CS 228 : Logic in Computer Science

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Basic Rules So Far

- $\wedge$i, $\wedge e_1$, $\wedge e_2$ (and introduction and elimination)
- $\neg\neg e$, $\neg\neg i$ (double negation elimination and introduction)
- MP (Modus Ponens)
- $\rightarrow i$ (Implies Introduction : remember opening boxes)
- $\vee i_1, \vee i_2, \vee e$ (Or introduction and elimination)

The Copy Rule

- $\vdash p \rightarrow (q \rightarrow p)$

1.	*true*	premise
2.		

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- $\perp \rightarrow \varphi$ for any formula φ.

Rules with $\perp$

The $\perp$ elimination rule $\perp e$

$$\frac{\perp}{\psi}$$

The $\perp$ introduction rule $\perp i$

$$\frac{\varphi \qquad \neg\varphi}{\perp}$$

An Example

- $\neg p \vee q \vdash p \rightarrow q$

1.	$\neg p \vee q$	premise
2.		

An Example

- $\neg p \vee q \vdash p \to q$

1.	$\neg p \vee q$	premise
2.	$\neg p$	$\vee e$ (1)
3.		

An Example

- $\neg p \vee q \vdash p \rightarrow q$

1.	$\neg p \vee q$	premise
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3.	p	assumption
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1.	$\neg p \vee q$	premise
2.	$\neg p$	$\vee e$ (1)
3.	p	assumption
4.	$\perp$	$\perp i$ 2,3
5.		

An Example

- $\neg p \lor q \vdash p \to q$

1.	$\neg p \lor q$	premise
2.	$\neg p$	$\lor e$ (1)
3.	p	assumption
4.	$\bot$	$\bot i$ 2,3
5.	q	$\bot e$ 4
6.		

An Example

- $\neg p \vee q \vdash p \to q$

1.	$\neg p \vee q$	premise
2.	$\neg p$	$\vee e$ (1)
3.	p	assumption
4.	$\perp$	$\perp i$ 2,3
5.	q	$\perp e$ 4
6.	$p \to q$	$\to i$ 3-5
7.	q	$\vee e$ (2)
8.		

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- $\neg p \vee q \vdash p \rightarrow q$

1.	$\neg p \vee q$	premise
2.	$\neg p$	$\vee e$ (1)
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6.	$p \rightarrow q$	$\rightarrow i$ 3-5
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5.	q	$\perp e$ 4
6.	$p \rightarrow q$	$\rightarrow i$ 3-5
7.	q	$\vee e$ (2)
8.	p	assumption
9.	q	copy 7
10.	$p \rightarrow q$	$\rightarrow i$ 8-9
11.	$p \rightarrow q$	$\vee e$ 1, 2-6, 7-10

Introducing Negations (PBC)

- In the course of a proof, if you assume φ (by opening a box) and obtain $\perp$ in the box, then we conclude $\neg\varphi$
- This rule is denoted $\neg i$ and is read as $\neg$ introduction.
- Also known as Proof By Contradiction

An Example

- $p \to \neg p \vdash \neg p$

1.	$p \to \neg p$	premise
2.		

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2.	p	assumption
3.	$\neg p$	MP 1,2
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An Example

- $p \rightarrow \neg p \vdash \neg p$

1.	$p \rightarrow \neg p$	premise
2.	p	assumption
3.	$\neg p$	MP 1,2
4.	$\perp$	$\perp i$ 2,3
5.	$\neg p$	$\neg i$ 2-4

The Last One

Law of the Excluded Middle (LEM)

$$\frac{}{\varphi \vee \neg\varphi}$$

Summary of Basic Rules

- $\wedge i$, $\wedge e_1$, $\wedge e_2$,
- $\neg\neg e$
- MP
- $\rightarrow i$
- $\vee i_1, \vee i_2, \vee e$
- Copy, $\neg i$ or PBC
- $\perp e, \perp i$

Derived Rules

- MT (derive using MP, $\perp i$ and $\neg i$)
- $\neg\neg i$ (derive using $\perp i$ and $\neg i$)
- LEM (derive using $\vee i_1, \perp i, \neg i, \vee i_2, \neg\neg e$)

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- So far, the "proof" we have seen is purely syntactic, no true/false meanings were attached
- Intuitively, $p \to q \vdash \neg p \vee q$ makes sense because you think semantically. However, we never used any semantics so far.
- Now we show that whatever can be proved makes sense semantically too.

Semantics

- Each propositional variable is assigned values true/false. Truth tables for each of the operators $\vee, \wedge, \neg, \rightarrow$ to determine truth values of complex formulae.

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  - Recall $\vdash$, and compare with $\models$
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- Formulae φ and ψ are semantically equivalent iff $\varphi \models \psi$ and $\psi \models \varphi$

Soundness of Propositional Logic

$\varphi_1, \ldots, \varphi_n \vdash \psi \Rightarrow \varphi_1, \ldots, \varphi_n \models \psi$

Whenever ψ can be proved from $\varphi_1, \ldots, \varphi_n$, then ψ evaluates to true whenever $\varphi_1, \ldots, \varphi_n$ evauate to true

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- Consider now a proof with k lines.

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- By inductive hypothesis, we have $\varphi_1, \ldots, \varphi_n \models \psi_1$ and $\varphi_1, \ldots, \varphi_n \models \psi_2$. By semantics, we have $\varphi_1, \ldots, \varphi_n \models \psi_1 \wedge \psi_2$.

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- Assume ψ was obtained using $\rightarrow i$. Then ψ is of the form $\psi_1 \rightarrow \psi_2$.
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- The line just after the box was ψ.
- Consider adding ψ_1 in the premises along with $\varphi_1, \ldots, \varphi_n$. Then we will get a proof $\varphi_1, \ldots, \varphi_n, \psi_1 \vdash \psi_2$, of length $k - 1$. By inductive hypothesis, $\varphi_1, \ldots, \varphi_n, \psi_1 \models \psi_2$. By semantics, this is same as $\varphi_1, \ldots, \varphi_n \models \psi_1 \rightarrow \psi_2$

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- The equivalence of $\varphi_1, \dots, \varphi_n \vdash \psi_1 \rightarrow \psi_2$ and $\varphi_1, \dots, \varphi_n, \psi_1 \vdash \psi_2$ gives the proof.

Soundness : Other cases

Do this as homework