Logic in CS Autumn 2024

Problem Sheet 3

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1. For each of the following conditions, give an example of an unsatisfiable set of formulae, Γ that meets the condition.
   (a) Each member of the set is—by itself—satisfiable.
   (b) For any two members γ_1 and γ_2 of Γ, the set $\{\gamma_1, \gamma_2\}$ is satisfiable.
   (c) For any three members γ_1, γ_2, and γ_3 of Γ, the set $\{\gamma_1, \gamma_2, \gamma_3\}$ is satisfiable.

2. Let α be a wff whose only connective symbols are $\wedge$, $\vee$, and $\neg$. Let α^* be the result of interchanging $\wedge$ and $\vee$ and replacing each propositional variable by its negation. Show that α^* is tautologically equivalent to $\neg\alpha$. Observe that this result is a generalization and a stronger form of De Morgan's laws, which deal with the negation of conjunctions and disjunctions.

3. Let $\mathcal{F}$ and $\mathcal{G}$ be two sets of formulae. We say $\mathcal{F} \equiv \mathcal{G}$ iff for any assignment α, $\alpha \models \mathcal{F}$ iff $\alpha \models \mathcal{G}$ ($\alpha \models \mathcal{F}$ iff $\alpha \models F_i$ for every $F_i \in \mathcal{F}$). Prove or disprove: For any $\mathcal{F}$ and $\mathcal{G}$, $\mathcal{F} \equiv \mathcal{G}$ iff
   (a) For each $G \in \mathcal{G}$, there exists $F \in \mathcal{F}$ such that $G \models F$, and
   (b) For each $F \in \mathcal{F}$, there exists $G \in \mathcal{G}$ such that $F \models G$,

4. A set of sentences $\mathcal{F}$ is said to be closed under conjunction if for any F and G in $\mathcal{F}$, $F \wedge G$ is also in $\mathcal{F}$. Suppose $\mathcal{F}$ is closed under conjunction and is inconsistent ($\mathcal{F} \vdash \perp$). Prove that for any $G \in \mathcal{F}$, there exists $F \in \mathcal{F}$ such that $\{F\} \vdash \neg G$.

5. Suppose $\models (F \to G)$ and F is satisfiable and G is not valid. Show that there exists a formula H such that the atomic propositions in H are in both F and G and $\models F \to H$ and $\models H \to G$.

6. Consider the parity function, $\mathsf{PARITY} : \{0,1\}^n \to \{0,1\}$, where PARITY evaluates to 1 iff an odd number of inputs is 1. In all of the CNFs below, we assume that each clause contains any variable at most once, i.e. no clause contains expressions of the form $p \wedge \neg p$ or $p \vee \neg p$. Furthermore, all clauses are assumed to be distinct.
   (a) Prove that any CNF representation of PARITY must have n literals (from distinct variables) in every clause.
   (b) Prove that any CNF representation of PARITY must have at least 2^{n-1} clauses.

7. Using resolution, or otherwise, show that there is a polynomial-time algorithm to decide satisfiability of those CNF formulas F in which each propositional variable occurs at most twice. Justify your answer. Note that this question is not the same as 2-SAT.

8. Say that a set Σ_1 of wffs is equivalent to a set Σ_2 of wffs iff for any wff α, we have $\Sigma_1 \models \alpha$ iff $\Sigma_2 \models \alpha$. A set Σ is independent iff no member of Σ is tautologically implied by the remaining members in Σ. Show that a finite set of wffs has an independent equivalent subset by describing an algorithm to compute this independent equivalent subset. Prove that your algorithm returns a subset that is independent and equivalent.