17 The core

Chapter summary

This chapter presents the *core*, which is the most important set solution concept for coalitional games. The core consists of all coalitionally rational imputations: for every imputation in the core and every coalition, the total amount that the members of the coalition receive according to that imputation is at least the worth of the coalition.

The core of a coalitional game may be empty. A condition that characterizes coalitional games with a nonempty core is provided in Section 17.3. A game satisfying this condition is called a *balanced* game and the Bondareva–Shapley Theorem states that the core of a coalitional game is nonempty if and only if the game is balanced. This characterization is used in Section 17.4 to prove that every market game has a nonempty core. A game is called *totally balanced* if the cores of all its subgames are nonempty. It is proved that a game is totally balanced if and only if it is a market game. Similarly, a game is totally balanced if and only if it is the minimum of finitely many additive games.

In Section 17.6 it is proved that the core is a consistent solution concept with respect to the Davis–Maschler definition of a reduced game; that is, for every imputation in the core and every coalition, the restriction of the imputation to that coalition is in the core of the Davis–Maschler reduced game to that coalition.

We introduce two families of coalitional games, *spanning tree games* and *flow games*, and by identifying imputations in the core we prove that games in both families possess a nonempty core.

Finally, in Section 17.10 the notion of the core is extended to any coalitional structure, and we establish a relation between the core of the coalitional game for a coalition structure and the core of the superadditive cover of the game.

Having previously defined what a solution concept in the context of coalitional games means, we proceed by introducing the core, which is a central solution concept for this class of games.

Suppose that the coalition formed by the players is the grand coalition N, and that the players now need to decide how to divide among themselves the worth of the coalition, $v(N)$. As explained on page 674, it is reasonable to assume that this will lead to an imputation in $X(N; v) = \{x \in \mathbb{R}^N : x(N) = v(N), \quad x_i \geq v(i) \quad \forall i \in N\}$. In most cases, there will be a continuum of alternative imputations, and it is natural to ask which imputations are more likely to be implemented. If x is an imputation according to which

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the players divide up $v(N)$, one reasonable assumption is that $x(S) \geq v(S)$ for every[1] coalition S; in words, the total sum received by the players in S should be at least $v(S)$. If this inequality does not hold, the members of S have an incentive to form their own separate coalition and attain $v(S)$, which they can then divide among themselves in such a way that every member i of S receives more than x_i, for example, by dividing the excess $v(S) - x(S)$ equally among the members of the coalition. The concept of the core is based on this idea: the core contains all the imputations x satisfying the property that for every coalition S, the members of S collectively receive at least $v(S)$.

17.1 Definition of the core

Definition 17.1 *Let* $(N; v)$ *be a coalitional game. An imputation* $x \in X(N; v)$ *is* coalitionally rational *if for every coalition* $S \subseteq N$

$$x(S) \geq v(S). \tag{17.1}$$

Definition 17.2 *The* core *of a coalitional game* $(N; v)$*, denoted by* $\mathcal{C}(N; v)$*, is the collection of all coalitionally rational imputations,*

$$\mathcal{C}(N; v) := \{x \in X(N; v) \colon x(S) \geq v(S), \quad \forall S \subseteq N\}. \tag{17.2}$$

The case where the players do not form the grand coalition N but instead divide up into several coalitions is dealt with in Section 17.10 (page 732).

When working with cost games, we reverse the inequalities in the definition of the core.

$$\mathcal{C}(N; c) := \{x \in X(N; c) \colon x(S) \leq c(S), \quad \forall S \subseteq N\}, \tag{17.3}$$

where $X(N; c)$ is the set of imputations in the game $(N; c)$,

$$X(N; c) := \{x \in \mathbb{R}^N \colon x(N) = c(N), x_i \leq c(i), \quad \forall i \in N\}. \tag{17.4}$$

Some simple properties of the core are detailed in the next theorem.

Theorem 17.3 *The core of a coalitional game is the intersection of a finite number of half-spaces, and is therefore a convex set. In addition, the core is a compact set.*

A compact set that is the intersection of a finite number of half-spaces is called a *polytope*. It follows from Theorem 17.3 that the core of a coalitional game is a polytope.

1 Recall that for $x \in \mathbb{R}^N$ we defined $x(S) := \sum_{i \in S} x_i$ for every nonempty coalition S, and $x(\emptyset) := 0$.

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Proof: For each coalition S the set $\{x \in \mathbb{R}^N : x(S) \geq v(S)\}$ is a closed half-space. The core is the intersection of $2^n - 1$ half-spaces $\{x \in \mathbb{R}^N : x(S) \geq v(S)\}$, for all $\emptyset \neq S \subseteq N$, and the half-spaces $\{x \in \mathbb{R}^N : x(N) \leq v(N)\}$.

Every half-space is convex, and the intersection of convex sets is convex. The core is therefore a convex set. Every half-space is closed, and the intersection of closed sets is closed. The core is therefore a closed set. Finally, since the core is a subset of $X(N; v)$, it is bounded. A closed and bounded set is compact. □

Example 17.4 In this example we consider four coalitional games, each with the set of players $N = \{1, 2, 3\}$, that are distinguished from each other solely by the worth of coalition $\{1, 3\}$.

First game

$$v(1) = v(2) = v(3) = 0, \quad v(1,2) = v(2,3) = 1, \quad v(1,3) = 2, \quad v(1,2,3) = 3.$$

The set of imputations is the triangle whose vertices are $(3, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 3)$. An imputation $x = (x_1, x_2, x_3)$ is in the core of this game if and only if

$$x_1 + x_2 + x_3 = 3, \tag{17.5}$$

$$x_1 + x_2 \geq 1, \tag{17.6}$$

$$x_1 + x_3 \geq 2, \tag{17.7}$$

$$x_2 + x_3 \geq 1, \tag{17.8}$$

$$x_1, x_2, x_3 \geq 0. \tag{17.9}$$

The set of solutions to this system of equations, which forms a trapezoid, is depicted in Figure 17.1. The plane in which the figure lies is given by $x_1 + x_2 + x_3 = 3$ in $\mathbb{R}^3$, and the labels in the figure refer to coordinates in $\mathbb{R}^3$.

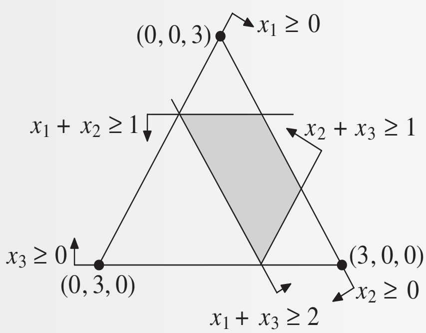

Figure 17.1 First game: the core of the game, and the inequalities defining it

The condition $x_1 + x_2 \geq 1$ is equivalent to the condition $x_3 \leq 2$, because $x_1 + x_2 + x_3 = 3$ and this corresponds to a line parallel to the side $(3, 0, 0) - (0, 3, 0)$ of the triangle in Figure 17.1. The rest of the inequalities can be treated similarly.

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Second game

If the worth of the coalition $\{1, 3\}$ is changed to $v(1, 3) = 1$, the core becomes the hexagon appearing in Figure 17.2.

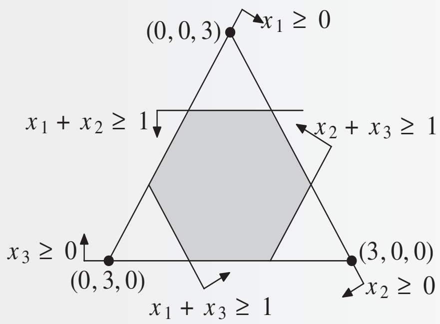

Figure 17.2 Second game: the core of the game, and the inequalities defining it

Third game

If the worth of the coalition $\{1, 3\}$ is changed to $v(1, 3) = 3$, the core becomes the one-dimensional line segment appearing in Figure 17.3.

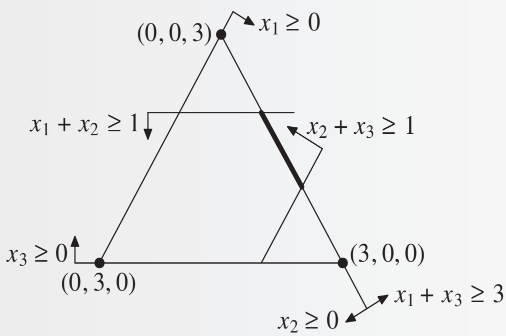

Figure 17.3 Third game the core of the game, and the inequalities defining it

Fourth game

If the worth of the coalition $\{1, 3\}$ is changed to $v(1, 3) = 4$, the core is the empty set.

The above examples do not exhaust all the geometric possibilities of the core. In the following example we exhibit a coalitional game whose core includes a single point. The core of a three-player game may also be a triangle, a parallelogram, or a pentagon (Exercise 17.2). ◀

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Example 17.5 **The gloves game** Consider a three-player game in which the coalitional function is defined as follows.

$$v(1) = v(2) = v(3) = v(1, 2) = 0, \quad v(1, 3) = v(2, 3) = v(1, 2, 3) = 1.$$

This game is called the "gloves game" because it corresponds to a situation in which Players 1 and 2 each have only a right-handed glove, Player 3 has only a left-handed glove, and the worth of a coalition equals the number of complementary pairs of gloves that it can form.

In this game, the set of imputations is given by the triangle whose vertices are $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, and the core contains a single imputation, $(0, 0, 1)$. To see this, if $x = (x_1, x_2, x_3)$ is in the core, then in particular $x_1 + x_2 + x_3 = 1$, $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$. In addition, $x_2 + x_3 \geq v(2, 3) = 1$, and using the efficiency condition $x_1 + x_2 + x_3 = 1$, one deduces that $x_1 \leq 0$. This implies that $x_1 = 0$. Similarly, $x_2 = 0$.

For an intuitive explanation of why the core contains only the imputation $(0, 0, 1)$, note that there is a surplus of right-handed gloves. This leads to a competition that greatly reduces their value: if, for example, Player 3 and Player 1 form a coalition in which Player 1 receives $\alpha > 0$ out of the quantity of 1 that the coalition can attain, Player 2 receives nothing. Player 2 will therefore be willing to form a coalition with Player 3 in return for a payoff that is less than α, say, $\frac{\alpha}{2}$. Knowing this, Player 1 will express readiness to form a coalition with Player 3 for an even smaller payoff, such as $\frac{\alpha}{4}$, and so on. ◀

Example 17.6 **The simple majority game** Consider a simple majority game with three players, where the coalitional function is defined as follows (see Example 16.12 on page 673):

$$v(1) = v(2) = v(3) = 0, \quad v(1, 2) = v(1, 3) = v(2, 3) = v(1, 2, 3) = 1.$$

In this game, again, the set of imputations is given by the triangle whose vertices are $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. In this case, the core is empty. To see this, for x to be in the core, it must be the case that $x_1 + x_2 \geq 1$, $x_1 + x_3 \geq 1$, and $x_2 + x_3 \geq 1$. Summing these three inequalities we deduce that $x_1 + x_2 + x_3 \geq \frac{3}{2}$, which contradicts the efficiency requirement $x_1 + x_2 + x_3 = 1$. ◀

Theorem 17.7 *The core of a coalitional game is covariant under strategic equivalence,*[2] *i.e., for every $a > 0$, and every $b \in \mathbb{R}^N$,*

$$\mathcal{C}(N; av + b) = a\mathcal{C}(N; v) + b. \tag{17.10}$$

As a corollary of Theorem 17.7, we deduce the following corollary.

Corollary 17.8 *The existence of a nonempty core is invariant under strategic equivalence.*

Proof of Theorem 17.7: Let $(N; u)$ be a coalitional game that is strategically equivalent to $(N; v)$: there exist $a > 0$ and $b \in \mathbb{R}^N$ such that

$$u(S) = av(S) + b(S), \quad \forall S \subseteq N. \tag{17.11}$$

2 Recall that for every set $C \subseteq \mathbb{R}^N$, every $a > 0$ and every $b \in \mathbb{R}^N$, the sets aC and $C + b$ are defined by $aC := \{ax : x \in C\}$ and $C + b := \{x + b : x \in C\}$.

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Let $x \in \mathcal{C}(N; v)$. We will show that $ax + b \in \mathcal{C}(N; u)$. Since $x \in \mathcal{C}(N; v)$,

$$x(S) \geq v(S), \quad \forall S \subseteq N, \tag{17.12}$$
$$x(N) = v(N). \tag{17.13}$$

Since $a > 0$,

$$ax(S) + b(S) \geq av(S) + b(S) = u(S), \quad \forall S \subseteq N, \tag{17.14}$$
$$ax(N) + b(N) = av(N) + b(N) = u(N). \tag{17.15}$$

In other words, $ax + b \in \mathcal{C}(N; u)$, and we have therefore shown that $\mathcal{C}(N; u) \subseteq a\mathcal{C}(N; v) + b$.

To show the opposite inclusion, note that since the strategic equivalence relation is symmetric, $(N; v)$ is strategically equivalent to $(N; u)$: indeed, $v = \frac{1}{a}u - \frac{b}{a}$. From the first part, $\mathcal{C}(N; v) \subseteq \frac{1}{a}\mathcal{C}(N; u) - \frac{b}{a}$. By multiplying both sides of the equal sign by a and adding b to both sides, we get $a\mathcal{C}(N; v) + b \subseteq \mathcal{C}(N; u)$. □

Since the core may in some cases be empty, it is natural to ask whether it is possible to characterize the games that have nonempty cores, or at least identify interesting families of games whose core is nonempty. This question is answered in the following sections.

17.2 Balanced collections of coalitions

We begin by seeking a necessary condition for the existence of a nonempty core in three-player games. Suppose $(N; v)$ is a coalitional game with three players, $N = \{1, 2, 3\}$. An imputation x is in the core if and only if the following inequalities hold:

$$x_1 + x_2 + x_3 = v(1, 2, 3), \tag{17.16}$$
$$x_1 + x_2 \geq v(1, 2), \tag{17.17}$$
$$x_1 + x_3 \geq v(1, 3), \tag{17.18}$$
$$x_2 + x_3 \geq v(2, 3), \tag{17.19}$$
$$x_1 \geq v(1), \tag{17.20}$$
$$x_2 \geq v(2), \tag{17.21}$$
$$x_3 \geq v(3). \tag{17.22}$$

We look for necessary conditions that the function v must meet for this system to have a solution. Suppose, therefore, that the core is not empty; i.e., Equations (17.16)–(17.22) have a solution. Combining the inequalities in Equations (17.20), (17.21), and (17.22) and using Equation (17.16) yields the following necessary condition:

$$v(1, 2, 3) \geq v(1) + v(2) + v(3). \tag{17.23}$$

Combining the inequalities in Equations (17.17) and (17.22), and using Equation (17.16), yields the following necessary condition:

$$v(1, 2, 3) \geq v(1, 2) + v(3). \tag{17.24}$$

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We similarly derive the following two inequalities:

$$v(1,2,3) \geq v(1,3) + v(2), \tag{17.25}$$
$$v(1,2,3) \geq v(2,3) + v(1). \tag{17.26}$$

Combining the inequalities in Equations (17.17), (17.18), and (17.19), and using Equation (17.16), yields the following necessary condition:

$$v(1,2,3) \geq \tfrac{1}{2}v(1,2) + \tfrac{1}{2}v(1,3) + \tfrac{1}{2}v(2,3). \tag{17.27}$$

With a little effort (Exercise 17.19) we can prove that if Equations (17.23)–(17.27) hold then there is a solution to the system of Equations (17.16)–(17.22), and therefore the core is nonempty. In other words, these equations constitute necessary and sufficient conditions for the core of a three-player game to be nonempty. Our goal now is to generalize this result to any number of players. To this end, let us look closely at these inequalities.

The inequalities in Equations (17.23)–(17.27) impose the requirement that $v(N)$ be "sufficiently large." The right-hand sides of these equations all contain collections of coalitions multiplied by various coefficients. This is summarized in the following table:

	Collection of coalitions	Coefficients
Equation (17.23)	$\{\{1\}, \{2\}, \{3\}\}$	1, 1, 1
Equation (17.24)	$\{\{1,2\}, \{3\}\}$	1, 1
Equation (17.25)	$\{\{1,3\}, \{2\}\}$	1, 1
Equation (17.26)	$\{\{2,3\}, \{1\}\}$	1, 1
Equation (17.27)	$\{\{1,2\}, \{1,3\}, \{2,3\}\}$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$

The first four collections are partitions of the set of players $\{1, 2, 3\}$, that is, a collection of disjoint coalitions whose union is $\{1, 2, 3\}$. The fifth collection, in contrast, does not satisfy this condition; it corresponds Inequality (17.27), which at this stage in our exposition does not yet have a clear interpretation. From the way that the first four inequalities in Equations (17.23), (17.24), (17.25), and (17.26) are obtained here, we can perceive a necessary condition for the nonemptiness of the core: for every partition $\{S_1, S_2, \ldots, S_k\}$ of N, it must be the case that

$$v(N) \geq v(S_1) + v(S_2) + \cdots + v(S_k). \tag{17.28}$$

Indeed, this inequality holds from the combination of the inequalities $x(S_i) \geq v(S_i)$ for $i = 1, \ldots, k$, and the use of the efficiency requirement $x(N) = v(N)$. This is a form of superadditivity, but it does not imply that $(N; v)$ is necessarily superadditive, because the condition applies only to partitions of N. Focusing again on the five collections in the above table, we will try to find a property common to all of them. To this end, we define the following two concepts that will be used in the sequel.

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Equation	Coalition	Incidence matrix	Coefficients
Equation (1274)	$\{1\}$ $\{2\}$ $\{3\}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	1 1 1
Equation (1275)	$\{1, 2\}$ $\{3\}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	1 1
Equation (1276)	$\{1, 3\}$ $\{2\}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	1 1
Equation (1277)	$\{2, 3\}$ $\{1\}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	1 1
Equation (1278)	$\{1, 2\}$ $\{1, 3\}$ $\{2, 3\}$	$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$	$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

Figure 17.4 The incidence matrices for balanced collections of coalitions in three-player games

Definition 17.9 *For every coalition S, the* incidence vector *of the coalition is the vector $\chi^S \in \mathbb{R}^N$ defined as follows:*

$$\chi_i^S = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases} \tag{17.29}$$

Definition 17.10 *Let $\mathcal{D} = \{S_1, S_2, \ldots, S_k\}$ be a collection of nonempty coalitions. The* incidence matrix *of $\mathcal{D}$ is the matrix with k rows (one for each coalition in the collection) and n columns (one per player), such that the i-th row is the incidence vector of coalition S_i.*

In words, the (i, j)-th entry of the incidence matrix contains 1 if player j is a member of coalition S_i, and 0 if he is not a member of the coalition.

The incidence matrices of the five collections of coalitions mentioned earlier appear in Figure 17.4, with the corresponding coefficients appearing alongside each matrix. For example, the second matrix, corresponding to Equation (17.24) and the collection of coalitions $\{\{1, 2\}, \{3\}\}$, has two rows: the first is the vector $(1, 1, 0)$, which is the incidence vector of the coalition $\{1, 2\}$, and the second is the vector $(0, 0, 1)$, which is the incidence vector of the coalition $\{3\}$.

One can see that in each collection of coalitions, the inner product of each column of the incidence matrix with the column of coefficients is 1. A collection of coalitions that has a vector of positive coefficients satisfying this property is called a *balanced collection*. The coefficients are then called *balancing coefficients*.

Definition 17.11 *A collection of coalitions $\mathcal{D}$ is a* balanced collection *if there exists a vector of positive numbers $(\delta_S)_{S \in \mathcal{D}}$ such that*

$$\sum_{\{S \in \mathcal{D}:\, i \in S\}} \delta_S = 1, \quad \forall i \in N. \tag{17.30}$$

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The vector $(\delta_S)_{S\in\mathcal{D}}$ *is a* vector of balancing weights of the collection. *If all we require is for the coefficients* $(\delta_S)_{S\in\mathcal{D}}$ *to be nonnegative, the corresponding collection of coalitions is called* weakly balanced, *and the coefficients* $(\delta_S)_{S\in\mathcal{D}}$ *are called* weakly balancing weights.

In vector notation, we can define a balanced collection of coalitions using the incidence vectors, as follows: a collection $\mathcal{D}$ of coalitions is balanced with balancing weights $(\delta_S)_{S\in\mathcal{D}}$ if

$$\sum_{S\in\mathcal{D}} \delta_S \chi^S = \chi^N, \quad \text{and} \quad \delta_S > 0 \quad \forall S \in \mathcal{D}. \tag{17.31}$$

The collection is weakly balanced if $\delta_S > 0$ is replaced by $\delta_S \geq 0$ in Equation (17.31). By definition it follows that every balanced collection is weakly balanced, and therefore a collection that is not weakly balanced is not balanced. Note that every partition of N constitutes a balanced collection with weight 1 for every coalition in the partition. This is because Equation (17.30) holds in this case, since every player $i \in N$ appears in one and only one coalition in the partition, and therefore the sum on the left-hand side of the equation is 1 for every player $i \in N$. The other direction does not obtain: it is not true that every balanced collection is a partition. For example, the fifth collection in Figure 17.4, corresponding to Equation (17.27), is a balanced collection that is not a partition.

The concept of a balanced collection, in fact, may be regarded as a generalization of the concept of a partition: suppose that the players could divide their time among the various coalitions; each player i can determine what part of his time he will devote to each coalition to which he belongs. For example, a partition $\mathcal{B}$ corresponds to the situation in which every player i belonging to a coalition S in $\mathcal{B}$ devotes all his time to this coalition. The balanced collection $\mathcal{D}$, with balancing coefficients $(\delta_S)_{S\in\mathcal{D}}$, corresponds to a situation in which every player i devotes δ_S of his time to the coalition S, for every coalition S in $\mathcal{D}$ to which he belongs. The condition $\sum_{\{S\in\mathcal{D}:\, i\in S\}} \delta_S = 1$ guarantees that no player will be idle (it will not be the case that $\sum_{\{S\in\mathcal{D}:\, i\in S\}} \delta_S < 1$), and that no player "puts in overtime hours" (it will not be the case that $\sum_{\{S\in\mathcal{D}:\, i\in S\}} \delta_S > 1$). Every coalition S in the collection will form for a time period δ_S (out of 1), which is the amount of time that each member of the coalition devotes to the coalition.

Example 17.12 **Balanced collections in three-player games** Suppose that $N = \{1, 2, 3\}$. As we saw above, the following collections are balanced collections, $\mathcal{D}_1 = \{\{1\}, \{2\}, \{3\}\}$ with the balancing weights $\delta_{\{1\}} = \delta_{\{2\}} = \delta_{\{3\}} = 1$; $\mathcal{D}_2 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ with the balancing weights $\delta_{\{1,2\}} = \delta_{\{1,3\}} = \delta_{\{2,3\}} = \frac{1}{2}$. We will show that the collection $\mathcal{D} = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$, which is the union of the collections $\mathcal{D}_1$ and $\mathcal{D}_2$, is also a balanced collection. In fact, this collection has an infinite number of vectors of balancing weights,

$$\delta_{\{1\}} = \delta_{\{2\}} = \delta_{\{3\}} = \lambda, \qquad \delta_{\{1,2\}} = \delta_{\{1,3\}} = \delta_{\{2,3\}} = \tfrac{1}{2}(1 - \lambda), \tag{17.32}$$

where $0 < \lambda < 1$.

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This indicates a general way to construct a balanced collection out of two other balanced collections: if $\mathcal{D}_1$ and $\mathcal{D}_2$ are two balanced collections, then their union $\mathcal{D}_1 \cup \mathcal{D}_2$ is also a balanced collection. This follows because if $(\delta^1_S)_{S\in\mathcal{D}_1}$ and $(\delta^2_S)_{S\in\mathcal{D}_2}$ are vectors of balancing weights of the two collections, then for every $0 < \lambda < 1$, the vector of weights $(\delta^*_S)_{S\in\mathcal{D}_1\cup\mathcal{D}_2}$ defined by

$$\delta^*_S = \begin{cases} \lambda\delta^1_S & \text{if } S \in \mathcal{D}_1 \setminus \mathcal{D}_2, \\ (1-\lambda)\delta^2_S & \text{if } S \in \mathcal{D}_2 \setminus \mathcal{D}_1, \\ \lambda\delta^1_S + (1-\lambda)\delta^2_S & \text{if } S \in \mathcal{D}_1 \cap \mathcal{D}_2, \end{cases} \tag{17.33}$$

is a vector of balancing weights for $\mathcal{D}_1 \cup \mathcal{D}_2$ (Exercise 17.20).

The collection $\mathcal{D} = \{\{1,2\},\{1,3\}\}$ is neither balanced, nor weakly balanced. To see this, note that if $(\delta_S)_{S\in\mathcal{D}}$ were a vector of weakly balancing weights for $\mathcal{D}$, then by Equation (17.30), for $i = 1$ one has $\delta_{\{1,2\}} + \delta_{\{1,3\}} = 1$ while for $i = 2$ one has $\delta_{\{1,2\}} = 1$, and for $i = 3$ one has $\delta_{\{1,3\}} = 1$. There is no nonnegative solution for these three equations.

The collection $\mathcal{D} = \{\{1,3\},\{2,3\},\{1\}\}$ is not balanced, but it is weakly balanced. To see this, note that if $(\delta_S)_{S\in\mathcal{D}}$ were a vector of balancing weights for $\mathcal{D}$, then by Equation (17.30) applied to $i = 2$ one has $\delta_{\{2,3\}} = 1$. From Equation (17.30) applied to $i = 3$ it then follows that one has $\delta_{\{1,3\}} = 0$, but for a balanced collection, all the coefficients must be positive numbers. This collection is, however, weakly balanced, with balancing weights $\delta_{\{2,3\}} = \delta_{\{1\}} = 1$, $\delta_{\{1,3\}} = 0$. ◀

Example 17.13 When $N = \{1,2,3,4\}$, the collection $\{\{1,2\},\{2,3\},\{1,3,4\},\{4\}\}$ is a balanced collection, with balancing weights

$$\delta_{\{1,2\}} = \delta_{\{1,3\}} = \delta_{\{1,3,4\}} = \delta_{\{4\}} = \tfrac{1}{2}. \tag{17.34}$$

In contrast, the collection $\{\{1,2\},\{1,3\},\{1,3,4\},\{4\}\}$ is not weakly balanced (why?). ◀

As previously noted, every balanced collection is in particular weakly balanced. Given a weakly balanced collection, one can obtain a balanced collection by removing every coalition whose weight is 0. If $(\delta_S)_{S\in\mathcal{D}}$ is a vector of weakly balancing weights for the collection $\mathcal{D}$, then the collection $\mathcal{T}$ defined as follows is a balanced collection, with balancing weights $(\delta_S)_{S\in\mathcal{T}}$.

$$\mathcal{T} = \{S \in \mathcal{D} \colon \delta_S > 0\}. \tag{17.35}$$

17.3 The Bondareva–Shapley Theorem

Bondareva [1963] and Shapley [1967] independently proved the following theorem, which provides a necessary and sufficient condition for the existence of a nonempty core.

Theorem 17.14 (Bondareva, Shapley) *A necessary and sufficient condition for the core of a coalitional game $(N;v)$ to be nonempty is for every balanced collection $\mathcal{D}$ of coalitions, and every vector of balancing weights $(\delta_S)_{S\in\mathcal{D}}$ for $\mathcal{D}$, to satisfy*

$$v(N) \geq \sum_{S\in\mathcal{D}} \delta_S v(S). \tag{17.36}$$

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This condition is called the *Bondareva–Shapley condition*, or the *balancing condition*. A coalitional game satisfying the Bondareva–Shapley condition is called a *balanced game*, and the Bondareva–Shapley Theorem can therefore be reformulated as stating that the core of a coalitional game is nonempty if and only if the game is balanced.

Remark 17.15 *The Bondareva–Shapley Theorem holds when the words "balanced collection" are replaced by "weakly balanced collection" in its statement, because Inequality (17.36) holds for every balanced collection if and only if it holds for every weakly balanced collection (explain why).* ♦

The Bondareva–Shapley Theorem is not very useful for checking whether the core of a particular game is empty; it is usually more convenient to solve directly the inequalities defining the core. The theorem is useful when one wishes to prove that all the games in a particular class of games have nonempty cores. An example of such a class of games is that of market games, which we will see in Section 17.4 (page 702).

The following claim will be useful for the proof.

Lemma 17.16 *A collection of coalitions $\mathcal{D}$ is balanced with balancing weights $(\delta_S)_{S\in\mathcal{D}}$ if and only if for every vector $x \in \mathbb{R}^N$*

$$\sum_{S\in\mathcal{D}} \delta_S x(S) = x(N). \tag{17.37}$$

Proof: Assume first that $\mathcal{D}$ is a balanced collection of coalitions with balanced weights $(\delta_S)_{S\in\mathcal{D}}$, and let $x \in \mathbb{R}^N$. Then

$$\sum_{S\in\mathcal{D}} \delta_S x(S) = \sum_{S\in\mathcal{D}} \left(\delta_S \sum_{i\in S} x_i \right) \tag{17.38}$$

$$= \sum_{i\in N} \left(x_i \sum_{\{S\in\mathcal{D}:\, i\in S\}} \delta_S \right) \tag{17.39}$$

$$= \sum_{i\in N} x_i = x(N), \tag{17.40}$$

where Equation (17.38) follows from the definition of $x(S)$, Equation (17.39) follows from changing the order of summation, and Equation (17.40) holds because $(\delta_S)_{S\in\mathcal{D}}$ is a vector of balanced weights of $\mathcal{D}$ (Equation (17.30)).

Suppose now that there exists a vector of positive numbers $(\delta_S)_{S\in\mathcal{D}}$ such that Equation (17.37) holds for every $x \in \mathbb{R}^N$. We will show that $\mathcal{D}$ is a balanced collection of coalitions with balancing weights $(\delta_S)_{S\in\mathcal{D}}$. To do so, we need to show that $\sum_{\{S\in\mathcal{D}:\, i\in S\}} \delta_S = 1$ for every player $i \in N$. This equality holds by setting $x = \chi^{\{i\}}$ in Equation (17.37). □

As we showed above, the property of having a nonempty core is invariant under strategic equivalence (Corollary 17.8). The following theorem states that the balancing property is also invariant under strategic equivalence.

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Theorem 17.17 *Let $(N;v)$ and $(N;u)$ be two coalitional games with the same set of players satisfying the condition that $u = av + b$, where $a > 0$ and $b \in \mathbb{R}^N$. The game $(N;v)$ is balanced if and only if the game $(N;u)$ is balanced.*

Proof: It suffices to prove that if the game $(N;v)$ is balanced then the game $(N;u)$ is also balanced (why?). Suppose that $(N;v)$ is balanced, and let $\mathcal{D}$ be a balanced collection with balancing weights $(\delta_S)_{S\in\mathcal{D}}$. Then

$$v(N) \geq \sum_{s\in\mathcal{D}} \delta_S v(S). \tag{17.41}$$

Lemma 17.16 implies that $\sum_{S\in\mathcal{D}} \delta_S b(S) = b(N)$. Since $a > 0$, for every balanced collection $\mathcal{D}$ with balancing weights $(\delta_S)_{S\in\mathcal{D}}$ one has

$$\sum_{s\in\mathcal{D}} \delta_S u(S) = \sum_{s\in\mathcal{D}} \delta_S (av(S) + b(S)) \tag{17.42}$$

$$= a \sum_{s\in\mathcal{D}} \delta_S v(S) + \sum_{s\in\mathcal{D}} \delta_S b(S) \tag{17.43}$$

$$\leq av(N) + b(N) = u(N). \tag{17.44}$$

Since this inequality holds for every balanced collection of coalitions $\mathcal{D}$ with balancing weights $(\delta_S)_{S\in\mathcal{D}}$, it follows that the game $(N;u)$ is balanced. □

We will present two different proofs of the Bondareva–Shapley Theorem: one proof is based on the Minmax Theorem, and the other proof is based on the Duality Theorem from linear programming. Sections 17.3.1 and 17.3.2 are devoted to proving Theorem 17.14 using the Minmax Theorem. The other proof will be presented in Section 17.3.3.

17.3.1 The Bondareva–Shapley condition is a necessary condition for the nonemptiness of the core

Let $x \in \mathcal{C}(N;v)$ be an imputation in the core of the coalitional game $(N;v)$. In particular, $x(N) = v(N)$ and $x(S) \geq v(S)$ for every coalition $S \subseteq N$.

Let $\mathcal{D}$ be a balanced collection of coalitions with balancing weights $(\delta_S)_{S\in\mathcal{D}}$. We will show that Equation (17.36) holds.

The balancing weights are nonnegative (they are, in fact, positive), and therefore one has

$$\delta_S v(S) \leq \delta_S x(S), \quad \forall S \in \mathcal{D}. \tag{17.45}$$

Summing this equation over all $S \in \mathcal{D}$ and making use of Lemma 17.16 and the fact that $x(N) = v(N)$ yields

$$\sum_{S\in\mathcal{D}} \delta_S v(S) \leq \sum_{S\in\mathcal{D}} \delta_S x(S) = x(N) = v(N). \tag{17.46}$$

This means that Equation (17.36) holds.

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17.3.2 The Bondareva–Shapley condition is a sufficient condition for the nonemptiness of the core

The proof of the other direction of the Bondareva–Shapley Theorem is more complicated, and relies on the Minmax Theorem. By Theorem 17.17, the balancing condition is invariant under strategic equivalence, and by Theorem 17.7, the property of having a nonempty core is also invariant under strategic equivalence. Since every game is strategically equivalent to either a $0-1$, $0-0$, or $0-(-1)$ normalized game (Theorem 16.7 on page 670), it suffices to prove this direction of the theorem for each of these three cases. In each case, the proof is by contradiction; we will assume that the game has an empty core, and prove that it is not balanced.

The proof for 0–1 normalized games:
Step 1: Defining the auxiliary game.
We will define an auxiliary two-player zero-sum strategic-form game that will be used throughout the proof of this case. The players in this game are denoted Player I and Player II. The set of (pure) strategies of Player I is $\{1, 2, \ldots, n\}$. We interpret this as Player I choosing a player in the coalitional game $(N; v)$. The set of (pure) strategies of Player II is $\{S \subseteq N : v(S) > 0\}$, the collection of coalitions of positive worth in the coalitional game $(N; v)$. Since the game is $0-1$ normalized, $v(N) = 1$, and therefore there is at least one such coalition. The payoffs in the auxiliary game are

$$u(i, S) = \begin{cases} \frac{1}{v(S)} & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases} \tag{17.47}$$

In words, Player II pays Player I the sum $\frac{1}{v(S)}$ if the player i chosen by Player I is in the coalition S chosen by Player II; otherwise he pays him 0. □

Example 17.18 Suppose that $N = \{1, 2, 3\}$, and that the coalitional function is

$$v(1) = v(2) = v(3) = 0, \quad v(1, 2) = \tfrac{1}{3}, \quad v(1, 3) = v(2, 3) = \tfrac{1}{2}, \quad v(1, 2, 3) = 1.$$

Then the payoff matrix in the auxiliary game is shown in Figure 17.5.

		Player II			
		$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
Player I	1	3	2	0	1
	2	3	0	2	1
	3	0	2	2	1

Figure 17.5 The payoff matrix for Example 17.18 ◀

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The game defined above is a zero-sum game, in which each player has a finite number of pure strategies. It therefore follows from the Minmax Theorem (see Theorem 5.11 on page 151), that the game has a value in mixed strategies. Denote this value by λ.

Step 2: Proving that the value λ of the auxiliary game is positive.
Since every column contains at least one positive entry, Player I's mixed strategy $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, in which he chooses every pure strategy with equal probability, guarantees him a positive payoff that is at least $\frac{1}{n} \min \left\{ \frac{1}{v(S)} : v(S) > 0 \right\}$. It follows that the value of the game in mixed strategies is positive.

Step 3: Proving that the value λ of the auxiliary game is less than 1.
We will prove this by showing that Player I cannot guarantee himself an expected payoff of 1. Let $x = (x_1, x_2, \ldots, x_n)$ be a mixed strategy of Player I. Since the game $(N; v)$ is $0-1$ normalized, the strategy x is also an imputation in this game. Because the core is empty, x cannot be contained in the core, and therefore there exists a coalition S such that $v(S) > x(S) \geq 0$ (the inequality $x(S) \geq 0$ holds because x is a probability distribution, and therefore every coordinate of x is nonnegative).

Since $v(S) > 0$, it follows that S is a pure strategy for Player II in the auxiliary game that we have constructed. The expected payoff, when Player I plays the mixed strategy x and Player II plays the pure strategy S, is

$$u(x, S) = \sum_{i \in S} x_i u(i, S) = \sum_{i \in S} \frac{x_i}{v(S)} = \frac{x(S)}{v(S)} < 1. \tag{17.48}$$

We deduce from this that for every mixed strategy of Player I, there is a pure strategy of Player II guaranteeing that the payoff will be less than 1. By Equation (5.25) (page 151) it then follows that $\lambda < 1$, as claimed.

Step 4: The coalitional game $(N; v)$ is not balanced. That is, there exists a balanced collection $\mathcal{D}$, with balancing weights $(\delta_S)_{S \in \mathcal{D}}$, satisfying

$$v(N) < \sum_{S \in \mathcal{D}} \delta_S v(S). \tag{17.49}$$

A mixed strategy of Player II in the auxiliary game is a probability distribution over coalitions with positive worth. Let $y = (y_S)_{\{S:\, v(S)>0\}}$ be an optimal strategy of Player II in the auxiliary game. Such a strategy guarantees that the expected payoff will be at most λ, for every mixed strategy of Player I.

Consider the following collection of coalitions,

$$\mathcal{D} = \{S \subseteq N : v(S) > 0\} \cup \{\{1\}, \{2\}, \ldots, \{n\}\}, \tag{17.50}$$

with weights

$$\delta_S = \frac{y_S}{\lambda v(S)}, \qquad v(S) > 0, \tag{17.51}$$

$$\delta_{\{i\}} = 1 - \sum_{\{S \subseteq N :\, i \in S, v(S) > 0\}} \delta_S, \qquad \forall i \in N. \tag{17.52}$$

Since the game is $0-1$ normalized, $v(i) = 0$, and therefore no coalition appears more than once in $\mathcal{D}$.

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We next show that $\mathcal{D}$ is a weakly balanced collection with balancing weights $(\delta_S)_{S\in\mathcal{D}}$ defined by Equations (17.51)–(17.52). To do so, we need to show that the sum of the weights of the coalitions containing each player i equals 1, and that the weights are nonnegative.

The definition of δ_S implies that for every player $i \in N$

$$\sum_{\{S\subseteq N:\, i\in S, v(S)>0\}} \delta_S + \delta_{\{i\}} = 1, \tag{17.53}$$

and the first property therefore obtains. We next show that the weights are nonnegative. By Equation (17.51), for every coalition S such that $v(S) > 0$, the weight δ_S is the quotient of the nonnegative number y_S and the positive number $\lambda v(S)$, and is therefore nonnegative. As for the weights $(\delta_{\{i\}})_{i\in N}$,

$$\sum_{\{S\subseteq N:\, i\in S, v(S)>0\}} \delta_S = \frac{1}{\lambda} \sum_{\{S\subseteq N:\, i\in S, v(S)>0\}} \frac{y_S}{v(S)} = \frac{1}{\lambda} u(i, y) \le \frac{\lambda}{\lambda} = 1, \tag{17.54}$$

where the inequality follows from the fact that y is an optimal strategy for Player II in the auxiliary game whose value is λ. By definition of $\delta_{\{i\}}$ (see Equation (17.52)), we deduce that $\delta_{\{i\}} \ge 0$, which is what we wanted to show.

We next show that the Bondareva–Shapley condition is not satisfied by the weakly balanced collection $\mathcal{D}$ and by the weights $(\delta_S)_{S\in\mathcal{D}}$. Indeed,

$$\sum_{S\in\mathcal{D}} \delta_S v(S) = \sum_{j\in N} \delta_{\{j\}} v(j) + \sum_{\{S\subseteq N:\, v(S)>0\}} \delta_S v(S) \tag{17.55}$$

$$= 0 + \sum_{\{S\subseteq N:\, v(S)>0\}} \frac{y_S}{\lambda v(S)} v(S) \tag{17.56}$$

$$= \frac{1}{\lambda} \sum_{\{S\subseteq N:\, v(S)>0\}} y_S = \frac{1}{\lambda} \cdot 1 > 1 = v(N). \tag{17.57}$$

In other words, $v(N) < \sum_{S\in\mathcal{D}} \delta_S v(S)$. Removing from $\mathcal{D}$ all the coalitions whose weight is 0, we are left with a balanced collection that does not satisfy the Bondareva–Shapley condition. We have shown that if the core is empty, the game is not balanced, and this completes the proof that the Bondareva–Shapley condition is a necessary condition for the nonemptiness of the core, in the case of $0-1$ normalized games.

The proof for 0–0 normalized games: We will now show that if a coalitional game $(N; v)$ is $0-0$ normalized and has an empty core, then it is not balanced. When a coalitional game is $0-0$ normalized, the only imputation is $x = (0, 0, \dots, 0)$. It therefore follows that if the core is empty, there exists a coalition S satisfying $v(S) > 0$. Denote by $\{i_1, i_2, \dots, i_{n-|S|}\}$ the set of players who are not in S, i.e., $N = S \cup \{i_1, i_2, \dots, i_{n-|S|}\}$. Define the collection of coalitions

$$\mathcal{D} = \{S, \{i_1\}, \{i_2\}, \dots, \{i_{n-|S|}\}\}. \tag{17.58}$$

This is a partition of N, and the collection is therefore balanced with balancing weights

$$\delta_R = 1, \quad \forall R \in \mathcal{D}. \tag{17.59}$$

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But

$$\sum_{R\in\mathcal{D}} \delta_R v(R) = v(S) + \sum_{i\notin S} v(i) = v(S) > 0 = v(N), \tag{17.60}$$

and the game is therefore not balanced.

The proof for $0-(-1)$ normalized games

The core of a $0-(-1)$ normalized game is empty, because there is no vector $x \in \mathbb{R}^N$ satisfying $x_i \geq v(i) = 0$ and $\sum_{i\in N} x_i = v(N) = -1$. It follows that to prove the sufficiency of the Bondareva–Shapley condition in this case, we need to show that every $0-(-1)$ normalized game is not balanced. Indeed, consider the balanced collection $\{\{1\}, \{2\}, \dots, \{n\}\}$, with balancing weights $\delta_{\{i\}} = 1$ for every $i \in N$. Since $\sum_{i\in N} \delta_{\{i\}} v(i) = 0 > -1 = v(N)$, the game is not balanced. We have shown that the Bondareva–Shapley Theorem holds for $0-0$ normalized games, $0-1$ normalized games, and $0-(-1)$ normalized games, thus concluding the proof of the theorem. □

17.3.3 A proof of the Bondareva–Shapley Theorem using linear programming

The Bondareva–Shapley Theorem (Theorem 17.14) can be proved using the Duality Theorem of linear programming. A brief review of linear programming appears in Section 23.3 (page 945).

Denote by $\mathcal{P}(N) := \{S \subseteq N, S \neq \emptyset\}$ the collection of nonempty coalitions. Denote by P the collection of all weights weakly balancing $\mathcal{P}(N)$,

$$P := \left\{ \delta = (\delta_S)_{S\in\mathcal{P}(N)} \colon \delta_S \geq 0 \quad \forall S \in \mathcal{P}(N), \sum_{S\in\mathcal{P}(N)} \delta_S \chi^S = \chi^N \right\}. \tag{17.61}$$

This set is a polytope in the space $\mathbb{R}^{2^n-1}$, and is nonempty: it contains, for example, the vector δ in which $\delta_{\{i\}} = 1$ for every $i \in N$, and $\delta_S = 0$ for every S containing at least two players.

The following theorem is equivalent to Theorem 17.14 (Exercise 17.28).

Theorem 17.19 (Bondareva, Shapley, second formulation) *A necessary and sufficient condition for the nonemptiness of the core of a coalitional game $(N; v)$ is that*

$$v(N) \geq \sum_{S\in\mathcal{P}(N)} \delta_S v(S), \qquad \forall \delta = (\delta_S)_{S\in\mathcal{P}(N)} \in P. \tag{17.62}$$

Proof of Theorem 17.19 using linear programming: The proof is conducted in steps. We will define a linear program and show that its set of feasible solutions is bounded and nonempty. By the Duality Theorem of linear programming we will deduce that the value of the linear program, Z_P, is equal to the value of its dual program, Z_D. We will prove that the core is nonempty if and only if $Z_D \leq v(N)$, and conclude by proving that $Z_P \leq v(N)$ if and only if Equation (17.62) holds.

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Step 1: Defining a linear program.
Consider the following linear program with the variables $(\delta_S)_{S\in\mathcal{P}(N)}$.

$$\begin{array}{ll} \text{Compute:} & Z_P := \max \sum_{S\in\mathcal{P}(N)} \delta_S v(S), \\ \text{subject to:} & \sum_{\{S:\, i\in S\}} \delta_S = 1, \quad \forall i \in N, \\ & \delta_S \geq 0, \quad \forall S \in \mathcal{P}(N). \end{array}$$

The set of feasible solutions of this linear program is the set P defined in Equation (17.61). As previously noted, this set is compact and nonempty; hence Z_P is finite.

Step 2: The dual problem.
The dual problem is the following problem with the variables $(x_i)_{i\in N}$ (verify!).

$$\begin{array}{ll} \text{Compute:} & Z_D := \min x(N), \\ \text{subject to:} & x(S) \geq v(S), \quad \forall S \in \mathcal{P}(N). \end{array}$$

As already shown Z_P is finite, and therefore the Duality Theorem of linear programming (Theorem 23.46 on page 950) implies that Z_D is also finite, and equals Z_P.

Step 3: If the core is not empty then $Z_D \leq v(N)$.
Let x be a vector in the core. Then $x(S) \geq v(S)$ for every coalition S, and therefore x satisfies all the constraints of the dual problem. The value of the objective function at x is $x(N) = v(N)$; hence $Z_D \leq v(N)$.

Step 4: If $Z_D \leq v(N)$ then the core is not empty.
Let x be a feasible solution of the dual problem at which the minimum is attained, i.e., $x(N) = Z_D$. Since x satisfies the constraints of the dual problem, it is coalitionally rational. We show that $x(N) = v(N)$. Since $Z_D \leq v(N)$, it follows that $x(N) = \sum_{i\in N} x_i = Z_D \leq v(N)$. For $S = N$, the constraint $x(S) \geq v(S)$ is $x(N) \geq v(N)$, so that we deduce that $x(N) = v(N)$. It follows that x is in the core, and therefore the core is not empty.

Step 5: $Z_P \leq v(N)$ if and only if Equation (17.62) holds.
$Z_P \leq v(N)$ if and only if $\sum_{\{S\in\mathcal{P}(N):\, i\in S\}} \delta_S v(S) \leq v(N)$ for every feasible solution $\delta = (\delta_S)_{S\in\mathcal{P}(N)}$, i.e., if and only if Equation (17.62) holds. □

17.4 Market games

In this section, we will concentrate on coalitional games that naturally arise in the study of economics and apply the Bondareva–Shapley Theorem to prove the nonemptiness of the core of these games. The economic model we will study is the model of a market with a set of producers $N = \{1, 2, \dots, n\}$ who trade l commodities. The set of commodities is denoted by $L = \{1, 2, \dots, l\}$. The goods produced can be of different types: metals, water, human resources, consultation hours, etc. We will assume that the final goods that the

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producers offer have fixed prices in the market, and it is therefore convenient to analyze production as if it involves the production of money.

Denote the nonnegative real numbers by $\mathbb{R}_+ := [0, \infty)$. A vector of commodities is denoted by $x = (x_j)_{j=1}^L \in \mathbb{R}_+^L$; i.e., we assume that the quantity of each commodity is nonnegative. Such a vector x is called a *bundle*. The bundle of producer i will be denoted by x_i, and the quantity of commodity j in this bundle will be denoted by $x_{i,j}$.

Each producer has at his disposal a "production technology" represented by a production function $u_i : \mathbb{R}_+^L \to \mathbb{R}$: if $x_i \in \mathbb{R}_+^L$ is the bundle of commodities owned by producer i, then that producer can produce the sum of money $u_i(x_i)$. Since the production technologies may differ from one producer to another, their production functions may also differ. We also assume that every producer i has an initial endowment that is a bundle $a_i \in \mathbb{R}_+^L$ of goods, and the producers can trade goods between each other.

If coalition S is formed, the members of S trade commodities among themselves, with the goal of maximizing the money that they can produce. In other words, if the coalition S is formed, the total bundle of goods available to the coalition is $a(S) := \sum_{i \in S} a_i \in \mathbb{R}_+^L$. The coalition can allocate to each of its members a bundle $x_i \in \mathbb{R}_+^L$, subject to the constraint

$$x(S) = \sum_{i \in S} x_i = \sum_{i \in S} a_i = a(S). \tag{17.63}$$

Hence, by this reallocation of commodities, the members of the coalition can together produce an amount of money equal to $\sum_{i \in S} u_i(x_i)$.

Formally, a market is defined as follows.

Definition 17.20 *A* market *is given by a vector* $(N, L, (a_i, u_i)_{i \in N})$ *where:*

- $N = \{1, 2, \ldots, n\}$ *is the set of producers.*
- $L = \{1, 2, \ldots, l\}$ *is the set of commodities.*
- *For every* $i \in N$, $a_i \in \mathbb{R}_+^L$ *is the initial endowment of producer* i.
- *For every* $i \in N$, $u_i : \mathbb{R}_+^L \to \mathbb{R}$ *is the production function of producer* i.

The assumption that $a_i \in \mathbb{R}_+^L$ for every $i \in N$ implies that there is a finite amount of each commodity in the market.

Definition 17.21 *An* allocation *for a coalition* S *is a collection of bundles of commodities* $(x_i)_{i \in S}$, *where* $x_i \in \mathbb{R}_+^L$ *for every producer* $i \in N$, *satisfying* $x(S) = a(S)$.

In words, an allocation is a redistribution of the commodities available to the members of S in their initial endowments. Denote by X^S the set of allocations for coalition S:

$$X^S := \{(x_i)_{i \in S} \colon x_i \in \mathbb{R}_+^L \ \ \forall i \in S, \quad x(S) = a(S)\} \subseteq \mathbb{R}_+^{S \times L}. \tag{17.64}$$

Theorem 17.22 *For every coalition* S, *the set* X^S *is compact.*

Proof: We need to show that X^S is a closed and bounded set. The set X^S is bounded because the total quantity of commodities in the market is bounded: if $x \in X^S$, then

$$0 \leq x_{i,j} \leq \sum_{i \in N} a_{i,j}, \quad \forall i \in N, \forall j \in L. \tag{17.65}$$

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To show that X^S is a closed set, note that every half-space is closed. The set X^S defined by Equation (17.64) is the intersection of half-spaces and therefore closed. □

Every market can be associated with a coalitional game, in which the set of players is the set of producers $N = \{1, 2, \ldots, n\}$, and the worth of each nonempty coalition $S \subseteq N$ is

$$v(S) = \max \left\{ \sum_{i \in S} u_i(x_i) : x = (x_i)_{i \in S} \in X^S \right\}. \tag{17.66}$$

In words, the worth of coalition S is the maximal sum of money that its members can produce if they trade commodities among themselves (without involving the players who are not in S). The coalitional game $(N; v)$ so defined is *the market game derived from the market* $(N, L, (a_i, u_i)_{i \in N})$.

The first question to answer is whether $v(S)$ is well defined, i.e., whether the maximum in Equation (17.66) is attained.

Theorem 17.23 *If for every $i \in N$ the production function u_i is continuous, the maximum at Equation (17.66) is attained for every coalition S.*

Proof: Since all production functions $(u_i)_{i \in N}$ are continuous, the function $\sum_{i \in S} u_i$, as the sum of a finite number of continuous functions, is also a continuous function. Since the maximum of a continuous function over a compact set is always attained, and since the set X^S is compact (Theorem 17.22), we deduce that the maximum at Equation (17.66) is attained. □

Example 17.24 Consider the following market:

- $N = \{1, 2, 3\}$; the market contains three producers.
- $L = \{1, 2\}$; there are two commodities.
- The initial endowments of the producers are

$$a_1 = (1, 0), \qquad a_2 = (0, 1), \qquad a_3 = (2, 2).$$

- The production functions of the producers are

$$u_1(x_1) = x_{1,1} + x_{1,2}, \qquad u_2(x_2) = x_{2,1} + 2x_{2,2}, \qquad u_3(x_3) = \sqrt{x_{3,1}} + \sqrt{x_{3,2}}.$$

The game derived from this market is given as follows. If a coalition contains only one producer, $S = \{i\}$, then the only bundle in X^S is a_i, the initial endowment of producer i. Therefore,

$$v(1) = 1, \qquad v(2) = 2, \qquad v(3) = 2\sqrt{2}.$$

We will compute $v(1, 2, 3)$, and leave the computations of $v(1, 2)$, $v(1, 3)$, and $v(2, 3)$ to the reader (Exercise 17.36). Note that $a_1 + a_2 + a_3 = (3, 3)$. Every unit of commodity 1 contributes equally to the production functions of producers 1 and 2, and every unit of commodity 2 contributes to the production function of producer 2 twice as much as it contributes to producer 1. No production

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loss therefore occurs if nothing is given to producer 1; every quantity of commodities we give him can be given instead to producer 2 without lessening total production at all. If we therefore set the bundle of producer 1 as $x_1 = (0,0)$, and denote by $x_2 = (x_{2,1}, x_{2,2})$ the bundle to producer 2, and by $x_3 = (3 - x_{2,1}, 3 - x_{2,2})$ the bundle to producer 3, then

$$\begin{aligned} &v(1,2,3) \\ &\quad = \max\left\{x_{2,1} + 2x_{2,2} + \sqrt{3 - x_{2,1}} + \sqrt{3 - x_{2,2}} \colon 0 \le x_{2,1} \le 3, 0 \le x_{2,2} \le 3\right\}. \end{aligned}$$

By differentiating the function $x_{2,1} + 2x_{2,2} + \sqrt{3 - x_{2,1}} + \sqrt{3 - x_{2,2}}$ and equating its directional derivatives to 0, we deduce that the optimal allocation is

$$x_1 = (0,0), \qquad x_2 = \left(2\tfrac{3}{4}, 2\tfrac{15}{16}\right), \qquad x_3 = \left(\tfrac{1}{4}, \tfrac{1}{4}\right),$$

and that the worth of the grand coalition $\{1,2,3\}$ is the value of the maximum, which is $v(1,2,3) = 9\frac{5}{8}$. It can be shown (Exercise 17.36) that the coalitional function of the market game derived from the market in this example is

$$\begin{aligned} &v(1) = 1, \quad v(2) = 2, \quad v(3) = 2\sqrt{2}, \quad v(1,2) = 3, \quad v(1,3) = 5\tfrac{1}{2}, \\ &v(2,3) = 8\tfrac{3}{8}, \quad v(1,2,3) = 9\tfrac{5}{8}. \end{aligned}$$

◀

Definition 17.25 *A coalitional game* $(N; v)$ *is a* market game *if there exist a positive number* l*, and for every player* $i \in N$ *an initial endowment* $a_i \in \mathbb{R}^L_+$*, and a continuous and concave production function* $u_i : \mathbb{R}^L_+ \to \mathbb{R}$*, where* $L = \{1, 2, \ldots, l\}$*, such that Equation (17.66) is satisfied*[3] *for every coalition* $S \in \mathcal{P}(N)$.

In other words, a market game is a coalitional game derived from a market in which the production functions are continuous and concave. The assumption that the production functions are continuous and concave is part of the definition of a market game. The proof of the following theorem is left to the reader (Exercise 17.38).

Theorem 17.26 *If* $(N; v)$ *is a market game, then every coalitional game that is strategically equivalent to* $(N; v)$ *is also a market game.*

Theorem 17.27 (Shapley and Shubik [1969]) *The core of a market game is nonempty.*

Proof: The proof of this theorem relies on the Bondareva–Shapley Theorem; we will prove that every market game is a balanced game. For every coalition S, choose an imputation $x^S = (x_i^S)_{i\in S} \in \mathbb{R}^{S\times L}_+$ at which the maximum in Equation (17.66) is attained. This is possible due to Theorem 17.23. Then:

- $x_i^S \in \mathbb{R}^L_+$ for every player i,
- $x^S(S) = \sum_{i\in S} x_i^S = a(S)$,
- and $\sum_{i\in S} u_i(x_i^S) = v(S)$.

3 Recall that $\mathcal{P}(N) = \{S \subseteq N \colon S \neq \emptyset\}$ is the collection of all the nonempty coalitions in N.

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Let $\delta = (\delta_S)_{S \in \mathcal{P}(N)} \in P$ be a vector[4] that weakly balances $\mathcal{P}(N)$. We need to show that

$$v(N) \geq \sum_{S \in \mathcal{P}(N)} \delta_S v(S). \tag{17.67}$$

For every $i \in N$ denote

$$z_i := \sum_{\{S \in \mathcal{P}(N) \colon i \in S\}} \delta_S x_i^S \in \mathbb{R}_+^L. \tag{17.68}$$

Equation (17.68) is a weighted average with weights $(\delta_S)_{\{S \in \mathcal{P}(N) \colon i \in S\}}$ of the bundles $(x_i^S)_{\{S \in \mathcal{P}(N) \colon i \in S\}}$ allocated to producer i in each coalition S in which he participates. We first show that $z = (z_i)_{i \in N}$ is a feasible bundle, i.e., that $z(N) = a(N)$. Note that by the definition of z_i,

$$z(N) = \sum_{i \in N} z_i = \sum_{i \in N} \sum_{\{S \in \mathcal{P}(N) \colon i \in S\}} \delta_S x_i^S. \tag{17.69}$$

By changing the order of summation,

$$z(N) = \sum_{S \in \mathcal{P}(N)} \sum_{i \in S} \delta_S x_i^S = \sum_{S \in \mathcal{P}(N)} \left(\delta_S \sum_{i \in S} x_i^S \right) = \sum_{S \in \mathcal{P}(N)} \delta_S x^S(S). \tag{17.70}$$

Since $x^S(S) = a(S)$, by changing the order of summation, one has

$$z(N) = \sum_{S \in \mathcal{P}(N)} \delta_S a(S) = \sum_{S \in \mathcal{P}(N)} \left(\delta_S \sum_{i \in S} a_i \right) = \sum_{i \in N} \left(a_i \sum_{\{S \in \mathcal{P}(N) \colon i \in S\}} \delta_S \right). \tag{17.71}$$

Since δ is a vector of balancing weights, $\sum_{\{S \in \mathcal{P}(N) \colon i \in S\}} \delta_S = 1$ for every player $i \in N$, and therefore

$$z(N) = \sum_{i \in N} a_i = a(N); \tag{17.72}$$

that is, z is indeed a feasible bundle. By this, and from the definition of the function v (Equation (17.66)) we deduce that

$$v(N) \geq \sum_{i \in N} u_i(z_i). \tag{17.73}$$

4 Recall that P is the set of all vectors that weakly balance $\mathcal{P}(N)$ (see Equation (17.61) on page 701).

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Next, based on Equation (17.68),

$$v(N) \geq \sum_{i \in N} u_i(z_i) \tag{17.74}$$

$$= \sum_{i \in N} u_i \left(\sum_{\{S \in \mathcal{P}(N) \colon i \in S\}} \delta_S x_i^S \right) \tag{17.75}$$

$$\geq \sum_{i \in N} \sum_{\{S \in \mathcal{P}(N) \colon i \in S\}} \delta_S u_i \left(x_i^S\right) \tag{17.76}$$

$$= \sum_{S \in \mathcal{P}(N)} \sum_{i \in S} \delta_S u_i \left(x_i^S\right) \tag{17.77}$$

$$= \sum_{S \in \mathcal{P}(N)} \left(\delta_S \sum_{i \in S} u_i \left(x_i^S\right) \right) \tag{17.78}$$

$$= \sum_{S \in \mathcal{P}(N)} \delta_S v(S), \tag{17.79}$$

where Equation (17.76) follows from the concavity of the functions $(u_i)_{i \in N}$, Equation (17.78) follows from changing the order of summation, and Equation (17.79) holds because $v(S) = \sum_{i \in N} u_i(x_i^S)$. It follows that the game is balanced, and therefore the core is nonempty, which is what we wanted to show. □

Let $(N, L, (a_i, u_i)_{i \in N})$ be a market, and let $(N; v)$ be the market game derived from it. Suppose that some of the producers leave the market, and that the only producers left are the members of the coalition S. This yields a new market, $(S, L, (a_i, u_i)_{i \in S})$. What is the market game derived from this market? If we denote this game by $(S; \widetilde{v})$, then for every coalition $T \subseteq S$ one has

$$\widetilde{v}(T) = \max \left\{ \sum_{i \in T} u_i(x_i) \colon x_i \in \mathbb{R}^L_+ \ \ \forall i \in T, x(T) = a(T) \right\} = v(T). \tag{17.80}$$

In other words, $\widetilde{v}$ is the function v restricted to the members of S. This distinction motivates the following definition.

Definition 17.28 *Let $(N; v)$ be a coalitional game, and let $S \subseteq N$ be a nonempty set of players. The* subgame *$(S; v)$ is the coalitional game where:*

- *The set of players is S.*
- *The coalitional function is the function v restricted to the coalitions contained in S.*

The game $(S; v)$ is also called the game $(N; v)$ restricted to S.

An immediate corollary of the above discussion and Theorem 17.27 is:

Corollary 17.29 *If $(N; v)$ is a market game, then every subgame $(S; v)$ of $(N; v)$ is a market game, and in particular its core is nonempty.*

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Indeed, if $(N, L, (a_i, u_i)_{i\in N})$ is a market from which the coalitional game $(N; v)$ is derived, then the same market restricted to the members of S, that is, the market $(S, L, (a_i, u_i)_{i\in S})$, is a market from which the game $(S; v)$ is derived.

Definition 17.30 *A coalitional game* $(N; v)$ *is* totally balanced *if the core of every subgame of* $(N; v)$ *is nonempty.*

We can reformulate Corollary 17.29 as follows.

Theorem 17.31 (Shapley and Shubik [1969]) *Every market game is totally balanced.*

Example 17.32 Let $(N; v)$ be a coalitional game, where $N = \{1, 2, 3\}$ and the coalitional function v is given by

$$v(1) = v(2) = v(3) = 10, \quad v(1, 2) = v(1, 3) = v(2, 3) = 15, \quad v(1, 2, 3) = 90.$$

This game has a nonempty core. To see this, note, for example, that the vector $x = (30, 30, 30)$ is in the core of the game. On the other hand, the game restricted to the coalition $\{1, 2\}$ is the game $(\{1, 2\}, \widetilde{v})$ where

$$\widetilde{v}(1) = \widetilde{v}(2) = 10, \quad \widetilde{v}(1, 2) = 15,$$

and the core of this game is empty. If follows therefore that the game $(N; v)$ is balanced, but not totally balanced. By Corollary 17.29, we deduce that $(N; v)$ is not a market game: there is no market from which this game can be derived. ◀

17.4.1 The balanced cover of a coalitional game

Let $(N; v)$ be a coalitional game. The inequalities in the Bondareva–Shapley condition indicate that if $v(N)$ is sufficiently large, the core of the game is not empty. It follows that when the core is empty, by enlarging $v(N)$ we may obtain a new game whose core is not empty, and differs from $(N; v)$ only in the worth of the grand coalition N. How large must the worth of the grand coalition N be for the core of the new game to be nonempty? By the Bondareva–Shapley Theorem (Theorem 17.19 on page 701), it suffices to increase to be at least $\max\{\sum_{S\in\mathcal{P}(N)} \delta_S v(S) \colon \delta \in P\}$.

Definition 17.33 *The* balanced cover *of a coalitional game* $(N; v)$ *is the coalitional game* $(N; \widetilde{v})$ *defined by*

$$\widetilde{v}(S) := \begin{cases} v(S) & \text{if } S \neq N, \\ \max\left\{\sum_{S\in\mathcal{P}(N)} \delta_S v(S) \colon \delta \in P\right\} & \text{if } S = N. \end{cases} \tag{17.81}$$

Theorem 17.34 *The coalitional game* $(N; v)$ *has a nonempty core if and only if* $\widetilde{v}(N) = v(N)$*, where* $(N; \widetilde{v})$ *is the balanced cover of* $(N; v)$*.*

Proof: We first show that $\widetilde{v}(N) \geq v(N)$, whether or not the core is empty: $\{N\}$ is a balanced collection with the balancing weight $\delta_N = 1$; hence $\widetilde{v}(N) \geq 1 \cdot v(N) = v(N)$.

To complete the proof, we will show that $v(N) \geq \widetilde{v}(N)$ if and only if the core is not empty. By the Bondareva–Shapley Theorem (see Theorem 17.14) the core of a coalitional game is nonempty if and only if $v(N) \geq \sum_{S\in\mathcal{P}(N)} \delta_S v(S)$ for every $\delta \in P$, i.e., if and only

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if

$$v(N) \geq \max\left\{ \sum_{S \in \mathcal{P}(N)} \delta_S v(S) \colon \delta \in P \right\} = \widetilde{v}(N), \tag{17.82}$$

which is what we wanted to show. □

Example 17.32 shows that a balanced game need not be totally balanced. To obtain a totally balanced game, one needs to guarantee that for every coalition S the worth $v(S)$ is sufficiently large so that the game restricted to S is balanced. How much must the worth of each coalition be increased for the resulting game to be totally balanced, or equivalently for the core of every subgame to be nonempty? Applying the same reasoning we applied to the coalition N to every nonempty coalition S, we deduce that the worth of every coalition S must be at least

$$\max\left\{ \sum_{\{R \subseteq S, R \neq \emptyset\}} \delta_R v(R) \colon \sum_{\{R \subseteq S, R \neq \emptyset\}} \delta_R \chi^R = \chi^S, \delta_R \geq 0 \quad \forall R \subseteq S \right\}. \tag{17.83}$$

Definition 17.35 *The* totally balanced cover *of a coalitional game* $(N; v)$ *is the coalitional game* $(N; \widehat{v})$ *defined as follows. For every nonempty coalition* $S \subseteq N$,

$$\widehat{v}(S) := \max\left\{ \sum_{\{R \subseteq S, R \neq \emptyset\}} \delta_R v(R) \colon \sum_{\{R \subseteq S, R \neq \emptyset\}} \delta_R \chi^R = \chi^S, \delta_R \geq 0 \quad \forall R \subseteq S \right\} \tag{17.84}$$

and $\widehat{v}(\emptyset) := 0$.

We can now characterize when a coalitional game is totally balanced.

Theorem 17.36 *A coalitional game* $(N; v)$ *is totally balanced if and only if* $\widehat{v}(S) = v(S)$ *for every coalition* $S \subseteq N$*, where the function* $\widehat{v}$ *is defined by Equation (17.84).*

Proof: The game $(N; v)$ is totally balanced if and only if for every nonempty coalition $S \subseteq N$ the coalitional game $(S; v)$ is balanced. Suppose that $S \subseteq N$ is a nonempty coalition. Theorem 17.34 implies that the game $(S; v)$ is balanced if and only if

$$v(S) = \max\left\{ \sum_{R \in \mathcal{P}(N)} \delta_R v(R) \colon \delta \in P \right\} = \widehat{v}(S). \tag{17.85}$$

Since $v(\emptyset) = 0 = \widehat{v}(\emptyset)$, if follows that the game $(N; v)$ is totally balanced if and only if $v(S) = \widehat{v}(S)$ for every coalition $S \subseteq N$, as claimed. □

17.4.2 Every totally balanced game is a market game

Corollary 17.29 states that the core of every subgame of a market game is nonempty. As Example 17.32 shows, there are games with nonempty cores that are not market games. We can now prove the following theorem, which is the converse to Corollary 17.29.

Theorem 17.37 *Every totally balanced game is a market game.*

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Proof: Let $(N; v)$ be a totally balanced game. We need to show that $(N; v)$ is a market game. To do so, we will define a market and show that $(N; v)$ is the market game derived from this market.

If $(N; v)$ is a totally balanced game, then every coalitional game strategically equivalent to $(N; v)$ is also totally balanced. It therefore suffices to prove the theorem for 0-normalized games. Recall that a coalitional game $(N; v)$ is 0-normalized if $v(i) = 0$ for every player $i \in N$.

Step 1: Defining the market.
Let $(N; v)$ be a totally balanced and 0-normalized game. We will construct a market in which both the set of producers and the set of commodities are the set of players N; i.e., $N = L = \{1, 2, \dots, n\}$. The space of bundles is therefore $\mathbb{R}^N_+$, with every coordinate associated with the name of a player. The initial endowment of player i is one unit of the commodity "associated with him," that is,

$$a_i := \chi^{\{i\}}, \quad \forall i \in N. \tag{17.86}$$

If we interpret the commodities as the labor time of the various players (over a certain time period), then the initial endowment of player i is the amount of labor time that he can give, namely one unit. In particular, we deduce that the sum total of commodities that the members of each coalition S have is

$$a(S) = \chi^S, \quad \forall S \subseteq N. \tag{17.87}$$

Define a production function $u \colon \mathbb{R}^N_+ \to \mathbb{R}$, as follows:

$$u(x) := \max\left\{ \sum_{S \in \mathcal{P}(N)} \delta_S v(S) \colon \sum_{S \in \mathcal{P}(N)} \delta_S \chi^S = x, \delta_S \geq 0 \quad \forall S \subseteq N \right\}. \tag{17.88}$$

We can interpret this production function as follows. For every coalition S, there exists an economic activity yielding the income $v(S)$ for every unit of time that the members of S (all together) give to that activity. If every player $i \in S$ gives δ_S of his time to the economic activity of coalition S, the coalition S is active δ_S of the time, and produces $\delta_S v(S)$. In this case, all the coalitions together can produce the total profit of $\sum_{S \in \mathcal{P}(N)} \delta_S v(S)$. If the vector $x = (x_i)_{i \in N}$ represents the amount of time that each player has, then the constraint on the amount of time that the players of the different coalitions have is $\sum_{S \in \mathcal{P}(N)} \delta_S \chi^S = x$. Under this interpretation, $u(x)$ is the maximum that the players can produce together when the amount of time available to each player i is x_i.

Note that the function u is well defined: the set

$$\left\{ \sum_{S \in \mathcal{P}(N)} \delta_S \chi^S = x, \delta_S \geq 0 \quad \forall S \subseteq N \right\}$$

is not empty since it contains the vector $(\delta_S)_{S \in \mathcal{P}(N)}$ that is defined by

$$\delta_S := \begin{cases} x_i & S = \{i\}, \\ 0 & |S| \geq 2. \end{cases} \tag{17.89}$$

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The function u is defined to be the production function of every player:

$$u_i(x) := u(x), \quad \forall i \in N. \tag{17.90}$$

A useful property of the function u is:

Lemma 17.38 $u(x+y) \geq u(x) + u(y)$ *for every* $x, y \in \mathbb{R}^N_+$.

Proof: Denote by $\alpha = (\alpha_T)_{T \in \mathcal{P}(N)}$ the weight vector at which the maximum is attained in the definition of $u(x)$,

$$u(x) = \sum_{T \subseteq N} \alpha_T v(T), \quad \alpha_T \geq 0 \ \forall T \subseteq N, \quad \sum_{T \subseteq N} \alpha_T \chi^T = x,$$

and by $\beta = (\beta_T)_{T \in \mathcal{P}(N)}$ the weight vector at which the maximum is attained in the definition of $u(y)$,

$$u(y) = \sum_{T \subseteq N} \beta_T v(T), \quad \beta_T \geq 0 \ \forall T \subseteq N, \quad \sum_{T \subseteq N} \beta_T \chi^T = y.$$

Denote

$$\gamma_T := \alpha_T + \beta_T.$$

Since for every coalition T, $\alpha_T, \beta_T \geq 0$, it follows that $\gamma_T \geq 0$. Moreover,

$$\sum_{T \subseteq N} \gamma_T v(T) = \sum_{T \subseteq N} \alpha_T v(T) + \sum_{T \subseteq N} \beta_T v(T) = x + y.$$

We therefore have that $\gamma = (\gamma_T)_{T \subseteq N}$ is one of the elements in the maximization in the definition of $u(x+y)$, and therefore

$$\begin{aligned} u(x+y) \geq \sum_{T \subseteq N} \gamma_T v(T) &= \sum_{T \subseteq N} \alpha_T v(T) + \sum_{T \subseteq N} \beta_T v(T) \\ &= u(x) + u(y), \end{aligned} \tag{17.91}$$

which is what we wanted to show. □

The production function u is a homogeneous function: for all $\alpha > 0$ and all $x \in \mathbb{R}^N_+$,

$$u(\alpha x) = \alpha u(x) \tag{17.92}$$

(check that this is true). Using this fact and Lemma 17.38 we deduce the following corollary.

Corollary 17.39 *The production function u defined in Equation (17.88) is a concave function.*

The market we have constructed is therefore the market in which:

- The set of players is N.
- The set of commodities is N.
- The initial endowment of player i is $a_i = \chi^{\{i\}}$.
- The production function $u_i(x)$ is the same for all players $i \in N$, and is given by Equation (17.88).

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This market is called the *direct market* corresponding to the coalitional game $(N; v)$. The definition of a market game requires the production function to be continuous and concave. The concavity of the function u follows from Corollary 17.39 while the proof of its continuity is left to the reader (Exercise 17.40).

Step 2: $u(\chi^S) = v(S)$ for every coalition S.
For $x = \chi^S$, Equation (17.88) is equivalent to the definition of $\widehat{v}(S)$ (see Equation (17.84)), i.e., $u(\chi^S) = \widehat{v}(S)$. We have assumed that $(N; v)$ is a totally balanced game, and therefore by Theorem 17.36, $\widehat{v} = v$. It follows that $u(\chi^S) = v(S)$ for every coalition S.

Step 3: Deriving the market game corresponding to the market we have defined.
Having defined the market, we need to prove that the market game derived from it, denoted by $(N; w)$, is the coalitional game $(N; v)$ we started with. In other words, we need to prove that $w(S) = v(S)$ for every $S \subseteq N$, where $w(S)$ is given by

$$w(S) := \max \left\{ \sum_{i \in S} u(x_i) \colon x(S) = a(S) = \chi^S, x_i \in \mathbb{R}^N_+ \right\}. \tag{17.93}$$

We will show that $w(S) \geq v(S)$ and $w(S) \leq v(S)$ for every coalition S.

One possible allocation of $a(S)$ among the members of S is to give the entire set of commodities to one of the players, i.e., $\widehat{x} = (\widehat{x}_i)_{i \in S}$, where $\widehat{x}_{i_0} = \chi^S$ for some player $i_0 \in S$ and $\widehat{x}_i = \vec{0}$ for every $i \in S \setminus \{i_0\}$, where $\vec{0}$ is the vector in $\mathbb{R}^L_+$ all of whose coordinates are 0. By Step 2 this leads to

$$w(S) \geq \sum_{i \in S} u(\widehat{x}_i) = u(\chi^S) = v(S). \tag{17.94}$$

To prove that $w(S) \leq v(S)$, let $x^* = (x_i^*)_{i \in S}$ be an allocation under which the maximum in the definition of $w(S)$ is attained. By Lemma 17.38 (generalized to any finite number of sums), and by Step 2,

$$w(S) = \sum_{i \in S} u(x_i^*) \leq u\left(\sum_{i \in S} x_i^* \right) = u(\chi^S) = v(S). \tag{17.95}$$

This completes the proof of Theorem 17.37. □

Theorems 17.31 and 17.37 imply the following theorem.

Theorem 17.40 *A coalitional game $(N; v)$ is a market game if and only if it is totally balanced.*

17.5 Additive games

In this section we study a family of coalitional games called *additive games* (also called *inessential games*), and show how they are related to totally balanced games.

Definition 17.41 *A coalitional game $(N; v)$ is* additive *if*

$$v(S) = \sum_{i \in S} v(i) \tag{17.96}$$

for every nonempty coalition S.

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Theorem 17.42 *Every additive game is totally balanced.*

Proof: We will show that for every coalition S, the vector $x = (x_i)_{i \in S}$ defined by $x_i = v(i)$ is in the core of the subgame $(S; v)$, and therefore the core of every subgame is nonempty; hence the game is totally balanced.

For every coalition $R \subseteq S$,

$$x(R) = \sum_{i \in R} x_i = v(R), \tag{17.97}$$

and therefore the vector x is coalitionally rational. By plugging $R = S$ into Equation (17.97), we deduce that it is also an efficient vector in the game $(S; v)$. This implies that x is indeed in the core of the game $(S; v)$. □

Theorem 17.43 *Let $(N; v)$ and $(N; u)$ be totally balanced games over the same set of players. Define a coalitional game $(N; w)$ by*

$$w(S) = \min\{v(S), u(S)\}, \quad \forall S \subseteq N. \tag{17.98}$$

Then $(N; w)$ is also a totally balanced game.

Proof: Let $S \subseteq N$ be a nonempty coalition. We will prove that the core of the game $(S; w)$ is nonempty. Suppose without loss of generality that $u(S) \leq v(S)$, and therefore $w(S) = u(S)$. Since the game $(N; u)$ is totally balanced, the core of the subgame $(S; u)$ is nonempty. Let $x \in \mathcal{C}(S; u)$ be an imputation in the core of this game. We will show that $x \in \mathcal{C}(S; w)$. Since x is an imputation in $(S; u)$,

$$x(S) = u(S) = w(S). \tag{17.99}$$

Since x is a coalitionally rational imputation in the game $(S; u)$, for every $R \subseteq S$,

$$x(R) \geq u(R) \geq w(R). \tag{17.100}$$

Hence x is coalitionally rational in $(S; w)$, and therefore $x \in \mathcal{C}(S; w)$; the core of the game $(S; w)$ is nonempty, which is what we wanted to show. □

Theorem 17.44 *A coalitional game $(N; v)$ is totally balanced if and only if it is the minimum of a finite number of additive games.*

Proof:
Step 1: The minimum of a finite number of additive games is a totally balanced game.
By Theorem 17.42 every additive game is totally balanced, and using Theorem 17.43, one obtains by induction over k that the minimum of k totally balanced games is totally balanced.

We now show that every totally balanced game is the minimum of a finite number of additive games. Let $(N; v)$ be a totally balanced game. We will define, for every coalition $S \subseteq N$, a corresponding additive game $(N; v^S)$, and we we will show that $v = \min_{S \subseteq N} v^S$.

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Step 2: The definition of the games $(N; v^S)$ for $S \subseteq N$.
Since $(N; v)$ is a totally balanced game, for every coalition S, the core of $(S; v)$ is nonempty. Choose an imputation $x^S = (x_i^S)_{i \in S} \in \mathcal{C}(S; v)$. In particular,

$$x_i^S \geq v(i), \quad \forall i \in S, \tag{17.101}$$

and

$$\sum_{i \in S} x_i^S = v(S). \tag{17.102}$$

Denote $M = 2 \max\{|v(S)|, S \subseteq N\}$. For every coalition $S \subseteq N$, expand the vector x^S to a vector in $\mathbb{R}^N$ by defining

$$x_i^S = M, \quad \forall i \notin S. \tag{17.103}$$

Note that for $S = \emptyset$, one has $x_i^\emptyset = M$, for every player $i \in N$.

For every coalition $S \subseteq N$, construct an additive game $(N; v^S)$ using the vector x^S, as follows:

$$v^S(R) := \sum_{i \in R} x_i^S, \quad R \subseteq N, R \neq \emptyset, \tag{17.104}$$

$$v^S(\emptyset) := 0. \tag{17.105}$$

By its definition, the game $(N; v^S)$ is additive.

Step 3: $v^S(R) \geq v(R)$ for every pair of coalitions S and R in N.
Consider the following chain of equalities and inequalities:

$$v^S(R) = \sum_{i \in R} x_i^S \tag{17.106}$$

$$= \sum_{i \in R \cap S} x_i^S + \sum_{i \in R \setminus S} x_i^S \tag{17.107}$$

$$\geq v(R \cap S) + \sum_{i \in R \setminus S} x_i^S \tag{17.108}$$

$$= v(R \cap S) + |R \setminus S| M. \tag{17.109}$$

Inequality (17.108) follows from the fact that x^S is in the core of the subgame $(S; v)$, and Equation (17.109) follows from the fact that $x_i^S = M$ for every $i \notin S$.

If $R \subseteq S$, then $R \cap S = R$ and $|R \setminus S| = 0$, and it follows from Equations (17.106)–(17.109) that $v^S(R) \geq v(R)$. If $R \not\subseteq S$ then $|R \setminus S| \geq 1$. By the choice of M, one has $M \geq v(R) - v(R \cap S)$, and, therefore,

$$v^S(R) \geq v(R \cap S) + |R \setminus S| M \geq v(R \cap S) + M \geq v(R). \tag{17.110}$$

Step 4: $\min_{S \subseteq N} v^S(R) = v(R)$ for every coalition $R \subseteq N$.
From Step 3 we know that $\min_{S \subseteq N} v^S(R) \geq v(R)$, and by Equations (17.104) and (17.102),

$$v^R(R) = \sum_{i \in R} x_i^R = v(R). \tag{17.111}$$

Therefore, we also have that $\min_{S\subseteq N} v^S(R) \le v^R(R) = v(R)$. We deduce that $\min_{S\subseteq N} v^S(R) = v(R)$, which completes the proof. □

By Theorems 17.40 and 17.44, we deduce the following corollary.

Corollary 17.45 *A coalitional game $(N; v)$ is a market game if and only if it is the minimum of a finite number of additive games.*

17.6 The consistency property of the core

Consider the three-player coalitional game appearing in Example 17.4:

$$v(1) = v(2) = v(3) = 0, \quad v(1,2) = v(2,3) = 1, \quad v(1,3) = 2, \quad v(1,2,3) = 3.$$

As we saw in Figure 17.1, the imputation $(2, \frac{1}{2}, \frac{1}{2})$ is in the core of this game. Suppose that the players decide to divide the worth of the grand coalition, 3, on the basis of this vector. Suppose now that Player 3 leaves with his share, $\frac{1}{2}$. Can the issue of dividing the total share of $2\frac{1}{2}$ by Players 1 and 2 be open to rediscussion? To answer this question, we will attempt to describe the new situation between players $\{1, 2\}$ as a new game. What is this new game? What is its core? Is $(2, \frac{1}{2})$ in it?

One way to define the new game is as follows.

Definition 17.46 *Let $(N; v)$ be a coalitional game, let S be a nonempty coalition, and let x be an efficient vector in $\mathbb{R}^N$ (so that $x(N) = v(N)$).* The Davis–Maschler reduced game to S relative to x*, denoted by $(S; w_S^x)$, is the coalitional game with the set of players S and a coalitional function*

$$w_S^x(R) = \begin{cases} \max_{Q\subseteq N\setminus S}(v(R\cup Q) - x(Q)) & \emptyset \neq R \subset S, \\ x(S) & R = S, \\ 0 & R = \emptyset. \end{cases} \tag{17.112}$$

The idea behind this definition is the following. $w_S^x(S)$, the sum that the players in S divide among themselves, should equal $x(S)$, which is the total sum that they receive according to x. Since we are defining a coalitional game, we must require that $w_S^x(\emptyset) = 0$. For each coalition R, $\emptyset \neq R \subset S$, when the members of R come to assess what their coalition is worth, they may add partners outside S, as long as they give these partners what they are allocated according to the original vector x. If they choose the set of partners Q, they will generate a worth $v(R \cup Q)$, will pay the members of Q the amount $x(Q)$, and hence be left with $v(R \cup Q) - x(Q)$. The definition assumes that the members of R will choose those partners so as to maximize this amount.

The amount $w_S^x(R)$ is a virtual worth associated with each coalition separately: if two coalitions R_1 and R_2 try to realize their worths under w_S^x and choose as partners Q_1 and Q_2, they may discover that Q_1 and Q_2 are not disjoint sets, which would mean that at least one of the coalitions will be unable to realize its worth. For this reason, it is important in every application to consider carefully the details of the reduced game, and check whether it fits the intended application.

Definition 17.47 *A set solution concept φ satisfies the* Davis–Maschler reduced game property *if for every coalitional game $(N; v)$, for every nonempty coalition $S \subseteq N$, and for every vector $x \in \varphi(N; v)$,*

$$(x_i)_{i \in S} \in \varphi\big(S; w_S^x\big). \tag{17.113}$$

The reduced game property is a consistency property: if the players believe in the solution concept φ, then every set of players S considering redistributing $\sum_{i \in S} x_i$ among its members will refrain from doing so, because the vector $(x_i)_{i \in S}$ is in the solution φ of the game reduced to S.

Theorem 17.48 *The core satisfies the Davis–Maschler reduced game property.*

Proof: Let x be a point in the core of the coalitional game $(N; v)$, and let S be a nonempty coalition. We will show that $(x_i)_{i \in S}$ is in the core of $(S; w_S^x)$. To do so, we need to show that $x(R) \geq w_S^x(R)$ for every $\emptyset \neq R \subset S$, and that $w_S^x(S) = x(S)$.

The second requirement is satisfied by the definition of the Davis–Maschler reduced game. To prove the first requirement, let $R \subset S$ be a nonempty coalition. We want to show that $x(R) \geq w_S^x(R)$. By the definition of $w_S^x(R)$, there exists a coalition $Q \subseteq N \setminus S$ such that $w_S^x(R) = v(R \cup Q) - x(Q)$. Then we have

$$w_S^x(R) = v(R \cup Q) - x(Q) = v(R \cup Q) - x(R \cup Q) + x(R). \tag{17.114}$$

The vector x is in the core of $(N; v)$, hence $x(R \cup Q) \geq v(R \cup Q)$, and therefore

$$x(R) \geq w_S^x(R), \tag{17.115}$$

which is what we wanted to prove. □

Given a solution concept φ, we can ask the converse question: let $x \in \mathbb{R}^N$ be an efficient vector in the game $(N; v)$. If it is known that $(x_i, x_j) \in \varphi(\{i, j\}, w_{\{i,j\}}^x)$ for every pair of players $i \neq j$, does it follow that $x \in \varphi(N; v)$. If the answer to this question is always affirmative, the solution concept φ is said to satisfy the converse reduced game property.

Definition 17.49 *A set-valued solution concept φ satisfies the Davis–Maschler* converse reduced game property *if for every coalitional game $(N; v)$, every preimputation $x \in X^0(N; v)$ that satisfies*

$$(x_i, x_j) \in \varphi\left(\{i, j\}; w_{\{i,j\}}^x\right), \quad \forall i, j \in N, i \neq j \tag{17.116}$$

also satisfies $x \in \varphi(N; v)$.

Having this property satisfied is useful in many cases, when one is seeking to calculate the solution φ to some game by considering two-player games (which are simpler than games involving more than two players).

Theorem 17.50 *The core satisfies the Davis–Maschler converse reduced game property.*

Proof: Let $(N; v)$ be a coalitional game, and let $x \in X^0(N; v)$ be a preimputation satisfying, for every pair of players $i \neq j$,

$$(x_i, x_j) \in \mathcal{C}\left(\{i, j\}, w_{\{i,j\}}^x\right). \tag{17.117}$$

To show that x is in the core of the game $(N; v)$ we need to prove that $x(S) \geq v(S)$ for every coalition $S \subseteq N$. Note that $x(\emptyset) = 0 = v(\emptyset)$. Since x is a preimputation, $x(N) = v(N)$. Let $S \subset N$ be a nonempty coalition. Let $i \in S$ and $j \notin S$. Since (x_i, x_j) is in the core of $(\{i, j\}, w^x_{\{i,j\}})$,

$$x_i \geq w^x_{\{i,j\}}(i). \tag{17.118}$$

By the definition of $w^x_{\{i,j\}}$,

$$x_i \geq w^x_{\{i,j\}}(i) = \max_{Q \subseteq N \setminus \{i,j\}} (v(\{i\} \cup Q) - x(Q)). \tag{17.119}$$

Set $Q = S \setminus \{i\}$. Then Q contains neither i nor j, and it is therefore one of the elements of the maximization in Equation (17.119). This further yields

$$x_i \geq v(\{i\} \cup Q) - x(Q) = v(S) - x(S \setminus \{i\}). \tag{17.120}$$

We deduce from this that

$$x(S) = x_i + x(S \setminus \{i\}) \geq v(S), \tag{17.121}$$

which is what we needed to show. □

17.7 Convex games

The class of convex games was first defined in Shapley [1971].

Definition 17.51 *A coalitional game $(N; v)$ is* convex *if for every pair of coalitions S and T the following holds:*

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T). \tag{17.122}$$

Recall that a superadditive game (Definition 16.8 on page 671) is a coalitional game in which Equation (17.122) holds for every pair of disjoint coalitions S and T, while in a convex game this equation holds for every pair of coalitions S and T. It follows that every convex game is superadditive. This means that the set of convex games is a subset of the set of superadditive games. In fact, it is a proper subset of the set of superadditive games (Exercise 17.45).

The corresponding definition for cost games is the following.

Definition 17.52 *A cost game $(N; c)$ is* convex *if for every pair of coalitions S and T the following holds:*

$$c(S) + c(T) \geq c(S \cup T) + c(S \cap T). \tag{17.123}$$

Remark 17.53 *If $(N; v)$ is a convex game then for every coalition $S \subseteq N$, the subgame $(S; v)$ restricted to the players in S is also a convex game (Exercise 17.46).* ♦

Convex games are characterized by the property that players have an incentive to join large coalitions. The mathematical formulation of this idea is expressed in the following theorem.

Theorem 17.54 *For any coalitional game $(N; v)$ the following conditions are equivalent:*

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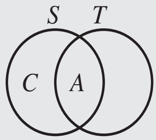

Figure 17.6 The sets S and T in the proof of Theorem 17.54

1. *$(N; v)$ is a convex game.*
2. *For every $S \subseteq T \subseteq N$ and every $R \subseteq N \setminus T$,*

$$v(S \cup R) - v(S) \leq v(T \cup R) - v(T). \tag{17.124}$$

3. *For every $S \subseteq T \subseteq N$ and every $i \in N \setminus T$,*

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T). \tag{17.125}$$

In words, the theorem states that a coalitional game is convex if and only if the marginal contribution of any fixed player i, or of any fixed set of players R, to coalition S rises as more players join S.

Proof: We first prove that Condition 1 implies Condition 2. Suppose that $(N; v)$ is a convex game, that S and T are two coalitions satisfying $S \subseteq T \subseteq N$, and that $R \subseteq N \setminus T$. By Condition 1 the game is convex, and therefore

$$v(S \cup R) + v(T) \leq v(S \cup T \cup R) + v((S \cup R) \cap T). \tag{17.126}$$

Since $S \cup R \cup T = T \cup R$ and $(S \cup R) \cap T = S$ we have

$$v(S \cup R) + v(T) \leq v(T \cup R) + v(S), \tag{17.127}$$

and hence

$$v(S \cup R) - v(S) \leq v(T \cup R) - v(T), \tag{17.128}$$

which is what we needed to show.

That Condition 2 implies Condition 3 is clear: set $R = \{i\}$.

Finally, we show that Condition 3 implies Condition 1. Let S and T be two coalitions. If $S \subseteq T$, then Equation (17.122) holds with equality (explain why). Suppose now that S is not contained in T. Define $A := S \cap T$ and $C := S \setminus T$ (see Figure 17.6). Since S is not contained in T, the set C is nonempty. Let $C = \{i_1, i_2, \ldots, i_k\}$.

Since $T \supseteq A$, $T \cup \{i_1, \ldots, i_l\} \supseteq A \cup \{i_1, \ldots, i_l\}$ for every $l = 0, 1, \ldots, k-1$. Moreover, $i_{l+1} \notin T \cup \{i_1, \ldots, i_l\}$. By Condition 3, for every $l = 0, 1, \ldots, k-1$,

$$\begin{aligned} v(T \cup \{i_1, \ldots, i_l, i_{l+1}\}) - v(T \cup \{i_1, \ldots, i_l\}) &\geq v(A \cup \{i_1, \ldots, i_l, i_{l+1}\}) \\ &\quad - v(A \cup \{i_1, \ldots, i_l\}). \end{aligned}$$

Summing this equation for $l = 0, 1, \ldots, k-1$ yields

$$v(T \cup C) - v(T) \geq v(A \cup C) - v(A). \tag{17.129}$$

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Since $T \cup C = T \cup S$, $A \cup C = S$, and $A = S \cap T$, we obtain

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T). \tag{17.130}$$

Since this inequality holds for every two coalitions S and T, the game is convex. This concludes the proof of Theorem 17.54. □

We next show that the core of a convex game is nonempty. We will show this by identifying a particular imputation that is in the core (actually, we will identify several imputations in the core).

Theorem 17.55 *Let $(N; v)$ be a convex game, and let x be the imputation:*

$$x_1 = v(1), \tag{17.131}$$
$$x_2 = v(1, 2) - v(1) \tag{17.132}$$
$$\dots$$
$$x_n = v(1, 2, \dots, n) - v(1, 2, \dots, n-1). \tag{17.133}$$

Then the vector x is in the core of $(N; v)$.

Proof: First, note that x is an efficient vector:

$$\begin{aligned}\sum_{i \in N} x_i &= v(1) + (v(1, 2) - v(1)) + (v(1, 2, 3) - v(1, 2)) + \cdots + (v(1, 2, \dots, n) \\ &\quad - v(1, 2, \dots, n-1)) \\ &= v(1, 2, \dots, n) = v(N).\end{aligned}$$

We next show that $x(S) \geq v(S)$ for every coalition $S \subseteq N$. Let $S = \{i_1, i_2, \dots, i_k\}$ be a coalition and suppose that $i_1 < i_2 < \cdots < i_k$. Then $\{i_1, i_2, \dots, i_{j-1}\} \subseteq \{1, 2, \dots, i_j - 1\}$ for every $j \in \{1, 2, \dots, k\}$. Theorem 17.54 implies that

$$v(1, 2, \dots, i_j) - v(1, 2, \dots, i_j - 1) \geq v(i_1, i_2, \dots, i_j) - v(i_1, i_2, \dots, i_{j-1}).$$

Hence

$$x(S) = \sum_{j=1}^{k} x_{i_j} \tag{17.134}$$
$$\begin{aligned}&= (v(1, 2, \dots, i_1) - v(1, 2, \dots, i_1 - 1)) + (v(1, 2, \dots, i_2) \\ &\quad - v(1, 2, \dots, i_2 - 1)) + \cdots + (v(1, 2, \dots, i_k) - v(1, 2, \dots, i_k - 1))\end{aligned} \tag{17.135}$$
$$\begin{aligned}&\geq (v(i_1) - v(\emptyset)) + (v(i_1, i_2) - v(i_1)) + \cdots + (v(i_1, i_2, \dots, i_k) \\ &\quad - v(i_1, i_2, \dots, i_{k-1}))\end{aligned} \tag{17.136}$$
$$= v(i_1, i_2, \dots, i_k) = v(S), \tag{17.137}$$

as claimed. □

Remark 17.56 *In the proof of Theorem 17.55, we proved that*

$$(v(1), v(1, 2) - v(1), \dots, v(1, 2, \dots, n) - v(1, 2, \dots, n-1))$$

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is an imputation in the core of the coalitional game $(N; v)$. In that case, we considered the players according to the ordering $1, 2, \ldots, n$. But the same result obtains under any ordering of the players. In other words, given any ordering $\pi = (i_1, i_2, \ldots, i_n)$ of the players, the following is an imputation in the core of the game $(N; v)$:

$$w^\pi := (v(i_1), v(i_1, i_2) - v(i_1), v(i_1, i_2, i_3) - v(i_1, i_2), \ldots, v(N) - v(N \setminus \{i_n\})). \tag{17.138}$$

This imputation corresponds to the following description: the players enter a room one after the other, according to the ordering π. Each player receives the marginal contribution that he provides to the coalition of players who have entered the room before him. The imputation that is arrived at through this process is w^π, and it is in the core of the game. ♦

Definition 17.57 *The convex hull of the imputations $\{w^\pi : \pi$ is a permutation of $N\}$ is called the* Weber set *of the coalitional game $(N; v)$.*

Since the core is a convex set, Theorem 17.55 and Remark 17.56 imply the following theorem.

Theorem 17.58 *In a convex game, the core contains the Weber set.*

Remark 17.59 *Using the Separating Hyperplane Theorem (Theorem 23.39 on page 944), one can prove that the Weber set always contains the core (see Weber [1988] and Derks [1992]). It therefore follows that in a convex game the core coincides with the Weber set.* ♦

The Weber set is a polytope in which there are at most $n!$ vertices, equal to the number of permutations of n players. When $w^{\pi_1} = w^{\pi_2}$ for two different permutations, the number of vertices in this polytope less than $n!$.

Theorem 17.55 has implications for the geometry of the core in convex games. The core is defined as the intersection of the half-spaces $\{x \in \mathbb{R}^N : x(S) \geq v(S)\}$ for each coalition S, and the hyperplane $\{x \in \mathbb{R}^N : x(N) = v(N)\}$. There are games in which the core does not touch some of the hyperplanes $\{x \in \mathbb{R}^N : x(S) = v(S)\}$ defining the half-spaces (see, for example, Figure 17.3, in which the core does not touch the hyperplane $x_1 = 0$). As the next theorem shows, this cannot happen in convex games.

Theorem 17.60 *Let $(N; v)$ be a convex game. Then for every coalition S there exists an imputation $x \in \mathcal{C}(N; v)$ satisfying $x(S) = v(S)$.*

Proof: Order the players such that the elements of S appear first. In other words, denote $S = \{i_1, i_2, \ldots, i_k\}$ and consider the following ordering, in which the players in S appear before the players not in S:

$$\pi = (i_1, i_2, \ldots, i_s, i_{s+1}, \ldots, i_n). \tag{17.139}$$

The imputation w^π is given by

$$w^\pi = (v(i_1), v(i_1, i_2) - v(i_1), \cdots, v(N) - v(N \setminus \{i_n\}))$$

and it satisfies

$$w^{\pi}(S) = v(S). \tag{17.140}$$

As argued in Remark 17.56, w^{π} is in the core, thus completing the proof. □

We have so far presented two families of games in which the core is nonempty. Using the Bondareva–Shapley Theorem we proved that the core of a market game is never empty. In convex games we explicitly find points that are in the core. The next section deals with another family of games with a nonempty core, and we will again explicitly find points in the core.

17.8 Spanning tree games

Spanning tree games were introduced in Section 16.1.7 (page 666). Denote by $\mathbb{R}_{++} := (0, \infty)$ the set of positive real numbers.

Definition 17.61 *A* spanning tree system *is a vector* (N, V, E, v^0, a)*, where:*

- $N = \{1, 2, \ldots, n\}$ *is the set of players.*
- (V, E) *is a finite connected (undirected) graph*[5] *where the set of vertices is* $V = N \cup \{v^0\}$ *and the set of edges is* E*. The vertex* v^0 *is called the* initial vertex *or the* source.
- $a : E \to \mathbb{R}_{++}$ *is a function associating each edge* $e \in E$ *with a cost* $a(e)$ *that is greater than* 0.

Example 17.62 Figure 17.7 depicts a spanning tree system with four players, Detroit, Lansing, Grand Rapids, and Ann Arbor. The graph has five vertices and five edges. The cost associated with each edge is indicated near that edge, and the player associated with each vertex is similarly indicated near the vertex.

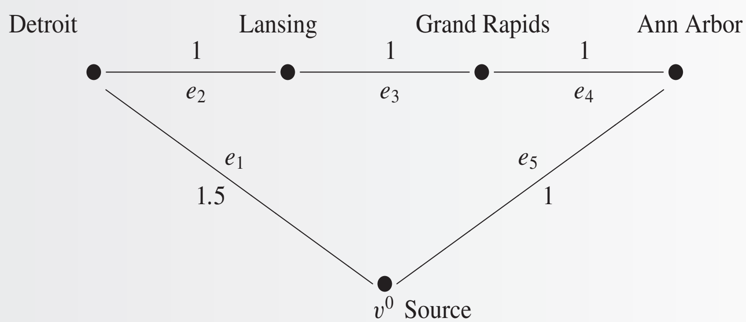

Figure 17.7 The spanning tree system in Example 17.62 ◀

5 Directed graphs were defined in Definition 3.2 (page 41). An *undirected* graph is a pair (V, E) where V is a finite set of vertices and E is a set of undirected edges; each edge is a subset of V of size 2.

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A spanning tree system can describe various networks, such as a road network system connecting towns, or a computer network system. Under this interpretation, every player is either a town or a computer that needs to be connected to a source, which is a central city or a central computer connected to the Internet. The cost associated with each edge is the cost of constructing or maintaining the corresponding connection in the network.

Every spanning tree system can be associated with a *spanning tree game* $(N; c)$, in which the set of players is N and $c(S)$ is the minimal cost of connecting all the members of S to the source v^0, defined as follows. For each coalition S, a spanning graph for S is a collection of edges that contains, for each i in S, a path leading from vertex i to the source v^0. The total cost of this collection is the sum of costs of the edges in the collection. From among all the spanning graphs for S we choose a spanning graph whose cost is minimal (there may be several such collections). This graph must be a tree, i.e., an acyclic graph (why?). We will denote this spanning tree by (V^S, E^S) and call it the *minimal-cost spanning tree of coalition S*. The total cost of this collection is the cost $c(S)$ of coalition S in the game $(N; c)$.

Example 17.62 (*Continued*) The following table (Figure 17.8) depicts several coalitions, the minimal-cost spanning tree corresponding to each coalition, and the cost of constructing the tree.

Coalition	Minimal-cost tree	Total cost
{Detroit}	e_1	1.5
{Grand Rapids}	e_4, e_5	2
{Detroit, Lansing}	e_1, e_2	2.5
{Detroit, Ann Arbor}	e_1, e_5	2.5
{Detroit, Grand Rapids}	e_1, e_2, e_3	3.5
{Detroit, Grand Rapids}	e_1, e_4, e_5	3.5
{Detroit, Grand Rapids, Lansing, Ann Arbor}	e_2, e_3, e_4, e_5	4

Figure 17.8 The minimal-cost tree and its total cost for several coalitions in Example 17.62

The minimal-cost spanning tree for the coalition {Grand Rapids} contains Ann Arbor; Ann Arbor is therefore a “free rider” i.e., it gains connection to the source despite not being a member of the coalition. As the case of the coalition {Detroit} shows, the set of edges in the graph (V^S, E^S) may be disjoint from the set of edges in the graph (V^N, E^N). ◀

A connected graph over a set V of $n+1$ vertices is a tree (i.e., an acyclic graph) if and only if it contains n edges (Exercise 17.56). The minimal-cost tree (V^N, E^N) therefore contains n edges. Such a tree contains a path from each vertex i to the source. Denote by $e(i)$ the first edge in the path from vertex i to the source. Then $e(i) \neq e(j)$ for every pair of distinct vertices i and j, and hence $E^N = \{e(i), i \in N\}$ and $V^N = N \cup \{v^0\}$. In Example 17.62, $e(\text{Detroit}) = e_2$, $e(\text{Lansing}) = e_3$, $e(\text{Grand Rapids}) = e_4$, and $e(\text{Ann Arbor}) = e_5$.

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Coalition S	E^S	$\{e(i)\colon i \notin S\}$	E^*
{Detroit}	e_1	e_3, e_4, e_5	$\{e_1, e_3, e_4, e_5\}$
{Grand Rapids}	e_4, e_5	e_2, e_3, e_5	$\{e_2, e_3, e_4, e_5\}$
{Detroit, Lansing}	e_1, e_2	e_4, e_5	$\{e_1, e_2, e_4, e_5\}$
{Detroit, Ann Arbor}	e_1, e_5	e_3, e_4	$\{e_1, e_3, e_4, e_5\}$
{Detroit, Grand Rapids}	e_1, e_2, e_3	e_3, e_5	$\{e_1, e_2, e_3, e_5\}$
{Detroit, Grand Rapids, Lansing, Ann Arbor}	e_2, e_3, e_4, e_5	$\emptyset$	$\{e_2, e_3, e_4, e_5\}$

Figure 17.9 The set of edges E^* for several coalitions in Example 17.62

Theorem 17.63 *Let (N, V, E, v^0, a) be a spanning tree system. Then the core of the corresponding spanning tree game is nonempty. Moreover, the imputation x defined by*

$$x_i := a(e(i)), \qquad \forall i \in N \tag{17.141}$$

is in the core of the game, where for each player $i \in N$, $e(i)$ is the first edge on the path from vertex i to the source in the minimal[6] *cost spanning tree-for the coalition N.*

Proof: We will show that x is efficient and coalitionally rational. Since $E^N = \{e(i)\}_{i\in N}$, one has $c(N) = \sum_{i\in N} a(e(i)) = x(N)$, and therefore x is an efficient imputation. We next show that $x(S) \leq c(S)$ for every coalition S. Let S be a coalition. Consider the set of edges

$$E^* = E^S \cup \{e(i)\colon i \in N \setminus S\}. \tag{17.142}$$

This set contains all the edges of the minimal-cost spanning tree of coalition S, and for every player who is not in S, it contains the edge emanating from him in the direction of the source in the minimal-cost spanning tree of the coalition N.

The table in Figure 17.9 illustrates the set E^* in the spanning tree system depicted in Figure 17.6 for various coalitions.

The coalition {Detroit, Grand Rapids} has two minimal-cost spanning trees. We have chosen one of them arbitrarily.

We will show that (V, E^*) is a spanning graph; in such a graph, every vertex is connected to the source.[7] Indeed, since the collection of edges E^* contains all the edges of E^S, and since (V^S, E^S) is the minimal-cost spanning tree for S, every vertex in S is connected to the source by a path in E^S (and therefore by a path in E^*). Let v_1 be a player in $N \setminus S$. The edge $e(v_1)$ is in E^*, and connects v_1 to another vertex in the graph, v_2. If $v_2 = v^0$ is the source, v_1 is connected to the source by an edge in E^*. If v_2 is a player in S, then since every vertex in S is connected to the source by a path in E^*, v_1 is also connected to the source by a path in E^*. If not, then neither of these possibilities holds, the edge $e(v_2)$ is in E^*, and it connects v_2 to vertex v_3. Continue the process with v_3. In the k-th stage of the

6 Since the definition of $\{e(i)\}_{i\in N}$ depends on the minimal-cost spanning tree, when there are several minimal-cost spanning trees, one particular minimal-cost spanning tree needs to be chosen, and with respect to that tree one defines $\{e(i)\}_{i\in N}$. The theorem holds true for any choice of a minimal-cost spanning tree.

7 In fact, $(V; E^*)$ is a spanning tree, because it is a connected graph containing n edges and $n+1$ vertices, and hence a tree (Exercise 17.56).

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process, a sequence of vertices $v_1, v_2, v_3, \ldots, v_k$, which are players in $N \setminus S$, is obtained such that the edge $e(v_l)$ is in E^* and connects v_l with v_{l+1} for every $l = 1, 2, \ldots, k-1$. If v_k is the source, or is contained in S, then v_1 is connected to the source via edges in E^*. Otherwise, the sequence can be extended by adding the vertex v_{k+1} that $e(v_k)$ leads to from v_k.

Since the graph has a finite number of vertices, either this process has an end, or is cyclical. If the process is cyclical, there exist positive integers l and k, $l < k$, such that $v_k = v_l$. Then the set of edges $\{e(v_l), e(v_{l+1}), \ldots, e(v_{k-1})\}$ contained in E^N is a cycle, in contradiction to E^N being a tree. The process must therefore end, meaning that for some k the vertex v_k is either the source or is in S. Since every vertex in S is connected to the source by edges in E^*, it follows that the vertex v_1 is connected to the source by edges in E^*.

Since the graph (V, E^*) is a spanning tree for the coalition N, its cost is greater than or equal to $c(N)$. The cost of the edges of this graph equal $c(S) + \sum_{i \notin S} a(e(i))$, and, therefore,

$$x(N) = c(N) \leq c(S) + \sum_{i \notin S} a(e(i)) = c(S) + \sum_{i \notin S} x_i. \tag{17.143}$$

This implies,

$$x(S) = \sum_{i \in S} x_i \leq c(S), \tag{17.144}$$

which is what we needed to show. □

17.9 Flow games

In the previous section we studied spanning tree games in which the worth of each coalition is given by the cost of the minimal-cost spanning tree connecting all the members of the coalition to the source. Another class of games derived from graphs is the class of flow games. A flow game is given by a directed graph in which every edge has a maximal capacity and is controlled by one of the players. The graph contains two distinguished vertices, a source and a sink, and the goal of the players is to direct as great a flow as possible from the source to the sink.

Definition 17.64 *A* flow problem *is described by a vector $F = (V, E, v^0, v^1, c, N, I)$ where:*

- *(V, E) is a directed graph: V is a set of vertices, and E is a set of directed edges, i.e., a set of pairs of vertices $(v, v') \in V \times V$.*
- *$v^0, v^1 \in V$ are two distinguished vertices. v^0 is called the* source *and v^1 is called the* sink.
- *$c : E \to \mathbb{R}_{++}$ is a function associating each edge with a positive number, which represents the maximal* capacity *of the edge.*
- *N is the set of players.*
- *$I : E \to N$ is a function associating each edge with a player who controls it.*

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A flow problem can be thought of as follows. The directed graph describes a toll road system (consisting of one-way roads) leading from one point (a residential area) to another point (a commercial district). Each road has a maximal capacity, and different roads are controlled by different operators.

Example 17.65 Figure 17.10 depicts a flow problem. In this problem, the set of players is $N = \{1, 2, 3\}$, and each edge is labeled (in an adjacent circle) with the player who controls that edge, along with its maximal capacity.

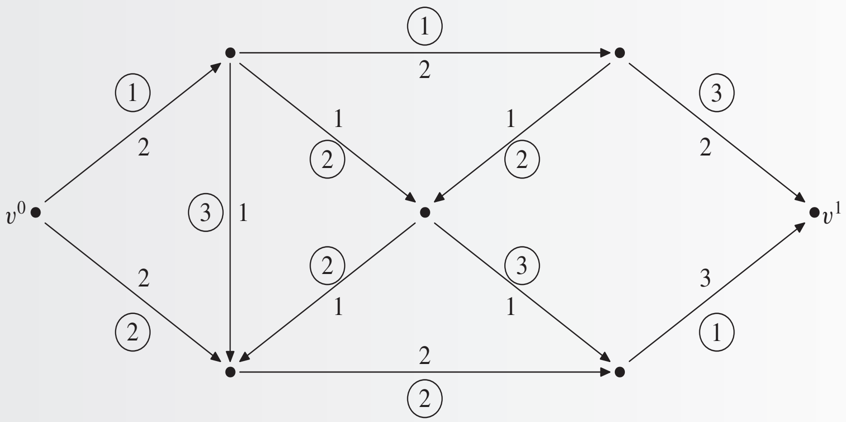

Figure 17.10 The flow problem in Example 17.65 ◀

Definition 17.66 *Let $F = (V, E, v^0, v^1, c, N, I)$ be a flow problem. A* flow *is a function $f : E \to \mathbb{R}_+$ associating each edge in the graph with a positive nonnegative real number and satisfying the following conditions:*

1. *$f(e) \le c(e)$ for every edge $f \in E$: the flow in each edge is not greater than the maximal capacity of the edge.*
2. *$\sum_{\{u \in V \setminus \{v\} :\, (u,v) \in E\}} f(u, v) = \sum_{\{u \in V \setminus \{v\} :\, (v,u) \in E\}} f(v, u)$ for every vertex $v \in V \setminus \{v^0, v^1\}$.*

The magnitude[8] *of a flow f, denoted by $M(f)$, is the total flow arriving at the sink:*

$$M(f) := \sum_{\{u \in V \setminus \{v^1\} :\, (u,v^1) \in E\}} f(u, v^1). \tag{17.145}$$

Since the capacity of each edge is finite, and because one cannot push more flow through an edge than its maximal capacity, the magnitude of the flow is bounded by $\sum_{(u,v^1) \in E} c(u, v^1)$. The flow whose magnitude is maximal (among all possible flows) is called the *maximal flow*.

8 We use the term “magnitude of a flow” instead of the term “value of a flow” because the term “value” has several other meanings in the game theory literature.

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Every flow problem may be described as a coalitional game in which the worth of a coalition S is the maximal amount of flow that the members of S can carry from source to sink without the assistance of players who are not members of S, i.e., the maximal magnitude in the flow problem $F_{|S} = (V, E_{|S}, v^0, v^1, c, S, I)$, where the set of players is S and the set of edges $E_{|S}$ consists of the edges in E controlled by the members of S, that is, $E_{|S} = I^{-1}(S)$.

Definition 17.67 *The coalitional game $(N; v)$ corresponding to the flow problem $F = (V, E, v^0, v^1, c, N, I)$ is the game in which the worth $v(S)$ of a coalition S is the magnitude of the maximal flow of the flow problem $F_{|S}$. A coalitional game $(N; v)$ corresponding to some flow problem is called a* flow game.

Returning to our interpretation of a flow problem as a system of toll roads, and of the capacity as the maximal number of cars that can pass through a road per hour, the flow that the members of S can carry is equal to the maximal number of cars that can pass from v^0 to v^1 per hour by using only roads controlled by the members of S.

Example 17.65 (*Continued*) The flow game corresponding to the flow problem is the following game $(N; u)$ (verify!):

$$\begin{array}{llll} u(1) = 0, & u(2) = 0, & u(3) = 0, & \\ u(1,2) = 2, & u(1,3) = 2, & u(2,3) = 0, & u(1,2,3) = 4. \end{array}$$

◀

The next theorem states that if $(N; v)$ is a flow game corresponding to a flow problem F, then $(S; v)$, the game restricted to coalition S, is the flow game corresponding to the flow problem $F_{|S}$. The proof of the theorem is left to the reader (Exercise 17.63).

Theorem 17.68 *Let $F = (V, E, v^0, v^1, c, N, I)$ be a flow problem, let $(N; v)$ be the corresponding flow game, and let S be a coalition. Then the flow game corresponding to $F_{|S}$ is $(S; v)$.*

In particular, it follows from this theorem that the subgame of a flow game is a flow game. The next theorem is the main result of this section.

Theorem 17.69 *A coalitional game is totally balanced if and only if it is a flow game.*

The theorem is proved in several steps. To prove that every totally balanced game is a flow game, we will prove that every additive game is a flow game (Theorem 17.70), and that the minimum of every two flow games is a flow game (Theorem 17.71). Since every totally balanced game is the minimum of additive games (Theorem 17.44), it follows that every totally balanced game is a flow game (Corollary 17.72). To prove the converse direction, which states that every flow game is a totally balanced game, we will make use of the Ford–Fulkerson Theorem from graph theory (Theorem 17.74).

Theorem 17.70 *Every additive game is a flow game.*

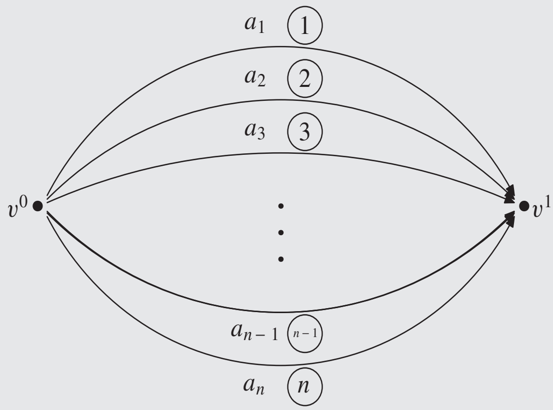

Figure 17.11 The flow problem corresponding to an additive game

Proof: Let $a = (a_i)_{i \in N} \in \mathbb{R}^N$, and let $(N; v)$ be the additive game corresponding to a, i.e., $v(S) = \sum_{i \in S} a_i$ for every nonempty coalition S. Then $(N; v)$ is the flow game corresponding to the flow problem depicted in Figure 17.11.

This flow problem has two nodes, the source v^0 and the sink v^1, and n edges, with each edge corresponding to one player, and the capacity of the edge corresponding to player i is equal to a_i. The magnitude of the maximal flow from v^0 to v^1 that use only edges controlled by coalition S is

$$\sum_{i \in S} a_i = v(S), \tag{17.146}$$

and therefore $(N; v)$ is indeed the game corresponding to this flow problem. □

Theorem 17.71 *The minimum of two flow games over the same set of players is a flow game.*

Proof: Let $(N; v)$ and $(N; \widehat{v})$ be two flow games, and consider two flow problems corresponding to these flow games: $F = (V, E, v^0, v^1, c, N, I)$ and $\widehat{F} = (\widehat{V}, \widehat{E}, \widehat{v}^0, \widehat{v}^1, \widehat{c}, N, \widehat{I})$. Construct a new flow problem by connecting in series these two flow problems (and identifying v^1 with $\widehat{v}^0$).

If, e.g., F is the flow problem depicted in Figure 17.11 (for $n = 3$) and $\widehat{F}$ is the flow problem depicted in Figure 17.10, the resulting new flow problem is depicted in Figure 17.12.

Since $v(S)$ is the magnitude of the maximal flow from v^0 to v^1 that uses only the edges controlled by members of coalition S, and $\widehat{v}(S)$ is the magnitude of the maximal flow from $\widehat{v}^0$ to $\widehat{v}^1$ that uses only the edges controlled by members of this coalition, it follows that the magnitude of the maximal flow from v^0 to $\widehat{v}^1$ that uses only the edges controlled by members of S is $\min\{v(S), \widehat{v}(S)\}$. We deduce that the flow game corresponding to this flow problem is $(N; u)$, in which $u(S) = \min\{v(S), \widehat{v}(S)\}$ for every coalition $S \subseteq N$, which is what we wanted to show. □

Corollary 17.72 *Every totally balanced game is a flow game.*

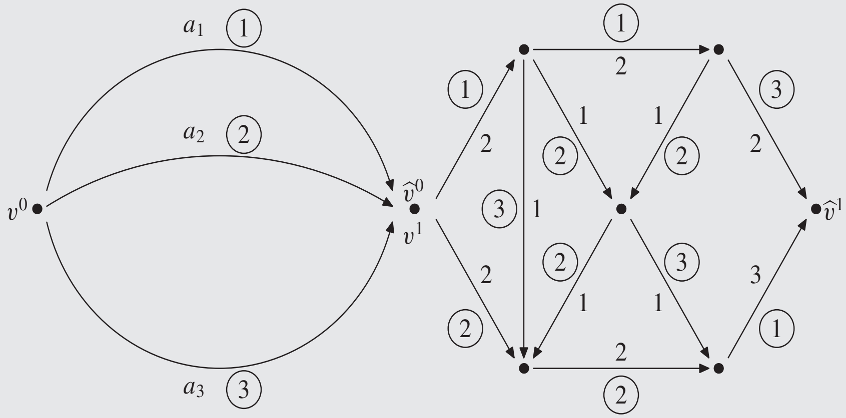

Figure 17.12 The flow problem in Figures 17.11 (with $n = 3$) and 17.10 connected in series

Proof: By Theorem 17.44, every totally balanced game is the minimum of a finite number of additive games. By Theorems 17.70 and 17.71 (using induction over the number of players) the minimum of a finite number of additive games is a flow game. It follows that every totally balanced game is a flow game. □

To complete the proof of Theorem 17.69, we need to prove the converse, i.e., to show that every flow game is totally balanced. This is accomplished using a new definition, and two more theorems.

Definition 17.73 *Let $F = (V, E, v^0, v^1, c, N, I)$ be a flow problem. A* cut *is a set of edges whose removal from the graph will prevent a flow with positive magnitude between v^0 and v^1. The* capacity of a cut *is the sum of the capacities of the edges in the cut.*

In other words, a subset A of E is a cut if the maximal magnitude in the flow problem $F = (V, E \setminus A, v^0, v^1, c, N, I)$ is 0. The capacity of a cut A will be denoted by $C(A) := \sum_{(u,v)\in A} c(u, v)$.

If $A \subseteq E$ is a set of edges, and if there exists a path from v^0 to v^1 that does not contain any edge in A, then A is not a cut. It follows that if a set of edges A is a cut, then every path from v^0 to v^1 contains at least one edge of A.

Theorem 17.74 (Ford and Fulkerson [1956]) *Let $(V, E, v^0, v^1, c, N, I)$ be a flow problem. The capacity of any cut is greater than or equal to the magnitude of the maximal flow of the flow problem. Moreover, there exists a cut whose capacity equals the magnitude of the maximal flow of the flow problem.*

Proof: For every flow f and every two sets of vertices $X, Y \subseteq V$ define

$$f(X, Y) = \sum\{f(u, v) : (u, v) \in E, u \in X, v \in Y\} - \sum\{f(u, v) : (u, v) \in E, u \in Y, v \in X\}. \quad (17.147)$$

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This is the difference between the total flow in the edges connecting the vertices in X to the vertices Y to the total flow in the edges connecting the vertices in Y to the vertices in X. This function has several useful properties:

1. $f(X, X) = 0$ for every set of vertices $X \subseteq V$.
2. $f(X_1 \cup X_2, Y) = f(X_1, Y) + f(X_2, Y)$ for all $Y \subseteq V$ and every pair of disjoint sets $X_1, X_2 \subseteq V$.
3. $f(V, \{v^1\}) = M(f)$.
4. $f(V, \{v\}) = 0$ for every vertex $v \in V \setminus \{v^0, v^1\}$.

The first two properties follow from the definition of $f(\cdot, \cdot)$ and the last two properties follow from the definition of a flow. Recall that for every set $X \subseteq V$, the complement of X is $X^c := V \setminus X$.

We next prove that for every flow f and every set X satisfying $v^0 \in X$ and $v^1 \notin X$,

$$M(f) = f(X, X^c). \tag{17.148}$$

Indeed, from Property (2) of $f(\cdot, \cdot)$,

$$\begin{aligned} f(X, X^c) &= f(V, X^c) - f(X^c, X^c) && (17.149)\\ &= \sum_{v \in X^c \setminus \{v^1\}} f(V, \{v\}) + f(V, \{v^1\}) - f(X^c, X^c) && (17.150)\\ &= M(f). && (17.151) \end{aligned}$$

To obtain Equation (17.151) note that $f(X^c, X^c)$ equals 0 by Property (1) of $f(\cdot, \cdot)$, and $\sum_{v \in X^c \setminus \{v^1\}} f(V, \{v\})$ equals 0 by Properties (4) and (2) of $f(\cdot, \cdot)$, since $X^c \setminus \{v^1\} \subseteq V \setminus \{v^0, v^1\}$.

For any cut A let $X(A)$ be the set of vertices containing the source v^0 and every vertex $v \in V$ such that there exists a path from v^0 to v that does not contain any edge in A. By the definition of a cut, we deduce that $v^1 \notin X(A)$.

Note that if $u \in X(A)$, $v \notin X(A)$, and $(u, v) \in E$, then $(u, v) \in A$. Indeed, since $u \in X(A)$ there exists a path from the source v^0 to u that does not contain any edge in A. If the edge (u, v) were not in A, there would be a path from v^0 to v that does not contain any edge in A, but this contradicts the fact that $v \notin X(A)$. In particular,

$$\{(u, v) \in E : u \in X(A), v \notin X(A)\} \subseteq A. \tag{17.152}$$

Therefore,

$$\begin{aligned} f(X(A), (X(A))^c) &= \sum \{f(u, v) : (u, v) \in E, u \in X(A), v \notin X(A)\}\\ &\quad - \sum \{f(u, v) : (u, v) \in E, u \notin X(A), v \in X(A)\} && (17.153)\\ &\le \sum \{f(u, v) : (u, v) \in E, u \in X(A), v \notin X(A)\} && (17.154)\\ &\le \sum_{(u,v) \in A} f(u, v) && (17.155)\\ &\le \sum_{(u,v) \in A} c(u, v) = C(A), && (17.156) \end{aligned}$$

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where Inequality (17.155) follows from Equation (17.152). Using Equations (17.149)–(17.151) and (17.153)–(17.156), we deduce that for every flow f and every cut A,

$$M(f) \leq C(A). \tag{17.157}$$

This leads to

$$\max_f M(f) \leq \min_A C(A), \tag{17.158}$$

where the maximum is taken over all flows and the minimum is over all cuts. The minimum is attained because the number of cuts in the graph is finite, and the maximum is attained because the set of flows is compact and the function $f \mapsto M(f)$ is continuous. That completes the proof of the first part of Theorem 17.74.

To prove the second part of the theorem, let f^* be a maximal flow. We will show that there exists a cut A such that $M(f^*) = C(A)$. Such a cut A is in particular a cut with minimal capacity.

Define a graph $\widehat{G} = (V, \widehat{E})$ as follows: The edge (u, v) is in $\widehat{E}$ if and only if at least one of the following two conditions holds:

- The edge (u, v) is in E, and $f^*(u, v) < c(u, v)$.
- The edge (v, u) is in E, and $f^*(v, u) > 0$.

In words, if the flow in edge (u, v) is less than its maximal capacity (and therefore more flow can be added to that edge), then we add the edge (u, v) to $\widehat{E}$. If the flow in edges (u, v) is positive (and therefore flow can be removed from that edge), we add the edge in the opposite direction, (v, u), to $\widehat{E}$. In particular, it follows that the set of edges $\widehat{E}$ is nonempty.

We first claim that the graph $\widehat{G}$ contains no path from v^0 to v^1. Suppose by contradiction that there exists such a path σ. Denote

$$\begin{aligned}\varepsilon := \min\Big\{&\min\{c(u, v) - f(u, v) \colon (u, v) \in E, (u, v) \in \widehat{E}, f^*(u, v) < c(u, v)\},\\ &\min\{f^*(v, u) \colon (v, u) \in E, (u, v) \in \widehat{E}, f^*(v, u) > 0\}\Big\} > 0.\end{aligned} \tag{17.159}$$

If the set $\{(u, v) \in E, (u, v) \in \widehat{E}, f^*(u, v) < c(u, v)\}$ is empty, then the first internal minimum on the right-hand side of Equation (17.159) is ∞, and if the set $\{(v, u) \in E, (u, v) \in \widehat{E}, f^*(v, u) > 0\}$ is empty, then the second internal minimum is ∞. Since at least one of these two sets is nonempty, ε is a positive number.

Define a function $\widehat{f} : E \to \mathbb{R}_+$ as follows:

$$\widehat{f}(u, v) = \begin{cases} f^*(u, v) & (u, v) \text{ is not in the path } \sigma,\\ f^*(u, v) + \varepsilon & f^*(u, v) < c(u, v) \text{ and } (u, v) \text{ in the path } \sigma,\\ f^*(u, v) - \varepsilon & f^*(u, v) > 0 \text{ and } (v, u) \text{ in the path } \sigma.\end{cases} \tag{17.160}$$

Since σ is a path from v^0 to v^1 in $\widehat{G}$, the function $\widehat{f}$ is a flow in G, and its magnitude is $M(f^*) + \varepsilon$ (Exercise 17.64). This contradicts the fact that f^* is a maximal flow.

Let X be a set containing v^0 and every vertex v such that there is a path from v^0 to v in the graph $\widehat{G}$. Since there is no path from v^0 to v^1 we deduce that $v^1 \notin X$. Let A be the following cut (why is this a cut?):

$$A := \{(u, v) \in E \colon u \in X, v \notin X\}. \tag{17.161}$$

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Let $u \in X$ and $v \notin X$. Note that $(u, v) \notin \widehat{E}$ by the definition of X. Therefore, if $(u, v) \in E$, then necessarily $f^*(u, v) = c(u, v)$, and if $(v, u) \in E$, then necessarily $f^*(v, u) = 0$. Therefore,

$$M(f^*) = f^*(X, X^c) \tag{17.162}$$
$$= \sum\{f^*(u, v)\colon (u, v) \in E, u \in X, v \in X^c\}$$
$$- \sum\{f^*(u, v)\colon (u, v) \in E, u \in X^c, v \in X\} \tag{17.163}$$
$$= \sum\{c(u, v)\colon (u, v) \in E, u \in X, v \in X^c\} = C(A). \tag{17.164}$$

This completes the proof of Theorem 17.74. □

If $(N; v)$ is a flow game corresponding to the flow problem $(V, E, v^0, v^1, c, N, I)$, then the magnitude of the maximal flow in the graph is $v(N)$. The Ford–Fulkerson Theorem implies that the minimal capacity of a cut in the problem equals $v(N)$.

We now complete the proof of Theorem 17.69 by proving the following theorem.

Theorem 17.75 *Every flow game is a totally balanced game.*

Proof: Let $(N; v)$ be a flow game. We first show that the core of the game $(N; v)$ is nonempty. Let $F = (V, E, v^0, v^1, c, N, I)$ be a flow problem corresponding to $(N; v)$. Let A be a cut of minimal capacity in the flow problem F. For each player $i \in N$, denote by c_i the sum of the capacities of all the edges in A controlled by player i,

$$c_i = \sum_{\{e \in A\colon I(e) = i\}} c(e). \tag{17.165}$$

We will show that the imputation $c = (c_i)_{i \in N}$ is in the core of $(N; v)$. The worth $v(N)$ of coalition N is the magnitude of the maximal flow, which equals the capacity of the minimal cut A by the Ford–Fulkerson Theorem (Theorem 17.74):

$$v(N) = \sum_{i \in N} c_i. \tag{17.166}$$

Thus, c is an efficient vector. We next show that $v(S) \geq \sum_{i \in S} c_i$ for every nonempty coalition $S \subseteq N$. Fix then a nonempty coalition S, and define $A_{|S} = \{e \in A\colon I(e) \in S\}$. These are all the edges in the cut that are controlled by the players in S. The collection $A_{|S}$ is a cut of $F_{|S}$, because every path from v^0 to v^1 using only edges controlled by the members of S must use an edge in $A_{|S}$. By definition, $v(S)$ is the magnitude of the maximal flow in $F_{|S}$, and by the first part of the Ford–Fulkerson Theorem, this quantity is at most the capacity of any cut. It follows that

$$v(S) \leq \sum_{e \in A_{|S}} c(e) = \sum_{i \in S} c_i. \tag{17.167}$$

Since this inequality holds for every nonempty coalition $S \subseteq N$, the vector c is coalitionally rational, and since it is efficient, it is in the core of the game $(N; v)$.

We have therefore proved that the core of every flow game is nonempty. Since every subgame of a flow game is a flow game (Theorem 17.68), it follows that the core of every subgame of a flow game is nonempty, and therefore the game $(N; v)$ is totally balanced. □

Using Theorem 17.40 (page 712), Theorem 17.44 (page 713), and Theorem 17.69 (page 726), we deduce the following corollary.

Corollary 17.76 *The following statements are equivalent for a coalitional game $(N; v)$:*

- $(N; v)$ *is totally balanced.*
- $(N; v)$ *is a market game.*
- $(N; v)$ *is the minimum of a finite number of additive games.*
- $(N; v)$ *is a flow game.*

17.10 The core for general coalitional structures

In this section, we extend the solution concept of the core to cover cases in which the grand coalition N is not formed, and instead players are partitioned into several disjoint coalitions. When several disjoint coalitions are formed, the members of each coalition divide the worth of that coalition among themselves.

Recall that a coalitional structure is a partition $\mathcal{B}$ of the set of players N. In other words, $\mathcal{B}$ is a set of disjoint sets whose union is N. The set of imputations for a coalitional structure $\mathcal{B}$ is the set

$$X(\mathcal{B}; v) := \{x \in \mathbb{R}^N : x(B) = v(B) \quad \forall B \in \mathcal{B}, \quad x_i \geq v(i) \quad \forall i \in N\}. \tag{17.168}$$

In words, an imputation for the coalitional structure $\mathcal{B}$ is an individually rational vector at which the total payoff to the members of each coalition B in the coalitional structure is equal to $v(B)$, the amount that the members of B can obtain on their own.

Definition 17.77 *The* core of a coalitional game $(N; v)$ for a coalitional structure $\mathcal{B}$ *is the set*

$$\mathcal{C}(N; v; \mathcal{B}) := \{x \in X(\mathcal{B}; v) : x(S) \geq v(S) \quad \forall S \subseteq N\}. \tag{17.169}$$

This is the set of imputations for $\mathcal{B}$, such that no coalition of players can profit by forming and producing its worth. The coalitional rationality condition applies to all coalitions, not only to the subcoalitions of the coalitions in $\mathcal{B}$. For $\mathcal{B} = \{N\}$, the set $\mathcal{C}(N; v; \mathcal{B})$ is the core of the game $(N; v)$ defined in Equation (17.2) on page 687.

A useful concept for the characterization of the core for coalitional structures is the superadditive cover of a coalitional game.

Definition 17.78 *Let $(N; v)$ be a coalitional game. The* superadditive cover *of $(N; v)$ is the game $(N; v^*)$ defined by*

$$v^*(S) := \max_{\mathcal{T}} \sum_{T \in \mathcal{T}} v(T), \tag{17.170}$$

where the maximization is taken over all the partitions $\mathcal{T}$ of S.

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The idea behind this definition is as follows. Suppose that the members of S are interested in maximizing the total amount they generate, without taking into account the players who are not in S. In that case they will separate into several subcoalitions in a way that maximizes the sum total of the worth of these subcoalitions. The quantity $v^*(S)$ is precisely this maximal sum.

Note that $v(S) \leq v^*(S)$ for every coalition S, with equality between these two worths for every $S \subseteq N$, if the game is superadditive. Also, $v^*(i) = v(i)$ for every player $i \in N$.

Theorem 17.79 *Let $(N; v)$ be a coalitional game.*

1. *The game $(N; v^*)$ is superadditive.*
2. *The game $(N; v^*)$ is the smallest superadditive game that is larger than $(N; v)$: for every superadditive game $(N; w)$ satisfying $w(S) \geq v(S)$ for every coalition S, $w(S) \geq v^*(S)$ for every coalition S.*
3. *The game $(N; v)$ is superadditive if and only if $v^* = v$.*

The proof of the theorem is left to the reader (Exercise 17.68). The next theorem characterizes the core for a general coalitional structure $\mathcal{B}$.

Theorem 17.80 *Let $(N; v)$ be a coalitional game with a coalitional structure $\mathcal{B}$. Then,*

$$\mathcal{C}(N; v; \mathcal{B}) = \mathcal{C}(N; v^*) \cap X(\mathcal{B}; v). \tag{17.171}$$

The theorem states that to compute the core for a given coalitional structure, and in particular for the coalitional structure $\mathcal{B} = \{N\}$, it suffices to compute the core of the superadditive cover of $(N; v)$, and the set of imputations for $\mathcal{B}$, and to take the intersection of these two sets. We deduce from this that it suffices to compute the core only for superadditive games.

Proof: We first prove that $\mathcal{C}(N; v; \mathcal{B}) \supseteq \mathcal{C}(N; v^*) \cap X(\mathcal{B}; v)$. Let $x \in \mathcal{C}(N; v^*) \cap X(\mathcal{B}; v)$. In particular, $x \in X(\mathcal{B}; v)$, and therefore to prove that $x \in \mathcal{C}(N; v; \mathcal{B})$, we need to prove that for every coalition S,

$$x(S) \geq v(S). \tag{17.172}$$

Since $x \in \mathcal{C}(N; v^*)$,

$$x(S) \geq v^*(S) \geq v(S), \tag{17.173}$$

which is what we needed to show.

We next prove that $\mathcal{C}(N; v; \mathcal{B}) \subseteq \mathcal{C}(N; v^*) \cap X(\mathcal{B}; v)$. Let $x \in \mathcal{C}(N; v; \mathcal{B})$. In particular, $x \in X(\mathcal{B}; v)$. To show that $x \in \mathcal{C}(N; v^*)$, we need to show that (a) $x(N) = v^*(N)$, and (b) $x(S) \geq v^*(S)$ for every coalition S.

Let $S \subseteq N$ be a coalition. Then there is a partition $\mathcal{T} = \{T_1, \ldots, T_K\}$ of S such that

$$v^*(S) = \sum_{k=1}^{K} v(T_K). \tag{17.174}$$

Since $x \in \mathcal{C}(N; v; \mathcal{B})$,

$$x(T_K) \geq v(T_K), \quad \forall k \in \{1, 2, \ldots, K\}, \tag{17.175}$$

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and, therefore,

$$x(S) = \sum_{k=1}^{K} x(T_K) \geq \sum_{k=1}^{K} v(T_K) = v^*(S). \qquad (17.176)$$

It follows that $x(S) \geq v^*(S)$ for every coalition $S \subseteq N$. We set $S = N$, to deduce

$$x(N) \geq v^*(N). \qquad (17.177)$$

Since $x \in X(\mathcal{B}; v)$, and since v^* is the superadditive cover of $(N; v)$,

$$x(N) = \sum_{B \in \mathcal{B}} x(B) = \sum_{B \in \mathcal{B}} v(B) \leq v^*(N). \qquad (17.178)$$

Equations (17.177) and (17.178) imply that $x(N) = v^*(N)$. This completes the proof. □

The last theorems lead to the following corollary.

Corollary 17.81 *Let $(N; v)$ be a coalitional game with a coalitional structure $\mathcal{B}$. If $v^*(N) = \sum_{B \in \mathcal{B}} v(B)$, then $\mathcal{C}(N; v; \mathcal{B}) = \mathcal{C}(N; v^*)$. If $v^*(N) > \sum_{B \in \mathcal{B}} v(B)$, then $\mathcal{C}(N; v; \mathcal{B})$ is empty.*

This corollary further leads to the conclusion that if the core of the superadditive cover $(N; v^*)$ is empty, then the core relative to any coalitional structure is also empty. This fact can also be deduced from Theorem 17.80. Similarly, if the core of $(N; v^*)$ is nonempty, then the core is nonempty only for coalitional structures $\mathcal{B}$ satisfying $v^*(N) = \sum_{B \in \mathcal{B}} v(B)$. The coalitional structures in which the core is nonempty are precisely those structures in which the sum $\sum_{B \in \mathcal{B}} v(B)$ attains its maximum, i.e., the "optimal" partition of the set of all players.

Proof of Corollary 17.81: If $v^*(N) > \sum_{B \in \mathcal{B}} v(B)$, then $\mathcal{C}(N; v^*)$ and $X(\mathcal{B}; v)$ are disjoint, since every imputation $x \in \mathcal{C}(N; v^*)$ satisfies $\sum_{i \in N} x_i = v^*(N)$, while every imputation $x \in X(\mathcal{B}; v)$ satisfies $\sum_{i \in N} x_i = \sum_{B \in \mathcal{B}} x(B) = \sum_{B \in \mathcal{B}} v(B)$. By Theorem 17.80, $\mathcal{C}(N; v; \mathcal{B})$ is empty.

If $v^*(N) = \sum_{B \in \mathcal{B}} v(B)$, then $\mathcal{C}(N; v^*) \subseteq X(\mathcal{B}; v)$, because every imputation $x \in \mathcal{C}(N; v^*)$ satisfies

$$\sum_{B \in \mathcal{B}} x(B) = \sum_{i \in N} x_i = v^*(N) = \sum_{B \in \mathcal{B}} v(B), \qquad (17.179)$$

and

$$x_i \geq v^*(i) = v(i), \quad \forall i \in N. \qquad (17.180)$$

Since $x \in \mathcal{C}(N; v^*)$,

$$x(B) \geq v^*(B) \geq v(B). \qquad (17.181)$$

Equations (17.179) and (17.181) imply that $x(B) = v(B)$ for every $B \in \mathcal{B}$, and therefore $x \in X(\mathcal{B}; v)$. Since $\mathcal{C}(N; v^*) \subseteq X(\mathcal{B}; v)$, Theorem 17.80 then implies that $\mathcal{C}(N; v; \mathcal{B}) = \mathcal{C}(N; v^*)$. □

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17.11 Remarks

The concept of a balanced collection was introduced in Shapley [1967]. The proof appearing in Section 17.3.2 showing that the Bondareva–Shapley condition implies that the core is nonempty is due to Robert J. Aumann. Other proofs of this result appear in Bondareva [1963] and Shapley [1967]. The Weber set was introduced in Weber [1988].

Theorem 17.63 was first proved in Bird [1976]. The proof presented in this chapter is from Granot and Huberman [1981]. The results in Section 17.9 (page 724) and Exercise 17.65 are from Kalai and Zemel [1982b].

The definition of the reduced game to coalition S relative to preimputation x was introduced in Davis and Maschler [1965]. A different definition of the concept of a reduced game was introduced by Hart and Mas-Colell; we study the Hart–Mas-Colell reduced game in Chapter 18. The concept of a reasonable solution (Exercise 17.16) was introduced in Milnor [1952]. Exercise 17.17 is based on Huberman [1980]. The result in the exercise first appeared in Gillies [1953, 1959]. Exercise 17.18 is based on Schmeidler [1972]. The ε-core appearing in Exercise 17.33 was introduced in Shapley and Shubik [1966]. The intuitive meaning of this concept is that a deviation by members of a coalition S that leads to its formation requires information and imposes a cost, and the players will therefore not deviate to form a coalition S unless the profit from deviating is greater than this cost. The least core, and its geometric analysis, were introduced in Maschler, Peleg, and Shapley [1979]. Exercise 17.41 is from Kalai and Zemel [1982a]. Exercise 17.43 is from Aumann and Drèze [1975]. Exercise 17.58 is from Tamir [1991].

17.12 Exercises

17.1 Prove that the core is a convex set. That is, show that for any two imputations x, y in the core of a coalitional game $(N; v)$, and for all $\alpha \in [0, 1]$, the imputation $\alpha x + (1 - \alpha)y$ is also in the core of the game $(N; v)$.

17.2 (a) Give an example of a three-player coalitional game whose core is a triangle.
(b) Give an example of a three-player coalitional game whose core is a parallelogram.
(c) Give an example of a three-player coalitional game whose core is a pentagon.

17.3 Give an example of a monotonic game with an empty core.

17.4 Give an example of a superadditive game with an empty core.

17.5 Draw the cores of the following coalitional games. These games are 0-normalized, and in all of them $N = \{1, 2, 3\}$ and $v(N) = 90$.

(a) $v(1, 2) = 20$, $v(1, 3) = 30$, $v(2, 3) = 10$.
(b) $v(1, 2) = 30$, $v(1, 3) = 10$, $v(2, 3) = 80$.
(c) $v(1, 2) = 10$, $v(1, 3) = 20$, $v(2, 3) = 70$.
(d) $v(1, 2) = 50$, $v(1, 3) = 50$, $v(2, 3) = 50$.
(e) $v(1, 2) = 70$, $v(1, 3) = 80$, $v(2, 3) = 60$.

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17.6 Draw the core of the following three-player coalitional game:

$$v(1) = 5, \quad v(2) = 10, \quad v(3) = 20, \quad v(1,2) = 50, \quad v(1,3) = 70,$$
$$v(2,3) = 50, \quad v(1,2,3) = 90.$$

17.7 Players i, j are *symmetric players* if for every coalition S that does not include any one of them,

$$v(S \cup \{i\}) = v(S \cup \{j\}). \tag{17.182}$$

(a) Prove that the symmetry relation between two players is transitive: if i and j are symmetric players, and j and k are symmetric players, then i and k are symmetric players.
(b) Show that if the core is nonempty, then there exists an imputation x in the core that grants every pair of symmetric players the same payoff, i.e., $x_i = x_j$ for every pair of symmetric players i, j.

17.8 Let $(a_i)_{i \in N}$ be nonnegative real numbers. Let v be the coalitional function

$$v(S) = \begin{cases} 0 & \text{if } |S| \leq k, \\ \sum_{i \in S} a_i & \text{if } |S| > k. \end{cases} \tag{17.183}$$

Compute the core of the game $(N; v)$ for every $k = 0, 1, \ldots, n$.

17.9 Prove that a three-player 0-normalized game whose core is nonempty, and satisfying $v(S) \geq 0$ for every coalition S, is monotonic. Is this true also for games with more than three players? Justify your answer. Does it hold true without the condition that $v(S) \geq 0$ for every coalition? Justify your answer.

17.10 A player i in a coalitional game $(N; v)$ is a *null player* if for every coalition S,

$$v(S \cup \{i\}) = v(S). \tag{17.184}$$

In particular, by setting $S = \emptyset$, this implies that if player i is a null player then $v(i) = 0$. Show that if the core is nonempty, then $x_i = 0$ for every imputation x in the core, and every null player i.

17.11 Let $(N; v)$ be a coalitional game satisfying the strong symmetry property: for every permutation π over the set of players, and every coalition $S \subseteq N$,

$$v(S) = v(\pi(S)), \tag{17.185}$$

where

$$\pi(S) = \{\pi(i) \colon i \in S\}. \tag{17.186}$$

Prove the following claims:

(a) The core of the game is nonempty if and only if for every coalition $S \subseteq N$,

$$v(S) \leq \frac{|S|}{n} v(N). \tag{17.187}$$

(b) If the core of the game is nonempty, and there exists a coalition $\emptyset \neq S \subset N$ satisfying $v(S) = \frac{|S|}{n} v(N)$, then the core contains only the imputation

$$\left(\frac{v(N)}{n}, \ldots, \frac{v(N)}{n}\right). \tag{17.188}$$

17.12 A player i in a simple game is a *veto player* if $v(S) = 0$ for every coalition S that does not contain i.

(a) Show that the core of a simple game satisfying $v(N) = 1$ contains every imputation x satisfying $x_i = 0$ for every player i who is not a veto player, and does not contain any other imputation. In other words, the only imputations in the core are those in which the set of veto players divide the worth of the grand coalition, $v(N)$, between them.
(b) Using part (a), find the core of the gloves game (Example 17.5 on page 690).
(c) Consider a simple majority game in which a coalition wins if and only if it has at least $\frac{n+1}{2}$ votes; that is, for every coalition $S \subseteq N$,

$$v(S) = \begin{cases} 1 & \text{if } |S| \geq \frac{n+1}{2}, \\ 0 & \text{if } |S| < \frac{n+1}{2}. \end{cases} \tag{17.189}$$

What is the core of this game?
(d) What is the core of a simple coalitional game without veto players?

17.13 A *buyer–seller game* is a coalitional game in which the set of players N is the union of a set of buyers B and a set of sellers S (with these two sets disjoint from each other). The payoff function is defined by

$$v(T) := \min\{|T \cap B|, |T \cap S|\}, \qquad \forall T \subseteq N. \tag{17.190}$$

Compute the core of this game. Check your answer against the gloves game (Example 17.5 on page 690).

17.14 Compute the core of the cost game $(N; c)$ in which $N = \{1, 2, 3, 4\}$ and the coalitional function c is

$$c(S) = \begin{cases} 0 & S = \emptyset, \\ 2 & \text{if } |S| = 2 \text{ or } |S| = 1, \\ 4 & \text{if } |S| = 3 \text{ or } |S| = 4. \end{cases} \tag{17.191}$$

17.15 Define the *dual game* of a coalitional game $(N; v)$ to be the coalitional game $(N; v^*)$ where

$$v^*(S) = v(N) - v(N \setminus S), \quad \forall S \subseteq N. \tag{17.192}$$

Is the core of a coalitional game $(N; v)$ nonempty if and only if the core of its dual $(N; v^*)$ is nonempty? Either prove this claim, or provide a counterexample.

17.16 Prove that every imputation x in the core of a coalitional game $(N; v)$ satisfies

$$x_i \leq \max_{S \subseteq N \setminus \{i\}} \{v(S \cup \{i\}) - v(S)\}, \quad \forall i \in N. \tag{17.193}$$

A solution satisfying this property is called a *reasonable solution*.

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17.17 In this exercise, we will show that to compute the core of a coalitional game it suffices to know the worth of only some of the coalitions.

A coalition S is *inessential* in a coalitional game $(N; v)$ if there exists a partition $S_1, S_2, \dots, S_r$ of S into nonempty coalitions such that $r \geq 2$ and $v(S) \leq \sum_{j=1}^{r} v(S_j)$. A coalition S that is not inessential is an *essential* coalition.

(a) Prove that if S is an inessential coalition, then there exists a partition $(S_j)_{j=1}^{r}$ of S into essential coalitions such that $v(S) \leq \sum_{j=1}^{r} v(S_j)$.
(b) Prove that an imputation x is in the core of the game $(N; v)$ if and only if (a) $x(N) = v(N)$, and (b) $x(S) \geq v(S)$ for every essential coalition S.

Let $(N; v)$ and $(N; u)$ be two coalitional games satisfying $v(S) = u(S)$ for every essential coalition S in $(N; v)$ or in $(N; u)$. Prove the following claims:

(c) A coalition S is essential in the game $(N; v)$ if and only if it is essential in the game $(N; u)$.
(d) Deduce that if $v(N) = u(N)$, then $\mathcal{C}(N; v) = \mathcal{C}(N; u)$.
(e) Prove that if the cores of the games $(N; v)$ and $(N; u)$ are nonempty, then $v(N) = u(N)$, and therefore by part (d), $\mathcal{C}(N; v) = \mathcal{C}(N; u)$.
(f) Show by example that it is possible for the core of the game $(N; v)$ to be nonempty while the core of the game $(N; u)$ is empty. In this case show, using part (d) above, that $v(N) \neq u(N)$.

17.18 A coalitional game $(N; v)$ with a nonempty core $\mathcal{C}(N; v)$ is an *exact game* if every coalition S satisfies $v(S) = \min_{x \in \mathcal{C}(N;v)} x(S)$. In other words, the worth of every coalition S equals the minimal total payoff, among the imputations in the core, that the members of S can get working together. In this exercise, we will show that for every game $(N; v)$ with a nonempty core there exists an exact game whose core equals the core of the original game $(N; v)$. In other words, the core of a coalitional game is also the core of an exact game. Moreover, we will show that every convex game is an exact game.

Let $(N; v)$ be a coalitional game with a nonempty core $\mathcal{C}(N; v)$. For every coalition $S \subseteq N$, define

$$v^E(S) := \min_{x \in \mathcal{C}(N;v)} x(S). \qquad (17.194)$$

Answer the following questions:

(a) Prove that $v^E(S) \geq v(S)$ for every coalition $S \subseteq N$.
(b) Prove that $v^E(N) = v(N)$.
(c) Prove that $\mathcal{C}(N; v) = \mathcal{C}(N; v^E)$. Deduce that the coalitional game $(N; v^E)$ is exact.

17.19 Prove that if Equations (17.23)–(17.27) hold for a coalitional game $(N; v)$, where $N = \{1, 2, 3\}$, then the game has a nonempty core.
Guidance: Show that if $v(1, 2) + v(1, 3) \geq v(N) + v(1)$, then the imputation

$$(v(1, 2) + v(1, 3) - v(N), v(N) - v(1, 3), v(N) - v(1, 2)) \qquad (17.195)$$

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is in the core. If $v(1,2)+v(1,3) < v(N)+v(1)$ and $v(1,3) \geq v(1)+v(3)$, then the imputation

$$(v(1), v(N)-v(1,3), v(1,3)-v(1)) \tag{17.196}$$

is in the core. If $v(1,2)+v(1,3) < v(N)+v(1)$ and $v(1,3) < v(1)+v(3)$, then the imputation

$$(v(1), v(N)-v(1,3), v(1,3)-v(1)) \tag{17.197}$$

is in the core.

17.20 Prove that if $\mathcal{D}_1$ and $\mathcal{D}_2$ are two balanced collections, then their union $\mathcal{D}_1 \cup \mathcal{D}_2$ is also a balanced collection.

17.21 Let $\mathcal{D}$ be a balanced collection of coalitions. Suppose that there is a player i contained in every coalition in $\mathcal{D}$. Prove that $\mathcal{D}$ contains a single coalition, $\mathcal{D} = \{N\}$.

17.22 Let $\mathcal{D}$ be a balanced collection of coalitions, and let $S \in \mathcal{D}$. Prove that there is a minimal balanced collection $\mathcal{T} \subseteq \mathcal{D}$ containing S. Deduce that $\mathcal{D}$ is the union of all the minimal balanced collections contained in $\mathcal{D}$.

17.23 Given a balanced collection $\mathcal{D}$ that is not minimal, and any coalition $S \in \mathcal{D}$, does there exist a minimal balanced collection $\mathcal{T} \subseteq \mathcal{D}$ that does not contain S? If so, prove it. If not, provide a counterexample.

17.24 Show that if $\mathcal{D}$ is a minimal balanced collection of coalitions, then the vectors $\{\chi^S, S \in \mathcal{D}\}$ (which are vectors in $\mathbb{R}^N$) are linearly independent.

17.25 Suppose that $|N| = 4$.

(a) Prove that $\{\{1\},\{2\},\{3\},\{3,4\},\{1,3,4\}\}$ is not a balanced collection.
(b) Prove that $\{\{1,2\},\{1,3\},\{1,4\},\{3\},\{4\},\{2,3,4\}\}$ is a balanced collection, but is not a minimal balanced collection.

17.26 Prove or disprove the following:

(a) If $\mathcal{D}$ is a balanced collection of coalitions that is not minimal, then it has an infinite set of balancing weights.
(b) If $\mathcal{D}$ is a weakly balanced collection of coalitions that is not minimal, then it has an infinite set of balancing weights.

17.27 Show that if $N = \{1,2,3\}$, then the only minimal balanced collections of coalitions are: (a) $\{\{1,2,3\}\}$, (b) $\{\{1\},\{2\},\{3\}\}$, (c) $\{\{1,2\},\{3\}\}$, (d) $\{\{1,3\},\{2\}\}$, (e) $\{\{2,3\},\{1\}\}$, (f) $\{\{1,2\},\{1,3\},\{2,3\}\}$.

17.28 Prove that the two formulations of the Bondareva–Shapley Theorem, Theorem 17.14 (page 695) and Theorem 17.19 (page 701), are equivalent. To do so, show that the following two conditions are equivalent:

- Equation (17.36) holds for every balanced collection of coalitions $\mathcal{D}$ with balancing weights $(\delta_S)_{S\in\mathcal{D}}$.
- Equation (17.62) holds for every $\delta \in P$.

17.29 Prove that the coalitional game $(\{1, 2, 3, 4\}; v)$, in which v is defined by

$$v(S) = \begin{cases} 0 & \text{if } |S| = 1, \\ 30 & \text{if } |S| = 2, \\ 0 & \text{if } |S| = 3, \\ 50 & \text{if } |S| = 4, \end{cases} \tag{17.198}$$

has an empty core.
Guidance: Find a balanced collection of coalitions that does not satisfy the Bondareva–Shapley condition.

17.30 In the following games, how large must $v(1, 2, 3)$ be for the game $(\{1, 2, 3\}, v)$ to have a nonempty core?

(a) $v(1) = 12$, $v(2) = 10$, $v(3) = 20$, $v(1, 2) = 20$, $v(1, 3) = 50$, $v(2, 3) = 70$.
(b) $v(1) = 30$, $v(2) = 40$, $v(3) = 70$, $v(1, 2) = 10$, $v(1, 3) = 20$, $v(2, 3) = 5$.

17.31 The totally balanced cover of a coalitional game $(N; v)$ is the minimal totally balanced coalitional game $(N; w)$ greater than or equal to $(N; v)$. In other words:

(a) $(N; w)$ is a totally balanced game.
(b) $w(S) \geq v(S)$ for every coalition $S \subseteq N$.
(c) Every totally balanced coalition $(N; u)$ satisfying $u(S) \geq v(S)$ for every coalition $S \subseteq N$ also satisfies $u(S) \geq w(S)$ for every coalition $S \subseteq N$.

What is the totally balanced cover of the two games in Exercise 17.30? (For $v(1, 2, 3)$ insert the worths you found in Exercise 17.30.)

17.32 Prove that a minimally balanced collection of coalitions contains at most n coalitions.

17.33 Let $(N; v)$ be a coalitional game. For any real number ε, define the ε-*core* of the game as follows.

$$\mathcal{C}_\varepsilon(N; v) = \{x \in \mathbb{R}^N : x(N) = v(N), x(S) \geq v(S) - \varepsilon \quad \forall S \subset N; S \neq \emptyset\}.$$

Note that for $\varepsilon = 0$, the ε-core $\mathcal{C}_0(N; v)$ is the core of the game. Denote $\varepsilon_0 = \inf\{\varepsilon \in \mathbb{R} : \mathcal{C}_\varepsilon(N; v) \neq \emptyset\}$. The set $\mathcal{C}_{\varepsilon_0}(N; v)$ is *the least core* of the game $(N; v)$.

(a) Prove that for every $\varepsilon \in \mathbb{R}$, the set $\mathcal{C}_\varepsilon(N; v)$ is a polytope; i.e., it is a compact set defined by the intersection of a finite number of half-spaces.
(b) Prove that the least core $\mathcal{C}_{\varepsilon_0}(N; v)$ is nonempty.

Finding the least core of a three-player coalitional game can be accomplished graphically as follows. Draw the space of payoffs, and the half-spaces $\{x \in \mathbb{R}^N : x(S) \geq v(S)\}$ defining the core. For example, the game

$$v(1) = v(2) = v(3) = 0, \quad v(1, 2) = 2, \quad v(1, 3) = 3, \quad v(2, 3) = 7, \quad v(1, 2, 3) = 9$$

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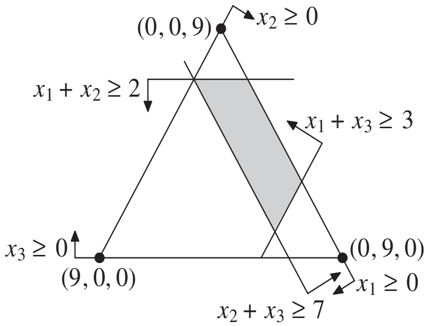
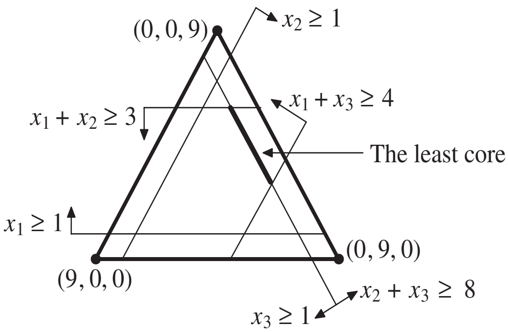
yields the following picture.



The core (which is the ε-core for $\varepsilon = 0$) is nonempty, and therefore to obtain the least core, we need to decrease ε. All the half-spaces appearing in the figure will then be moved in parallel in the direction of the arrow at a constant rate (with all the half-spaces moving at the same rate). As can be seen in the following figure, the least core is obtained at $\varepsilon_* = -1$, and is the interval whose ends are $(1, 5, 3)$ and $(1, 2, 6)$.



(c) Find the least core of each of the following games. Note that if the core is empty, then the least core is obtained at a positive ε.

(i) $v(1) = v(2) = v(3) = 0, \quad v(1, 2) = 2, \quad v(1, 3) = 8, \quad v(2, 3) = 3, \quad v(1, 2, 3) = 12.$

(ii) $v(1) = 6, \quad v(2) = 5, \quad v(3) = 4, \quad v(1, 2) = 2, \quad v(1, 3) = 4, \quad v(2, 3) = 3, \quad v(1, 2, 3) = 12.$

(iii) $v(1) = v(2) = v(3) = 0, \quad v(1, 2) = 8, \quad v(1, 3) = 8, \quad v(2, 3) = 8, \quad v(1, 2, 3) = 12.$

(iv) $v(1) = v(2) = v(3) = 0, \quad v(1, 2) = 3, \quad v(1, 3) = 6, \quad v(2, 3) = 2, \quad v(1, 2, 3) = 12.$

(v) $v(1) = v(2) = v(3) = 0, \quad v(1, 2) = 12, \quad v(1, 3) = 15, \quad v(2, 3) = 12, \quad v(1, 2, 3) = 12.$

17.34 Let $(N; v)$ and $(N; w)$ be two coalitional games satisfying $v(N) = w(N)$ and $v(S) \geq w(S)$ for every coalition $S \subseteq N$. Prove or disprove each of the following two claims:

(a) If $\mathcal{C}(N; v) \neq \emptyset$ then $\mathcal{C}(N; w) \neq \emptyset$.
(b) If $\mathcal{C}(N; w) \neq \emptyset$ then $\mathcal{C}(N; v) \neq \emptyset$.

17.35 Prove that if $(N; v)$ is a market game, then every game that is strategically equivalent to $(N; v)$ is also a market game.

17.36 Complete the computation of the market game derived from the market in Example 17.24 (page 704).

17.37 (a) Prove that a coalitional game $(N; v)$ where $N = \{1, 2\}$ and the coalitional function v is defined by

$$v(1) = v(2) = 1, v(1, 2) = 3 \tag{17.199}$$

is a market game.
(b) Find a market such that $(N; v)$ is the market game derived from it.

17.38 Let $(N; v)$ be a coalitional game, and let $(N; w)$ be a coalitional game strategically equivalent to it. Give direct proofs of the following claims:

(a) The game $(N; v)$ is a market game if and only if the game $(N; w)$ is a market game.
(b) The game $(N; v)$ is totally balanced if and only if the game $(N; w)$ is totally balanced.

17.39 Consider the following coalitional game $(N; v)$ where the set of players is $N = \{1, 2, 3\}$, and the coalitional function is

$$v(S) = \begin{cases} 0 & \text{if } |S| = 1, \\ 1 & \text{if } |S| = 2, \\ 2 & \text{if } |S| = 3. \end{cases} \tag{17.200}$$

Show in a direct way that this game is totally balanced, and find a market from which this game can be derived.

17.40 Let $\varepsilon > 0$, let $x \in \mathbb{R}^N$, and let $(\delta_S)_{\{S \subseteq N, S \neq \emptyset\}}$ be nonnegative weights satisfying $\sum_{\{S \subseteq N, S \neq \emptyset\}} \delta_S \chi^S = x$. Let $y \in \mathbb{R}^N$ be a vector satisfying $|x_i - y_i| < \varepsilon$ for all $i \in N$. Prove that there exists a collection of nonnegative weights $(\mu_S)_{\{S \subseteq N, S \neq \emptyset\}}$ satisfying (a) $\sum_{\{S \subseteq N, S \neq \emptyset\}} \mu_S \chi^S = y$ and (b) $|\delta_S - \mu_S| < 2^{|N|}\varepsilon$.

Deduce that the function u defined in Equation (17.88) (page 710) is a continuous function.

Guidance: First, define $\widehat{\mu}_S := \max\{\delta_S - \varepsilon, 0\}$ for every nonempty coalition S, and show that $\sum_{\{S \subseteq N, S \neq \emptyset\}} \widehat{\mu}_S \chi^S$ is approximately x. The vector $(\mu_S)_{\{S \subseteq N, S \neq \emptyset\}}$ equals the vector $(\widehat{\mu}_S)_{\{S \subseteq N, S \neq \emptyset\}}$, except for coalitions containing only one player.

17.41 Let $N = \{1, 2, \ldots, n\}$ be a set of players. A collection $(Y_S)_{\{S \subseteq N, S \neq \emptyset\}}$ of subsets of $\mathbb{R}^d$ is *balanced* relative to N if for every balanced collection of coalitions $\mathcal{D}$ with

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balancing weights $(\delta_S)_{S\in\mathcal{D}}$, and every list of points $(y_S)_{\{S\subseteq N, S\neq\emptyset\}}$ such that $y_S \in Y_S$ for every $S \in \mathcal{D}$, one has $\sum_{S\in\mathcal{D}} \delta_S y_S \in Y_N$. The collection $(Y_S)_{\{S\subseteq N,S\neq\emptyset\}}$ is *totally balanced* if for every nonempty coalition T, the collection $(Y_S)_{S\subseteq T,S\neq\emptyset}$ is balanced relative to T.

Let $\{B_1, B_2, \dots, B_n\}$ be a partition of the set $\{1, 2, \dots, d\}$. For each coalition S define

$$R_S := \{y \in \mathbb{R}^d : y_j = 0 \quad \forall j \notin \cup_{i\in S} B_i\}. \tag{17.201}$$

Regard the variables $(y_j)_{j\in B_i}$ as variables under the control of player i. The set R_S denotes the possible values of the variables $(y_j)_{j=1}^d$, when the players who are not in S set all the variables under their control to 0.

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a concave function, and let $(Y_S)_{\{S\subseteq N,S\neq\emptyset\}}$ be a collection of compact, nonempty subsets of $\mathbb{R}^d$ satisfying $R_S \subseteq Y_S$ for every nonempty coalition S. Define a coalitional game $(N; v)$ by

$$v(S) := \max_{y\in Y_S} f(y). \tag{17.202}$$

(a) Prove that if the collection $(Y_S)_{\{S\subseteq N,S\neq\emptyset\}}$ is totally balanced, then $(N; v)$ is a totally balanced game.

(b) Show that for every market game $(N; v)$ there exist a natural number $d \in \mathbb{N}$, a partition $\{B_1, B_2, \dots, B_n\}$ of $\{1, 2, \dots, d\}$, a concave function $f : \mathbb{R}^d \to \mathbb{R}$, and a totally balanced collection $(Y_S)_{\{S\subseteq N,S\neq\emptyset\}}$ of compact, nonempty subsets of $\mathbb{R}^d$ satisfying $R_S \subseteq Y_S$ for every nonempty coalition S, such that $v(S) = \max_{y\in Y_S} f(y)$.

17.42 Find the cores for all the coalitional structures of all the games in Exercise 17.5, under the assumption that $v(N) = 60$ (and not $v(N) = 90$, as stated in the exercise).

17.43 Let $(N; v)$ be a coalitional game with a coalitional structure $\mathcal{B}$. Let k and l be two players who are members of different coalitions in $\mathcal{B}$. Prove that if k and l are symmetric players, i.e., $v(S \cup \{k\}) = v(S \cup \{l\})$ for every coalition S that does not contain either of them, then for every imputation x in the core of the game with coalitional structure $\mathcal{B}$, one has $x_k = x_l$.

17.44 Prove that every additive game is convex.

17.45 Find a superadditive game that is not convex.

17.46 Prove that a subgame of a convex game is a convex game. Deduce that every subgame of a convex game has a nonempty core, and that every convex game is totally balanced.

17.47 Prove or disprove: the core of a superadditive game is not empty.

17.48 Let $(N; v)$ be a convex game whose core contains exactly one imputation. Prove that $(N; v)$ is an additive game; i.e., $v(S) = \sum_{i\in S} v(i)$ for every coalition S.
Hint: Make use of Remark 17.56 on page 719.

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17.49 Let N be a set of players, let p_0 be a probability distribution over N, and let $\mathcal{B}$ be a partition of N into disjoint sets. Define a coalitional game $(N; v)$ by

$$v(S) := \sum_{\{B \in \mathcal{B}, B \subseteq S\}} \sum_{i \in B} p_0(i). \tag{17.203}$$

In words, $v(S)$ is the sum of the probabilities associated with the atoms of $\mathcal{B}$ that are contained in S. Let $C(p_0)$ be a set of probability distributions over N that are identical with p_0 over the elements of $\mathcal{B}$,

$$C(p_0) := \left\{ p \in \Delta(N) \colon \sum_{i \in B} p_i = \sum_{i \in B} p_0(i) \quad \forall B \in \mathcal{B} \right\}. \tag{17.204}$$

Prove the following claims:

(a) $v(N) = 1$ and $v(i) \geq 0$ for all $i \in N$. Deduce that the set of imputations $X(\mathcal{B}; v)$ is a subset of $\Delta(N)$. When does this inclusion hold as an equality?
(b) $(N; v)$ is a convex game.
(c) The core of $(N; v)$ equals $C(p_0)$.

17.50 Prove that if the coalitional game $(N; v)$ is strategically equivalent to the coalitional game $(N; w)$, and if $(N; v)$ is a convex game, then $(N; w)$ is also a convex game.

17.51 Let N be a set of players, and let $f : \mathbb{N} \to \mathbb{R}$ be a function. The function f is a *convex* function[9] if for every three natural numbers k, m, l satisfying $k \leq m \leq l$ and $k < l$,

$$f(m) \leq \frac{l-m}{l-k} f(k) + \frac{m-k}{l-k} f(l). \tag{17.205}$$

Let N be a set of players, and let $f : \mathbb{N} \to \mathbb{R}$ be a function. Define a coalitional game $(N; v)$ by

$$v(S) := f(|S|). \tag{17.206}$$

Prove that $(N; v)$ is a convex game if and only if f is a convex function.

17.52 The *monotonic cover* of a coalitional game $(N; v)$ is the coalitional game $(N; \widetilde{v})$ defined by

$$\widetilde{v}(S) := \max_{R \subseteq S} v(R). \tag{17.207}$$

Prove that the monotonic cover of a convex game is a convex game.

17.53 Find a coalitional game that is not convex, and has a nonempty core that does not contain the Weber set.

17.54 Find a coalitional game in which all the vectors w^π defined in Equation (17.138) (page 720) are identical: $w^{\pi_1} = w^{\pi_2}$ for every pair of permuations π_1 and π_2 of the set of players N.

---

9 This is the discrete analogue to the definition of a convex function over $\mathbb{R}$, since $\frac{l-m}{l-k}k + \frac{m-k}{l-k}l = m$. Recall that a real-valued function g is convex if $g(\alpha x + (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)g(y)$ for all x, y and for all $\alpha \in [0, 1]$.

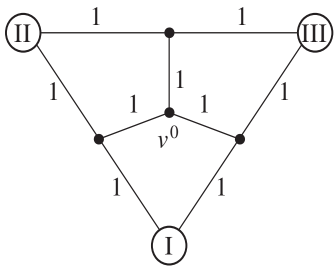
17.55 Write out the coalitional function of the spanning tree game corresponding to Example 17.62 (page 721). Is this a convex game?

17.56 Prove that a connected graph over a set V with $n+1$ vertices is a tree (that is, it is acyclic) if and only if it contains n edges.

17.57 A spanning tree system (N, V, E, v^0, a), where (V, E) is a tree (i.e., a connected acyclic graph), is called a *tree system*. Prove that the spanning tree game corresponding to such a system is a convex game.

17.58 In Section 17.8 we defined a spanning tree system (N, V, E, v^0, a) in which every vertex that is not the source is associated with a player in N. In this exercise we will assume that some of the vertices are unmanned, i.e., $V \supseteq N \cup \{v^0\}$. In this case, as in the standard case, the spanning tree game $(N; c)$ corresponding to the spanning tree system is a cost game in which the worth $c(S)$ of each coalition S is the cost of the minimal-cost spanning tree of the coalition.

(a) Write out the spanning tree game corresponding to the following spanning tree system.



(b) Prove that the core of this game is empty.

17.59 A *bankruptcy problem* is given by $n+1$ nonnegative numbers $[E; d_1, d_2, \ldots, d_n]$. Here E represents the assets of a bankrupt individual or corporation, and $N = \{1, 2, \ldots, n\}$ is the set of creditors, with each creditor i owed a debt of d_i. It is assumed that $E < \sum_{i=1}^n d_i$ (otherwise every creditor can be paid off in full, and there is no bankruptcy problem to be considered).

This problem can be analyzed using a mathematical model in several different ways. One way, as presented in O'Neill [1982], depicts the problem as a coalitional game $(N; v)$, where the set of players is the set of creditors, and the coalitional function is

$$v(S) := \max\left\{E - \sum_{i \notin S} d_i, 0\right\}. \tag{17.208}$$

Prove that this game is convex, and deduce that it has a nonempty core.

Explanation: The intuition behind this definition is as follows. When a coalition S intends on forming, it computes what it can gain for its members. The coalition

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takes into account the worst-case scenario, in which all the creditors who are not members of S get paid all the debt they are owed in full, if the worth of E enables this.

17.60 Let $[E; d_1, d_2, \ldots, d_n]$ be a bankruptcy problem, and let x be an imputation that divides the assets of the bankrupt estate among the creditors proportionally to the debt that is owned to each of them:

$$x_i = \frac{d_i}{\sum_{j=1}^n d_j} \cdot E. \qquad (17.209)$$

Is x in the core of the game defined in Exercise 17.59?

17.61 Let $[E; d_1, d_2, \ldots, d_n]$ be a bankruptcy problem, and let $(N; v)$ be the coalitional game corresponding to this problem, as defined in Exercise 17.59. Let $x = (x_1, x_2, \ldots, x_n)$ be an imputation in the core of $(N; v)$, and let S be a nonempty coalition. Let $\Gamma = [x(S); (d_i)_{i \in S}]$ be the bankruptcy problem restricted to the creditors in S, given the imputation x. Denote by $(S; x_S^w)$ the reduced game of $(N; v)$ to coalition S relative to x. Prove that $(S; w_S^x)$ is the game corresponding to the bankruptcy problem Γ.

In other words, this exercise states that for every imputation x in the core, the game corresponding to the bankruptcy problem for coalition S at the point x is the reduced game to S relative to x.

17.62 Let φ be the solution concept to the collection of bankruptcy problems that divides the asset E among the creditors proportionally to the debt owed to them:

$$\varphi_i(E; d_1, d_2, \ldots, d_n) := \frac{d_i}{\sum_{j=1}^n d_j} \cdot E. \qquad (17.210)$$

Does this solution concept satisfy the Davis–Maschler reduced game property (see page 715)?

17.63 Prove Theorem 17.68 on page 726.

17.64 Prove that the function $\widehat{f}$ that is defined in Equation (17.160) (page 730) is a flow in the graph G, and its magnitude is $M(f^*) + \varepsilon$.

17.65 In Section 17.9 we defined a flow problem in which each edge is controlled by one of the players. In this exercise we consider a flow problem $(V, E, v^0, v^1, c, N, I)$ in which some of the edges are "public edges"; that is, the function I is a function from E to $N \cup \{*\}$. If $I(e) = *$ we say that e is a *public edge*.

The worth of a coalition S is the maximal magnitude of the flow that can pass from v^0 to v^1 using the edges controlled by the members of S and the public edges. Answer the following questions.

(a) Construct a flow problem with public edges such that the core of the corresponding flow game is empty.
(b) Prove that every flow game corresponding to a flow problem with public edges is a monotonic game. In addition, show that for every monotonic game $(N; v)$

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there is a flow problem in which there may be public edges such that $(N; v)$ is the flow game corresponding to this flow problem.

17.66 Write out the superadditive cover of the games in Exercise 17.5.

17.67 Draw the core of the game in Exercise 17.6, with respect to each possible coalitional structure.

17.68 Prove Theorem 17.79 on page 733.

17.69 Let $x \in \mathcal{C}(N; v; \mathcal{B})$ and $y \in \mathcal{C}(N; w; \mathcal{B})$. Is $x + y \in \mathcal{C}(N; v + w; \mathcal{B})$? Is $x - y \in \mathcal{C}(N; v - w; \mathcal{B})$? For each claim, either prove it, or provide a counterexample.

17.70 Prove or disprove: for every coalitional game $(N; v)$ there exists a coalitional structure $\mathcal{B}$ such that $\mathcal{C}(N; v; \mathcal{B}) \neq \emptyset$.

17.71 Let $(N; v)$ be a coalitional game, let (N, v^*) be its superadditive cover, and let $(N; \widetilde{v})$ be its monotonic cover (see Definition 17.78 for the definition of the superadditive cover, and Exercise 17.52 for the definition of the monotonic cover). Is one of the quantities $v(N)$, $v^*(N)$, and $\widetilde{v}(N)$ always greater than or equal to one of the other quantities? Justify your answer.