CS 228 : Logic in Computer Science

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Recap

- Completeness of Propositional Logic
- Normal forms

Satisfiability Checking : Horn Formulae

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- $p \wedge (\neg p \vee \neg q \vee r) \wedge (\neg a \vee \neg b)$ is Horn, but $a \vee b$ is not Horn.
- A basic Horn formula is one which has no $\wedge$. Every Horn formula is a conjunction of basic Horn formulae.

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- Thus, a Horn formula is written as a conjunction of implications.

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- Consider subformulae of the form $(p_1 \wedge \cdots \wedge p_m) \rightarrow \bot$. If there is one such subformula with all p_i marked, then say Unsat, otherwise say Sat.

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$(\top \rightarrow A) \wedge (C \rightarrow D) \wedge ((A \wedge B) \rightarrow C) \wedge ((C \wedge D) \rightarrow \perp) \wedge (\top \rightarrow B).$

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- If B has the form $(C_1 \wedge \cdots \wedge C_n) \to D$, where each $\alpha(C_i) = 1$, then $\alpha(D) = 1$.
- Hence, $\alpha(C_i)$ agrees with the marking of the algo.

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- Assume the algo says H is unsat. Then there is a subformula B of the form $(A_1 \wedge \cdots \wedge A_m) \rightarrow \perp$, where each A_i is marked. Hence, $\alpha(A_i) = 1$ for each A_i. By semantics, $\alpha(B) = 0$, a contradiction to our assumption that $\alpha(B) = 1$ for each B.

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- Show that $\alpha \models B$ for each basic Horn formula B of H.

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- Assume all the A_i's were marked. Then $\alpha(A_i) = 1$ for all i. Since the algo said *Sat*, $G \neq \perp$. Then G is also marked (step 2 of algo). Hence, $\alpha(G) = 1$, and we have $\alpha(B) = 1$.

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- Thus, the markings of the algorithm gives rise to a satisfying assignment α if the algorithm said *Sat*.

Complexity of Horn

- Given a Horn formula ψ with *n* propositions, how many times do you have to read ψ?
- Step 1: Read once
- Step 2: Read atmost *n* times
- Step 3: Read once

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- Let $C_1 = \{A_1, \neg A_2, A_3\}$ and $C_2 = \{A_2, \neg A_3, A_4\}$. As $A_3 \in C_1$ and $\neg A_3 \in C_2$, we can find the resolvent. The resolvent is $\{A_1, A_2, \neg A_2, A_4\}$.

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- Resolvent not unique : $\{A_1, A_3, \neg A_3, A_4\}$ is also a resolvent.

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- Let F be a formula in CNF. Let R be a resolvent of two clauses of F. Then $F \vdash R$ (Prove!)

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- There is some m such that $Res^m(F) = Res^{m+1}(F)$. Denote it by $Res^*(F)$.

Example

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