4 Strategic-form games

Chapter summary

In this chapter we present the model of *strategic-form games*. A game in strategic form consists of a set of players, a strategy set for each player, and an outcome to each vector of strategies, which is usually given by the vector of utilities the players enjoy from the outcome. The strategic-form description ignores dynamic aspects of the game, such as the order of the moves by the players, chance moves, and the informational structure of the game.

The goal of the theory is to suggest which strategies are more likely to be played by the players, or to recommend to players which strategy to implement (or not to implement). We present several concepts that allow one to achieve these goals. The first concept introduced is *domination* (strict or weak), which provides a partial ordering of strategies of the same player; it tells when one strategy is "better" than another strategy. Under the hypothesis that it is commonly known that "rational" players do not implement a dominated strategy we can then introduce the process of *iterated elimination of dominated strategies*, also called *rationalizability*. In this process, dominated strategies are successively eliminated from the game, thereby simplifying it. We go on to introduce the notion of *stability*, captured by the concept of *Nash equilibrium*, and the notion of *security*, captured by the concept of the maxmin value and maxmin strategies. The important class of *two-player zero-sum* games is introduced along with its solution called the *value* (or the *minmax value*). This solution concept shares both properties of security and stability. When the game is not zero-sum, security and stability lead typically to different predictions.

We prove that every extensive-form game with perfect information has a Nash equilibrium. This is actually a generalization of the theorem on the game of chess proved in Chapter 1.

To better understand the relationships between the various concepts, we study the effects of elimination of dominated strategies on the maxmin value and on equilibrium payoffs. Finally, as a precursor to mixed strategies introduced in the next chapter, we look at an example of a two-player game on the unit square and compute its Nash equilibrium.

As we saw in Chapter 3, a player's strategy in an extensive-form game is a decision rule that determines that player's action in each and every one of his information sets. When there are no chance moves in the game, each vector of strategies – one strategy per player – determines the play of the game and therefore also the outcome. If there are chance moves, a vector of strategies determines a probability distribution over possible plays of the game, and therefore also over the outcomes of the game. The strategy chosen by a player therefore influences the outcome (or the probability distribution of outcomes, if there are chance moves).

If all we are interested in is the outcomes of the game and not the specific actions that brought about those outcomes, then it suffices to describe the game as the set of strategies available to each player, along with the distribution over the outcomes that each vector of strategies brings about.

4.1 Examples and definition of strategic-form games

For the analysis of games, every player must have preferences with respect to the set of outcomes. This subject was covered in detail in Chapter 2 on utility theory, where we saw that if player i's preference relation $\succsim_i$ satisfies the von Neumann–Morgenstern axioms, then it can be represented by a linear utility function u_i. In other words, to every possible outcome o, we can associate a real number $u_i(o)$ representing the utility that player i ascribes to o, with the player preferring one outcome to another if and only if the utility of the first outcome is higher than the utility of the second outcome. The player prefers one lottery to another lottery if and only if the expected utility of the outcomes according to the first lottery is greater than the expected utility of the outcomes according to the second lottery.

In most games we analyze in this book, we assume that the preference relations of the players satisfy the von Neumann–Morgenstern axioms. We will also assume that the outcomes of plays of games are given in utility terms. This means that the outcome of a play of a game is an n-dimensional vector, where the i-th coordinate is player i's utility from that play of the game.[1]

Example 4.1 Consider the following two-player game (Figure 4.1) presented in extensive form with six possible outcomes.

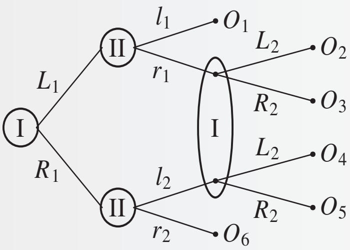

Figure 4.1 A two-player game in extensive form

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[1] This is equivalent to the situation where the outcomes are monetary payoffs and the players are risk neutral, in which case every lottery over payoffs is equivalent to the expected monetary payoff in the lottery drawing.

Player I has four strategies: L_1L_2, L_1R_2, R_1L_2, R_1R_2, and Player II has four strategies: l_1l_2, l_1r_2, r_1l_2, r_1r_2. The extensive-form description presents in detail what each player knows at each of his decision points. But we can ignore all that information, and present the players' strategies, along with the outcomes they lead to, in the table in Figure 4.2:

		Player II			
		l_1l_2	l_1r_2	r_1l_2	r_1r_2
Player I	L_1L_2	O_1	O_1	O_2	O_2
	L_1R_2	O_1	O_1	O_3	O_3
	R_1L_2	O_4	O_6	O_4	O_6
	R_1R_2	O_5	O_6	O_5	O_6

Figure 4.2 The game in Figure 4.1 in strategic form

In this description of the game, the rows represent the strategies of Player I and the columns those of Player II. In each cell of the table appears the outcome that arises if the two players choose the pair of strategies associated with that cell. For example, if Player I chooses strategy L_1L_2 and Player II chooses strategy l_1l_2, we will be in the upper-leftmost cell of the table, leading to outcome O_1. ◀

A game presented in this way is called a game in *strategic form* or a game in *normal form*.

Definition 4.2 *A game in* strategic form *(or in* normal form*) is an ordered triple* $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$*, in which:*

- $N = \{1, 2, \dots, n\}$ *is a finite set of players.*
- S_i *is the set of strategies of player* i*, for every player* $i \in N$.

We denote the set of all vectors of strategies by $S = S_1 \times S_2 \times \cdots \times S_n$.

- $u_i : S \to \mathbb{R}$ *is a function associating each vector of strategies* $s = (s_i)_{i\in N}$ *with the payoff (= utility)* $u_i(s)$ *to player* i*, for every player* $i \in N$.

In this definition, the sets of strategies available to the players are not required to be finite, and in fact we will see games with infinite strategy sets in this book. A game in which the strategy set of each player is finite is termed a *finite game*. The fact that u_i is a function of the vector of strategies s, and not solely of player i's strategy s_i, is what makes this a *game*, i.e., a situation of interactive decisions in which the outcome for each player depends not on his strategy alone, but on the strategies chosen by all the players.

Example 4.3 **Rock, Paper, Scissors** In the game "Rock, Paper, Scissors," each one of two players chooses an action from three alternatives: Rock, Paper, or Scissors. The actions are selected by the players simultaneously, with a circular dominance relationship obtaining between the three alternatives: rock smashes scissors, which cut paper, which in turn covers rock. The game in extensive form is described in Figure 4.3 in which the terminal vertices are labeled by the outcomes "I wins," "II wins," or D (for draw).

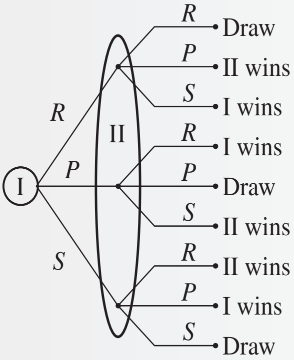

Figure 4.3 Rock, Paper, Scissors as a game in extensive form

Setting the payoff to a player to be 1 for a win, −1 for a loss, and 0 for a draw, we obtain the game in strategic form appearing in Figure 4.4. In each cell in Figure 4.4 the left number denotes the payoff to Player I and the right number denotes the payoff to Player II.

		Player II		
		Rock	Paper	Scissors
	Rock	0, 0	1, −1	1, −1
Player I	Paper	1, −1	0, 0	− 1, 1
	Scissors	−1, 1	−1, 1	0, 0

Figure 4.4 Rock, Paper, Scissors as a strategic-form game

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Games in strategic form are sometimes called *matrix games* because they are described by matrices.[2] When the number of players n is greater than 2, the corresponding matrix is n-dimensional, and each cell of the matrix contains a vector with n coordinates, representing the payoffs to the n players.

When there are no chance moves, a game in strategic form is derived from a game in extensive form in the following way:

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[2] When $n = 2$ it is customary to call these games *bimatrix games*, as they are given by two matrices, one for the payoff of each player.

- List the set of all strategies S_i available to each player i in the extensive-form game.
- For each vector of strategies $s = (s_i)_{i \in N}$ find the play determined by this vector of strategies, and then derive the payoffs induced by this play:

$$u(s) := (u_1(s), u_2(s), \dots, u_n(s)).$$

- Draw the appropriate n-dimensional matrix. When there are two players, the number of rows in the matrix equals the number of strategies of Player I, the number of columns equals the number of strategies of Player II, and the pair of numbers appearing in each cell is the pair of payoffs defined by the pair of strategies associated with that cell. When there are more than two players, the matrix is multi-dimensional (see Exercises 4.17 and 4.18 for examples of games with three players).

How is a strategic-form game derived from an extensive-form game when there are chance moves? In that case, every strategy vector $s = (s_i)_{i \in N}$ determines a probability distribution μ_s over the set O of the game's possible outcomes, where for each $o \in O$ the value of $\mu_s(o)$ is the probability that if the players play according to strategy vector s the outcome will be o. The cell corresponding to strategy vector s contains the average of the payoffs corresponding to the possible outcomes according to this probability distribution, i.e., the vector $u(s) = (u_i(s))_{i \in N} \in \mathbb{R}^N$ defined by

$$u_i(s) := \sum_{o \in O} \mu_s(o) \times u_i(o). \tag{4.1}$$

Since we are assuming that the preference relations of all player satisfy the von Neumann–Morgenstern axioms, $u_i(s)$ is the utility that player i receives from the lottery over the outcomes of the game that is induced when the players play according to strategy vector s.

Example 4.4 Consider the game in extensive form presented in Figure 4.5.

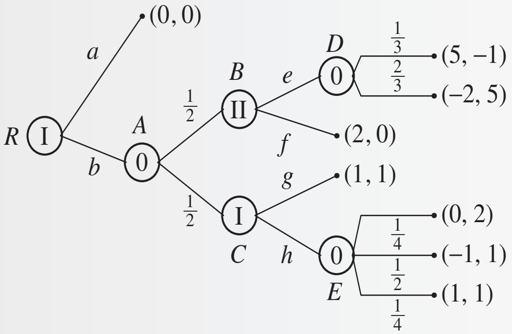

Figure 4.5 An extensive-form game with chance moves

In this game the outcome is a payoff to each of the players. This is a game of perfect information. Player I has two decision nodes, in each of which he has two possible actions. Player I's strategy set is therefore

$$S_{\mathrm{I}} = \{(a, g), (a, h), (b, g), (b, h)\}. \tag{4.2}$$

Player II has one decision node with two possible actions, so that Player II's strategy set is

$$S_{\mathrm{II}} = \{e, f\}. \tag{4.3}$$

To see how the payoffs are calculated, look, for example, at Player I's strategy (b, g) and at Player II's strategy e. If the players choose these strategies, three possible plays can occur with positive probability:

- The play $R \to A \to B \to D \to (5, -1)$, with probability $\frac{1}{6}$.
- The play $R \to A \to B \to D \to (-2, 5)$, with probability $\frac{1}{3}$.
- The play $R \to A \to C \to (1, 1)$, with probability $\frac{1}{2}$.

It follows that the expected payoff is

$$\tfrac{1}{6}(5, -1) + \tfrac{1}{3}(-2, 5) + \tfrac{1}{2}(1, 1) = \left(\tfrac{2}{3}, 2\right). \tag{4.4}$$

We can similarly calculate the payoffs to each pair of strategies. The resulting strategic-form game appears in Figure 4.6 (verify!).

Player II

		f	e
Player I	(a, g)	$0, 0$	$0, 0$
	(a, h)	$0, 0$	$0, 0$
	(b, g)	$\frac{3}{2}, \frac{1}{2}$	$\frac{2}{3}, 2$
	(b, h)	$\frac{7}{8}, \frac{5}{8}$	$\frac{1}{24}, \frac{17}{8}$

Figure 4.6 The strategic form of the game in Figure 4.5

◀

In the game in Figure 4.6, Player I's two strategies (a, g) and (a, h) correspond to the same row of payoffs. This means that, independently of Player II's strategy, the strategy (a, g) leads to the same payoffs as does the strategy (a, h). We say that these two strategies are *equivalent*. This equivalence can be understood by considering the corresponding game in extensive form (Figure 4.5): when Player I chooses R (at vertex a), the choice between g and h has no effect on the outcome of the game, because the play never arrives at vertex C. We can therefore represent the two strategies (a, g) and (a, h) by one strategy, (a), and derive the strategic-form game described in Figure 4.7.

A strategic-form game in which every set of equivalent strategies is represented by a single strategy ("the equivalence set") is called *a game in reduced strategic form*. This is essentially the form of the game that is arrived at when we take into account the fact that a particular action by a player excludes reaching some information sets of that player. In that case, there is no need to specify his strategies at those information sets.

		Player II	
		f	*e*
	(*a*)	0, 0	0, 0
Player I	(*b*, *g*)	$\frac{3}{2}, \frac{1}{2}$	$\frac{2}{3}, 2$
	(*b*, *h*)	$\frac{7}{8}, \frac{5}{8}$	$\frac{1}{24}, \frac{17}{8}$

Figure 4.7 The reduced strategic form of the game in Figure 4.5

Example 4.5 **The game of chess in strategic form** The number of strategies in the game of chess is immense even if we impose a maximal (large) number of moves after which the outcome is declared as a draw. There is no practical way to write down its game matrix (just as there is no practical way to present the game in extensive form). But it is significant that in principle the game can be represented by a finite matrix (even if its size is astronomic) (see Figure 4.8). The only possible outcomes of the game appearing in the cells of the matrix are W (victory for White), B (victory for Black), and D (draw).

		Black					
		1	2	3	·	·	·
	1	*W*	*D*	*W*	·	·	·
	2	*D*	*D*	*B*	·	·	·
White	3	*B*	*B*	*D*	·	·	·
	·	·	·	·	·	·	·
	·	·	·	·	·	·	·
	·	·	·	·	·	·	·

Figure 4.8 The game of chess in strategic form

A winning strategy for the White player (if one exists) would be represented by a row all of whose elements are W. A winning strategy for the Black player (if one exists) would be represented in this matrix by a column all of whose elements are B. A strategy ensuring at least a draw for White (or Black) is a row (or a column) all of whose elements are D or W (or B or D).

It follows from Theorem 1.4 (page 3) that in the matrix representing the game of chess, one and only one of the following alternatives holds:

1. There is a row all of whose elements are W.
2. There is a column all of whose elements are B.
3. There is a row all of whose elements are W or D, and a column all of whose elements are B or D.

If the third possibility obtains, then the cell at the intersection of the row ensuring at least a draw for White and the column guaranteeing at least a draw for Black must contain D: if both players are playing a strategy guaranteeing at least a draw, then the outcome of the play must necessarily be a draw. ◀

4.2 The relationship between the extensive form and the strategic form

We have shown that every extensive-form game can be associated with a unique reduced strategic-form game (meaning that every set of equivalent strategies in the extensive-form game is represented by one strategy in the strategic-form game). We have also exhibited a way of deriving a strategic-form game from an extensive-form game. There are two natural questions that arise with respect to the inverse operation: Does every strategic-form game have an extensive-form game from which it is derived? Is there a unique extensive-form game associated with a strategic-form game? The answer to the first question is affirmative, while the answer to the second question is negative. To show that the first question has an affirmative answer, we will now describe how to associate an extensive-form game with a given strategic-form game.

Let $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a game in strategic form, and denote the strategies of each player i by $S_i = \{s_i^1, \dots, s_i^{m_i}\}$. The reader can verify that G is the strategic-form game associated with the extensive-form game that appears in Figure 4.9.

This is a natural description that is also called "the canonical representation" of the game. It captures the main characteristic of a strategic-form game: in essence, the players choose their strategies simultaneously. This property is expressed in Figure 4.9 by the fact that each player has a single information set. For example, despite the fact that in the extensive-form game Player 1 chooses his strategy first, none of the other players, when coming to choose their strategies, know which strategy Player 1 has chosen. Clearly, the order of the players that appear in Figure 4.10 can be selected arbitrarily. Since there are $n!$ permutations over the set of n players, and each permutation defines a different ordering of the players, there are $n!$ such extensive-form canonical descriptions of the same strategic-form game.

Are there other, significantly different, ways of describing the same strategic-form game? The answer is positive. For example, each one of the three games in Figure 4.11 yields the two-player strategic-form game of Figure 4.10.

Representation A in Figure 4.11 is the canonical representation of the game. In representation C we have changed the order of the players: instead of Player I playing first followed by Player II, we have divided the choice made by Player II into two parts: one choice is made before Player I makes his selection, and one afterwards. As neither player knows which strategy was selected by the other player, the difference is immaterial to the game. Representation B is more interesting, because in that game Player II knows Player I's selection before he makes his selection (verify that the strategic form of each of the extensive-form games in Figure 4.11 is identical to the strategic-form game in Figure 4.10.)

The fact that a single strategic-form game can be derived from several different extensive-form games is not surprising, because the strategic-form description of a game is a condensed description of the extensive-form game. It ignores many of the dynamic aspects of the extensive-form description. An interesting mathematical question that arises here is "what is the extent of the difference" between two extensive-form games associated with the same strategic-form game? Given two extensive-form games, is it possible

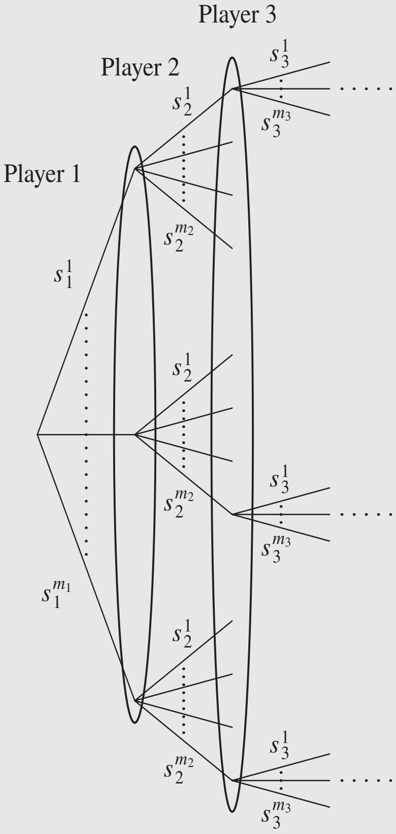

Figure 4.9 A canonical representation of a strategic-form game as an extensive-form game

		Player II			
		$r\tilde{r}$	$r\tilde{l}$	$l\tilde{r}$	$l\tilde{l}$
Player I	L	$2, -1$	$2, -1$	$2, 0$	$2, 0$
	R	$3, 1$	$1, 0$	$3, 1$	$1, 0$

Figure 4.10 The strategic-form game derived from each of the three games in Figure 4.11

to identify whether or not they yield the same strategic-form game, without explicitly calculating their strategic-form representation? This subject was studied by Thompson [1952], who defined three elementary operations that do not change the "essence" of a game. He then proved that if two games in extensive form with the same set of players can be transformed into each other by a finite number of these three elementary operations,

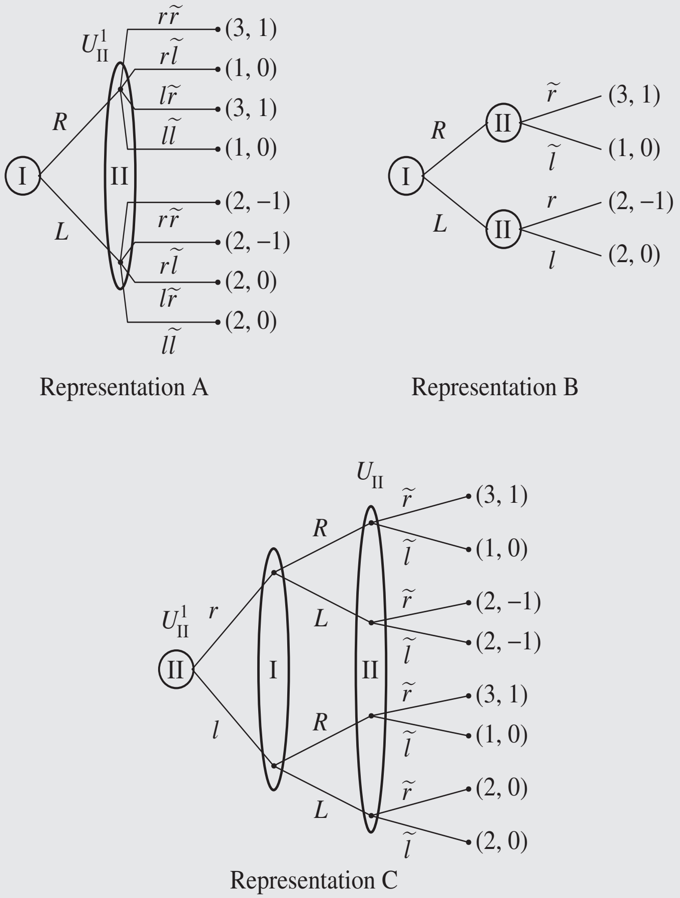

Figure 4.11 Three extensive-form games corresponding to the same strategic-form game in Figure 4.10

then those two extensive-form games correspond to the same strategic-form game. He also showed that the other direction obtains: if two games in extensive form yield the same strategic-form game, then they can be transformed into each other by a finite number of these three elementary operations.

4.3 Strategic-form games: solution concepts

We have dealt so far only with the different ways of describing games in extensive and strategic form. We discussed von Neumann's theorem in the special case of two players and three possible outcomes: victory for White, a draw, or victory for Black. Now we will look at more general games, and consider the central question of game theory: What can we say about what "will happen" in a given game? First of all, note that this question has at least three different possible interpretations:

1. An empirical, descriptive interpretation: How do players, in fact, play in a given game?
2. A normative interpretation: How “should” the players play in a given game?
3. A theoretical interpretation: What can we predict will happen in a game given certain assumptions regarding “reasonable” or “rational” behavior on the part of the players?

Descriptive game theory deals with the first interpretation. This field of study involves observations of the actual behavior of players, both in real-life situations and in artificial laboratory conditions where they are asked to play games and their behavior is recorded. This book will not address that area of game theory.

The second interpretation would be appropriate for a judge, legislator, or arbitrator called upon to determine the outcome of a game based on several agreed-upon principles, such as justice, efficiency, nondiscrimination, and fairness. This approach is best suited for the study of cooperative games, in which binding agreements are possible, enabling outcomes to be derived from “norms” or agreed-upon principles, or determined by an arbitrator who bases his decision on those principles. This is indeed the approach used for the study of bargaining games (see Chapter 15) and the Shapley value (see Chapter 18).

In this chapter we will address the third interpretation, the theoretical approach. After we have described a game, in either extensive or strategic form, what can we expect to happen? What outcomes, or set of outcomes, will reasonably ensue, given certain assumptions regarding the behavior of the players?

4.4 Notation

Let $N = \{1, \ldots, n\}$ be a finite set, and for each $i \in N$ let X_i be any set. Denote $X := \times_{i \in N} X_i$, and for each $i \in N$ define $X_{-i} := \times_{j \neq i} X_j$. For each $i \in N$ we will denote by

$$X_{-i} = \bigtimes_{j \neq i} X_j \tag{4.5}$$

the Cartesian product of all the sets X_j except for the set X_i. In other words,

$$X_{-i} = \{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \colon x_j \in X_j, \quad \forall_j \neq i\}. \tag{4.6}$$

An element in X_{-i} will be denoted $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, which is the $(n-1)$-dimensional vector derived from $(x_1, \ldots, x_n) \in X$ by suppressing the i-th coordinate.

4.5 Domination

Consider the two-player game that appears in Figure 4.12, in which Player I chooses a row and Player II chooses a column.

Comparing Player II’s strategies M and R, we find that:

- If Player I plays T, the payoff to Player II under strategy M is 2, compared to only 1 under strategy R.
- If Player I plays B, the payoff to Player II under strategy M is 1, compared to only 0 under strategy R.

			Player II	
		L	M	R
Player I	T	1, 0	1, 2	0, 1
	B	0, 3	0, 1	2, 0

Figure 4.12 Strategy M dominates strategy R

We see that independently of whichever strategy is played by Player I, strategy M always yields a higher payoff to Player II than strategy R. This motivates the following definition:

Definition 4.6 *A strategy s_i of player i is* strictly dominated *if there exists another strategy t_i of player i such that for each strategy vector $s_{-i} \in S_{-i}$ of the other players,*

$$u_i(s_i, s_{-i}) < u_i(t_i, s_{-i}). \tag{4.7}$$

If this is the case, we say that s_i is strictly dominated *by t_i, or that t_i* strictly dominates *s_i.*

In the example in Figure 4.12 strategy R is strictly dominated by strategy M. It is therefore reasonable to assume that if Player II is "rational," he will not choose R, because under any scenario in which he might consider selecting R, the strategy M would be a better choice. This is the first rationality property that we assume.

Assumption 4.7 *A rational player will not choose a strictly dominated strategy.*

We will assume that all the players are rational.

Assumption 4.8 *All players in a game are rational.*

Can a strictly dominated strategy (such as strategy R in Figure 4.12) be eliminated, under these two assumptions? The answer is: not necessarily. It is true that if Player II is rational he will not choose strategy R, but if Player I does not know that Player II is rational, he is liable to believe that Player II may choose strategy R, in which case it would be in Player I's interest to play strategy B. So, in order to eliminate the strictly dominated strategies one needs to postulate that:

- Player II is rational, and
- Player I knows that Player II is rational.

On further reflection, it becomes clear that this, too, is insufficient, and we also need to assume that:

- Player II knows that Player I knows that Player II is rational.

Otherwise, Player II would need to consider the possibility that Player I may play B, considering R to be a strategy contemplated by Player II, in which case Player II may be tempted to play L. Once again, further scrutiny reveals that this is still insufficient, and we need to assume that:

		Player II	
		L	M
Player I	T	1, 0	1, 2
	B	0, 3	0, 1

Figure 4.13 The game in Figure 4.12 after the elimination of strategy R

		Player II	
		L	M
Player I	T	1, 0	1, 2

Figure 4.14 The game in Figure 4.12 following the elimination of strategies R and B

- Player I knows that Player II knows that Player I knows that Player II is rational.
- Player II knows that Player I knows that Player II knows that Player I knows that Player II is rational.
- And so forth.

If all the infinite conditions implied by the above are satisfied, we say that the fact that Player II is rational is *common knowledge* among the players. This is an important concept underlying most of our presentation. Here we will give only an informal presentation of the concept of common knowledge. A formal definition appears in Chapter 9, where we extensively study common knowledge.

Definition 4.9 *A fact is* common knowledge *among the players of a game if for any finite chain of players* $i_1, i_2, \ldots, i_k$ *the following holds: player* i_1 *knows that player* i_2 *knows that player* i_3 *knows . . . that player* i_k *knows the fact.*

The chain in Definition 4.9 may contain several instances of the same player (as indeed happens in the above example). Definition 4.9 is an informal definition since we have not formally defined what the term “fact” means, nor have we defined the significance of knowing a fact. We will now add a further assumption to the two assumptions listed above:

Assumption 4.10 *The fact that all players are rational (Assumption 4.8) is common knowledge among the players.*

Strictly dominated strategies can be eliminated under Assumptions 4.7, 4.8, and 4.10 (we will not provide a formal proof of this claim). In the example in Figure 4.12, this means that, given the assumptions, we should focus on the game obtained by the elimination of strategy R, which appears in Figure 4.13.

In this game strategy B of Player I is strictly dominated by strategy T. Because the rationality of Player I is common knowledge, as is the fact that B is a strictly dominated strategy, after the elimination of strategy R, strategy B can also be eliminated. The players therefore need to consider a game with even fewer strategies, which is given in Figure 4.14.

Because in this game strategy L is strictly dominated (for Player II) by strategy M, after its elimination only one result remains, (1, 2), which obtains when Player I plays T and Player II plays M.

The process we have just described is called *iterated elimination of strictly dominated strategies*. When this process yields a single strategy vector (one strategy per player), as in the example above, then, under Assumptions 4.7, 4.8, and 4.10, that is the strategy vector that will obtain, and it may be regarded as the *solution* of the game.

A special case in which such a solution is guaranteed to exist is the family of games in which every player has a strategy that strictly dominates all of his other strategies, that is, a *strictly dominant strategy*. Clearly, in that case, the elimination of all strictly dominated strategies leaves each player with only one strategy: his strictly dominant strategy. When this occurs we say that the game has a solution in *strictly dominant strategies*.

Example 4.11 **The Prisoner's Dilemma** The "Prisoner's Dilemma" is a very simple game that is interesting in several respects. It appears in the literature in the form of the following story.

Two individuals who have committed a serious crime are apprehended. Lacking incriminating evidence, the prosecution can obtain an indictment only by persuading one (or both) of the prisoners to confess to the crime. Interrogators give each of the prisoners – both of whom are isolated in separate cells and unable to communicate with each other – the following choices:

1. If you confess and your friend refuses to confess, you will be released from custody and receive immunity as a state's witness.
2. If you refuse to confess and your friend confesses, you will receive the maximum penalty for your crime (ten years of incarceration).
3. If both of you sit tight and refuse to confess, we will make use of evidence that you have committed tax evasion to ensure that both of you are sentenced to a year in prison.
4. If both of you confess, it will count in your favor and we will reduce each of your prison terms to six years.

This situation defines a two-player strategic-form game in which each player has two strategies: D, which stands for Defection, betraying your fellow criminal by confessing, and C, which stands for Cooperation, cooperating with your fellow criminal and not confessing the crime. In this notation, the outcome of the game (in prison years) is shown in Figure 4.15.

		Player II	
		D	C
Player I	D	6, 6	0, 10
	C	10, 0	1, 1

Figure 4.15 The Prisoner's Dilemma in prison years

As usual, the left-hand number in each cell of the matrix represents the outcome (in prison years) for Player I, and the right-hand number represents the outcome for Player II.

We now present the game in utility units. For example, suppose the utility of both players is given by the following function u:

$$u(\text{release}) = 5, \qquad u(\text{one year in prison}) = 4,$$
$$u(\text{6 years in prison}) = 1, \qquad u(\text{10 years in prison}) = 0.$$

The game in utility terms appears in Figure 4.16.

		Player II	
		D	C
Player I	D	1, 1	5, 0
	C	0, 5	4, 4

Figure 4.16 The Prisoner's Dilemma in utility units

For both players, strategy D (Defect) strictly dominates strategy C (Cooperate). Elimination of strictly dominated strategies leads to the single solution (D, D) in which both prisoners confess, resulting in the payoff $(1, 1)$. ◀

What makes the Prisoner's Dilemma interesting is the fact that if both players choose strategy C, the payoff they receive is $(4, 4)$, which is preferable for both of them. The solution derived from Assumptions 4.7, 4.8, and 4.10, which appear to be quite reasonable assumptions, is "inefficient": The pair of strategies (C, C) is unstable, because each individual player can deviate (by defecting) and gain an even better payoff of 5 (instead of 4) for himself (at the expense of the other player, who would receive 0).

In the last example, two strictly dominated strategies were eliminated (one per player), but there was no specification regarding the order in which these strategies were eliminated: was Player I's strategy C eliminated first, or Player II's, or were they both eliminated simultaneously? In this case, a direct verification reveals that the order of elimination makes no difference. It turns out that this is a general result: whenever iterated elimination of strictly dominated strategies leads to a single strategy vector, that outcome is independent of the order of elimination. In fact, we can make an even stronger statement: even if iterated elimination of strictly dominated strategies yields a set of strategies (not necessarily a single strategy), that set does not depend on the order of elimination (see Exercise 4.10).

There are games in which iterated elimination of strictly dominated strategies does not yield a single strategy vector. For example, in a game that has no strictly dominated strategies, the process fails to eliminate any strategy. The game in Figure 4.17 provides an example of such a game.

Although there are no strictly dominated strategies in this game, strategy B does have a special attribute: although it does not always guarantee a higher payoff to Player I relative to strategy T, in all cases it does grant him a payoff at least as high, and in the special case in which Player II chooses strategy L, B is a strictly better choice than T. In this case we say that strategy B *weakly dominates* strategy T (and strategy T is *weakly dominated* by strategy B).

		Player II	
		L	R
Player I	T	1, 2	2, 3
	B	2, 2	2, 0

Figure 4.17 A game with no strictly dominated strategies

Definition 4.12 *Strategy s_i of player i is termed* weakly dominated *if there exists another strategy t_i of player i satisfying the following two conditions:*

(a) For every strategy vector $s_{-i} \in S_{-i}$ of the other players,

$$u_i(s_i, s_{-i}) \le u_i(t_i, s_{-i}). \tag{4.8}$$

(b) There exists a strategy vector $t_{-i} \in S_{-i}$ of the other players such that

$$u_i(s_i, t_{-i}) < u_i(t_i, t_{-i}). \tag{4.9}$$

In this case we say that strategy s_i is weakly dominated *by strategy t_i, and that strategy t_i* weakly dominates *strategy s_i.*

If strategy t_i dominates (weakly or strictly) strategy s_i, then s_i does not (weakly or strictly) dominate t_i. Clearly, strict domination implies weak domination. Because we will refer henceforth almost exclusively to weak domination, we use the term "domination" to mean "weak domination," unless the term "strict domination" is explicitly used. The following rationality assumption is stronger than Assumption 4.7.

Assumption 4.13 *A rational player does not use a dominated strategy.*

Under Assumptions 4.8, 4.10, and 4.13 we may eliminate strategy T in the game in Figure 4.17 (as it is weakly dominated), and then proceed to eliminate strategy R (which is strictly dominated after the elimination of T). The only remaining strategy vector is (B, L), with a payoff of (2, 2). Such a strategy vector is called *rational*, and the process of iterative elimination of weakly dominated strategies is called *rationalizability*. The meaning of "rationalizability" is that a player who expects a certain strategy vector to obtain can explain to himself why that strategy vector will be reached, based on the assumption of rationality.

Definition 4.14 *A strategy vector $s \in S$ is termed* rational *if it is the unique result of a process of iterative elimination of weakly dominated strategies.*

Whereas Assumption 4.7 looks reasonable, Assumption 4.13 is quite strong. Reinhard Selten, in trying to justify Assumption 4.13, suggested a concept he termed the *trembling hand principle*. The basic postulate of this principle is that every single strategy available to a player may be used with positive probability, which may well be extremely small. This may happen simply by mistake (the player's hand might tremble as he reaches to press the button setting in motion his chosen strategy, so that by mistake the button associated with a different strategy is activated instead), by irrationality on the part of the player, or because the player chose a wrong strategy due to miscalculations. This topic will be explored in greater depth in Section 7.3 (page 262).

To illustrate the trembling hand principle, suppose that Player II in the example of Figure 4.17 chooses strategies L and R with respective probabilities x and $1 - x$, where $0 < x < 1$. The expected payoff to Player I in that case is $x + 2(1 - x) = 2 - x$ if he chooses strategy T, as opposed to a payoff of 2 if he chooses strategy B. It follows that strategy B grants him a strictly higher expected payoff than T, so that a rational Player I facing Player II who has a trembling hand will choose B and not T; i.e., he will not choose the weakly dominated strategy.

The fact that strategy s_i of player i (weakly or strictly) dominates his strategy t_i depends only on player i's payoff function, and is independent of the payoff functions of the other players. Therefore, a player can eliminate his dominated strategies even when he does not know the payoff functions of the other players. This property will be useful in Section 4.6. In the process of rationalizability we eliminate dominated strategies one after the other. Eliminating strategy s_i of player i after strategy s_j of player j means that we assume that player i believes that player j will not implement s_j. This assumption is reasonable only if player i knows player j's payoff function. Therefore, the process of iterative elimination of dominated strategies can be justified only if the payoff functions of the players are common knowledge among them; if this condition does not hold, this process is harder to justify.

The process of rationalizability – iterated elimination of dominated strategies – is an efficient tool that leads, sometimes surprisingly, to significant results. The following example, taken from the theory of auctions, provides an illustration.

4.6 Second-price auctions

A detailed study of auction theory is presented in Chapter 12. In this section we will concentrate on the relevance of the concept of dominance to auctions known as *sealed-bid second-price auctions*, which are conducted as follows:

- An indivisible object is offered for sale.
- The set of buyers in the auction is denoted by N. Each buyer i attaches a value v_i to the object; that is, he is willing to pay at most v_i for the object (and is indifferent between walking away without the object and obtaining it at price v_i). The value v_i is buyer i's *private value*, which may arise from entirely subjective considerations, such as his preference for certain types of artistic objects or styles, or from potential profits (for example, the auctioned object might be a license to operate a television channel). This state of affairs motivates our additional assumption that each buyer i knows his own private value v_i but not the values that the other buyers attach to the object. This does not, however, prevent him from assessing the private values of the other buyers, or from believing that he knows their private values with some level of certainty.
- Each buyer i bids a price b_i (presented to the auctioneer in a sealed envelope).

- The winner of the object is the buyer who makes the highest bid. That may not be surprising, but in contrast to the auctions most of us usually see, the winner does not proceed to pay the bid he submitted. Instead he pays the *second-highest* price offered (hence the name second-price auction). If several buyers bid the same maximal price, a fair lottery is conducted between them to determine who will receive the object in exchange for paying that amount (which in this case is also the second-highest price offered.)

Let us take a concrete example. Suppose there are four buyers respectively bidding 5, 15, 2, and 21. The buyer bidding 21 is the winner, paying 15 in exchange for the object. In general, the winner of the auction is a buyer i for which

$$b_i = \max_{j \in N} b_j. \tag{4.10}$$

If buyer i is the winner, the amount he pays is $\max_{j \neq i} b_j$. We now proceed to describe a sealed-bid second-price auction as a strategic-form game:[3]

1. The set of players is the set N of buyers in the auction.
2. The set of strategies available to buyer i is the set of possible bids $S_i = [0, \infty)$.
3. The payoff to buyer i, when the strategy vector is $b = (b_1, \ldots, b_n)$, is

$$u_i(b) = \begin{cases} 0 & \text{if } b_i < \max_{j \in N} b_j, \\ \frac{v_i - \max_{j \neq i} b_j}{|\{k \colon b_k = \max_{j \in N} b_j\}|} & \text{if } b_i = \max_{j \in N} b_j. \end{cases} \tag{4.11}$$

How should we expect a rational buyer to act in this auction? At first glance, this appears to be a very difficult problem to solve, because no buyer knows the private values of his competitors, let alone the prices they will bid. He may not even know how many other buyers are participating in the auction. So what price b_i will buyer i bid? Will he bid a price lower than v_i, in order to ensure that he does not lose money in the auction, or higher than v_i, in order to increase his probability of winning, all the while hoping that the second-highest bid will be lower than v_i? The process of rationalizability leads to the following result:

Theorem 4.15 *In a second-price sealed-bid auction, the strategy $b_i = v_i$ weakly dominates all other strategies.*

In other words, under Assumptions 4.8, 4.10, and 4.13, the auction will proceed as follows:

- Every buyer will bid $b_i = v_i$.
- The winner will be the buyer whose private valuation of the object is the highest.[4] The price paid by the winning buyer (i.e., the object's sale price) is the second-highest private value. If several buyers share the same maximal bid, one of them, selected randomly by a fair lottery, will get the object, and will pay his private value (which in this special case is also the second-highest bid, and his profit will therefore be 0).

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[3] The relation between this auction method and other, more familiar, auction methods is discussed in Chapter 12.
[4] This property is termed *efficiency* in the game theory and economics literature.

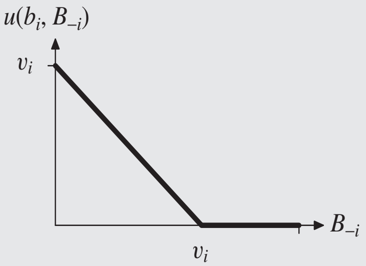

Figure 4.18 The payoff function for strategy $b_i = v_i$

Each buyer knows his private value v_i and therefore he also knows his payoff function. Since buyers do not necessarily know each other's private value, they do not necessarily know each other's payoff functions. Nevertheless, as we mentioned on page 91, the concept of domination is defined also when a player does not know the other players' payoff functions.

Proof: Consider a buyer i whose private value is v_i. Divide the set of strategies available to him, $S_i = [0, \infty)$, into three subsets:

- The strategies in which his bid is less than v_i: $[0, v_i)$.
- The strategy in which his bid is equal to v_i: $\{v_i\}$.
- The strategies in which his bid is higher than v_i: (v_i, ∞).

We now show that strategy $b_i = v_i$ dominates all the strategies in the other two subsets.

Given the procedure of the auction, the payment eventually made by buyer i depends on the strategies selected by the other buyers, through their highest bid, and the number of buyers bidding that highest bid. Denote the maximal bid put forward by the other buyers by

$$B_{-i} = \max_{j \neq i} b_j, \tag{4.12}$$

and the number of buyers who offered this bid by

$$N_{-i} = \left| \left\{ k \neq i : b_k = \max_{j \neq i} b_j \right\} \right|. \tag{4.13}$$

The payoff function of buyer i, as a function of the strategy vector b (i.e., the vector of all the bids made by the buyers) is

$$u_i(b) = \begin{cases} 0 & \text{if } b_i < B_{-i}, \\ \frac{v_i - B_{-i}}{N_{-i}+1} & \text{if } b_i = B_{-i}, \\ v_i - B_{-i} & \text{if } b_i > B_{-i}. \end{cases} \tag{4.14}$$

Since the only dependence that the payoff function $u_i(b)$ has on the bids b_{-i} of the other buyers is via the highest bid, B_{-i}, we sometimes denote this function by $u_i(b_i, B_{-i})$. If buyer i chooses strategy $b_i = v_i$, his payoff as a function of B_{-i} is given in Figure 4.18.

If buyer i chooses strategy b_i satisfying $b_i < v_i$, his payoff function is given by Figure 4.19.

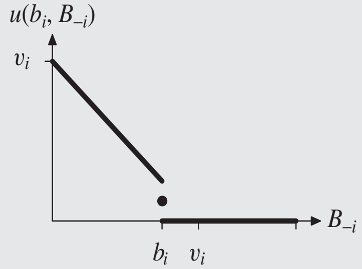

Figure 4.19 The payoff function for strategy $b_i < v_i$

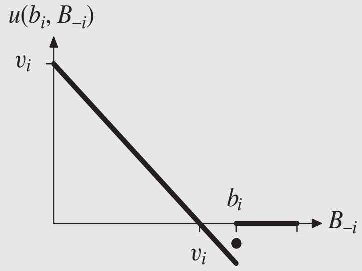

Figure 4.20 The payoff function for strategy $b_i > v_i$

The height of the dot in Figure 4.19, when $b_i = B_{-i}$, depends on the number of buyers who bid B_{-i}.

The payoff function in Figure 4.18 (which corresponds to the strategy $b_i = v_i$) is (weakly) greater than the one in Figure 4.19 (corresponding to a strategy b_i with $b_i < v_i$). The former is strictly greater than the latter when $b_i \leq B_{-i} < v_i$. It follows that strategy $b_i = v_i$ dominates all strategies in which the bid is lower than buyer i's private value.

The payoff function for a strategy b_i satisfying $b_i > v_i$ is displayed in Figure 4.20.

Again, we see that the payoff function in Figure 4.18 is (weakly) greater than the payoff function in Figure 4.20. The former is strictly greater than the latter when $v_i < B_{-i} \leq b_i$. It follows that the strategy in which the bid is equal to the private value weakly dominates all other strategies, as claimed. □

Theorem 4.15 holds also when some buyers do not know the number of buyers participating in the auction, their private values, their beliefs (about the number of buyers and the private values of the other buyers), and their utility functions (for example, information on whether the other players are risk seekers, risk averse, or risk neutral; see Section 2.7). The only condition needed for Theorem 4.15 to hold is that each buyer know the rules of the auction.

4.7 The order of elimination of dominated strategies

As we have argued, when only strictly dominated strategies are involved in a process of iterated elimination, the result is independent of the order in which strategies are eliminated (Exercise 4.10). In iterated elimination of weakly dominated strategies, the result may be sensitive to the order of elimination. This phenomenon occurs for example in the following game.

Example 4.16 Consider the strategic-form game that appears in Figure 4.21.

Player II

		L	C	R
	T	1, 2	2, 3	0, 3
Player I	M	2, 2	2, 1	3, 2
	B	2, 1	0, 0	1, 0

Figure 4.21 A game in which the order of the elimination of dominated strategies influences the yielded result

In the table below, we present three strategy elimination procedures, each leading to a different result (verify!).

	Order of elimination from left to right	Result	Payoff
(1)	T, R, B, C	ML	2, 2
(2)	B, L, C, T	MR	3, 2
(3)	T, C, R	ML or BL	2, 2 or 2, 1

The last line shows that eliminating strategies in the order T, C, R leaves two results ML and BL, with no possibility for further elimination because Player I is indifferent between the two results. This means that the order of elimination may determine not only the yielded strategy vector, but also whether or not the process yields a single strategy vector. ◀

4.8 Stability: Nash equilibrium

Dominance is a very important concept in game theory. As we saw in the previous section, it has several limitations, and it is insufficient for predicting a rational result in every game. In this section we present another important principle, *stability*.

Consider the following two-player game in strategic form (Figure 4.22).

There is no dominance relationship between the strategies in this game. For example, if we compare the strategies T and M of Player I, it turns out that neither of them is always preferable to the other: M is better than T if Player II chooses L, and T is better than M if Player II chooses C. In fact, M is the best reply of Player I to L, while T is his best reply

		Player II		
		L	C	R
Player I	T	0, 6	6, 0	4, 3
	M	6, 0	0, 6	4, 3
	B	3, 3	3, 3	5, 5

Figure 4.22 A two-player game with no dominated strategies

to C, and B is his best reply to R. Similarly, for Player II, L is the best reply to T and C is the best reply to M, while strategy R is the best reply to B.

A player who knows the strategies used by the other players is in effect playing a game in which only he is called upon to choose a strategy. If that player is rational, he will choose the best reply to those strategies used by the other players. For example, in the game in Figure 4.22:

- If Player II knows that Player I will choose T, he will choose L (his best reply to T).
- If Player I knows that Player II will choose L, he will choose M (his best reply to L).
- If Player II knows that Player I will choose M, he will choose C (his best reply to M).
- If Player I knows that Player II will choose C, he will choose T (his best reply to C).
- If Player II knows that Player I will choose B, he will choose R (his best reply to B).
- If Player I knows that Player II will choose R, he will choose B (his best reply to R).

The pair of strategies (B, R) satisfies a stability property: each strategy in this pair is the best reply to the other strategy. Alternatively, we can state this property in the following way: assuming the players choose (B, R), neither player has a *profitable deviation*; that is, under the assumption that the other player indeed chooses his strategy according to (B, R), neither player has a strategy that grants a higher payoff than sticking to (B, R). This stability property was defined by John Nash, who invented the equilibrium concept that bears his name.

Definition 4.17 *A strategy vector* $s^* = (s_1^*, \ldots, s_n^*)$ *is a* Nash equilibrium *if for each player* $i \in N$ *and each strategy* $s_i \in S_i$ *the following is satisfied:*

$$u_i(s^*) \geq u_i(s_i, s_{-i}^*). \tag{4.15}$$

The payoff vector $u(s^*)$ *is the* equilibrium payoff *corresponding to the Nash equilibrium* s^*.

The strategy $\widehat{s}_i \in S_i$ is *a profitable deviation* of player i at a strategy vector $s \in S$ if $u_i(\widehat{s}_i, s_{-i}) > u_i(s)$. A Nash equilibrium is a strategy vector at which no player has a profitable deviation.

The Nash equilibrium is often simply referred to as an *equilibrium*, and sometimes as an *equilibrium point*. As defined above, it says that no player i has a profitable unilateral deviation from s^*. The Nash equilibrium can be equivalently expressed in terms of the best-reply concept, which we first define.

Definition 4.18 *Let s_{-i} be a strategy vector of all the players not including player i. Player i's strategy s_i is termed a* best reply *to s_{-i} if*

$$u_i(s_i, s_{-i}) = \max_{t_i \in S_i} u_i(t_i, s_{-i}). \tag{4.16}$$

The next definition, based on the best-reply concept, is equivalent to the definition of Nash equilibrium in Definition 4.17 (Exercise 4.15).

Definition 4.19 *The strategy vector $s^* = (s_1^*, \ldots, s_n^*)$ is a* Nash equilibrium *if s_i^* is a best reply to s_{-i}^* for every player $i \in N$.*

In the example in Figure 4.22, the pair of strategies (B, R) is the unique Nash equilibrium (verify!). For example, the pair (T, L) is not an equilibrium, because T is not a best reply to L; Player I has a profitable deviation from T to M or to B. Out of all the nine strategy pairs, (B, R) is the only equilibrium (verify!).

Social behavioral norms may be viewed as Nash equilibria. If a norm were not an equilibrium, some individuals in society would find some deviation from that behavioral norm to be profitable, and it would cease to be a norm.

A great deal of research in game theory is devoted to identifying equilibria and studying the properties of equilibria in various games. One important research direction that has been emerging in recent years studies processes (such as learning, imitation, or regret) leading to equilibrium behavior, along with the development of algorithms for calculating equilibria.

Example 4.11 **The Prisoner's Dilemma (*continued*)** The Prisoner's Dilemma is presented in the matrix in Figure 4.23.

		Player II	
		D	C
Player I	D	1, 1	5, 0
	C	0, 5	4, 4

Figure 4.23 The Prisoner's Dilemma

The unique equilibrium is (D, D), in which both prisoners confess to the crime, resulting in payoff (1, 1). Recall that this is the same result that is obtained by elimination of strictly dominated strategies. ◀

Example 4.20 **Coordination game** The game presented in Figure 4.24 is an example of a broad class of games called "coordination games." In a coordination game, it is in the interests of both players to coordinate their strategies. In this example both (A, a) and (B, b) are equilibrium points. The equilibrium payoff associated with (A, a) is (1, 1), and the equilibrium payoff of (B, b) is (3, 3). In both cases, and for both players, the payoff is better than (0, 0), which is the payoff for "miscoordinated" strategies (A, b) or (B, a).

Player II

		a	*b*
Player I	*A*	1, 1	0, 0
	B	0, 0	3, 3

Figure 4.24 A coordination game

◀

Example 4.21 **Battle of the Sexes** The game in Figure 4.25 is called the "Battle of the Sexes."

Woman

		F	*C*
Man	*F*	2, 1	0, 0
	C	0, 0	1, 2

Figure 4.25 Battle of the Sexes

The name of the game is derived from the following description. A couple is trying to plan what they will be doing on the weekend. The alternatives are going to a concert (C) or watching a football match (F). The man prefers football and the woman prefers the concert, but both prefer being together to being alone, even if that means agreeing to the less-preferred recreational pastime.

There are two equilibrium points: (F, F) with a payoff of (2, 1) and (C, C) with a payoff of (1, 2). The woman would prefer the strategy pair (C, C) while the man would rather see (F, F) chosen. However, either one is an equilibrium. ◀

Example 4.22 **The Security Dilemma** The game illustrated in Figure 4.26 is also a coordination game called the "Security Dilemma." The game describes the situation involving the Union of Soviet Socialist Republics (USSR, Player 1) and the United States (US, Player 2) after the Second World War. Each of these countries had the capacity to produce nuclear weapons. The best outcome for each country (4 utility units in the figure) was the one in which neither country had nuclear weapons, because producing nuclear weapons is expensive and possession of such weapons is liable to lead to war with severe consequences. A less desirable outcome for each country (3 utility units in the figure) is for it to have nuclear weapons while the other country lacks nuclear weapons. Even less desirable for each country (2 utility units in the figure) is for both countries to have nuclear weapons. The worst outcome for a country (1 utility unit in the figure) is for it to lack nuclear weapons while the other country has nuclear weapons.

		US	
		Don't produce nuclear weapons	Produce nuclear weapons
USSR	Produce nuclear weapons	3, 1	2, 2
	Don't produce nuclear weapons	4, 4	1, 3

Figure 4.26 The Security Dilemma

There are two Nash equilibria in this game: in one equilibrium neither country produces nuclear weapons and in the other equilibrium both countries produce nuclear weapons. If the US believes that the USSR is not going to produce nuclear weapons then it has no reason to produce nuclear weapons, while if the US believes that the USSR is going to produce nuclear weapons then it would be better off producing nuclear weapons. In the first equilibrium each country runs the risk that the other country will produce nuclear weapons, but in the second equilibrium there is no such risk: if the US does produce nuclear weapons then if the USSR also produces nuclear weapons then the US has implemented the best strategy under the circumstances, while if the USSR does not produce nuclear weapons then the outcome for the US has improved from 2 to 3. In other words, the more desirable equilibrium for both players is also the more risky one. This is why this game got the name the Security Dilemma. Some have claimed that the equilibrium under which both countries produce nuclear weapons is the more reasonable equilibrium (and that is in fact the equilibrium that has obtained historically). Note that the maxmin strategy of each country is to produce nuclear weapons; that strategy guarantees a country implementing it at least 2, while a country implementing the strategy of not producing nuclear weapons runs the risk of getting only 1. ◀

Example 4.23 **Cournot[5] duopoly competition** Two manufacturers, labeled 1 and 2, produce the same product and compete for the same market of potential customers. The manufacturers simultaneously select their production quantities, with demand determining the market price of the product, which is identical for both manufacturers. Denote by q_1 and q_2 the quantities respectively produced by

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[5] Antoine Augustin Cournot, August 28, 1801–March 31, 1877, was a French philosopher and mathematician. In his book *Researches on the Mathematical Principles of the Theory of Wealth*, published in 1838, he presented the first systematic application of mathematical tools for studying economic theory. The book marks the beginning of modern economic analysis.

Manufacturers 1 and 2. The total quantity of products in the market is therefore $q_1 + q_2$. Assume that when the supply is $q_1 + q_2$ the price of each item is $2 - q_1 - q_2$. Assume also that the per-item production cost for the first manufacturer is $c_1 > 0$ and that for the second manufacturer it is $c_2 > 0$. Does there exist an equilibrium in this game? If so, what is it?

This is a two-player game (Manufacturers 1 and 2 are the players), and the strategy set of each player is $[0, \infty)$. If Player 1 chooses strategy q_1 and Player 2 chooses strategy q_2, the payoff for Player 1 is

$$u_1(q_1, q_2) = q_1(2 - q_1 - q_2) - q_1 c_1 = q_1(2 - c_1 - q_1 - q_2), \tag{4.17}$$

and the payoff for Player 2 is

$$u_2(q_1, q_2) = q_2(2 - q_1 - q_2) - q_1 c_2 = q_2(2 - c_2 - q_1 - q_2). \tag{4.18}$$

Player 1's best reply to Player 2's strategy q_2 is the value q_1 maximizing $u_1(q_1, q_2)$. The function $q_1 \mapsto u_1(q_1, q_2)$ is a quadratic function that attains its maximum at the point where the derivative of the function is zero:

$$\frac{\partial u_1}{\partial q_1}(q_1, q_2) = 0. \tag{4.19}$$

Differentiating the right-hand side of Equation (4.17) yields the first-order condition $2 - c_1 - 2q_1 - q_2 = 0$, or

$$q_1 = \frac{2 - c_1 - q_2}{2}. \tag{4.20}$$

Similarly, Player 2's best reply to Player 1's strategy q_1 is given by the point where the derivative of $u_2(q_1, q_2)$ with respect to q_2 is zero. Taking the derivative, we get

$$q_2 = \frac{2 - c_2 - q_1}{2}. \tag{4.21}$$

Solving equations (4.20) and (4.21) yields

$$q_1^* = \frac{2 - 2c_1 + c_2}{3}, \quad q_2^* = \frac{2 - 2c_2 + c_1}{3}. \tag{4.22}$$

A careful check indicates that this is an equilibrium (Exercise 4.24) and this is the only equilibrium of the game. The payoffs to the players at equilibrium are

$$u_1(q_1^*, q_2^*) = \left(\frac{2 - 2c_1 + c_2}{3}\right)^2 = (q_1^*)^2, \tag{4.23}$$

$$u_2(q_1^*, q_2^*) = \left(\frac{2 - 2c_2 + c_1}{3}\right)^2 = (q_2^*)^2. \tag{4.24}$$

For example, if the two players have identical production costs $c_1 = c_2 = 1$, then the equilibrium production quantities will be $q_1^* = q_2^* = \frac{1}{3}$, and the payoff to each player is $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$. ◀

4.9 Properties of the Nash equilibrium

The Nash equilibrium is the most central and important solution concept for games in strategic form or extensive form. To understand why, it is worthwhile to consider both the advantages and limitations of Nash's seminal concept.

		Player II	
		L	*R*
Player I	*T*	0, 0	4, 2
	B	3, 5	0, 0

Figure 4.27 A coordination game

4.9.1 Stability

The most important property expressed by the Nash equilibrium is stability: under Nash equilibrium, each player acts to his best possible advantage with respect to the behavior of the other players. Indeed, this would appear to be a requirement for any solution concept: if there is to be any "expected" result (by any conceivable theory predicting the result of a game), that result must be in equilibrium, because otherwise there will be at least one player with a profitable deviation and the "expected" result will not materialize. From this perspective, the Nash equilibrium is not a solution concept but rather a meta-solution: the stability property is one that we would like every "expected" or "reasonable" solution to exhibit.

4.9.2 A self-fulfilling agreement

Another way to express the property of stability is to require that if there is "agreement" to play a particular equilibrium, then, even if the agreement is not binding, it will not be breached: no player will deviate from the equilibrium point, because there is no way to profit from any unilateral violation of the agreement. This appears to be particularly convincing in games of coordination, as in the example in Figure 4.27.

This game has two Nash equilibria, (T, R) and (B, L), and it is reasonable to suppose that if the players were to communicate they would "agree" (probably after a certain amount of debate) to play one of them. The properties of the equilibrium concept imply that whether they choose (T, R) or (B, L) they will both fulfill the agreement and not deviate from it, because any unilateral deviation will bring about a loss to the deviator (and to the other player).

4.9.3 Equilibrium and evolution

The principle of the *survival of the fittest* is one of the fundamental principles of Darwin's Theory of Evolution. The principle postulates that our world is populated by a multitude of species of plants and animals, including many mutations, but only those whose inherited traits are fitter than those of others to withstand the test of survival will pass on their genes to posterity. For example, if an animal that has been endowed with certain inherited characteristics has on average four offspring who manage to live to adulthood, while a mutation of the animal with a different set of traits has on average only three offspring living to adulthood, then, over several generations, the descendants of the first animal will overwhelm the descendants of the mutation in absolute numbers.

A picturesque way of expressing Darwin's principle depicts an animal (or plant) being granted the capacity of rational intelligence prior to birth and selecting the genetic traits with which it will enter the world. Under that imaginary scenario we would expect the animal (or plant) to choose those traits that grant the greatest possible advantages in the struggle for survival. Animals, of course, are not typically endowed with rational thought and no animal can choose its own genetic inheritance. What actually happens is that those individuals born with traits that are a poor fit relative to the conditions for survival will pass those same characteristics on to their progeny, and over time their numbers will dwindle.

In other words, the surviving and prevailing traits are a kind of "best reply" to the environment – from which the relationship to the concept of Nash equilibrium follows. Section 5.8 (page 186) presents in greater detail how evolutionary processes can be modeled in game-theoretic terms and the role played by the Nash equilibrium in the theory of evolution.

4.9.4 Equilibrium from the normative perspective

Consider the concept of equilibrium from the normative perspective of an arbitrator or judge recommending a certain course of action (hopefully based on reasonable and acceptable principles). In that case we should expect the arbitrator's recommendation to be an equilibrium point. Otherwise (since it is a recommendation and not a binding agreement) there will be at least one agent who will be tempted to benefit from not following his end of the recommendation. Seeking equilibrium alone, however, is not enough for the arbitrator to arrive at a decision. If, for example, there is more than one equilibrium point, as in the coordination game in Figure 4.27, choosing between them requires more considerations and principles. A rich literature, in fact, deals with "refinements" of the concept of equilibrium, which seek to choose (or "invalidate") certain equilibria within the set of all possible equilibria. This subject will be discussed in Chapter 7.

Despite all its advantages, the Nash equilibrium is not the final be-all and end-all in the study of strategic- or extensive-form games. Beyond the fact that in some games there is no equilibrium and in others there may be a multiplicity of equilibria, even when there is a single Nash equilibrium it is not always entirely clear that the equilibrium will be the strategy vector that is "recommended" or "predicted" by a specific theory. There are many who believe, for example, that the unique equilibrium of the Prisoner's Dilemma does not constitute a "good recommendation" or a "good prediction" of the outcome of the game. We will later see additional examples in which it is unclear that an equilibrium will necessarily be the outcome of a game (cf. the first example in Section 4.10, the repeated Prisoner's Dilemma in Example 7.15 (page 259), the Centipede game (Examples 7.16, page 259), and Example 7.17 on page 261).

4.10 Security: the maxmin concept

As we have already pointed out, the concept of equilibrium, despite its advantages, does not always describe the expected behavior of rational players, even in those cases where an equilibrium exists and is unique. Consider, for example, the game described in Figure 4.28.

		Player II	
		L	R
Player I	T	2, 1	2, −20
	M	3, 0	−10, 1
	B	−100, 2	3, 3

Figure 4.28 A game with a unique but "dangerous" equilibrium

The unique equilibrium in this game is (B, R), with a payoff of $(3, 3)$. But thinking this over carefully, can we really expect this result to obtain with high probability? One can imagine Player I hesitating to choose B: what if Player II were to choose L (whether by accident, due to irrationality, or for any other reason)? Given that the result (B, L) is catastrophic for Player I, he may prefer strategy T, guaranteeing a payoff of only 2 (compared to the equilibrium payoff of 3), but also guaranteeing that he will avoid getting -100 instead. If Player II is aware of this hesitation, and believes that there is a reasonable chance that Player I will flee to the safety of T, he will also be wary of choosing the equilibrium strategy R (and risking the -20 payoff), and will likely choose strategy L instead. This, in turn, increases Player I's motivation to choose T.

This underscores an additional aspect of rational behavior that exists to some extent in the behavior of every player: guaranteeing the best possible result without "relying" on the rationality of the other players, and even making the most pessimistic assessment of their potential behavior.

So what can player i, in a general game, guarantee for himself? If he chooses strategy s_i, the worst possible payoff he can get is

$$\min_{t_{-i} \in S_{-i}} u_i(s_i, t_{-i}). \tag{4.25}$$

Player i can choose the strategy s_i that maximizes this value. In other words, disregarding the possible rationality (or irrationality) of the other players, he can guarantee for himself a payoff of

$$\underline{v}_i := \max_{s_i \in S_i} \min_{t_{-i} \in S_{-i}} u_i(s_i, t_{-i}). \tag{4.26}$$

The quantity $\underline{v}_i$ is called the *maxmin value* of player i, which is sometimes also called the player's *security level*. A strategy s_i^* that guarantees this value is called a *maxmin strategy*. Such a strategy satisfies

$$\min_{t_{-i} \in S_{-i}} u_i(s_i^*, t_{-i}) \geq \min_{t_{-i} \in S_{-i}} u_i(s_i, t_{-i}), \quad \forall s_i \in S_i, \tag{4.27}$$

which is equivalent to

$$u_i(s_i^*, t_{-i}) \geq \underline{v}_i, \quad \forall t_{-i} \in S_{-i}. \tag{4.28}$$

		Player II		
		L	R	$\min_{s_{\mathrm{II}} \in S_{\mathrm{II}}} u_{\mathrm{I}}(s_{\mathrm{I}}, s_{\mathrm{II}})$
	T	2, 1	2, −20	2
Player I	M	3, 0	−10, 1	−10
	B	−100, 2	3, 3	−100
$\min_{s_{\mathrm{I}} \in S_{\mathrm{I}}} u_{\mathrm{II}}(s_{\mathrm{I}}, s_{\mathrm{II}})$		0	−20	2, 0

Figure 4.29 The game in Figure 4.28 with the security value of each player

Remark 4.24 *The definition of a game in strategic form does not include a requirement that the set of strategies available to any of the players be finite. When the strategy set is infinite, the minimum in Equation (4.25) may not exist for certain strategies $s_i \in S_i$. Even if the minimum in Equation (4.25) is attained for every strategy $s_i \in S_i$, the maximum in Equation (4.26) may not exist. It follows that when the strategy set of one or more players is infinite, we need to replace the minimum and maximum in the definition of the maxmin value by infimum and supremum, respectively:*

$$\underline{v}_i := \sup_{s_i \in S_i} \inf_{t_{-i} \in S_{-i}} u_i(s_i, t_{-i}). \tag{4.29}$$

If the supremum is never attained there is no maxmin strategy: for each $\varepsilon > 0$ the player can guarantee for himself at least $\underline{v}_i - \varepsilon$, but not at least $\underline{v}_i$.

A continuous function defined over a compact domain always attains a maximum and a minimum. Moreover, when X and Y are compact sets in $\mathbb{R}^m$ and $f : X \times Y \to \mathbb{R}$ is a continuous function, the function $x \mapsto \min_{y \in Y} f(x, y)$ is also continuous (Exercise 4.22). It follows that when the strategy sets of the players are compact and the payoff functions are continuous, the maxmin strategies of the players are well defined. ◆

We will now proceed to calculate the value guaranteed by each strategy in the example in Figure 4.28. In Figure 4.29, the numbers in the right-most column (outside the payoff matrix) indicate the the worst payoff to Player I if he chooses the strategy of the corresponding row. Similarly, the numbers in the bottom-most row (outside the payoff matrix) indicate the worst payoff to Player II if he chooses the strategy of the corresponding column. Finally, the oval contains the maxmin value of both players.

The maxmin value of Player I is 2 and the strategy that guarantees this value is T. The maxmin value of Player II is 0 with maxmin strategy L. If the two players choose their maxmin strategies, the result is (T, L) with payoff (2, 1), in which Player II's payoff of 1 is greater than his maxmin value.

As the next example illustrates, a player may have several maxmin strategies. In such a case, when the players use maxmin strategies the payoff depends on which strategies they have chosen.

Example 4.25 Consider the two-player game appearing in Figure 4.30.

		Player II		
		L	R	$\min_{s_{\mathrm{II}} \in S_{\mathrm{II}}} u_{\mathrm{I}}(s_{\mathrm{I}}, s_{\mathrm{II}})$
Player I	T	3, 1	0, 4	0
	B	2, 3	1, 1	1
$\min_{s_{\mathrm{I}} \in S_{\mathrm{I}}} u_{\mathrm{II}}(s_{\mathrm{I}}, s_{\mathrm{II}})$		1	1	1, 1

Figure 4.30 A game with the maxmin values of the players

The maxmin value of Player I is 1 and his unique maxmin strategy is B. The maxmin value of Player II is 1, and both L and R are his maxmin strategies. It follows that when the two players implement maxmin strategies the payoff might be (2, 3), or (1, 1), depending on which maxmin strategy is implemented by Player II. ◀

We next explore the connection between the maxmin strategy and dominant strategies.

Theorem 4.26 *A strategy of player i that dominates all his other strategies is a maxmin strategy for that player. Such a strategy, furthermore, is a best reply of player i to any strategy vector of the other players.*

The proof of this theorem is left to the reader (Exercise 4.25). The theorem implies the following conclusion.

Corollary 4.27 *In a game in which every player has a strategy that dominates all of his other strategies, the vector of dominant strategies is an equilibrium point and a vector of maxmin strategies.*

An example of this kind of game is a sealed-bid second-price auction, as we saw in Section 4.6. The next theorem constitutes a strengthening of Corollary 4.27 in the case of strict domination (for the proof see Exercise 4.26).

Theorem 4.28 *In a game in which every player i has a strategy s_i^* that strictly dominates all of his other strategies, the strategy vector $(s_1^*, \ldots, s_n^*)$ is the unique equilibrium point of the game as well as the unique vector of maxmin strategies.*

Is there a relation between the maxmin value of a player and his payoff in a Nash equilibrium? As the next theorem states, the payoff of each player in a Nash equilibrium is at least his maxmin value.

Theorem 4.29 *Every Nash equilibrium σ^* of a strategic-form game satisfies $u_i(\sigma^*) \geq \underline{v}_i$ for every player i.*

Proof: For every strategy $s_i \in S_i$ we have

$$u_i(s_i, s_{-i}^*) \geq \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}). \qquad (4.30)$$

Since the definition of an equilibrium implies that $u_i(s^*) = \max_{s_i \in S_i} u_i(s_i, s^*_{-i})$, we deduce that

$$u_i(s^*) = \max_{s_i \in S_i} u_i(s_i, s^*_{-i}) \geq \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) = \underline{v}_i, \tag{4.31}$$

as required. □

4.11 The effect of elimination of dominated strategies

Elimination of dominated strategies was discussed in Section 4.5 (page 85). A natural question that arises is how does the process of iterative elimination of dominated strategies change the maxmin values and the set of equilibria of the game? We will show here that the elimination of strictly dominated strategies has no effect on a game's set of equilibria. The iterated elimination of weakly dominated strategies can reduce the set of equilibria, but it cannot create new equilibria. On the other hand, the maxmin value of any particular player is unaffected by the elimination of his dominated strategies, whether those strategies are weakly or strictly dominated.

Theorem 4.30 *Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a strategic-form game, and let $\widehat{s}_j \in S_j$ be a dominated strategy of player j. Let $\widehat{G}$ be the game derived from G by the elimination of strategy $\widehat{s}_j$. Then the maxmin value of player j in $\widehat{G}$ is equal to his maxmin value in G.*

Proof: The maxmin value of player j in G is

$$\underline{v}_j = \max_{s_j \in S_j} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j}), \tag{4.32}$$

and his maxmin value in $\widehat{G}$ is

$$\widehat{\underline{v}}_j = \max_{\{s_j \in S_j, s_j \neq \widehat{s}_j\}} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j}). \tag{4.33}$$

Let t_j be a strategy of player j that dominates $\widehat{s}_j$ in G. Then the following is satisfied:

$$u_j(\widehat{s}_j, s_{-j}) \leq u_j(t_j, s_{-j}), \quad \forall s_{-j} \in S_{-j}, \tag{4.34}$$

and therefore

$$\min_{s_{-j} \in S_{-j}} u_j(\widehat{s}_j, s_{-j}) \leq \min_{s_{-j} \in S_{-j}} u_j(t_j, s_{-j}) \leq \max_{\{s_j \in S_j, s_j \neq \widehat{s}_j\}} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j}). \tag{4.35}$$

This leads to the conclusion that

$$\underline{v}_j = \max_{s_j \in S_j} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j}) \tag{4.36}$$

$$= \max \left\{ \max_{\{s_j \in S_j, s_j \neq \widehat{s}_j\}} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j}), \min_{s_{-j} \in S_{-j}} u_j(\widehat{s}_j, s_{-j}) \right\} \tag{4.37}$$

$$= \max_{\{s_j \in S_j, s_j \neq \widehat{s}_j\}} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j}) = \widehat{\underline{v}}_j, \tag{4.38}$$

which is what we wanted to prove. □

Note that the elimination of a (strictly or weakly) dominated strategy of one player may increase the maxmin values of other players (but not decrease them; see Exercise 4.27).

It follows that when calculating the maxmin value of player i we can eliminate his dominated strategies, but we must not eliminate dominated strategies of other players, since this may result in increasing player i's maxmin value. Therefore, iterated elimination of (weakly or strictly) dominated strategies may increase the maxmin value of some players.

The next theorem states that if we eliminate some of the strategies of each player (whether or not they are dominated), then every equilibrium of the original game (the game prior to the elimination of strategies) is also an equilibrium of the game resulting from the elimination process, provided that none of the strategies of that equilibrium were eliminated.

Theorem 4.31 *Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a game in strategic form, and let $\widehat{G} = (N, (\widehat{S}_i)_{i \in N}, (u_i)_{i \in N})$ be the game derived from G through the elimination of some of the strategies, namely, $\widehat{S}_i \subseteq S_i$ for each player $i \in N$. If s^* is an equilibrium in game G, and if $s^*_i \in \widehat{S}_i$ for each player i, then s^* is an equilibrium in the game $\widehat{G}$.*

Proof: Because s^* is an equilibrium of the game G, it follows that for each player i,

$$u_i(s_i, s^*_{-i}) \leq u_i(s^*), \quad \forall s_i \in S_i. \tag{4.39}$$

Because $\widehat{S}_i \subseteq S_i$ for each player $i \in N$, it is the case that

$$u_i(s_i, s^*_{-i}) \leq u_i(s^*), \quad \forall s_i \in \widehat{S}_i. \tag{4.40}$$

Because s^* is a vector of strategies in the game $\widehat{G}$, we conclude that it is an equilibrium of $\widehat{G}$. □

It should be noted that in general the post-elimination game $\widehat{G}$ may contain *new* equilibria that were not equilibria in the original game (Exercise 4.28). The next theorem shows that this cannot happen if the eliminated strategies are weakly dominated – that is, no new equilibria are created if a weakly dominated strategy of a particular player is eliminated. Repeated application of the theorem then yields the fact that the process of iterated elimination of weakly dominated strategies does not lead to the creation of new equilibria (Corollary 4.33).

Theorem 4.32 *Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a game in strategic form, let $j \in N$, and let $\widehat{s}_j \in S_j$ be a weakly dominated strategy of player j in this game. Denote by $\widehat{G}$ the game derived from G by the elimination of the strategy $\widehat{s}_j$. Then every equilibrium of $\widehat{G}$ is also an equilibrium of G.*

Proof: The strategy sets of the game $\widehat{G}$ are

$$\widehat{S}_i = \begin{cases} S_i & \text{if } i \neq j, \\ S_j \setminus \{\widehat{s}_j\} & \text{if } i = j. \end{cases} \tag{4.41}$$

Let $s^* = (s_i^*)_{i \in N}$ be an equilibrium strategy vector of the game $\widehat{G}$. Then

$$u_i(s_i, s_{-i}^*) \leq u_i(s^*), \qquad \forall i \neq j, \quad \forall s_i \in \widehat{S}_i = S_i, \tag{4.42}$$

$$u_j(s_j, s_{-j}^*) \leq u_j(s^*), \qquad \forall s_j \in \widehat{S}_j. \tag{4.43}$$

To show that s^* is an equilibrium of the game G we must show that no player i can profit in G by deviating to a strategy that differs from s_i^*. First we will show that this is true of every player i, $i \neq j$. Let i be a player who is not player j. Since $\widehat{S}_i = S_i$, by Equation (4.42) player i has no deviation from s_i^* that is profitable for him. As for player j, Equation (4.43) implies that he cannot profit from deviating to any strategy in $\widehat{S}_j = S_j \setminus \{\widehat{s}_j\}$. It only remains, then, to check that player j sees no gain from switching from strategy s_j^* to strategy $\widehat{s}_j$.

Because $\widehat{s}_j$ is a dominated strategy, there exists a strategy $t_j \in S_j$ that dominates it. It follows that $t_j \neq \widehat{s}_j$, and in particular that $t_j \in \widehat{S}_j$, so that

$$u_j(\widehat{s}_j, s_{-j}) \leq u_j(t_j, s_{-j}), \quad \forall s_{-j} \in S_{-j}. \tag{4.44}$$

Inserting $s_{-j} = s_{-j}^*$ in Equation (4.44) and $s_j = t_j$ in Equation (4.43), we get

$$u_j(\widehat{s}_j, s_{-j}^*) \leq u_j(t_j, s_{-j}^*) \leq u_j(s_j^*, s_{-j}^*), \tag{4.45}$$

which shows that deviating to strategy $\widehat{s}_j$ is indeed not profitable for player j. ☐

The following corollary (whose proof is left to the reader in Exercise 4.29) is implied by Theorem 4.32.

Corollary 4.33 *Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a game in strategic form, and let $\widehat{G}$ be the game derived from the game G by iterative elimination of dominated strategies. Then every equilibrium s^* of $\widehat{G}$ is also an equilibrium of G. In particular, if the iterative elimination results in a single vector s^*, then s^* is an equilibrium of the game G.*

Iterated elimination of dominated strategies, therefore, cannot create new equilibria. However, as the next example shows, it can result in the loss of some of the equilibria of the original game. This can happen even when there is only one elimination process possible.

Example 4.34 Consider the two-player game given by the matrix in Figure 4.31.

Player II

		L	R
Player I	T	0, 0	2, 1
	B	3, 2	1, 2

Figure 4.31 Elimination of dominated strategies may eliminate an equilibrium point

The game has two equilibria: (T, R) and (B, L). The only dominated strategy in the game is L (dominated by R). The elimination of strategy L results in a game in which B is dominated, and its elimination in turn yields the result (T, R). Thus, the elimination of L also eliminates the strategy vector (B, L) – an equilibrium point in the original game. The payoff corresponding to the eliminated equilibrium is $(3, 2)$, which for both players is preferable to $(2, 1)$, the payoff corresponding to (T, R), the equilibrium of the post-elimination game. ◀

In fact, the iterative elimination of weakly dominated strategies can result in the elimination of all the equilibria of the original game (Exercise 4.12). But this cannot happen under iterative elimination of strictly dominated strategies, which preserves the set of equilibrium points. That is the content of the following theorem.

Theorem 4.35 *Let $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a game in strategic form, let $j \in N$, and let $\widehat{s}_j \in S_j$ be a strictly dominated strategy of player j. Let $\widehat{G}$ be the game derived from G by the elimination of strategy $\widehat{s}_j$. Then the set of equilibria in the game $\widehat{G}$ is identical to the set of equilibria of the game G.*

Theorem 4.35 leads to the next corollary.

Corollary 4.36 *A strictly dominated strategy cannot be an element of a game's equilibrium.*

The conclusion of the last corollary is not true for weakly dominated strategies. As can be seen in Example 4.34, a weakly dominated strategy can be an element of an equilibrium. Indeed, there are cases in which an equilibrium strategy vector s^* is comprised of a weakly dominated strategy s_i^* for each player $i \in N$ (Exercise 4.30).

Proof of Theorem 4.35 Denote by E the set of equilibria of the game G, and by $\widehat{E}$ the set of equilibria of the game $\widehat{G}$. Theorem 4.32 implies that $\widehat{E} \subseteq E$, because every strictly dominated strategy is also a weakly dominated strategy. It remains to show that $E \subseteq \widehat{E}$.

Let $s^* \in E$ be an equilibrium of the game G. To show that $s^* \in \widehat{E}$, we will show that s^* is a strategy vector in the game $\widehat{G}$, which by Theorem 4.31 then implies that $s^* \in \widehat{E}$. As the game $\widehat{G}$ was derived from the game G by elimination of player j's strategy $\widehat{s}_j$, it suffices to show that $s_j^* \neq \widehat{s}_j$. Strategy $\widehat{s}_j$ is strictly dominated in the game G, so that there exists a strategy $t_j \in S_j$ that strictly dominates it:

$$u_j(\widehat{s}_j, s_{-j}) < u_j(t_j, s_{-j}), \quad \forall s_{-j} \in S_{-j}. \tag{4.46}$$

Because s^* is an equilibrium point, by setting $s_{-j} = s_{-j}^*$ in Equation (4.46) we get

$$u_j(\widehat{s}_j, s_{-j}^*) < u_j(t_j, s_{-j}^*) \leq u_i(s_j^*, s_{-j}^*), \tag{4.47}$$

thus yielding the conclusion that $\widehat{s}_j \neq s_j^*$, which is what we needed to show. □

When we put together Corollary 4.33 and Theorem 4.35, the following picture emerges: in implementing a process of iterated elimination of dominated strategies we may lose equilibria, but no new equilibria are created. If the elimination is of only strictly dominated strategies, the set of equilibria remains unchanged throughout the process. In particular, if the process of eliminating strictly dominated strategies results in a single strategy vector, this strategy vector is the unique equilibrium point of the original game (because it is the equilibrium of the game at the end of the process in which each player has only one strategy remaining). The uniqueness of the equilibrium constitutes a strengthening of Corollary 4.33 in the case in which only strictly dominated strategies are eliminated.

Corollary 4.37 *If iterative elimination of strictly dominated strategies yields a unique strategy vector s^*, then s^* is the unique Nash equilibrium of the game.*

In summary, to find a player's maxmin values we can first eliminate his (strictly or weakly) dominated strategies. In implementing this elimination process we may eliminate some of his maxmin strategies and also change the maxmin values of some other players. For finding equilibria we can also eliminate strictly dominated strategies without changing the set of equilibria of the game. Elimination of weakly dominated strategies may eliminate some equilibria of the game. The process of iterated elimination of weakly dominated strategies is useful for cases in which finding all equilibrium points is a difficult problem and we can be content with finding at least one equilibrium.

4.12 Two-player zero-sum games

As we have seen, the Nash equilibrium and the maxmin are two different concepts that reflect different behavioral aspects: the first is an expression of stability, while the second captures the notion of security. Despite the different roots of the two concepts, there are cases in which both lead to the same results. A special case where this occurs is in the class of two-player zero-sum games, which is the subject of this section.

In a given two-player game, denote, as we have done so far, the set of players by $N = \{\text{I}, \text{II}\}$ and the set of strategies respectively by S_{I} and S_{II}.

Example 4.38 Consider the two-player game appearing in Figure 4.32.

			Player II		
		L	C	R	$\min_{s_{\text{II}} \in S_{\text{II}}} u_{\text{I}}(s_{\text{I}}, s_{\text{II}})$
	T	3, −3	−5, 5	−2, 2	−5
Player I	M	1, −1	4, −4	1, −1	1
	B	6, −6	−3, 3	−5, 5	−5
$\min_{s_{\text{I}} \in S_{\text{I}}} u_{\text{II}}(s_{\text{I}}, s_{\text{II}})$		−6	−4	−1	1, −1

Figure 4.32 A two-player zero-sum game

In this example, $\underline{v}_{\text{I}} = 1$ and $\underline{v}_{\text{II}} = -1$. The maxmin strategy of Player I is M and that of Player II is R. The strategy pair (M, R) is also the equilibrium of this game (check!). In other words, here we have a case where the vector of maxmin strategies is also an equilibrium point: the two concepts lead to the same result. ◀

In the game in Example 4.38, for each pair of strategies the sum of the payoffs that the two players receive is zero. In other words, in any possible outcome of the game the payoff one player receives is exactly equal to the payoff the other player has to pay.

Definition 4.39 *A two-player game is a* zero-sum game *if for each pair of strategies* $(s_{\mathrm{I}}, s_{\mathrm{II}})$ *one has*

$$u_{\mathrm{I}}(s_{\mathrm{I}}, s_{\mathrm{II}}) + u_{\mathrm{II}}(s_{\mathrm{I}}, s_{\mathrm{II}}) = 0. \tag{4.48}$$

In other words, a two-player game is a zero-sum game if it is a closed system from the perspective of the payoffs: each player gains what the other player loses. It is clear that in such a game the two players have diametrically opposed interests.

Remark 4.40 *As we saw in Chapter 2, assuming that the players have von Neumann–Morgenstern linear utilities, any player's utility function is determined only up to a positive affine transformation. Therefore, if the payoffs represent the players' utilities from the various outcomes of the game, then they are determined up to a positive affine transformation. Changing the representation of the utility function of the players can then transform a zero-sum game into a non-zero-sum game. We will return to this issue in Section 5.5 (page 172); it will be proved there that the results of this chapter are independent of the particular representation of utility functions, and they hold true in two-player non-zero-sum games that are obtained from two-player zero-sum games by applying positive affine transformations to the players' payoffs.* ◆

Most real-life situations analyzed using game theory are not two-player zero-sum games, because even though the interests of the players diverge in many cases, they are often not completely diametrically opposed. Despite this, two-player zero-sum games have a special importance that justifies studying them carefully, as we do in this section. Here are some of the reasons:

1. Many classical games, such as chess, backgammon, checkers, and a plethora of dice games, are two-player zero-sum games. These were the first games to be studied mathematically and the first to yield formal results, results that spawned and shaped game theory as a young field of study in the early part of the twentieth century.
2. Given their special and highly restrictive properties, these games are generally simpler and easier to analyze mathematically than many other games. As is usually the case in mathematics, this makes them convenient objects for the initial exploration of ideas and possible directions for research in game theory.
3. Because of the fact that two-player zero-sum games leave no room for cooperation between the players, they are useful for isolating certain aspects of games and checking which results stem from cooperative considerations and which stem from other aspects of the game (information flows, repetitions, and so on).
4. In every situation, no matter how complicated, a natural benchmark for each player is his "security level": what he can guarantee for himself based solely on his own efforts, without relying on the behavior of other players. In practice, calculating the security level means assuming a worst-case scenario in which all other players are acting as an adversary. This means that the player is considering an auxiliary zero-sum game, in which all the other players act as if they were one opponent whose payoff is the opposite

		Player II		
		L	C	R
	T	3	−5	−2
Player I	M	1	4	1
	B	6	−3	−5

Figure 4.33 The payoff function u of the zero-sum game in Example 4.38

of his own payoff. In other words, even when analyzing a game that is non-zero-sum, the analysis of auxiliary zero-sum games can prove useful.

5. Two-player zero-sum games emerge naturally in other models. One example is games involving only a single player, which are often termed *decision problems*. They involve a decision maker choosing an action from among a set of alternatives, with the resultant payoff dependent both on his choice of action and on certain, often unknown, parameters over which he has no control. To calculate what the decision maker can guarantee for himself, we model the player's environment as if it were a second player who controls the unknown parameters and whose intent is to minimize the decision maker's payoff. This in effect yields a two-player zero-sum game. This approach is used in statistics, and we will return to it in Section 14.8 (page 600).

Let us now turn to the study of two-player zero-sum games. Since the payoffs u_{I} and u_{II} satisfy $u_{\mathrm{I}} + u_{\mathrm{II}} = 0$, we can confine our attention to one function, $u_{\mathrm{I}} = u$, with $u_{\mathrm{II}} = -u$. The function u will be termed the *payoff function* of the game, and it represents the payment that Player II makes to Player I. Note that this creates an artificial asymmetry (albeit only with respect to the symbols being used) between the two players: Player I, who is usually the row player, seeks to maximize $u(s)$ (his payoff) and Player II, who is usually the column player, is trying to minimize $u(s)$, which is what he is paying (since his payoff is $-u(s)$).

The game in Example 4.38 (page 110) can therefore be represented as shown in Figure 4.33.

The game of Matching Pennies (Example 3.20, page 52) can also be represented as a zero-sum game (see Figure 4.34).

Consider now the maxmin values of the players in a two-player zero-sum game. Player I's maxmin value is given by

$$\underline{v}_{\mathrm{I}} = \max_{s_{\mathrm{I}} \in S_{\mathrm{I}}} \min_{s_{\mathrm{II}} \in S_{\mathrm{II}}} u(s_{\mathrm{I}}, s_{\mathrm{II}}), \tag{4.49}$$

and Player II's maxmin value is

$$\underline{v}_{\mathrm{II}} = \max_{s_{\mathrm{II}} \in S_{\mathrm{II}}} \min_{s_{\mathrm{I}} \in S_{\mathrm{I}}} (-u(s_{\mathrm{I}}, s_{\mathrm{II}})) = -\min_{s_{\mathrm{II}} \in S_{\mathrm{II}}} \max_{s_{\mathrm{I}} \in S_{\mathrm{I}}} u(s_{\mathrm{I}}, s_{\mathrm{II}}). \tag{4.50}$$

		Player II	
		H	T
Player I	H	1	−1
	T	−1	1

Figure 4.34 The payoff function u of the game Matching Pennies

		Player II		
		L	R	$\min_{s_{\mathrm{II}} \in S_{\mathrm{II}}} u_{\mathrm{I}}(s_{\mathrm{I}}, s_{\mathrm{II}})$
Player I	T	−2	5	−2
	B	3	0	0
$\max_{s_{\mathrm{I}} \in S_{\mathrm{I}}} u_{\mathrm{II}}(s_{\mathrm{I}}, s_{\mathrm{II}})$		3	5	0, 3

Figure 4.35 A game in strategic form with the maxmin and minmax values

Denote

$$\underline{v} := \max_{s_{\mathrm{I}} \in S_{\mathrm{I}}} \min_{s_{\mathrm{II}} \in S_{\mathrm{II}}} u(s_{\mathrm{I}}, s_{\mathrm{II}}), \tag{4.51}$$

$$\overline{v} := \min_{s_{\mathrm{II}} \in S_{\mathrm{II}}} \max_{s_{\mathrm{I}} \in S_{\mathrm{I}}} u(s_{\mathrm{I}}, s_{\mathrm{II}}). \tag{4.52}$$

The value $\underline{v}$ is called the *maxmin value* of the game, and $\overline{v}$ is called the *minmax value*. Player I can guarantee that he will get at least $\underline{v}$, and Player II can guarantee that he will pay no more than $\overline{v}$. A strategy of Player I that guarantees $\underline{v}$ is termed a *maxmin strategy*. A strategy of Player II that guarantees $\overline{v}$ is called a *minmax strategy*.

We next calculate the maxmin value and minmax value in various examples of games. In Example 4.38, $\underline{v} = 1$ and $\overline{v} = 1$. In other words, Player I can guarantee that he will get a payoff of at least 1 (using the maxmin strategy M), while Player II can guarantee that he will pay at most 1 (by way of the minmax strategy R).

Consider the game shown in Figure 4.35. In this figure we have indicated on the right of each row the minimal payoff that the corresponding strategy of Player I guarantees him. Beneath each column we have indicated the maximal amount that Player II will pay if he implements the corresponding strategy.

In this game $\underline{v} = 0$ but $\overline{v} = 3$. Player I cannot guarantee that he will get a payoff higher than 0 (which he can guarantee using his maxmin strategy B) and Player II cannot guarantee that he will pay less than 3 (which he can guarantee using his minmax strategy L).

Finally, look again at the game of Matching Pennies (Figure 4.36).

		Player II		
		H	T	$\min_{s_{\mathrm{II}} \in S_{\mathrm{II}}} u_{\mathrm{I}}(s_{\mathrm{I}}, s_{\mathrm{II}})$
Player I	H	1	−1	−1
	T	−1	1	−1
$\max_{s_{\mathrm{I}} \in S_{\mathrm{I}}} u_{\mathrm{II}}(s_{\mathrm{I}}, s_{\mathrm{II}})$		1	1	−1, 1

Figure 4.36 Matching Pennies with the maxmin and minmax values

In this game, $\underline{v} = -1$ and $\overline{v} = 1$. Neither of the two players can guarantee a result that is better than the loss of one dollar (the strategies H and T of Player I are both maxmin strategies, and the strategies H and T of Player II are both minmax strategies).

As these examples indicate, the maxmin value $\underline{v}$ and the minmax value $\overline{v}$ may be unequal, but it is always the case that $\underline{v} \leq \overline{v}$. The inequality is clear from the definitions of the maxmin and minmax: Player I can guarantee that he will get at least $\underline{v}$, while Player II can guarantee that he will not pay more than $\overline{v}$. As the game is a zero-sum game, the inequality $\underline{v} \leq \overline{v}$ must hold. A formal proof of this fact can of course also be given (Exercise 4.34).

Definition 4.41 *A two-player game has a* value *if* $\underline{v} = \overline{v}$. *The quantity* $v := \underline{v} = \overline{v}$ *is then called the* value of the game.[6] *Any maxmin and minmax strategies of Player I and Player II respectively are then called* optimal strategies.

Consider again the game shown in Figure 4.33. This game has a value equal to 1. Player I can guarantee that he will get at least 1 for himself by selecting the optimal strategy M, and Player II can guarantee that he will not pay more than 1 by choosing the optimal strategy R. Note that the strategy pair (M, R) is also a Nash equilibrium.

Another example of a game that has a value is the game of chess, assuming that if the play does not end after a predetermined number of moves, it terminates in a draw. We do not know what that value is, but the existence of a value follows from Theorem 1.4 (page 3). Since it is manifestly a two-player game in which the interests of the players are diametrically opposed, we describe chess as a zero-sum game where White is the maximizer and Black is the minimizer by use of the following payoff function:

$$\begin{aligned} u(\text{White wins}) &= 1, \\ u(\text{Black wins}) &= -1, \\ u(\text{Draw}) &= 0. \end{aligned} \tag{4.53}$$

---

[6] The value of a game is sometimes also called the *minmax value of the game*.

Theorem 1.4 (page 3) implies that one and only one of the following must occur:

(i) White has a strategy guaranteeing a payoff of 1.
(ii) Black has a strategy guaranteeing a payoff of -1.
(iii) Each of the two players has a strategy guaranteeing a payoff of 0; that is, White can guarantee a payoff in the set $\{0, 1\}$, and Black can guarantee a payoff in the set $\{0, -1\}$.

If case (i) holds, then $\underline{v} \geq 1$. As the maximal payoff is 1, it must be true that $\overline{v} \leq 1$. Since we always have $\underline{v} \leq \overline{v}$, we deduce that $1 \leq \underline{v} \leq \overline{v} \leq 1$, which means that $\underline{v} = \overline{v} = 1$. Thus, the game has a value and $\underline{v} = \overline{v} = 1$ is its value.

If case (ii) holds, then $\overline{v} \leq -1$. Since the minimal payoff is -1, it follows that $\underline{v} \geq -1$. Hence $-1 \leq \underline{v} \leq \overline{v} \leq -1$, leading to $\underline{v} = \overline{v} = -1$, and the game has a value of -1.

Finally, suppose case (iii) holds. Then $\underline{v} \geq 0$ and $\overline{v} \leq 0$. So $0 \leq \underline{v} \leq \overline{v} \leq 0$, leading to $\underline{v} = \overline{v} = 0$, and the game has a value of 0.

Note that in chess each pair of optimal strategies is again a Nash equilibrium. For example, if case (i) above holds, then White's strategy is optimal if and only if it is a winning strategy. On the other hand, any strategy of Black guarantees him a payoff of at least -1 and therefore all his strategies are optimal. Every pair consisting of a winning strategy for White and any strategy for Black is an equilibrium. Since White can guarantee victory for himself, he certainly has no profitable deviation; since Black will lose no matter what, no deviation is strictly profitable for him either. The following conclusion has therefore been proved.

Corollary 4.42 *The game of chess has a value that is either* 1 *(if case (i) holds), or* -1 *(if case (ii) holds), or* 0 *(if case (iii) holds).*

The following theorem can be proven in the same way that Theorem 3.13 (page 46) was proved. Later in this book, a more general result is shown to be true, for games that are not zero-sum (see Theorem 4.49 on page 118).

Theorem 4.43 *Every finite two-player zero-sum extensive-form game with perfect information has a value.*

In every example we have considered so far, every zero-sum game with a value also has an equilibrium. The following two theorems establish a close relationship between the concepts of the value and of Nash equilibrium in two-player zero-sum games.

Theorem 4.44 *If a two-player zero-sum game has a value* v*, and if* s^*_{I} *and* s^*_{II} *are optimal strategies of the two players, then* $s^* = (s^*_{\mathrm{I}}, s^*_{\mathrm{II}})$ *is an equilibrium with payoff* $(v, -v)$.

Theorem 4.45 *If* $s^* = (s^*_{\mathrm{I}}, s^*_{\mathrm{II}})$ *is an equilibrium of a two-player zero-sum game, then the game has a value* $v = u(s^*_{\mathrm{I}}, s^*_{\mathrm{II}})$*, and the strategies* s^*_{I} *and* s^*_{II} *are optimal strategies.*

Before we prove Theorems 4.44 and 4.45, we wish to stress that these theorems show that in two-player zero-sum games the concept of equilibrium, which is based on stability, and the concept of minmax, which is based on security levels, coincide. If security level considerations are important factors in determining players' behavior, one may expect that the concept of equilibrium will have greater predictive power in two-player zero-sum games (where equilibrium strategies are also minmax strategies) than in more general games in which the two concepts lead to different predictions regarding players' behavior.

Note that despite the fact that the strategic form of the game is implicitly a simultaneously played game in which each player, in selecting his strategy, does not know the strategy selected by the other player, if the game has a value then each player can reveal the optimal strategy that he intends to play to the other player and still guarantees his maxmin value. Suppose that s_{I}^* is an optimal strategy for Player I in a game with value v. Then

$$\min_{s_{\mathrm{II}} \in S_{\mathrm{II}}} u(s_{\mathrm{I}}^*, s_{\mathrm{II}}) = v, \tag{4.54}$$

and therefore for each $s_{\mathrm{II}} \in S_{\mathrm{II}}$ the following inequality is satisfied:

$$u(s_{\mathrm{I}}^*, s_{\mathrm{II}}) \geq v. \tag{4.55}$$

In other words, even if Player I were to "announce" to Player II that he intends to play s_{I}^*, Player II cannot bring about a situation in which the payoff (to Player I) will be less than the value. This simple observation has technical implications for the search for optimal strategies: in order to check whether or not a particular strategy, say of Player I, is optimal, we check what it can guarantee, that is, what the payoff will be when Player II knows that this is the strategy chosen by Player I and does his best to counter it.

Proof of Theorem 4.44: From the fact that both s_{I}^* and s_{II}^* are optimal strategies, we deduce that

$$u(s_{\mathrm{I}}^*, s_{\mathrm{II}}) \geq v, \quad \forall s_{\mathrm{II}} \in S_{\mathrm{II}}, \tag{4.56}$$

$$u(s_{\mathrm{I}}, s_{\mathrm{II}}^*) \leq v, \quad \forall s_{\mathrm{I}} \in S_{\mathrm{I}}. \tag{4.57}$$

Inserting $s_{\mathrm{II}} = s_{\mathrm{II}}^*$ into Equation (4.56) we deduce $u(s_{\mathrm{I}}^*, s_{\mathrm{II}}^*) \geq v$, and inserting $s_{\mathrm{I}} = s_{\mathrm{I}}^*$ into Equation (4.57) we get $u(s_{\mathrm{I}}^*, s_{\mathrm{II}}^*) \leq v$. The equation $v = u(s_{\mathrm{I}}^*, s_{\mathrm{II}}^*)$ follows. Equations (4.56) and (4.57) can now be written as

$$u(s_{\mathrm{I}}^*, s_{\mathrm{II}}) \geq u(s_{\mathrm{I}}^*, s_{\mathrm{II}}^*), \quad \forall s_{\mathrm{II}} \in S_{\mathrm{II}}, \tag{4.58}$$

$$u(s_{\mathrm{I}}, s_{\mathrm{II}}^*) \leq u(s_{\mathrm{I}}^*, s_{\mathrm{II}}^*), \quad \forall s_{\mathrm{I}} \in S_{\mathrm{I}}, \tag{4.59}$$

and therefore $(s_{\mathrm{I}}^*, s_{\mathrm{II}}^*)$ is an equilibrium with payoff $(v, -v)$. □

Proof of Theorem 4.45: Since $(s_{\mathrm{I}}^*, s_{\mathrm{II}}^*)$ is an equilibrium, no player can benefit by a unilateral deviation:

$$u(s_{\mathrm{I}}, s_{\mathrm{II}}^*) \leq u(s_{\mathrm{I}}^*, s_{\mathrm{II}}^*), \quad \forall s_{\mathrm{I}} \in S_{\mathrm{I}} \tag{4.60}$$

$$u(s_{\mathrm{I}}^*, s_{\mathrm{II}}) \geq u(s_{\mathrm{I}}^*, s_{\mathrm{II}}^*), \quad \forall s_{\mathrm{II}} \in S_{\mathrm{II}}. \tag{4.61}$$

Let $v = u(s_{\mathrm{I}}^*, s_{\mathrm{II}}^*)$. We will prove that v is indeed the value of the game. From Equation (4.60) we get

$$u(s_{\mathrm{I}}^*, s_{\mathrm{II}}) \geq v, \quad \forall s_{\mathrm{II}} \in S_{\mathrm{II}}, \tag{4.62}$$

and therefore $\underline{v} \geq v$. From Equation (4.60) we deduce that

$$u(s_{\mathrm{I}}, s_{\mathrm{II}}^*) \leq v, \quad \forall s_{\mathrm{I}} \in S_{\mathrm{I}}, \tag{4.63}$$

		Player II	
		a	b
Player I	A	1, 1	0, 0
	B	0, 0	3, 3

Figure 4.37 Coordination game

and therefore $\overline{v} \leq v$. Because it is always the case that $\underline{v} \leq \overline{v}$ we get

$$v \leq \underline{v} \leq \overline{v} \leq v, \tag{4.64}$$

which implies that the value exists and is equal to v. Furthermore, from Equation (4.62) we deduce that s_{I}^{*} is an optimal strategy for Player I, and from Equation (4.63) we deduce that s_{II}^{*} is an optimal strategy for Player II. ☐

Corollary 4.46 *In a two-player zero-sum game, if* $(s_{\mathrm{I}}^{*}, s_{\mathrm{II}}^{*})$ *and* $(s_{\mathrm{I}}^{**}, s_{\mathrm{II}}^{**})$ *are two equilibria, then it follows that*

1. *Both equilibria yield the same payoff:* $u(s_{\mathrm{I}}^{*}, s_{\mathrm{II}}^{*}) = u(s_{\mathrm{I}}^{**}, s_{\mathrm{II}}^{**})$.
2. *Both* $(s_{\mathrm{I}}^{*}, s_{\mathrm{II}}^{**})$ *and* $(s_{\mathrm{I}}^{**}, s_{\mathrm{II}}^{*})$ *are also equilibria (and, given the above, they also yield the same payoff).*

Proof: The first part follows from Theorem 4.45, because the payoff of each one of the equilibria is necessarily equal to the value of the game. For the second part, note that Theorem 4.45 implies that all the strategies $s_{\mathrm{I}}^{*}, s_{\mathrm{I}}^{**}, s_{\mathrm{II}}^{*}, s_{\mathrm{II}}^{**}$ are optimal strategies. By Theorem 4.44 we conclude that $(s_{\mathrm{I}}^{*}, s_{\mathrm{II}}^{**})$ and $(s_{\mathrm{I}}^{**}, s_{\mathrm{II}}^{*})$ are equilibria. ☐

Neither of the two conclusions of Corollary 4.46 is necessarily true in a two-player game that is not zero-sum. Consider, for example, the coordination game in Example 4.20, shown in Figure 4.37.

(A, a) and (B, b) are two equilibria with different payoffs (thus, the first part of Corollary 4.46 does not hold in this example) and (A, b) and (B, a) are not equilibria (thus the second part of the corollary does not hold).

The most important conclusion to take away from this section is that in two-player zero-sum games the value and Nash equilibrium, two different solution concepts, actually coincide and lead to the same results. Put another way, in two-player zero-sum games, the goals of security and stability are unified. John Nash regarded his concept of equilibrium to be a generalization of the value. But while the concept of the value expresses both the aspects of security and stability, the Nash equilibrium expresses only the aspect of stability. In games that are not zero-sum games, security and stability are different concepts, as we saw in the game depicted in Figure 4.28.

There is a geometric interpretation to the value of a two-player zero-sum game, which finds expression in the concept of the saddle point.

Definition 4.47 *A pair of strategies* $(s_{\mathrm{I}}^*, s_{\mathrm{II}}^*)$ *is a* saddle point *of the function* $u : S_{\mathrm{I}} \times S_{\mathrm{II}} \to \mathbb{R}$ *if*

$$u(s_{\mathrm{I}}^*, s_{\mathrm{II}}^*) \geq u(s_{\mathrm{I}}, s_{\mathrm{II}}^*), \quad \forall s_{\mathrm{I}} \in S_{\mathrm{I}}, \tag{4.65}$$

$$u(s_{\mathrm{I}}^*, s_{\mathrm{II}}^*) \leq u(s_{\mathrm{I}}^*, s_{\mathrm{II}}), \quad \forall s_{\mathrm{II}} \in S_{\mathrm{II}}. \tag{4.66}$$

In other words, $u(s_{\mathrm{I}}^*, s_{\mathrm{II}}^*)$ *is the highest value in column* s_{II}^**, and the smallest in the row* s_{I}^**.*

The name "saddle point" stems from the shape of a horse's saddle, whose center is perceived to be the minimal point of the saddle from one direction and the maximal point from the other direction.

The proof of the next theorem is left to the reader (Exercise 4.36).

Theorem 4.48 *In a two-player zero-sum game,* $(s_{\mathrm{I}}^*, s_{\mathrm{II}}^*)$ *is a saddle point of the payoff function* u *if and only if* s_{I}^* *is an optimal strategy for Player I and* s_{II}^* *is an optimal strategy for Player II. In that case,* $u(s_{\mathrm{I}}^*, s_{\mathrm{II}}^*)$ *is the value of the game.*

4.13 Games with perfect information

As we have shown, there are games in which there exists no Nash equilibrium. These games will be treated in Chapter 5. In this section we focus instead on a large class of widely applicable games that all have Nash equilibria. These games are best characterized in extensive form. We will show that if an extensive-form game satisfies a particular characteristic, then it always has a Nash equilibrium. Furthermore, there are even equilibria that can be calculated directly from the game tree, without requiring that the game first be transformed into strategic form. Because it is often more convenient to work directly with the extensive form of a game, this way of calculating equilibria has a significant advantage.

In this section we study extensive-form games with perfect information. Recall that an extensive-form game is of perfect information if every information set of every player consists of only one vertex.

Theorem 4.49 (Kuhn) *Every finite game with perfect information has at least one Nash equilibrium.*

Kuhn's Theorem constitutes a generalization of Theorem 4.43, which states that every two-player zero-sum game with perfect information has a value. The proof of the theorem is similar to the proof of Theorem 1.4 (page 3), and involves induction on the number of vertices in the game tree. Every child of the root of a game tree defines a subgame containing fewer vertices than the original game (a fact that follows from the assumption that the game has perfect recall) and the induction hypothesis then implies that the subgame has an equilibrium. Choose one equilibrium for each such subgame. If the root of the original game involves a chance move, then the union of the equilibria of all the subgames defines an equilibrium for the entire game. If the root involves a decision taken

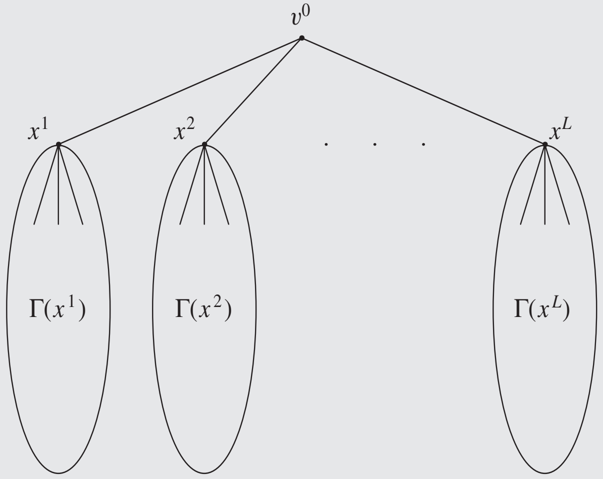

Figure 4.38 The game tree and subgames starting at the children of the root

by player i, then that player will survey the subgames that will be played (one for each child that he may choose), calculate the payoff he will receive under the chosen equilibrium in each of those subgames, and choose the vertex leading to the subgame that grants him the maximal payoff. These intuitive ideas will now be turned into a formal proof.

Proof of Theorem 4.49: It is convenient to assume that if a player in any particular game has no action available in any vertex in the game tree, then his strategy set consists of a single strategy denoted by $\emptyset$.

The proof of the theorem is by induction on the number of vertices in the game tree. If the game tree is comprised of a single vertex, then the unique strategy vector is $(\emptyset, \dots, \emptyset)$ (so a fortiori there are no available deviations), and it is therefore the unique Nash equilibrium.

Assume by induction that the claim is true for each game in extensive form containing fewer than K vertices, and consider a game Γ with K vertices. Denote by $x^1, \dots, x^L$ the children of the root v^0, and by $\Gamma(x^l)$ the subgame whose root is x^l and whose vertices are those following x^l in the tree (see Figure 4.38). Because the game is one with perfect information, $\Gamma(x^l)$ is indeed a subgame. If we had not assumed this then $\Gamma(x^l)$ would not necessarily be a subgame, because there could be an information set containing vertices that are descendants of both x^{l_1} and x^{l_2} (where $l_1 \neq l_2$) and we would be unable to make use of the induction hypothesis.

The payoff functions of the game Γ are, as usual, $u_i : \times_{i \in N} S_i \to \mathbb{R}$. For each $l \in 1, 2, \dots, L$, the payoff functions in the subgame $\Gamma(x^l)$ are $u_i^l : \times_{i \in N} S_i^l \to \mathbb{R}$, where S_i^l is player i's set of strategies in the subgame $\Gamma(x^l)$.

For any $l \in \{1, \dots, L\}$, the root v^0 of the original game Γ is not a vertex of $\Gamma(x^l)$, and therefore the number of vertices in $\Gamma(x^l)$ is less than K. By the induction hypothesis, for each $l \in \{1, 2, \dots, L\}$ the game $\Gamma(x^l)$ has an equilibrium $s^{*l} = (s_i^{*l})_{i \in N}$ (if there are several such equilibria we arbitrarily choose one of them).

Case 1: The root v^0 is a chance move.

For each $l \in \{1, 2, \ldots, L\}$ denote by p^l the probability that child x^l is chosen. For each player i consider the strategy s_i^* in the game Γ defined as follows. If vertex x^l is chosen in the first move of the play of the game, implement strategy s_i^{*l} in the subgame $\Gamma(x^l)$. By definition it follows that $u_i(s^*) = \sum_{l=1}^{L} p^l u_i^l(s^{*l})$.

We will show that the strategy vector $s^* = (s_i^*)_{i \in N}$ is a Nash equilibrium. Suppose that player j deviates to a different strategy s_j. Let s_j^l be the restriction of s_j to the subgame $\Gamma(x^l)$. The expected payoff to player j under the strategy vector (s_j^l, s_{-j}^{*l}) is $\sum_{l=1}^{L} p^l u_j^l(s_j^l, s_{-j}^{*l})$.

Since s^{*l} is an equilibrium of $\Gamma(x^l)$, $u_j^l(s_j^l, s_{-j}^{*l}) \leq u_j^l(s^{*l})$ for all $l = 1, \ldots, L$, and therefore

$$u_j(s_j, s_{-j}^*) = \sum_{l=1}^{L} p^l u_j^l \left(s_j^l, s_{-j}^{*l}\right) \leq \sum_{l=1}^{L} p^l u_j^l(s^{*l}) = u_j(s^*). \tag{4.67}$$

In other words, player j does not profit by deviating from s_j^* to s_j. Since this holds true for every player $j \in N$, the strategy vector s^* is indeed a Nash equilibrium.

Case 2: The root is a decision vertex for player i_0.

We first define a strategy vector $s^* = (s_i^*)_{i \in N}$ and then show that it is a Nash equilibrium. For each player i, $i \neq i_0$, consider the strategy s_i^* defined as follows. If vertex x^l is chosen in the first move of the play of the game, in the subgame $\Gamma(x^l)$ implement strategy s_i^{*l}. For player i_0 define the following strategy $s_{i_0}^*$: at the root choose the child x^{l_0} at which the maximum $\max_{1 \leq l \leq L} u_i^l(s^{*l})$ is attained. For each $l \in \{1, 2, \ldots, L\}$, in the subgame $\Gamma(x^l)$ implement[7] the strategy s_i^{*l}. The payoff under the strategy vector $s^* = (s_i^*)_{i \in N}$ is $u^{l_0}(s^{*l_0})$.

The proof that each player i, except for player i_0, cannot profit from a deviation from s_i^* is similar to the proof in Case 1 above. We will show that player i_0 also cannot profit by deviating from $s_{i_0}^*$, thus completing the proof that the strategy vector s^* is a Nash equilibrium.

Suppose that player i_0 deviates by selecting strategy s_{i_0}. Let $x^{\widehat{l}}$ be the child of the root selected by this strategy, and for each child x^l of the root let $s_{i_0}^l$ be the strategy s_{i_0} restricted to the subgame $\Gamma(x^l)$.

- If $\widehat{l} = l_0$, since s^{*l_0} is an equilibrium of the subgame $\Gamma(x^{l_0})$, the payoff to player i_0 is

$$u_{i_0}(s_{i_0}, s_{-i_0}^*) = u_{i_0}^{l_0}\left(s_{i_0}^{l_0}, s_{-i_0}^{*l_0}\right) \leq u_{i_0}^{l_0}(s^{*l_0}) = u_{i_0}(s^*). \tag{4.68}$$

In other words, the deviation is not a profitable one.

- If $\widehat{l} \neq l_0$, since $s^{*\widehat{l}}$ is an equilibrium of the subgame $\Gamma(x^{\widehat{l}})$ and using the definition of l_0 we obtain

$$u_{i_0}(s_{i_0}, s_{-i_0}^*) = u_{i_0}^{\widehat{l}}\left(s_{i_0}^{\widehat{l}}, s_{-i_0}^{*\widehat{l}}\right) \leq u_{i_0}^{\widehat{l}}(s^{*\widehat{l}}) \leq u_i^{l_0}(s^{*l_0}) = u_{i_0}(s^*). \tag{4.69}$$

This too is not a profitable deviation, which completes the proof. □

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[7] Since defining a strategy requires defining how a player plays at each node at which he chooses an action, we also need to define $s_{i_0}^*$ in the subgames $\Gamma(x^l)$ which the first move of the play of the game does not lead to ($l \neq l_0$).

Remark 4.50 *In the course of the last proof, we proceeded by induction from the root to its children and beyond. This is called* forward induction. *We can prove the theorem by backward induction, as follows. Let x be a vertex all of whose children are leaves. Since the game has perfect recall, the player choosing an action at vertex x knows that the play of the game has arrived at that vertex (and not at a larger information set containing x) and he therefore chooses the leaf l giving him the maximal payoff. We can imagine erasing the leaves following x and thus turning x into a leaf with a payoff equal to the payoff of l. The resulting game tree has fewer vertices than the original tree, so we can apply the induction hypothesis to it. The reader is asked to complete this proof in Exercise 4.39. This process is called* backward induction. *It yields a practical algorithm for finding an equilibrium in finite games with perfect information: start at vertices leading immediately to leaves. Assuming the play of the game gets to such a vertex, the player at that vertex will presumably choose the leaf granting him the maximal payoff (if there are two or more such vertices, the player may arbitrarily choose any one of them). We then attach that payoff to such a vertex. If one of these vertices is the vertex of a chance move, the payoff at that vertex is the expectation of the payoff at the leaf reached by the chance move. From here we proceed in stages: at each stage, we attach payoffs to vertices leading immediately to vertices that had payoffs attached to them in previous stages. At each such vertex, the player controlling that vertex will make a selection leading to the maximal possible payoff to him, and that is the payoff associated with the vertex. We continue by this process to climb the tree until we reach the root. In some cases this process leads to multiple equilibria. As shown in Exercise 4.40 some equilibria cannot be obtained by this process.* ◆

4.14 Games on the unit square

In this section we analyze two examples of two-player games in which the set of strategies is infinite, namely, the unit interval [0, 1]. These examples will be referred to in Chapter 5, where we introduce mixed strategies.

4.14.1 A two-player zero-sum game on the unit square

Consider the two-player zero-sum strategic-form game in which:[8]

- the strategy set of Player I is $X = [0, 1]$;
- the strategy set of Player II is $Y = [0, 1]$;
- the payoff function (which is what Player II pays Player I) is

$$u(x, y) = 4xy - 2x - y + 3, \quad \forall x \in [0, 1], \forall y \in [0, 1]. \tag{4.70}$$

This game is called a *game on the unit square*, because the set of strategy vectors is the unit square in $\mathbb{R}^2$. We can check whether or not this game has a value, and if it does, we

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[8] In games on the unit square it is convenient to represent a strategy as a continuous variable, and we therefore denote player strategies by x and y (rather than s_{I} and s_{II}), and the sets of strategies are denoted by X and Y respectively (rather than S_{I} and S_{II}).

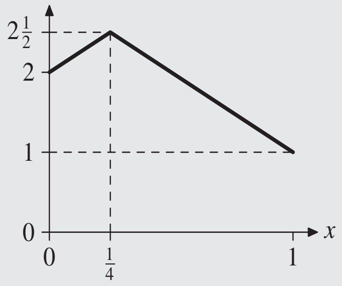

Figure 4.39 The function $x \mapsto \min_{y\in[0,1]} u(x, y)$

can identify optimal strategies for the two players, as follows. First we calculate

$$\underline{v} = \max_{x\in[0,1]} \min_{y\in[0,1]} u(x, y), \tag{4.71}$$

and

$$\overline{v} = \min_{y\in[0,1]} \max_{x\in[0,1]} u(x, y), \tag{4.72}$$

and check whether or not they are equal. For each $x \in [0, 1]$,

$$\min_{y\in[0,1]} u(x, y) = \min_{y\in[0,1]} (4xy - 2x - y + 3) = \min_{y\in[0,1]} (y(4x - 1) - 2x + 3). \tag{4.73}$$

For each fixed x, this is a linear function in y, and therefore the point at which the minimum is attained is determined by the slope $4x - 1$: if the slope is positive the function is increasing and the minimum is attained at $y = 0$; if the slope is negative this is a decreasing function and the minimum is attained at $y = 1$; if the slope is 0 the function is constant in y and every point is a minimum point. This leads to the following (see Figure 4.39):

$$\min_{y\in[0,1]} u(x, y) = \begin{cases} 2x + 2 & \text{if } x \leq \frac{1}{4}, \\ -2x + 3 & \text{if } x \geq \frac{1}{4}. \end{cases} \tag{4.74}$$

This function of x attains a unique maximum at $x = \frac{1}{4}$, and its value there is $2\frac{1}{2}$. Therefore,

$$\underline{v} = \max_{x\in[0,1]} \min_{y\in[0,1]} u(x, y) = 2\tfrac{1}{2}. \tag{4.75}$$

We similarly calculate the following (see Figure 4.40):

$$\max_{x\in[0,1]} u(x, y) = \max_{x\in[0,1]} (4xy - 2x - y + 3) = \max_{x\in[0,1]} (x(4y - 2) - y + 3) \tag{4.76}$$

$$= \begin{cases} -y + 3 & \text{if } y \leq \frac{1}{2}, \\ 3y + 1 & \text{if } y \geq \frac{1}{2}. \end{cases} \tag{4.77}$$

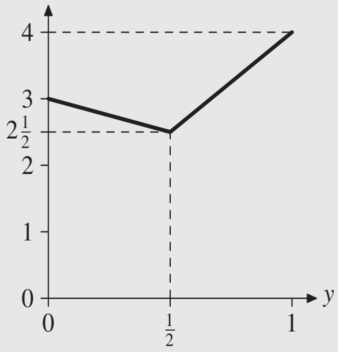

Figure 4.40 The function $y \mapsto \max_{x\in[0,1]} u(x, y)$

This function of y attains a unique minimum at $y = \frac{1}{2}$, and its value there is $2\frac{1}{2}$.

$$\overline{v} = \min_{y\in[0,1]} \max_{x\in[0,1]} u(x, y) = 2\tfrac{1}{2}. \tag{4.78}$$

In other words, the game has a value $v = 2\frac{1}{2}$, and $x^* = \frac{1}{4}$ and $y^* = \frac{1}{2}$ are optimal strategies (in fact the only optimal strategies in this game).

Since x^* and y^* are the only optimal strategies of the players, we deduce from Theorems 4.44 and 4.45 that (x^*, y^*) is the only equilibrium of the game.

4.14.2 A two-player non-zero-sum game on the unit square

Consider the following two-player non-zero-sum game in strategic form:

- the strategy set of Player I is $X = [0, 1]$;
- the strategy set of Player II is $Y = [0, 1]$;
- the payoff function of Player I is

$$u_{\mathrm{I}}(x, y) = 3xy - 2x - 2y + 2, \quad \forall x \in [0, 1], \forall y \in [0, 1]; \tag{4.79}$$

- the payoff function of Player II is

$$u_{\mathrm{II}}(x, y) = -4xy + 2x + y, \quad \forall x \in [0, 1], \forall y \in [0, 1]. \tag{4.80}$$

Even though this is not a zero-sum game, the maxmin concept, reflecting the security level of a player, is still well defined (see Equation (4.26)). Player I can guarantee

$$\underline{v}_{\mathrm{I}} = \max_{x\in[0,1]} \min_{y\in[0,1]} u_{\mathrm{I}}(x, y), \tag{4.81}$$

and Player II can guarantee

$$\underline{v}_{\mathrm{II}} = \max_{y\in[0,1]} \min_{x\in[0,1]} u_{\mathrm{II}}(x, y). \tag{4.82}$$

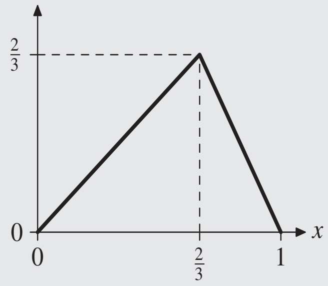

Figure 4.41 The function $x \mapsto \min_{y\in[0,1]} u_{\mathrm{I}}(x, y)$

Similarly to the calculations carried out in Section 4.14.1, we derive the following (see Figure 4.41):

$$\min_{y\in[0,1]} u_{\mathrm{I}}(x, y) = \min_{y\in[0,1]} (3xy - 2x - 2y + 2) = \min_{y\in[0,1]} (y(3x - 2) - 2x + 2) \tag{4.83}$$

$$= \begin{cases} x & \text{for } x \leq \frac{2}{3}, \\ -2x + 2 & \text{for } x \geq \frac{2}{3}. \end{cases} \tag{4.84}$$

This function of x has a single maximum, attained at $x = \frac{2}{3}$, with the value $\frac{2}{3}$. We therefore have

$$\underline{v}_{\mathrm{I}} = \max_{x\in[0,1]} \min_{y\in[0,1]} u_{\mathrm{I}}(x, y) = \frac{2}{3}. \tag{4.85}$$

The sole maxmin strategy available to Player I is $\widehat{x} = \frac{2}{3}$. We similarly calculate for Player II (see Figure 4.42):

$$\min_{x\in[0,1]} u\mathrm{II}(x, y) = \min_{x\in[0,1]} (-4xy + 2x + y) = \min_{x\in[0,1]} (x(2 - 4y) + y) \tag{4.86}$$

$$= \begin{cases} y & \text{for } y \leq \frac{1}{2}, \\ 2 - 3y & \text{for } y \geq \frac{1}{2}. \end{cases} \tag{4.87}$$

This function of y has a single maximum, attained at $y = \frac{1}{2}$, with value $\frac{1}{2}$. We therefore have

$$\underline{v}_{\mathrm{II}} = \max_{y\in[0,1]} \min_{x\in[0,1]} u_{\mathrm{II}}(x, y) = \frac{1}{2}, \tag{4.88}$$

and the sole maxmin strategy of Player II is $\widehat{y} = \frac{1}{2}$.

The next step is to calculate a Nash equilibrium of this game, assuming that there is one. The most convenient way to do so is to use the definition of the Nash equilibrium based on the "best reply" concept (Definition 4.18 on page 97): a pair of strategies (x^*, y^*) is a Nash equilibrium if x^* is Player I's best reply to y^*, and y^* is Player II's best reply to x^*.

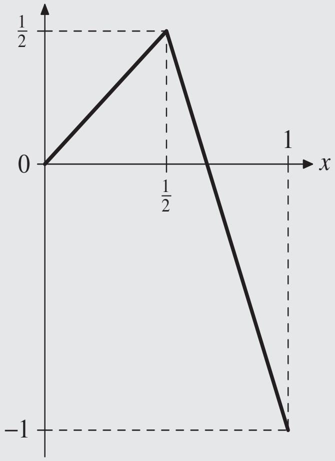

Figure 4.42 The function $y \mapsto \min_{x\in[0,1]} u_{\mathrm{II}}(x, y)$

For each $x \in [0, 1]$, denote by $\mathrm{br}_{\mathrm{II}}(x)$ the collection of best replies[9] of Player II to the strategy x:

$$\mathrm{br}_{\mathrm{II}}(x) := \mathrm{argmax}_{y\in[0,1]} u_{\mathrm{II}}(x, y) = \{y \in [0, 1] \colon u_{\mathrm{II}}(x, y) \geq u_{\mathrm{II}}(x, z) \quad \forall z \in [0, 1]\}. \tag{4.89}$$

In other words, $\mathrm{br}_{\mathrm{II}}(x)$ is the collection of values y at which the maximum of $u_{\mathrm{II}}(x, y)$ is attained. To calculate $\mathrm{br}_{\mathrm{II}}(x)$ in this example, we will write $u_{\mathrm{II}}(x, y)$ as

$$u_{\mathrm{II}}(x, y) = y(1 - 4x) + 2x. \tag{4.90}$$

For each fixed x, this is a linear function of y: if it has a positive slope the function is increasing and attains its maximum at $y = 1$. If the slope is negative, the function is decreasing and the maximum point is $y = 0$. If the slope of the function is 0, then the function is constant and every point $y \in [0, 1]$ is a maximum point. The slope turns from positive to negative at $x = \frac{1}{4}$, and the graph of $\mathrm{br}_{\mathrm{II}}(x)$ is given in Figure 4.43.

Note that $\mathrm{br}_{\mathrm{II}}$ is not a function, because $\mathrm{br}_{\mathrm{II}}(\frac{1}{4})$ is not a single point but the interval $[0, 1]$.

The calculation of $\mathrm{br}_{\mathrm{I}}(y)$ is carried out similarly. The best reply of Player I to each $y \in [0, 1]$ is

$$\mathrm{br}_{\mathrm{I}}(y) := \mathrm{argmax}_{x\in[0,1]} u_{\mathrm{I}}(x, y) = \{x \in [0, 1] \colon u_{\mathrm{I}}(x, y) \geq u_{\mathrm{I}}(z, y) \quad \forall z \in [0, 1]\}. \tag{4.91}$$

Writing $u_{\mathrm{I}}(x, y)$ as

$$u_{\mathrm{I}}(x, y) = x(3y - 2) - 2y + 2 \tag{4.92}$$

shows that, for each fixed y, this is a linear function in x: if it has a positive slope the function is increasing and attains its maximum at $x = 1$. A negative slope implies that the function is decreasing and its maximum point is $x = 0$, and a slope of 0 indicates a

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[9] br stands for best reply.

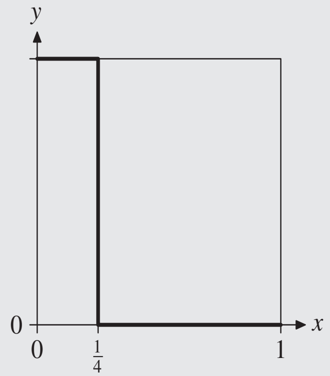


Figure 4.43 The graph of $\mathrm{br}_{\mathrm{II}}(x)$

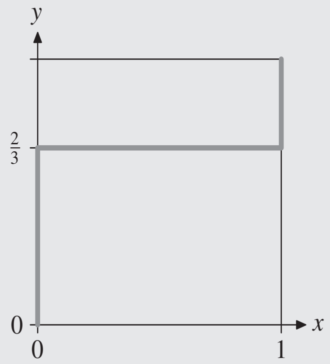


Figure 4.44 The graph of $\mathrm{br}_{\mathrm{I}}(y)$

constant function where every point $x \in [0, 1]$ is a maximum point. The slope turns from negative to positive at $y = \frac{2}{3}$, and the graph of $\mathrm{br}_{\mathrm{I}}(y)$ is given in Figure 4.44.

Note that the variable y is represented by the vertical axis, even though it is the variable of the function $\mathrm{br}_{\mathrm{I}}(y)$. This is done so that both graphs, $\mathrm{br}_{\mathrm{I}}(y)$ and $\mathrm{br}_{\mathrm{II}}(x)$, can be conveniently depicted within the same system of axes, as follows (Figure 4.45):

In terms of the best-reply concept, the pair of strategies (x^*, y^*) is an equilibrium point if and only if $x^* \in \mathrm{br}_{\mathrm{I}}(y^*)$ and $y^* \in \mathrm{br}_{\mathrm{II}}(x^*)$. In other words, we require (x^*, y^*) to be on *both* graphs $\mathrm{br}_{\mathrm{II}}(x)$ and $\mathrm{br}_{\mathrm{I}}(y)$. As is clear from Figure 4.40, the only point satisfying this condition is $(x^* = \frac{1}{4}, y^* = \frac{2}{3})$.

We conclude that the game has a single Nash equilibrium (x^*, y^*) where $x^* = \frac{1}{4}$ and $y^* = \frac{2}{3}$, with the equilibrium payoff of $u_{\mathrm{I}}(x^*, y^*) = \frac{2}{3}$ to Player I and $u_{\mathrm{II}}(x^*, y^*) = \frac{1}{2}$ to Player II.

This example shows, again, that in games that are not zero-sum the concepts of Nash equilibrium and optimal strategies differ; despite the fact that for both players the equilibrium payoff is equal to the security level ($\frac{2}{3}$ for Player I and $\frac{1}{2}$ for Player II), the maxmin strategies are not the equilibrium strategies. The maxmin strategies are $\widehat{x} = \frac{2}{3}$ and $\widehat{y} = \frac{1}{2}$, while the equilibrium strategies are $x^* = \frac{1}{4}$ and $y^* = \frac{2}{3}$.

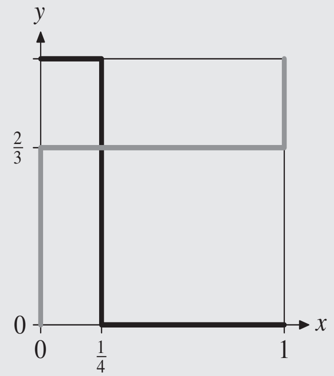


Figure 4.45 The graphs of $x \mapsto \mathrm{br}_{\mathrm{II}}(x)$ (darker line) and $y \mapsto \mathrm{br}_{\mathrm{I}}(y)$ (lighter line)

- The pair of maxmin strategies, $\widehat{x} = \frac{2}{3}$ and $\widehat{y} = \frac{1}{2}$, is not an equilibrium. The payoff to Player I is $\frac{2}{3}$, but he can increase his payoff by deviating to $x = 0$ because

$$u_{\mathrm{I}}(0, \widehat{y}) = u_{\mathrm{I}}\left(0, \tfrac{1}{2}\right) = 1 > \tfrac{2}{3} = u_{\mathrm{I}}(\widehat{x}, \widehat{y}). \tag{4.93}$$

  The payoff to Player II is $\frac{1}{2}$, and he can also increase his payoff by deviating to $y = 0$ because

$$u_{\mathrm{II}}(\widehat{x}, 0) = u_{\mathrm{II}}\left(\tfrac{2}{3}, 0\right) = \tfrac{4}{3} > \tfrac{1}{2} = u_{\mathrm{II}}(\widehat{x}, \widehat{y}). \tag{4.94}$$

- The equilibrium strategies $x^* = \frac{1}{4}$ and $y^* = \frac{2}{3}$ are not optimal strategies. If Player I chooses strategy $x^* = \frac{1}{4}$ and Player II plays $y = 1$, the payoff to Player I is less than his security level $\frac{2}{3}$:

$$u_{\mathrm{I}}\left(\tfrac{1}{4}, 1\right) = \tfrac{1}{4} < \tfrac{2}{3} = \underline{v}_{\mathrm{I}}. \tag{4.95}$$

  Similarly, when Player II plays $y^* = \frac{2}{3}$, if Player I plays $x = 1$ then the payoff to Player II is less than his security level $\frac{1}{2}$:

$$u_{\mathrm{II}}\left(1, \tfrac{2}{3}\right) = 0 < \tfrac{1}{2} = \underline{v}_{\mathrm{II}}. \tag{4.96}$$

Note that $u_{\mathrm{I}}(x, \frac{2}{3}) = \frac{2}{3}$ for all $x \in [0, 1]$. It follows that when Player II implements the strategy $y^* = \frac{2}{3}$, Player I is "indifferent" between all of his strategies. Similarly, $u_{\mathrm{II}}(\frac{1}{4}, y) = \frac{1}{2}$ for all $y \in [0, 1]$. It follows that when Player I implements the strategy $x^* = \frac{1}{4}$, Player II is "indifferent" between all of his strategies. This outcome occurs in every two-player game on the unit square when the payoff functions are bilinear functions with a unique equilibrium (x^*, y^*) satisfying $0 < x^*, y^* < 1$. This is not a coincidence: it is the result of a general game-theoretic principle called the indifference principle, which is studied in Chapter 5 in Section 5.2.3.

4.15 Remarks

Mathematician John Nash received the Nobel Memorial Prize in Economics in 1994 for the equilibrium concept that is named after him. The Nash equilibrium is a central concept in mathematical economics.

The Prisoner's Dilemma game was first defined and studied by Merrill Flood and Melvin Dresher in 1950. The name commonly given to that game, as well as the accompanying story, was first suggested by Albert Tucker. The version of the Prisoner's Dilemma appearing in Exercise 4.1 was suggested by Reinhard Selten.

The name "Security Dilemma" (see Example 4.22 on page 98) was coined by Herz [1950]. The dilemma was extensively studied in the political science literature (see, for example, Jervis [1978]). Alain Ledoux [1985] was the first to present the Guessing Game appearing in Exercise 4.44. Many experiments have been based on this game, including experiments conducted by Rosmarie Nagel. Exercise 4.47 describes the Braess Paradox, which first appeared in Braess [1968]. Exercise 4.48 is a variation of the Braess Paradox, due to Kameda and Hosokawa [2000]. The authors wish to thank Hisao Kameda for bringing this example to their attention. Exercise 4.49 is an example of a location "game," a concept that was first introduced and studied in Hotelling [1929].

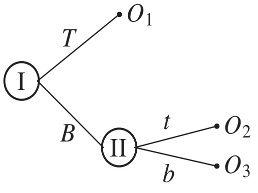
4.16 Exercises

4.1 William and Henry are participants in a televised game show, seated in separate booths with no possibility of communicating with each other. Each one of them is asked to submit, in a sealed envelope, one of the following two requests (requests that are guaranteed to be honored):

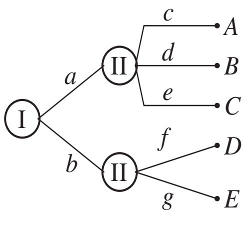
- Give me $1,000.
- Give the other participant $4,000.

Describe this situation as strategic-form game. What is the resulting game? What will the players do, and why?

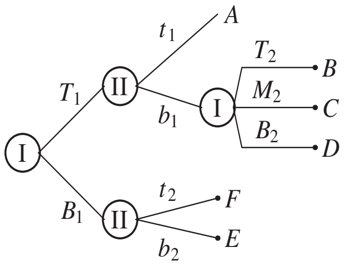
4.2 Describe the following games in strategic form.



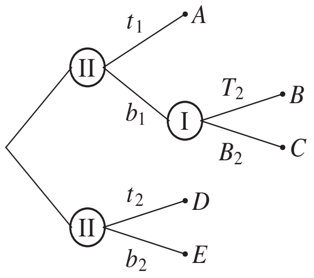
Game A



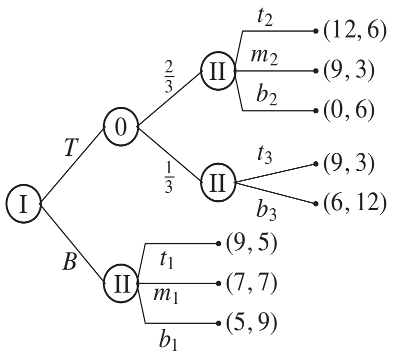
Game B



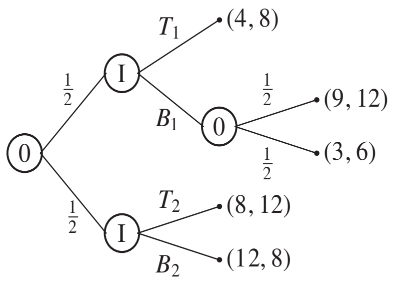
Game C



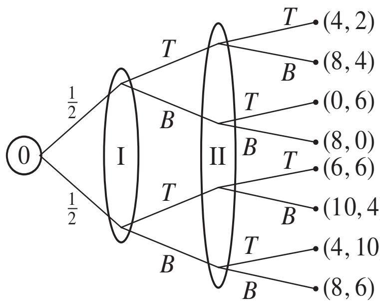
Game D



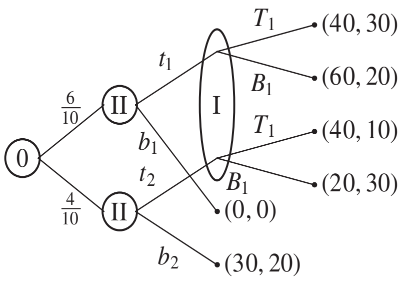
Game E



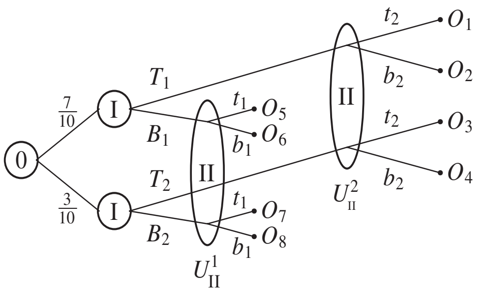
Game F



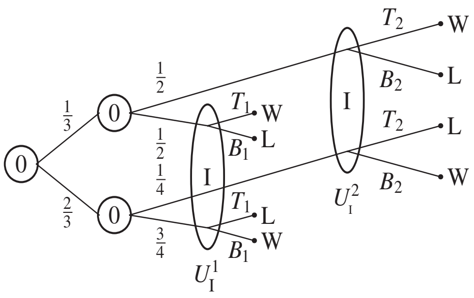
Game G



Game H

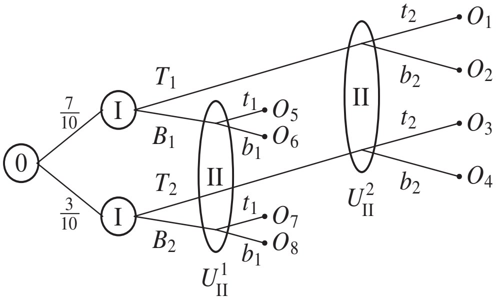


Game I



Game J

4.3 Consider the following two-player game.



(a) What does Player II know in each one of his information sets? What does he not know?

(b) Describe as a game in extensive form the version of this game in which Player II knows the result of the chance move but does not know the action chosen by Player I.

(c) Describe as a game in extensive form the version of this game in which Player II knows both the result of the chance move and the action chosen by Player I.

(d) Convert all three games into strategic-form games. Are all the matrices you derived in this way identical?

4.4 In the game of Hex (Exercise 3.19 on page 64) the two players eliminate (weakly) dominated strategies. What remains of the game once the elimination process ends?

4.5 Establish whether there exists a two-player game in extensive form with perfect information, and possible outcomes I (Player I wins), II (Player II wins), and *D* (a draw), whose strategic-form description is

Player II

		s_{II}^1	s_{II}^2	s_{II}^3	s_{II}^4
	s_I^1	*D*	I	II	I
Player I	s_I^2	I	II	I	*D*
	s_I^3	I	I	II	II

If the answer is yes, describe the game. If not, explain why not.

4.6 Repeat Exercise 4.5, with respect to the following strategic-form game:

		Player II			
		s_{II}^1	s_{II}^2	s_{II}^3	s_{II}^4
Player I	s_I^1	*D*	*D*	*D*	*D*
	s_I^2	I	I	II	II
	s_I^3	II	II	*D*	*D*

4.7 In each of the following games, where Player I is the row player and Player II is the column player, determine whether the process of iterated elimination of strictly dominated strategies yields a single strategy vector when it is completed. If so, what is that vector? Verify that it is the only Nash equilibrium of the game.

	L	*R*
H	4, 2	0, 1
T	1, 1	3, 3

Game A

	L	*R*
H	1, 3	2, 3
T	0, 4	0, 2

Game B

	a	*b*	*c*
γ	1, 0	3, 0	2, 1
β	3, 1	0, 1	1, 2
α	2, 1	1, 6	0, 2

Game C

4.8 What advice would you give to the players in each of the following four games? Provide a detailed justification for all advice given.

		Player II	
		L	*R*
Player I	*H*	2, 3	1, 5
	T	0, 0	4, 1

Game A

		Player II	
		L	*R*
Player I	*H*	4, 8	5, 10
	T	3, 7	6, 20

Game B

		Player II			
		a	*b*	*c*	*d*
Player I	δ	2, 13	4, 8	6, 5	8, 2
	γ	6, 4	2, 3	3, 8	8, 4
	β	0, 9	7, 7	2, 7	14, 8
	α	4, 0	0, 4	4, 6	6, 0

Game C

		Player II			
		a	*b*	*c*	*d*
Player I	δ	–5, 5	–4, 4	–5, 1	0, 4
	γ	–4, 1	–3, 3	–2, 2	1, 1
	β	0, 0	1, –1	–4, 4	–3, 3
	α	–1, 6	2, –2	7, 1	5, 2

Game D

4.9 A Nash equilibrium s^* is termed *strict* if every deviation undertaken by a player yields a definite loss for that player, i.e., $u_i(s^*) > u_i(s_i, s^*_{-i})$ for each player $i \in N$ and each strategy $s_i \in S_i \setminus \{s^*_i\}$.

(a) Prove that if the process of iterative elimination of strictly dominated strategies results in a unique strategy vector s^*, then s^* is a strict Nash equilibrium, and it is the only Nash equilibrium of the game.

(b) Prove that if $s^* = (s^*_i)^n_{i=1}$ is a strict Nash equilibrium, then none of the strategies s^*_i can be eliminated by iterative elimination of dominated strategies (under either strict or weak domination).[10]

4.10 Prove that the result of iterated elimination of strictly dominated strategies (that is, the set of strategies remaining after the elimination process has been completed) is independent of the order of elimination. Deduce that if the result of the elimination process is a single vector s^*, then that same vector will be obtained under every possible order of the elimination of strictly dominated strategies.

4.11 Find all rational strategy vectors in the following games.

Player II

		a	b	c	d
Player I	α	6, 2	6, 3	7, 6	2, 8
	β	8, 5	6, 9	4, 6	4, 7

Game A

Player II

		a	b
Player I	α	9, 5	5, 3
	β	8, 6	8, 4

Game B

Player II

		a	b	c	d
Player I	α	−1, 20	−7, −7	−1, 2	−5, 8
	β	27, 20	13, −1	21, 2	13, −1
	γ	−5, 20	−3, 5	7, −1	3, −4

Game C

Player II

		a	b	c	d
Player I	α	3, 7	0, 13	4, 5	5, 3
	β	5, 3	4, 5	4, 5	3, 7
	γ	4, 5	3, 7	4, 5	5, 3
	δ	4, 5	4, 5	4, 5	4, 5

Game D

4.12 Find a game that has at least one equilibrium, but in which iterative elimination of dominated strategies yields a game with no equilibria.

4.13 Prove directly that a strictly dominated strategy cannot be an element of a game's equilibrium (Corollary 4.36, page 109). In other words, show that in every strategy vector in which there is a player using a strictly dominated strategy, that player can deviate and increase his payoff.

---

[10] This is not true of equilibria that are not strict. See Example 4.16, where there are four nonstrict Nash equilibria (T, C), (M, L), (M, R), and (B, L).

4.14 In a first-price auction, each buyer submits his bid in a sealed envelope. The winner of the auction is the buyer who submits the highest bid, and the amount he pays is equal to what he bid. If several buyers have submitted bids equal to the highest bid, a fair lottery is conducted among them to choose one winner, who then pays his bid.

(a) In this situation, does the strategy β_i^* of buyer i, in which he bids his private value for the item, weakly dominate all his other strategies?
(b) Find a strategy of buyer i that weakly dominates strategy β_i^*.

Does the strategy under which each buyer bids his private value weakly dominate all the other strategies? Justify your answer.

4.15 Prove that the two definitions of the Nash equilibrium, presented in Definitions 4.17 and 4.19, are equivalent to each other.

4.16 Find all the equilibria in the following games.

Player II

		a	*b*	*c*	*d*
	γ	7, 3	6, 3	5, 5	4, 7
Player I	*β*	4, 2	5, 8	8, 6	5, 8
	α	6, 1	3, 8	2, 4	6, 9

Game A

Player II

		a	*b*	*c*	*d*
	δ	5, 2	3, 1	2, 2	4, 5
Player I	*γ*	0, 3	2, 2	0, 1	−1, 3
	β	8, 4	7, 0	6, −1	5, 2
	α	0, 5	1, −2	2, 2	3, 4

Game B

Player II

		a	*b*	*c*	*d*
	ϵ	0, 0	−1, 1	1, 1	0, −1
	δ	1, −1	1, 0	0, 1	0, 0
Player I	*γ*	0, 1	−1, −1	1, 0	1, −1
	β	−1, 1	0, −1	−1, 1	0, 0
	α	1, 1	0, 0	−1, −1	0, 0

Game C

4.17 In the following three-player game, Player I chooses a row (A or B), Player II chooses a column (a or b), and Player III chooses a matrix (α, β, or γ). Find all the equilibria of this game.

	a	*b*
A	0, 0, 5	0, 0, 0
B	2, 0, 0	0, 0, 0

α

	a	*b*
A	1, 2, 3	0, 0, 0
B	0, 0, 0	1, 2, 3

β

	a	*b*
A	0, 0, 0	0, 0, 0
B	0, 5, 0	0, 0, 4

γ

4.18 Find the equilibria of the following three-player game (Player I chooses row T, C, or B, Player II a column L, M, or R, and Player III chooses matrix P or Q).

	L	M	R
T	3, 10, 8	8, 14, 6	4, 12, 7
C	4, 7, 2	5, 5, 2	2, 2, 8
B	3, –5, 0	0, 3, 4	–3, 5, 0

P

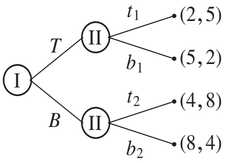
	L	M	R
T	4, 9, 3	7, 8, 10	5, 7, –1
C	3, 4, 5	17, 3, 12	3, 5, 2
B	9, 7, 2	20, 0, 13	0, 15, 0

Q

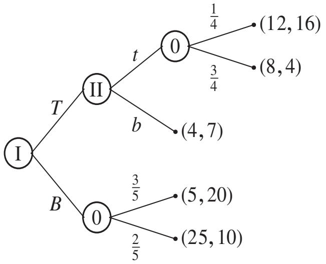
4.19 Prove that in the Centipede game (see Exercise 3.12 on page 61), at every Nash equilibrium, Player I chooses S at the first move in the game.

4.20 A two-player game is *symmetric* if the two players have the same strategy set $S_1 = S_2$ and the payoff functions satisfy $u_1(s_1, s_2) = u_2(s_2, s_1)$ for each $s_1, s_2 \in S_1$. Prove that the set of equilibria of a two-player symmetric game is a symmetric set: if (s_1, s_2) is an equilibrium, then (s_2, s_1) is also an equilibrium.

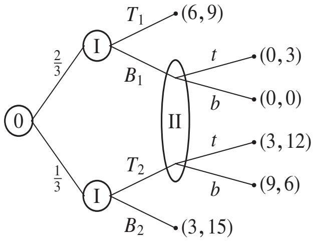
4.21 Describe the following games in strategic form (in three-player games, let Player I choose the row, Player II choose the column, and Player III choose the matrix). In each game, find all the equilibria, if any exist.



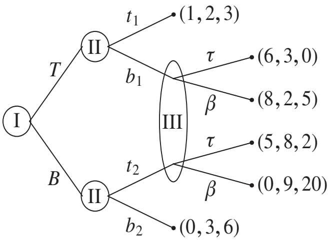
Game A



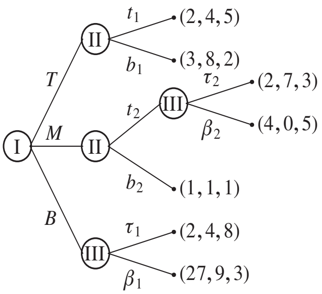
Game B



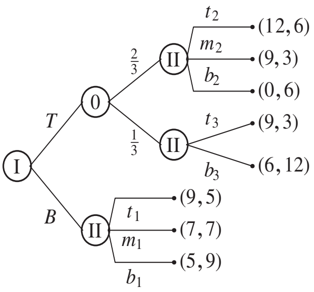
Game C



Game D



Game E



Game F

4.22 Let X and Y be two compact sets in $\mathbb{R}^m$, and let $f : X \times Y \to \mathbb{R}$ be a continuous function. Prove that the function $x \mapsto \min_{y \in Y} f(x, y)$ is also a continuous function.

4.23 In each of the following two-player zero-sum games, implement a process of iterative elimination of dominated strategies. For each game list the strategies you have eliminated and find the maxmin strategy of Player I and the minmax strategy of Player II.

Player II

		a	b	c	d
	γ	8	4	8	4
Player I	β	2	5	3	8
	α	6	1	4	5

Game A

Player II

		a	b	c	d
	δ	6	4	2	1
Player I	γ	5	3	3	0
	β	1	0	5	4
	α	2	–3	2	3

Game B

Player II

		a	b	c	d
	δ	3	6	5	5
Player I	γ	5	5	5	5
	β	5	3	5	6
	α	6	5	5	3

Game C

4.24 Prove that in Example 4.23 on page 99 (duopoly competition) the pair of strategies (q_1^*, q_2^*) defined by

$$q_1^* = \frac{2 - 2c_1 + c_2}{3}, \qquad q_2^* = \frac{2 - 2c_2 + c_1}{3} \tag{4.97}$$

is an equilibrium.

4.25 Prove Theorem 4.26 (page 105): if player i has a (weakly) dominant strategy, then it is his (not necessarily unique) maxmin strategy. Moreover, this strategy is his best reply to every strategy vector of the other players.

4.26 Prove Theorem 4.28 (page 105): in a game in which every player i has a strategy s_i^* that strictly dominates all of his other strategies, the strategy vector $(s_1^*, \ldots, s_n^*)$ is the unique equilibrium point of the game as well as the unique vector of maxmin strategies.

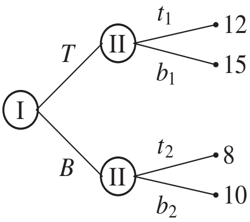
4.27 Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a game in strategic form, and let $\widehat{s}_i \in S_i$ be an arbitrary strategy of player i in this game. Let $\widehat{G}$ be the game derived from G by the elimination of strategy $\widehat{s}_i$. Prove that for each player j, $j \neq i$, the maxmin value of player j in the game $\widehat{G}$ is greater than or equal to his maxmin value in G. Is the maxmin value of player i in game $\widehat{G}$ necessarily less than his maxmin value in G? Prove this last statement, or find a counterexample.

4.28 Find an example of a game $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ in strategic form such that the game $\widehat{G}$ derived from G by elimination of one strategy in one player's strategy set has an equilibrium that is not an equilibrium in the game G.

4.29 Prove Corollary 4.33 on page 108: let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a strategic form game and let $\widehat{G}$ be the game derived from G by iterative elimination of dominated strategies. Then every equilibrium s^* in the game $\widehat{G}$ is also an equilibrium in the game G.

4.30 Find an example of a strategic form game G and of an equilibrium s^* of that game such that for each player $i \in N$ the strategy s_i^* is dominated.

4.31 The following questions relate to the following two-player zero-sum game.



(a) Find an optimal strategy for each player by applying backward induction.
(b) Describe this game in strategic form.
(c) Find all the optimal strategies of the two players.
(d) Explain why there are optimal strategies in addition to the one you identified by backward induction.

4.32 (a) Let $A = (a_{ij})$ be an $n \times m$ matrix representing a two-player zero-sum game, where the row player is Ann and the column player is Bill. Let $B = (b_{ji})$ be a new $m \times n$ matrix in which the row player is Bill and the column player is Ann. What is the relation between the matrices A and B?

(b) Conduct a similar transformation of the names of the players in the following matrix and write down the new matrix.

		Player II		
		L	M	R
Player I	T	3	−5	7
	B	−2	8	4

4.33 The value of the two-player zero-sum game given by the matrix A is 0. Is it necessarily true that the value of the two-player zero-sum game given by the matrix $-A$ is also 0? If your answer is yes, prove this. If your answer is no, provide a counterexample.

4.34 Let A and B be two finite sets, and let $u : A \times B \to \mathbb{R}$ be an arbitrary function.[11] Prove that

$$\max_{a\in A}\min_{b\in B} u(a,b) \le \min_{b\in B}\max_{a\in A} u(a,b). \tag{4.98}$$

4.35 Show whether or not the value exists in each of the following games. If the value exists, find it and find all the optimal strategies for each player. As usual, Player I is the row player and Player II is the column player.

	a	*b*
A	2	2
B	1	3

Game A

	a	*b*	*c*
A	1	2	3
B	4	3	0

Game B

	a	*b*	*c*	*d*
A	$3\frac{1}{2}$	3	4	12
B	7	5	6	13
C	4	2	3	0

Game C

	a	*b*
A	3	0
B	2	2
C	0	3

Game D

4.36 Prove Theorem 4.48 (page 118): in a two-player zero-sum game, $(s_{\mathrm{I}}^*, s_{\mathrm{II}}^*)$ is a saddle point if and only if s_{I}^* is an optimal strategy for Player I and s_{II}^* is an optimal strategy for Player II.

4.37 Let A and B be two finite-dimensional matrices with positive payoffs. Show that the game

A	0
0	*B*

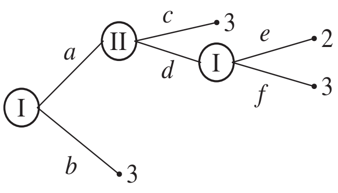
has no value. (Each 0 here represents a matrix of the proper dimensions, such that all of its entries are 0.)

---

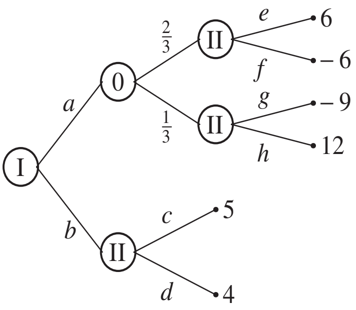
[11] The finiteness of A and B is needed to ensure the existence of a minimum and maximum in Equation (4.98). The claim holds (using the same proof) for each pair of sets A and B and function u for which the min and the max of the function in Equation (4.98) exist (for example, if A and B are compact sets and u is a continuous function; see Exercise 4.22). Alternatively, we may remove all restrictions on A, B, and u and replace min by inf and max by sup.

4.38 Answer the following questions with reference to Game A and Game B that appear in the diagram below.

(a) Find all equilibria obtained by backward induction.
(b) Describe the games in strategic form.
(c) Check whether there are other Nash equilibria in addition to those found by backward induction.



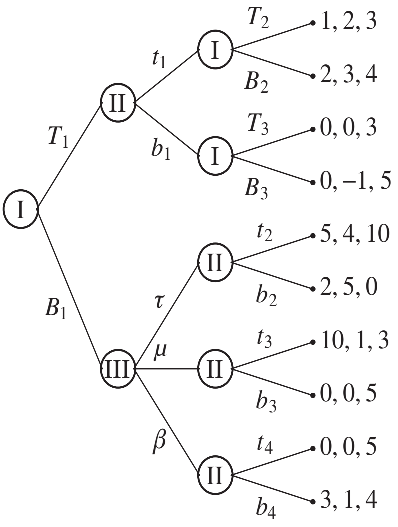
Game A



Game B

4.39 Prove Theorem 4.49 (on page 118) using backward induction (a general outline for the proof can be found in Remark 4.50 on page 121).

4.40 Find a Nash equilibrium in the following game using backward induction:



Find an additional Nash equilibrium of this game.

4.41 In a two-player zero-sum game on the unit square where Player I's strategy set is $X = [0, 1]$ and Player II's strategy set is $Y = [0, 1]$, check whether or not the game associated with each of the following payoff functions has a value, and if so, find the value and optimal strategies for the two players:

(a) $u(x, y) = 1 + 4x + y - 5xy$.
(b) $u(x, y) = 4 + 2y - 4xy$.

4.42 Consider a two-player non-zero-sum game on the unit square in which Player I's strategy set is $X = [0, 1]$, Player II's strategy set is $Y = [0, 1]$, and the payoff functions for the players are given below. Find the maxmin value and the maxmin strategy (or strategies) of the players. Does this game have an equilibrium? If so, find it.

$$u_{\mathrm{I}}(x, y) = 2x - xy,$$
$$u_{\mathrm{II}}(x, y) = 2 + 3x + 3y - 3xy.$$

4.43 Consider a two-player non-zero-sum game on the unit square in which Player I's strategy set is $X = [0, 1]$, and Player II's strategy set is $Y = [0, 1]$, which has a unique equilibrium (x^*, y^*), where $x^*, y^* \in (0, 1)$. Prove that the equilibrium payoff to each player equals his maxmin value.

4.44 Fifty people are playing the following game. Each player writes down, on a separate slip of paper, one integer in the set $\{0, 1, \ldots, 100\}$, alongside his name. The game-master then reads the numbers on each slip of paper, and calculates the average x of all the numbers written by the players. The winner of the game is the player (or players) who wrote down the number that is closest to $\frac{2}{3}x$. The winners equally divide the prize of $1,000 between them.

Describe this as a strategic-form game, and find all the Nash equilibria of the game. What would be your strategy in this game? Why?

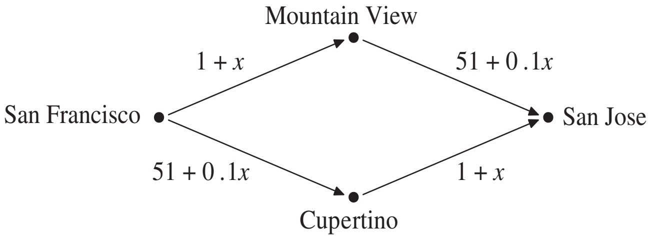
4.45 Peter, Andrew, and James are playing the following game in which the winner is awarded M dollars. Each of the three players receives a coupon and is to decide whether or not to bet on it. If a player chooses to bet, he or she loses the coupon with probability $\frac{1}{2}$ and wins an additional coupon with probability $\frac{1}{2}$ (thus resulting in two coupons in total). The success of each player in the bet is independent of the results of the bets of the other players. The winner of the prize is the player with the greatest number of coupons. If there is more than one such player, the winner is selected from among them in a lottery where each has an equal chance of winning. The goal of each player is to maximize the probability of winning the award.

(a) Describe this game as a game in strategic form and find all its Nash equilibria.
(b) Now assume that the wins and losses of the players are perfectly correlated: a single coin flip determines whether all the players who decided to bid either all win an additional coupon or all lose their coupons. Describe this new situation as a game in strategic form and find all its Nash equilibria.

4.46 **Partnership Game** Lee (Player 1), and Julie (Player 2), are business partners. Each of the partners has to determine the amount of effort he or she will put into the business, which is denoted by e_i, $i = 1, 2$, and may be any nonnegative real number. The cost of effort e_i for Player i is ce_i, where $c > 0$ is equal for both players. The success of the business depends on the amount of effort put in by the players; the business's profit is denoted by $r(e_1, e_2) = e_1^{\alpha_1} e_2^{\alpha_2}$, where $\alpha_1, \alpha_2 \in (0, 1)$ are fixed constants known by Lee and Julie, and the profit is shared equally between the two partners. Each player's utility is given by the difference between the share of the profit received by that player and the cost of the effort he or she put into the business. Answer the following questions:

(a) Describe this situation as a strategic-form game. Note that the set of strategies of each player is the continuum.
(b) Find all the Nash equilibria of the game.

4.47 Braess Paradox There are two main roads connecting San Francisco and San Jose, a northern road via Mountain View and a southern road via Cupertino. Travel time on each of the roads depends on the number x of cars using the road per minute, as indicated in the following diagram.

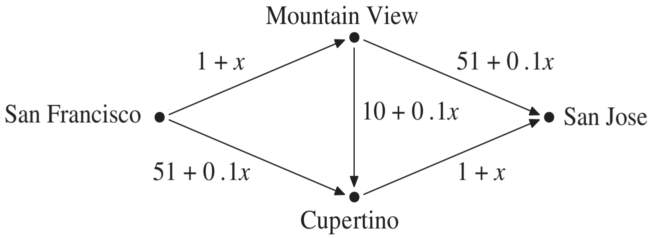


For example, the travel time between San Francisco and Mountain View is $1 + x$, where x is the number of cars per minute using the road connecting these cities, and the travel time between Mountain View and San Jose is $51 + 0.1x$, where x is the number of cars per minute using the road connecting those two cities. Each driver chooses which road to take in going from San Francisco to San Jose, with the goal of reducing to a minimum the amount of travel time. Early in the morning, 60 cars per minute get on the road from San Francisco to San Jose (where we assume the travellers leave early enough in the morning so that they are the only ones on the road at that hour).

(a) Describe this situation as a strategic-form game, in which each driver chooses the route he will take.
(b) What are all the Nash equilibria of this game? At these equilibria, how much time does the trip take at an early morning hour?
(c) The California Department of Transportation constructs a new road between Mountain View and Cupertino, with travel time between these cities $10 + 0.1x$ (see the diagram below). This road is one way, enabling travel solely from Mountain View to Cupertino.

Find a Nash equilibrium in the new game. Under this equilibrium how much time does it take to get to San Jose from San Francisco at an early morning hour?

(d) Does the construction of the additional road improve travel time?

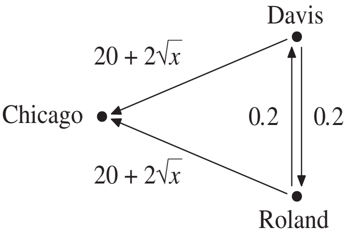


This phenomenon is “paradoxical” because, as you discovered in the answers to (b) and (c), the construction of a new road increases the travel time for all travellers. This is because when the new road is opened, travel along the San Francisco–Mountain View–Cupertino–San Jose route takes less time than along the San Francisco–Mountain View–San Jose route and the San Francisco–Cupertino–San Jose route, causing drivers to take the new route. But that causes the total number of cars along the two routes San Francisco–Mountain View–San Jose and San Francisco–Cupertino–San Jose to increase: travel time along each stretch of road increases.

Such a phenomenon was in fact noted in New York (where the closure of a road for construction work had the effect of decreasing travel time) and in Stuttgart (where the opening of a new road increased travel time).

4.48 The Davis Removal Company and its main rival, Roland Ltd, have fleets of ten trucks each, which leave the companies’ headquarters for Chicago each morning at 5 am for their daily assignments. At that early hour, these trucks are the only vehicles on the roads. Travel time along the road between the Davis Removal Company and Chicago is $20 + 2\sqrt{x}$, where x is the number of cars on the road, and it is similarly $20 + 2\sqrt{x}$ on the road connecting the headquarters of Roland Ltd with Chicago, where x is the number of cars on the road.

The Illinois Department of Transportation paves a new two-way road between the companies’ headquarters, where travel time on this new road is 0.2, independent of the number of cars on the road. This situation is described in the following diagram.



Answer the following questions:

(a) Before the new road is constructed, what is the travel time of each truck between its headquarters and Chicago?

(b) Describe the situation after the construction of the new road as a two-player strategic-form game, in which the players are the managers of the removal companies and each player must determine the number of trucks to send on the road connecting his company with Chicago (with the rest traveling on the newly opened road and the road connecting the other company's headquarters and Chicago), with the goal of keeping to a minimum the total travel time to Chicago of all the trucks in its fleet. Note that if Davis, for example, instructs all its drivers to go on the road between company headquarters and Chicago, and Roland sends seven of its trucks directly to Chicago and three first to the Davis headquarters and then to Chicago, the total time racked up by the fleet of Roland Ltd is

$$7 \times (20 + 2\sqrt{7}) + 3 \times (0.2 + 20 + 2\sqrt{13}). \tag{4.99}$$

(c) Is the strategy vector in which both Davis and Roland send their entire fleets directly to Chicago, ignoring the new road, a Nash equilibrium?

(d) Show that the strategy vector in which both Davis and Roland send six drivers directly to Chicago and four via the new road is an equilibrium. What is the total travel time of the trucks of the two companies in this equilibrium? Did the construction of a new road decrease or increase total travel time?

(e) Construct the payoff matrix of this game, with the aid of a spreadsheet program. Are there any additional equilibria in this game?

4.49 Location games Two competing coffee house chains, Pete's Coffee and Caribou Coffee, are seeking locations for new branch stores in Cambridge. The town is comprised of only one street, along which all the residents live. Each of the two chains therefore needs to choose a single point within the interval [0, 1], which represents the exact location of the branch store along the road. It is assumed that each resident will go to the coffee house that is nearest to his place of residence. If the two chains choose the exact same location, they will each attract an equal number of customers. Each chain, of course, seeks to maximize its number of customers.

To simplify the analysis required here, suppose that each point along the interval [0, 1] represents a town resident, and that the fraction of residents who frequent each coffee house is the fraction of points closer to one store than to the other.

(a) Describe this situation as a two-player strategic-form game.

(b) Prove that the only equilibrium in this game is that given by both chains selecting the location $x = \frac{1}{2}$.

(c) Prove that if three chains were to compete for a location in Cambridge, the resulting game would have no equilibrium. (Under this scenario, if two or three of the chains choose the same location, they will split the points closest to them equally between them.)

4.50 For each of the following two games, determine whether or not it can represent a strategic-form game corresponding to an extensive-form game with perfect information. If so, describe a corresponding extensive-form game; if not, justify your answer.

		Player II	
		a	b
Player I	A	1, 1	5, 3
	B	3, 0	5, 3
	C	1, 1	0, 4
	D	3, 0	5, 3

Game A

		Player II	
		a	b
Player I	A	3, 0	0, 4
	B	3, 0	0, 4
	C	3, 0	1, 1
	D	3, 0	5, 3

Game B

4.51 Let Γ be a game in extensive form. The *agent-form game* derived from Γ is a strategic-form game where each player i in Γ is split into several players: for each information set $U_i \in \mathcal{U}_i$ of player i we define a player (i, U_i) in the agent-form game. Thus, if each player i has k_i information sets in Γ, then there are $\sum_{i \in N} k_i$ players in the agent-form game. The set of strategies of player (i, U_i) is $A(U_i)$. There is a bijection between the set of strategy vectors in the game Γ and the set of strategy vectors in the agent-form game: the strategy vector $\sigma = (\sigma_i)_{i \in N}$ in Γ corresponds to the strategy vector $(\sigma_i(U_i))_{\{i \in N, U_i \in \mathcal{U}_i\}}$ in the agent-form game. The payoff function of player (i, U_i) in the agent-form game is the payoff function of player i in the game Γ.

Prove that if $\sigma = (\sigma_i)_{i \in N}$ is a Nash equilibrium in the game Γ, then the strategy vector $(\sigma_i(U_i))_{\{i \in N, U_i \in \mathcal{U}_i\}}$ is a Nash equilibrium in the agent-form game derived from Γ.