6 Behavior strategies and Kuhn's Theorem

Chapter summary

In strategic-form games, a mixed strategy extends the player's possibilities by allowing him to choose a pure strategy randomly. In extensive-form games random choices can be executed in two ways. The player can randomly choose a pure strategy for the whole play at the outset of the game; this type of randomization yields in fact the concept of mixed strategy in an extensive-form game. Alternatively, at every one of his information sets, the player can randomly choose one of his available actions; this type of randomization yields the concept of *behavior strategy*, which is the subject of this chapter.

We study the relationship between behavior strategies and mixed strategies in extensive-form games. To this end we define an *equivalence* relation between strategies and we show by examples that there are games in which some mixed strategies do not have equivalent behavior strategies, and there are games in which some behavior strategies do not have equivalent mixed strategies. We then introduce the concept of *perfect recall*: a player has perfect recall in an extensive-form game if along the play of the game he does not forget any information that he knew in the past (regarding his moves, the other players' moves, or chance moves). We prove Kuhn's Theorem, which states that if a player has perfect recall, then any one of his behavior strategies is equivalent to a mixed strategy, and vice versa. It follows that a game in which all players have perfect recall possesses an equilibrium in behavior strategies.

As noted in previous chapters, extensive-form games and strategic-form games are not related in a one-to-one manner. In general, the extensive form is richer in detail, and incorporates "dynamic aspects" of the game that are not expressed in strategic form. Strategic-form games focus exclusively on strategies and outcomes. Given this, it is worthwhile to take a closer look at the concepts developed for the two forms of games and detect differences between them, if there are any, due to the different representations of the game. We have already seen that the concept of pure strategy, which is a fundamental element of strategic-form games, is also well defined in extensive-form games, where a pure strategy of a player is a function that maps each of his information sets to an action that is feasible at that information set.

In this chapter (only), we will denote the multilinear extension (expectation) of player i's payoff function by u_i, rather than U_i, because U_i will denote an information set of player i.

Example 6.1 Consider the two-player extensive-form game given in Figure 6.1.

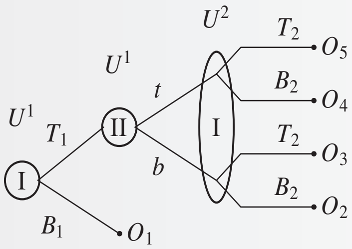

Figure 6.1 The game in Example 6.1

In this game, Player I has two information sets, U_{I}^1 and U_{I}^2, and four pure strategies:

$$S_{\mathrm{I}} = \{T_1T_2, T_1B_2, B_1T_2, B_1B_2\}. \tag{6.1}$$

Player II has one information set, U_{II}^1, and two pure strategies:

$$S_{\mathrm{II}} = \{t, b\}. \tag{6.2}$$

Mixed strategies are defined as probability distributions over sets of pure strategies. The concept of mixed strategy is therefore well defined in every game in which the set of pure strategies is a finite or countable set, whether the game is an extensive-form game or a strategic-form game. The sets of mixed strategies are[1]

$$\Sigma_{\mathrm{I}} = \Delta(S_{\mathrm{I}}), \qquad \Sigma_{\mathrm{II}} = \Delta(S_{\mathrm{II}}). \tag{6.3}$$

One of the interpretations of the concept of mixed strategy is that it is a random choice of how to play the game. But there may be different ways of attaining such randomness. Clearly, if a player has only one move (only one information set), such as Player II in the game in Figure 6.1, there is only one way to implement a random choice of an action: to pick t with probability α, and b with probability $1-\alpha$. That does indeed define a mixed strategy.

What about Player I, who has two information sets in the game in Figure 6.1? Suppose that he implements a mixed strategy, such as, for example, $\sigma_{\mathrm{I}} = [\frac{1}{3}(T_1T_2), 0(T_1B_2), \frac{1}{3}(B_1T_2), \frac{1}{3}(B_1B_2)]$. Then he is essentially conducting a lottery at the start of the game, and then implementing the pure strategy that has been chosen by lottery.

However, Player I has another, equally natural, alternative way to attain randomness: he can choose randomly between T_1 and B_1 when the play of the game arrives at his information set U_{I}^1, and then choose randomly between T_2 and B_2 when the play of the game arrives at his information set U_{I}^2. Such a strategy is described by two lotteries: $[\alpha(T_1), (1-\alpha)(B_1)]$ at U_{I}^1, and $[\beta(T_2), (1-\beta)(B_2)]$ at U_{I}^2. In other words, instead of randomly choosing a grand plan (a pure strategy) that determines his actions at each of his information sets, the player randomly chooses his action every time he is at a particular information set. Such a strategy is called a *behavior strategy*. ◀

Is there an essential difference between these two strategies? Can a player attain a higher payoff by using a behavior strategy instead of a mixed strategy? Alternatively, can he attain a higher payoff by using a mixed strategy instead of a behavior strategy? We

1 Recall that for every finite set S, $\Delta(S)$ is the set of all probability distributions over S, (Definition 5.1, page 146).

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will answer these questions in this chapter, and find conditions under which it makes no difference which of these alternative strategy concepts is used.

6.1 Behavior strategies

Definition 6.2 *A* behavior strategy *of a player in an extensive-form game is a function mapping each of his information sets to a probability distribution over the set of possible actions at that information set.*

Recall that we denote by $\mathcal{U}_i$ the collection of information sets of player i, and for every information set $U_i \in \mathcal{U}_i$, we denote by $A(U_i)$ the set of possible actions at U_i. A behavior strategy of player i in an extensive-form game is a function $b_i : \mathcal{U}_i \to \cup_{U_i \in \mathcal{U}_i} \Delta(A(U_i))$ such that $b_i(U_i) \in \Delta(A(U_i))$ for all $U_i \in \mathcal{U}_i$. Equivalently, a behavior strategy is a vector of probability distributions (lotteries), one per information set. This is in contrast with the single probability distribution (single lottery) defining a mixed strategy. The probability that a behavior strategy b_i will choose an action $a_i \in A(U_i)$ at an information set U_i is denoted by $b_i(a_i; U_i)$.

Recall that Σ_i is player i's set of mixed strategies; player i's set of behavior strategies is denoted by $\mathcal{B}_i$. What is the relationship between $\mathcal{B}_i$ and Σ_i? Note first that in every case in which player i has at least two information sets at which he has at least two possible actions, the sets $\mathcal{B}_i$ and Σ_i are different mathematical structures – two sets in different spaces. This is illustrated in Example 6.1.

Example 6.1 (*Continued*) As noted above, in this example Player I's behavior strategy is described by two lotteries: $b_{\mathrm{I}} = ([\alpha(T_1), (1-\alpha)(B_1)], [\beta(T_2), (1-\beta)(B_2)])$. Equivalently, we can describe this behavior strategy by a pair of real numbers α, β in the unit interval. The set $\mathcal{B}_{\mathrm{I}}$ is thus equivalent to the set

$$\{(\alpha, \beta) \colon 0 \le \alpha \le 1, 0 \le \beta \le 1\}, \tag{6.4}$$

while Σ_{I} is equivalent to the set

$$\left\{ (x_1, x_2, x_3, x_4) \colon x_j \ge 0, \sum_{j=1}^{4} x_j = 1 \right\}. \tag{6.5}$$

In other words, Σ_{I} is equivalent to a subset of $\mathbb{R}^4$ (which is three-dimensional, due to the constraint $\sum_{j=1}^4 x_j = 1$). By contrast, $\mathcal{B}_{\mathrm{I}}$ is equivalent to a subset of $\mathbb{R}^2$: the unit square $[0, 1]^2$. The fact that Σ_{I} is of higher dimension than $\mathcal{B}_{\mathrm{I}}$ (three dimensions versus two dimensions) suggests that Σ_{I} may be a "richer," or a "larger," set.

In fact, in this example, for every behavior strategy one can define an "equivalent" mixed strategy: the behavior strategy

$$([\alpha(T_1), (1-\alpha)(B_1)], [\beta(T_2), (1-\beta)(B_2)]) \tag{6.6}$$

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is equivalent to the mixed strategy

$$[\alpha\beta(T_1T_2), \alpha(1-\beta)T_1B_2, (1-\alpha)\beta B_1T_2, (1-\alpha)(1-\beta)B_1B_2]. \tag{6.7}$$

The sense in which these two strategies are equivalent is as follows: for each one of Player II's mixed strategies, the probability of reaching a particular vertex of the tree when Player I uses the behavior strategy (α, β) of Equation (6.6) equals the probability of reaching that vertex when Player I uses the mixed strategy of Equation (6.7). ◀

To define formally the equivalence between a mixed strategy and a behavior strategy, we consider strategy vectors that consist of both mixed strategies and behavior strategies.

Definition 6.3 *A* mixed/behavior strategy vector *is a vector of strategies $\sigma = (\sigma_i)_i$ in which σ_i can be either a mixed strategy or a behavior strategy of player i, for each i.*

For every mixed/behavior strategy vector $\sigma = (\sigma_i)_{i\in N}$ and every vertex x in the game tree, denote by $\rho(x;\sigma)$ the probability that vertex x will be visited during the course of the play of the game when the players implement strategies $(\sigma_i)_{i\in N}$.

Example 6.4 Consider the two-player game depicted in Figure 6.2. In this figure, the vertices of the tree are denoted by $x_1, x_2, \ldots, x_{17}$.

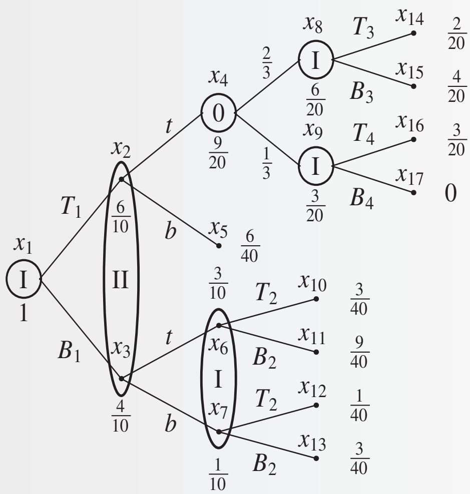

Figure 6.2 A two-player game, including the probabilities of arriving at each vertex

Suppose that the players implement the following mixed strategies:

$$\sigma_{\mathrm{I}} = \left[\frac{3}{10}(B_1B_2B_3B_4), \frac{1}{10}(B_1T_2B_3B_4), \frac{4}{10}(T_1B_2B_3T_4), \frac{2}{10}(T_1B_2T_3T_4)\right], \tag{6.8}$$

$$\sigma_{\mathrm{II}} = \left[\frac{3}{4}(t), \frac{1}{4}(b)\right]. \tag{6.9}$$

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Given these mixed strategies, we have computed the probabilities that the play of the game will reach the various vertices of the game tree, and these probabilities are listed alongside the vertices in Figure 6.2. We began at the leaves of the tree; for example, the probability of arriving at leaf x_{13} is the probability that Player I will play B_1 at vertex x_1, and B_2 at information set $\{x_6, x_7\}$, and that Player II will play b at information set $\{x_2, x_3\}$. From among the four pure strategies of Player I for which σ_{I} assigns positive probability, Player I will play B_1 at vertex x_1 and B_2 at information set $\{x_6, x_7\}$ only at the pure strategy $(B_1 B_2 B_3 B_4)$, with this pure strategy chosen by σ_{I} with probability $\frac{3}{10}$. Since the mixed strategies of the two players (which are probability distributions over their pure strategy sets) are independent, the probability that the play of the game will reach the leaf x_{13} is $\frac{3}{10} \times \frac{1}{4} = \frac{3}{40}$. We compute the probability of getting to a vertex that is not a leaf by recursion from the leaves to the root: the probability of getting to a vertex x is the sum of the probabilities of getting to one of the children of x. ◀

Definition 6.5 *A mixed strategy σ_i and a behavior strategy b_i of player i in an extensive-form game are* equivalent *to each other if for every mixed/behavior strategy vector σ_{-i} of the players $N \setminus \{i\}$ and every vertex x in the game tree*

$$\rho(x; \sigma_i, \sigma_{-i}) = \rho(x; b_i, \sigma_{-i}). \tag{6.10}$$

In other words, the mixed strategy σ_i and the behavior strategy b_i are equivalent if for every mixed/behavior strategy vector σ_{-i}, the two strategy vectors (σ_i, σ_{-i}) and (b_i, σ_{-i}) induce the same probability of arriving at each vertex in the game tree. In particular, $\rho(x; \sigma_i, \sigma_{-i}) = \rho(x; b_i, \sigma_{-i})$ for every leaf x. The probability $\rho(x; \sigma)$ that the vertex x will be visited during a play of the game equals the sum of the probabilities that the leaves that are descendants of x will be visited. It follows that to check that Equation (6.10) holds for every vertex x it suffices to check that it holds for every leaf of the game tree. It further follows from the definition that when the behavior strategy b_i is equivalent to the mixed strategy σ_i, then for every mixed/behavior strategy vector σ_{-i} of the other players the two strategy vectors (σ_i, σ_{-i}) and (b_i, σ_{-i}) lead to the same expected payoff (Exercise 6.6).

Theorem 6.6 *If a mixed strategy σ_i of player i is equivalent to a behavior strategy b_i, then for every mixed/behavior strategy vector σ_{-i} of the other players and every player $j \in N$,*

$$u_j(\sigma_i, \sigma_{-i}) = u_j(b_i, \sigma_{-i}). \tag{6.11}$$

Repeated application of Theorem 6.6 leads to the following corollary.

Corollary 6.7 *Let $\sigma = (\sigma_i)_{i \in N}$ be a vector of mixed strategies. For each player i, let b_i be a behavior strategy that is equivalent to σ_i, and denote $b = (b_i)_{i \in N}$. Then, for each player i,*

$$u_i(\sigma) = u_i(b). \tag{6.12}$$

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Example 6.4 (*Continued*) Given the probabilities calculated in Figure 6.2, the behavior strategy b_{I} defined by

$$b_{\mathrm{I}} = \left(\left[\tfrac{3}{5}(T_1), \tfrac{2}{5}(B_1)\right], \left[\tfrac{1}{4}(T_2), \tfrac{3}{4}(B_2)\right], \left[\tfrac{1}{3}(T_3), \tfrac{2}{3}(B_3)\right], \left[1(T_4), 0(B_4)\right]\right) \tag{6.13}$$

is equivalent to the mixed strategy σ_{I} defined in Equation (6.8),

$$\sigma_{\mathrm{I}} = \left[\tfrac{3}{10}(B_1B_2B_3B_4), \tfrac{1}{10}(B_1T_2B_3B_4), \tfrac{4}{10}(T_1B_2B_3T_4), \tfrac{2}{10}(T_1B_2T_3T_4)\right]. \tag{6.14}$$

To see how behavior strategy b_{I} was computed from the mixed strategy σ_{I}, suppose that Player II implements strategy $\sigma_{\mathrm{I}} = [\frac{3}{4}(t), \frac{1}{4}(b)]$. The probability that the play of the game will arrive at each vertex x appears in the game tree in Figure 6.2. If behavior strategy b_{I} is equivalent to the mixed strategy σ_{I}, then the probability that an action in a particular information set is chosen is the ratio between the probability of arriving at the vertex that leads to that action and the probability of arriving at the vertex at which the action is chosen. For example, in order to compute the probability at which the action B_2 is chosen in the information set $\{x_6, x_7\}$, we divide the probability $\frac{3}{40}$ of reaching vertex x_{13} by the probability $\frac{1}{10}$ of reaching vertex x_7, to obtain $\frac{3/40}{1/10} = \frac{3}{4}$, corresponding to $[\frac{1}{4}(T_2), \frac{3}{4}(B_2)]$ in strategy b_{I} (we obtain a similar result, of course, if we divide the probability $\frac{9}{40}$ of reaching vertex x_{11} by the probability $\frac{3}{10}$ of reaching vertex x_6). To complete the construction of b_{I} from the mixed strategy σ_{I}, similar computations need to be conducted at Player I's other information sets, and it must be shown that these computations lead to the same outcome for all strategies $[\alpha(t), (1-\alpha)(b)]$ of Player II (Exercise 6.7). ◀

Using a behavior strategy, instead of a mixed strategy, may be advantageous for two reasons: first, the set $\mathcal{B}_i$ is "smaller," and defined by fewer parameters, than the set Σ_i. For example, if the player has four information sets, with two actions at each information set (as happens in Example 6.4), the total number of pure strategies available is $2^4 = 16$, so that a mixed strategy involves 15 variables, as opposed to a behavior strategy, which involves only four variables (namely, the probability of selecting the first action in each one of the information sets). Secondly, in large extensive-form games, behavior strategies appear to be "more natural," because in behavior strategies, players choose randomly between their actions at each information set at which they find themselves, rather than making one grand random choice of a "master plan" (i.e., a pure strategy) for the entire game, all at once. This motivates the questions of whether each mixed strategy has an equivalent behavior strategy, and whether each behavior strategy has an equivalent mixed strategy. As the next two examples show, the answers to both questions may, in general, be negative.

Example 6.8 **A mixed strategy that has no equivalent behavior strategy** Consider the game in Figure 6.3, involving only one player.

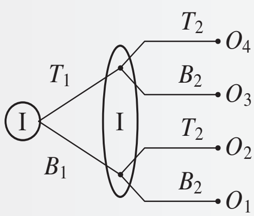

Figure 6.3 A game with a mixed strategy that has no equivalent behavior strategy

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There are four pure strategies, $\{T_1T_2, T_1B_2, B_1T_2, B_1B_2\}$. We will show that there is no behavior strategy that is equivalent to the mixed strategy $\sigma_{\mathrm{I}} = [\frac{1}{2}(T_1T_2), 0(T_1B_2), 0(T_2B_1), \frac{1}{2}(B_1B_2)]$. This mixed strategy induces the following probability distribution over the outcomes of the game:

$$\left[\tfrac{1}{2}(O_1), 0(O_2), 0(O_3), \tfrac{1}{2}(O_4)\right]. \tag{6.15}$$

A behavior strategy $\big([\alpha(T_1), (1-\alpha)(B_1)], [\beta(T_2), (1-\beta)(B_2)]\big)$ induces the following probability distribution over the outcomes of the game:

$$[(1-\alpha)(1-\beta)(O_1), (1-\alpha)\beta(O_2), \alpha(1-\beta)(O_3), \alpha\beta(O_4)]. \tag{6.16}$$

If this behavior strategy were equivalent to the mixed strategy σ_{I}, they would both induce the same probability distributions over the outcomes of the game, so that the following equalities would have to obtain:

$$\alpha\beta = \tfrac{1}{2}, \tag{6.17}$$

$$\alpha(1-\beta) = 0, \tag{6.18}$$

$$(1-\alpha)\beta = 0, \tag{6.19}$$

$$(1-\alpha)(1-\beta) = \tfrac{1}{2}. \tag{6.20}$$

But this system of equations has no solution: Equation (6.18) implies that either $\alpha = 0$ or $\beta = 1$. If $\alpha = 0$, Equation (6.17) does not hold, and if $\beta = 1$, Equation (6.20) does not hold. ◀

Example 6.9 **The Absent-Minded Driver: a game with a behavior strategy that has no equivalent mixed strategy** Consider the game in Figure 6.4, involving only one player, Player I. In this game, the player, when he comes to choosing an action, cannot recall whether or not he has chosen an action in the past. An illustrative story that often accompanies this example is that of an absent-minded driver, motoring down a road with two exits. When the driver arrives at an exit, he cannot recall whether it is the first exit on the road, or the second exit.

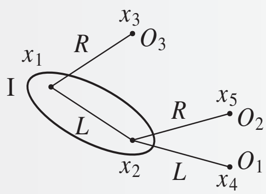

Figure 6.4 The Absent-Minded Driver game

There are two pure strategies: T and B. The pure strategy T yields the outcome O_3, while the pure strategy B yields the outcome O_1. Since a mixed strategy is a probability distribution over the set of pure strategies, no mixed strategy can yield the outcome O_2 with positive probability.

In contrast, the behavior strategy $[\frac{1}{2}(T), \frac{1}{2}(B)]$, where the player chooses one of the two actions with equal probability at each of the two vertices in his information set, leads to the following probability distribution over outcomes:

$$\left[\tfrac{1}{4}(O_1), \tfrac{1}{4}(O_2), \tfrac{1}{2}(O_3)\right]. \tag{6.21}$$

Since this probability distribution can never be the result of implementing a mixed strategy, we conclude that there is no mixed strategy equivalent to this behavior strategy. ◀

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6.2 Kuhn's Theorem

Let us note that the player suffers from forgetfulness of a different kind in each of the above examples: in Example 6.8, when the player is about to take an action the second time, he cannot recall what action he chose the first time; he knows that he has made a previous move, but cannot recall what action he took. In Example 6.9, the player does not even recall whether or not he has made a move in the past (although he does know that if he did make a prior move, he necessarily must have chosen action B). What happens when the player is not forgetful? Will this ensure that every behavior strategy has an equivalent mixed strategy, and that every mixed strategy has an equivalent behavior strategy? As we will show in this section, the answer to these questions is affirmative.

6.2.1 Conditions for the existence of an equivalent mixed strategy to any behavior strategy

Let x be a vertex in the game tree that is not the root, and let x_1 be a vertex on the path from the root to x. The (unique) edge emanating from x_1 on the path from the root to x is called *the action at* x_1 *leading to* x.

A pure strategy selects the same action at every vertex in each one of the corresponding player's information sets. It follows that if the path from the root to x passes through two vertices x_1 and $\widehat{x}_1$ that are in the same information set of player i, and if the action at x_1 leading to x differs from the action at $\widehat{x}_1$ leading to x, then when player i implements a pure strategy the play of the game cannot arrive at x. For this reason, in Example 6.9 there is no pure strategy leading to the vertex x_5. Since a mixed strategy is a probability distribution over pure strategies, the probability that a play of the game will arrive at such a vertex x is 0 when player i implements any mixed strategy. In contrast, if all the players implement behavior strategies in which at every information set every possible action is played with positive probability, then for each vertex in the game tree there is a positive probability that the play of the game will reach that vertex. This leads to the following conclusion (Exercise 6.8).

Corollary 6.10 *If there exists a path from the root to some vertex x that passes at least twice through the same information set U_i of player i, and if the action leading in the direction of x is not the same action at each of these information sets, then player i has a behavior strategy that has no equivalent mixed strategy.*

The last corollary will be used to prove the next theorem, which gives a necessary and sufficient condition for the existence of a mixed strategy equivalent to every behavior strategy. If every path emanating from the root passes through each information set at most once, then every behavior strategy has an equivalent mixed strategy.

Theorem 6.11 *Let $\Gamma = (N, V, E, v_0, (V_i)_{i \in N \cup \{0\}}, (p_x)_{x \in V_0}, (\mathcal{U}_i)_{i \in N}, O, u)$ be an extensive-form game that satisfies the condition that at every vertex there are at least two actions. Every behavior strategy of player i has an equivalent mixed strategy if and only if each information set of player i intersects every path emanating from the root at most once.*

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In the game in Example 6.9, there is a path that twice intersects the same information set, and we indeed identified a behavior strategy of that game that has no equivalent mixed strategy. The theorem does not hold without the condition that there are at least two actions at each vertex (Exercise 6.9). We first prove that the condition in the statement of the theorem is necessary.

Proof of Theorem 6.11: the condition is necessary Suppose that there exists a path from the root to a vertex x that intersects the same information set U_i of player i at least twice. We will prove that there is a behavior strategy of player i that has no mixed strategy equivalent to it. Let x_1 and $\widehat{x}_1$ be two distinct vertices in the above-mentioned information set that are located along the path (see Figure 6.5). Denote by a the action at x_1 leading to x, and by b an action at x_1 that differs from a. Let x_2 be the vertex that the play of the game reaches if at vertex $\widehat{x}_1$ player i chooses action b.

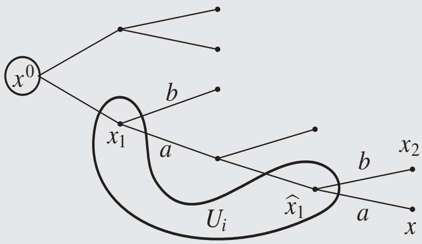

Figure 6.5 The game tree in the proof of Theorem 6.11

The path from the root to x_2 passes through the vertices x_1 and $\widehat{x}_1$, the action at x_1 leading to x_2 is a, and the action at $\widehat{x}_1$ leading to x_2 is b. By Corollary 6.10 it follows that there is a behavior strategy of player i that has no mixed strategy equivalent to it, which is what we needed to show. □

We now explain the idea underlying the proof of the second direction. The proof itself will be presented in Section 6.2.3 after we introduce several definitions. Let b_i be a behavior strategy of player i. When the play of the game arrives at information set U_i, player i conducts a lottery based on the probability distribution $b_i(U_i)$ to choose one of the actions available at information set U_i. Player i could just as easily conduct this lottery at the start of the game, instead of waiting until he gets to the information set U_i. In other words, at the start of the game, the player can conduct a lottery for each one of his information sets U_i, using in each case the probability distribution $b_i(U_i)$, and then play the action thus chosen at each information set, respectively, if and when the play of the game reaches it. Since all the lotteries are conducted at the start of the game, we have essentially defined a mixed strategy that is equivalent to b_i.

This construction would not be possible without the condition that any path from the root intersects every information set at most once. Indeed, if there were a path intersecting the same information set of player i several times, then the mechanism described in the previous paragraph would require player i to choose the same action every time he gets

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to that information set. In contrast, a behavior strategy enables the player to choose his actions at the information sets independently every time the play of the game arrives at the information set. It follows that in this case the mixed strategy that the process defines is not equivalent to the behavior strategies b_i. Before we prove the other direction of Theorem 6.11 (sufficiency), we present $\rho(x;\sigma)$, the probability that the play of the game reaches vertex x, as the product of probabilities, each of which depends solely on one player.

This representation will serve us in several proofs in this section, as will the notation that we now introduce.

6.2.2 Representing $\rho(x;\sigma)$ as a product of probabilities

For each decision vertex x of player i, denote by $U_i(x) \in \mathcal{U}_i$ the information set of player i containing x. For each descendant $\widehat{x}$ of x denote by $a_i(x \to \widehat{x}) \in A(U_i(x))$ the equivalence class containing the action leading from x to $\widehat{x}$. This is the action that player i must choose at vertex x for the play of the game to continue in the direction of vertex $\widehat{x}$.

For each vertex x (not necessarily a decision vertex of player i) denote the number of vertices along the path from the root to x (not including x) at which player i is the decision maker by L_i^x, and denote these nodes by $x_i^1, x_i^2, \dots, x_i^{L_i^x}$. In Example 6.4, $L_{\mathrm{I}}^{x_{10}} = 2$, $x_{\mathrm{I}}^1 = x_1$, $x_{\mathrm{I}}^2 = x_6$, and

$$U_{\mathrm{I}}(x_1) = \{x_1\}, \qquad U_{\mathrm{I}}(x_{10}) = \{x_6, x_7\}. \tag{6.22}$$

Since an information set can contain several vertices on the path from the root to x, as happens in the Absent-Minded Driver game (Example 6.9), it is possible that $U_i(x_i^{l_1}) = U_i(x_i^{l_2})$ even when $l_1 \neq l_2$. In Example 6.9, $L_{\mathrm{I}}^{x_4} = 2$ and

$$U_{\mathrm{I}}\left(x_4^1\right) = U_{\mathrm{I}}\left(x_4^2\right) = \{x_1, x_2\}. \tag{6.23}$$

What is the probability that under the strategy implemented by player i, he will choose the action leading to x at each one of the information sets preceding x? If player i implements behavior strategy b_i, this probability equals

$$\rho_i(x; b_i) := \begin{cases} \prod_{l=1}^{L_i^x} b_i\left(a_i\left(x_i^l \to x\right); U_i\left(x_i^l\right)\right) & \text{if } L_i^x > 0, \\ 1 & \text{if } L_i^x = 0. \end{cases} \tag{6.24}$$

If player i implements the mixed strategy σ_i, then $\sigma_i(s_i)$ is the probability that he chooses pure strategy s_i. Denote by $S_i^*(x) \subseteq S_i$ all of player i's pure strategies under which at each information set $U_i(x_i^l)$, $1 \le l \le L_i^x$, he chooses the action $a(U_i(x_i^l) \to x)$. The set $S_i^*(x)$ may be empty; since a pure strategy cannot choose two different actions at the same information set, this happens when the path from the root to x passes at least twice through the same information set of player i, and the action leading to x is not the same action in every case.

When $S_i^*(x) \neq \emptyset$, the probability that player i chooses the actions leading to vertex x is

$$\rho_i(x;\sigma_i) := \sum_{s_i \in \mathcal{S}_i^*(x)} \sigma_i(s_i). \tag{6.25}$$

When $S_i^*(x) = \emptyset$, this probability is defined by $\rho_i(x;\sigma_i) := 0$. Because the lotteries conducted by the players are independent, we get that for each mixed/behavior strategy

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vector σ and every vertex x,

$$\rho(x;\sigma) = \prod_{i \in N} \rho_i(x;\sigma_i). \tag{6.26}$$

We turn now to the proof of the second direction of Theorem 6.11.

6.2.3 Proof of the second direction of Theorem 6.11: sufficiency

We want to prove that if every path intersects each information set of player i at most once, then every mixed strategy of player i has an equivalent behavior strategy.

A pure strategy of player i is a choice of an action from his action set at each of his information sets. Hence the set of pure strategies of player i is

$$S_i = \bigtimes_{U_i \in \mathcal{U}_i} A(U_i). \tag{6.27}$$

For every pure strategy s_i of player i, and every information set U_i, the action that the player chooses at U_i is $s_i(U_i)$. It follows that for every behavior strategy b_i and every pure strategy s_i of player i, $b_i(s_i(U_i);U_i)$ is the probability that under behavior strategy b_i, at each time that the play of the game reaches a vertex in information set U_i, player i chooses the same action that s_i chooses at this information set.

Given a behavior strategy b_i of player i, we will now define a mixed strategy σ_i that is equivalent to b_i. For every pure strategy s_i of player i define the "probability that this strategy is chosen according to b_i" as

$$\sigma_i(s_i) := \prod_{U_i \in \mathcal{U}_i} b_i(s_i(U_i);U_i). \tag{6.28}$$

First, we will show that $\sigma_i := (\sigma_i(s_i))_{s_i \in S_i}$ is a probability distribution over S_i, and hence it defines a mixed strategy for player i. Since $\sigma_i(s_i)$ is a product of nonnegative numbers, $\sigma_i(s_i) \geq 0$ for every pure strategy $s_i \in S_i$. We now verify that $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$. Indeed,

$$\sum_{s_i \in S_i} \sigma_i(s_i) = \sum_{s_i \in S_i} \left(\prod_{U_i \in \mathcal{U}_i} b_i(s_i(U_i);U_i) \right) \tag{6.29}$$

$$= \prod_{U_i \in \mathcal{U}_i} \sum_{a_i \in A(U_i)} b_i(a_i;U_i) \tag{6.30}$$

$$= \prod_{U_i \in \mathcal{U}_i} 1 = 1. \tag{6.31}$$

Equation (6.30) follows from changing the order of the product and the summation and from the assumption that every path intersects every information set at most once.

Finally, we need to check that the mixed strategy σ_i is equivalent to b_i. Let x be a vertex. We will show that for each mixed/behavior strategy vector σ_{-i} of players $N \setminus \{i\}$,

$$\rho(x;b_i,\sigma_{-i}) = \rho(x;\sigma_i,\sigma_{-i}). \tag{6.32}$$

From Equation (6.26), we deduce that

$$\rho(x;b_i,\sigma_{-i}) = \rho_i(x;b_i) \times \prod_{j \neq i} \rho_j(x;\sigma_j), \tag{6.33}$$

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and

$$\rho(x;\sigma_i,\sigma_{-i}) = \rho_i(x;\sigma_i) \times \prod_{j\neq i} \rho_j(x;\sigma_j). \tag{6.34}$$

It follows that in order to show that Equation (6.32) is satisfied, it suffices to show that

$$\rho_i(x;b_i) = \rho_i(x;\sigma_i). \tag{6.35}$$

Divide player i's collection of information sets into two: $\mathcal{U}_i^1$, containing all the information sets intersected by the path from the root to x, and $\mathcal{U}_i^2$, containing all the information sets that are not intersected by this path. Since $S_i^*(x)$ is the set of pure strategies of player i in which he implements the action leading to vertex x in all information sets intersected by the path from the root to x,

$$\begin{aligned}
\rho_i(x;\sigma_i) &= \sum_{s_i\in S_i^*(x)} \sigma_i(s_i) && (6.36)\\
&= \sum_{s_i\in S_i^*(x)} \prod_{U_i\in\mathcal{U}_i} b_i(s_i(U_i);U_i) && (6.37)\\
&= \sum_{s_i\in S_i^*(x)} \left(\prod_{U_i\in\mathcal{U}_i^1} b_i(s_i(U_i);U_i) \times \prod_{U_i\in\mathcal{U}_i^2} b_i(s_i(U_i);U_i)\right). && (6.38)
\end{aligned}$$

Since $\mathcal{U}_i^1$ contains only the information sets $U_i(x_i^1), U_i(x_i^2), \ldots, U_i(x_i^{L_i^x})$, and since for every $l \in \{1,2,\ldots,L_i^x\}$ the pure strategy $s_i \in S_i^*(x)$ instructs player i to play action $a(U_i(x_i^l) \to x)$ at information set $U_i(x_i^l)$, we deduce, using Equation (6.24), that

$$\prod_{U_i\in\mathcal{U}_i^1} b_i(s_i(U_i);U_i) = \prod_{l=1}^{L_i^x} b_i(a_i(x_i^l \to x); U_i(x_i^l)) = \rho_i(x;b_i). \tag{6.39}$$

In particular, this product is independent of $s_i \in S_i^*(s)$. We can therefore move the product outside of the sum in Equation (6.38), yielding

$$\rho_i(x;\sigma_i) = \rho_i(x;b_i) \times \left(\sum_{s_i\in S_i^*(x)} \prod_{U_i\in\mathcal{U}_i^2} b_i(s_i(U_i);U_i)\right). \tag{6.40}$$

We will now show that the second element on the right-hand side of Equation (6.40) equals 1. The fact that s_i is contained in $S_i^*(x)$ does not impose any constraints on the actions implemented by player i at the information sets in $\mathcal{U}_i^2$. For every sequence $(a_{U_i})_{U_i\in\mathcal{U}_i^2}$ at which $a_{U_i} \in A(U_i)$ is a possible action for player i at information set U_i for all $U_i \in \mathcal{U}_i^2$, there is a pure strategy $s_i \in S_i^*(x)$ such that $a_{U_i} = s_i(U_i)$ for all $U_i \in \mathcal{U}_i^2$. Moreover, there is an injective mapping between the set of pure strategies $S_i^*(x)$ and the set of the sequences

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$(a_{U_i})_{U_i \in \mathcal{U}_i^2} \in \times_{U_i \in \mathcal{U}_i^2} A(U_i)$. Therefore,

$$\sum_{s_i \in S_i^*(x)} \prod_{U_i \in \mathcal{U}_i^2} b_i(s_i(U_i); U_i) = \sum_{\left\{(a_{U_i})_{U_i \in \mathcal{U}_i^2} \in \times_{U_i \in \mathcal{U}_i^2} A(U_i)\right\}} \prod_{U_i \in \mathcal{U}_i^2} b_i(a_{U_i}; U_i)$$
$$= \prod_{U_i \in \mathcal{U}_i^2} \sum_{a_{U_i} \in A(U_i)} b_i(a_{U_i}; U_i) = \prod_{U_i \in \mathcal{U}_i^2} 1 = 1. \quad (6.41)$$

Equation (6.40) therefore implies that

$$\rho_i(x; \sigma_i) = \rho_i(x; b_i) \quad (6.42)$$

which is what we wanted to prove.

6.2.4 Conditions guaranteeing the existence of a behavior strategy equivalent to a mixed strategy

In this section, we present a condition guaranteeing that every mixed strategy has an equivalent behavior strategy. This requires formalizing when a player never forgets anything. During the play of a game, a player can forget many things:

- He can forget what moves he made in the past (as in Example 6.8).
- He can forget whether or not he made a move at all in the past (as in Example 6.9).
- He can forget things he knew at earlier stages of the games, such as the result of a chance move, what actions another player has played, which players acted in the past, or how many times a particular player played in the past.

The next definition guarantees that a player never forgets any of the items in the above list (Exercises 6.11–6.15). Recall that all the vertices in the same information set must have the same associated action set (Definition 3.23 on page 54).

Definition 6.12 *Let $\mathcal{X} = (x^0 \to x^1 \to \ldots \to x^K)$ and $\widehat{\mathcal{X}} = (x^0 \to \widehat{x}^1 \to \ldots \to \widehat{x}^L)$ be two paths*[2] *in the game tree. Let U_i be an information set of player i, which intersects each of these two paths at only one vertex: $\mathcal{X}$ at x^k, and $\widehat{\mathcal{X}}$ at $\widehat{x}^l$. We say that these two paths* choose the same action at information set U_i *if $k < K$, $l < L$, and the action at x^k leading to x^{k+1} is identical to the action at $\widehat{x}^l$ leading to $\widehat{x}^{l+1}$, i.e., $a_i(x^k \to x^{k+1}) = a_i(\widehat{x}^l \to \widehat{x}^{l+1})$.*

Definition 6.13 *Player i has* perfect recall *if the following conditions are satisfied:*

(a) Every information set of player i intersects every path from the root to a leaf at most once.
(b) Every two paths from the root that end in the same information set of player i pass through the same information sets of player i, and in the same order, and in every such information set the two paths choose the same action. In other words, for every information set U_i of player i and every pair of vertices $x, \widehat{x}$ in U_i, if the decision vertices of player i on the path from the root to x are $x_i^1, x_i^2, \ldots, x_i^L = x$ and his

2 In the description of a path in a game tree we list only the vertices, because the edges along the path are uniquely determined by those vertices.

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decision vertices on the path from the root to $\widehat{x}$ are $\widehat{x}_i^1, \widehat{x}_i^2, \ldots, \widehat{x}_i^{\widehat{L}} = \widehat{x}$, then $L = \widehat{L}$, and $U_i(x_i^l) = U_i(\widehat{x}_i^l)$, and $a_i(x_i^l \to x) = a_i(\widehat{x}_i^l \to \widehat{x})$ for all $l \in \{1, 2, \ldots, L\}$.

A game is called a game with perfect recall *if all the players have perfect recall.*

Two games are shown in Figure 6.6. In Game A, every player has a single information set, and all the players have perfect recall. In Game B, in contrast, Player I has imperfect recall, because the two paths connecting the root to the vertices in information set $\{x_3, x_4\}$ do not choose the same action in information set $\{x_1\}$. Player II, however, has perfect recall in this game.

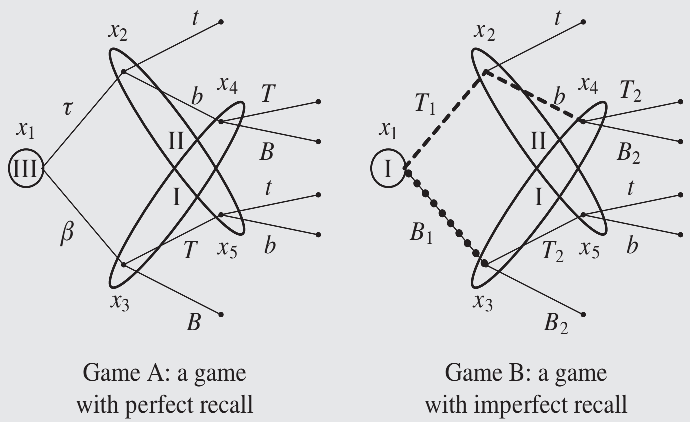

Figure 6.6 Two games in extensive form

Recall that $S_i^*(x)$ is the set of pure strategies of player i at which he chooses the actions leading to vertex x (see page 228). The definition of perfect recall implies the following corollary (Exercise 6.16).

Theorem 6.14 *Let i be a player with perfect recall in an extensive-form game, and let x and x' be two vertices in the same information set of player i. Then $S_i^*(x) = S_i^*(x')$.*

Theorem 6.15 (Kuhn [1957]) *In every game in extensive form, if player i has perfect recall, then for every mixed strategy of player i there exists an equivalent behavior strategy.*

Proof: We make use of the following notation: for each vertex x of player i, and each possible action a in $A(U_i(x))$, we denote by x^a the vertex in the game tree that the play of the game reaches if player i chooses action a at vertex x.

Let σ_i be a mixed strategy of player i. Our goal is to define a behavior strategy b_i equivalent to σ_i.

Step 1: Defining a behavior strategy b_i.
To define a behavior strategy b_i we have to define, for each information set U_i of player i, a probability distribution over the set of possible actions at U_i.

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So suppose U_i is an information set of player i, and let x be a vertex in U_i. For each action $a_i \in A(U_i)$, the collection $S_i^*(x^{a_i})$ contains all the pure strategies s_i in $S_i^*(x)$ satisfying $s_i(U_i) = a_i$.

If $\sum_{s_i \in S_i^*(x)} \sigma_i(s_i) > 0$ define

$$b_i(a_i; U_i) := \frac{\sum_{s_i \in S_i^*(x^{a_i})} \sigma_i(s_i)}{\sum_{s_i \in S_i^*(x)} \sigma_i(s_i)}, \quad \forall a_i \in A(x). \tag{6.43}$$

The numerator on the right-hand side of Equation (6.43) is the probability that player i will play the actions leading to x^a, and the denominator is the probability that player i will play the actions leading to x. It follows that the ratio between the two values equals the conditional probability that player i plays action a if the play reaches vertex x.

If $\sum_{s_i \in S_i^*(x)} \sigma_i(s_i) = 0$, by Theorem 6.14 it follows that $\sum_{s_i \in S_i^*(x')} \sigma_i(s_i)$ for each vertex x' in the information set of x. Therefore, when player i implements σ_i the probability that the play of the game will visit the information set containing x is 0, i.e., $\rho_i(x; \sigma_i) = 0$. In this case, the definition of b_i, for information set U_i, makes no difference. For the definition of b_i to be complete, we define in this case

$$b_i(a_i; U_i) = \frac{1}{|A(U_i)|}, \quad \forall a_i \in A(x). \tag{6.44}$$

We now show that the definition of b_i is independent of the vertex x chosen in information set U_i, so that the behavior strategy b_i is well defined. It suffices to check the case $\sum_{s_i \in S_i^*(x)} \sigma_i(s_i) > 0$, because when $\sum_{s_i \in S_i^*(x)} \sigma_i(s_i) = 0$, the definition of b_i (see Equation (6.44)) is independent of x; it depends only on U_i. Let then x_1 and x_2 be two different vertices in U_i. Since player i has perfect recall, Theorem 6.14 implies that $S_i^*(x_1) = S_i^*(x_2)$. Since x_1 and x_2 are in the same information set, the set of possible actions at x_1 equals the set of possible actions at x_2: $A(x_1) = A(x_2)$. If a is a possible action at these vertices, x_1^a and x_2^a are the vertices reached by the play of the game from x_1 and from x_2 respectively, if player i implements action a at these vertices. Using Theorem 6.14 we deduce that $S_i^*(x_1^a) = S_i^*(x_2^a)$. In particular, it follows that the numerator and denominator of Equation (6.43) are independent of the choice of vertex x in U_i.

Step 2: Showing that b_i is a behavior strategy.
We need to prove that for every information set U_i of player i, $b_i(U_i)$ is a probability distribution over $A(U_i)$, i.e., that $b_i(U_i)$ is a vector of nonnegative numbers summing to one. Equation (6.44) defines a probability distribution over $A(U_i)$ for the case $\sum_{s_i \in S_i^*(x)} \sigma_i(s_i) = 0$. We show now that when $\sum_{s_i \in S_i^*(x)} \sigma_i(s_i) > 0$, Equation (6.43) defines a probability distribution over $A(U_i)$. Since $\sigma_i(s_i) \geq 0$ for every pure strategy s_i, the numerator in Equation (6.43) is nonnegative, and hence $b_i(a_i; U_i) \geq 0$ for every action $a_i \in A(U_i)$. The sets $\{S_i^*(x^a) \colon a \in A(U_i)\}$ are disjoint, and their union is $S_i^*(x)$. It follows that

$$\sum_{a \in A(U_i)} \sum_{s_i \in S_i^*(x^a)} \sigma_i(s_i) = \sum_{s_i \in S_i^*(x)} \sigma_i(s_i). \tag{6.45}$$

We deduce from Equations (6.43) and (6.45) that in this case $\sum_{a_i \in A(U_i)} b_i(a_i; U_i) = 1$.

Step 3: Showing that b_i is equivalent to σ_i.
Let σ_{-i} be a mixed/behavior strategy vector of the other players, and let x be a vertex in the game tree (not necessarily a decision vertex of player i). We need to show that

$$\rho(x; b_i, \sigma_{-i}) = \rho(x; \sigma_i, \sigma_{-i}). \tag{6.46}$$

As we saw previously, Equation (6.26) implies that

$$\rho(x; b_i, \sigma_{-i}) = \rho_i(x; b_i) \times \prod_{j \neq i} \rho_j(x; \sigma_j), \tag{6.47}$$

and

$$\rho(x; \sigma_i, \sigma_{-i}) = \rho_i(x; \sigma_i) \times \prod_{j \neq i} \rho_j(x; \sigma_j). \tag{6.48}$$

To show that Equation (6.46) is satisfied, it therefore suffices to show that

$$\rho_i(x; b_i) = \rho_i(x; \sigma_i). \tag{6.49}$$

In words, we need to show that the probability that player i will play actions leading to x under σ_i equals the probability that player i will do the same under b_i. Recall that $x_i^1, x_i^2, \ldots, x_i^{L_i^x}$ is the sequence of decision vertices of player i along the path from the root to x (not including the vertex x if player i is the decision maker there). If $L_i^x = 0$, then player i has no information set intersected by the path from the root to x, so $S_i^*(x) = S_i$. In this case, we have defined $\rho_i(x; b_i) = 1$ (see Equation (6.24)), and also

$$\rho_i(x; \sigma_i) = \sum_{s_i \in S_i^*(x)} \sigma_i(s_i) = \sum_{s_i \in S_i} \sigma_i(s_i) = 1. \tag{6.50}$$

Hence Equation (6.49) is satisfied.

Suppose, then, that $L_i^x > 0$. Every strategy of player i that chooses, at each information set $U_i(x_i^1), U_i(x_i^2), \ldots, U_i(x_i^l)$, the action leading to x is a strategy that does so at each information set $U_i(x_i^1), U_i(x_i^2), \ldots, U_i(x_i^{l-1})$ and at information set $U_i(x_i^l)$ chooses the action $a_l := a_i(x_i^l \to x)$. In other words,

$$S_i^*\left(x_i^{l+1}\right) = S_i^*\left(x_i^{l,a_l}\right). \tag{6.51}$$

Since b_i is a behavior strategy, Equation (6.24) implies that

$$\rho_i(x; b_i) = \prod_{l=1}^{L_i^x} b_i\left(a_l; U_i\left(x_i^l\right)\right). \tag{6.52}$$

If $\rho_i(x; b_i) \neq 0$, then the definition of b_i (Equation (6.43)) implies that

$$\rho_i(x; b_i) = \prod_{l=1}^{L_i^x} \frac{\sum_{s_i \in S_i^*(x_l^{a_l})} \sigma_i(s_i)}{\sum_{s_i \in S_i^*(x_l)} \sigma_i(s_i)}. \tag{6.53}$$

From Equation (6.51) we deduce that

$$\sum_{s_i \in S_i^*(x_l^{a_l})} \sigma_i(s_i) = \sum_{s_i \in S_i^*(x_{l+1})} \sigma_i(s_i). \tag{6.54}$$

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It follows that the product on the right-hand side of Equation (6.53) is a telescopic product: the numerator in the l-th element of the product equals the denominator in the $(l+1)$-th element of the product. This means that adjacent product elements cancel each other out. Note that $S_i^*(x_l^{a_l}) = S_i^*(x)$ is satisfied for $l = L_i^x$, so that canceling adjacent product elements in Equation (6.53) yields

$$\rho_i(x; b_i) = \frac{\sum_{s_i \in S_i^*(x)} \sigma_i(s_i)}{\sum_{s_i \in S_i^*(x_i^1)} \sigma_i(s_i)}. \tag{6.55}$$

Recall that x_i^1 is player i's first decision vertex on the path from the root to x. Since player i has no information set prior to x_i^1, every strategy of player i is in $S_i^*(x_i^1)$, i.e., $S_i^*(x_i^1) = S_i$. The denominator in Equation (6.55) therefore equals 1, so that

$$\rho_i(x; b_i) = \sum_{s_i \in S_i^*(x)} \sigma_i(s_i) = \rho_i(x; \sigma_i), \tag{6.56}$$

which is what we claimed.

To wrap up, we turn our attention to the case $\rho_i(x; b_i) = 0$. From Equation (6.24), we deduce that $\rho_i(x; b_i)$ is given by a product of elements and therefore one of those elements vanishes: there exists l, $1 \le l \le L_i^x$, such that $b_i(a_l; U_i(x_i^l)) = 0$. From the definition of b_i (Equation (6.43)) we deduce that $\sum_{s_i \in S_i^*(x_i^{l,a_l})} \sigma_i(s_i) = 0$. On the other hand, $S_i^*(x_i^{l,a_l}) \supseteq S_i^*(x)$ and therefore by Equation (6.25)

$$\rho_i(x; \sigma_i) = \sum_{s_i \in S_i^*(x)} \sigma_i(s_i) \le \sum_{s_i \in S_i^*(x_i^{l,a_l})} \sigma_i(s_i) = 0. \tag{6.57}$$

Hence Equation (6.49) is satisfied in this case. □

6.3 Equilibria in behavior strategies

By Nash's Theorem (Theorem 5.10 on page 151) every finite extensive-form game has a Nash equilibrium in mixed strategies. In other words, there exists a vector of mixed strategies under which no player has a profitable deviation to another mixed strategy. An equilibrium in behavior strategies is a vector of behavior strategies under which no player has a profitable deviation to another behavior strategy.

The next theorem states that to ensure the existence of a Nash equilibrium in behavior strategies, it suffices that all the players have perfect recall.

Theorem 6.16 *If all the players in an extensive-form game have perfect recall then the game has a Nash equilibrium in behavior strategies.*

Proof: Since an extensive-form game is by definition a finite game, Nash's Theorem (Theorem 5.10 on page 151) implies that the game has a Nash equilibrium in mixed strategies $\sigma^* = (\sigma_i^*)_{i \in N}$. Since all the players in the game have perfect recall, we know from Kuhn's Theorem (Theorem 6.15) that for each player i there exists a behavior strategy b_i^* equivalent to σ_i^*. Corollary 6.7 then implies that

$$u_i(\sigma^*) = u_i(b^*), \quad \forall i \in N, \tag{6.58}$$

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where $b^* = (b_i^*)_{i \in N}$. We show now that no player can increase his expected payoff by deviating to another behavior strategy. Let b_i be a behavior strategy of player i. From Theorem 6.11, there exists a mixed strategy σ_i equivalent to b_i. Since σ^* is an equilibrium in mixed strategies,

$$u_i(\sigma^*) \geq u_i(\sigma_i, \sigma_{-i}^*). \tag{6.59}$$

Since σ_i is equivalent to b_i, and for each $j \neq i$ the strategy σ_j^* is equivalent to b_j^*, Corollary 6.7 implies that

$$u_i(\sigma_i, \sigma_{-i}^*) = u_i(b_i, b_{-i}^*). \tag{6.60}$$

From Equations (6.58)–(6.60) we then have

$$u_i(b^*) = u_i(\sigma^*) \geq u_i(\sigma_i, \sigma_{-i}^*) = u_i(b_i, b_{-i}^*). \tag{6.61}$$

In other words, player i cannot profit by deviating from b_i^* to b_i, so that the strategy vector b^* is an equilibrium in behavior strategies. □

As the proof of the theorem shows, when a game has perfect recall, at each equilibrium in mixed strategies no player has a profitable deviation to a behavior strategy, and at each equilibrium in behavior strategies no player has a profitable deviation to a mixed strategy. Moreover, there exist equilibria at which some players implement mixed strategies and some players implement behavior strategies, and at each such equilibrium no player has a profitable deviation to either a mixed strategy or a behavior strategy.

The next example shows that when it is not the case that all players have perfect recall, the game may not have a pure strategy equilibrium.

Example 6.17 Figure 6.7 depicts a two-player zero-sum game.

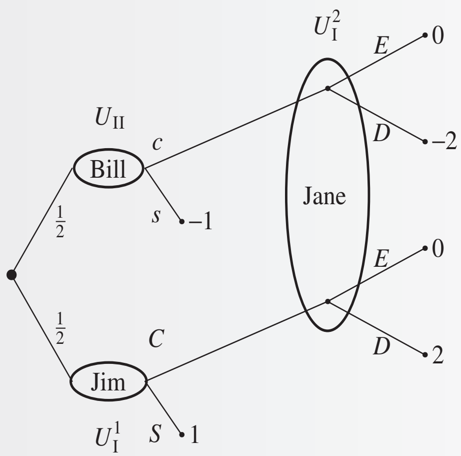

Figure 6.7 The game in Example 6.17, in extensive form

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This game may be interpreted as follows: Player I represents a couple, Jim and Jane. Player II is named Bill. At the first stage of the game, a winning card is handed either to Jim or to Bill, with equal probability. The player who receives the card may choose to show ("S" or "s") his card, and receive a payoff of 1 from the other player (thus ending the play of the game), or to continue ("C" or "c"). If the player holding the winning card chooses to continue, Jane (who does not know who has the card) is called upon to choose between declaring end ("E"), and thus putting an end to the play of the game without any player receiving a payoff, or declaring double ("D"), which results in the player holding the winning card receiving a payoff of 2 from the other player.

In this game, Player I has imperfect recall, because the paths from the root to the two vertices in information set U_{I}^2 do not intersect the same information set of Player I: one path intersects U_{I}^1, while the other does not intersect it.

Player I's set of pure strategies is $\{SD, SE, CD, CE\}$, and Player II's set of pure strategies is $\{S, C\}$. The strategic form of this game is given in the matrix in Figure 6.8 (in terms of payments from Player II to I):

Player II

Player I	c	s
CE	0	$-\frac{1}{2}$
CD	0	$\frac{1}{2}$
SE	$\frac{1}{2}$	0
SD	$-\frac{1}{2}$	0

Figure 6.8 The game in Example 6.17 in strategic form

The value of the game in mixed strategies is $v = \frac{1}{4}$, and an optimal mixed strategy guaranteeing this payoff to Player I is $\sigma_{\mathrm{I}} = [0(CE), \frac{1}{2}(CD), \frac{1}{2}(SE), 0(SD)]$. Player II's only optimal (mixed) strategy is $\sigma_{\mathrm{II}} = [\frac{1}{2}(c), \frac{1}{2}(s)]$.

To check whether the game has a value in behavior strategies we compute the minmax value $\overline{v}_b$ and the maxmin value $\underline{v}_b$ in behavior strategies. The maxmin value in behavior strategies equals the maxmin value in mixed strategies, since Player II has one information set. It follows that his set of behavior strategies $\mathcal{B}_{\mathrm{II}}$ equals his set of mixed strategies Σ_{II} (Exercise 6.4). Since he can guarantee $\frac{1}{4}$ in mixed strategies he can guarantee $\frac{1}{4}$ in behavior strategies. Formally,

$$\overline{v}_b = \min_{b_{\mathrm{II}}\in\mathcal{B}_{\mathrm{II}}} \max_{b_{\mathrm{I}}\in\mathcal{B}_{\mathrm{I}}} U(b_{\mathrm{I}}, b_{\mathrm{II}}) \tag{6.62}$$

$$= \min_{\sigma_{\mathrm{II}}\in\mathcal{B}_{\mathrm{II}}} \max_{b_{\mathrm{I}}\in\mathcal{B}_{\mathrm{I}}} U(b_{\mathrm{I}}, b_{\mathrm{II}}) \tag{6.63}$$

$$= \min_{\sigma_{\mathrm{II}}\in\mathcal{B}_{\mathrm{II}}} \max_{s_{\mathrm{I}}\in S_{\mathrm{I}}} U(b_{\mathrm{I}}, b_{\mathrm{II}}) \tag{6.64}$$

$$= \min_{\sigma_{\mathrm{II}}\in\mathcal{B}_{\mathrm{II}}} \max_{\sigma_{\mathrm{I}}\in\Sigma_{\mathrm{I}}} U(b_{\mathrm{I}}, b_{\mathrm{II}}) = v = \frac{1}{4} \tag{6.65}$$

Equation (6.64) holds because, as explained on page 179, it suffices to conduct maximization on the right-hand side of Equation (6.63) over the pure strategies of Player I, and Equation (6.65) holds because the function U is bilinear.

We now compute the maxmin value in behavior strategies $\underline{v}_b$. In other words, we will calculate Player I's maxmin value when he is restricted to using only behavior strategies. A behavior strategy of Player I can be written as $b_{\mathrm{I}} = ([\alpha(S), (1-\alpha)(C)], [\beta(D), (1-\beta)(E)])$. His expected payoff, when he plays b_{I}, depends on Player II's strategy:

- If Player II plays s, Player I's expected payoff is
$$\tfrac{1}{2}(\alpha + (1-\alpha)(2\beta + 0(1-\beta))) + \tfrac{1}{2}(-1) = (1-\alpha)\left(\beta - \tfrac{1}{2}\right). \tag{6.66}$$
- If Player II plays c, Player I's expected payoff is
$$\tfrac{1}{2}(\alpha + (1-\alpha)(2\beta + 0(1-\beta))) + \tfrac{1}{2}(\beta(-2) + 0(1-\beta)) = \alpha\left(\tfrac{1}{2} - \beta\right). \tag{6.67}$$

Player I's maxmin value in behavior strategies is therefore[3]
$$\underline{v}_b = \max_{\alpha,\beta} \min\left\{(1-\alpha)\left(\beta - \tfrac{1}{2}\right), \alpha\left(\tfrac{1}{2} - \beta\right)\right\} = 0. \tag{6.68}$$

To see that indeed $\underline{v}_b = 0$, note that if $\beta \leq \frac{1}{2}$, then the first element in the minimization in Equation (6.68) is nonpositive; if $\beta \geq \frac{1}{2}$, then the second element is nonpositive; and if $\beta = \frac{1}{2}$, both elements are zero. We conclude that $\overline{v}_b = \frac{1}{4} \neq 0 = \underline{v}_b$, and therefore the game has no value in behavior strategies.

Since the strategy $\sigma_{\mathrm{I}} = [0(CE), \frac{1}{2}(CD), \frac{1}{2}(SE), 0(SD)]$ guarantees Player I an expected payoff of $\frac{1}{4}$, while any behavior strategy guarantees him at most 0, we confirm that there does not exist a behavior strategy equivalent to σ_{I}, which can also be proved directly (prove it!).

The source of the difference between the two types of strategies in this case lies in the fact that Player I wants to coordinate his actions at his two information sets: ideally, Jane should play E if Jim plays S, and should play D if Jim plays C. This coordination is possible using a mixed strategy, but cannot be achieved with a behavior strategy, because in any behavior strategy the lotteries $[\alpha(S), (1-\alpha)(C)]$ and $[\beta(D), (1-\beta)(E)]$ are independent lotteries. ◀

6.4 Kuhn's Theorem for infinite games

In Section 6.2 we proved Kuhn's Theorem when the game tree is finite. There are extensive-form games with infinite game trees. This can happen in two ways: when there is a vertex with an infinite number of children, and when there are infinitely long paths in the game tree. In this section we generalize the theorem to the case in which each vertex has a finite number of children and the game tree has infinitely long paths. Infinitely long paths exist in games that may never end, such as backgammon and Monopoly. In Chapters 13 and 14 we present models of games that may not end. Generalizing Kuhn's Theorem to infinite games involves several technical challenges:

- The set of pure strategies has the cardinality of the continuum. Indeed, if for example player i has a countable number of information sets and in each of his information sets there are only two possible actions, a pure strategy of player i is equivalent to an infinite sequence of zeros and ones. The collection of all such sequences is equivalent

3 Again, it suffices to conduct maximization over the pure strategies of Player II, which are c and s.

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to the interval $[0, 1]$ of real numbers, which has the cardinality of the continuum. Since a mixed strategy is a probability distribution over pure strategies, we need to define a σ-algebra over the collection of all pure strategies in order to be able to define probability distributions over this set.

- In finite games, the equivalence of mixed strategies and behavior strategies was defined using the equivalence between the probabilities that they induce over the vertices of the game tree, and in particular over the set of leaves, which determines the outcome of the game. In infinite games, the outcome of the game may be determined by an infinitely long path in the game tree that corresponds to an infinitely long play of the game. It follows that instead of probability distributions induced over a finite set of leaves, in the case of an infinite game we need to deal with probability distributions induced over the set of paths in the game tree, which as we showed above has the cardinality of the continuum. This requires defining a measurable space over the set of plays of the game, that is, over the set of paths (finite and infinite) starting at the root of the tree.

We first introduce several definitions that will be used in this section.

Definition 6.18 *Let X be a set. A collection $\mathcal{Y}$ of subsets of X is a* σ-algebra *over X if (a) $\emptyset \in \mathcal{Y}$, (b) $X \setminus Y \in \mathcal{Y}$ for all $Y \in \mathcal{Y}$, and (c) $\cup_{i\in\mathbb{N}} Y_i \in \mathcal{Y}$ for every sequence $(Y_i)_{i\in\mathbb{N}}$ of elements in $\mathcal{Y}$.*

De Morgan's Laws imply that a σ-algebra is also closed under countable intersections: if $(Y_i)_{i\in\mathbb{N}}$ is a sequence of elements of $\mathcal{Y}$, then $\cap_{i\in\mathbb{N}} Y_i \in \mathcal{Y}$. For each family $\widehat{\mathcal{Y}}$ of subsets of X, the *σ-algebra generated by* $\widehat{\mathcal{Y}}$ is the smallest σ-algebra of $\mathcal{Y}$ (with respect to set inclusion) satisfying $\widehat{\mathcal{Y}} \subseteq \mathcal{Y}$. The σ-algebra that we will use in the rest of this section is the σ-algebra of cylinder sets.

Definition 6.19 *Let $(X_n)_{n\in\mathbb{N}}$ be sequence of finite sets, and let $X^\infty := \times_{n\in\mathbb{N}} X_n$. A set $B \in X^\infty$ is called a* cylinder set *if there exist $N \in \mathbb{N}$ and $(A_n)_{n=1}^N$, $A_n \subseteq X_n$ for all $n \in \{1, 2, \ldots, N\}$, such that $B = (\times_{n=1}^N A_n) \times (\times_{n=N+1}^\infty X_n)$. The* σ-algebra of cylinder sets *is the σ-algebra $\mathcal{Y}$ generated by the cylinder sets in X^∞.*

Definition 6.20 *A* measurable space *is a pair $(X, \mathcal{Y})$ such that X is a set and $\mathcal{Y}$ is a σ-algebra over X. A* probability distribution *over a measurable space $(X, \mathcal{Y})$ is a function $p : \mathcal{Y} \to [0, 1]$ satisfying:*

- *$p(\emptyset) = 0$.*
- *$p(X \setminus Y) = 1 - p(Y)$ for every $Y \in \mathcal{Y}$.*
- *$p(\cup_{n\in\mathbb{N}} Y_n) = \sum_{n\in\mathbb{N}} p(Y_n)$ for any sequence $(Y_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets in $\mathcal{Y}$.*

The third property in the definition of a probability distribution is called *σ-additivity*.

The next theorem follows from the Kolmogorov Extension Theorem (see, for example, Theorem $A.3.1$ in Durrett [2004]) and the Carathéodory Extension Theorem (see, for example, Theorem $13.A$ in Halmos [1994]). Given an infinite product of spaces $X^\infty = \times_{n\in\mathbb{N}} X_n$ and a sequence $(p^N)_{N\in\mathbb{N}}$ of probability distributions, where each p^N is a probability distribution over the finite product $X^N := \times_{n=1}^N X_n$, the theorem presents a condition guaranteeing the existence of an extension of the probability distributions $(p^N)_{N\in\mathbb{N}}$ to X^∞, i.e., a probability distribution p over X^∞ whose marginal distribution over X^N is p^N, for each $N \in \mathbb{N}$.

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Theorem 6.21 *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of finite sets. Suppose that for each $N \in \mathbb{N}$ there exists a probability distribution p^N over $X^N := \times_{n=1}^N X_n$ that satisfies*

$$p^N(A) = p^{N+1}(A \times X_{N+1}), \quad \forall N \in \mathbb{N}, \forall A \subseteq X^N. \tag{6.69}$$

Let $X^\infty := \times_{n\in\mathbb{N}} X_n$ and let $\mathcal{Y}$ be the σ-algebra of cylinder sets over X^∞. Then there exists a unique probability distribution p over $(X^\infty, \mathcal{Y})$ extending $(p^N)_{N\in\mathbb{N}}$, i.e.,

$$p^N(A) = p(A \times X_{N+1} \times X_{N+2} \times \cdots), \quad \forall N \in \mathbb{N}, \forall A \subseteq X^N. \tag{6.70}$$

When (V, E, x^0) is a (finite or infinite) game tree, denote by H the set of *maximal paths* in the tree, meaning paths from the root to a leaf, and infinite paths from the root. For each vertex x denote by $H(x)$ the set of paths in H passing through x. Let $\mathcal{H}$ be the σ-algebra generated by the sets $H(x)$ for all $x \in V$. Recall that for each vertex x that is not a leaf the set of children of x is denoted by $C(x)$.

In this section we also make use of the following version of Theorem 6.21, which states that if there is an infinite tree such that each vertex x in the tree has an associated probability distribution $p(x)$, and if these probability distributions are consistent in the sense that the probability associated with a vertex equals the sum of the probabilities associated with its children, then there is a unique probability distribution $\widehat{p}$ over the set of maximal paths satisfying the property that the probability that the set of paths passing through vertex x equals $p(x)$. The proof of the theorem is left to the reader (Exercise 6.24).

Theorem 6.22 *Let (V, E, x^0) be a (finite or infinite) game tree such that $|C(x)| < \infty$ for each vertex x. Denote by H the set of maximal paths. Let $p : V \to [0, 1]$ be a function satisfying $p(x) = \sum_{x'\in C(x)} p(x')$ for each vertex $x \in V$ that is not a leaf. Then there exists a unique probability distribution $\widehat{p}$ over $(H, \mathcal{H})$ satisfying $\widehat{p}(H(x)) = p(x)$ for all $x \in V$.*

6.4.1 Definitions of pure strategy, mixed strategy, and behavior strategy

Let G be an extensive-form game with an infinite game tree such that each vertex has a finite number of children. In such a game, as in the finite case, a pure strategy of player i is a function that associates each information set of player i with a possible action at that information set. A behavior strategy of player i is a function associating each one of his information sets with a probability distribution over the set of possible actions at that information set. Denote by $S_i = \times_{U_i\in\mathcal{U}_i} A(U_i)$ player i's set of pure strategies and by $\mathcal{B}_i = \times_{U_i\in\mathcal{U}_i} \Delta(A(U_i))$ his set of behavior strategies.

A mixed strategy is a probability distribution over the collection of pure strategies. When the game has finite depth,[4] the set of pure strategies is a finite set and the set of player i's mixed strategies Σ_i is a simplex. When player i has an infinite number of information sets at which he has at least two possible actions the set of pure strategies S_i has the cardinality of the continuum. To define a probability distribution over this set we need to define a σ-algebra over it. Let $\mathcal{S}_i$ be the σ-algebra of cylinder sets of S_i. The pair $(S_i, \mathcal{S}_i)$ is a measurable space and the set of probability distributions over it is the set of mixed strategies Σ_i of player i.

4 The *depth* of a vertex is the number of edges in the path from the root to the vertex. The *depth of a game* is the maximum (or supremum) of the depth of all vertices in the game tree.

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6.4.2 Equivalence between mixed strategies and behavior strategies

In a finite game of depth T, a mixed strategy σ_i^T is equivalent to a behavior strategy b_i^T if $\rho(x; \sigma_i^T, \sigma_{-i}^T) = \rho(x; b_i^T, \sigma_{-i}^T)$ for every mixed/behavior strategy vector σ_{-i}^T of the other players and every vertex x in the game tree. In this section we extend the definition of equivalence between mixed and behavior strategies to infinite games.

We begin by defining $\rho_i(x; \sigma_i)$ and $\rho_i(x; b_i)$, the probability that player i implementing either mixed strategy σ_i or behavior strategy b_i will choose actions leading to the vertex x at each vertex along the path from the root to x that is in his information sets.

For each behavior strategy b_i of player i, and each vertex x in the game tree,

$$\rho_i(x; b_i) := \Pi_{l=1}^{L_i^x} b_i\left(a_i; U_i^l\right), \tag{6.71}$$

where L_i^x is the number of vertices along the path from the root to x that are in player i's information sets (not including the vertex x, if at vertex x player i chooses an action), and $U_i^1, U_i^2, \ldots, U_1^{L_i^x}$ are the information sets containing these vertices (if there are several vertices along the path to x in the same information set U_i of player i, then this information set will appear more than once in the list $U_i^1, U_i^2, \ldots, U_1^{L_i^x}$).

We now define the probability $\rho_i(x; \sigma_i)$ for a mixed strategy σ_i. For any $T \in \mathbb{N}$ let G^T be the game that includes the first T stages of the game G.

- The set of vertices V^T of G^T contains all vertices of G with depth at most T.
- The information sets of each player i in G^T are all nonempty subsets of V^T that are obtained as the intersection of an information set in $\mathcal{U}_i^T$ with V^{T-1}; that is, an information set in G^T contains only vertices whose depth is strictly less than T. This is because the vertices whose depth is T are leaves of G^T. Denote by $\mathcal{U}_i^T$ the collection of player i's information sets in the game G that have a nonempty intersection with V^{T-1}. With this notation, player i's collection of information sets in the game G^T is all the nonempty intersections of V^{T-1} with a set in $\mathcal{U}_i^T$. Below, for any $T \in \mathbb{N}$, we identify each information set U_i^T of player i in the game G^T with the information set $U_i \in \mathcal{U}_i^T$ for which $U_i^T = V^{T-1} \cap U_i$.

Since each vertex has a finite number of children, the set V^T contains a finite number of vertices. To simplify the notation, an information set of player i in the game G^T, which is the intersection of V^T and an information set U_i of player i in the game G, will also be denoted by U_i. Since Kuhn's Theorem does not involve the payoffs of a game, we will not specify the payoffs in the game G^T.

Player i's set of pure strategies in the game G^T is $S_i^T := \times_{U_i \in \mathcal{U}_i^T} A(U_i)$. For each mixed strategy σ_i in the game G, let σ_i^T be its marginal distribution over S_i^T. Then σ_i^T is a mixed strategy in the game G^T. The sequence of probability distributions $(\sigma_i^T)_{T \in \mathbb{N}}$ satisfies the following property: the marginal distribution of σ_i^T over S_i^{T-1} is σ_i^{T-1}. It follows that for each vertex x whose depth is less than or equal to T we have $\rho_i(x; \sigma_i^{T_1}) = \rho_i(x; \sigma_i^{T_2})$ for all $T_1, T_2 \geq T$. Define for each vertex x

$$\rho_i(x; \sigma_i) := \rho_i\left(x; \sigma_i^T\right), \tag{6.72}$$

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where T is greater than or equal to the depth of x. Finally, define, for each mixed/behavior strategy vector σ,

$$\rho(x;\sigma) := \Pi_{i\in N}\rho_i(x;\sigma_i). \tag{6.73}$$

This is the probability that the play of the game reaches vertex x when the players implement the strategy vector σ.

The following theorem, which states that every vector of strategies uniquely defines a probability distribution over the set of infinite plays, follows from Theorem 6.22 and the definition of ρ (Exercise 6.25).

Theorem 6.23 *Let σ be a mixed/behavior strategy vector in a (finite or infinite) extensive-form game. Then there exists a unique probability distribution μ_σ over $(H,\mathcal{H})$ satisfying $\mu_\sigma(H(x)) = \rho(x;\sigma)$ for every vertex x.*

Definition 6.24 *A mixed strategy σ_i of player i is* equivalent *to a behavior strategy b_i of player i if, for every mixed/behavior strategy vector σ_{-i} of the other players, $\mu_{(\sigma_i,\sigma_{-i})} = \mu_{(b_i,\sigma_{-i})}$.*

Theorem 6.23 implies the following theorem.

Theorem 6.25 *A mixed strategy σ_i of player i is* equivalent *to his behavior strategy b_i if for every mixed/behavior strategy vector σ_{-i} of the other players and every vertex x we have $\rho(x;\sigma_i,\sigma_{-i}) = \rho(x;b_i,\sigma_{-i})$.*

6.4.3 Statement of Kuhn's Theorem for infinite games and its proof

The definition of a player with perfect recall in an infinite extensive-form game is identical to the definition for finite games (Definition 6.13 on page 231). If player i has perfect recall in a game G, then he also has perfect recall in the game G^T for all $T \in \mathbb{N}$ (verify!).

Theorem 6.26 *Let G be an extensive-form game with an infinite game tree such that each vertex in the game tree has a finite number of children. If player i has perfect recall, then for each mixed strategy of player i there is an equivalent behavior strategy and for each behavior strategy of player i there is an equivalent mixed strategy.*

Proof: Let G be an extensive-form game with an infinite game tree such that each vertex in the game tree has a finite number of children. Let i be a player with perfect recall. We begin by proving one direction of the statement of the theorem: for each mixed strategy of player i there is an equivalent behavior strategy. Let σ_i be a mixed strategy of player i in the game G. For each $T \in \mathbb{N}$, let σ_i^T be the restriction of σ_i to the game G^T; in other words, σ_i^T is the marginal distribution of σ_i over S_i^T. In the proof of Kuhn's Theorem (Theorem 6.15 on page 232), we constructed an equivalent behavior strategy for any given mixed strategy in a finite extensive-form game. Let b_i^T be the behavior strategy equivalent to the mixed strategy σ_i^T in the game G^T, constructed according to that theorem. Since σ_i^{T+1} is equivalent to b_i^{T+1} in the game G_i^{T+1}, since the marginal distribution of σ_i^{T+1} over S_i^T is σ_i^T, and since σ_i^T is equivalent to b_i^T in the game G_i^T, it follows that for each vertex x whose depth is less than or equal to T,

$$\rho_i\left(x;b_i^{T+1}\right) = \rho_i\left(x;\sigma_i^{T+1}\right) = \rho_i\left(x;\sigma_i^T\right) = \rho_i\left(x;b_i^T\right). \tag{6.74}$$

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It follows that

$$b_i^{T+1}(U_i) = b_i^T(U_i), \quad \forall U_i \in \mathcal{U}_i^T. \tag{6.75}$$

In other words, the behavior strategies $(b_i^T)_{T\in\mathbb{N}}$ are consistent, in the sense that every two of them coincide on information sets that are in the domain of both. Define a behavior strategy b_i of player i by

$$b_i(U_i) := b_i^T(U_i), \quad \forall U_i \in \mathcal{U}_i, \tag{6.76}$$

where T satisfies $U_i \in \mathcal{U}_i^T$. By Equation (6.75) it follows that $b_i(U_i)$ is well defined. We will prove that σ_i and b_i are equivalent in the game G. Let $\sigma_{-i} = (\sigma_j)_{j\neq i}$ be a mixed/behavior strategy vector of the other players. For each $T \in \mathbb{N}$, let σ_j^T be the strategy σ_j restricted to the game G^T. Denote $\sigma_{-i}^T = (\sigma_j^T)_{j\neq i}$. Since the strategies σ_i^T and b_i^T are equivalent in the game G^T,

$$\rho\left(x; \sigma_i^T, \sigma_{-i}^T\right) = \rho\left(x; b_i^T, \sigma_{-i}^T\right) \tag{6.77}$$

for each vertex whose depth is less than or equal to T. By definition, it follows that for each vertex x

$$\rho(x; \sigma_i, \sigma_{-i}) = \rho(x; b_i, \sigma_{-i}). \tag{6.78}$$

Theorem 6.25 implies that σ_i and b_i are equivalent strategies.

We now prove the other direction of the statement of the theorem. Let b_i be a behavior strategy of player i. For each $T \in \mathbb{N}$ let b_i^T be the restriction of b_i to the collection of information sets $\mathcal{U}_i^T$. It follows that b_i^T is a behavior strategy of player i in the game G^T. Since player i has perfect recall in the game G^T, and since the game G^T is a finite game, there exists a mixed strategy σ_i^T equivalent to b_i^T in the game G^T.

Since σ_i^{T+1} is equivalent to b_i^{T+1} in the game G_i^{T+1}, since the restriction of b_i^{T+1} to $\mathcal{U}_i^T$ is b_i^T, and since σ_i^T is equivalent to b_i^T in the game G_i^T, it follows that σ_i^T is the marginal distribution of σ_i^{T+1} on S_i^T. By Theorem 6.21 (with respect to the product space $S_i = \times_{U_i\in\mathcal{U}_i} A(U_i)$) we deduce that there exists a mixed strategy σ_i whose projection over S_i^T is σ_i^T for all $T \in \mathbb{N}$. Reasoning similar to that used in the first part of this proof shows that σ_i and b_i are equivalent strategies in the game G (Exercise 6.26). □

Using methods similar to those presented in this section one can prove Kuhn's Theorem for extensive-form games with game trees of finite depth in which every vertex has a finite or countable number of children. Combining that result with the proof of Theorem 6.26 shows that Kuhn's Theorem holds in extensive-form games with game trees of infinite depth in which every vertex has a finite or countable number of children.

6.5 Remarks

The Absent-Minded Driver game appearing in Example 6.9 (page 225) was first introduced in Piccione and Rubinstein [1997], and an entire issue of the journal *Games and Economic Behavior* (1997, issue 1) was devoted to analyzing it. Item (b) of Exercise 6.17 is taken from von Stengel and Forges [2008].

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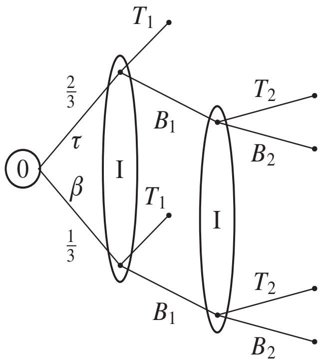
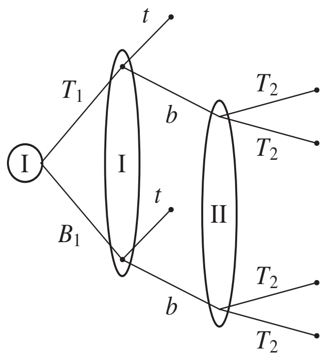
6.6 Exercises

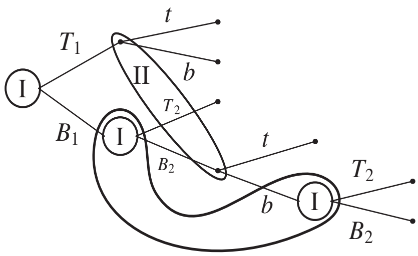
6.1 In each of the games in the following diagrams, identify which players have perfect recall. In each case in which there is a player with imperfect recall, indicate what the player may forget during a play of the game, and in what way the condition in Definition 6.13 (page 231) fails to obtain.



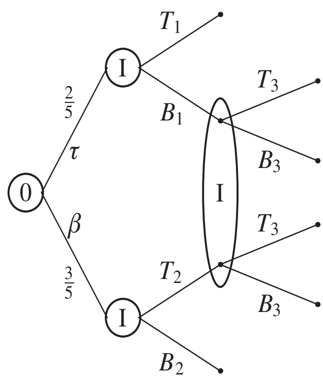
Game A



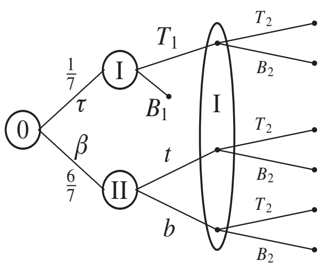
Game B



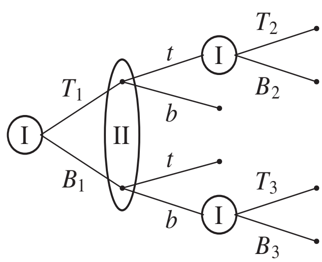
Game C



Game D



Game E



Game F

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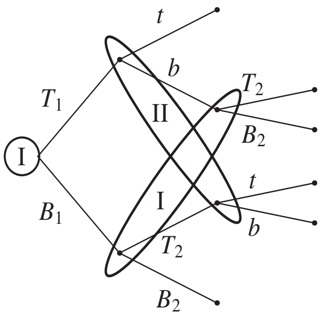
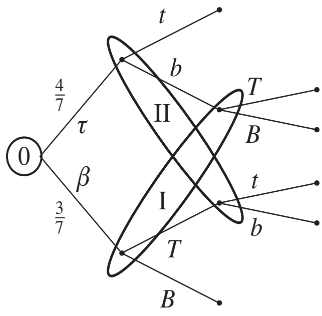


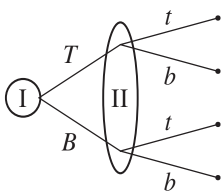
Game G



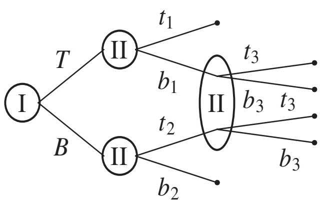
Game H

6.2 In each of the following games, find a mixed strategy equivalent to the noted behavior strategy.

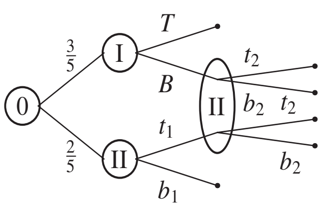
(a) $b_{\mathrm{I}} = \left[\frac{1}{3}(T), \frac{2}{3}(B)\right]$, in the game



(b) $b_{\mathrm{II}} = \left(\left[\frac{4}{9}(t_1), \frac{5}{9}(b_1)\right], \left[\frac{3}{5}(t_2), \frac{2}{5}(b_2)\right], \left[\frac{2}{3}(t_3), \frac{1}{3}(b_3)\right]\right)$, in the game



(c) $b_{\mathrm{II}} = \left(\left[\frac{4}{9}(t_1), \frac{5}{9}(b_1)\right], \left[\frac{1}{4}(t_2), \frac{3}{4}(b_2)\right]\right)$, in the game



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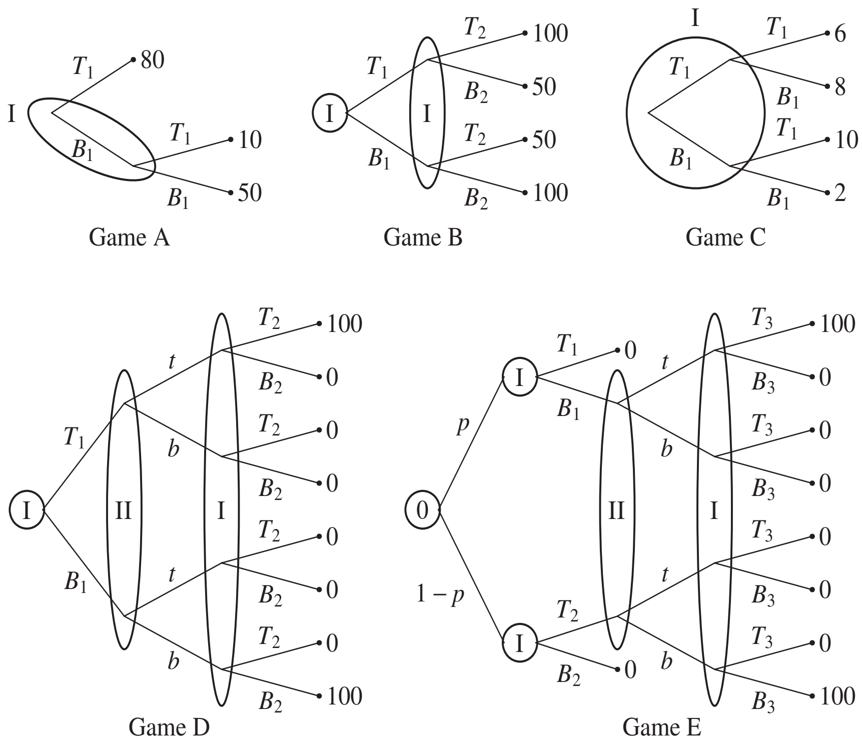
6.3 Identify the payoff that each player can guarantee for himself in each of the following two-player zero-sum game using mixed strategies and using behavior strategies.



6.4 Prove that if a player in an extensive-form game has only one information set, then his set of mixed strategies equals his set of behavior strategies.

6.5 Does there exist a two-player zero-sum extensive-form game that has a value in mixed strategies and a value in behavior strategies, but these two values are not equal to each other? Either prove that such a game exists or provide a counterexample.

6.6 Prove Theorem 6.6 (page 223): if b_i is a behavior strategy equivalent to the mixed strategy σ_i, then for every strategy vector σ_{-i},

$$u_j(\sigma_i, \sigma_{-i}) = u_j(b_i, \sigma_{-i}), \quad \forall j \in N. \tag{6.79}$$

6.7 Prove that in Example 6.4 (page 222) for every mixed strategy $\widehat{\sigma}_{\mathrm{II}}$ of Player II the probability distribution induced by $(\sigma_{\mathrm{I}}, \widehat{\sigma}_{\mathrm{II}})$ over the leaves of the game tree is identical with the probability distribution induced by $(b_{\mathrm{I}}, \widehat{\sigma}_{\mathrm{II}})$ over the leaves of the game tree. The mixed strategy σ_{I} defined in Equation (6.14) and the behavior strategy b_{I} defined in Equation (6.13) (page 224).

6.8 Prove Corollary 6.10 (page 226): if there exists a path from the root to the vertex x that passes at least twice through the same information set U_i of player i, and if the action leading to x is not the same action in each of these passes through the information set, then player i has a behavior strategy that has no equivalent mixed strategy.

6.9 Show that Theorem 6.11 does not hold without the condition that there are at least two possible actions at each vertex.

6.10 Explain why Equation (6.37) in the proof of Theorem 6.11 (page 230) does not necessarily hold when a game does not have perfect recall.

6.11 Prove that if a player does not know whether or not he has previously made a move during the play of a game, then he does not have perfect recall (according to Definition 6.13 on page 231).

6.12 Prove that if a player knows during the play of a game how many moves he has previously made, but later forgets this, then he does not have perfect recall (according to Definition 6.13 on page 231).

6.13 Prove that if a player knows during the play of a game which action another player has chosen at a particular information set, but later forgets this, then he does not have perfect recall (according to Definition 6.13 on page 231).

6.14 Prove that if a player does not know what action he chose at a previous information set in a game, then he has imperfect recall in that game (according to Definition 6.13 on page 231).

6.15 Prove that if at a particular information set in a game a player knows which player made the move leading to that information set, but later forgets this, then he does not have perfect recall (according to Definition 6.13 on page 231).

6.16 Prove that if x_1 and x_2 are two vertices in the same information set of player i, and if player i has perfect recall in the game, then $S_i^*(x_1) = S_i^*(x_2)$. (See page 228 for the definition of the set $S_i^*(x)$.)

6.17 Let U and $\widehat{U}$ be two information sets (they may both be the information sets of the same player, or of two different players). U will be said to *precede* $\widehat{U}$ if there exist a vertex $x \in U$ and a vertex $\widehat{x} \in \widehat{U}$ such that the path from the root to $\widehat{x}$ passes through x.

(a) Prove that if U is an information set of a player with perfect recall, then U does not precede U.
(b) Prove that in a two-player game without chance moves, where both players have perfect recall, if U precedes $\widehat{U}$, then $\widehat{U}$ does not precede U.
(c) Find a two-player game with chance moves, where both players have perfect recall and there exist two information sets U and $\widehat{U}$ such that U precedes $\widehat{U}$, and $\widehat{U}$ precedes U.

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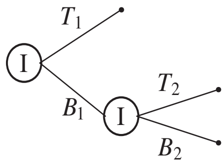
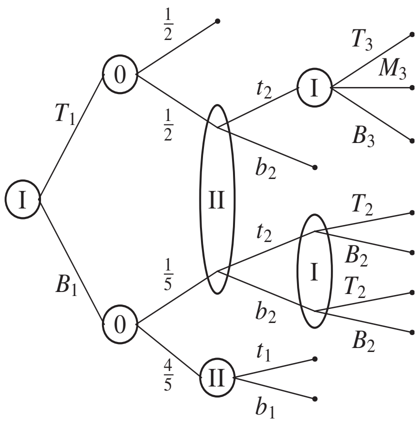
(d) Find a three-player game without chance moves, where all the players have perfect recall and there exist two information sets U and $\widehat{U}$ such that U precedes $\widehat{U}$, and $\widehat{U}$ precedes U.

6.18 Find a behavior strategy equivalent to the given mixed strategies in each of the following games.

(a) $s_{\mathrm{I}} = \left[\frac{1}{2}(B_1, B_2), \frac{1}{2}(T_1, T_2)\right]$, in the game



(b) $s_{\mathrm{I}} = \left[\frac{3}{7}(B_1 B_2 M_3), \frac{1}{7}(B_1 T_2 B_3), \frac{2}{7}(T_1 B_2 M_3), \frac{1}{7}(T_1 T_2 T_3)\right]$ and $s_{\mathrm{II}} = \left[\frac{3}{7}(b_1 b_2), \frac{1}{7}(b_1 t_2), \frac{1}{7}(t_1 b_2), \frac{2}{7}(t_1 t_2)\right]$, in the game



6.19 (a) Let i be a player with perfect recall in an extensive-form game and let σ_i be a mixed strategy of player i. Suppose that there is a strategy vector σ_{-i} of the other players such that $\rho(x; \sigma_i, \sigma_{-i}) > 0$ for each leaf x in the game tree. Prove that there exists a unique behavior strategy b_i equivalent to σ_i.

(b) Give an example of an extensive-form game in which player i has perfect recall and there is a mixed strategy σ_i with more than one behavior strategy equivalent to it.

6.20 Let i be a player with perfect recall in an extensive-form game and let b_i be a behavior strategy of player i. Suppose that there is a strategy vector σ_{-i} of the other

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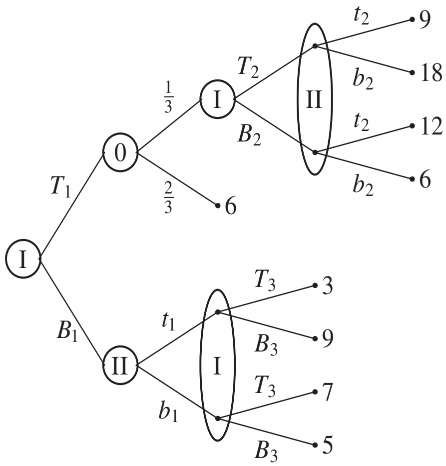
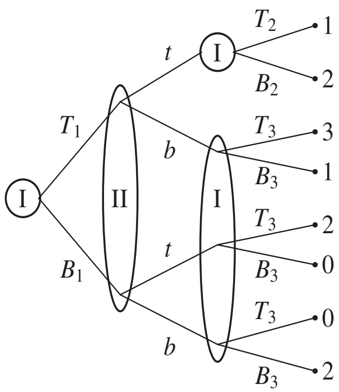
players such that $\rho(x; b_i, \sigma_{-i}) > 0$ for each leaf in the game tree. Prove that there exists a unique mixed strategy σ_i equivalent to b_i.

6.21 In the following two-player zero-sum game, find the optimal behavior strategies of the two players. (Why must such strategies exist?)



6.22 Compute the value of the following game, in mixed strategies, and in behavior strategies, if these values exist.



6.23 (a) Compute the value in mixed strategies of the game below.
(b) Compute what each player can guarantee using behavior strategies (in other words, compute each player's security value in behavior strategies).

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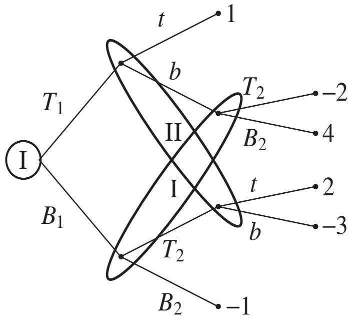

6.24 Prove Theorem 6.22 (page 240).

6.25 Prove Theorem 6.23 (page 242): let σ be a mixed/behavior strategy vector in a (finite or infinite) extensive-form game. Then there exists a unique probability distribution μ_σ over $(H, \mathcal{H})$ satisfying $\mu_\sigma(H(x)) = \rho(x; \sigma)$ for each vertex x.

6.26 Complete the proof of the second direction of Kuhn's Theorem for infinite games (Theorem 6.26, page 242): prove that the mixed strategy σ_i constructed in the proof is equivalent to the given behavior strategy b_i.

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