability 0.6 to the Philadelphia Phillies not winning the World Series and to me ascribing probability 0.2 that they will win the World Series."
- I ascribe probability 0.2 to "the Philadelphia Phillies will win the World Series next year, Chris ascribes probability 0.4 to their winning the World Series and to me ascribing probability 0.5 that they will win the World Series, and Chris ascribes probability 0.6 to the Philadelphia Phillies not winning the World Series and to me ascribing probability 0.8 that they will win the World Series."
- I ascribe probability 0.4 to "the Philadelphia Phillies will win the World Series next year, Chris ascribes probability 0.2 to their winning the World Series and to me ascribing probability 0.1 that they will win the World Series, and Chris ascribes probability 0.8 to the Philadelphia Phillies not winning the World Series and to me ascribing probability 0.3 that they will win the World Series."
- I ascribe probability 0.1 to "the Philadelphia Phillies will win the World Series next year, Chris ascribes probability 0.6 to their winning the World Series and to me ascribing probability 0.4 that they will win the World Series, and Chris ascribes probability 0.4 to the Philadelphia Phillies not winning the World Series and to me ascribing probability 0.7 that they will win the World Series."

Answer the following questions:

(a) Construct a space of states of nature corresponding to the above description.
(b) What is Roger's first-order belief? What is his second-order belief? What is his third-order belief?

11.6 Prove Theorem 11.7: let $\mu_k = (\mu_1; \nu_1, \nu_2, \ldots, \nu_{k-1}) \in Z_k$ be a coherent belief hierarchy of order k and let l_1, l_2 be two integers such that $1 \leq l_1 \leq l_2 \leq k$. Then:

(a) The marginal distribution of ν_1 over S equals μ_1.
(b) The marginal distribution of ν_{l_2} over $S \times (Z_{l_1})^{n-1}$ is ν_{l_1}.

11.7 Let $s_0 \in S$ be a state of nature. Prove that the following sentence defines a coherent belief hierarchy of order k: "I ascribe probability 1 to the state of nature being s_0,

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I ascribe probability 1 to all the other players ascribing probability 1 to the state of nature being s_0, I ascribe probability 1 to each of the other players ascribing probability 1 to every other player ascribing probability 1 to the state of nature being s_0, and so on, to level k."

11.8 There are two players, $N = \{I, II\}$, and the space of states of nature is $S = \{s_1, s_2\}$. Ascertain for each of the following belief hierarchies of Player I whether or not it is a coherent belief hierarchy (of some order). Justify your answer.

(a) I ascribe probability $\frac{1}{9}$ to the state of nature being s_1.

(b)
- I ascribe probability $\frac{1}{2}$ to the state of nature being s_1.
- I also ascribe probability $\frac{2}{3}$ to the state of nature being s_1 and to me ascribing probability $\frac{3}{4}$ to the state of nature being s_1.
- I also ascribe probability $\frac{1}{3}$ to the state of nature being s_1 and to me ascribing probability 0 to the state of nature being s_1.

(c)
- I ascribe probability $\frac{1}{2}$ to the state of nature being s_1.
- I also ascribe probability $\frac{1}{2}$ to the state of nature being s_1 and me ascribing probability $\frac{1}{3}$ to the state of nature being s_1.
- I also ascribe probability $\frac{1}{2}$ to the state of nature being s_1 and me ascribing probability $\frac{2}{3}$ to the state of nature being s_1.

(d)
- I ascribe probability $\frac{1}{2}$ to the state of nature being s_1.
- I also ascribe probability $\frac{2}{3}$ to the state of nature being s_1 and to Player II ascribing probability $\frac{3}{4}$ to the state of nature being s_1.
- I also ascribe probability $\frac{1}{3}$ to the state of nature being s_1 and to Player II ascribing probability 0 to the state of nature being s_1.

(e)
- I ascribe probability $\frac{1}{2}$ to the state of nature being s_1.
- I also ascribe probability $\frac{1}{3}$ to the state of nature being s_1 and to Player II ascribing probability $\frac{1}{4}$ to the state of nature being s_1.
- I also ascribe probability $\frac{1}{6}$ to the state of nature being s_1 and to Player II ascribing probability $\frac{1}{2}$ to the state of nature being s_1.
- I also ascribe probability $\frac{1}{2}$ to the state of nature being s_2 and to Player II ascribing probability $\frac{2}{3}$ to the state of nature being s_1.

(f)
- I ascribe probability $\frac{1}{2}$ to the state of nature being s_1.
- I also ascribe probability $\frac{1}{3}$ to the state of nature being s_1 and to Player II ascribing probability $\frac{1}{4}$ to the state of nature being s_1.
- I also ascribe probability $\frac{1}{6}$ to the state of nature being s_1 and to Player II ascribing probability $\frac{1}{4}$ to the state of nature being s_1.
- I also ascribe probability $\frac{1}{2}$ to the state of nature being s_2 and to Player II ascribing probability $\frac{2}{3}$ to the state of nature being s_1.
- I also ascribe probability $\frac{1}{3}$ to the state of nature being s_2 and to Player II ascribing probability $\frac{1}{4}$ to the state of nature being s_1, and to me ascribing probability $\frac{2}{5}$ to the state of nature being s_1, and to Player II ascribing probability $\frac{3}{4}$ to the state of nature being s_2, and to me ascribing probability $\frac{4}{5}$ to the state of nature being s_1.

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- I also ascribe probability $\frac{1}{6}$ to the state of nature being s_1 and to Player II ascribing probability $\frac{1}{4}$ to the state of nature being s_1, and to me ascribing probability $\frac{3}{5}$ to the state of nature being s_1, and to Player II ascribing probability $\frac{3}{4}$ to the state of nature being s_2, and to me ascribing probability $\frac{3}{7}$ to the state of nature being s_1.
- I also ascribe probability $\frac{1}{2}$ to the state of nature being s_2 and to Player II ascribing probability $\frac{1}{4}$ to the state of nature being s_1, and to me ascribing probability $\frac{4}{5}$ to the state of nature being s_1, and to Player II ascribing probability $\frac{3}{4}$ to the state of nature being s_2, and to me ascribing probability $\frac{2}{5}$ to the state of nature being s_1.

(g)
- I ascribe probability $\frac{1}{2}$ to the state of nature being s_1.
- I also ascribe probability $\frac{1}{3}$ to the state of nature being s_1 and to Player II ascribing probability $\frac{1}{4}$ to the state of nature being s_1.
- I also ascribe probability $\frac{1}{6}$ to the state of nature being s_1 and to Player II ascribing probability $\frac{1}{2}$ to the state of nature being s_1.
- I also ascribe probability $\frac{1}{2}$ to the state of nature being s_1 and to Player II ascribing probability $\frac{2}{3}$ to the state of nature being s_1.
- I also ascribe probability $\frac{1}{2}$ to the state of nature being s_1 and to Player II ascribing probability $\frac{1}{3}$ to the state of nature being s_1, and to me ascribing probability $\frac{2}{5}$ to the state of nature being s_1, and to Player II ascribing probability $\frac{2}{3}$ to the state of nature being s_2, and to me ascribing probability $\frac{4}{5}$ to the state of nature being s_1.
- I also ascribe probability $\frac{1}{2}$ to the state of nature being s_2 and to Player II ascribing probability $\frac{2}{3}$ to the state of nature being s_1, and to me ascribing probability $\frac{4}{5}$ to the state of nature being s_1, and to Player II ascribing probability $\frac{1}{3}$ to the state of nature being s_2, and to me ascribing probability $\frac{1}{5}$ to the state of nature being s_1.

(h)
- I ascribe probability $\frac{1}{2}$ to the state of nature being s_1.
- I also ascribe probability $\frac{1}{3}$ to the state of nature being s_1 and to Player II ascribing probability $\frac{1}{4}$ to the state of nature being s_1.
- I also ascribe probability $\frac{1}{6}$ to the state of nature being s_1 and to Player II ascribing probability $\frac{1}{2}$ to the state of nature being s_1.
- I also ascribe probability $\frac{1}{2}$ to the state of nature being s_2 and to Player II ascribing probability $\frac{2}{3}$ to the state of nature being s_1.
- I also ascribe probability $\frac{1}{3}$ to the state of nature being s_1, and to Player II ascribing probability $\frac{1}{4}$ to the state of nature being s_1 and to me ascribing probability $\frac{2}{5}$ to the state of nature being s_1, and to Player II ascribing probability $\frac{3}{4}$ to the state of nature being s_2 and to me ascribing probability $\frac{4}{5}$ to the state of nature being s_1.
- I also ascribe probability $\frac{1}{6}$ to the state of nature being s_1, and to Player II ascribing probability $\frac{1}{2}$ to the state of nature being s_1 and to me ascribing probability $\frac{4}{5}$ to the state of nature being s_1, and to Player II ascribing

probability $\frac{1}{2}$ to the state of nature being s_2 and to me ascribing probability $\frac{3}{5}$ to the state of nature being s_1.

- I also ascribe probability $\frac{1}{2}$ to the state of nature being s_2, and to Player II ascribing probability $\frac{2}{3}$ to the state of nature being s_1 and to me ascribing probability $\frac{2}{5}$ to the state of nature being s_1, and to Player II ascribing probability $\frac{1}{3}$ to the state of nature being s_2 and to me ascribing probability $\frac{1}{5}$ to the state of nature being s_1.

11.9 There is a single player $N = \{\mathrm{I}\}$ and the set of states of nature is a finite set S. What is the universal type space in this case? What is the universal belief space $\Omega(N, S)$?

11.10 Complete the proof of Theorem 11.20 on page 455: prove that the set $\widehat{Y}$ that was defined in the proof of the theorem is a belief subspace of Π^*.

11.11 Prove that the universal type space T is a Hausdorff space.

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