CS 228 : Logic in Computer Science

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Basic Rules So Far

- $ightharpoonup \land i, \land e_1, \land e_2$ (and introduction and elimination)
- $\rightarrow \neg \neg e, \neg \neg i$ (double negation elimination and introduction)
- ► MP (Modus Ponens)
- $ightharpoonup \rightarrow i$ (Implies Introduction : remember opening boxes)
- \lor $\lor i_1, \lor i_2, \lor e$ (Or introduction and elimination)

▶
$$\vdash p \rightarrow (q \rightarrow p)$$

1. true

premise

$$\blacktriangleright \vdash p \rightarrow (q \rightarrow p)$$

1.	true	premise
2.	р	assumption
3.		

▶
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1.	true	premise
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1.	true	premise
2.	р	assumption
3.	q	assumption
4.	p	copy 2
5.	$oldsymbol{q} ightarrow oldsymbol{p}$	<i>→ i</i> 3-4

6.

▶
$$\vdash p \rightarrow (q \rightarrow p)$$

1.	true	premise
2.	р	assumption
3.	q	assumption
4.	р	copy 2
5.	$oldsymbol{q} ightarrow oldsymbol{p}$	→ <i>i</i> 3-4
6.	$p \rightarrow (q \rightarrow p)$	\rightarrow i 2-5

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- $ightharpoonup \perp \to \varphi$ for any formula φ .

Rules with \bot

The \perp elimination rule $\perp e$

$$\frac{\perp}{\psi}$$

The \perp introduction rule $\perp i$

$$\frac{\varphi \qquad \neg \varphi}{\bot}$$

- 1. $\neg p \lor q$ premise
- 2.

▶
$$\neg p \lor q \vdash p \rightarrow q$$

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- 2. $\neg p \lor e(1)$
- 3.

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1. $\neg p \lor q$ premise

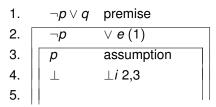
2. $\neg p \lor e(1)$

3. p

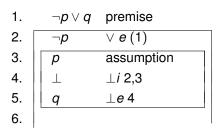
4.

p assumption

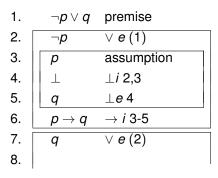
▶
$$\neg p \lor q \vdash p \rightarrow q$$



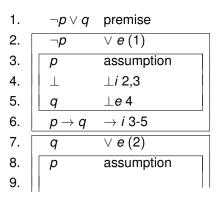
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1.	$\neg p \lor q$	premise
2.	$\neg p$	∨ <i>e</i> (1)
3.	р	assumption
4.		<i>⊥i</i> 2,3
5.	q	⊥ <i>e</i> 4
6.	p o q	→ <i>i</i> 3-5
7.	q	∨ e (2)
8.	р	assumption
9.	q	copy 7
0.	p o q	→ <i>i</i> 8-9
1.	$oldsymbol{ ho} ightarrow oldsymbol{q}$	∨ <i>e</i> 1, 2-6, 7-10

Introducing Negations (PBC)

- In the course of a proof, if you assume φ (by opening a box) and obtain \bot in the box, then we conclude $\neg \varphi$
- ▶ This rule is denoted $\neg i$ and is read as \neg introduction.
- ► Also known as Proof By Contradiction

- 1. $p \rightarrow \neg p$ premise
- 2.

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- 2. *p* assumption
- 3.

$$\blacktriangleright \ p \to \neg p \vdash \neg p$$

1.	p ightarrow eg p	premise
2.	р	assumption
3.	eg p	MP 1,2
4.		

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2.	р	assumption
3.	$\neg p$	MP 1,2
4.		<i>⊥i</i> 2,3
5.	$\neg p$	¬ <i>i</i> 2-4

The Last One

Law of the Excluded Middle (LEM)



Summary of Basic Rules

- \blacktriangleright $\land i$, $\land e_1$, $\land e_2$,
- ¬¬e
- ► MP
- $\rightarrow i$
- $\triangleright \forall i_1, \forall i_2, \forall e$
- ▶ Copy, $\neg i$ or PBC
- ► *⊥e*, *⊥i*

Derived Rules

- ▶ MT (derive using MP, $\perp i$ and $\neg i$)
- $ightharpoonup \neg \neg i$ (derive using $\bot i$ and $\neg i$)
- ▶ LEM (derive using $\forall i_1, \bot i, \neg i, \forall i_2, \neg \neg e$)

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- So far, the "proof" we have seen is purely syntactic, no true/false meanings were attached
- ▶ Intuitively, $p \rightarrow q \vdash \neg p \lor q$ makes sense because you think semantically. However, we never used any semantics so far.
- Now we show that whatever can be proved makes sense semantically too.

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- Formulae φ and ψ are semantically equivalent iff $\varphi \models \psi$ and $\psi \models \varphi$

Soundness of Propositional Logic

$$\varphi_1, \ldots, \varphi_n \vdash \psi \Rightarrow \varphi_1, \ldots, \varphi_n \models \psi$$

Whenever ψ can be proved from $\varphi_1, \dots, \varphi_n$, then ψ evaluates to true whenever $\varphi_1, \dots, \varphi_n$ evaluate to true

▶ Assume $\varphi_1, \ldots, \varphi_n \vdash \psi$.

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- Assume that whenever $\varphi_1, \dots, \varphi_n \vdash \psi$ using a proof of length $\leq k 1$, we have $\varphi_1, \dots, \varphi_n \models \psi$.
- ► Consider now a proof with *k* lines.

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- ▶ By inductive hypothesis, we have $\varphi_1, \dots, \varphi_n \models \psi_1$ and $\varphi_1, \dots, \varphi_n \models \psi_2$. By semantics, we have $\varphi_1, \dots, \varphi_n \models \psi_1 \land \psi_2$.

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- ▶ A box starting with ψ_1 was opened at some line $k_1 < k$.
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- ▶ The line just after the box was ψ .
- ▶ Consider adding ψ_1 in the premises along with $\varphi_1, \ldots, \varphi_n$. Then we will get a proof $\varphi_1, \ldots, \varphi_n, \psi_1 \vdash \psi_2$, of length k-1. By inductive hypothesis, $\varphi_1, \ldots, \varphi_n, \psi_1 \models \psi_2$. By semantics, this is same as $\varphi_1, \ldots, \varphi_n \models \psi_1 \rightarrow \psi_2$

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- ▶ The equivalence of $\varphi_1, \ldots, \varphi_n \vdash \psi_1 \rightarrow \psi_2$ and $\varphi_1, \ldots, \varphi_n, \psi_1 \vdash \psi_2$ gives the proof.

Soundness: Other cases

Do this as homework

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