10 Games with incomplete information: the general model

Chapter summary

In this chapter we extend Aumann's model of incomplete information with beliefs in two ways. First, we do not assume that the set of states of the world is finite, and allow it to be any measurable set. Second, we do not assume that the players share a common prior, but rather that the players' beliefs at the interim stage are part of the data of the game. These extensions lead to the concept of a *belief space*. We also define the concept of a *minimal belief subspace* of a player, which represents the model that the player "constructs in his mind" when facing the situation with incomplete information. The notion of games with incomplete information is extended to this setup, along with the concept of Bayesian equilibrium. We finally discuss in detail the concept of *consistent beliefs*, which are beliefs derived from a common prior and thus lead to an Aumann or Harsanyi model of incomplete information.

Chapter 9 focused on the Aumann model of incomplete information, and on Harsanyi games with incomplete information. In both of those models, players share a common prior distribution, either over the set of states of the world or over the set of type vectors. As noted in that chapter, there is no compelling reason to assume that such a common prior exists. In this chapter, we will expand the Aumann model of incomplete information to deal with the case where players may have heterogeneous priors, instead of a common prior.

The equilibrium concept we presented for analyzing Harsanyi games with incomplete information and a common prior was the Nash equilibrium. This is an equilibrium in a game that begins with a chance move that chooses the type vector. As shown in Chapter 9, every Nash equilibrium in a Harsanyi game is a Bayesian equilibrium, and conversely every Bayesian equilibrium is a Nash equilibrium. When there is no common prior, we cannot postulate a chance move choosing a type vector; hence the concept of Nash equilibrium is not applicable in this case. However, as we will show, the concept of Bayesian equilibrium is still applicable. We will study the properties of this concept in Section 10.5 (page 407).

10.1 Belief spaces

Recall that an Aumann model of incomplete information is given by a set of players N, a finite set Y of states of the world, a partition $\mathcal{F}_i$ of Y for each player $i \in N$, a set of states of nature S, a function $\mathfrak{s} : Y \to S$ mapping each state of the world to a state of nature,

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and a common prior $\mathbf{P}$ over Y. The next definition extends this model to the case in which there is no common prior.

Definition 10.1 *Let N be a finite set of players, and let $(S, \mathcal{S})$ be a measurable space of states of nature.*[1] *A* belief space *of the set of players N over the set of states of nature is an ordered vector $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i \in N})$, where:*

- *$(Y, \mathcal{Y})$ is a measurable space of states of the world.*
- *$\mathfrak{s} : Y \to S$ is a measurable function,*[2] *mapping each state of the world to a state of nature.*
- *For each player $i \in N$, a function $\pi_i : Y \to \Delta(Y)$ mapping each state of the world $\omega \in Y$ to a probability distribution over Y. We will denote the probability that player i ascribes to event E, according to the probability distribution $\pi_i(\omega)$, by $\pi_i(E \mid \omega)$. We require the function $(\pi_i)_{i \in N}$ to satisfy the following conditions:*
  1. *Coherency: for each player $i \in N$ and each $\omega \in Y$, the set $\{\omega' \in Y : \pi_i(\omega') = \pi_i(\omega)\}$ is measurable in Y, and*

$$\pi_i(\{\omega' \in Y : \pi_i(\omega') = \pi_i(\omega)\} \mid \omega) = 1. \tag{10.1}$$

  2. *Measurability: for each player $i \in N$ and each measurable set $E \in \mathcal{Y}$, the function $\pi_i(E \mid \cdot) : Y \to [0, 1]$ is a measurable function.*

As in the Aumann model of incomplete information, belief spaces describe situations in which there is a true state of the world ω_* but the players may not know which state is the true state. At the true state of the world ω_* each player $i \in N$ believes that the true state of the world is distributed according to the probability distribution $\pi_i(\omega^*)$. This probability distribution is called player i's belief at the state of the world ω_*. We assume that each player knows his own belief and therefore if at the state of the world ω_* player i believes that the state of the world might be ω, then his beliefs at ω_* and ω must coincide. Indeed, if his beliefs at ω differed from his beliefs at ω_*, then he would be able to distinguish between these states, and therefore at ω_* he could not ascribe a positive probability to the state of the world ω. It follows that at the state of the world ω_* player i ascribes probability 1 to the set of states of the world at which his beliefs equal his belief at ω_*, and therefore the support of $\pi_i(\omega)$ is contained in the set $\{\omega' \in Y : \pi_i(\omega') = \pi_i(\omega)\}$, for each state of the world $\omega \in Y$. This is the reason we demand coherency in Definition 10.1. The measurability condition is a technical condition that is required for computing the expected payment in games in which incomplete information games are modeled using belief spaces.

The concept "belief space" generalizes the concept "Aumann model of incomplete information" that was presented in Definition 9.27 (page 334). Every Aumann model of incomplete information is a belief space. To see this, let $\Pi = (N, Y, (\mathcal{F}_i)_{i \in N}, \mathfrak{s}, \mathbf{P})$ be an Aumann model of incomplete information. Let $\mathcal{Y} = 2^Y$ be the collection of all subsets

1 A *measurable space* is a pair $(X, \mathcal{X})$, where X is a set and $\mathcal{X}$ is a σ-algebra over X; i.e., $\mathcal{X}$ is a collection of subsets of X that includes the empty set, is closed under complementation, and is closed under countable intersections. A set in $\mathcal{X}$ is called a *measurable* set. This definition was mentioned on page 344.

2 A function $f : X \to Y$ is *measurable* if the inverse image under f of every measurable set in Y is a measurable set in X. In other words, for each measurable set C in Y, the set $f^{-1}(C) := \{x \in X : f(x) \in C\}$ is measurable in X.

of Y. For each player $i \in N$ and every $\omega \in Y$, let $\pi_i(\omega) = \mathbf{P}(\cdot \mid F_i(\omega))$; i.e., player i's belief at state ω is the common prior $\mathbf{P}$, conditioned on his information. It follows that $(Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$ is a belief space equivalent to the original Aumann model: for every event $A \subseteq Y$, the probability that player i ascribes at every state of the world ω to event A is equal in both models (verify!).

Since every Harsanyi model of incomplete information is equivalent to an Aumann model of incomplete information (see page 350), every Harsanyi model of incomplete information can be represented by a belief space.

Belief spaces generalize Aumann models of incomplete information with belief in the following ways:

1. The set of states of the world in an Aumann model of incomplete information is finite, while the set of states of the world in a belief space may be any measurable space.
2. The beliefs $(\pi_i)_{i\in N}$ in a belief space are not necessarily derived from a prior $\mathbf{P}$ common to all the players.

In most of the examples in this chapter, the set of states of the world Y is finite. In those examples, we assume that $\mathcal{Y} = 2^Y$: the σ-algebra over Y is the collection of all the subsets of Y.

Example 10.2 Let the set of players be $N = \{\mathrm{I}, \mathrm{II}\}$, and let the set of states of nature be $S = \{s_1, s_2\}$. Consider a belief space $Y = \{\omega_1, \omega_2, \omega_3\}$, where:

State of the world	$\mathfrak{s}(\cdot)$	$\pi_{\mathrm{I}}(\cdot)$	$\pi_{\mathrm{II}}(\cdot)$
ω_1	s_1	$\left[\frac{2}{3}(\omega_1), \frac{1}{3}(\omega_2)\right]$	$[1(\omega_1)]$
ω_2	s_1	$\left[\frac{2}{3}(\omega_1), \frac{1}{3}(\omega_2)\right]$	$\left[\frac{1}{2}(\omega_2), \frac{1}{2}(\omega_3)\right]$
ω_3	s_2	$[1(\omega_3)]$	$\left[\frac{1}{2}(\omega_2), \frac{1}{2}(\omega_3)\right]$

The states of the world appear in the left-hand column of the table, the next column displays the state of nature associated with each state of the world, and the two right-hand columns display the beliefs of the players at each state of the world.

At the state of the world ω_1, Player II ascribes probability 1 to the state of nature being s_1, while at the states ω_2 and ω_3 he ascribes probability $\frac{1}{2}$ to each of the two states of nature. At each state of the world, Player I ascribes probability 1 to the true state of nature. As for the beliefs of Player I about the beliefs of Player II about the state of nature, at the state of the world ω_3 he ascribes probability 1 to Player II ascribing equal probabilities to the two states of nature, while at the states of the world ω_1 and ω_2 he ascribes probability $\frac{2}{3}$ to Player II ascribing probability 1 to the true state of nature, and probability $\frac{1}{3}$ to Player II ascribing probability $\frac{1}{2}$ to the true state of nature.

The beliefs of the players can be calculated from the following common prior $\mathbf{P}$:

$$\mathbf{P}(\omega_1) = \tfrac{1}{2}, \quad \mathbf{P}(\omega_2) = \tfrac{1}{4}, \quad \mathbf{P}(\omega_3) = \tfrac{1}{4}, \tag{10.2}$$

and the following partitions of the two players (verify that this is true)

$$\mathcal{F}_{\mathrm{I}} = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}, \qquad \mathcal{F}_{\mathrm{II}} = \{\{\omega_1\}, \{\omega_2, \omega_3\}\}. \tag{10.3}$$

It follows that the belief space of this example is equivalent to an Aumann model of incomplete information. ◀

As the next example shows, however, it is not true that every belief space is equivalent to an Aumann model of incomplete information; in other words, there are cases in which the beliefs of the players $(\pi_i)_{i \in N}$ cannot be calculated as conditional probabilities of a common prior.

Example 10.3 Let the set of players be $N = \{\mathrm{I}, \mathrm{II}\}$, and let the set of states of nature be $S = \{s_1, s_2\}$. Consider a belief space $Y = \{\omega_1, \omega_2\}$ where:

State of the world	$\mathfrak{s}(\cdot)$	$\pi_{\mathrm{I}}(\cdot)$	$\pi_{\mathrm{II}}(\cdot)$
ω_1	s_1	$\left[\frac{2}{3}(\omega_1), \frac{1}{3}(\omega_2)\right]$	$\left[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)\right]$
ω_2	s_2	$\left[\frac{2}{3}(\omega_1), \frac{1}{3}(\omega_2)\right]$	$\left[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)\right]$

In this space, at every state of the world Player I ascribes probability $\frac{2}{3}$ to the state of nature being s_1, while Player II ascribes probability $\frac{1}{2}$ to the state of nature being s_1. There is no common prior over Y that enables both of these statements to be true (verify that this is true).

Since each player has the same belief at both states of the world, if there is an Aumann model of incomplete information describing this situation, the partition of each player must be the trivial partition: $\mathcal{F}_i = \{Y\}$ for all $i \in N$. ◀

As the next example shows, it is possible for the support of $\pi_i(\omega)$ to be contained in the set $\{\omega' \in Y : \pi_i(\omega') = \pi_i(\omega)\}$, but not equal to it.

Example 10.4 Let the set of players be $N = \{\mathrm{I}, \mathrm{II}\}$, and let the set of states of nature be $S = \{s_1, s_2\}$. Consider a belief space $Y = \{\omega_1, \omega_2, \omega_3\}$, where:

State of the world	$\mathfrak{s}(\cdot)$	$\pi_{\mathrm{I}}(\cdot)$	$\pi_{\mathrm{II}}(\cdot)$
ω_1	s_1	$[1(\omega_1)]$	$\left[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)\right]$
ω_2	s_2	$[1(\omega_1)]$	$\left[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)\right]$
ω_3	s_2	$[1(\omega_3)]$	$\left[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)\right]$

At both states of the world ω_1 and ω_2, Player I believes that the true state is ω_1: the support of $\pi_{\mathrm{I}}(\omega_1)$ is the set $\{\omega_1\}$, which is a proper subset of $\{\omega' \in Y : \pi_{\mathrm{I}}(\omega') = \pi_{\mathrm{I}}(\omega_1)\} = \{\omega_1, \omega_2\}$. Note that at the state of the world ω_2, the state of nature is s_2, but Player I believes that the state of nature is s_1. ◀

The belief spaces described in Examples 10.3 and 10.4 are not equivalent to Aumann models of incomplete information, but they can be described as Aumann models in which every player has a prior distribution of his own. In Example 10.3, in both states of the world, Player I has a prior distribution $[\frac{2}{3}(\omega_1), \frac{1}{3}(\omega_2)]$, and Player II has a prior distribution $[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)]$. The beliefs of the players in the belief space of Example 10.4 can also be computed as being derived from prior distributions in the following way (verify!). The beliefs of Player II can be derived from the prior

$$\mathbf{P}_{\mathrm{II}}(\omega_1) = \tfrac{1}{2}, \quad \mathbf{P}_{\mathrm{II}}(\omega_2) = \tfrac{1}{2}, \quad \mathbf{P}_{\mathrm{II}}(\omega_3) = 0 \tag{10.4}$$

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and the partition

$$\mathcal{F}_{\mathrm{II}} = \{Y\}. \tag{10.5}$$

The beliefs of Player I can be derived from any prior of the form

$$\mathbf{P}_{\mathrm{I}}(\omega_1) = x, \quad \mathbf{P}_{\mathrm{I}}(\omega_2) = 0, \quad \mathbf{P}_{\mathrm{I}}(\omega_3) = 1 - x, \tag{10.6}$$

where $x \in (0, 1)$, and the partition

$$\mathcal{F}_{\mathrm{I}} = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}. \tag{10.7}$$

This is not coincidental: every belief space with a finite set of states of the world is an Aumann model of incomplete information in which every player has a prior distribution whose support is not necessarily all of Y, and the priors of the players may be heterogeneous.[3] To see this, for the case that Y is finite define, for each player i, a partition $\mathcal{F}_i$ of Y based on his beliefs:

$$F_i(\omega) = \{\omega' \in Y : \pi_i(\omega') = \pi_i(\omega)\}. \tag{10.8}$$

For each ω, the partition element $F_i(\omega)$ is the set of all states of the world at which the beliefs of player i equal his beliefs at ω: player i's beliefs do not distinguish between the states of the world in $F_i(\omega)$.

Define, for each player $i \in N$ a probability distribution $\mathbf{P}_i \in \Delta(Y)$ as follows (verify that this is indeed a probability distribution):

$$\mathbf{P}_i(A) = \sum_{\omega \in Y} \frac{1}{|Y|} \pi_i(A \mid \omega). \tag{10.9}$$

Then the belief $\pi_i(\omega)$ of player i at the state of the world ω is the probability distribution $\mathbf{P}_i$, conditioned on $F_i(\omega)$, which is his information at that state of the world (see Exercise 10.3):

$$\pi_i(A \mid \omega) = \mathbf{P}_i(A \mid F_i(\omega)), \quad \forall \omega \in Y, \forall A \in \mathcal{Y}. \tag{10.10}$$

It follows that every belief space $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i \in N})$, where Y is a finite set, is equivalent to an Aumann model of incomplete information $(N, Y, (\mathcal{F}_i)_{i \in N}, \mathfrak{s}, (\mathbf{P}_i)_{i \in N})$ in which every player has a prior of his own.

Example 10.4 (*Continued*) Using Equation (10.9), we have

$$\mathbf{P}_{\mathrm{I}} = \left[\tfrac{2}{3}(\omega_1), 0(\omega_2), \tfrac{1}{3}(\omega_3)\right], \quad \mathbf{P}_{\mathrm{II}} = \left[\tfrac{1}{2}(\omega_1), \tfrac{1}{2}(\omega_2), 0(\omega_3)\right]. \tag{10.11}$$

In fact, the definition of $\mathbf{P}_i$ in Equation (10.9) can be replaced with any weighted average of the beliefs $(\pi_i(\cdot \mid \omega))_{\omega \in Y}$, where all the weights are positive. The probability distribution of Equation (10.6) corresponds to the weights $(y, x - y, 1 - x)$, where $y \in (0, x)$ (verify!). ◀

3 When the space of states of the world is infinite, additional technical assumptions are needed to ensure the existence of a prior distribution from which each player's beliefs can be derived.

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Just as in an Aumann model of incomplete information, we can trace all levels of beliefs for each player at any state of the world in a belief space. For example, consider Example 10.4 and write out Player 1's beliefs at the state of the world ω_3. At that state, Player 1 ascribes probability 1 to the state of nature being s_2; this is his first-order belief. He ascribes probability 1 to the state of nature being s_2 and to Player 2 ascribing equal probability to the two possible states of nature; this is his second-order belief. Player 1's third-order belief at the state of the world ω_3 is as follows: Player 1 ascribes probability 1 to the state of nature being s_2, to Player 2 ascribing equal probability to the two states of nature, and to Player 2 believing that Player 1 ascribes probability 1 to the state of nature s_1. We can similarly describe the beliefs of every player, at any order, at every state of the world.

10.2 Belief and knowledge

One of the main elements of the Aumann model of incomplete information is the partitions $(\mathcal{F}_i)_{i\in N}$ defining the players' knowledge operators. In an Aumann model, the players' beliefs are derived from a common prior, given the information that the player has (i.e., the partition element $F_i(\omega)$). In contrast, in a belief space, a player's beliefs are given by the model itself. Since an Aumann model of incomplete information is a special case of a belief space, it is natural to ask whether a knowledge operator can be defined generally, in all belief spaces. As we saw in Equation (10.8), the beliefs $(\pi_i)_{i\in N}$ of the players define partitions $(\mathcal{F}_i)_{i\in N}$ of Y. A knowledge operator can then be defined using these partitions. When player i knows what the belief space Π is, he can indeed compute his partitions $(\mathcal{F}_i)_{i\in N}$ and the knowledge operators corresponding to these partitions. As the next example shows, knowledge based on these knowledge operators is not equivalent to belief with probability 1.

Example 10.5 Consider the belief space Π of a single player $N = \{\mathrm{I}\}$ over a set of states of the world $S = \{s_1, s_2\}$, shown in Figure 10.1.

Π	State of the world	$\mathfrak{s}(\cdot)$	$\pi_{\mathrm{I}}(\cdot)$
	ω_1	s_1	$[1(\omega_1)]$
	ω_2	s_2	$[1(\omega_1)]$

Figure 10.1 The belief space Π in Example 10.5

In the belief space Π, the partition defined by Equation (10.8) contains a single element, and therefore the minimal knowledge element of Player I at every state of the world is $\{\omega_1, \omega_2\}$. In other words, at the state of the world ω_1 Player I does not know that the state of the world is ω_1. Thus, despite the fact that at the state of the world ω_1 Player I ascribes probability 1 to the state of the world ω_1, he does not know that this is the true state of the world. ◀

The assumption that a player knows the belief space Π is a strong assumption: at the state of the world ω_1 in Example 10.5 the player ascribes probability 1 to the state of the

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world ω_1. Perhaps he does not know that there is a state of the world ω_2? We will assume that the only information that a player has is his belief, and that he does not know what the belief space is. In particular, different players may have different realities. In such a case, when a player does not know Π, he cannot compute the partitions $(\mathcal{F}_i)_{i\in N}$ and the knowledge operators corresponding to these partitions, and therefore cannot compute the events that he knows.

Under these assumptions the natural operator to use in belief spaces is a belief operator and not a knowledge operator. Under a knowledge operator, if a player knows a certain fact it must be true. This requirement may not be satisfied by belief operators; a player may ascribe probability 1 to a 'fact' that is actually false. After we define this operator and study its properties we will relate it to the knowledge operator in Aumann models of incomplete information.

Definition 10.6 *At the state of the world $\omega \in Y$, player $i \in N$* believes *that an event A obtains if $\pi_i(A \mid \omega) = 1$. Denote*

$$B_i A := \{\omega \in Y : \pi_i(A \mid \omega) = 1\}. \tag{10.12}$$

At the state of the world ω player i believes that event A obtains if he ascribes probability 1 to A. The event $B_i A$ is the set of all states of the world at which player i believes event A obtains. The belief operator B_i satisfies four of the five properties of Kripke that a knowledge operator must satisfy (see page 327 and Exercise 10.8).

Theorem 10.7 *For each player $i \in N$, the belief operator B_i satisfies the following four properties:*

1. *$B_i Y = Y$: At each state of the world, player i believes that Y is the set of states of the world.*
2. *$B_i A \cap B_i C = B_i(A \cap C)$: If player i believes that event A obtains and he believes that event C obtains, then he believes that event $A \cap C$ obtains.*
3. *$B_i(B_i A) = B_i A$: If player i believes that event A obtains, then he believes that he believes that event A obtains.*
4. *$(B_i A)^c = B_i((B_i A)^c)$: If player i does not believe that event A obtains, then he believes that he does not believe that event A obtains.*

The knowledge operator K_i satisfies a fifth property: $K_i A \subseteq A$. This property is not necessarily satisfied by a belief operator: it is not always the case that $B_i A \subseteq A$. In other words, it is possible that $\omega \in B_i A$ but $\omega \notin A$. This means that a player may believe that the event A obtains despite the fact that the true state of the world is not in A; i.e., A does not obtain. This is the case in Example 10.4: for $A = \{\omega_1\}$, $B_1 A = \{\omega_1, \omega_2\}$: at the state of the world ω_2 the player believes that A obtains, despite the fact that $\omega_2 \notin A$.

The belief operator does satisfy the following additional property (Exercise 10.13). The analogous property for the knowledge operator is stated in Theorem 9.10 (page 326).

Theorem 10.8 *For each player $i \in N$, and any pair of events $A, C \subseteq Y$, if $A \subseteq C$, then $B_i A \subseteq B_i C$.*

In words, when a player believes that event A obtains, he also believes that every event containing A obtains.

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Just as we defined the concept of common knowledge (Definition 9.2 on page 321), we can define the concept of common belief.

Definition 10.9 *Let $A \subseteq Y$ be an event and let $\omega \in Y$. The event A is* common belief *among the players at the state of the world ω if at that state of the world every player believes that A obtains, every player believes that every player believes that A obtains, and so on. In other words, for every finite sequence $i_1, i_2, \ldots, i_l$ of players in N,*

$$\omega \in B_{i_1} B_{i_2} \ldots B_{i_{l-1}} B_{i_l} A. \tag{10.13}$$

It follows from Definition 10.9, and Theorem 10.7(1) that, in particular, the event Y is common belief among the players at every state of the world $\omega \in Y$. The next theorem presents a sufficient condition for an event to be common belief among the players at a particular state of the world.

Theorem 10.10 *Let $\omega \in Y$ be a state of the world. Let $A \in \mathcal{Y}$ be an event satisfying the following two conditions:*

- $\pi_i(A \mid \omega) = 1$ *for every player $i \in N$.*
- $\pi_i(A \mid \omega') = 1$ *for every player $i \in N$ and every $\omega' \in A$.*

Then A is common belief among the players at ω.

Proof: The first condition implies that $\omega \in B_i A$, and the second condition implies that $A \subseteq B_i A$, for each player $i \in N$. From this, and from repeated application of Theorem 10.8, we get for every finite sequence $i_1, i_2, \ldots, i_l$ of players:

$$\omega \in B_{i_1} A \subseteq B_{i_1} B_{i_2} A \subseteq \cdots \subseteq B_{i_1} B_{i_2} \ldots B_{i_{l-1}} A \subseteq B_{i_1} B_{i_2} \ldots B_{i_{l-1}} B_{i_l} A.$$

It follows that at the state of the world ω event A is common belief among the players. □

When a belief space is equivalent to an Aumann model of incomplete information, the concept of knowledge is a meaningful one, and the question naturally arises as to whether there is a relation between knowledge and belief in this case. As we now show, the answer to this question is positive. Let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i \in N})$ (where Y is a finite set of states of the world) be a belief space that is equivalent to an Aumann model of incomplete information. In particular, there exists a probability distribution $\mathbf{P}$ over Y satisfying $\mathbf{P}(\omega) > 0$ for all $\omega \in Y$, and there exist partitions $(\mathcal{F}_i)_{i \in N}$ of Y such that

$$\pi_i(\omega) = \mathbf{P}(\cdot \mid F_i(\omega)), \quad \forall i \in N, \forall \omega \in Y. \tag{10.14}$$

The partition $\mathcal{F}_i$ in the Aumann model coincides with the partition defined by Equation (10.8) for the belief space Π (Exercise 10.14); hence the knowledge operator in the Aumann model is the same operator as the belief operator in the belief space. We therefore have the following theorem:

Theorem 10.11 *Let Π be a belief space equivalent to an Aumann model of incomplete information. Then the belief operator in the belief space is the same operator as the knowledge operator in the Aumann model: For every $i \in N$, at the state of the world ω player i believes that event A obtains (in the belief space) if and only if he knows that event A obtains (in the Aumann model).*

Note that for this result to obtain, it must be the case that $P(\omega) > 0$ for every state of the world $\omega \in Y$ (Exercise 10.11).

If $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$ is a belief space satisfying the condition that for player $i \in N$, $B_i A \subseteq A$ for every event $A \subseteq Y$, then the operator B_i satisfies the five properties of Kripke, and is therefore a knowledge operator: there exists a partition $\mathcal{G}_i$ of Y such that B_i is the knowledge operator defined by $\mathcal{G}_i$ via Equation (9.2) on page 325 (Exercise 9.2, Chapter 9). Since this partition is simply the partition defined by Equation (10.8) (Exercise 10.9), the conclusion of Theorem 10.11 obtains in this case as well. We stress that this case is more general than the case in which a belief space is equivalent to an Aumann model of incomplete information, because this condition can be met in an Aumann model of incomplete information in which the players do not share a common prior (see Example 10.3). Nevertheless, in this case, the belief operator is also the same operator as the knowledge operator.

10.3 Examples of belief spaces

As stated above, the information that player i has at the state of the world ω is given by his belief $\pi_i(\omega)$. We shall refer to this belief at the player's *type*. A player's type is thus a probability distribution over Y.

Definition 10.12 *Let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$ be a belief space. The* type *of player i at the state of the world ω is $\pi_i(\omega)$. The set of all types of player i in a belief space Π is denoted by T_i and called the* type set *of player i.*

$$T_i := \{\pi_i(\omega)\colon \omega \in Y\} \subseteq \Delta(Y). \tag{10.15}$$

The coherency requirement in Definition 10.1, and the definition of the belief operator B_i, together imply that at each state of the world ω, every player $i \in N$ believes that his type is $\pi_i(\omega)$:

$$\pi_i(\{\omega' \in Y \colon \pi_i(\omega') = \pi_i(\omega)\} \mid \omega) = 1. \tag{10.16}$$

We next present examples showing how situations of incomplete information can be modeled by belief spaces. We start with situations that can be modeled both by Aumann models of incomplete information, and by belief spaces.

Example 10.13 **Complete information** Suppose that a state of nature $s_0 \in S$ is common belief among the players in a finite set $N = \{1, 2, \ldots, n\}$. The following belief space corresponds to this situation, where the set of states of the world, $Y = \{\omega\}$, contains only one state, and all the players have the same beliefs:

State of the world	$\mathfrak{s}(\cdot)$	$\pi_1(\cdot), \cdots, \pi_n(\cdot)$
ω	s_0	$[1(\omega)]$

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In this space, every player $i \in N$ has only one type, $T_i = \{[1(\omega)]\}$. The beliefs of the players can be calculated from the common prior $\mathbf{P}$ defined by $\mathbf{P}(\omega) = 1$, hence this belief space is also an Aumann model of incomplete information, with the trivial partition $\mathcal{F}_i = \{\{\omega\}\}$, for every player $i \in N$.

This situation can also be modeled by the following belief space, where $Y = \{\omega_1, \omega_2\}$:

State of the world	$\mathfrak{s}(\cdot)$	$\pi_1(\cdot), \cdots, \pi_n(\cdot)$
ω_1	s_0	$[1(\omega_1)]$
ω_2	s_0	$[1(\omega_2)]$

The two states of the world ω_1 and ω_2 are distinguished only by their names; they are identical from the perspective both of the state of nature associated with them and of the beliefs of the players about the state of nature: at both states of the world, the state of nature is s_0, and at both states that fact is common belief. This is an instance of *redundancy*: two states of the world describe the same situation. If we eliminate the redundancy, we recapitulate the former belief space. ◀

Example 10.14 **Known lottery** The set of players is $N = \{1, \ldots, n\}$. The set of states of nature, $S = \{s_1, s_2\}$, contains two elements; a chance move chooses s_1 with probability $\frac{1}{3}$, and s_2 with probability $\frac{2}{3}$. This probability distribution is common belief among the players. The following belief space, where $Y = \{\omega_1, \omega_2\}$, corresponds to this situation:

State of the world	$\mathfrak{s}(\cdot)$	$\pi_1(\cdot), \cdots, \pi_n(\cdot)$
ω_1	s_1	$\left[\frac{1}{3}(\omega_1), \frac{2}{3}(\omega_2)\right]$
ω_2	s_2	$\left[\frac{1}{3}(\omega_1), \frac{2}{3}(\omega_2)\right]$

In this space, each player $i \in N$ has only one type, $T_i = \{[\frac{1}{3}(\omega_1), \frac{2}{3}(\omega_2)]\}$, and this fact is therefore common belief among the players. The beliefs of the players can be calculated from a common prior $\mathbf{P}$ defined by $\mathbf{P}(\omega_1) = \frac{1}{3}$ and $\mathbf{P}(\omega_2) = \frac{2}{3}$, and the partitions derived from the beliefs of the types, i.e., $\mathcal{F}_i = \{\{\omega_1, \omega_2\}\}$ for every player $i \in N$; hence this belief space is also an Aumann model of incomplete information. ◀

Example 10.15 **Incomplete information on one side** There are two players, $N = \{\mathrm{I}, \mathrm{II}\}$, and two states of nature, $S = \{s_1, s_2\}$; a chance move chooses s_1 with probability p, and s_2 with probability $1 - p$. The state of the world that is chosen is known to Player I, but not to Player II. This description of the situation is common belief among the players. The following belief space, where $Y = \{\omega_1, \omega_2\}$, corresponds to this situation:

State of the world	$\mathfrak{s}(\cdot)$	$\pi_{\mathrm{I}}(\cdot)$	$\pi_{\mathrm{II}}(\cdot)$
ω_1	s_1	$[1(\omega_1)]$	$[p(\omega_1), (1-p)(\omega_2)]$
ω_2	s_2	$[1(\omega_2)]$	$[p(\omega_1), (1-p)(\omega_2)]$

In this belief space, Player II has only one type, $T_{\mathrm{II}} = \{[p(\omega_1), (1-p)(\omega_2)]\}$, while Player I has two possible types, $T_{\mathrm{I}} = \{[1(\omega_1)], [1(\omega_2)]\}$, because he knows the true state of nature. The beliefs of the players can be calculated from a common prior $\mathbf{P}$ given by $\mathbf{P}(\omega_1) = p$ and $\mathbf{P}(\omega_2) = 1 - p$, and the partition $\mathcal{F}_{\mathrm{I}} = \{\{\omega_1\}, \{\omega_2\}\}$ (Player I knows which state of the world has been chosen) and $\mathcal{F}_{\mathrm{II}} = \{Y\}$ (Player II does not know which state of the world has been chosen). Note that in this example, the belief operator is the same as the knowledge operator (in accordance with Theorem 10.11). ◀

Example 10.16 **Incomplete information about the information of the other player** This example, which is similar to Example 10.2 (on page 388), describes a situation in which one of the players knows the true state of nature, but is uncertain whether the other player knows the true state of nature. Consider a situation with two players, $N = \{I, II\}$. A state of nature s_1 or s_2 is chosen by tossing a fair coin. Player I is informed which state of nature has been chosen. If the chosen state of nature is s_2, only Player I is informed of that fact. If the chosen state of nature is s_1, the coin is tossed again, in order to determine whether or not Player II is to be informed that the chosen state of nature is s_1; Player I is not informed of the result of the second coin toss; hence in this situation, even though he knows the state of nature, he does not know whether or not Player II knows the state of nature. The belief space corresponding to this situation contains three states of the world $Y = \{\omega_1, \omega_2, \omega_3\}$ and is given by:

State of the world	$\mathfrak{s}(\cdot)$	$\pi_{\mathrm{I}}(\cdot)$	$\pi_{\mathrm{II}}(\cdot)$
ω_1	s_1	$\left[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)\right]$	$[1(\omega_1)]$
ω_2	s_1	$\left[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)\right]$	$\left[\frac{1}{3}(\omega_2), \frac{2}{3}(\omega_3)\right]$
ω_3	s_2	$[1(\omega_3)]$	$\left[\frac{1}{3}(\omega_2), \frac{2}{3}(\omega_3)\right]$

At the state of the world ω_1, the state of nature is s_1, and both Player I and Player II know this. At the state of the world ω_2, the state of nature is s_1, and Player I knows this, but Player II does not know this. The state of the world ω_3 corresponds to the situation in which the state of nature is s_2. Player I cannot distinguish between the states of the world ω_1 and ω_2. Player II cannot distinguish between the states of the world ω_2 and ω_3. The beliefs of the players can be derived from the probability distribution **P**,

$$\mathbf{P}(\omega_1) = \tfrac{1}{4}, \quad \mathbf{P}(\omega_2) = \tfrac{1}{4}, \quad \mathbf{P}(\omega_3) = \tfrac{1}{2}, \tag{10.17}$$

given the partitions $\mathcal{F}_{\mathrm{I}} = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$ and $\mathcal{F}_{\mathrm{II}} = \{\{\omega_1\}, \{\omega_2, \omega_3\}\}$. Notice that, as required, at every state of the world ω, every player ascribes probability 1 to the states of the world at which his beliefs coincide with his beliefs at ω. In this belief space, each player has two possible types:

$$T_{\mathrm{I}} = \left\{\left[\tfrac{1}{2}(\omega_1), \tfrac{1}{2}(\omega_2)\right], [1(\omega_3)]\right\}, \tag{10.18}$$

$$T_{\mathrm{II}} = \left\{[1(\omega_1)], \left[\tfrac{1}{3}(\omega_2), \tfrac{2}{3}(\omega_3)\right]\right\}. \tag{10.19}$$

What information does Player I of type $[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)]$ lack (at the states of the world ω_1 and ω_2)? He knows that the state of nature is s_1, but he does not know whether Player II knows this: Player I ascribes probability $\frac{1}{2}$ to the state of the world being ω_1, at which Player II knows that the state of nature is s_1, and he ascribes probability $\frac{1}{2}$ to the state of the world being ω_2, at which Player II does not know what the true state of nature is. Player I's lack of information involves the information that Player II has. ◀

Example 10.17 **Incomplete information on two sides (the independent case)** There are two players, $N = \{I, II\}$, and four states of nature, $S = \{s_{11}, s_{12}, s_{21}, s_{22}\}$. One of the states of nature is chosen, using the probability distribution p defined by $p(s_{11}) = p(s_{12}) = \frac{1}{6}$, $p(s_{21}) = p(s_{22}) = \frac{1}{3}$.

Each player has partial information about the chosen state of nature: Player I knows only the first coordinate of the chosen state, while Player II knows only the second coordinate. The belief space corresponding to this situation contains four states of the world, $Y = \{\omega_{11}, \omega_{12}, \omega_{21}, \omega_{22}\}$, and is given by:

State of the world	$\mathfrak{s}(\cdot)$	$\pi_{\mathrm{I}}(\cdot)$	$\pi_{\mathrm{II}}(\cdot)$
ω_{11}	s_{11}	$\left[\frac{1}{2}(\omega_{11}), \frac{1}{2}(\omega_{12})\right]$	$\left[\frac{1}{3}(\omega_{11}), \frac{2}{3}(\omega_{21})\right]$
ω_{12}	s_{12}	$\left[\frac{1}{2}(\omega_{11}), \frac{1}{2}(\omega_{12})\right]$	$\left[\frac{1}{3}(\omega_{12}), \frac{2}{3}(\omega_{22})\right]$
ω_{21}	s_{21}	$\left[\frac{1}{2}(\omega_{21}), \frac{1}{2}(\omega_{22})\right]$	$\left[\frac{1}{3}(\omega_{11}), \frac{2}{3}(\omega_{21})\right]$
ω_{22}	s_{22}	$\left[\frac{1}{2}(\omega_{21}), \frac{1}{2}(\omega_{22})\right]$	$\left[\frac{1}{3}(\omega_{12}), \frac{2}{3}(\omega_{22})\right]$

In this case, each player has two possible types:

$$T_{\mathrm{I}} = \{\mathrm{I}_1, \mathrm{I}_2\} = \left\{\left[\tfrac{1}{2}(\omega_{11}), \tfrac{1}{2}(\omega_{12})\right], \left[\tfrac{1}{2}(\omega_{21}), \tfrac{1}{2}(\omega_{22})\right]\right\}, \tag{10.20}$$

$$T_{\mathrm{II}} = \{\mathrm{II}_1, \mathrm{II}_2\} = \left\{\left[\tfrac{1}{3}(\omega_{11}), \tfrac{2}{3}(\omega_{21})\right], \left[\tfrac{1}{3}(\omega_{12}), \tfrac{2}{3}(\omega_{22})\right]\right\}. \tag{10.21}$$

Note that each player knows his own type: at each state of the world, each player ascribes positive probability only to those states of the world in which he has the same type. The beliefs of each player about the type of the other player are described in Figure 10.2.

	II_1	II_2
I_1	1/2	1/2
I_2	1/2	1/2

The beliefs of Player I

	II_1	II_2
I_1	1/3	1/3
I_2	2/3	2/3

The beliefs of Player II

Figure 10.2 The beliefs of each player about the type of the other player

The tables in Figure 10.2 state, for example, that Player I of type I_2 ascribes probability $\frac{1}{2}$ to Player II being of type II_1, and probability $\frac{1}{2}$ to his being of type II_2.

The beliefs of each player about the types of the other player do not depend on his own type, which is why this model is termed the "independent case." This is a Harsanyi model of incomplete information, in which the common prior p over the set of type vectors, $T = T_{\mathrm{I}} \times T_{\mathrm{II}}$ is the product distribution shown in Figure 10.3.

	II_1	II_2
I_1	1/6	1/6
I_2	1/3	1/3

Figure 10.3 The common prior in Example 10.17

The independence in this example is expressed in the fact that p is a product distribution over T_{I} and T_{II}. In summary, in this case the belief space Π is equivalent to a Harsanyi model of incomplete information. ◀

Example 10.18 **Incomplete information on two sides (the dependent case)** This example is similar to Example 10.17, but here the probability distribution p according to which the state of nature is chosen is given by $p(s_{11}) = 0.3$, $p(s_{12}) = 0.4$, $p(s_{21}) = 0.2$, $p(s_{22}) = 0.1$. As in Example 10.17, the corresponding belief space has four states of the world, $Y = \{\omega_{11}, \omega_{12}, \omega_{21}, \omega_{22}\}$, and is given by:

State of the world	$\mathfrak{s}(\cdot)$	$\pi_{\mathrm{I}}(\cdot)$	$\pi_{\mathrm{II}}(\cdot)$
ω_{11}	s_{11}	$\left[\frac{3}{7}(\omega_{11}), \frac{4}{7}(\omega_{12})\right]$	$\left[\frac{3}{5}(\omega_{11}), \frac{2}{5}(\omega_{21})\right]$
ω_{12}	s_{12}	$\left[\frac{3}{7}(\omega_{11}), \frac{4}{7}(\omega_{12})\right]$	$\left[\frac{4}{5}(\omega_{12}), \frac{1}{5}(\omega_{22})\right]$
ω_{21}	s_{21}	$\left[\frac{2}{3}(\omega_{21}), \frac{1}{3}(\omega_{22})\right]$	$\left[\frac{3}{5}(\omega_{11}), \frac{2}{5}(\omega_{21})\right]$
ω_{22}	s_{22}	$\left[\frac{2}{3}(\omega_{21}), \frac{1}{3}(\omega_{22})\right]$	$\left[\frac{4}{5}(\omega_{12}), \frac{1}{5}(\omega_{22})\right]$

The sets of type sets are

$$T_{\mathrm{I}} = \{\mathrm{I}_1, \mathrm{I}_2\} = \left\{\left[\tfrac{3}{7}(\omega_{11}), \tfrac{4}{7}(\omega_{12})\right], \left[\tfrac{2}{3}(\omega_{21}), \tfrac{1}{3}(\omega_{22})\right]\right\},$$
$$T_{\mathrm{II}} = \{\mathrm{II}_1, \mathrm{II}_2\} = \left\{\left[\tfrac{3}{5}(\omega_{11}), \tfrac{2}{5}(\omega_{21})\right], \left[\tfrac{4}{5}(\omega_{12})), \tfrac{1}{5}(\omega_{22})\right]\right\}.$$

The mutual beliefs of the players are described in Figure 10.4.

	II_1	II_2
I_1	3/7	4/7
I_2	2/3	1/3

The beliefs of Player I

	II_1	II_2
I_1	3/5	4/5
I_2	2/5	1/5

The beliefs of Player II

Figure 10.4 The beliefs of each player about the types of the other player

These beliefs correspond to a Harsanyi model with incomplete information, with the common prior p described in Figure 10.5.

	II_1	II_2
I_1	0.3	0.4
I_2	0.2	0.1

Figure 10.5 The common prior in Example 10.18

This prior distribution can be calculated from the mutual beliefs described in Figure 10.4 as follows. Denote $x = p(\mathrm{I}_1, \mathrm{II}_1)$. From the beliefs of type I_1, we get $p(\mathrm{I}_1, \mathrm{II}_2) = \frac{4}{3}x$; from the beliefs of type II_2, we get $p(\mathrm{I}_2, \mathrm{II}_2) = \frac{1}{3}x$; from the beliefs of type I_2, we get $p(\mathrm{I}_2, \mathrm{II}_1) = \frac{2}{3}x$. From the beliefs of type II_1, we get $p(\mathrm{I}_1, \mathrm{II}_1) = x$, which is what we started with. Since p is a probability distribution, $x + \frac{4}{3}x + \frac{1}{3}x + \frac{2}{3}x = 1$. Then $x = \frac{3}{10}$, and we have indeed shown that the common prior of the Harsanyi model is the probability distribution appearing in Figure 10.5.

The difference between this example and Example 10.17 is that in this case the common prior is not a product distribution over $T = T_{\mathrm{I}} \times T_{\mathrm{II}}$. Equivalently, the beliefs of one player about the types of the other player depend on his own type: Player I of type I_1 ascribes probability $\frac{3}{7}$ to Player II being of type II_1, while Player I of type I_2 ascribes probability $\frac{2}{3}$ to Player II being of type II_1. ◀

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Example 10.19 **Inconsistent Beliefs.** This example studies a belief space in which the players hold inconsistent beliefs. This means that the beliefs cannot be derived from a common prior. Such a situation cannot be described by an Aumann model of incomplete information.

There are two players, $N = \{\mathrm{I}, \mathrm{II}\}$, and four states of nature, $S = \{s_{11}, s_{12}, s_{21}, s_{22}\}$. The corresponding belief space has four states of the world, $Y = \{\omega_{11}, \omega_{12}, \omega_{21}, \omega_{22}\}$:

State of the world	$\mathfrak{s}(\cdot)$	$\pi_{\mathrm{I}}(\cdot)$	$\pi_{\mathrm{II}}(\cdot)$
ω_{11}	s_{11}	$\left[\frac{3}{7}(\omega_{11}), \frac{4}{7}(\omega_{12})\right]$	$\left[\frac{1}{2}(\omega_{11}), \frac{1}{2}(\omega_{21})\right]$
ω_{12}	s_{12}	$\left[\frac{3}{7}(\omega_{11}), \frac{4}{7}(\omega_{12})\right]$	$\left[\frac{4}{5}(\omega_{12}), \frac{1}{5}(\omega_{22})\right]$
ω_{21}	s_{21}	$\left[\frac{2}{3}(\omega_{21}), \frac{1}{3}(\omega_{22})\right]$	$\left[\frac{1}{2}(\omega_{11}), \frac{1}{2}(\omega_{21})\right]$
ω_{22}	s_{22}	$\left[\frac{2}{3}(\omega_{21}), \frac{1}{3}(\omega_{22})\right]$	$\left[\frac{4}{5}(\omega_{12}), \frac{1}{5}(\omega_{22})\right]$

The type sets are

$$T_{\mathrm{I}} = \{\mathrm{I}_1, \mathrm{I}_2\} = \left\{\left[\tfrac{3}{7}(\omega_{11}), \tfrac{4}{7}(\omega_{12})\right], \left[\tfrac{2}{3}(\omega_{21}), \tfrac{1}{3}(\omega_{22})\right]\right\},$$
$$T_{\mathrm{II}} = \{\mathrm{II}_1, \mathrm{II}_2\} = \left\{\left[\tfrac{1}{2}(\omega_{11}), \tfrac{1}{2}(\omega_{21})\right], \left[\tfrac{4}{5}(\omega_{12}), \tfrac{1}{5}(\omega_{22})\right]\right\}.$$

The mutual beliefs of the players are described in Figure 10.6.

	II_1	II_2
I_1	3/7	4/7
I_2	2/3	1/3

The beliefs of Player I

	II_1	II_2
I_1	1/2	4/5
I_2	1/2	1/5

The beliefs of Player II

Figure 10.6 The beliefs of each player about the types of the other player

These mutual beliefs are the same beliefs as in Example 9.56 (page 366). As shown there, there does not exist a common prior in the Harsanyi model with these beliefs. Note that this example resembles Example 10.18, the only difference being the change of one of the types of Player II, namely, type II_1, from $[\frac{3}{5}(\omega_{11}), \frac{2}{5}(\omega_{21})]$ to $[\frac{1}{2}(\omega_{11}), \frac{1}{2}(\omega_{21})]$. These two situations, which are similar in their presentations as belief spaces, are in fact significantly different: one can be modeled by an Aumann or Harsanyi model of incomplete information, while the other cannot. ◀

In general, if there exists a probability distribution p such that at any state of the world ω in the support of p, the beliefs of the player are calculated as a conditional probability via

$$\pi_i(\omega) = p(\cdot \mid \{\omega' \in Y : \pi_i(\omega') = \pi_i(\omega)\}), \tag{10.22}$$

then p is called a *consistent distribution*, and every state of the world in the support of p is called a *consistent state of the world*. In that case, the collection of beliefs $(\pi_i)_{i\in N}$ is called a *consistent belief system* (see also Section 10.6 on page 415).

In the above example, all the states of the world in Y are inconsistent. Ensuring consistency requires the existence of certain relationships between the subjective probabilities of the players, and, therefore, the dimension of the set of consistent belief systems is lower than the dimension of the set of all mutual belief systems. For example, in the examples

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above containing two players and two types for each player, the mutual belief system of the types contains four probability distributions over $[0, 1]$; hence the set of mutual belief systems is isomorphic to $[0, 1]^4$. The consistency condition requires that any one of these four probability distributions be determined by the three others; hence the set of mutual belief systems is isomorphic to $[0, 1]^3$. In other words, within the set of all mutual belief systems, the relative dimension of the set of consistent belief systems is 0 (see Exercise 10.18).

Example 10.20 **Infinite type space** There are two players $N = \{\mathrm{I}, \mathrm{II}\}$ and the set of states of nature is $S = [0, 1]^2$. Player I is informed of the first coordinate of the chosen state of nature, while Player II is informed of the second coordinate. The beliefs of the players are as follows. If x is the value that Player I is informed of, he believes that the value Player II is informed of is taken from the uniform distribution over $[0.9x, 0.9x + 0.1]$. If y is the value that Player 2 is informed of, then if $y \leq \frac{1}{2}$, Player 2 believes that the value Player 1 is informed of is taken from the uniform distribution over $[0.7, 1]$, and if $y > \frac{1}{2}$, Player 2 believes that the value Player 1 is informed of is taken from the uniform distribution over $[0, 0.3]$.

A belief space that corresponds to this situation is:

- The set of states of the world is $Y = [0, 1]^2$. A state of the world is denoted by $\omega_{xy} = (x, y)$, where $0 \leq x, y \leq 1$. For every $(x, y) \in [0, 1]^2$, the equation $\mathfrak{s}(\omega_{xy}) = (x, y)$ holds.
- For every $x \in [0, 1]$, Player I's belief $\pi_{\mathrm{I}}(\omega_{xy})$ is a uniform distribution over the set $\{(x, y) \in [0, 1]^2 : 0.9x \leq y \leq 0.9x + 0.1\}$, which is the interval $[(x, 0.9x), (x, 0.9x + 0.1)]$.
- If $y \leq \frac{1}{2}$ then Player II's belief $\pi_{\mathrm{II}}(\omega_{xy})$ is the uniform distribution over the set $\{(x, y) \in [0, 1]^2 : 0.7 \leq x \leq 1\}$ (which is the interval $[(0.7, y), (1, y)]$), and if $y > \frac{1}{2}$ then the belief $\pi_{\mathrm{II}}(\omega_{xy})$ is the uniform distribution over the set $\{(x, y) \in [0, 1]^2 : 0 \leq x \leq 0.3\}$ (which is the interval $[(0, y), (0.3, y)]$).

The type sets of the players are[4]

$$T_{\mathrm{I}} = \{U[(x, 0.9x), (x, 0.9x + 0.1)] : 0 \leq x \leq 1\}, \tag{10.23}$$

$$T_{\mathrm{II}} = \left\{U[(0.7, y), (1, y)] : 0 \leq y \leq \tfrac{1}{2}\right\} \bigcup \left\{U[(0, y), (0.3, y)] : \tfrac{1}{2} < y \leq 1\right\}. \tag{10.24}$$

The beliefs of the players in this example are inconsistent. At every state of the world, Player I believes that $|x - y| \leq 0.1$, while Player II believes that $|x - y| \geq 0.2$; there cannot be a common prior from which these two beliefs are both derived (Exercise 10.19). ◀

10.4 Belief subspaces

Definition 10.21 *Let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i \in N})$ be a belief space and let $\widetilde{Y} \in \mathcal{Y}$ be a nonempty subset of the set of states of the world. The ordered vector $\widetilde{\Pi} = (\widetilde{Y}, \mathcal{Y}_{|\widetilde{Y}}, \mathfrak{s}, (\pi_i)_{i \in N})$ is called*[5] *a* belief subspace *if*

$$\pi_i(\widetilde{Y} \mid \omega) = 1, \quad \forall i \in N, \forall \omega \in \widetilde{Y}. \tag{10.25}$$

4 For $-\infty < a < b < \infty$, we denote the uniform distribution over $[a, b]$ by $U[a, b]$.

5 We denote the restriction of $\mathcal{Y}$ to $\widetilde{Y}$ by $\mathcal{Y}_{|\widetilde{Y}} = \{E \subseteq \widetilde{Y} : E \in \mathcal{Y}\}$, and $\mathfrak{s}$ and $(\pi_i)_{i \in N}$ are the functions appearing in the definition of Π, restricted to $\widetilde{Y}$.

In words, a belief subspace is a set of states of the world that is closed under the beliefs of the players. If the true state of the world ω is in the belief subspace $\widetilde{Y}$, then the states of the world that are not in $\widetilde{Y}$ are irrelevant to all the players, and this fact is common belief among them (Exercise 10.16). Later in this chapter, we will analyze games with incomplete information, where each player chooses his action as a function of his beliefs. In his strategic considerations, each player may ignore the states of the world that are not in the belief subspace that describes the situation he is in. For convenience, we will call both $\widetilde{Y}$ and $\widetilde{\Pi}$ belief subspaces of Π. If a belief space is derived from an Aumann model of incomplete information, then for every state of the world ω the common knowledge component $C(\omega)$ at the state of the world ω (see page 333) is a belief subspace (Exercise 10.17).

Example 10.22 Consider the following belief space, in which the set of players is $N = \{\mathrm{I}, \mathrm{II}\}$, the set of states of nature is $S = \{s_1, s_2, s_3\}$, and $Y = \{\omega_1, \omega_2, \omega_3\}$:

State of the world	$\mathfrak{s}(\cdot)$	$\pi_{\mathrm{I}}(\cdot)$	$\pi_{\mathrm{II}}(\cdot)$
ω_1	s_1	$\left[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)\right]$	$[1(\omega_1)]$
ω_2	s_2	$\left[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)\right]$	$[1(\omega_3)]$
ω_3	s_3	$[1(\omega_3)]$	$[1(\omega_3)]$

The subspace $\widetilde{Y} = \{\omega_3\}$ is a belief subspace, and it is the only strict subspace of Y that is a belief subspace (verify!). ◀

The next theorem states that the intersection of two belief subspaces of the same belief space is also a belief subspace. The proof of the theorem is left to the reader as an exercise (Exercise 10.29).

Theorem 10.23 *If $\widetilde{\Pi}_1 = (\widetilde{Y}_1, \mathcal{Y}_{|\widetilde{Y}_1}, \mathfrak{s}, (\pi_i)_{i \in N})$ and $\widetilde{\Pi}_2 = (\widetilde{Y}_2, \mathcal{Y}_{|\widetilde{Y}_2}, \mathfrak{s}, (\pi_i)_{i \in N})$ are two belief subspaces of the same belief space $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i \in N})$, and if $\widetilde{Y}_1 \cap \widetilde{Y}_2 \neq \emptyset$, then $(\widetilde{Y}_1 \cap \widetilde{Y}_2, \mathcal{Y}_{|\widetilde{Y}_1 \cap \widetilde{Y}_2}, \mathfrak{s}, (\pi_i)_{i \in N})$ is also a belief subspace.*

By definition, for every state of the world $\omega \in Y$, the space Y itself is a belief subspace containing ω; i.e., Y is a model of the situation: it can be used to describe the situation associated with ω. However, this may be a model that is "too large," in the sense that it contains states of the world that all the players deem to be irrelevant. The most "efficient" model is the smallest belief subspace (with respect to set inclusion) that contains ω.

Definition 10.24 *The* minimal belief subspace *at a state of the world ω is the smallest belief subspace (with respect to set inclusion) that contains ω. We will denote the minimal subspace at ω (if such a space exists) by $\widetilde{\Pi}(\omega)$, and the set of states of the world of $\widetilde{\Pi}(\omega)$ we will denote by $\widetilde{Y}(\omega)$.*

Theorem 10.23 implies that if there exists a minimal belief subspace, then it is unique (Exercise 10.30). Does every state of the world $\omega \in Y$ have a minimal belief subspace containing it? As the next theorem shows, the answer to this question is positive, when the set of states of the world is finite. The same holds true if the set of states of the world is countably infinite (Exercise 10.31).

Theorem 10.25 *Let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i \in N})$ be a belief space in which the set of states of the world Y is a finite set. For each state of the world $\omega \in Y$, there exists a unique minimal belief subspace containing ω.*

Proof: Π is a belief subspace; hence there exists at least one belief subspace containing ω. Since Y is a finite set, there is a finite number of belief subspaces containing ω. By repeated application of Theorem 10.23, the intersection of all the belief subspaces containing ω is a belief subspace containing ω, and it is the minimal belief subspace at ω. □

As the next example shows, in more general spaces there may not exist a minimal belief space.

Example 10.26 Consider the one-player belief space in which $N = \{\mathrm{I}\}$, the set of states of nature is $S = [0, 1]$, the set of states of the world is $Y = [0, 1]$, the σ-algebra $\mathcal{Y}$ is the σ-algebra of Borel sets, the function $\mathfrak{s}$ is given by $\mathfrak{s}(\omega) = \omega$ for each $\omega \in Y$, and the player's belief at each state of the world is the uniform distribution over [0, 1]. At each state of the world ω, every subset of states of the world $\widetilde{Y} \subseteq Y$ whose Lebesgue measure is 1 is a belief subspace. Since there does not exist a minimal set containing ω whose Lebesgue measure is 1, it follows that there is no minimal belief subspace at any state of the world in this example. ◀

Since player i does not know the true state of the world ω, and knows only his belief $\pi_i(\omega)$, he may not be able to calculate $\widetilde{Y}(\omega)$. This point is elucidated in the next example.

Example 10.27 Figure 10.7 depicts a belief space with a set of players $N = \{\mathrm{I}, \mathrm{II}\}$, and a set of states of nature $S = \{s_1, s_2, s_3\}$. The set of states of the world is $Y = \{\omega_1, \omega_2, \omega_3\}$.

State of the world	$\mathfrak{s}(\cdot)$	$\pi_{\mathrm{I}}(\cdot)$	$\pi_{\mathrm{II}}(\cdot)$	$\widetilde{Y}(\omega)$
ω_1	s_1	$[1(\omega_1)]$	$[1(\omega_1)]$	$\{\omega_1\}$
ω_2	s_2	$[1(\omega_1)]$	$[1(\omega_3)]$	Y
ω_3	s_3	$[1(\omega_3)]$	$[1(\omega_3)]$	$\{\omega_3\}$

Figure 10.7 The belief space in Example 10.27

At the state of the world ω_2, the players do not agree on the states of the world that are relevant to the situation: at this state of the world, Player I believes it is common belief that the state of nature is s_1, while Player II believes it is common belief that the state of nature is s_3. In fact, both of them are wrong, because the true state of nature is s_2.

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The sets $\widetilde{Y}_1 = \{\omega_1\}$ and $\widetilde{Y}_3 = \{\omega_3\}$ are belief subspaces. At the state of the world ω_2, Player I believes that the state of the world is ω_1, and he therefore ascribes probability 1 to the minimal belief subspace being $\widetilde{Y}_1$. Player II, in contrast, ascribes probability 1 at the state of the world ω_2 to the minimal belief subspace being $\widetilde{Y}_3$. If the situation is a game situation, then at the state of the world ω_2 Player I will ignore in his strategic considerations the states of the world ω_2 and ω_3 and Player II will ignore the states of the world ω_1 and ω_2. ◀

The last example leads to the following definition.

Definition 10.28 *Let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$ be a belief space, and let $\omega \in Y$ be a state of the world. A* belief subspace of player i *at a state of the world ω is a belief subspace $\widetilde{\Pi} = (\widetilde{Y}, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$ satisfying*

$$\pi_i(\widetilde{Y} \mid \omega) = 1. \tag{10.26}$$

The condition in Equation (10.26) guarantees that at the state of the world ω, player i ascribes probability 1 to the true state of the world being in $\widetilde{Y}$. Note that Y itself is a belief subspace for each player at each state of the world.

The next theorem states that the intersection of two belief subspaces of player i at the state of the world ω is a belief subspace of player i at ω.

Theorem 10.29 *If $\widetilde{\Pi}_1 = (\widetilde{Y}_1, \mathcal{Y}_{|\widetilde{Y}_1}, \mathfrak{s}, (\pi_i)_{i\in N})$ and $\widetilde{\Pi}_2 = (\widetilde{Y}_2, \mathcal{Y}_{|\widetilde{Y}_2}, \mathfrak{s}, (\pi_i)_{i\in N})$ are two belief subspaces of player i at a state of the world ω, then $(\widetilde{Y}_1 \cap \widetilde{Y}_2, \mathcal{Y}_{|\widetilde{Y}_1\cap\widetilde{Y}_2}, \mathfrak{s}, (\pi_i)_{i\in N})$ is a belief subspace of player i at ω.*

Proof: We first establish that $\widetilde{Y}_1 \cap \widetilde{Y}_2 \neq \emptyset$. Since $\widetilde{\Pi}_1$ is a belief subspace of player i at ω, it follows that $\pi_i(\widetilde{Y}_1 \mid \omega) = 1$. We similarly deduce that $\pi_i(\widetilde{Y}_2 \mid \omega) = 1$. Therefore,

$$\pi_i(\widetilde{Y}_1 \cap \widetilde{Y}_2 \mid \omega) = 1, \tag{10.27}$$

and, in particular, the set $\widetilde{Y}_1 \cap \widetilde{Y}_2$ is not empty. If follows from Theorem 10.23 that $(\widetilde{Y}_1 \cap \widetilde{Y}_2, \mathcal{Y}_{|\widetilde{Y}_1\cap\widetilde{Y}_2}, \mathfrak{s}, (\pi_i)_{i\in N})$ is a belief subspace; hence this is a belief subspace of player i at the state of the world ω. □

The smallest belief subspace of a player i (with respect to set inclusion) is called the *minimal belief subspace* of player i.

Definition 10.30 *Let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$ be a belief space, and let $\omega \in Y$ be a state of the world. The* minimal belief subspace of player i *at ω is the smallest belief subspace of player i (with respect to set inclusion) at the state of the world ω.*

The minimal belief subspace of player i at a state of the world ω (if such a subspace exists) will be denoted by $\widetilde{\Pi}_i(\omega)$, and the set of states of the world in $\widetilde{\Pi}_i(\omega)$ will be denoted by $\widetilde{Y}_i(\omega)$.

The next two examples show that the minimal belief subspaces of two players at a given state of the world may be different, and even disjoint.

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Example 10.22 (*Continued*) The table in Figure 10.8 presents the belief space of Example 10.22, along with the minimal belief subspaces of the players. At the state of the world ω_2, the minimal belief subspace of Player I is $\widetilde{Y}_{\mathrm{I}}(\omega_2) = \{\omega_1, \omega_2, \omega_3\}$. It properly contains the minimal belief subspace of Player II, which is $\widetilde{Y}_{\mathrm{II}}(\omega_2) = \{\omega_3\}$.

State of the world	$\mathfrak{s}(\cdot)$	$\pi_{\mathrm{I}}(\cdot)$	$\pi_{\mathrm{II}}(\cdot)$	$\widetilde{Y}_{\mathrm{I}}(\omega)$	$\widetilde{Y}_{\mathrm{II}}(\omega)$
ω_1	s_1	$[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)]$	$[1(\omega_1)]$	Y	Y
ω_2	s_2	$[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)]$	$[1(\omega_3)]$	Y	$\{\omega_3\}$
ω_3	s_3	$[1(\omega_3)]$	$[1(\omega_3)]$	$\{\omega_3\}$	$\{\omega_3\}$

Figure 10.8 The belief space in Example 10.22, and the minimal belief subspaces of the players

◄

Example 10.27 (*Continued*) The table in Figure 10.9 presents the belief space of Example 10.27, along with the minimal belief subspaces of the players.

State of the world	$\mathfrak{s}(\cdot)$	$\pi_{\mathrm{I}}(\cdot)$	$\pi_{\mathrm{II}}(\cdot)$	$\widetilde{Y}_{\mathrm{I}}(\omega)$	$\widetilde{Y}_{\mathrm{II}}(\omega)$
ω_1	s_1	$[1(\omega_1)]$	$[1(\omega_1)]$	$\{\omega_1\}$	$\{\omega_1\}$
ω_2	s_2	$[1(\omega_1)]$	$[1(\omega_3)]$	$\{\omega_1\}$	$\{\omega_3\}$
ω_3	s_3	$[1(\omega_3)]$	$[1(\omega_3)]$	$\{\omega_3\}$	$\{\omega_3\}$

Figure 10.9 The belief space in Example 10.27, and the minimal belief subspaces of the players

As consideration of the table shows, the belief subspaces of the players at the state of the world ω_2 are disjoint. Note in addition that the state of the world ω_2 is not contained in the minimal belief subspaces of the two players at the state of the world ω_2, $\omega_2 \notin \widetilde{Y}_{\mathrm{I}}(\omega_2)$ and $\omega_2 \notin \widetilde{Y}_{\mathrm{II}}(\omega_2)$. ◄

The significance of $\widetilde{Y}_i(\omega)$ is that, when player i is considering the situation that he faces at the state of the world ω, $\widetilde{Y}_i(\omega)$ is the belief subspace that he deems relevant: player i believes that it is common belief among all the players that the state of the world is contained in $\widetilde{Y}_i(\omega)$. Note the difference between the definitions of the minimal belief subspace $\widetilde{Y}(\omega)$ and the minimal belief subspace of player i, $\widetilde{Y}_i(\omega)$: while $\widetilde{Y}_i(\omega)$ does not necessarily contain ω, as shown in Example 10.27, $\widetilde{Y}(\omega)$ by definition must contain ω, because for an "objective" analysis of the game, the true state is of relevance for the situation; in particular, it affects the payoffs of the players.

The statement of the following theorem shows that there is a tight relationship between the minimal belief subspaces of different players.

Theorem 10.31 *Let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$ be a belief space in which Y is a finite set. If $\omega' \in \widetilde{Y}_i(\omega)$ then $\widetilde{Y}_j(\omega') \subseteq \widetilde{Y}_i(\omega)$ for every $j \in N$.*

In words, if at the state of the world ω, the belief subspace of player i contains another state of the world ω', then it also contains all the minimal belief subspaces of the other players at ω'.

Proof: We will show that for any player $j \in N$, every state of the world $\omega'' \in \widetilde{Y}_j(\omega')$ is also contained in $\widetilde{Y}_i(\omega)$. From the definition of the minimal belief subspace of player j, if $\omega'' \in \widetilde{Y}_j(\omega')$, then every belief subspace $\widetilde{Y}$ containing ω' and satisfying $\pi_j(\widetilde{Y} \mid \omega') = 1$ contains ω''. Since $\widetilde{Y}_i(\omega)$ is such a space, it too contains ω''. □

The next theorem states that if the set of states of the world is finite, and at the state of the world ω all the players ascribed positive probability to ω, then the minimal belief subspaces of all the players at ω are the same.

Theorem 10.32 *Let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$ be a belief space in which Y is a finite set, and let $\omega \in Y$. If $\pi_i(\{\omega\} \mid \omega) > 0$ for each player $i \in N$, then $\widetilde{Y}_i(\omega) = \widetilde{Y}_j(\omega)$ for every pair of players i and j (hence $\widetilde{Y}_i(\omega) = \widetilde{Y}(\omega)$ for every $i \in \mathbb{N}$).*

Proof: Let $i \in N$ be a player. Since $\pi_i(\{\omega\} \mid \omega) > 0$, it follows from Equation (10.26) in Definition 10.28 that $\omega \in \widetilde{Y}_i(\omega)$. Then Theorem 10.31 implies that $\widetilde{Y}_j(\omega) \subseteq \widetilde{Y}_i(\omega)$ for every $j \in N$. Since this is true for any pair of players $i, j \in N$, the proof of the theorem is complete. □

The next theorem states that the minimal belief subspace at a state of the world ω is simply the union of the true state of the world ω and the minimal belief subspaces of the players at that state. The proof of the theorem is left to the reader (Exercise 10.32).

Theorem 10.33 *Let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$ be a belief space in which Y is a finite set. Then for every state of the world $\omega \in Y$,*

$$\widetilde{Y}(\omega) = \{\omega\} \cup \left(\bigcup_{i\in N} \widetilde{Y}_i(\omega) \right). \tag{10.28}$$

Remark 10.34 *As shown in Example 10.26, when the set of states of the world has the cardinality of the continuum, the minimal belief subspace may not necessarily exist. If the set of states of the world is a topological space,*[6] *define the minimal belief subspace of a player as follows.*

A belief subspace is an ordered vector $\widetilde{\Pi} = (\widetilde{Y}, \mathcal{Y}_{|\widetilde{Y}}, \mathfrak{s}, (\pi_i)_{i\in N})$ satisfying Equation (10.25), and also satisfying the property that $\widetilde{Y}$ is a closed set. Player i's minimal belief subspace at the state of the world ω, is the belief subspace $\widetilde{\Pi} = (\widetilde{Y}, \mathcal{Y}|\widetilde{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$ in which the set $\widetilde{Y}$ is the smallest closed subset (with respect to set inclusion) among all the belief subspaces satisfying Equation (10.26). ♦

When the set of states of the world Y is finite, there exists a characterization of belief subspaces. Define a directed graph $G = (Y, E)$ in which the set of vertices is the set of states of the world Y, and there is a directed edge from ω_1 to ω_2 if and only if there exists a player $i \in N$ for whom $\pi_i(\{\omega_2\} \mid \omega_1) > 0$. A set of vertices C in a directed graph is called

6 A space Y is called a *topological space* if there exists a family of subsets $\mathcal{T}$ that are called *open sets*: the empty set is contained in $\mathcal{T}$, the set Y is contained in $\mathcal{T}$, the union of any set of elements of $\mathcal{T}$ is in $\mathcal{T}$, and the intersection of a finite number of elements in $\mathcal{T}$ is also a set in $\mathcal{T}$. A set A in a topological space is called *closed* if it is the complement of an open set.

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a *closed component* if for each vertex $\omega \in C$, every vertex connected to ω by a directed edge is also in C; i.e., if there exists an edge from ω to ω', then $\omega' \in C$.

The next theorem states that the set of belief subspaces is exactly the set of closed sets in the graph G. The proof of the theorem is left to the reader (Exercise 10.33).

Theorem 10.35 *Let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$ be a belief space in which Y is a finite set of states of the world. A subset $\widetilde{Y}$ of Y is a belief subspace if and only if $\widetilde{Y}$ is a closed component in the graph G.*

Denote the minimal closed set containing ω by $C(\omega)$. This set contains the vertex ω, all the vertices that are connected to ω by way of directed edges emanating from ω, all vertices that are connected to those vertices by directed edges, and so on. Since the graph G is finite, this is a finite process, and therefore the set $C(\omega)$ is well defined. Together with the construction of the set $C(\omega)$, Theorem 10.35 provides a practical method for calculating belief subspaces and minimal belief subspaces of the players.

The next theorem provides a practical method of computing minimal belief subspaces at a particular state of the world. The proof of the theorem is left to the reader (Exercise 10.34).

Theorem 10.36 *Let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$ be a belief space in which Y is a finite set, let $\omega \in Y$, and let $i \in N$. Then*

$$\widetilde{Y}_i(\omega) = \bigcup_{\{\omega' \colon \pi_i(\{\omega'\}|\omega)>0\}} C(\omega'). \tag{10.29}$$

Recognizing his own beliefs, player i can compute his own minimal belief subspace. To see this, note that since he knows $\pi_i(\omega)$, he knows which states of the world are in the support of this probability distribution. Knowing the states of the world in the support, player i knows the beliefs of the other players at these states of the world; hence he knows which states of the world are in the supports of those beliefs. Player i can thus recursively construct the portion of the graph G relevant for computing $\widetilde{Y}_i(\omega)$. The construction is completed in a finite number of steps because Y is a finite set.

While player i can compute his minimal belief subspace $\widetilde{Y}_i(\omega)$ using his own beliefs, in order to compute the belief subspaces of the other players, $(\widetilde{Y}_j(\omega))_{j\neq i}$, he needs to know their beliefs. Since player i does not know the true state of the world ω, he does not know the beliefs of the other players at that state, which means that he cannot compute the minimal belief subspaces of the other players. In Example 10.22 (page 404), at the states of the world ω_2 and ω_3, the belief of Player II is $[1(\omega_3)]$; hence Player II cannot distinguish between the two states of the world based on his beliefs. The minimal belief subspaces of Player I at the two states of the world are different:

$$\widetilde{Y}_{\mathrm{I}}(\omega_2) = Y, \qquad \widetilde{Y}_{\mathrm{I}}(\omega_3) = \{\omega_3\}. \tag{10.30}$$

It follows that, based on his beliefs, Player II cannot know whether the minimal belief subspace of Player I is Y or $\{\omega_3\}$.

10.5 Games with incomplete information

So far, we have discussed the structure of the mutual beliefs of players, and largely ignored the other components of a game, namely, the actions and the payoffs. In this section, we will define games with incomplete information without a common prior, and the concept of Bayesian equilibrium in such games.

Definition 10.37 *A* game with incomplete information *is an ordered vector $G = (N, S, (A_i)_{i \in N}, \Pi)$, where:*

- *N is a finite set of players.*
- *S is a measurable space of states of nature. To avoid a surfeit of symbols, we will not mention the σ-algebra over S, or over other measurable sets that are defined below.*
- *A_i is a measurable set of possible actions of player i, for every $i \in N$.*
- *Each state of nature in S is a state game $s = (N, (A_i(s))_{i \in N}, (u_i(s))_{i \in N})$, where $A_i(s) \subseteq A_i$ is a nonempty measurable set of possible actions of player i, for each $i \in N$. We denote by $A(s) = \times_{i \in N} A_i(s)$ the set of vectors of possible actions in s. For each player $i \in N$ the function $u_i(s) : A(s) \to \mathbb{R}$ is a measurable function assigning a payoff to player i in the state game s for each vector of possible actions.*
- *$\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i \in N})$ is a belief space of the players N over the set of states of nature S, satisfying the following condition: for every pair of states of the world $\omega, \omega' \in Y$, if $\pi_i(\omega) = \pi_i(\omega')$, then $A_i(\mathfrak{s}(\omega)) = A_i(\mathfrak{s}(\omega'))$.*

The last condition in Definition 10.37 implies that at each state of the world $\omega \in Y$, player i's set of possible actions $A_i(\mathfrak{s}(\omega))$ depends on ω, but only through his type $\pi_i(\omega)$. Since the player knows his own type, he knows the set of possible actions $A_i(\mathfrak{s}(\omega))$ available to him. Formally, consider the partition $\mathcal{F}_i$ of Y determined by player i's beliefs (see Equation (10.8) on page 390) given by the sets

$$F_i(\omega) = \{\omega' \in Y : \pi_i(\omega') = \pi_i(\omega)\}, \tag{10.31}$$

and the knowledge operator defined by this partition. Define the event $C_i(\omega)$ = "player i's set of actions is $A_i(\mathfrak{s}(\omega))$":

$$C_i(\omega) := \{\omega' \in Y : A_i(\mathfrak{s}(\omega')) = A_i(\mathfrak{s}(\omega))\}. \tag{10.32}$$

Then the last condition in Definition 10.37 guarantees that at each state of the world ω, player i knows $C_i(\omega)$, i.e., $F_i(\omega) \subseteq C_i(\omega)$ for each $\omega \in Y$. A game with incomplete information, therefore, is composed of a belief space Π, and a collection of state games, one for each state of nature in S. The information that each player i has at the state of the world ω is his type, $\pi_i(\omega)$. As required in the Harsanyi model of incomplete information, the set of actions available to a player must depend solely on his type. Every Harsanyi game with incomplete information (see Definition 9.39 on page 347) is a game with incomplete information according to Definition 10.37 (Exercise 10.48).

Player i's type set was denoted by

$$T_i := \{\pi_i(\omega) : \omega \in Y\}. \tag{10.33}$$

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To define the expected payoff, we assume that the graph of the function $s \mapsto A(s)$, defined by $\mathrm{Graph}(A) := \{(s,a)\colon s \in S, a \in A(s)\} \subseteq S \times A$, is a measurable set. We similarly assume that for each player $i \in N$, the function $u_i : \mathrm{Graph}(A) \to \mathbb{R}$ is a measurable function.

Definition 10.38 *A* behavior strategy *of player i in a game with incomplete information $G = (N, S, (A_i)_{i\in N}, \Pi)$ is a measurable function $\sigma_i : Y \to \Delta(A_i)$, mapping every state of the world to a mixed action available at the stage game that corresponds to that state of the world,*[7] *and dependent solely on the type of the player. In other words, for each $\omega, \omega' \in Y$,*

$$\sigma_i(\omega) \in \Delta(A_i(\mathfrak{s}(\omega))), \tag{10.34}$$

$$\pi_i(\omega) = \pi_i(\omega') \quad \Longrightarrow \quad \sigma_i(\omega) = \sigma_i(\omega'). \tag{10.35}$$

Since the mixed action $\sigma_i(\omega)$ of player i depends solely on his type $t_i = \pi_i(\omega)$, it can also be denoted by $\sigma_i(t_i)$. Because the type sets of the players may be infinite, strategies must be measurable functions in order for us to calculate the expected payoff of a player given his type. Let $\sigma = (\sigma_i)_{i\in N}$ be a strategy vector. Denote by

$$\sigma(\omega) := (\sigma_i(\omega))_{i\in N} \in \bigtimes_{i\in N} \Delta(A_i(\mathfrak{s}(\omega))) \tag{10.36}$$

the vector of mixed actions of the players when the state of the world is ω. Player i's *payoff* under σ at the state of the world ω is[8]

$$\gamma_i(\sigma \mid \omega) = \int_Y U_i(\mathfrak{s}(\omega'); \sigma(\omega'))\mathrm{d}\pi_i(\omega' \mid \omega). \tag{10.37}$$

Since $\pi_i(\omega)$ is player i's belief at the state of the world ω about the states of the world $\omega' \in Y$, the integral of the payoff function with respect to this probability distribution describes the expected payoff of the player at the state of the world ω, based on his subjective beliefs, and given the other players' strategies.

To emphasize that the expected payoff of player i at the state of world ω depends on the mixed action implemented by player i at ω, and is independent of mixed actions that he implements at other states of the world, we sometimes write $\gamma_i(\sigma_i(\omega), \sigma_{-i} \mid \omega)$ instead of $\gamma_i(\sigma_i, \sigma_{-i} \mid \omega)$.

We will now define the concept of Bayesian equilibrium in games with incomplete information.

Definition 10.39 *A* Bayesian equilibrium *is a strategy vector $\sigma^* = (\sigma_i^*)_{i\in N}$ satisfying*

$$\gamma_i(\sigma^* \mid \omega) \geq \gamma_i(\sigma_i(\omega), \sigma_{-i}^* \mid \omega), \quad \forall i \in N, \forall \sigma_i(\omega) \in \Delta(A_i(\mathfrak{s}(\omega))), \forall \omega \in Y. \tag{10.38}$$

7 Since a behavior strategy is a measurable function whose range is the space of mixed actions, we need to specify the σ-algebra over the space $\Delta(A_i)$ that we are using. The σ-algebra over this space is the σ-algebra induced by the weak topology (see Dunford and Schwartz [1999]). An alternative definition of a measurable function taking values in this space is: for each measurable set $C \subseteq A_i$, the function $\omega \mapsto \sigma_i(C \mid \omega)$ is a measurable function.

In an infinite space, the existence of a behavior strategy requires that the function $\omega \mapsto A_i(\mathfrak{s}(\omega))$ be measurable. Results appearing in Kuratowski and Ryll-Nardzewski [1965] imply that a sufficient condition for the existence of a behavior strategy is that (a) A_i is a complete metric space for every $i \in N$, (b) the function $\omega \mapsto A_i(\mathfrak{s}(\omega))$ is a measurable function, and (c) for each state of nature s, the set $A_i(s)$ is a closed set.

8 Recall that U_i is the multilinear extension of u_i; see Equation (5.9) on page 147.

In other words, a Bayesian equilibrium is a strategy vector that satisfies the condition that based on his subjective beliefs, at no state of the world can a player profit by deviating from his strategy.

As the next theorem states, a strategy vector is a Bayesian equilibrium if no player of any type can profit by deviating to any other action. The theorem is a generalization of Corollary 5.8 (page 149). In our formulation of the theorem, we will use the following notation. Let $\sigma = (\sigma_j)_{j\in N}$ be a strategy vector. For each player $i \in N$, each state of the world $\omega \in Y$, and for each action $a_{i,\omega} \in A_i(\mathfrak{s}(\omega))$, denote by $(\sigma; a_{i,\omega})$ the strategy vector at which every player $j \neq i$ implements strategy σ_j, and player i plays action $a_{i,\omega}$ when his type is $\pi_i(\omega)$.

Theorem 10.40 *A strategy vector $\sigma^* = (\sigma^*_i)_{i\in N}$ is a Bayesian equilibrium if and only if for each player $i \in N$, for each state of the world $\omega \in Y$, and each action $a_{i,\omega} \in A_i(\mathfrak{s}(\omega))$,*

$$\gamma_i(\sigma^* \mid \omega) \geq \gamma_i((\sigma^*; a_{i,\omega}) \mid \omega). \tag{10.39}$$

The proof of the theorem is left to the reader (Exercise 10.49).

Example 10.41 We consider a game that extends Example 10.19 (page 399), where beliefs are inconsistent.

There are two players $N = \{\mathrm{I}, \mathrm{II}\}$, four states of nature $S = \{s_{11}, s_{12}, s_{21}, s_{22}\}$, and four states of the world, $Y = \{\omega_{11}, \omega_{12}, \omega_{21}, \omega_{22}\}$. The beliefs of the players, and the function $\mathfrak{s}$ are given in Figure 10.10.

State of the world	$\mathfrak{s}(\cdot)$	$\pi_{\mathrm{I}}(\cdot)$	$\pi_{\mathrm{II}}(\cdot)$
ω_{11}	s_{11}	$[\frac{3}{7}(\omega_{11}), \frac{4}{7}(\omega_{12})]$	$[\frac{1}{2}(\omega_{11}), \frac{1}{2}(\omega_{21})]$
ω_{12}	s_{12}	$[\frac{3}{7}(\omega_{11}), \frac{4}{7}(\omega_{12})]$	$[\frac{4}{5}(\omega_{12}), \frac{1}{5}(\omega_{22})]$
ω_{21}	s_{21}	$[\frac{2}{3}(\omega_{21}), \frac{1}{3}(\omega_{22})]$	$[\frac{1}{2}(\omega_{11}), \frac{1}{2}(\omega_{21})]$
ω_{22}	s_{22}	$[\frac{2}{3}(\omega_{21}), \frac{1}{3}(\omega_{22})]$	$[\frac{4}{5}(\omega_{12}), \frac{1}{5}(\omega_{22})]$

Figure 10.10 The beliefs of the players and the function $\mathfrak{s}$ in Example 10.41

The players' type sets are

$$T_{\mathrm{I}} = \{\mathrm{I}_1, \mathrm{I}_2\} = \left\{\left[\tfrac{3}{7}(\omega_{11}), \tfrac{4}{7}(\omega_{12})\right], \left[\tfrac{2}{3}(\omega_{21}), \tfrac{1}{3}(\omega_{22})\right]\right\}, \tag{10.40}$$

$$T_{\mathrm{II}} = \{\mathrm{II}_1, \mathrm{II}_2\} = \left\{\left[\tfrac{1}{2}(\omega_{11}), \tfrac{1}{2}(\omega_{21})\right], \left[\tfrac{4}{5}(\omega_{12}), \tfrac{1}{5}(\omega_{22})\right]\right\}. \tag{10.41}$$

The state games s_{11}, s_{12}, s_{21}, and s_{22} are given in Figure 10.11. A behavior strategy of Player I is a pair (x, y), defined as:

- Play the mixed action $[x(T), (1-x)(B)]$ if your type is I_1.
- Play the mixed action $[y(T), (1-y)(B)]$ if your type is I_2.

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Similarly, a behavior strategy of Player II is a pair (z, t), defined as:

- Play the mixed action $[z(L), (1-z)(R)]$ if your type is II_1.
- Play the mixed action $[t(L), (1-t)(R)]$ if your type is II_2.

We will now find a Bayesian equilibrium satisfying $0 < x, y, z, t < 1$ (assuming one exists). In such an equilibrium, every player of each type must be indifferent between his two actions, given his beliefs about the type of the other player.

	L	R
T	2, 0	0, 1
B	0, 0	1, 0

State game s_{11}

	L	R
T	0, 0	0, 0
B	1, 1	1, 0

State game s_{12}

	L	R
T	0, 0	0, 0
B	1, 1	0, 0

State game s_{21}

	L	R
T	0, 0	2, 1
B	0, 0	0, 2

State game s_{22}

Figure 10.11 The payoff functions in Example 10.41

If the players are indifferent between their actions, then:

Player I of type I_1 is indifferent between B and T : $\quad \frac{3}{7}\cdot 2z = \frac{3}{7}(1-z)+\frac{4}{7};$

Player I of type I_2 is indifferent between B and T : $\quad \frac{1}{3}\cdot 2(1-t) = \frac{2}{3}z;$

Player II of type II_1 is indifferent between R and L : $\quad \frac{1}{2}(1-y) = \frac{1}{2}x;$

Player II of type II_2 is indifferent between R and L : $\quad \frac{4}{5}(1-x) = \frac{1}{5}(y+2(1-y)).$

The solution to this system of equations is (verify!)

$$x = \tfrac{3}{5}, \quad y = \tfrac{2}{5}, \quad z = \tfrac{7}{9}, \quad t = \tfrac{2}{9}. \tag{10.42}$$

The mixed actions $[\frac{3}{5}(T), \frac{2}{5}(B)]$ for Player I of type I_1, and $[\frac{2}{5}(T), \frac{3}{5}(B)]$ for Player I of type I_2, and $[\frac{7}{9}(L), \frac{2}{9}(R)]$ for Player II of type II_1, and $[\frac{2}{9}(L), \frac{7}{9}(R)]$ for Player II of type II_2 therefore form a Bayesian equilibrium of this game. This game has no "expected payoff," because there is no common prior distribution over Y. Nevertheless, one can speak about an "objective" expected payoff at each state of nature (calculated from the actions of the players at that state of nature). Denote by $\gamma_i(s)$ the payoff of player i at the state game s. Denote the payoff matrix of player i at the state game s_{kl} by $G_{i,kl}$. The payoff $\gamma_i(s_{kl})$, for example, can be represented in vector form:

$$\gamma_i(s_{11}) = (x, 1-x)G_{i,11}\begin{pmatrix} z \\ 1-z \end{pmatrix}. \tag{10.43}$$

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A simple calculation gives the payoff of each player at each state of nature:

$$\gamma_{\mathrm{I}}(\omega_{11}) = \left(\frac{3}{5}, \frac{2}{5}\right) G_{\mathrm{I},11} \begin{pmatrix} 7/9 \\ 2/9 \end{pmatrix} = \frac{46}{45}, \quad \gamma_{\mathrm{II}}(\omega_{11}) = \left(\frac{3}{5}, \frac{2}{5}\right) G_{\mathrm{I},11} \begin{pmatrix} 7/9 \\ 2/9 \end{pmatrix} = \frac{6}{45},$$

$$\gamma_{\mathrm{I}}(\omega_{12}) = \left(\frac{3}{5}, \frac{2}{5}\right) G_{\mathrm{I},12} \begin{pmatrix} 2/9 \\ 7/9 \end{pmatrix} = \frac{18}{45}, \quad \gamma_{\mathrm{II}}(\omega_{12}) = \left(\frac{3}{5}, \frac{2}{5}\right) G_{\mathrm{I},12} \begin{pmatrix} 2/9 \\ 7/9 \end{pmatrix} = \frac{4}{45},$$

$$\gamma_{\mathrm{I}}(\omega_{21}) = \left(\frac{2}{5}, \frac{3}{5}\right) G_{\mathrm{II},21} \begin{pmatrix} 7/9 \\ 2/9 \end{pmatrix} = \frac{21}{45}, \quad \gamma_{\mathrm{II}}(\omega_{21}) = \left(\frac{2}{5}, \frac{3}{5}\right) G_{\mathrm{II},21} \begin{pmatrix} 7/9 \\ 2/9 \end{pmatrix} = \frac{21}{45},$$

$$\gamma_{\mathrm{I}}(\omega_{22}) = \left(\frac{2}{5}, \frac{3}{5}\right) G_{\mathrm{II},22} \begin{pmatrix} 2/9 \\ 7/9 \end{pmatrix} = \frac{28}{45}, \quad \gamma_{\mathrm{II}}(\omega_{22}) = \left(\frac{2}{5}, \frac{3}{5}\right) G_{\mathrm{II},22} \begin{pmatrix} 2/9 \\ 7/9 \end{pmatrix} = \frac{70}{45}.$$

Because the players do not know the true state game, the relevant expected payoff for a player is the subjective payoff he receives given his beliefs. For example, at the state of the world ω_{11} (or ω_{12}) Player I believes that the state of the world is ω_{11} with probability $\frac{3}{7}$, and ω_{12} with probability $\frac{4}{7}$. Player I therefore believes that the state game is G_{11} with probability $\frac{3}{7}$, and G_{12} with probability $\frac{4}{7}$. His subjective expected payoff is therefore $\frac{3}{7} \times \frac{46}{45} + \frac{4}{7} \times \frac{18}{45} = \frac{2}{3}$, and it is this payoff that he "expects" to receive at the state of the world ω_{11} (or ω_{12}). Similarly, at the state of the world ω_{21} (or ω_{22}), Player I "expects" to receive $\frac{2}{3} \times \frac{21}{45} + \frac{1}{3} \times \frac{28}{45} = \frac{14}{27}$. At ω_{11} (or ω_{21}) Player II "expects" to receive $\frac{1}{2} \times \frac{6}{45} + \frac{1}{2} \times \frac{21}{45} = \frac{3}{10}$. At ω_{12} (or ω_{22}) Player II "expects" to receive $\frac{4}{5} \times \frac{4}{45} + \frac{1}{5} \times \frac{70}{45} = \frac{86}{225}$. ◀

There are no general results concerning the existence of Bayesian equilibria in inconsistent models, but we do have the following result.

Theorem 10.42 *Let $G = (N, S, (A_i)_{i \in N}, \Pi)$ be a game with incomplete information, where Y is a finite set of states of the world, and each player i has a finite set of actions A_i. Then G has a Bayesian equilibrium in behavior strategies.*

Proof: To prove the theorem, we will define the agent-form game corresponding to G (see Definition 9.50, on page 354), and show that every Nash equilibrium of the agent-form game is a Bayesian equilibrium of G. Since Nash's Theorem (Theorem 5.10 on page 151) implies that there exists an equilibrium in the agent-form game, we will deduce that the given game G has a Bayesian equilibrium.

Recall that the type set of player i is denoted $T_i = \{\pi_i(\omega) \colon \omega \in Y\}$. The agent-form game corresponding to G is a strategic-form game $\Gamma = (\widehat{N}, (\widehat{S}_k)_{k \in \widehat{N}}, (\widehat{u}_k)_{k \in \widehat{N}})$, where:

- The set of players is $\widehat{N} = \{(i, t_i) \colon i \in N, t_i \in T_i\}$. In other words, each type of each player is a player in the agent-form game.
- The set of pure strategies of player $(i, t_i) \in \widehat{N}$ is $\widehat{S}_{(i,t_i)} := A_i(\mathfrak{s}(\omega))$, where ω is any state of the world satisfying $t_i = \pi_i(\omega)$.

A pure strategy $\sigma_{(i,t_i)}$ of player (i, t_i) in the agent-form game is a possible action of player i's type t_i in G. It follows that a pure strategy vector $\sigma = (\sigma_{(i,t_i)})_{(i,t_i) \in \widehat{N}}$ is a prescription for what each type of each player should play; hence it is, in fact, also a pure strategy vector in G, in which for each $i \in N$, the vector $(\sigma_{(i,t_i)})_{t_i \in T_i}$ is a pure strategy of player i. This means that the set of pure strategy vectors in G is equal to the set of strategy vectors in

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Γ, and the set of behavior strategy vectors in the game G equals the set of mixed strategy vectors in the game Γ.

- The payoff function of player (i, t_i) is

$$\widehat{u}_{(i,t_i)}(\widehat{\sigma}) = \gamma_i(\widehat{\sigma} \mid \omega), \tag{10.44}$$

where ω is any state of the world satisfying $t_i = \pi_i(\omega)$. Since $\gamma_i(\widehat{\sigma} \mid \omega)$ depends on ω only via $\pi_i(\omega)$, this expression depends only on t_i, and therefore $u_{(i,t_i)}(\widehat{\sigma})$ is well defined.

Because the set of states of the world Y is finite, we deduce that the set of players $\widehat{N}$ in Γ is also finite. Since every player i's set of actions A_i is finite, the agent-form game Γ satisfies the conditions of Nash's Theorem (see Theorem 5.10 on page 151); hence it has an equilibrium $\sigma^* = (\sigma^*_{(i,t_i)})_{(i,t_i)\in\widehat{N}}$ in mixed strategies. Since the set of behavior strategy vectors of G equals the set of mixed strategies of Γ, we can regard $\sigma^* = (\sigma^*_i(\cdot \mid t_i))_{i\in N, t_i\in T_i}$ as a vector of behavior strategies in G. The fact that σ^* is a Bayesian equilibrium then follows from the definition of the agent-form game, and because σ^* is a mixed strategy equilibrium of the agent-form game Γ. □

The following examples look at games with incomplete information with infinite spaces of states of nature.

Example 10.43 **Sealed-bid first-price auction**[9] An original van Gogh painting is being offered in a first-price sealed-bid auction, meaning that every buyer writes his bid on a slip of paper that is placed in a sealed envelope, which is then inserted into a box. After all buyers have submitted their bids, all of the envelopes in the box are opened and read. The buyer who has made the highest bid wins the painting, paying for it the amount that he offered. If more than one buyer bids the highest bid, a fair lottery is conducted among them to choose the winner. Every buyer has a private evaluation for the painting, which will be referred to as his private value for the object. This is the subjective value he ascribes to the painting; private values may differ from one buyer to the next.

Only two buyers take part in this auction, Elizabeth and Charles. Each of them knows his or her private value, but not the private value of the other buyer. Each buyer believes that the private value of the other buyer is uniformly distributed in the interval $[0, 1]$, and this fact is common belief among the buyers. This situation can be modeled as a game with incomplete information, in the following way:

- The set of players is $N = \{\text{Elizabeth}, \text{Charles}\}$.
- The set of states of nature is $S = \{s_{x,y} : 0 \leq x, y \leq 1\}$; The subscript x corresponds to Elizabeth's private value, and subscript y corresponds to Charles's private value.
- Player i's set of actions is $A_i(s) = [0, \infty)$; hence each player i can submit any nonnegative bid a_i. The pair of bids submitted in the envelopes is therefore $(a_{\mathrm{E}}, a_{\mathrm{C}})$.

9 Auction theory is studied in greater detail in Chapter 12.

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- Elizabeth's payoff function is

$$u_E(s_{x,y}; a_E, a_C) = \begin{cases} x - a_E & \text{if } a_E > a_C, \\ \frac{1}{2}(x - a_E) & \text{if } a_E = a_C, \\ 0 & \text{if } a_E < a_C. \end{cases} \tag{10.45}$$

  If Elizabeth wins the auction, her payoff is the difference between her private value and the amount of money she pays for the painting; if she does not win the auction, her payoff is 0. If both buyers submit the same bid, the winner of the auction is chosen at random between them, where each buyer has a probability of $\frac{1}{2}$ of being chosen (this can be accomplished, for example, by tossing a fair coin). It follows that in this case Elizabeth's payoff is half of the difference between her private value and the sum of money she pays for the painting. Charles's payoff function is defined similarly.
- The space of the states of the world is $Y = [0, 1]^2$ with the σ-algebra generated by the Borel sets.
- The function $\mathfrak{s} : Y \to S$ is defined by $\mathfrak{s}(x, y) = s_{x,y}$ for all $(x, y) \in Y$.
- For each state of the world $\omega = (x, y)$, Elizabeth's belief, $\pi_E(x, y)$, is the uniform distribution over the set $\{(x, \widehat{y}) : \widehat{y} \in [0, 1]\}$ and Charles's belief, $\pi_C(x, y)$, is the uniform distribution over the set $\{(\widehat{x}, y) : \widehat{x} \in [0, 1]\}$.

We will show that this game has a symmetric Bayesian equilibrium $\sigma^* = (\sigma^*_E, \sigma^*_C)$, in which both buyers make use of the same strategy: player i's bid is half of his private value. That is,

$$\sigma^*_E(x, y) = \frac{x}{2}, \quad \sigma^*_C(x, y) = \frac{y}{2}, \quad \forall (x, y) \in [0, 1]^2. \tag{10.46}$$

Suppose that Charles uses strategy σ^*_C. We will show that Elizabeth's best reply is to bid half of her private value. Elizabeth's expected payoff if her private value is x, and her bid is a_E, is

$$\gamma_E(a_E, \sigma^*_C \mid x) = \mathbf{P}\left(a_E > \frac{y}{2}\right) \times (x - a_E) \tag{10.47}$$

$$= \mathbf{P}(2a_E > y) \times (x - a_E) \tag{10.48}$$

$$= \min\{2a_E, 1\} \times (x - a_E). \tag{10.49}$$

As a function of a_E, this is a quadratic function over $a_E \in [0, \frac{1}{2}]$ (attaining a maximum at $a_E = \frac{x}{2}$), and a linear function with a negative slope for $a_E \geq \frac{1}{2}$. The graph of this function is shown in Figure 10.12.

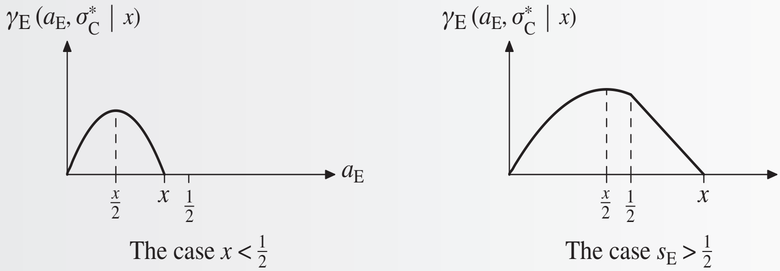

Figure 10.12 Elizabeth's payoff function

In both cases, the function attains a maximum at the point $a_E = \frac{x}{2}$. It follows that $a^*_E(x) = \frac{x}{2}$ is the best reply to σ^*_C. Thus, $\sigma^* = (\sigma^*_E, \sigma^*_C)$ is a Bayesian equilibrium. ◀

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Example 10.44 This is an example in which the beliefs of the players are inconsistent. There are two players $N = \{\mathrm{I}, \mathrm{II}\}$. The set of states of nature is $S = \{s_{x,y} : 0 < x, y < 1\}$; the state games in S are depicted in Figure 10.13.

	L	R
T	1, 0	0, 0
B	2, 1	1, –1

State game $s_{x,y}$ for $x > y$

	L	R
T	0, 0	0, 0
B	0, 0	0, 0

State game $s_{x,y}$ for $x = y$

	L	R
T	0, 1	1, 2
B	0, 0	–1, 1

State game $s_{x,y}$ for $x < y$

Figure 10.13 The state games in Example 10.44

The set of states of the world is $Y = (0, 1)^2$, and the function $\mathfrak{s} : Y \to S$ is defined by $\mathfrak{s}(x, y) = s_{x,y}$ for every $(x, y) \in Y$. Player I is told the first coordinate x of the state of world, and Player II is told the second coordinate y of the state of world. Given the value z that is told to a player, that player believes that the value told to the other player is uniformly distributed over the interval $(0, z)$. In other words, $\pi_{\mathrm{I}}(x, y)$ is the uniform distribution over the line segment $((x, 0), (x, x))$ and $\pi_{\mathrm{I}}(x, y)$ is the uniform distribution over the line segment $((0, y), (y, y))$.

At every state of the world, Player I believes that $x > y$; hence he believes that action B strictly dominates action T. In a similar way, at every state of the world Player II believes that $y > x$; hence he believes that action R strictly dominates action L. It follows that the only equilibrium is that where Player I, of any type, plays B, and Player II, of any type, plays R. The equilibrium payoff is then $(-1, 1)$ if $x < y$, $(1, -1)$ if $x > y$, and $(0, 0)$ if $x = y$. However, in every state of the world, each player believes that his payoff is 1. ◀

Example 10.45 We now consider a game similar to the game in Example 10.44, but with different state games, given in Figure 10.14; here, x represents the first coordinate of the state of nature, and y the second coordinate. Note that each player, after learning his type, knows his payoff function, but does not know the payoff function of the other player, even if he knows the strategy used by the other player, because he does not know the other player's type.

Player II

Player I		L	R
	T	x, 0	0, y
	B	0, 1	1, 0

Figure 10.14 The state game $s_{(x,y)}$ in Example 10.45

We will seek a Bayesian equilibrium in which both players, of each type, use a completely mixed action. At such an equilibrium, every player of every type is indifferent between his two actions. Denote by $\sigma_{\mathrm{I}}(x)$ the probability that Player I of type x, who has received the information x, will choose action T, and by $\sigma_{\mathrm{II}}(y)$ the probability that Player II of type y,

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who has received the information y, will choose action L. Denote by U_x the uniform distribution over $[0, x]$. The payoff to Player I of type x if he plays the action T is then

$$\gamma_{\mathrm{I}}(T, \sigma_{\mathrm{II}} \mid x) = \int_{y=0}^{x} x\sigma_{\mathrm{II}}(y)\mathrm{d}U_x(y) = x\int_{y=0}^{x} \sigma_{\mathrm{II}}(y)\mathrm{d}U_x(y), \tag{10.50}$$

and the payoff to Player I of type x if he plays the action B is then

$$\gamma_{\mathrm{I}}(B, \sigma_{\mathrm{II}} \mid x) = \int_{y=0}^{x} (1-\sigma_{\mathrm{II}}(y))\mathrm{d}U_x(y) = 1 - \int_{y=0}^{x} \sigma_{\mathrm{II}}(y)\mathrm{d}U_x(y). \tag{10.51}$$

Player I of type x is indifferent between T and B if these two quantities are equal to each other, i.e., if

$$(1+x)\int_{y=0}^{x} \sigma_{\mathrm{II}}(y)\mathrm{d}U_x(y) = 1. \tag{10.52}$$

The density function of the distribution U_x equals $\frac{1}{x}$ in the interval $[0, x]$, and it follows that in this interval $\mathrm{d}U_x(y) = \frac{\mathrm{d}y}{x}$. After inserting this equality in Equation (10.52) and moving terms from one side of the equal sign to the other, we get

$$\int_{y=0}^{x} \sigma_{\mathrm{II}}(y)\mathrm{d}y = \frac{x}{1+x}. \tag{10.53}$$

Differentiating by x yields

$$\sigma_{\mathrm{II}}(x) = \frac{1}{(1+x)^2}. \tag{10.54}$$

By replacing the variable x by y, which is the information that Player II receives, we deduce that $\sigma_{\mathrm{II}}(y) = \frac{1}{(1+y)^2}$ is a strategy of Player II that makes Player I of any type indifferent between his two actions. In Exercise 10.54, the reader is asked to conduct a similar calculation to find a strategy of Player I that makes Player II of any type indifferent between his two actions. When each player implements a strategy that makes the other player indifferent between his two actions, we obtain an equilibrium (why is this true?). ◀

10.6 The concept of consistency

The concept of consistency in belief spaces was defined on page 399. A consistent belief space is one in which the beliefs of the players are derived from a common prior p over the types. In this section, we will study this concept in greater detail. For simplicity, we will deal here only with finite belief spaces, but all the results of this section also hold in belief spaces with a countably infinite number of states of the world. The definitions and results can be generalized to infinite belief spaces, often requiring only adding appropriate technical conditions. Denote the support of $\pi_i(\omega)$ by $P_i(\omega)$,

$$P_i(\omega) := \mathrm{supp}(\pi_i(\omega)) \subseteq Y. \tag{10.55}$$

This is the set of states of the world that are possible, in player i's opinion, at the state of the world ω.

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Definition 10.46 *Let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i \in N})$ be a belief space, where Y is a finite set. A belief subspace $\widetilde{Y}$ is* consistent *if there exists a probability distribution p over $\widetilde{Y}$ such that, for every event $A \subseteq \widetilde{Y}$, for each player i, and for each $\omega \in \mathrm{supp}(p)$, $p(P_i(\omega)) > 0$ and*

$$\pi_i(A \mid \omega) = p(A \mid P_i(\omega)). \tag{10.56}$$

A probability distribution p satisfying Equation (10.56) for every event $A \subseteq \widetilde{Y}$, for every player i and for every $\omega \in \mathrm{supp}(p)$, is called a consistent distribution *(over $\widetilde{Y}$), or a* common prior.

In words, the belief of player i at the state of the world ω is given by the conditional probability of p given the information $P_i(\omega)$ that the player has at ω. A consistent distribution, therefore, plays the same role as the common prior in the Aumann model of incomplete information.

On page 399 we defined the concept of consistency by conditioning on the set $\{\omega' \in Y : \pi_i(\omega') = \pi_i(\omega)\}$ instead of the set $P_i(\omega)$ (see Equation (10.56)). The coherency requirement in the definition of a belief space guarantees that these two definitions are equivalent (Exercise 10.65).

If p is a consistent distribution, then every state of the world $\omega \in \mathrm{supp}(p)$ is called a *consistent state of the world*. Every state of the world that is not consistent is called *inconsistent*.

Remark 10.47 *Let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i \in N})$ be a belief space, and let $\widetilde{\Pi} = (\widetilde{Y}, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i \in N})$ be a consistent belief subspace of Π. Then Π is also a consistent belief space. To see this, note that since $\widetilde{\Pi}$ is a consistent belief subspace, there exists a consistent distribution $\widetilde{p}$ over $\widetilde{Y}$. In that case, define a probability distribution p over Y by*

$$p(\omega) = \begin{cases} \widetilde{p}(\omega) & \text{if } \omega \in \widetilde{Y}, \\ 0 & \text{if } \omega \notin \widetilde{Y}. \end{cases} \tag{10.57}$$

This is a consistent distribution over Π (Exercise 10.64). ◆

The next example elucidates the concepts of consistent distribution and consistent state of the world.

Example 10.48 Figure 10.15 depicts a belief space for the set of players $N = \{\mathrm{I}, \mathrm{II}\}$ over the set of states of nature $S = \{s_1, s_2\}$. In this belief space, the set of states of the world is $Y = \{\omega_1, \omega_2\}$.

State of the world	$\mathfrak{s}(\cdot)$	$\pi_{\mathrm{I}}(\cdot)$	$\pi_{\mathrm{II}}(\cdot)$
ω_1	s_1	$[1(\omega_1)]$	$[1(\omega_1)]$
ω_2	s_2	$[1(\omega_1)]$	$[1(\omega_2)]$

Figure 10.15 The belief space in Example 10.48

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At the state of the world ω_1, the fact that the state of nature is s_1 is common belief among the players. At the state of the world ω_2, Player I believes that the fact that the state of nature is s_1 is common belief among the players, while Player II believes that the state of nature is s_2, and believes that Player I believes that the fact that the state of nature is s_1 is common belief among the players.

The belief subspace $\widetilde{Y} = \{\omega_1\}$ is a consistent belief subspace, with a consistent distribution $\widetilde{p} = [1(\omega_1)]$. Remark 10.47 shows that Π is also a consistent belief space, with consistent distribution $p = [1(\omega_1)]$. The state of the world ω_1 is contained in the support of p; therefore it is a consistent state of the world.

The state of the world ω_2, however, is inconsistent. Indeed, at that state of the world Player I ascribes probability 1 to the state of nature being s_1, while Player II ascribes probability 1 to the state of nature being s_2. There cannot, then, exist a probability distribution p from which these two beliefs can be derived (verify!). ◀

Example 10.18 (*Continued*) Consider the event $A = \{\omega_{12}, \omega_{21}\}$ and the probability distribution p, as defined in Figure 10.5 on page 398. Then

$$\pi_{\mathrm{I}}(A \mid \omega_{11}) = \tfrac{4}{7}, \tag{10.58}$$

$$p(A \mid P_{\mathrm{I}}(\omega_{11})) = p(A \mid \{\omega_{11}, \omega_{12}\}) = \tfrac{4}{7}; \tag{10.59}$$

hence $\pi_{\mathrm{I}}(A \mid \omega_{11}) = p(A \mid P_{\mathrm{I}}(\omega))$. It can be shown that Equation (10.56) is satisfied for every event $A \subseteq Y$, for every player $i \in \{\mathrm{I}, \mathrm{II}\}$, and for every $\omega \in Y$ (Exercise 10.66); hence p is a consistent distribution. ◀

Example 10.19 (*Continued*) The beliefs of the players are shown in Figure 10.6 on page 399. Consider the event $A = \{\omega_{12}, \omega_{21}\}$ and the probability distribution p defined in Figure 10.5 on page 398. Then

$$\pi_{\mathrm{I}}(A \mid \omega_{11}) = \tfrac{4}{7}, \tag{10.60}$$

$$p(A \mid P_{\mathrm{I}}(\omega_{11})) = p(A \mid \{\omega_{11}, \omega_{12}\}) = \tfrac{4}{7}. \tag{10.61}$$

On the other hand,

$$\pi_{\mathrm{II}}(A \mid \omega_{11}) = \tfrac{1}{2}, \tag{10.62}$$

$$p(A \mid P_{\mathrm{II}}(\omega_{11})) = p(A \mid \{\omega_{11}, \omega_{21}\}) = \tfrac{2}{5}; \tag{10.63}$$

hence $\pi_{\mathrm{II}}(A \mid \omega_{11}) \neq p(A \mid P_{\mathrm{II}}(\omega_{11}))$. It follows that p is not a consistent distribution. This is not surprising, since we proved that this belief space is not consistent. ◀

It can be shown (Exercise 10.67) that the following definition is equivalent to the definition of a consistent distribution (see Definition 10.46).

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Definition 10.49 *A probability distribution $p \in \Delta(Y)$ (over a finite set of states of the world Y) is called* consistent *if for each player $i \in N$,*

$$p = \sum_{\omega \in Y} \pi_i(\omega) p(\omega). \tag{10.64}$$

In other words,

$$p(\omega') = \sum_{\omega \in Y} \pi_i(\{\omega'\} \mid \omega) p(\omega), \quad \forall \omega' \in Y. \tag{10.65}$$

A probability distribution p is consistent if, for every player i, it is the average according to p of player i's types (recall that a type is a probability distribution over Y). This gives us a new angle from which to regard the consistency condition: in any consistent system of beliefs, we can find weights for the states of the world (a probability distribution p), such that the weighted average of the type of each player is the same for all the players, and this average is exactly the probability distribution p. (Equations (10.64) and (10.65) refer to sums, and not integrals, because of the assumption that the set of states of the world is finite.)

Example 10.18 (*Continued*) To ascertain that p is a consistent distribution according to Definition 10.49, we need to ascertain that Equation (10.64) is satisfied for each player $i \in N$. Each row in the table in Figure 10.16 describes a type of Player I at a state of the world. The right column shows the common prior, and the bottom row describes the weighted average of Player I's types.

	ω_{11}	ω_{12}	ω_{21}	ω_{22}	Probability
$\pi_{\mathrm{I}}(\omega_{11})$	$\frac{3}{7}$	$\frac{4}{7}$	0	0	$\frac{3}{10}$
$\pi_{\mathrm{I}}(\omega_{12})$	$\frac{3}{7}$	$\frac{4}{7}$	0	0	$\frac{4}{10}$
$\pi_{\mathrm{I}}(\omega_{21})$	0	0	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{10}$
$\pi_{\mathrm{I}}(\omega_{22})$	0	0	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{10}$
Average	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{2}{10}$	$\frac{1}{10}$	

Figure 10.16 The probability that each type of Player I ascribes to each state of nature in Example 10.18

When we take the weighted average of each row of the table (i.e., compute the right-hand side of the equal sign in Equation (10.64)), we obtain the probability distribution p with which we started (listed in the left-most column). We obtain a similar result with respect to Player II (Exercise 10.68); hence p is a consistent distribution according to Definition 10.49. ◀

Definition 10.46 does not require the support of a consistent distribution p to be all of Y. The next theorem states, however, that the support of such a probability distribution is also a consistent belief subspace.

Theorem 10.50 *Let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i \in N})$ be a belief space in which Y is a finite set. If p is a consistent distribution over Y, then $\widetilde{Y} = \mathrm{supp}(p)$ is a consistent belief subspace.*

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The proof of Theorem 10.50 is left to the reader (Exercise 10.70). As the next example shows, it is possible for a consistent belief space to have several different consistent distributions.

Example 10.51 Consider the following belief space, with one player, $N = \{I\}$, two states of nature, $S = \{s_1, s_2\}$, and two states of the world, $Y = \{\omega_1, \omega_2\}$, where:

State of the world	$\mathfrak{s}(\cdot)$	$\pi_I(\cdot)$
ω_1	s_1	$[1(\omega_1)]$
ω_2	s_2	$[1(\omega_2)]$

That is, at ω_1 the player believes that the state of nature is s_1, and at ω_2 he believes that the state of nature is s_2. For each $\lambda \in [0, 1]$, the probability distribution p^λ, defined as follows, is consistent (verify!):

$$p^\lambda(\omega_1) = \lambda, \quad p^\lambda(\omega_2) = 1 - \lambda. \tag{10.66}$$

◀

The belief space $Y = \{\omega_1, \omega_2\}$ in Example 10.51 properly contains two belief subspaces: $\{\omega_1\}$ and $\{\omega_2\}$. As the next theorem states, this is the only possible way in which multiple consistent distributions can arise.

Theorem 10.52 *Let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$ be a consistent belief space in which Y is finite that does not properly contain a belief subspace. Then there exists a unique consistent probability distribution p whose support* $\text{supp}(p)$ *is contained in Y.*

By definition, for each consistent belief subspace $\widetilde{Y}$ there exists a consistent distribution p satisfying $\text{supp}(p) \subseteq \widetilde{Y}$. Theorem 10.52 states that if the belief subspace is minimal, then there exists a unique such probability distribution. It then follows from Theorem 10.50 that $\text{supp}(p) = \widetilde{Y}$.

In order to prove Theorem 10.52, we will first prove the following auxiliary theorem. We adopt here the convention that $\frac{0}{0} = 1$.

Theorem 10.53 *Let $(\alpha_k)_{k=1}^n$ be nonnegative numbers, let $(x_k, y_k)_{k=1}^n$ be a sequence of pairs of positive numbers, and let $C > 1$. If $\frac{x_k}{y_k} \le C$ for all $i \in \{1, 2, \ldots, n\}$, then $\frac{\sum_{k=1}^n \alpha_k x_k}{\sum_{k=1}^n \alpha_k y_k} \le C$. If, in addition, there exists $j \in \{1, 2, \ldots, n\}$ such that $\alpha_j > 0$, and $\frac{x_j}{y_j} < C$, then $\frac{\sum_{k=1}^n \alpha_k x_k}{\sum_{k=1}^n \alpha_k y_k} < C$.*

Proof: If $\alpha_k = 0$ for each k, then both the numerator and the denominator in the expression $\frac{\sum_{k=1}^n \alpha_k x_k}{\sum_{k=1}^n \alpha_k y_k}$ are zero; hence their ratio equals 1, which is smaller than C. Suppose, therefore, that at least one of the numbers in $(\alpha_k)_{k=1}^n$ is positive. This implies that both the numerator and the denominator in the expression $\frac{\sum_{k=1}^n \alpha_k x_k}{\sum_{k=1}^n \alpha_k y_k}$ are positive.

Since $\frac{x_k}{y_k} \le C$ for each $k \in \{1, 2, \ldots, n\}$, it follows that $x_k \le C y_k$. Because $(\alpha_k)_{k=1}^n$ are nonnegative numbers,

$$\sum_{k=1}^n \alpha_k x_k \le C \sum_{k=1}^n \alpha_k y_k. \tag{10.67}$$

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Since the sum on the right-hand side of the equation is positive, we can divide by that sum to get

$$\frac{\sum_{k=1}^{n}\alpha_k x_k}{\sum_{k=1}^{n}\alpha_k y_k} \leq C. \tag{10.68}$$

Next, suppose that there also exists $j \in \{1, 2, \ldots, n\}$ such that $\alpha_j > 0$ and $\frac{x_j}{y_j} < C$. In this case, we deduce that Equation (10.67) is satisfied as a strict inequality (verify!). It follows that Equation (10.68) is also satisfied as a strict inequality, which is what we sought to show. □

Proof of Theorem 10.52: Let p and $\widehat{p}$ be two distinct consistent probability distributions, where the supports $\mathrm{supp}(p)$ $\mathrm{supp}(\widehat{p})$ are contained in Y. Theorem 10.50 implies that $\mathrm{supp}(p)$ is a belief subspace. Since Y is a minimal belief subspace, it must be the case that $\mathrm{supp}(p) = Y$. We similarly deduce that $\mathrm{supp}(\widehat{p}) = Y$ and therefore in particular $\mathrm{supp}(\widehat{p}) = \mathrm{supp}(p)$. Denote $C := \max_{\omega\in\mathrm{supp}(p)} \frac{p(\omega)}{\widehat{p}(\omega)}$. Since $\mathrm{supp}(\widehat{p}) = Y$, the denominator $\widehat{p}(\omega)$ is positive for all $\omega \in Y$; therefore C is well defined.

Since p and $\widehat{p}$ are different probability distributions, it must be the case that $C > 1$. To see why, suppose that $C \leq 1$. Then $p(\omega) \leq \widehat{p}(\omega)$ for each $\omega \in Y$, and since $\sum_{\omega\in Y} p(\omega) = 1 = \sum_{\omega\in Y} \widehat{p}(\omega)$, we have $p(\omega) = \widehat{p}(\omega)$ for each $\omega \in \widehat{Y}$. Denote

$$A := \left\{\omega \in Y : \frac{p(\omega)}{\widehat{p}(\omega)} = C\right\}. \tag{10.69}$$

We now show that Theorem 10.53 implies that $\mathrm{supp}(\pi_i(\omega')) \subseteq A$ for each $\omega' \in A$. Let $\omega' \in A$ be an element of A, and write out the expressions in Theorem 10.53 with $(\pi_i(\{\omega'\} \mid \omega))_{\omega\in Y}$ as the set of nonnegative numbers $(\alpha_k)_{k=1}^n$ and $(p(\omega))_{\omega\in Y}$ and $(\widehat{p}(\omega))_{\omega\in Y}$ as, respectively, the sets of positive numbers $(x_k)_{k=1}^n$ and $(y_k)_{k=1}^n$. Since p and $\widehat{p}$ are consistent distributions, with the aid of Definition 10.49, we get

$$\frac{\sum_k \alpha_k x_k}{\sum_k \alpha_k y_k} = \frac{\sum_\omega \pi_i(\{\omega'\} \mid \omega)p(\omega)}{\sum_\omega \pi_i(\{\omega'\} \mid \omega)\widehat{p}(\omega)} = \frac{p(\omega')}{\widehat{p}(\omega')}. \tag{10.70}$$

If $\mathrm{supp}(\pi_i(\omega'))$ were not contained in A, then there would be a state $\omega \in \mathrm{supp}(\pi_i(\omega'))$ satisfying $\frac{p(\omega)}{\widehat{p}(\omega)} < C$. Then Theorem 10.53 would in turn imply that $\frac{p(\omega')}{\widehat{p}(\omega')} < C$, i.e., $\omega' \notin A$, which would be a contradiction.

If follows that $\mathrm{supp}(\pi_i(\omega')) \subseteq A$ for each $\omega' \in A$; hence A is a belief subspace (see Definition 10.21 on page 400). Since Y is a minimal belief subspace, $A = Y$. We then deduce that $\frac{p(\omega)}{\widehat{p}(\omega)} = C > 1$ for each $\omega \in Y$, i.e., $p(\omega) > \widehat{p}(\omega)$. Summing over $\omega \in Y$, we get

$$1 = \sum_{\omega\in Y} p(\omega) > \sum_{\omega\in Y} \widehat{p}(\omega) = 1. \tag{10.71}$$

This contradiction establishes that $p = \widehat{p}$. □

The consistency presented in this section is an "objective" concept, by which we mean objective from the perspective of an outside observer, who knows the belief space $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$ and can verify whether or not a given state of the world $\omega \in Y$ is consistent according to Definition 10.46. But what are the beliefs of the players about

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the consistency of a given state of the world? If a player believes that the state of the world is consistent, then he can describe the situation he is in as a Harsanyi game with incomplete information, and choose his actions by analyzing that game. The following theorem relates to this question.

Theorem 10.54 *Let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i \in N})$ be a belief space, where Y is a finite set, and let $\omega \in Y$ be a consistent state of the world. Then it is common belief among the players at ω that ω is consistent. In particular, every player at ω believes that ω is consistent.*

Proof: Since ω is a consistent state of the world, there exists a consistent distribution p over Y satisfying $p(\omega) > 0$. Since $p(\omega) > 0$, Equation (10.56) implies that for each player $i \in N$,

$$\pi_i(\{\omega\} \mid \omega) = p(\{\omega\} \mid P_i(\omega)) > 0. \tag{10.72}$$

It follows from Theorem 10.32 on page 405 that for each pair of players $i, j \in N$,

$$\widetilde{Y}_j(\omega) = \widetilde{Y}_i(\omega), \tag{10.73}$$

where $\widetilde{Y}_i(\omega)$ is the minimal belief space of player i at the state of the world ω (see Definition 10.30 on page 403). Note that $\omega \in \mathrm{supp}(\pi_i(\omega))$ for each player $i \in N$; hence $\omega \in Y_i(\omega)$. Theorem 10.33 implies that $\widetilde{Y}_i(\omega) = \widetilde{Y}(\omega)$ for each $i \in N$, where $\widetilde{Y}(\omega)$ is the minimal belief space at the state of the world ω (see Definition 10.24 on page 401). Let $\widetilde{p}$ be the probability distribution p, conditioned on the set $\widetilde{Y}(\omega)$,

$$\widetilde{p}(\omega') = \frac{p(\omega')}{p(\widetilde{Y}(\omega))}, \quad \forall \omega' \in \widetilde{Y}(\omega). \tag{10.74}$$

Then $\widetilde{p}$ is a consistent distribution (Exercise 10.72). It follows that $\widetilde{Y}(\omega)$ is a consistent belief subspace. As stated after Definition 10.21 (page 400), every belief subspace is common belief at every state of the world contained in it; hence the event $\widetilde{Y}(\omega)$ is common belief among the players at the state of the world ω. In particular, it is common belief among the players at ω that ω is consistent. □

Every player i, based only on his own private information (i.e., his type), can construct the minimal belief subspace $\widetilde{Y}_i(\omega)$ that includes, according to his beliefs, all the states of the world that are relevant to the situation he is in. If the state of the world ω is consistent, then the belief subspace $\widetilde{Y}_i(\omega)$ is also consistent, and, since it is a minimal belief subspace, there is a unique consistent distribution p over it (Theorem 10.52), which the player can compute. In this case, the situation, according to player i's beliefs, is equivalent to a Harsanyi model of incomplete information. That model is constructed by first selecting a state of the world according to the consistent distribution p over $\widetilde{Y}_i(\omega)$; hence the situation, according to player i's beliefs, is equivalent to the interim stage, which is the point in time at which every player knows his partition element, which contains the true state of the world. Every Aumann model, or equivalently Harsanyi model, is but an auxiliary construction that can be made by each of the players. The entire model thus constructed, including the space of types (which is computed from the belief subspace $\widetilde{Y}_i(\omega)$), and the probability distribution used to choose the types (which is derived from the consistent distribution p), is based on the private information of the player. In addition, when ω is a consistent state of the world, Theorem 10.54 states that every player computes the same minimal belief space, and computes the same consistent distribution p, and it is common belief among the players that this is the case. In particular, all the players arrive at the same Aumann (or Harsanyi) model of incomplete information, and this model is common belief among them.

The next question that naturally arises is what are the beliefs of the players about the consistency of a given state of the world when that state of the world is inconsistent. As we will now show, in that case there are two possibilities: it is possible for a player to believe that the state of the world is inconsistent, and in some cases this may even be common belief among the players. However, it is also possible for a player (mistakenly) to believe that the state of the world is consistent, and it is even possible for all the players to believe that the state of the world is consistent, and for that fact to be common belief among the players.

Theorem 10.55 *Let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$ be a belief space, where Y is a finite set, and let $\omega \in Y$ be an inconsistent state of the world. If $\pi_i(\{\omega\} \mid \omega) > 0$, then at the state of the world ω player i believes that the state of the world is inconsistent.*

Proof: The assumption that $\pi_i(\{\omega\} \mid \omega) > 0$ implies that $\omega \in \widetilde{Y}_i(\omega)$; hence $\widetilde{Y}_i(\omega)$ is a belief subspace containing ω. Since ω is an inconsistent state of the world, there does not exist a consistent distribution p over $\widetilde{Y}_i(\omega)$ satisfying $p(\omega) > 0$; hence player i, after calculating his minimal belief subspace $\widetilde{Y}_i(\omega)$, believes that the state of the world is inconsistent. □

It follows from Theorem 10.55 that at an inconsistent state of the world ω, player i is liable (mistakenly) to believe that the state of the world is consistent only if he ascribes probability 0 to ω; that is, $\pi_i(\{\omega\} \mid \omega) = 0$. This happens in fact in Example 10.27 on page 402, in which the state of the world ω_2 is inconsistent, but at this state of the world, both players believe that the actual state of the world is consistent. In fact, in Example 10.27, at the state of the world ω_2 it is common belief among the players that the state of the world is consistent, even though it is inconsistent (Exercise 10.80).

In Example 10.19 on page 399, every state of the world is inconsistent; hence the fact that any given state of the world is inconsistent is common belief among the players at every state of the world. The next theorem generalizes this example, and presents a sufficient condition that guarantees that the fact that a given state of the world is inconsistent is common belief among the players.

Theorem 10.56 *Let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$ be a belief space, where Y is a finite set, and let $\omega \in Y$ be an inconsistent state of the world. If $\pi_i(\{\omega'\} \mid \omega') > 0$ for every player i and every state of the world ω' in the minimal belief subspace $\widetilde{Y}(\omega)$ at ω, then the fact that the state of the world is inconsistent is common belief among the players at ω, and at every state of the world in $\widetilde{Y}(\omega)$.*

Proof: Because $\omega \in \widetilde{Y}(\omega)$, the assumption implies that $\pi_i(\{\omega\} \mid \omega) > 0$ for every player $i \in N$. By Theorem 10.32 (page 405), $\widetilde{Y}(\omega) = \widetilde{Y}_i(\omega)$ for each player $i \in N$. It then follows from Theorem 10.55 that at every state of the world $\omega' \in \widetilde{Y}(\omega)$, every player believes that the state of the world is inconsistent. In particular, it follows that at every state of the world $\omega' \in \widetilde{Y}(\omega)$ it is common belief that the state of the world is inconsistent. □

There are cases in which at a given state of the world, every player believes that the state of the world is inconsistent, but this fact is not common belief among the players (Exercise 10.81). There are also cases in which some of the players believe that the state of the world is consistent while others believe that the state of the world is inconsistent (Exercise 10.82).

Most of the models of incomplete information used in the game theory literature are Harsanyi games. This means that nearly every model in published papers is described using a consistent belief system, despite the fact that, as we have seen, not only do inconsistent belief spaces exist, they comprise an "absolute majority" of situations of incomplete information – the set of consistent situations is a set of measure zero within the set of belief spaces. The reason that consistent models are ubiquitous in the literature is mainly because consistent models are presentable extensive-form games, while situations of inconsistent beliefs cannot be presented as either extensive-form or strategic-form games. This makes the mathematical study of such situations difficult. It should, however, be reiterated that the central solution concept – Bayesian equilibrium – is computable and applicable in both consistent and inconsistent situations.

10.7 Remarks

Example 10.16 (page 396) appears in Sorin and Zamir [1985], under the name "Lack of information on one-and-a-half sides." Results on Nash equilibria and Bayesian equilibria in games with incomplete information and general space of states of the world can be found in many papers. A partial list includes Milgrom and Weber [1985], Milgrom and Roberts [1990], van Zandt and Vives [2007], van Zandt [2007], and Vives [1990].

Exercises 10.37–10.40 are taken from Mertens and Zamir [1985]. The notion "belief with probability at least p" that appears in Exercise 10.15 was defined in Monderer and Samet [1989]. Discussion about the subject of probabilistic beliefs appeared in Gaifman [1986]. Exercise 10.62 is taken from van Zandt [2007].

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10.8 Exercises

10.1 Let the set of players be $N = \{\mathrm{I}, \mathrm{II}\}$, the set of states of nature be $S = \{s_1, s_2\}$, and the set of states of the world be $Y = \{\omega_1, \omega_2, \omega_3\}$. The σ-algebra over Y is $\mathcal{Y} = \{\emptyset, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$, and the function $\mathfrak{s}$ mapping the states of the world to the states of nature is given by

$$\mathfrak{s}(\omega_1) = s_1, \quad \mathfrak{s}(\omega_2) = \mathfrak{s}(\omega_3) = s_2. \tag{10.75}$$

For each of the following three belief functions, determine whether $(Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$ is a belief space of N over S. Justify your answers.

(a) $\pi_{\mathrm{I}}(\omega_1) = [1(\omega_1)]$, $\pi_{\mathrm{I}}(\omega_2) = \pi_{\mathrm{I}}(\omega_3) = [\frac{1}{3}(\omega_2), \frac{2}{3}(\omega_3)]$, $\pi_{\mathrm{II}}(\omega_1) = \pi_{\mathrm{II}}(\omega_2) = \pi_{\mathrm{II}}(\omega_3) = [\frac{4}{7}(\omega_2), \frac{3}{7}(\omega_3)]$.
(b) $\pi_{\mathrm{I}}(\omega_1) = \pi_{\mathrm{I}}(\omega_2) = \pi_{\mathrm{I}}(\omega_3) = [\frac{1}{4}(\omega_1), \frac{3}{4}(\omega_2)]$, $\pi_{\mathrm{II}}(\omega_1) = [1(\omega_1)]$, $\pi_{\mathrm{II}}(\omega_2) = \pi_{\mathrm{II}}(\omega_3) = [\frac{1}{3}(\omega_2), \frac{2}{3}(\omega_3)]$.
(c) $\pi_{\mathrm{I}}(\omega_1) = [\frac{1}{3}(\omega_1), \frac{1}{3}(\omega_2), \frac{1}{3}(\omega_3)]$, $\pi_{\mathrm{I}}(\omega_2) = \pi_{\mathrm{I}}(\omega_3) = [\frac{1}{6}(\omega_2), \frac{5}{6}(\omega_3)]$, $\pi_{\mathrm{II}}(\omega_1) = [1(\omega_1)]$, $\pi_{\mathrm{II}}(\omega_2) = \pi_{\mathrm{II}}(\omega_3) = [1(\omega_3)]$.

10.2 Let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$ and $\widehat{\Pi} = (\widehat{Y}, \widehat{\mathcal{Y}}, \widehat{\mathfrak{s}}, (\widehat{\pi}_i)_{i\in N})$ be two belief spaces satisfying $Y \cap \widehat{Y} = \emptyset$. Prove that $\widetilde{\Pi} = (\widetilde{Y}, \widetilde{\mathcal{Y}}, \widetilde{\mathfrak{s}}, (\widetilde{\pi}_i)_{i\in N})$, as defined below, is a belief space.[10]

- $\widetilde{Y} := Y \cup \widehat{Y}$.
- $\widetilde{\mathcal{Y}} := \{F \cup \widehat{F} \colon F \in \mathcal{Y}, \widehat{F} \in \widehat{\mathcal{Y}}\}$.
- $\widetilde{\mathfrak{s}}(\omega) = \mathfrak{s}(\omega)$ for every $\omega \in Y$ and $\widetilde{\mathfrak{s}}(\widehat{\omega}) = \widehat{\mathfrak{s}}(\widehat{\omega})$ for every $\widehat{\omega} \in \widehat{Y}$.
- $\widetilde{\pi}_i(\omega) = \pi_i(\omega)$ for every $\omega \in Y$ and $\widetilde{\pi}_i(\widehat{\omega}) = \widehat{\pi}_i(\widehat{\omega})$ for every $\widehat{\omega} \in \widehat{Y}$.

10.3 Prove Equation (10.10) on page 390.

10.4 Minerva ascribes probability 0.7 to Hercules being able to lift a massive rock, and she believes that Hercules believes that he can lift the rock. Construct a belief space in which the described situation is represented by a state of the world and indicate that state (more than one answer is possible).

10.5 Minerva ascribes probability 0.7 to Hercules being able to lift a massive rock, and she believes that Hercules believes that he can lift the rock. Hercules, in contrast, believes that if he attempts to lift the rock he will fail to do so. Construct a belief space in which the described situation is represented by a state of the world and indicate that state (more than one answer is possible).

10.6 Eric believes that it is common belief among him and Jack that the New York Mets won the baseball World Series in 1969. Jack ascribes probability 0.5 to the New York Mets having won the World Series in 1969, and to Eric believing that it is common belief among the two of them that the New York Mets won the World Series in 1969. Jack also ascribes probability 0.5 to the New York Mets not having won the World Series in 1969, and to Eric believing that it is not common belief among the two of them that the New York Mets won the World Series in 1969. Construct a belief space in which the described situation is represented by a state of the world and indicate that state (more than one answer is possible).

10.7 (a) Using two states of the world describe the following situation, specifying how each state differs from the other: "Roger ascribes probability 0.4 to the Philadelphia Phillies winning the World Series."
(b) Add to the two states of the world that you listed above two more states of the world, and use the four states to construct a belief space in which the following situation is represented as a state of the world: "Jimmy ascribes probability 0.3

10 Every probability distribution π over the space Y can be regarded as a probability distribution over the space $Y \cup \widehat{Y}$ such that $\pi(\widehat{Y}) = 0$.

to the Philadelphia Phillies winning the World Series and to Roger ascribing probability 0.4 to the Philadelphia Phillies winning the World Series, and Jimmy ascribes probability 0.7 to the Philadelphia Phillies not winning the World Series and to Jimmy ascribing 0.4 to the Philadelphia Phillies winning the World Series."

(c) Construct a belief space in which the following situation is represented by a state of the world and indicate that state.

Roger says:

- I ascribe probability 0.3 to "the Philadelphia Phillies will win the World Series, and Jimmy ascribes probability 0.3 to the Philadelphia Phillies winning the World Series and to my ascribing 0.4 to the Philadelphia Phillies winning the World Series, and Jimmy ascribes probability 0.7 to the Philadelphia Phillies not winning the World Series and to my ascribing 0.4 to the Philadelphia Phillies winning the World Series."
- I ascribe probability 0.2 to "the Philadelphia Phillies will win the World Series, and Jimmy ascribes probability 0.7 to the Philadelphia Phillies winning the World Series and to my ascribing 0.5 to the Philadelphia Phillies winning the World Series, and Jimmy ascribes probability 0.3 to the Philadelphia Phillies not winning the World Series and to my ascribing 0.5 to the Philadelphia Phillies winning the World Series."
- I ascribe probability 0.4 to "the Philadelphia Phillies will win the World Series, and Jimmy ascribes probability 0.2 to the Philadelphia Phillies winning the World Series and to my ascribing 0.1 to the Philadelphia Phillies winning the World Series, and Jimmy ascribes probability 0.8 to the Philadelphia Phillies not winning the World Series and to my ascribing 0.1 to the Philadelphia Phillies winning the World Series."
- I ascribe probability 0.1 to "the Philadelphia Phillies will win the World Series, and Jimmy ascribes probability 0.6 to the Philadelphia Phillies winning the World Series and to my ascribing 0.4 to the Philadelphia Phillies winning the World Series, and Jimmy ascribes probability 0.4 to the Philadelphia Phillies not winning the World Series and to my ascribing 0.3 to the Philadelphia Phillies winning the World Series."

10.8 Prove Theorem 10.7: player i's belief operator B_i (see Definition 10.6 on page 392) satisfies the following properties:

(a) $B_i Y = Y$: player i believes that Y is the set of all states of the world.

(b) $B_i A \cap B_i C = B_i(A \cap C)$: if player i believes that event A obtains, and that event C obtains, then he believes that event $A \cap C$ obtains.

(c) $B_i(B_i A) = B_i A$: if player i believes that event A obtains, then he believes that he believes that event A obtains.

(d) $(B_i A)^c = B_i((B_i A)^c)$: if player i does not believe that event A obtains, then he believes that he does not believe that event A obtains.

10.9 Let Π be a belief space equivalent to an Aumann model of incomplete information and let B_i be player i's belief operator in Π (see Definition 10.6 on page 392).

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Prove that the knowledge operator K_i that is defined by the partition via Equation (10.8) (page 390) is the same operator as the belief operator B_i.

10.10 In this exercise we show that the converse of the statement of Theorem 10.10 does not hold. Find an example of a belief space in which there exists a state of the world $\omega \in Y$, an event A that is common belief among the players at the state of the world ω, a player $i \in N$, and a state of the world $\omega' \in A$, such that $\pi_i(A \mid \omega') < 1$.

10.11 In this exercise we show that Theorem 10.11 on page 393 does not hold without the assumption that $\mathbf{P}(\omega) > 0$ for every state of the world $\omega \in Y$.

Prove that there exists an Aumann model of incomplete information in which the common prior $\mathbf{P}$ satisfies $\mathbf{P}(\omega) = 0$ in at least one state of the world $\omega \in Y$, such that the following claim holds: the knowledge operator in the Aumann model is not identical to the belief operator in the belief space Π equivalent to the Aumann model.

10.12 Describe in words the beliefs of the players about the state of nature and the beliefs of the other players about the state of nature in each of the states of the world in Example 10.4 (page 389).

10.13 Prove Theorem 10.8 (page 392): for each player $i \in N$ and every pair of events $A, C \subseteq Y$, if $A \subseteq C$, then $B_i A \subseteq B_i C$.

10.14 Let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i \in N})$ be a belief space equivalent to an Aumann model of incomplete information. Prove that the partition defined by Equation (10.8) (page 390) for the belief space Π is the same partition as the partition $\mathcal{F}_i$ in the equivalent Aumann model.

10.15 For every player $i \in N$ and every real number $p \in [0, 1]$, define B_i^p to be the operator mapping each set $E \in \mathcal{Y}$ to the set of states of the world at which player i ascribes to E probability equal to or greater than p,

$$B_i^p(E) := \{\omega \in Y : \pi_i(E \mid \omega) \geq p\}. \tag{10.76}$$

Which of the following properties are satisfied by B_i^p for $p \in [0, 1]$? For each property, either prove that it is satisfied, or present a counterexample. There may be different answers for different values of p.

(a) $B_i^p(Y) = Y$.
(b) $B_i^p(A) \subseteq A$.
(c) If $A \subseteq C$ then $B_i^p(A) \subseteq B_i^p(C)$.
(d) $B_i^p(B_i^p(A)) = B_i^p(A)$.
(e) $B_i^p((B_i^p(A))^c) = (B_i^p(A))^c$.

10.16 Prove that if $\widetilde{Y}$ is a belief subspace of a belief space $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i \in N})$, then the event $\widetilde{Y}$ is common belief among the players at every state of the world $\omega \in \widetilde{Y}$.

10.17 Let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i \in N})$ be a belief space equivalent to an Aumann model of incomplete information. Prove that for each state of the world ω, the common knowledge component among the players at ω (see page 333) is a belief subspace.

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10.18 Consider Example 10.19 (page 399), and suppose that the beliefs of the types in that example are as follows, where $x, y, z, w \in [0, 1]$. For which values of x, y, z, and w is the belief system of the players consistent?

	II_1	II_2
I_1	x	$1 - x$
I_2	y	$1 - y$

The beliefs of Player I

	II_1	II_2
I_1	z	w
I_2	$1 - z$	$1 - w$

The beliefs of Player II

10.19 Prove that the beliefs of the players in Example 10.20 (page 400) are inconsistent. Recall that when the set of states of the world is a topological space, we require that a belief subspace be a closed set (see Remark 10.34).

10.20 Consider the following belief space, where the set of players is $N = \{\mathrm{I}, \mathrm{II}\}$, and the set of states of nature is $S = \{s_1, s_2\}$.

State of the world	$\mathfrak{s}(\cdot)$	$\pi_{\mathrm{I}}(\cdot)$	$\pi_{\mathrm{II}}(\cdot)$
ω_1	s_1	$\left[\frac{2}{5}(\omega_1), \frac{3}{5}(\omega_2)\right]$	$[1(\omega_1)]$
ω_2	s_2	$\left[\frac{2}{5}(\omega_1), \frac{3}{5}(\omega_2)\right]$	$\left[\frac{3}{4}(\omega_2), \frac{1}{4}(\omega_3)\right]$
ω_3	s_2	$[1(\omega_1)]$	$\left[\frac{3}{4}(\omega_2), \frac{1}{4}(\omega_3)\right]$

(a) List the types of the two players at each state of the world in Y.
(b) Can the beliefs of the players be derived from a common prior? If so, what is that common prior? If not, justify your answer.

10.21 Boris believes that "it is common belief among me and Alex that Bruce Jenner won a gold medal at the Montreal Olympics," while Alex believes that "it is common belief among me and Boris that Bruce Jenner won a silver medal at the Montreal Olympics."

(a) Construct a belief space in which the described situation is represented by a state of the world and indicate that state.
(b) Prove that, in any belief space in which the set of states of the world is a finite set and contains a state ω describing the situation in this exercise, ω is not contained in the support of the beliefs of either Boris or Alex, at any state of the world. In other words, $\omega \notin \mathrm{supp}(\pi_i(\omega'))$ for any state of the world ω', for $i \in \{\text{Boris}, \text{Alex}\}$.

10.22 Laocoön declares: "I ascribe probability 0.6 to the Greeks attacking us from within a wooden horse." Priam then declares: "I ascribe probability 0.7 to the Greeks attacking us from within a wooden horse." After Priam's declaration, is the fact that "Laocoön ascribes probability 0.6 to the Greeks attacking from within a wooden horse" common belief among the players? Justify your answer.

10.23 There are two players, $N = \{\mathrm{I}, \mathrm{II}\}$, and two states of nature $S = \{s_1, s_2\}$. A chance move chooses the state of nature, where s_1 is chosen with probability 0.4, and s_2 is

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chosen with probability 0.6. Player I knows the true state of nature that has been chosen. A chance move selects a signal that is received by Player II. The signal depends on the state of nature, as follows: if the true state of nature is s_1, Player II receives signal R with probability 0.6, and signal L with probability 0.4; if the true state of nature is s_2, Player II receives signal M with probability 0.7, and signal L with probability 0.3. It follows that if Player II receives signal L, he does not know with certainty which state of nature has been chosen. If the state of nature that has been chosen is s_2, and Player II has received signal M, then Player I is informed of this with probability 0.2, and Player I is not informed of this with probability 0.8. This description is common belief among the players. Construct a belief space in which the described situation is represented by a state of the world and indicate that state.

10.24 Repeat Exercise 10.23, under the assumption that the players do not agree on the probability distribution according to which the state of nature is chosen; that is, there is no common prior over S: Player I believes that s_1 is chosen with probability 0.4, while Player II believes that s_1 is chosen with probability 0.5. The rest of the description of the situation is as in Exercise 10.23, and this description is common belief among the players.

10.25 John, Bob, and Ted meet at a party in which all the invitees are either novelists or poets (but no one is both a novelist and a poet). Every poet knows all the other poets, but no novelist knows any other attendee at the party, whether novelist or poet. Every novelist believes that one-quarter of the attendees are novelists. Construct a belief space describing the beliefs of John, Bob, and Ted about the others' profession.

10.26 Walter, Karl, and Ferdinand are on the road to Dallas. They arrive at a fork in the road; should they turn right or left? Type t_1 believes that "we should turn right, everyone here believes that we should turn right, everyone here believes that everyone here believes that we should turn right, etc.": in other words, that type believes that turning right is called for, and believes that this is common belief among the three. Type t_2 believes that "we should turn right, the two others believe that we should turn left, the two others believe that everyone here believes that we should turn left, the two others believe that everyone here believes that everyone here believes that we should turn left, etc.": in other words, that type believes that turning right is called for, but believes that the other two believe that they should turn left and that this fact is common belief among the three. Type t_3 does not know which way to turn, but believes that the two others know the right way to turn, and believes that the others believe that everyone knows the right way to turn: he believes "the probability that we should turn right is $\frac{1}{2}$, and the probability that we should turn left is $\frac{1}{2}$; if we should turn right, then the two others believe that we should turn right, and that this is common belief among everyone here, and if we should turn left, then the two others believe that we should turn left, and that this is common belief among everyone here." Walter's type is t_1, Karl's type is t_2, and Ferdinand's type is t_3.

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(a) Construct a belief space in which the described situation is represented by a state of the world and indicate that state.
(b) What is the minimal belief subspace of each of the three players (at the state of the world in which Walter's type is t_1, Karl's type is t_2, and Ferdinand's type is t_3)?

10.27 Repeat Exercise 10.26, when all three players are of type t_3.

10.28 In this exercise we show that when there is no common prior, it is possible to find a lottery that satisfies the property that each player has a positive expectation of profiting from the lottery, using his subjective probability belief.

Let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$ be a belief space of the set of players $N = \{\mathrm{I}, \mathrm{II}\}$, where the set of states of the world Y is finite. For each $i \in N$, define a set P_i of probability distributions over Y as follows:

$$P_i := \left\{ \sum_{\omega\in Y} x_\omega \pi_i(\cdot \mid \omega) \colon \sum_{\omega\in Y} x_\omega = 1, x_\omega > 0 \quad \forall \omega \in Y \right\} \subset \Delta(Y). \quad (10.77)$$

This is the set of all convex combinations of the beliefs $(\pi_i(\cdot \mid \omega))_{\omega\in Y}$ of player i such that the weight given to every ω is positive.

(a) Prove that for every $p \in P_i$ and every $\omega \in Y$, the belief $\pi_i(\cdot \mid \omega)$ is the conditional probability distribution of p given $F_i(\omega)$ (for the definition of the set $F_i(\omega)$ see Equation (10.8) on page 390). In other words, if p were a common prior, the beliefs of player i would be given by π_i.
(b) Prove that the set P_i is an open and convex set in $\Delta(Y)$, for every $i \in N$.
(c) Prove that if there is no common prior, then P_{I} and P_{II} are disjoint sets.
(d) Using Exercise 23.46 (page 956) prove that there exist $\alpha \in \mathbb{R}^{|Y|}$ and $\beta \in \mathbb{R}$ such that[11]

$$\langle \alpha, p_{\mathrm{I}} \rangle > \beta > \langle \alpha, p_{\mathrm{II}} \rangle, \quad \forall p_{\mathrm{I}} \in P_{\mathrm{I}}, \forall p_{\mathrm{II}} \in P_{\mathrm{II}}. \quad (10.78)$$

(e) The beliefs of Players I and II about the state of the world are given by the probability distributions $(\pi_i)_{i\in N}$. The state of the world, while unknown to them today, will become known to them tomorrow. They decide that after the state of the world ω will be revealed to them, Player II will pay Player I the sum $\alpha(\omega) - \beta$. If this quantity is negative, the payment will be from Player I to Player II. Prove that, given his subjective beliefs, the expected payoff of each player under this procedure is positive.

10.29 Prove Theorem 10.23 (page 401): given two belief subspaces $\widetilde{\Pi}_1 = (\widetilde{Y}_1, \mathcal{Y}_{|\widetilde{Y}_1}, \mathfrak{s}, (\pi_i)_{i\in N})$ and $\widetilde{\Pi}_2 = (\widetilde{Y}_2, \mathcal{Y}_{|\widetilde{Y}_2}, \mathfrak{s}, (\pi_i)_{i\in N})$ of a belief space $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$ satisfying $\widetilde{Y}_1 \cap \widetilde{Y}_2 \neq \emptyset$, prove that $(\widetilde{Y}_1 \cap \widetilde{Y}_2, \mathcal{Y}_{|\widetilde{Y}_1\cap\widetilde{Y}_2}, \mathfrak{s}, (\pi_i)_{i\in N})$ is also a belief subspace of Π.

10.30 Prove that if there exists a minimal belief subspace, then it is unique.

11 The inner product is given by $\langle p, \alpha \rangle = \sum_{\omega\in Y} p(\omega)\alpha(\omega)$.

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10.31 Generalize Theorem 10.25 to the case in which the set of states of the world is countably infinite: let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$ be a belief space in which the set of states of the world Y is countably infinite. Prove that there exists a minimal belief subspace at each state of the world $\omega \in Y$.

10.32 Prove Theorem 10.33 (page 405): let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$ be a belief space, where Y is a finite set. Then for each state of the world $\omega \in Y$,

$$\widetilde{Y}(\omega) = \{\omega\} \cup \left(\bigcup_{i\in N} \widetilde{Y}_i(\omega)\right). \tag{10.79}$$

10.33 Prove Theorem 10.35 (page 406): let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$ be a belief space, where Y is a finite set. Then the subset $\widetilde{Y}$ of Y is a belief subspace if and only if $\widetilde{Y}$ is a closed component in the graph G defined by Π.

10.34 Prove Theorem 10.36 on page 406: let $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$ be a belief space, where Y is a finite set, let $\omega \in Y$, and let $i \in N$. For each state of the world ω, let $C(\omega)$ be the minimal closed component containing ω in the graph corresponding to Π. Prove that

$$\widetilde{Y}_i(\omega) = \bigcup_{\{\omega' :\, \pi_i(\{\omega'\}|\omega)>0\}} C(\omega'). \tag{10.80}$$

10.35 Present an example of a belief space Π with three players, and a state of the world ω satisfying $\widetilde{Y}_1(\omega) \supset \widetilde{Y}_2(\omega) \supset \widetilde{Y}_3(\omega)$ (where all the set inclusions are strict inclusions).

10.36 Prove or disprove the following. There exists a belief space $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i\in N})$, where Y is a finite set, and there are two players, $i, j \in N$, such that there exists a state of the world $\omega \in Y$ satisfying the property that $\widetilde{Y}_i(\omega) \cap \widetilde{Y}_j(\omega)$ is nonempty and strictly included in both $\widetilde{Y}_i(\omega)$ and $\widetilde{Y}_j(\omega)$.

10.37 In this exercise, suppose there are four states of nature, $S = \{s_{11}, s_{12}, s_{21}, s_{22}\}$. The information that Player I receives is the first coordinate of the state of nature chosen, while the information that Player II receives is the second coordinate. The conditional probabilities of the players, given their respective informations, are given by the following table (the conditional probability of Player I appears in the left column, while the conditional probability of Player II appears in the top row of the table):

	$\begin{pmatrix}\frac{3}{5}\\ \frac{2}{5}\end{pmatrix}$	$\begin{pmatrix}0\\ 1\end{pmatrix}$
$(1, 0)$	s_{11}	s_{12}
$(\frac{2}{3}, \frac{1}{3})$	s_{21}	s_{22}

The table is to be read as stating, e.g., that if Player I receives information indicating that the state of nature is contained in $\{s_{11}, s_{12}\}$, he believes with probability 1 that the state of nature is s_{11}.

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(a) Construct a belief space in which the described situation is represented by a state of the world and indicate that state.

Suppose that the state of nature is s_{12}, and that ω is the corresponding state of the world. Answer the following questions:

(b) What are the minimal belief subspaces $\widetilde{Y}_{\mathrm{I}}(\omega)$ and $\widetilde{Y}_{\mathrm{II}}(\omega)$ of the players?
(c) Is $\widetilde{Y}_{\mathrm{I}}(\omega) = \widetilde{Y}_{\mathrm{II}}(\omega)$?
(d) Is there a common prior p over S such that the players agree that the state of the world has been chosen according to p?
(e) Is the state of the world ω ascribed positive probability by p?

10.38 Repeat Exercise 10.37, where this time there are nine states of nature $S = \{s_{11}, s_{12}, s_{13}, s_{21}, s_{22}, s_{23}, s_{31}, s_{32}, s_{33}\}$ and the beliefs of the players, given their information, are presented in the following table:

	$\begin{pmatrix} \frac{3}{5} \\ \frac{1}{5} \\ \frac{1}{5} \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 0 \end{pmatrix}$
$(1, 0, 0)$	s_{11}	s_{12}	s_{13}
$(\frac{2}{3}, \frac{1}{3}, 0)$	s_{21}	s_{22}	s_{23}
$(0, 0, 1)$	s_{31}	s_{32}	s_{33}

10.39 Repeat Exercise 10.37, but this time suppose the beliefs of each player of each type are:

	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$(1, 0)$	s_{11}	s_{12}
$(0, 1)$	s_{21}	s_{22}

Parts (b)–(e) of Exercise 10.37 relate to a situation in which the true state of nature is s_{12}. Can each player calculate the minimal belief subspace of the other player at each state of the world? Justify your answer.

10.40 Repeat Exercise 10.37, where S includes 20 states of nature, the true state of nature is s_{13}, and the beliefs of the players are given in the following table:

	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} \frac{3}{5} \\ \frac{2}{5} \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$
$(1, 0, 0, 0, 0)$	s_{11}	s_{12}	s_{13}	s_{14}	s_{15}
$(\frac{1}{3}, \frac{2}{3}, 0, 0, 0)$	s_{21}	s_{22}	s_{23}	s_{24}	s_{25}
$(0, 0, 0, \frac{1}{4}, \frac{3}{4})$	s_{31}	s_{32}	s_{33}	s_{34}	s_{35}
$(0, 0, 0, \frac{1}{4}, \frac{3}{4})$	s_{41}	s_{42}	s_{43}	s_{44}	s_{45}

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10.41 Suppose there are two players, $N = \{\mathrm{I}, \mathrm{II}\}$, and four states of nature, $S = \{s_{11}, s_{12}, s_{21}, s_{22}\}$. Player I's information is the first coordinate of the state of nature, and Player II's information is the second coordinate. The beliefs of each player, given his information, about the other player's type are given by the following tables:

	II_1	II_2
I_1	$\frac{1}{4}$	$\frac{3}{4}$
I_2	$\frac{4}{9}$	$\frac{5}{9}$

The beliefs of Player I

	II_1	II_2
I_1	$\frac{1}{5}$	$\frac{3}{8}$
I_2	$\frac{4}{5}$	$\frac{5}{8}$

The beliefs of Player II

(a) Find a belief space describing this situation.
(b) Is this belief space consistent? If so, describe this situation as an Aumann model of incomplete information.

10.42 Repeat Exercise 10.41 for the following beliefs of the players:

	II_1	II_2
I_1	$\frac{1}{4}$	$\frac{3}{4}$
I_2	$\frac{1}{5}$	$\frac{4}{5}$

The beliefs of Player I

	II_1	II_2
I_1	$\frac{2}{3}$	$\frac{1}{2}$
I_2	$\frac{1}{3}$	$\frac{1}{2}$

The beliefs of Player II

10.43 Calculate the minimal belief subspaces of the two players at each state of the world in Example 10.20 (page 400). Recall that when the set of states of the world is a topological space, a belief subspace is required to be a closed set (Remark 10.34).

10.44 Prove or disprove: There exists a belief space in which the set of states of the world contains K states, and there are $2^K - 1$ different belief subspaces (in other words, every subset of states of the world, except for the empty set, constitutes a belief subspace).

10.45 Prove or disprove: There exists a belief space comprised of three states of the world and six different belief subspaces.

10.46 Prove or disprove: There exists a belief space with $N = \{\mathrm{I}, \mathrm{II}\}$ and a finite set of states of the world containing a state of the world ω such that $\widetilde{Y}_{\mathrm{I}}(\omega) \not\supseteq \widetilde{Y}_{\mathrm{II}}(\omega)$ and $\widetilde{Y}_{\mathrm{II}}(\omega) \not\supseteq \widetilde{Y}_{\mathrm{I}}(\omega)$.

10.47 Prove or disprove: For each $\omega \in Y$, and each player $i \in N$, the set $\widetilde{Y}_i(\omega) \cup \{\omega\}$ is a belief subspace.

10.48 Prove that every Harsanyi game with incomplete information (see Definition 9.39 on page 347) is a game with incomplete information according to Definition 10.37 on page 407.

10.49 Prove Theorem 10.40 (page 409): the strategy vector $\sigma^* = (\sigma_i^*)_{i \in N}$ is a Bayesian equilibrium if and only if for each player $i \in N$, each state of the world $\omega \in Y$,

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and each action $a_{i,\omega} \in A_i(\omega)$,

$$\gamma_i(\sigma^* \mid \omega) \geq \gamma_i((\sigma^*; a_{i,\omega}) \mid \omega). \tag{10.81}$$

10.50 In this exercise we show that for studying the set of Bayesian equilibria, one may assume that the action sets of the players are independent of the state of nature.

Let $G = (N, S, (A_i)_{i \in N}, \Pi)$ be a game with incomplete information such that the payoff functions $(u_i)_{i \in N}$ are uniformly bounded from below:

$$M := \inf_{i \in N} \inf_{s \in S} \inf_{a \in A(s)} u_i(s; a) > -\infty. \tag{10.82}$$

Let $\widehat{G} = (N, S, (A_i)_{i \in N}, \Pi)$ be the game with incomplete information defined as follows:

- The action sets of the players are independent of the state of nature: $A_i(s) = A_i$ for every player $i \in N$ and every state of nature $s \in S$.
- For each player $i \in N$, the payoff function $\widehat{u}_i$ is a real-valued function defined over the set $A = \times_{i \in N} A_i$ and given by

$$\widehat{u}_i(s; a) = \begin{cases} u_i(s; a) & a \in A(s), \\ M & a_i \in A_i(s), a \notin A(s), \\ M - 1 & a_i \notin A_i(s), \end{cases} \tag{10.83}$$

where $A(s) = \times_{i \in N} A_i(s)$. In other words, if at least one player $j \neq i$ chooses an action that is not in $A_j(s)$, while player i chooses an action in $A_i(s)$, player i receives payoff M, and if player i chooses an action that is not in $A_i(s)$, he receives a payoff that is less than M.

Prove that the set of Bayesian equilibria of the game G coincides with the set of Bayesian equilibria of the game $\widehat{G}$.

10.51 Prove that there exists a Bayesian equilibrium (in behavior strategies) in every game with incomplete information in which the set of players is finite, the number of types of each player is countable, the number of actions of each type is finite, and the payoff functions are uniformly bounded.

10.52 In this exercise we generalize Corollary 4.27 (page 105) to Bayesian equilibria. Suppose that for every player i in a game with incomplete information there exists a strategy σ_i^* that weakly dominates all his other strategies; in particular,

$$U_i(\mathfrak{s}(\omega); \sigma^*(\omega)) \geq U_i(\mathfrak{s}(\omega); \sigma^*_{-i}(\omega), a_i), \quad \forall i \in N, \forall \omega \in Y, \forall a_i \in A_i(\mathfrak{s}(\omega)).$$

Prove that the strategy vector $\sigma^* = (\sigma_i^*)_{i \in N}$ is a Bayesian equilibrium.

10.53 This exercise presents an alternative proof of Theorem 10.42 (page 411), regarding the existence of Bayesian equilibria in finite games.

Let $G = (N, S, (A_i)_{i \in N}, \Pi)$ be a game with incomplete information where the set of actions $(A_i)_{i \in N}$ is a finite set, and $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i \in N})$ is a belief space with a finite set of states of the world Y. Define a strategic-form game Γ, where the set of players is N, the set of player i's pure strategies is the set of functions σ_i that map each type of player i to an available action for that type, and player i's

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payoff function w_i is given by

$$w_i(\sigma) = \sum_{\omega \in Y} \gamma_i(\sigma \mid \omega). \tag{10.84}$$

(a) Prove that the game Γ has a Nash equilibrium in mixed strategies.
(b) Deduce that the game Γ has a Nash equilibrium in behavior strategies.
(c) Prove that the set of Nash equilibria in behavior strategies of the game Γ coincides with the set of Bayesian equilibria of the game G.

10.54 In Example 10.45 (page 414), find a strategy for Player I that guarantees that Player II, of any type, is indifferent between L and R.

10.55 Find a Bayesian equilibrium in pure strategies in the following two-player game. Are there any additional Bayesian equilibria?

The set of states of nature is $S = \{s_1, s_2\}$, the set of players is $N = \{\mathrm{I}, \mathrm{II}\}$, and the belief space is given by:

State of the world	$\mathfrak{s}(\cdot)$	$\pi_1(\cdot)$	$\pi_2(\cdot)$
ω_1	s_1	$[1(\omega_1)]$	$[1(\omega_1)]$
ω_2	s_1	$[1(\omega_1)]$	$[\frac{1}{2}(\omega_2), \frac{1}{2}(\omega_3)]$
ω_3	s_2	$[1(\omega_4)]$	$[\frac{1}{2}(\omega_2), \frac{1}{2}(\omega_3)]$
ω_4	s_2	$[1(\omega_4)]$	$[1(\omega_4)]$

The state games are as follows:

	L	R
T	$0, 0$	$0, 1$
B	$-10, 0$	$1, 1$

State game s_1

	L	R
T	$1, 2$	$-10, 0$
B	$0, 2$	$0, 0$

State game s_2

10.56 Find a Bayesian equilibrium in the game appearing in Exercise 9.39 (page 378, when each player has a different prior, as follows. The prior distribution of Player I is

$$p_{\mathrm{I}}(\mathrm{I}_1, \mathrm{II}_1) = 0.4, \quad p_{\mathrm{I}}(\mathrm{I}_1, \mathrm{II}_2) = 0.1, \quad p_{\mathrm{I}}(\mathrm{I}_2, \mathrm{II}_1) = 0.2, \quad p_{\mathrm{I}}(\mathrm{I}_2, \mathrm{II}_2) = 0.3.$$

The prior distribution of Player II is

$$p_{\mathrm{II}}(\mathrm{I}_1, \mathrm{II}_1) = 0.3, \quad p_{\mathrm{II}}(\mathrm{I}_1, \mathrm{II}_2) = 0.2,$$
$$p_{\mathrm{II}}(\mathrm{I}_2, \mathrm{II}_1) = 0.25, \quad p_{\mathrm{II}}(\mathrm{I}_2, \mathrm{II}_2) = 0.25.$$

Assume that these prior distributions are common knowledge among the players.

10.57 Ronald and Jimmy are betting on the result of a coin toss. Ronald ascribes probability $\frac{1}{3}$ to the event that the coin shows heads, while Jimmy ascribes probability $\frac{3}{4}$ to that event. The betting rules are as follows: Each of the two players writes on a slip of paper "heads" or "tails," with neither player knowing what the other player is writing. After they are done writing, they show each other what they have written. If both players wrote heads, or both players wrote tails, each of them receives a payoff of 0. If they have made mutually conflicting predictions, they toss the coin. The player who has made the correct prediction regarding the result of the coin toss receives $1 from the player who has made the incorrect prediction. This description is common knowledge among the two players.

(a) Depict this situation as a game with incomplete information.
(b) Are the beliefs of Ronald and Jimmy consistent? Justify your answer.
(c) If you answered the above question positively, find the common prior.
(d) Find a Bayesian equilibrium in this game (whether or not the beliefs of the players are consistent).

10.58 Find a Bayesian equilibrium in the game appearing in Exercise 9.50 (page 382).

10.59 In Exercise 9.42 (page 379), suppose that the players' beliefs are:

- Marc thinks that the probability of every possible value is $\frac{1}{3}$, and he believes that this is common belief among him and Nicolas.
- Nicolas knows that Marc's beliefs are as described above, but he also knows that the true value of the company is 11.

Answer the following questions:

(a) Can this situation be described as a Harsanyi game with incomplete information (with a common prior)? Justify your answer.
(b) Find a Bayesian equilibrium of the game.

10.60 Find all the Bayesian equilibria of the following two-player game with incomplete information. The set of states of nature is $S = \{s_1, s_2, s_3\}$, the set of players is $N = \{\mathrm{I}, \mathrm{II}\}$, the set of states of the world is $Y = \{\omega_1, \omega_2, \omega_3\}$, and the belief space is:

State of the world	$\mathfrak{s}(\cdot)$	$\pi_{\mathrm{I}}(\cdot)$	$\pi_{\mathrm{II}}(\cdot)$
ω_1	s_1	$\left[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)\right]$	$[1(\omega_1)]$
ω_2	s_2	$\left[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)\right]$	$[1(\omega_1)]$
ω_3	s_3	$[1(\omega_3)]$	$[1(\omega_3)]$

The state games are as follows:

	L	R
T	4, 0	1, 1
B	1, 2	3, 0

State game s_1

	L	R
T	0, 3	1, 5
B	1, 0	0, 2

State game s_2

	L	R
T	0, 1	6, 4
B	7, 5	2, 3

State game s_3

10.61 **A Cournot game with inconsistent beliefs** Each of two manufacturers $i \in \{\mathrm{I}, \mathrm{II}\}$ must determine the quantity of a product x_i to be manufactured (in thousands of

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units) for sale in the coming month. The unit price of the manufactured products depends on the total quantities both manufacturers produce, and is given by $p = 2 - x_{\mathrm{I}} - x_{\mathrm{II}}$. Each manufacturer knows his own unit production cost, but does not know the unit production cost of his rival. The unit production cost of manufacturer i may be high ($c_i = \frac{5}{4}$) or low ($c_i = \frac{3}{4}$). Manufacturer i's profit is $x_i(p - c_i)$.

The first manufacturer ascribes probability $\frac{2}{3}$ to the second manufacturer's costs being high (and probability $\frac{1}{3}$ to the second manufacturer's costs being low). The second manufacturer ascribes probability $\frac{3}{4}$ to the costs of both manufacturers being equal to each other (and probability $\frac{1}{4}$ to their costs being different).

Answer the following questions:

(a) Describe this situation as a game with incomplete information.
(b) Prove that the beliefs of the manufacturers are inconsistent.
(c) Find all Bayesian equilibria in pure strategies in this game.

10.62 This exercise shows that when the players agree that every state of the world may obtain, the game is equivalent to a game with a common prior.

Let $G = (N, S, (A_i)_{i \in N}, \Pi)$ be a game with incomplete information, where $s = (N, (A_i(s))_{i \in N}, (u_i(s))_{i \in N})$ for every $s \in S$, $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i \in N})$, the set of states of the world Y is finite, the set of states of nature S equals the set of states of the world, $S = Y$ with $\mathfrak{s}(\omega) = \omega$, and each player i has a prior distribution $\mathbf{P}_i$ whose support is Y, and a partition $\mathcal{F}_i$ of Y such that $\pi_i(\omega) = \mathbf{P}_i(\omega \mid F_i(\omega))$, for every player i and every state of the world ω.

Let $\mathbf{P}$ be a probability distribution over Y whose support is Y. For each $s \in S$ define a state game $\widehat{s} := (N, (A_i(s))_{i \in N}, (\widehat{u}_i(s))_{i \in N})$, where $\widehat{u}_i(a; \omega) := \frac{\mathbf{P}_i(\omega)}{\mathbf{P}(\omega)} u_i(a; \omega)$. Let $\widehat{S}$ be the collection of all state games $\widehat{s}$ defined in this way. Let $\widehat{G} = (N, \widehat{S}, (A_i)_{i \in N}, \widehat{\Pi})$ be a game with incomplete information, where $\widehat{\mathfrak{s}}(\omega) = \widehat{\mathfrak{s}(\omega)}$, $\widehat{\Pi} = (Y, \mathcal{Y}, \widehat{\mathfrak{s}}, (\widehat{\pi}_i)_{i \in N})$, and for every player $i \in N$ and every $\omega \in Y$, $\widehat{\pi}_i(\omega) := \mathbf{P}(\omega \mid F_i(\omega))$. In words, the game $\widehat{G}$ has a common prior equal to $\mathbf{P}$. Each player i's payoff function at the state of the world ω is his payoff function in the game G multiplied by the ratio $\frac{\mathbf{P}_i(\omega)}{\mathbf{P}(\omega)}$.

Prove that the set of Bayesian equilibria of the game G coincides with the set of Bayesian equilibria of the game $\widehat{G}$.

10.63 In this exercise, we relate the set of Bayesian equilibria in a game G with incomplete information to the set of Bayesian equilibria when we restrict the game to a belief subspace of the belief space of G.

Let $G = (N, S, (A_i)_{i \in N}, \Pi)$ be a game with incomplete information, where $\Pi = (Y, \mathcal{Y}, \mathfrak{s}, (\pi_i)_{i \in N})$ is a belief space with a finite set of states of the world Y, and let $\widetilde{\Pi} = (\widetilde{Y}, \mathcal{Y}_{|\widetilde{Y}}, \mathfrak{s}, (\pi_i)_{i \in N})$ be a belief subspace of Π.

(a) Prove that $\widetilde{G} = (N, S, (A_i)_{i \in N}, \widetilde{\Pi})$ is a game with incomplete information.
(b) Let σ^* be a Bayesian equilibrium of G. Prove that $\sigma^*_{|\widetilde{Y}}$, the strategy vector σ^* restricted to the states of the world in $\widetilde{Y}$, is a Bayesian equilibrium of the game $\widetilde{G}$.

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(c) Let $\widetilde{\sigma}$ be a Bayesian equilibrium of $\widetilde{G}$. Prove that there exists a Bayesian equilibrium σ^* of G satisfying $\widetilde{\sigma}_i(\omega) = \sigma_i^*(\omega)$ for each player $i \in N$ and each state of the world $\omega \in \widetilde{Y}$.

10.64 Prove that the probability distribution defined by Equation (10.57) on page 416 is consistent over the belief space described in Remark 10.47.

10.65 Prove that for probability distributions p over Y whose support is a finite set in Equation (10.56) one may condition on the set $\{\omega' \in Y : \pi_i(\omega') = \pi_i(\omega)\}$ instead of $P_i(\omega)$.

10.66 Prove that, in Example 10.18 (page 398), Equation (10.56) (page 416) is satisfied for each event $A \subseteq Y$, for each player i, and for each $\omega \in Y$.

10.67 Prove that the two definitions of a consistent distribution, Definition 10.46 (page 416) and Definition 10.49 (page 418), are equivalent.

10.68 Prove that Equation (10.64) on page 418 is satisfied for Player II in Example 10.18 (page 398).

10.69 Verify that Equation (10.64) on page 418 is satisfied in Examples 10.17 (page 396) and 10.18 (page 398).

10.70 Prove Theorem 10.50 (page 418): if the set of states of the world is finite, and if p is a consistent distribution, then $\widetilde{Y} = \operatorname{supp}(p)$ is a consistent belief subspace.

10.71 Let $\widetilde{Y}_1$ and $\widetilde{Y}_2$ be two consistent belief subspaces of the same belief space Π, and let p_1 and p_2 be consistent distributions over these two subspaces, respectively. Prove that the set $\widetilde{Y}_1 \cup \widetilde{Y}_2$ is also a consistent belief subspace, and that for each $\lambda \in [0, 1]$ the probability distribution $\lambda p_1 + (1 - \lambda) p_2$ is consistent. In addition, if for each $i \in \{1, 2\}$ we expand p_i to a probability distribution over Y by setting $p_i(\omega) = 0$ for every $\omega \notin \widetilde{Y}_i$, then for every $\lambda \in [0, 1]$ the probability distribution $\lambda p_1 + (1 - \lambda) p_2$ is consistent.

10.72 Let Π be a consistent belief space, and let p be a consistent distribution. Let $\omega \in Y$ be a state of the world satisfying $p(\widetilde{Y}(\omega)) > 0$. Prove that the probability distribution p conditioned on the set $\widetilde{Y}(\omega)$ is consistent. Deduce that $\widetilde{Y}(\omega)$ is a consistent belief subspace.

10.73 Prove or disprove: Every finite belief space has a consistent belief subspace.

10.74 Prove or disprove: If $\widetilde{Y}_i(\omega)$ is inconsistent for some player $i \in N$, then $\widetilde{Y}(\omega)$ is also inconsistent.

10.75 Prove or disprove: If $\widetilde{Y}_i(\omega)$ is inconsistent for every player $i \in N$, then $\widetilde{Y}(\omega)$ is also inconsistent.

10.76 Prove or disprove: If $\widetilde{Y}(\omega)$ is inconsistent then there exists a player $i \in N$ for whom $\widetilde{Y}_i(\omega)$ is inconsistent.

10.77 Provide an example of a belief space Π with three players, which contains a state of the world ω, such that the minimal belief subspaces of the players at ω are inconsistent, and differ from each other.

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10.78 Provide an example of a belief space Π with three players, which contains a state of the world ω, such that the minimal belief subspaces of two of the players at ω are consistent and differ from each other, but the minimal belief space of the third player is inconsistent.

10.79 Prove that if $\widetilde{Y}$ is a minimal consistent belief subspace, and $\widetilde{Y}'$ is an inconsistent belief subspace, then $\widetilde{Y} \cap \widetilde{Y}' = \emptyset$. Does the claim hold without the condition that $\widetilde{Y}$ is minimal?

10.80 In Example 10.27 on page 402, show that at the state of the world ω_2, the fact that the state of the world is consistent is common belief among the players.

10.81 Consider the following belief space, where the set of players is $N = \{\mathrm{I}, \mathrm{II}\}$, the set of states of nature is $S = \{s_1, s_2\}$, the set of states of the world is $Y = \{\omega_1, \omega_2, \omega_3\}$, and the beliefs of the players are given by the following table:

State of the world	$\mathfrak{s}(\cdot)$	$\pi_{\mathrm{I}}(\cdot)$	$\pi_{\mathrm{II}}(\cdot)$
ω_1	s_1	$\left[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)\right]$	$[1(\omega_1)]$
ω_2	s_1	$\left[\frac{1}{2}(\omega_1), \frac{1}{2}(\omega_2)\right]$	$[1(\omega_3)]$
ω_3	s_2	$[1(\omega_3)]$	$[1(\omega_3)]$

(a) Prove that the state of the world ω_3 is the only consistent state of the world in Y.
(b) Prove that at the state of the world ω_2, Player II believes that the state of the world is consistent.
(c) Prove that at the state of the world ω_1, both players believe that the state of the world is inconsistent.
(d) Prove that at the state of the world ω_1, the fact that the state of the world is inconsistent is not common belief among the players.

10.82 Find an example of a belief space, where the set of players is $N = \{\mathrm{I}, \mathrm{II}\}$, and there is a state of the world at which Player I believes that the state of the world is consistent, while Player II believes the state of the world is inconsistent.

10.83 At the state of the world ω, Player I believes that the state of the world is consistent, while Player II believes the state of the world is inconsistent. Is it possible that ω is a consistent state of the world? Justify your answer.

10.84 Prove or disprove: If it is common belief among the players at the state of the world ω that the state of the world is consistent, then $\widetilde{Y}_i(\omega) = \widetilde{Y}_j(\omega)$ for every pair of players $i, j \in N$.

10.85 Find an example of a belief space where the set of players consists of two players, and there exists an inconsistent state of the world ω at which $\pi_i(\{\omega\} \mid \omega) = 0$, and each player i believes that the state of the world is inconsistent.

10.86 Prove that if player i believes at the state of the world ω that the state of the world is consistent, then he believes that every player believes that the state of the world

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is consistent. Deduce that in this case player i believes that it is common belief that the state of the world is consistent.

10.87 Two buyers are participating in a first-price auction. Each of them has a private value, located in $[0, 1]$. With regards to each of the following belief situations, in which the two buyers have symmetric beliefs, answer the following questions:

- Ascertain whether the beliefs of the buyers are consistent. Prove your reply.
- Find a Bayesian equilibrium.

(a) The buyer whose private value is x believes:
- If $x \in [0, \frac{1}{2}]$, the buyer believes that the private value of the other buyer is given by the uniform distribution over $[0, \frac{1}{2}]$.
- If $x \in (\frac{1}{2}, 1]$, the buyer believes that the private value of the other buyer is given by the uniform distribution over $[\frac{1}{2}, 1]$.

(b) A buyer whose private value is x believes:
- If $x \in [0, \frac{1}{2}]$, the buyer believes that the private value of the other buyer is given by the uniform distribution over $[\frac{1}{2}, 1]$.
- If $x \in (\frac{1}{2}, 1]$, the buyer believes that the private value of the other buyer is given by the uniform distribution over $[0, \frac{1}{2}]$.

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