16 Coalitional games with transferable utility

Chapter summary

Coalitional games model situations in which players may cooperate to achieve their goals. It is assumed that every set of players can form a coalition and engage in a binding agreement that yields them a certain amount of profit. The maximal amount that a coalition can generate through cooperation is called the *worth* of the coalition.

In this and subsequent chapters we ask which coalitions of players will form, and, if the players are partitioned into a certain collection of coalitions, how each coalition will divide its worth. Specifically, in this chapter we present the model of coalitional games, and introduce various classes of coalitional games: revenue games, cost games, simple games, weighted majority games, market games, sequencing games, spanning tree games, and cost-sharing games. We define the notion of strategic equivalence between games.

We then turn to define the notion of a solution concept. A *single-valued solution concept* is a function that assigns to each coalitional game a vector in $\mathbb{R}^N$ indicating the amount each player receives. A *set solution concept* is a function that assigns to each coalitional game a set of vectors in $\mathbb{R}^N$. Single-valued solution concepts model a judge or an arbitrator who has to recommend to the players how to divide the worth of the coalition among its members. A set solution concept may indicate which divisions are more likely than others.

Finally, using barycentric coordinates, we introduce a graphic representation of three-player coalitional games.

The games we studied in the previous chapters, with the exception of Chapter 15, were characterized by the assumption that each player acted independently of the actions of the other players. Even if there is nothing preventing correlation of actions between the players and no limit on the information being exchanged among them, and even when there is an observer who can give the players recommendations on which actions to choose (as in the chapter on correlated equilibria), correlations and recommendations are not binding on the players: a player always has the option of choosing any action from the set of actions available to him. For that reason, such games are called *noncooperative games*. In this chapter and the following chapters we will study *cooperative games*. Cooperative games model situations in which the players may conclude binding agreements that impose a particular action or series of actions on each player. The bargaining games that we studied in Chapter 15 are examples of cooperative games, since the players can agree and commit themselves to choosing a particular outcome from the set of possible alternatives S: they

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can sign an agreement binding them to implement the actions required to obtain this outcome. The subject of this chapter and the next four chapters is cooperative games with a finite number of players. The concept of Nash equilibrium as it is applied to strategic-form and extensive-form games is insufficient for analyzing such games because every agreement, being a binding agreement, constitutes an equilibrium

Cooperative game theory therefore concentrates on questions such as which sets of players (coalitions) will agree to conclude binding agreements? Which agreements can reasonably be expected to be arrived at by players (and which are not reasonable)? Which agreements can reasonably be proposed to the players by an arbitrator or a judge? For this reason, cooperative games are also called *coalitional games*, to underscore the fact that the situations modeled by cooperative games involve issues related to the formation of coalitions by the players.

As in noncooperative game theory, in cooperative game theory a player may be a person, corporation, nation, and so on. The only requirement is that players be capable of arriving at decisions, and committing to those decisions. In addition, when a coalition is formed, the coalition must be able to undertake commitments binding on all its members. We will see that cooperative games can be used to study situations in which the "players" are more abstract objects, such as roads, flights, political parties, and so on.

In this chapter, and Chapters 17, 18, 19, and 20, we will concentrate on *finite games with transferable utilities*. Such games involve a finite number of players, and every coalition is associated with a sum that it can guarantee for itself.

Definition 16.1 *A* coalitional game with transferable utility (TU game) *is a pair* $(N; v)$ *such that:*[1]

- $N = \{1, 2, \ldots, n\}$ *is a finite set of players. A subset of* N *is called a* coalition. *The collection of all the coalitions is denoted by* 2^N.
- $v : 2^N \to \mathbb{R}$ *is a function associating every coalition* S *with a real number* $v(S)$, *satisfying* $v(\emptyset) = 0$. *This function is called the* coalitional function *of the game.*[2]

The real number $v(S)$ is called the *worth of the coalition* S. The significance of this is that if the members of S agree to form the coalition S, then as a result they can produce (or expect to receive) the sum of $v(S)$ units of money, independently of the actions of the players who are not members of S.

The fact that $v(S)$ is a real number is an expression of two assumptions of the model:

- The utilities of all the players can be measured in a common unit, such as monetary units.
- Utility (money) can be tranferred between the players.

Under these assumptions, we can summarize the worth of each coalition using a single number $v(S)$, which is the amount of utility (= money) that a coalition S can produce by cooperation among its members. Situations in which a coalition creates utilities that cannot be transferred between the members of the coalition, such as reputation, prestige,

1 Such games are also called *coalitional games with side payments* in the literature.
2 The function v is also called the *characteristic function* of the game.

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political influence, and so on, are not dealt with by this model, and are instead described by coalitional functions without transferable utility. An example of such a model is the bargaining game discussed in Chapter 15.

In contrast to the commonplace meaning of the term “coalition,” for mathematical convenience a set containing only one player, and similarly the empty set $\emptyset$, will each be called a coalition. As we saw in the definition of coalitional games, it is standard to set the worth of the empty coalition at 0.

Under the interpretation used here, a coalitional game is sometimes called a *profit game*. We may also be interested in what is termed a *cost game*, in which the worth of a coalition S is the sum that the members of the coalition must pay. An example of a cost game is given by a set of townships that wish to pave a network of roads, where the worth of a coalition is the cost of paving a network of roads connecting only the townships in that coalition. The coalitional function in a cost game is denoted by c: it is the function associating each coalition S with the costs $c(S)$ that the members of S will bear if they agree to form a coalition. It is again mathematically convenient to define $c(\emptyset) = 0$.

The term “forming a coalition” is given to varying interpretations, depending on the situation one wishes to model. The property common to all coalition formations is the fact that the members of a coalition agree to join the coalition, and commit to their roles within it. A coalition cannot form without the agreement of all its members.

We remark that when we say that coalitions S and T are formed, we mean that they are disjoint: $S \cap T = \emptyset$. If one or more players participate in two coalitions S and T, it is meaningless to say that S and T are formed.

A coalitional game model $(N; v)$ (or $(N; c)$) always includes a basic assumption: the money gained (or paid) by a coalition S, if it is formed, does not depend on the behavior of players who are not in S, nor on other coalitions that may form. This assumption greatly restricts the applicability of the model. For example, in an economic analysis of oligopoly, in which a set of manufacturers forms a coalition, the coalitional game model cannot take into account the effects of coalitions of players formed outside of the oligopoly on the oligopoly’s profits. There exists a more general model called a *game in partition function form*, which is not considered in this book, that takes into account the possibility that the profit of every coalition S may depend on the partition into coalitions of the players who are not members of S.

16.1 Examples

In this section, we present several situations that can be analyzed as coalitional games.

16.1.1 Profit games

Profit games are games in which players profit according to the coalitions they form. For example, imagine a situation with three entrepreneurs: Orville, Ron, and Wilbur. Orville has ideas for various new inventions and patents, and he estimates his profits from these inventions to be \$170,000 per year. Ron has a sharp business sense, and is interested in forming a business consultancy, which he estimates can yield profits of \$150,000 per year. Wilbur, an excellent salesman, is interested in forming a sales company, which he

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estimates can yield profits of \$180,000 per year. The three entrepreneurs recognize that their talents are complementary, and that if they work together, they can profit more than if they each work separately. Ron can advise Orville regarding which patents command the greatest market demand, so they estimate that together they can gain profits of \$350,000 per year. Wilbur can sell Orville's inventions, so they estimate that together they can gain profits of \$380,000 per year. Ron and Wilbur together can form a business consulting and sales corporation, which they estimate can gain profits of \$360,000 per year. If all three of them work together, Ron can tell Orville which inventions will enjoy the greatest market demand, and then Wilbur can sell those inventions; the estimated profits of all three working together is \$560,000 per year.

The entrepreneurs understand that it is to their advantage to work together, but it is not so immediately clear how they ought to divide among them the profits of a joint company, should they form one, because the contribution of each entrepreneur is different from that of the others, and the profit each one can earn working alone also differs from one entrepreneur to the next.

The coalitional game that corresponds to this situation (with payoffs in dollars) is as follows:[3]

$$v(\emptyset) = 0,$$
$$v(\text{Wilbur}) = 180{,}000,$$
$$v(\text{Orville}) = 170{,}000,$$
$$v(\text{Ron}) = 150{,}000,$$
$$v(\text{Orville, Wilbur}) = 380{,}000,$$
$$v(\text{Ron, Wilbur}) = 360{,}000,$$
$$v(\text{Orville, Ron}) = 350{,}000,$$
$$v(\text{Orville, Wilbur, Ron}) = 560{,}000.$$

16.1.2 Cost games

Cost games are similar to profit games, but the worth of a coalition in a cost game represents the price the coalition members must pay if the coalition were to form. We present an example of a cost game.

Sweden, Norway, and Finland are interested in constructing a dam for generating 3 gigawatts of electricity; each country will receive one-third of the electricity generated by the dam. The cost of constructing the dam is \$180 million. The question the leaders of the three nations need to decide is how to divide that cost between them. Since each country will receive one-third of the generated electricity, it is reasonable to require that each one also contribute one-third of the construction costs. But suppose that it is discovered that there is a river inside Sweden appropriate for a smaller dam, that can generate 2 gigwatts of electricity, at a construction cost of \$100 million. Sweden then claims that if the three countries agree to share the cost of the larger dam equally, with each paying \$60 million, then Sweden would be better off building a smaller dam with either Norway or Finland,

3 For simplicity, we will write $v(\text{Orville, Wilbur})$ instead of $v(\{\text{Orville, Wilbur}\})$.

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with each paying \$50 million. But Finland and Norway can also build a smaller dam generating 2 gigawatts together, at a cost of \$130 million. Any country that fails to join the two other countries in a joint dam-constructing project will have no choice but to build a 1 gigawatt electricity-generating plant alone. The cost of constructing such a plant is \$80 million for Sweden, \$90 million for Norway, and \$70 million for Finland.

Given this, how should the three countries divide the cost of constructing a large dam between them?

The coalitional game corresponding to this situation is the following. Let $c(S)$ represent the cost in millions of dollars for coalition S to provide a gigawatt of electricity to each member country (costs are listed in millions of dollars).

$$\begin{aligned}
&c(\emptyset) = 0,\\
&c(\text{Finland}) = 70,\\
&c(\text{Sweden}) = 80,\\
&c(\text{Norway}) = 90,\\
&c(\text{Sweden, Finland}) = 100,\\
&c(\text{Sweden, Norway}) = 100,\\
&c(\text{Norway, Finland}) = 130,\\
&c(\text{Sweden, Norway, Finland}) = 180.
\end{aligned}$$

16.1.3 Simple games

A coalitional game is simple if the worth of any coalition is either 0 or 1.

Definition 16.2 *A coalitional game* $(N; v)$ *is called* simple *if for each coalition* S*, either* $v(S) = 0$*, or* $v(S) = 1$.

In a simple game, a coalition S is called *winning* if $v(S) = 1$, and is called *losing* if $v(S) = 0$. It is sometimes convenient to represent simple games by indicating the family of winning coalitions $\mathcal{W} = \{S \subseteq N : v(S) = 1\}$.

Simple games can model committee votes, including cases in which the voting rule is not necessarily majority rule. We interpret $v(S) = 1$ as meaning that the coalition S can pass a motion, and $v(S) = 0$ as meaning that coalition S cannot pass a motion on its own. For example, the United Nations Security Council has 5 permanent members, and 10 nonpermanent members. Every permanent member can cast a veto on any Security Council resolution, and adopting a resolution requires the support of a majority of 9 council members. Ignoring the possibility of abstention in votes, this means that for a resolution to be adopted by the council it needs the support of all 5 permanent members, and at least 4 nonpermanent members. The coalitional function v corresponding to this game is

$$v(S) = \begin{cases} 1 & \text{if } |S| \geq 9 \text{ and } S \text{ contains all the permanent members,} \\ 0 & \text{for any other coalition } S. \end{cases} \tag{16.1}$$

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Another example is the legislative process in the United States. Passing a bill into law in the United States requires the signature of the President of the United States, and a simple majority in both the Senate (composed of 100 Senators) and the House of Representatives (composed of 435 members), or alternatively the two-thirds majority in both the Senate and the House of Representatives required to override a presidential veto. In Exercise 16.2, the reader is asked to write down the coalitional game corresponding to this situation. In this example, $v(S)$ is not measured in units of money, but rather in terms of "governance," or "victory," or "the power to make decisions," and $v(S)$ takes the values of either 0 or 1.

16.1.4 Weighted majority games

Weighted majority games are a special case of simple games. In the British Parliament's House of Commons, for example, which is comprised of 650 members, a coalition requires a majority of 326 members to form a government. Suppose there are three parties represented, the first with 282 seats, the second with 260 seats, and the third with 108 seats.

Denote by 1 the "worth" of being the governing coalition and by 0 the "worth" of being in the opposition. The coalitional game corresponding to this situation is a three-player game. Since no single party has 326 seats or more, no party alone can form a governing coalition. Therefore,

$$v(1) = v(2) = v(3) = 0.$$

Since every pair of parties together has more than 326 seats, each pair of parties may form a governing coalition, and all three parties together may form a governing coalition. Therefore,

$$v(1, 2) = v(1, 3) = v(2, 3) = v(1, 2, 3) = 1.$$

Note that although each party has a different number of parliament members, when this situation is presented as a coalitional game, the parties have entirely symmetric roles. We now define the family of weighted majority games.

Definition 16.3 *A coalitional game* $(N; v)$ *is a* weighted majority game *if there exists a quota* $q \geq 0$ *and nonnegative real weights* $(w_i)_{i \in N}$*, one for each player, such that the worth of each nonempty coalition* S *is*

$$v(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i \geq q, \\ 0 & \text{if } \sum_{i \in S} w_i < q. \end{cases} \tag{16.2}$$

A weighted majority game is denoted by $[q; w_1, w_2, \ldots, w_n]$. The example above with the three parties in the British House of Commons is the weighted majority game [326; 282, 260, 108]. Every weighted majority game is a simple game, but not every simple game can be represented as a weighted majority game (Exercise 16.10).

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16.1.5 Market games

Market games are games arising from economic situations. Consider five traders, three of whom arrive to market with 100 liters of gin apiece, and two others with 100 liters of tonic apiece. Assuming that customers are interested solely in cocktails of equal parts gin and tonic, the traders recognize that they must cooperate with each other. The price for a cocktail composed of equal parts gin and tonic is \$100 per liter.

Denoting the set of traders with gin by G, and the set of traders with tonic by T, the coalitional game corresponding to this situation is given by the following coalitional function:[4,5]

$$v(S) = 20{,}000 \times \min\{|S \cap G|, |S \cap T|\}. \tag{16.3}$$

The formal definition and analysis of market games appear in Section 17.4 (page 702).

16.1.6 Sequencing games

Sequencing games are games in which players are to be ordered in a particular sequence; every player prefers being placed earlier in the queue to being placed later in it.

Consider three customers, Eileen, Barbara, and Gail, who seek to hire an architect. The following table presents the placements in the queue of each customer in the architect's schedule book, the amount of time the architect needs to devote to each customer, and the financial loss of each customer from every day lost until the completion of the architectural job she wishes to accomplish.

Number in the Queue	Name	Time to completion of job	Loss per day (in dollars)
1	Eileen	3	2,000
2	Barbara	4	1,500
3	Gail	2	3,000

The current sequencing of the architect's work leads to a loss of $3 \times \$2{,}000 = \$6{,}000$ for Eileen, a loss of $7 \times \$1{,}500 = \$10{,}500$ for Barbara, and a loss of $9 \times \$3{,}000 = \$27{,}000$ for Gail. But if Barbara and Gail were to exchange places in the queue, Gail's loss would fall to $5 \times \$3{,}000 = \$15{,}000$, while Barbara's loss would rise to $9 \times \$1{,}500 = \$13{,}500$. The sum of their losses together, however, falls by \$9,000 (from \$37,500 to \$28,500); hence if Gail were to compensate Barbara for this by paying her, say, \$5,000, both of them would profit from the new scheduling. Eileen and Gail cannot switch places unilaterally: such a switch would affect Barbara, and could not be accomplished without her consent.

We can similarly compute the optimal scheduling of jobs that can be arrived at by changing the ordering of Eileen, Barbara, and Gail's jobs, as well as the gain achievable by each coalition of two or three players. The coalitional function corresponding to the

4 For every coalition S, $|S|$ denotes the number of members of coalition S.
5 The constant 20,000 is the worth of a cocktail composed of 100 liters of gin and 100 liters of tonic.

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	Amherst	Belchertown	Conway
Central drainage point	2	1	4
Amherst	0	4	3
Belchertown	4	0	2
Conway	3	2	0

Figure 16.1 Costs of laying sewage pipes

sequencing game is:

$$v(\text{Eileen}) = 0,$$
$$v(\text{Barbara}) = 0,$$
$$v(\text{Gail}) = 0,$$
$$v(\text{Eileen, Barbara}) = 0,$$
$$v(\text{Eileen, Gail}) = 0,$$
$$v(\text{Barbara, Gail}) = 9{,}000,$$
$$v(\text{Eileen, Gail, Barbara}) = 14{,}000.$$

In general, a sequencing game is given by an ordering of the sets of players, the amount of time that needs to be devoted to each player, and the cost per day borne by each player until the completion of the job he needs accomplished. We call a re-ordering of the players *feasible* for coalition S if under both orderings (the new ordering and the original ordering) the amount of time needed for completing the job of every player who is not a member of S is unchanged. For example, the ordering [1: Gail, 2: Barbara, 3: Eileen] is feasible for the coalition {Eileen, Barbara, Gail}, but is not feasible for the coalition {Eileen, Gail}. Every sequencing game corresponds to a coalitional game in which the worth of coalition S is the amount of money that the members of the coalition can save by forming the optimal feasible ordering (for S).

16.1.7 Spanning tree games

We will give a formal presentation of spanning tree games in Section 17.8. In this section, we present one example of such games.

Suppose that local authorities have decided to connect three villages, Amherst, Belchertown, and Conway, to a single sewage system. The chief engineer of the project knows that directly connecting each village to the central drainage point into the sewage line is not necessarily the most cost-effective method: for example, it may be more cost-effective to connect only Conway to the central drainage point, and connect the other two villages to Conway. The table in Figure 16.1 depicts the cost of laying a sewage pipe between every pair of villages, and the cost in millions of dollars of connecting each village to the drainage point.

These costs are graphically depicted in Figure 16.2.

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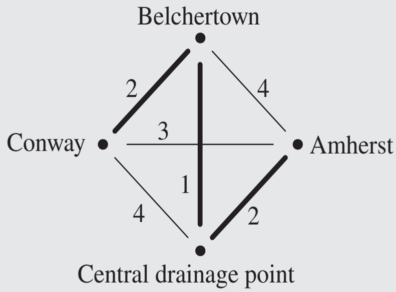

Figure 16.2 A graphical presentation of the costs of laying sewage pipes

After crunching the numbers, the chief engineer concludes that the most cost-effective solution is to connect Amherst and Belchertown directly to the main sewage line, and to connect Conway to Belchertown (see Figure 16.2, and check that this is the least expensive connection option). The total cost of this project is \$5,000,000. The next question is how to divide this cost between the three villages. If the residents of Belchertown are asked to bear a third of the cost, \$1,666,666, they may well claim that the cost of connecting them directly to the central drainage point is only \$1,000,000. Why should they then subsidize the other villages?

The coalitional game corresponding to this situation, in which the worth of each coalition is the minimal cost of connecting the members of that coalition, directly or indirectly via other villages, to the central drainage point, is:

$$\begin{aligned}
&c(\text{Amherst}) = 2,\\
&c(\text{Belchertown}) = 1,\\
&c(\text{Conway}) = 4,\\
&c(\text{Amherst, Belchertown}) = 3,\\
&c(\text{Amherst, Conway}) = 5,\\
&c(\text{Belchertown, Conway}) = 3,\\
&c(\text{Amherst, Belchertown, Conway}) = 5.
\end{aligned}$$

In general, spanning tree games are games based on a connected graph, one of whose vertices is called the *source*, each player is associated with a node, and every edge of the graph is associated with a nonnegative cost. The cost $c(S)$ of a coalition S is the minimal cost of a collection of edges connecting all the members of S to the source. Since the collection of edges that attains the minimal cost is a tree (containing no circular paths), these games are called *spanning tree games*.

16.1.8 Cost-sharing games

Cost-sharing games model situations in which the cost of a service is to be divided among different users, where different users need different amounts of that service.

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An example of such a situation, as presented in Littlechild [1974], is the construction of landing and takeoff runways in airports. Suppose an airport authority intends to construct a new landing runway in an international airport, and needs to decide:

1. The length of a new runway.
2. The price to charge every airplane using the new runway.

The longer the runway, the greater the construction cost, the more land that needs to be appropriated, and the greater the maintenance costs. On the other hand, if the runway is insufficiently long, large airplanes will be unable to land or take off. Charging every airplane an equal amount per landing and takeoff seems unfair: why should airplanes capable of landing on a shorter runway pay as much as airplanes needing a long runway? Why should the smaller airplanes subsidize the needs of the large airplanes? This situation can be depicted as a cost game, in which a set of players is a set of flights, and the worth of a coalition is the cost of a landing and takeoff runway that is sufficiently long to satisfy the needs of all the flights in the coalition.

This is an example in which the "players" are not decision makers in the usual sense, but rather takeoffs and landings. The real decision makers, of course, are airline executives, but it is more convenient to model takeoffs and landings as players in the game, because the relevant information for the problem we are solving is related to the price of takeoffs and landings. This game is discussed in greater detail in Exercise 18.16 (page 777).

16.2 Strategic equivalence

The descriptions presented in the examples above do not uniquely define the coalitional functions of their corresponding games. For example, the worth of a coalition in the cost game presented in Section 16.1.1 can be calculated in British pounds instead of American dollars, using an exchange rate. A different perspective on the situation takes into account all the incomes of all the entrepreneurs. For example, if Wilbur has an income of \$10,000 from renting a house he owns, while Orville and Ron have incomes of \$5,000 and \$4,000 from other endeavors, the coalitional function can be rewritten to take into account the total income of each member of every coalition. The different games obtained this way can all be considered to be equivalent from the perspectives of the players, because the income every entrepreneur draws from his other endeavors should not affect his income from joining a coalition with other players. We now present a formal definition of this sort of equivalence.

Let $(N; v)$ be a coalitional game, and let $a > 0$. The game $(N; w)$ defined by

$$w(S) = av(S), \quad \forall S \subseteq N \tag{16.4}$$

is the game derived from $(N; v)$ by changing the units of measurement of the worth of each coalition, by the exchange rate a. Suppose that in addition to changing the units of measurement, we give the sum b_i to every player i, independently of the coalition he joins (b_i can also be a negative value). If every coalition takes into account the extra sums thus received by every one of its members, we get the game $(N; u)$, whose coalitional function

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is given by

$$u(S) = av(S) + \sum_{i \in S} b_i, \quad \forall S \subseteq N. \tag{16.5}$$

We next introduce some convenient notation. For every coalition S, let $\mathbb{R}^S$ be an $|S|$-dimensional Euclidean space, where every axis is associated with one of the players in coalition S. In other words, if $x \in \mathbb{R}^S$ we denote the coordinates of x by $(x_i)_{i \in S}$.

For every vector $x \in \mathbb{R}^N$, define

$$x(S) := \sum_{i \in S} x_i, \quad \emptyset \neq S \subseteq N, \tag{16.6}$$

$$x(\emptyset) := 0. \tag{16.7}$$

In general, x_i denotes the amount of money that player i receives (or pays in a cost game). It follows that the quantity $x(S)$ is the sum of the payments received by members of coalition S.

Definition 16.4 *A coalitional game* $(N; w)$ *is* strategically equivalent *to the game* $(N; v)$ *if there exists a positive number a, and a vector $b \in \mathbb{R}^N$, such that for every coalition $S \subseteq N$:*

$$w(S) = av(S) + \sum_{i \in S} b_i = av(S) + b(S). \tag{16.8}$$

In other words, $(N; w)$ is derived from $(N; v)$ by changing the units of measurement, and adding a constant sum that every player receives or pays at the start of the game.

Theorem 16.5 *The strategic equivalence relation is an equivalence relation; i.e., it is reflexive, symmetric, and transitive.*[6]

Proof: To show that $(N; v)$ is equivalent to itself, in order to prove reflexivity, set $a = 1$ and $b = (0, 0, \ldots, 0)$.

We next show that the strategic equivalence relation is symmetric; i.e., if $(N; w)$ is strategically equivalent to $(N; v)$, then $(N; v)$ is strategically equivalent to $(N; w)$. To see this, if $w(S) = av(S) + b(S)$ for every $S \subseteq N$, where $a > 0$, then

$$v(S) = \frac{1}{a} w(S) + \sum_{i \in S} \frac{-b_i}{a}. \tag{16.9}$$

In other words, $v(S) = \widetilde{a} w(S) + \widetilde{b}(S)$, where $\widetilde{a} = \frac{1}{a} > 0$ and $\widetilde{b}_i = -\frac{b_i}{a}$ for every $i \in N$.

We complete the proof by showing that the strategic equivalence relation is transitive. Suppose that $(N; v)$ is strategically equivalent to $(N; w)$, and $(N; w)$ is strategically equivalent to $(N; u)$, i.e., there exist $a, a' > 0$ and $b, b' \in \mathbb{R}^N$ such that

$$v(S) = aw(S) + b(S), \qquad w(S) = a'u(S) + b'(S), \quad \forall S \subseteq N. \tag{16.10}$$

6 A binary relation P over a set X is *reflexive* if aPa for all $a \in X$. It is *symmetric* if aPb implies bPa. It is *transitive* if aPb and bPc imply aPc.

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Then

$$v(S) = aw(S) + b(S), \qquad w(S) = a'u(S) + b'(S), \tag{16.11}$$

where $a, a' > 0$. Then

$$v(S) = aw(S) + b(S) = aa'u(S) + (ab' + b)(S). \tag{16.12}$$

In other words, $v(S) = \widetilde{a}u(S) + \widetilde{b}(S)$, where $\widetilde{a} = aa' > 0$ and $\widetilde{b}_i = ab'_i + b_i$ for every $i \in N$. This means that $(N; v)$ is strategically equivalent to $(N; u)$. □

Definition 16.6 *The game* $(N; v)$ *is* $0 - 1$ normalized *if* $v(i) = 0$ *for every player* $i \in N$ *and* $v(N) = 1$. *The game is* $0 - 0$ normalized *if* $v(i) = 0$ *for every player* $i \in N$ *and* $v(N) = 0$. *The game is* $0 - (-1)$ normalized *if* $v(i) = 0$ *for every player* $i \in N$ *and* $v(N) = -1$.

The next theorem states that every coalitional game is strategically equivalent to a $0 - 1$, or $0 - 0$, or $0 - (-1)$ normalized game. The proof of the theorem is left to the reader (Exercise 16.17).

Theorem 16.7 *Let* $(N; v)$ *be a coalitional game.*

1. $(N; v)$ *is strategically equivalent to a* $0 - 1$ *normalized game if and only if* $v(N) > \sum_{i \in N} v(i)$.
2. $(N; v)$ *is strategically equivalent to a* $0 - 0$ *normalized game if and only if* $v(N) = \sum_{i \in N} v(i)$.
3. $(N; v)$ *is strategically equivalent to a* $0 - (-1)$ *normalized game if and only if* $v(N) < \sum_{i \in N} v(i)$.

A coalitional game $(N; v)$ is called 0-*normalized* if $v(i) = 0$ for every $i \in N$. By Theorem 16.7, every game is strategically equivalent to a 0-normalized game.

16.3 A game as a vector in a Euclidean space

Denote the set of nonempty coalitions by $\mathcal{P}(N) := \{S \subseteq N, S \neq \emptyset\}$. The space of coalitional games with the set of players N, G_N, is equivalent[7] to the $(2^n - 1)$-dimensional Euclidean space $\mathbb{R}^{\mathcal{P}(N)}$: a coalitional game $(N; v)$ is associated with a point $z = (z_S)_{S \in \mathcal{P}(N)} \in \mathbb{R}^{\mathcal{P}(N)}$ if and only if $v(S) = z_S$ for each coalition $S \in \mathcal{P}(N)$. The space $\mathbb{R}^{\mathcal{P}(N)}$ is a vector space: a game can be multiplied by a constant, and two games can be added together. The equivalence between $\mathbb{R}^{\mathcal{P}(N)}$ and G_N induces a vector space structure on G_N. For every coalitional function v and every real number α, define the coalitional function αv as follows:

$$(\alpha v)(S) := \alpha v(S), \quad \forall S \subseteq N. \tag{16.13}$$

7 Note that when a point in $\mathbb{R}^{\mathcal{P}(N)}$ is written as a vector with $2^n - 1$ elements, it is necessary to state which coalition is associated with each coordinate.

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For every pair of coalitional functions v and w, define the coalitional function $v + w$ as follows:

$$(v + w)(S) := v(S) + w(S), \quad \forall S \subseteq N. \tag{16.14}$$

When α is positive, multiplying by α can be interpreted as changing the units of measurement by α. The game $(N; v + w)$ can be interpreted as a situation in which the players are playing two games $(N; v)$ and $(N; w)$, and the worth of each coalition is the sum of the worth of the coalition in both games. This definition applies to situations in which (a) both the games $(N; v)$ and $(N; w)$ are related to each other, in the sense that every coalition formed in one is also formed in the other, and (b) the worth of a coalition formed in one game does not affect its worth in the other game, and therefore $(v + w)(S) := v(S) + w(S)$ for every coalition S.

16.4 Special families of games

Up until now we have not imposed any conditions on the coalitional function v. But these functions sometimes satisfy properties that have implications for the solutions of the game. We now present several such properties.

Definition 16.8 *A coalitional game $(N; v)$ is* superadditive *if for any pair of disjoint coalitions S and T,*

$$v(S) + v(T) \leq v(S \cup T). \tag{16.15}$$

A superadditive game is one in which two disjoint coalitions that choose to merge can obtain at least what they could obtain if they instead were to work separately. This property makes the formation of large coalitions an advantage in superadditive games; there is "positive pressure" to form the grand coalition N. Although this is not an unequivocal determination, as we will see in Example 16.12, superadditivity serves as a justification for the assumption that the grand coalition N will be formed, and many solution concepts therefore focus mainly on this case.

The corresponding definition for cost games is as follows.

Definition 16.9 *A cost game $(N; c)$ is* superadditive *if for every pair of disjoint coalitions S and T,*

$$c(S \cup T) \leq c(S) + c(T). \tag{16.16}$$

The assumption of superadditivity appears natural at first, but there are examples in which it does not obtain: in some cases, merging coalitions can be detrimental to the aims they seek to achieve, for political, legal, personal, or other reasons. For example, the merger of several large companies is liable to lead to a cartel, which is illegal, or to bureaucratic bloating that can reduce efficiency.

Definition 16.10 *A coalitional game* $(N; v)$ *is* monotonic *if for any pair of coalitions* S *and* T*, such that* $S \subseteq T$*,*

$$v(S) \le v(T). \tag{16.17}$$

In monotonic games, as a coalition grows larger its worth is not reduced. Although there may superficially appear to be a resemblance between the definitions of superadditive games and monotonic games, the two concepts are significantly different.

A monotonic game is not necessarily superadditive, and a superadditive game is not necessarily monotonic (Exercise 16.21). In addition, while superadditivity is invariant under strategic equivalence (Exercise 16.22), monotonicity is not invariant under strategic invariance, since every game is strategically equivalent to a monotonic game (Exercise 16.24). The following example depicts two games that are strategically equivalent; one is monotonic, and the other is not monotonic.

Example 16.11 Consider two three-player games $(N; v)$ and $(N; w)$ with the following coalitional functions:

$$v(S) = |S|, \forall S \subseteq N, \tag{16.18}$$

$$w(S) = -|S|, \forall S \subseteq N. \tag{16.19}$$

The coalitional game $(N; v)$ is monotonic, and the coalitional game $(N; w)$ is not monotonic. Yet the two games are strategically equivalent. To see this, let $a = 1$ and let $b \in \mathbb{R}^3$ be defined by $b = (-2, -2, -2)$. Then

$$w(S) = -|S| = |S| - 2|S| = av(S) + b(S). \tag{16.20}$$

◀

16.5 Solution concepts

The main questions that are the focus of coalitional game theory include:

1. What happens when the players play the game? What coalitions will form, and if a coalition S is formed, how does it divide the worth $v(S)$ among its members?
2. What would a judge or an arbitrator recommend that the players do?

The answers to these two questions are quite different. The question regarding the coalitional structure that the players can be expected to form is a difficult one, and will not be addressed in this book. We will often assume that the grand coalition N is formed and ask how will the players divide among them the worth $v(N)$. The answer to this question is generally a set solution, i.e., a solution that contains several possible payoff vectors, according to which the players may choose to divide $v(N)$ among them. On the other hand, a recommendation of a judge or an arbitrator is usually a point solution, i.e., a single payoff vector.

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Example 16.12 **Majority game** Let $(N; v)$ be the three-player coalitional game in which $N = \{1, 2, 3\}$, and the coalitional function v is given by

$$v(1) = v(2) = v(3) = 0, \qquad v(1, 2) = v(1, 3) = v(2, 3) = v(1, 2, 3) = 1.$$

What can we expect to happen in this game? It is reasonable to suppose that a coalition of two players will form and that, given the symmetry between the players, the members of the coalition will divide the worth equally among themselves. This leads to one of the following payoff vectors in the set

$$\left\{\left(\tfrac{1}{2}, \tfrac{1}{2}, 0\right), \left(\tfrac{1}{2}, 0, \tfrac{1}{2}\right), \left(0, \tfrac{1}{2}, \tfrac{1}{2}\right)\right\}. \qquad (16.21)$$

Without more information about the players, it is not possible to know which payoff vector will finally be chosen.

In contrast, if the players were to approach an arbitrator and ask him to determine how to divide the sum 1 that they can achieve, given the symmetry between the players it is reasonable to expect that the arbitrator will recommend a split of $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

We can therefore regard the three outcomes $\left\{(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2}), (0, \frac{1}{2}, \frac{1}{2})\right\}$ as a set solution to the game, while $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a point solution. ◀

The simple majority game in Example 16.12 is superadditive (verify!), but there is a high likelihood in this game that the grand coalition N will not form.

Definition 16.13 *Let $\mathcal{U}$ be a family of coalitional games (over any set of players). A* solution concept *(over $\mathcal{U}$) is a function φ associating every game $(N; v) \in \mathcal{U}$ with a subset $\varphi(N; v)$ of $\mathbb{R}^N$. A solution concept is called a* point solution *if for every coalitional game $(N; v) \in \mathcal{U}$, the set $\varphi(N; v)$ contains only one element.*

Note that in Definition 16.13, it is possible for a particular game $(N; v)$ to satisfy $\varphi(N; v) = \emptyset$.

Sometimes the players form the grand coalition N, and sometimes several coalitions are formed instead. Both the set solutions and the point solutions that we will see in the examples below depend on the coalitional structures that are formed.

Definition 16.14 *A* coalitional structure *is a partition $\mathcal{B}$ of the set of players N.*

In other words, a coalitional structure is a collection of disjoint and nonempty sets whose union is N. Examples of coalitional structures for a set of players $N = \{1, 2, 3, 4\}$ include:

All the players play as isolated individuals: $\mathcal{B} = \{\{1\}, \{2\}, \{3\}, \{4\}\}$.
Two two-player coalitions form: $\mathcal{B} = \{\{1, 2\}, \{3, 4\}\}$.
Players 2 and 3 form a coalition: $\mathcal{B} = \{\{1\}, \{2, 3\}, \{4\}\}$.
All the players form the grand coalition: $\mathcal{B} = \{\{1, 2, 3, 4\}\}$.

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A point solution offers a single solution to every game, given the coalitional structure: for every coalition S in a coalitional structure, it offers one and only one way to divide the worth $v(S)$ among the members of the coalition. A set solution offers a set of vector payoffs for every game and every coalitional structure. We use the notation $\varphi(N; v; \mathcal{B})$ to denote a (set or point) solution, when $\mathcal{B}$ is the coalitional structure that is formed. When the coalitional structure that is formed contains only the coalition N, i.e., $\mathcal{B} = \{N\}$, we will omit the explicit denotation of the coalitional structure and instead write $\varphi(N; v)$.

Definition 16.15 *Let $(N; v)$ be a game, and let $\mathcal{B}$ be a coalitional structure. A vector $x \in \mathbb{R}^N$ is called* efficient[8] *for the coalitional structure $\mathcal{B}$ if for every coalition $S \in \mathcal{B}$,*

$$x(S) = v(S). \tag{16.22}$$

A vector x is called individually rational *if for every player $i \in N$,*

$$x_i \geq v(i). \tag{16.23}$$

When the players divide into coalitions, forming a coalitional structure $\mathcal{B}$, it is reasonable to suppose that each coalition $S \in \mathcal{B}$ will divide its worth $v(S)$ among its members: the members of the coalition cannot divide among themselves more than the sum $v(S)$ available to the coalition, and there is no point in dividing less than $v(S)$, because then part of the worth $v(S)$ available to the coalition is wasted. Therefore, if x_i is the sum that player i receives (or pays), it is reasonable that $\sum_{i \in S} x_i = v(S)$ holds for every coalition $S \in \mathcal{B}$. In that case, the vector $x = (x_i)_{i \in N}$ is efficient for a coalitional structure $\mathcal{B}$. Since every player can guarantee for himself $v(i)$ if he does not join any coalition, it is reasonable to suppose that every player will demand at least that sum. In other words, $x_i \geq v(i)$: this is the *individual rationality* property of the vector x.

This leads to the definition of the set of imputations as the set of all efficient and individually rational payoffs.

Definition 16.16 *Let $(N; v)$ be a coalitional game, and let $\mathcal{B}$ be a coalitional structure. An* imputation *for the coalitional structure $\mathcal{B}$ is a vector $x \in \mathbb{R}^N$ that is efficient for the coalitional structure $\mathcal{B}$, and individually rational. The set of all imputations for the coalitional structure $\mathcal{B}$ is denoted by $X(\mathcal{B}; v)$.*

When the coalitional structure is $\mathcal{B} = \{N\}$ we will say for short "imputation" instead of "imputation for the coalitional structure $\{N\}$,"and write $X(N; v)$ instead of $X(\{N\}; v)$. Note that the set $X(\mathcal{B}; v)$ is compact in $\mathbb{R}^N$ (see Exercise 16.31).

When there are two players $N = \{1, 2\}$, the set of imputations is

$$X(N; v) = \{x \in \mathbb{R}^N : x_1 \geq v(1), x_2 \geq v(2), x_1 + x_2 = v(1, 2)\}. \tag{16.24}$$

8 The efficiency property is sometimes also called *social rationality*.

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This set can be empty (if $v(1)+v(2) > v(1,2)$), an isolated point (if $v(1)+v(2) = v(1,2)$), or an interval (if $v(1)+v(2) < v(1,2)$). In the last case, the extreme points of the interval are $(v(1), v(1,2)-v(1))$ and $(v(1,2)-v(2), v(2))$.

Example 16.17 Let $N = \{1,2,3\}$, and let the coalitional function be

$$v(1) = v(2) = v(3) = 0, \quad v(1,2) = 2, \quad v(1,3) = 3, \quad v(2,3) = 4, \quad v(1,2,3) = 7.$$

The set of imputations is given by the triangle whose vertices are $(7,0,0)$, $(0,7,0)$, and $(0,0,7)$ (see Figure 16.3(a)). The sets of imputations for the coalitional structures containing two sets are intervals (see Figure 16.3(b)).

For the coalitional structure $\{\{1\},\{2\},\{3\}\}$, the only imputation is $(0,0,0)$. The collections of imputations for coalitional structures containing two sets are the interval (see Figure 16.3(b)):

$$X(\{\{1,2\},\{3\}\}; v) = [(0,2,0),(2,0,0)], \tag{16.25}$$

$$X(\{\{1,3\},\{2\}\}; v) = [(0,0,3),(3,0,0)], \tag{16.26}$$

$$X(\{\{2,3\},\{1\}\}; v) = [(0,4,0),(0,0,4)]. \tag{16.27}$$

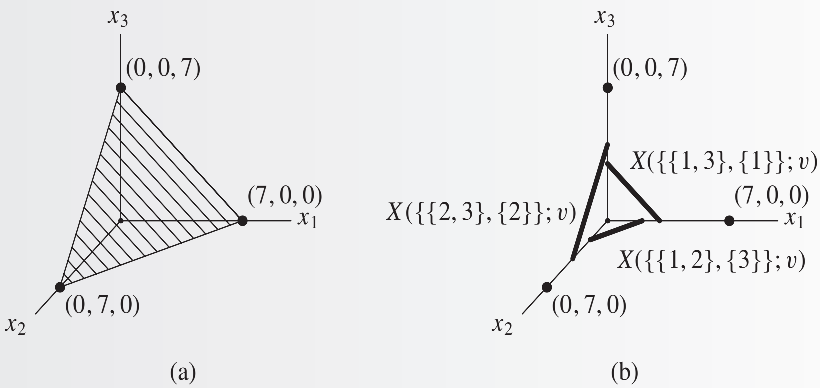

Figure 16.3 The set of imputations of all the coalitional structures in Example 16.17 ◀

The set of imputations may be empty: this happens if there is a coalition $S \in \mathcal{B}$ such that $\sum_{i \in S} v(i) > v(S)$.

Let N be a set of players, and $\mathcal{B}$ be a coalitional structure. One possible solution concept is

$$\varphi(N; v; \mathcal{B}) := X(\mathcal{B}; v). \tag{16.28}$$

This is the set of all possible individually rational and efficient outcomes. In other words, if $x \notin X(\mathcal{B}; v)$, it is not reasonable for the players to divide among themselves the sum they receive according to x. This is a weak solution: in many games the set $X(\mathcal{B}; v)$ is very large (as in Example 16.17). When the set $X(\mathcal{B}; v)$ is empty, it is unclear whether or not the coalitional structure $\mathcal{B}$ will be formed, and if it is formed, this solution concept provides no prediction regarding the outcome of the game.

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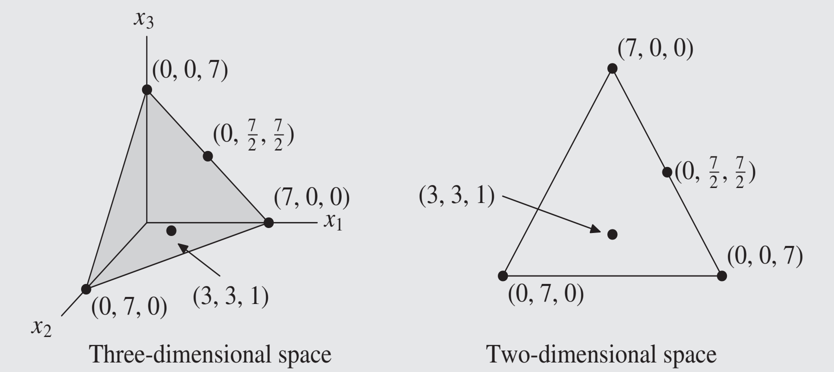

Figure 16.4 The set of imputations in Example 16.17

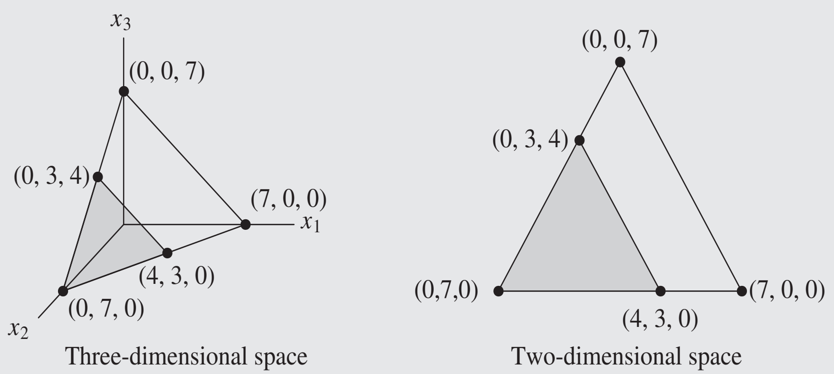

Figure 16.5 Imputations at which $x_2 \geq 3$, or, equivalently, $x_1 + x_3 \leq 4$

16.6 Geometric representation of the set of imputations

As we saw in Example 16.17, although the set of imputations is a subset of $\mathbb{R}^N$, the efficiency constraint ($\sum_{i=1}^n x_i = x(N) = v(N)$) implies that this set is located in an $(n-1)$-dimensional subspace. Since a large part of our intuition comes from graphical representations of figures, and the smaller the dimension of the space in which they are displayed, the easier it is to present them, it is more convenient to present the set of imputations in $\mathbb{R}^{n-1}$. This advantage is especially pronounced in three-player games, because in that case the set of imputations is a triangle in $\mathbb{R}^3$, which can be more conveniently presented in $\mathbb{R}^2$ (see Figure 16.4).

The set of all imputations in which one coordinate is fixed is a straight line parallel to the corresponding side of the triangle. Figure 16.5 depicts (in both $\mathbb{R}^3$ and $\mathbb{R}^2$) the set of imputations x in Example 16.17 satisfying $x_2 \geq 3$. This is also the set of imputations x in Example 16.17 satisfying $x_1 + x_3 \leq 4$.

In a 0-normalized three-player game it is convenient to present the set of imputations as an equilateral triangle in $\mathbb{R}^2$ whose height is $v(N)$ (as opposed to its height in $\mathbb{R}^3$, which is

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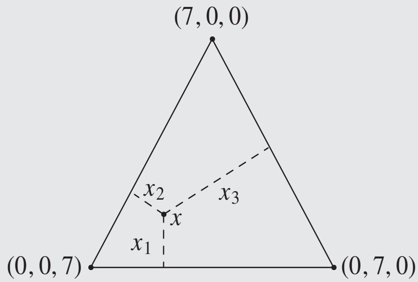

Figure 16.6 The barycentric coordinates of x: the distances of the point x from the three sides of the triangle

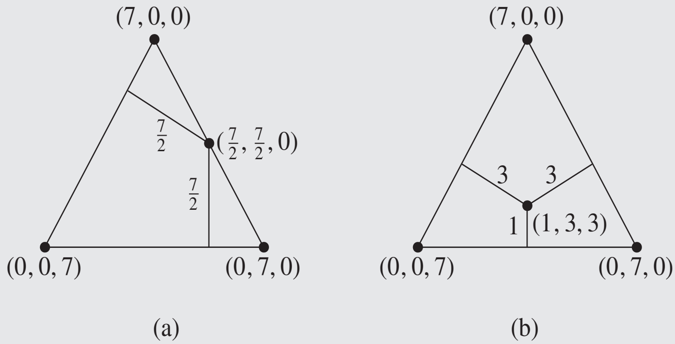

Figure 16.7 The sum of the distances of every point from the sides of a triangle equals the height of the triangle

$\frac{\sqrt{3}}{2}v(N)$; check that this is true). The sides of the triangle will be labeled by the names of the three players, 1, 2, and 3. A point in the triangle will be denoted by $x = (x_1, x_2, x_3)$, where x_i is the distance of the point from the side labeled i, for $i \in \{1, 2, 3\}$. Recall that in an equilateral triangle, the sum of the distances of each point from the three sides of the triangle equals the height of the triangle (see Exercise 16.35). It follows that the vertices of the triangle are $(v(N), 0, 0)$, $(0, v(N), 0)$, and $(0, 0, v(N))$, and every point in the triangle satisfies $x_1 + x_2 + x_3 = v(N)$ (see Figures 16.6 and 16.7). Similarly, every point in the triangle corresponds to an efficient imputation because the game is 0-normalized and the distance of the point from each one of the sides of the triangle is nonnegative, hence $x_i \geq 0 = v(i)$. This coordinate system is called the *barycentric coordinate* system, and it can be generalized to any number of players. For further discussion of this topic, see Section 23.1 (page 916). Barycentric coordinates have the following physical interpretation. If we place weights x_1, x_2, and x_3 respectively at the three vertices $(v(N), 0, 0)$, $(0, v(N), 0)$, and $(0, 0, v(N))$ of the triangle, the center of mass of the system will be the point $x = (x_1, x_2, x_3)$. The word barycenter means "center of mass," which is where the term barycentric coordinates comes from.

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16.7 Remarks

Simple games and weighted majority games were first introduced in von Neumann and Morgenstern [1944]. Market games were first introduced in Shapley and Shubik [1963]. Sequencing games were first introduced in Curiel, Pederzoli, and Tijs [1989]. Spanning tree games were first introduced in Claus and Kleitman [1973] and in Bird [1976]. Exercise 16.6 is taken from Tamir [1991].

16.8 Exercises

16.1 By finding the appropriate weights, show that the United Nations Security Council game described by Equation (16.1) (page 663) is a weighted majority game.

16.2 Passing a bill into law in the United States requires the signature of the President of the United States, and a simple majority in both the Senate (composed of 100 Senators) and the House of Representatives (composed of 435 members), or alternatively a two-thirds majority in both the Senate and the House of Representatives required to override a presidential veto.

(a) Write down the coalitional function of the corresponding game.
(b) Is this a weighted majority game? If you answer yes, write down the quota and weights of the game. If you answer no, prove that it is not a weighted majority game.

16.3 **Sequencing game** Don, a painter, is hired to paint the houses of Henry, Ethan, and Tom. The following table depicts the sequential ordering of each client in Don's schedule book, the amount of time required to paint his house, and the loss he suffers from every day that passes until work is completed.

Sequential ordering	Name	Time required	Daily loss in dollars
1	Henry	5	200
2	Ethan	3	550
3	Tom	4	400

Write down the coalitional function of the corresponding sequencing game in which the worth of a coalition is the sum of money that the members of the coalition can save by changing their ordering in a feasible way in Don's schedule book.

16.4 Write down the coalitional functions in each of the following weighted majority games:

(a) $[4; 3, 2, 1, 1]$.
(b) $[5; 3, 2, 1, 1]$.
(c) $[4; 3, 1, 1, 1]$.
(d) $[39; 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$.

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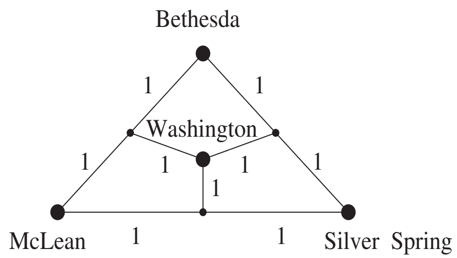
16.5 The board of directors of a certain company contains four members (including the chairman of the board). A motion is passed by the board only if the chairman approves it, and it is supported by a majority of the board (i.e., gets at least three votes). Write down the coalitional function of the corresponding game.

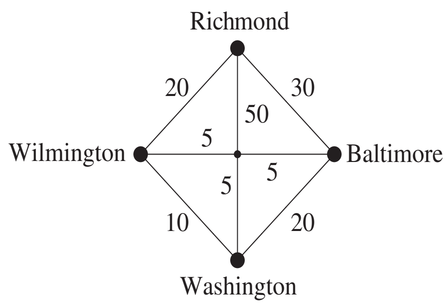
16.6 **Spanning tree games** The following figure depicts a network of roads connecting the capital city, Washington, with three nearby towns, Bethesda, Silver Spring, and McLean. The towns are responsible for maintaining the roads between themselves and the capital. The maintenance cost of every segment of road is listed as a unit.

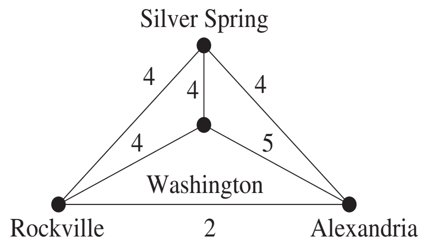
In the figure, a large dot indicates a vertex at which a town is located, and a small dot indicates a vertex at which no town is located.

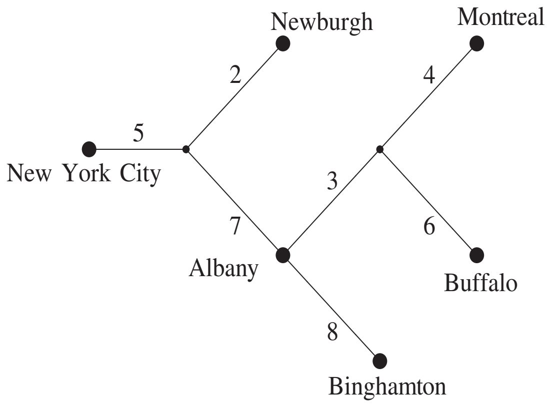
Define $c(S)$ as the minimal cost required for all the towns in coalition S to be connected with the capital. Write down the coalitional function.



16.7 Repeat Exercise 16.6 for each of the following networks. The maintenance cost of each road segment is indicated next to it.







16.8 Prove that the following weighted majority games share the same coalitional function, and therefore they are different representations of the same game.

$$[2; 1, 1, 1], \qquad [9; 8, 2, 7], \qquad [9; 8, 1, 8].$$

16.9 The representation $[2; 1, 1, 1]$ of the game in Exercise 16.8 has the property that the sum of the weights equals the quota 2 in every *minimal winning* coalition, that is, a winning coalition such that every one of its proper subsets is not winning. These weights are called *homogeneous weights*, and the representation is called a *homogeneous representation*.

In general, weights $w_1, w_2, \ldots, w_n$ are called *homogeneous weights* if there exists a real number q such that in the weighted majority game $[q; w_1, w_2, \ldots, w_n]$ the equality $\sum_{i \in S} w_i = q$ holds for every minimal winning coalition S. This representation is called a *homogeneous representation*.

For each of the following weighted majority games determine whether it has a homogeneous representation. If yes, write it down. If no, explain why.

(a) $[10; 9, 1, 2, 3, 4]$.
(b) $[8; 5, 4, 2]$.
(c) $[9; 7, 5, 3, 1]$.
(d) $[10; 7, 5, 3, 1]$.

16.10 **A projective game with seven players** Consider a simple game with seven players, with winning coalitions:

$$\{1, 2, 4\}, \{2, 3, 5\}, \{1, 3, 6\}, \{3, 4, 7\}, \{1, 5, 7\}, \{2, 6, 7\}, \{4, 5, 6\},$$

and every coalition containing at least one of these winning coalitions. The game is presented graphically in the following figure. Each winning coalition must contain three players who are either all along one of the straight lines, or all on the circle.

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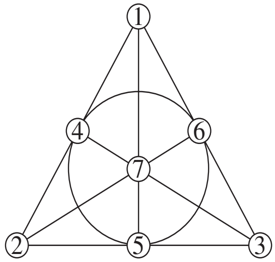

Prove that this game cannot be represented as a weighted majority game.

16.11 A player i in a simple game $(N; v)$ is called a *veto player* if $v(S) = 0$ for every coalition S that does not contain i. The player is called a *dictator* if $v(S) = 1$ if and only if $i \in S$.

(a) Prove that in a simple game satisfying the property that $v(S) + v(N \setminus S) = 1$ for every coalition $S \subseteq N$, there exists at most one veto player, and that player is a dictator.
(b) Find a simple three-player game satisfying $v(S) + v(N \setminus S) = 1$ for every coalition $S \subseteq N$ that has no veto player.

16.12 **Market game** A set of merchants $N_1 = \{1, 2\}$ sell their wares in the market. Each merchant has an "initial endowment" $(0, \frac{1}{2})$: the first number represents the amount of gin the merchant has and the second number the amount of tonic the merchant has. A second set of merchants $N_2 = \{3, 4, 5\}$ also sells in the same market, and each member of this set has an initial endowment of $(1, 0)$. The total bundle available to a coalition $S \subseteq N_1 \cup N_2$ is

$$\left(|S \cap N_2|, \tfrac{1}{2}|S \cap N_1|\right),$$

where $|R|$ denotes the number of members of coalition R.

Consumers will only buy cocktails containing equal parts gin and tonic. The net profit from selling α units of cocktail is α dollars. Describe this situation as a coalitional game, and write down in detail the coalitional function.

16.13 Repeat Exercise 16.12, but assume that the initial endowment of the merchants in N_1 is $(0, \frac{2}{3})$.

16.14 Are the following three-player games strategically equivalent? Justify your answer.

$$v(1) = 6, \quad v(2) = 5, \quad v(3) = 8, v(1,2) = 10, \quad v(1,3) = 20,$$
$$v(2,3) = 50, \quad v(1,2,3) = 80,$$
$$w(1) = 13, \quad w(2) = 10, \quad w(3) = 19, w(1,2) = 25, \quad w(1,3) = 55,$$
$$w(2,3) = 140, \quad w(1,2,3) = 235.$$

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16.15 Let $(N; v)$ be the coalitional game with $N = \{1, 2, 3\}$ and the following coalitional function:

$$v(1) = 3, \quad v(2) = 6, \quad v(3) = 8, \quad v(1, 2) = 12, \quad v(1, 3) = 15,$$
$$v(2, 3) = 18, \quad v(1, 2, 3) = 80.$$

Write down a $0 - 1$ normalized coalitional game $(N; w)$ that is strategically equivalent to $(N; v)$.

16.16 What is the coalitional function of the game derived from

$$v(1) = 20, \quad v(2) = 30, \quad v(3) = 50, \quad v(1, 2) = 10, \quad v(1, 3) = 15,$$
$$v(2, 3) = 40, \quad v(1, 2, 3) = 5,$$

if each player is given an initial sum of \$1,000?

16.17 Prove Theorem 16.7 (page 670): let $(N; v)$ be a coalitional game. Then

(a) $(N; v)$ is strategically equivalent to a $0 - 1$ normalized game if and only if $v(N) > \sum_{i \in N} v(i)$.
(b) $(N; v)$ is strategically equivalent to a $0 - 0$ normalized game if and only if $v(N) = \sum_{i \in N} v(i)$.
(c) $(N; v)$ is strategically equivalent to a $0 - (-1)$ normalized game if and only if $v(N) < \sum_{i \in N} v(i)$.

16.18 Describe the family of all superadditive games in which the set of players is $N = \{1, 2\}$.

16.19 Let $(N; v)$ be a coalitional game with a set of players $N = \{1, 2, 3\}$ and coalitional function

$$v(S) = \begin{cases} 0 & \text{if } S = \emptyset, \\ 1 & \text{if } S = \{1\}, \{2\}, \\ 2 & \text{if } S = \{3\}, \\ 4 & \text{if } |S| = 2, \\ 5 & \text{if } |S| = 3. \end{cases}$$

(a) Is $(N; v)$ a superadditive game?
(b) What is the set of imputations of this game?

16.20 Prove that the convex combination of superadditive games is also superadditive. In other words, if $(N; v)$ and $(N; w)$ are superadditive games, and if $0 \leq \lambda \leq 1$, then the game $(N, \lambda v + (1 - \lambda)w)$ defined by

$$(\lambda v + (1 - \lambda)w)(S) := \lambda v(S) + (1 - \lambda)w(S) \tag{16.29}$$

is also superadditive.

16.21 Give an example of a monotonic game that is not superadditive, and an example of a superadditive game that is not monotonic.

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16.22 Prove that every game that is strategically equivalent to a superadditive game is itself superadditive.

16.23 Prove that a convex combination of monotonic games is also monotonic. In other words, if $(N; v)$ and $(N; w)$ are monotonic games, and if $0 \le \lambda \le 1$, then the game $(N, \lambda v + (1-\lambda)w)$ defined by

$$(\lambda v + (1-\lambda)w)(S) := \lambda v(S) + (1-\lambda)w(S) \tag{16.30}$$

is also monotonic.

16.24 (a) Is the three-player game $(N; v)$ in which v is given by

$$v(1) = 3, \quad v(2) = 13, \quad v(3) = 4, \quad v(1,2) = 12, \quad v(1,3) = 15,$$
$$v(2,3) = 1, \quad v(1,2,3) = 10$$

monotonic? Justify your answer.

(b) Find a monotonic game that is strategically equivalent to $(N; v)$.

(c) Prove that every game is strategically equivalent to a monotonic game. It follows that the property of monotonicity is not invariant under strategic equivalence.

16.25 Let $(N; v)$ be a nonnegative coalitional game, i.e., $v(S) \ge 0$ for every coalition $S \subseteq N$.

(a) Prove that if $(N; v)$ is superadditive then $(N; v)$ is monotonic.

(b) Show by example that the converse does not hold: it is possible for $(N; v)$ to be monotonic but not superadditive.

16.26 The 0-*normalization* of a coalitional game $(N; v)$ is a coalitional game $(N; w)$ that is strategically equivalent to $(N; v)$ and satisfies $w(i) = 0$ for every player $i \in N$. A coalitional game is called 0-*monotonic* if its 0-normalization is a monotonic game.

(a) Which of the following monotonic games with set of players $N = \{1, 2, 3\}$ is 0-monotonic?

(i) $v(1) = 5, \quad v(2) = 8, \quad v(3) = 15, v(1,2) = 10, \quad v(1,3) = 30, \quad v(2,3) = 50, \quad v(1,2,3) = 80.$

(ii) $v(1) = 5, \quad v(2) = -2, \quad v(3) = 7, v(1,2) = 9, \quad v(1,3) = 30, \quad v(2,3) = 17, \quad v(1,2,3) = 30.$

(b) Give an example of a coalitional game that is not monotonic, but is 0-monotonic.

16.27 Prove that a coalitional game $(N; v)$ is 0-monotonic if and only if $v(S \cup \{i\}) \ge v(S) + v(i)$ for every coalition S and every player $i \notin S$.

16.28 Let $(N; v)$ be a coalitional game. The *superadditive cover* of $(N; v)$ is the coalitional game $(N; w)$ satisfying the properties:

- $(N; w)$ is a superadditive game.
- $w(S) \ge v(S)$ for every coalition S.
- Every game $(N; u)$ satisfying the previous two properties also satisfies $u(S) \ge w(S)$ for every coalition S.

Find the superadditive cover of the following game, whose set of players is $N = \{1, 2, 3\}$:

$$v(1) = 3, \quad v(2) = 5, \quad v(3) = 7, \quad v(1, 2) = 6, \quad v(1, 3) = 8,$$
$$v(2, 3) = 10, \quad v(1, 2, 3) = 13.$$

16.29 Let $(N; v)$ be a coalitional game. The *monotonic cover* of $(N; v)$ is the coalitional game $(N; w)$ satisfying the properties:

- $(N; w)$ is a monotonic game.
- $w(S) \geq v(S)$ for every coalition S.
- Every game $(N; u)$ satisfying the above two properties also satisfies $u(S) \geq w(S)$ for every coalition S.

Prove that the monotonic cover $(N; w)$ of the game $(N; v)$ satisfies

$$w(S) = \max_{T \subseteq S} v(S). \tag{16.31}$$

16.30 (a) How many different coalitional structures can there be in a three-player game? Write down all of them.
(b) How many different coalitional structures can there be in a four-player game? Write down all of them.

16.31 Prove that for every coalitional game $(N; v)$ and every coalitional structure $\mathcal{B}$, the set of imputations $X(\mathcal{B}; v)$ is convex and compact.

16.32 Prove that for every coalitional game $(N; v)$ there exists a coalitional structure $\mathcal{B}$ for which the set of imputations $X(\mathcal{B}; v)$ is nonempty.

16.33 (a) Write down the set of imputations of the three-player game in which

$$v(1) = 3, \quad v(2) = 5, \quad v(3) = 7, \quad v(1, 2) = 6, \quad v(1, 3) = 12,$$
$$v(2, 3) = 15, \quad v(1, 2, 3) = 10,$$

for all coalitional structures.
(b) Repeat part (a) when $v(1, 2, 3) = 13$.
(c) Repeat part (a) when $v(1, 2, 3) = 34$.

16.34 Let $(N; v)$ and $(N; w)$ be two coalitional games with the same set of players. Let $x \in X(\mathcal{B}; v)$ and $y \in X(\mathcal{B}; w)$. Does $x + y \in X(\mathcal{B}; v + w)$ necessarily hold? Does $x - y \in X(\mathcal{B}; v - w)$ necessarily hold? If you answer yes to either question, provide a proof. If you answer no, present a counterexample.

16.35 Suppose that you are given an equilateral triangle, with x being a point in the triangle. Denote by x_1, x_2, x_3 the distance of the point x from each side of the triangle, respectively (see accompanying figure).

(a) Prove that $x_1 + x_2 + x_3 = k$, where k is the height of the triangle.
(b) Prove that this is true even if the point is located in the plane of the triangle, but not necessarily in the triangle, where the distance from the point to the side of

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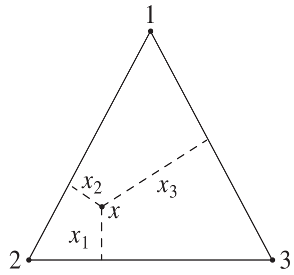
the triangle is negative if the line on which the side lies separates the triangle from the point (in the accompanying diagram, y_1 and y_2 are positive and y_3 is negative).

(c) Describe a similar property in one-dimensional line segments.



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