CS 228: Logic in CS

The Completeness Theorem of Propositional Logic

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Theorem 1 (Soundness Theorem)**.** *Given a set Σ of propositional logic formulae over variables $p_0, p_1, \ldots$, and any propositional logic formula φ over those variables, if $\Sigma \vdash \varphi$, then $\Sigma \models \varphi$.*

Proof. The proof of this has been covered in class for the finite case. It proceeds by induction on the length of the proof of $\Sigma \vdash \varphi$. Extending this to the infinite case requires only cosmetic changes to the proof. $\square$

Theorem 2 (Completeness Theorem)**.** *Given a set Σ of propositional logic formulae over variables $p_0, p_1, \ldots$, and any propositional logic formula φ over those variables, if $\Sigma \models \varphi$, then $\Sigma \vdash \varphi$.*

Proof. The proof of the case where Σ is finite has already been covered in class. We will cover the general case. Firstly, note that since the number of propositional variables is countably infinite, the number of possible propositional logic formulae over these variables is also countably infinite. Therefore, Σ is a countable set. Since the set of all propositional logic formulae is countably infinite, let us enumerate them as $\psi_0, \psi_1, \ldots$.
We will prove the completeness theorem by showing that if $\Sigma \nvdash \varphi$ then $\Sigma \not\models \varphi$. Assume $\Sigma \nvdash \varphi$.
Now, we define the following sequence of sets inductively:

1. $\Sigma_0 = \Sigma$
2. $\forall n \in \mathbb{N}, \Sigma_{n+1} = \Sigma_n \cup \{\psi_n\}$ if $\Sigma_n \cup \{\psi_n\} \nvdash \varphi$
3. $\forall n \in \mathbb{N}, \Sigma_{n+1} = \Sigma_n$ if $\Sigma_n \cup \{\psi_n\} \vdash \varphi$

Set $\Sigma^* = \bigcup\limits_{n=0}^{\infty} \Sigma_n$. Note the following facts:

1. $\Sigma \subseteq \Sigma^*$ (since $\Sigma_0 = \Sigma$)
2. $\Sigma^* \nvdash \varphi$. This is due to the fact that for each $k \in \mathbb{N}, \Sigma_k \nvdash \varphi$. Note that if $\Sigma^* \vdash \varphi$, then since the proof would use only a finite number of formulae in Σ^*, we would have $\Sigma_{k+1} \vdash \varphi$ where k is the largest natural such that ψ_k is involved in the proof of $\Sigma^* \vdash \varphi$, contradicting how Σ_{k+1} was defined. Note that this means that $\varphi \notin \Sigma^*$.
3. If $\psi \notin \Sigma^*$, then $\Sigma^* \cup \{\psi\} \vdash \varphi$. This is as, if $\psi = \psi_k$, then we must have $\Sigma_k \cup \{\psi\} \vdash \varphi$, as otherwise we would have $\psi \in \Sigma^*$. Therefore we would have $\Sigma^* \cup \{\psi\} \vdash \varphi$, since $\Sigma_k \subseteq \Sigma^*$. This means that Σ^* is the maximal set of formulae containing Σ that cannot prove φ[1].
4. For each formula ψ, exactly one of ψ and $\neg\psi$ lies in Σ^*. This is as if both ψ and $\neg\psi$ lie in Σ^*, then we would have $\Sigma^* \vdash \varphi$ and $\Sigma^* \vdash \neg\varphi$, from which we would get $\Sigma^* \vdash \perp$ by $\perp$-introduction, from which we would get $\Sigma^* \vdash \varphi$ by $\perp$-elimination, a contradiction.

   On the other hand if neither ψ nor $\neg\psi$ lie in Σ^*, this means that (assuming $\psi = \psi_{k_1}$ and $\neg\psi = \psi_{k_2}$) $\Sigma_{k_1} \cup \{\psi\} \vdash \varphi$ and $\Sigma_{k_2} \cup \{\neg\psi\} \vdash \varphi$, which means $\Sigma^* \cup \{\psi\} \vdash \varphi$ and $\Sigma^* \cup \{\neg\psi\} \vdash \varphi$. By the law of the excluded middle, $\Sigma^* \vdash \psi \vee \neg\psi$. Applying $\vee$-elimination gives us $\Sigma^* \vdash \varphi$, a contradiction.

Consider the assignment $\alpha : \{p_0, p_1, \ldots\} \to \{0, 1\}$ where $\alpha(p_i) = 1$ if $p_i \in \Sigma^*$ and $\alpha(p_i) = 0$ if $p_i \notin \Sigma^*$ (which would mean $\neg p_i \in \Sigma^*$). We claim that $\alpha \models \Sigma^*$.
If this were not the case, then there would be some $\psi \in \Sigma^*$ such that $\alpha \not\models \psi$. Say all the propositional variables in ψ are from $p_0, \ldots p_n$. Let $\hat{p}_i$ be p_i if $\alpha(p_i) = 1$ and $\neg p_i$ if $\alpha(p_i)$ for each $i \in \mathbb{N}$. We have $\{\hat{p}_0, \ldots \hat{p}_n\} \vdash \neg\psi$. This is proven by structural induction on ψ, and was crucial to the proof of the finite case of the completeness theorem, as covered in class. Now, note that for each $i \in \mathbb{N}$, $\hat{p} \in \Sigma^*$. Therefore, we have $\Sigma^* \vdash \neg\psi$. But since $\psi \in \Sigma^*$ we also have $\Sigma^* \vdash \psi$, and by $\perp$-introduction we get $\Sigma^* \vdash \perp$, after which $\perp$-elimination gives us $\Sigma^* \vdash \varphi$, a contradiction. Therefore, $\alpha \models \Sigma^*$.
Now, for any formula $\psi \notin \Sigma^*$, we will have $\neg\psi \in \Sigma^*$, and hence $\alpha \models \neg\psi$, ie $\alpha \not\models \psi$. Since $\varphi \notin \Sigma^*$, we have $\alpha \not\models \varphi$. Now, since $\alpha \models \Sigma^*$ and $\Sigma \subseteq \Sigma^*$ we also have $\alpha \models \Sigma$. Therefore, we cannot have $\Sigma \models \varphi$, ie $\Sigma \not\models \varphi$. $\square$

[1] One can prove the completeness theorem even in the case where the number of propositional variables is uncountable by showing the existence of such a maximal set.