Roll No : 1000XYZW
e.g., 190040001

Dept.:
e.g., CSE

IIT Bombay
CS 6001: GT&AMD
Midsem Exam, 2023-24-I
Date: September 21, 2023

CS6001: Game Theory and Algorithmic Mechanism Design

Total: **10 × 4 = 40 points,** *Duration:* **2 hours, ATTEMPT ALL QUESTIONS**

Instructions:

1. This question-and-answersheet booklet contains a total of **6 sheets** of paper **(11 pages, pages 2 and 12 are blank)**. Please verify.

2. Write your roll number and department on **every side of every sheet** (except the blank sheet) of this booklet. Use only **black/blue ball-point pen**. The first 5 minutes of additional time is given exclusively for this activity.

3. Write final answers neatly with a pen **only in the given boxes**.

4. Use the rough sheets for scratch works / attempts to solution. **Write only the final solution (which may be a sequence of logical arguments) in a precise and succinct manner in the boxes provided**. Do not provide unnecessarily elaborate steps. The space within the boxes are sufficient for the correct and precise answers.

5. Submit your answerscripts to the teaching staff when you leave the exam hall or the time runs out (whichever is earlier). **Your exam will not be graded if you fail to return the paper**.

6. **This is a closed book, notes, internet exam. No communication device, e.g., cellphones, iPad, etc., is allowed**. Keep it switched off in your bag and keep the bag away from you. If anyone is found in possession of such devices during the exam, that answerscript may be disqualified for evaluation and DADAC may be invoked.

7. One A4 assistance sheet (text **on both sides**) is allowed for the exam.

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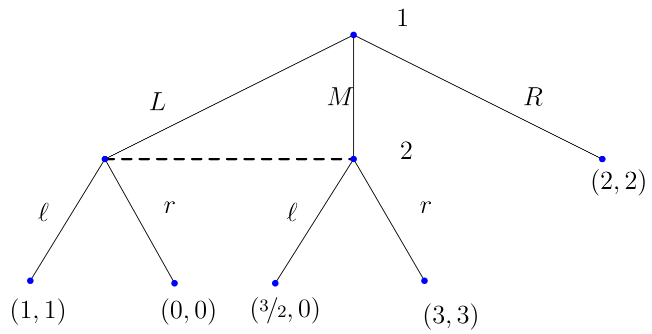
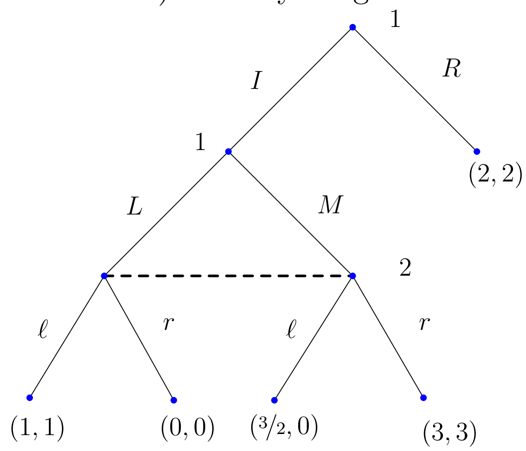
Problem 1 (2 + 2 + 2 + 4 points). Consider the two extensive form games as shown below. A *subgame* of an imperfect information extensive-form game (IIEFG) is a subgame where a player has a singleton information set. Subgame perfection in IIEFGs are similar to the perfect information EFGs: the subgame perfect Nash equilibrium of an IIEFG has to be an NE (pure or mixed) at every subgame of the IIEFG.



Game A



Game B

(a) How many subgames do these games have?

Game A:

1 , at history ϕ.

Game B:

2 , at histories ϕ and (I).

(b) Are these games with perfect recall? (write Yes/No for each case below)

Game A:

Yes

Game B:

Yes

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(c) What are the strategy sets of each of the players?

Game A:

$S_1 = \{L, M, R\}$, $S_2 = \{l, r\}$

Game B:

$S_1 = \{IL, IM, RL, RM\}$, $S_2 = \{l, r\}$

(d) Find all the subgame-perfect Nash equilibria (SPNEs) of these games. Write the probabilities w.r.t. each strategy in the form $(p_1\ (s_i^1), p_2\ (s_i^2), \ldots, p_k\ (s_i^k))$ for each player where s_i^j's are the strategies of player i and p_j's are the probabilities.

Game A:

Since there is only one subgame, the game matrix looks like

	l	r
~~L~~	~~1,1~~	~~0,0~~
M	3/2, 0	3, 3
R	2, 2	2, 2

L is removed since it is strictly dominated by M. In the remaining game, solving for the PSNEs and MSNE, we find

$((1(M)),(1(r)))$ as a PSNE and

$((1\ (R)), (q\ (l), (1-q)\ (r)))$, $q \geq \frac{2}{3}$ as MSNEs

the second one subsumes the other PSNE $((1\ (R)),(1(l)))$.

Game B:

In this game, using backward induction, at history I, the game has a unique NE (M, r), since L is strictly dominated. Using this, player 1's best response is I since that gives 1 a utility of 3. Hence the unique SPNE is

$((1\ (IM)), (1\ (r)))$.

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Problem 2 (1 + 1 + 2 + 2 + 1 + 2 + 1 points). The following result may useful for this problem.

Definition 1. A normal form game $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is called **symmetric** if

1. each player has the same set of strategies: $S_i = S_j$ for each $i, j \in N$, and
2. the payoff functions satisfy

$$u_i(s_1, s_2, \ldots, s_n) = u_j(s_1, \ldots, s_{i-1}, s_j, s_{i+1}, \ldots, s_{j-1}, s_i, s_{j+1}, \ldots, s_n),$$

   for any vector of pure strategies $s = (s_1, s_2, \ldots, s_n) \in S$ and for each pair of players i, j satisfying $i < j$. In other words, if the players swap their strategies, their utilities also get swapped.

Result 1. In every symmetric game, there exists a **symmetric equilibrium** in mixed strategies: an equilibrium $\sigma = (\sigma_i)_{i \in N}$ satisfying $\sigma_i = \sigma_j$ for each $i, j \in N$.

Consider the following 'lottery' game with n participants competing for a prize worth ₹$M, (M > 1)$. Every player may purchase as many numbers of tickets numbered $\{1, 2, \ldots, K\}$ as she wishes at a cost of ₹1 per number. The set of all the *numbers that have been purchased by only one of the players* is then identified, and the winning number is *the smallest number* in that set. The (necessarily only) player who purchased that number is the lottery winner, receiving the full prize. If no number is purchased by only one player, no player receives a prize.

(a) Write down every player's set of pure strategies and payoff function. Use the following notation: For every vector of pure strategies $(s_1, s_2, \ldots, s_n)$ let $D(s_1, s_2, \ldots, s_n)$ be the set of of all the numbers that have been purchased by only one of the players, i.e., $D(s_1, s_2, \ldots, s_n) = \cup_{i=1}^{n} (s_i \setminus (\cup_{j \neq i} s_j))$.

Set of strategies of player i

$S_i = \{ s_i : s_i \subseteq \{1, 2, \ldots, K\} \}$ all possible subsets of $\{1, \ldots, K\}$

define, $D(s_1, s_2, \ldots, s_n) = \bigcup_{i=1}^{n} (s_i \setminus \bigcup_{j \neq i} s_j)$ as given.

The set of numbers purchased by exactly one player.

Utility of player i :

$$u_i(s_1, \ldots, s_n) = \begin{cases} M - |s_i| & \text{if } \min_{k \in D(s_1, \ldots, s_n)} k \in s_i \\ -|s_i| & \text{otherwise} \end{cases}$$

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(b) Consider the statement: there exists an equilibrium in which every player uses the same mixed strategy that is not a pure strategy. (True/False)?

True

(c) Justify the previous answer. Write only the reasoning in not more than *three* sentences, each sentence arguing one implication from the previous statement.

This is a symmetric game. Hence, by the result above, there must exist a symmetric mixed strategy equilibrium. However, no symmetric pure strategy can be an equilibrium, since each of the players will have a profitable deviation. Hence, the above statement is true.

(d) If at equilibrium there is a positive probability that player i will not purchase any number, then what will be her expected payoff? Provide arguments along with the answer. (Hint: use the fact from the MSNE characterization theorem: if $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE, then $u_i(\sigma_i^*, \sigma_{-i}^*) = u_i(s_i, \sigma_{-i}^*), \forall s_i \in \delta(\sigma_i^*)$, the support of σ_i^*)

At equilibrium, if the strategy of a player i puts positive mass on ϕ, that implies that ϕ is in the support of the equilibrium. Hence, for that equilibrium

$$u_i(\sigma_i^*, \sigma_{-i}^*) = u_i(\phi, \sigma_{-i}^*) = 0.$$

expected payoff $= 0$.

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(e) If $M < n$, meaning that the number of participants is greater than the value of the prize, then at equilibrium, every player w.p. 1 purchases a number, i.e., there does not exist any player who does not purchase any number with positive probability. Is this statement True/False?

False

(f) Justify the previous answer. Write only the reasoning in not more than *six* sentences, each sentence arguing one implication from the previous statement.

Suppose the statement is true: at equilibrium, each player purchases a number w.p. 1. Then the expected cost of all the numbers purchased is $\geq n > M$. Since the reward is smaller than the expected total cost, there must exist at least one player whose expected payoff is negative. This player is strictly better off not purchasing any number – a contradiction to the equilibrium. Hence proved.

(g) What will be the expected utility of each player in a **symmetric** equilibrium of this game? Explain why in at most *two* sentences. You may use any result from the previous parts of this problem.

By (f), in a symmetric equilibrium, each player does not purchase any number with positive probability. Hence, the expected utility of each player must be zero (by (d)).

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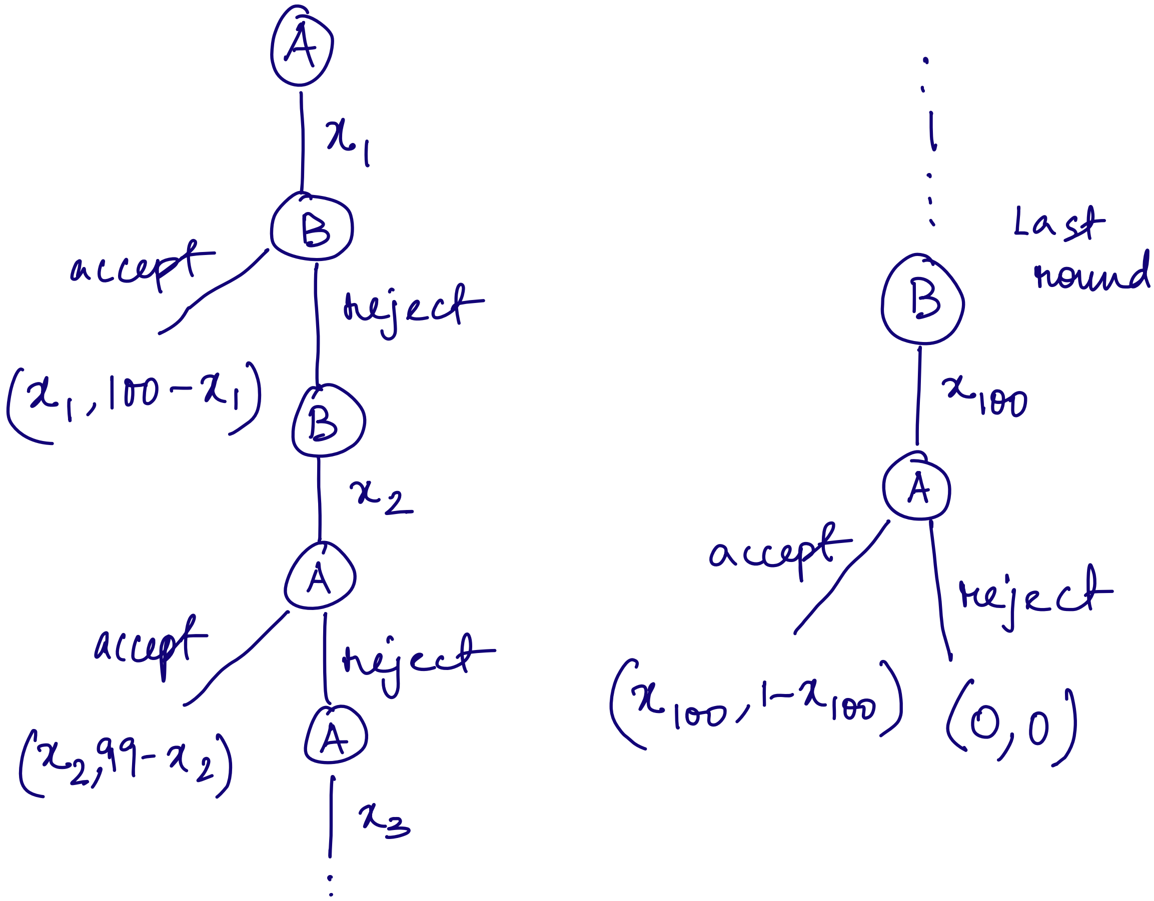
Problem 3 (4 + (2 + 4) points). Ankita and Bharat are jointly conducting a project that will pay them a total payoff of ₹100. Every delay in implementing the project reduces payment for completing the project. How should they divide this money between them? The two decide to implement the following procedure: A starts by offering a division $(x_A, 100 - x_A)$, where x_A is a number in ₹$[0, 100]$ representing the amount of money that A receives under the terms of this offer, while B receives $100 - x_A$. B may **accept** or **reject** A's offer. If he rejects the offer, he has to propose a counteroffer $(y_A, 99 - y_A)$ where y_A is a number in ₹$[0, 99]$ representing the amount of money that A receives under the terms of this offer, while B receives ₹$99 - y_A$. B's offer can only divide ₹99 between the two players, because the delay caused by his rejection of A's offer has reduced the payment for completing the project by ₹1. A may accept or reject B's offer. If she rejects the offer, she may then propose yet another counteroffer, and so on. Each additional round of offers, however, reduces the amount of money available by ₹1: if the two players come to an agreement on a division after the kth offer has been passed between them, then they can divide only $(101 - k)$ rupees between them. If the two players cannot come to any agreement, after 100 rounds of alternating offers, they drop plans to conduct the project jointly, and each receives 0.

Answer the following sub-questions.

(a) Describe this situation as an extensive-form game



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(b) Find all its subgame-perfect Nash equilibria (SPNE). Write concise arguments on why these are the SPNEs of this game.

Use backward induction.

In the last offer (round 100), A accepts any offer $(x_{100}, 1-x_{100})$, if $x_{100}=0$, she is indifferent between accept and reject. Thus B offers $(0,1)$. Hence in the previous round (round 99) A offers $(1,1)$. Hence in round 98, B must offer A $(1,2)$ and so on.

Hence, in a SPNE, for every odd k-th round, B offers $\left(\frac{100-k}{2}, 1+\frac{100-k}{2}\right)$ and A accept, and every even k-th round A offers $\left(\frac{101-k}{2}, \frac{101-k}{2}\right)$ and B accepts.

In particular, in the first round A offers $(50,50)$ and B accepts becomes the outcome of the game in the unique SPNE.

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Problem 4 (4 + 2 + 2 + 2 points). Consider the following (Harsanyi) game with incomplete information.

- $N = \{1, 2\}$.
- $\Theta_1 = \{\theta_1^1, \theta_1^2\}, \Theta_2 = \{\theta_2^1, \theta_2^2\}$, i.e., both players 1 and 2 have two types each.
- The interim beliefs (on the type of the other player given her own type) of the players in this game before actions are chosen, are given by

	θ_2^1	θ_2^2
θ_1^1	$1/4$	$3/4$
θ_1^2	$2/3$	$1/3$

Player 1's beliefs

	θ_2^1	θ_2^2
θ_1^1	$3/11$	$9/13$
θ_1^2	$8/11$	$4/13$

Player 2's beliefs

- In this problem, we will find if the beliefs of the players Bayesian consistent, i.e., if they can be derived from a common prior distribution of $P(\theta_1, \theta_2)$ using Bayes rule.
- Define the variables

$$x_1 := P(\theta_1^1, \theta_2^1), \qquad x_2 := P(\theta_1^1, \theta_2^2), \qquad x_3 := P(\theta_1^2, \theta_2^1), \qquad x_4 := P(\theta_1^2, \theta_2^2).$$

(a) If the beliefs were consistent, then what set of conditions should they have satisfied? Write only the independent conditions (that cannot be derived from the previous ones) in a separate line, and the number the conditions. The conditions should be in terms of the variables defined above.

$$P(\theta_2^1 \mid \theta_1^1) = \frac{P(\theta_1^1, \theta_2^1)}{P(\theta_1^1, \theta_2^1) + P(\theta_1^1, \theta_2^2)} = \frac{x_1}{x_1 + x_2} = \frac{1}{4} \quad \text{--- } 1$$

$$P(\theta_2^1 \mid \theta_1^2) = \frac{P(\theta_1^2, \theta_2^1)}{P(\theta_1^2, \theta_2^1) + P(\theta_1^2, \theta_2^2)} = \frac{x_3}{x_3 + x_4} = \frac{2}{3} \quad \text{--- } 2$$

$$P(\theta_1^1 \mid \theta_2^1) = \frac{P(\theta_1^1, \theta_2^1)}{P(\theta_1^1, \theta_2^1) + P(\theta_1^2, \theta_2^1)} = \frac{x_1}{x_1 + x_3} = \frac{3}{11} \quad \text{--- } 3$$

$$P(\theta_1^1 \mid \theta_2^2) = \frac{P(\theta_1^1, \theta_2^2)}{P(\theta_1^1, \theta_2^2) + P(\theta_1^2, \theta_2^2)} = \frac{x_2}{x_2 + x_4} = \frac{9}{13} \quad \text{--- } 4$$

finally $x_1 + x_2 + x_3 + x_4 = 1$ --- 5

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(b) Based on the conditions derived in the previous part, what conclusion can you reach about the consistency of beliefs? Is it **consistent** / **not consistent**?

It is consistent. [Because the system of linear equations above has a unique solution]

(c) If it is consistent, fill the values of x_k's, $k = 1, \ldots, 4$ (a fractional form will be preferred). Write 'NA' if the beliefs were not consistent.

$x_1 =$ 1/8 $x_3 =$ 1/3

$x_2 =$ 3/8 $x_4 =$ 1/6

(d) If the state games are **matrix games** and are given as follows. If the beliefs are consistent, find the Bayesian equilibrium of this game. Note that equilibrium should be a strategy profile, i.e., what each agent would do at each of her types. Write 'NA' if the beliefs were not consistent.

Player 2

Player 1	L	R
T	1	0
B	0	0

The state game for (θ_1^1, θ_2^1)

Player 2

Player 1	L	R
T	0	1
B	0	0

The state game for (θ_1^1, θ_2^2)

Player 2

Player 1	L	R
T	0	0
B	1	0

The state game for (θ_1^2, θ_2^1)

Player 2

Player 1	L	R
T	0	0
B	0	1

The state game for (θ_1^2, θ_2^2)

Easy to see that player 1 at type θ_1^1 finds B as strictly dominated. Hence $s_1^*(\theta_1^1) = T$. Similarly, arguing for the players for every type, we find the following Bayesian equilibrium [skipping the steps here, but you should show each step]

$s_1^*(\theta_1^1) = T$, $s_1^*(\theta_1^2) = B$, $\underbrace{s_2^*(\theta_2^1) = R, \ s_2^*(\theta_2^2) = L}$

note: this is a matrix game for player 2, utilities are negative of the matrix.

END OF QUESTION PAPER. GOOD LUCK!