Lectures 3-4

Definition 0.1 *(Probability space).*

Let $\Omega, \mathcal{F}, P$ are respectively a nonempty set (sample space); σ-field and probability measure. Then the triplet $(\Omega, \mathcal{F}, P)$ is called a probability space.

Example 0.1 *Let $\Omega = \{H,\ T\}$, $\mathcal{F} = \mathcal{P}(\Omega)$. Define P on $\mathcal{F}$ as follows.*

$$P(\emptyset) = 0,\ P\{H\} = P\{T\} = \frac{1}{2},\ P(\Omega) = 1.$$

Then $(\Omega,\ \mathcal{F},\ P)$ is a probability space. This probability space corresponds to the random experiment of tossing an unbiased coin and noting the face.

Example 0.2 *Let $\Omega = \{0,\ 1,\ 2, \cdots\}$, $\mathcal{F} = \mathcal{P}(\Omega)$. Define P on $\mathcal{F}$ as follows.*

$$P(A) = \sum_{k \in A} e^{-5} \frac{5^k}{k!}\, , A \in \mathcal{F}\, .$$

Then $(\Omega,\ \mathcal{F},\ P)$ is a probability space.

Solution

$$P(\Omega) = \sum_{k\in\Omega} e^{-5}\frac{5^k}{k!} = e^{-5}\sum_{k=0}^{\infty}\frac{5^k}{k!} = e^{-5}e^{5} = 1\, .$$

If $A_1\ \ A_2 \cdots, \in \mathcal{F}$ are pairwise disjoint. Then

$$\begin{aligned}
\sum_{k\in\cup_{n=1}^{\infty}A_n} e^{-5}\frac{5^k}{k!} &= \lim_{m\to\infty}\sum_{k\in\cup_{n=1}^{m}A_n} e^{-5}\frac{5^k}{k!}\\
&= \lim_{m\to\infty}\sum_{n=1}^{m}\sum_{k\in A_n} e^{-5}\frac{5^k}{k!}\\
&= \lim_{m\to\infty}\sum_{n=1}^{m} P(A_n)\\
&= \sum_{n=1}^{\infty} P(A_n).
\end{aligned}$$

Here first and the last equlity follows from the definition of convergence of series, second from the definition of finite sum of convergent series.

Therefore properties (i) , (ii) are satisfied. Hence P is a probability measure.

The above probability space is a model for a random phenomenon where one is observing the occurence of a special event over a fixed period of time. Say for example, counting the number of arrivals of customers to a counter.

Observe that probability measures are finite additivity, i.e., if P is a probability measure, then for pairwise disjoint $A, B \in \mathcal{F}$, then

$$P(A \cup B) = P(A) + P(B).$$

This follows easily from $P(\emptyset) = 0$ and applying countable additivity for $A, B, \emptyset, \emptyset, \cdots$.

Theorem 0.1 *(Properties of probability measure) Let* $(\Omega,\, \mathcal{F},\, P)$ *a probability space and* $A,\, B,\, A_1,\, A_2, \ldots$ *are in* $\mathcal{F}$*. Then*

(1) $P(A^c) \;=\; 1 \;-\; P(A)$.

(2) Monotonicity: if $A \subseteq\, B$*, then*

$$P(A) \;\leq\; P(B)\,.$$

(3) Inclusion - exclusion formula:

$$\begin{aligned} P\big(\textstyle\bigcup_{k=1}^{n} A_k\big) \;&=\; \sum_{k=1}^{n} P(A_k) \;-\; \sum_{1 \leq i < j \leq n} P(A_i \cap A_j) \\ &\quad + \sum_{1 \leq i < j < k \leq n} P(A_i \cap A_j \cap A_k) + \ldots \\ &\quad + \,(-1)^{n+1} P(A_1 \cap A_2 \cap \ldots A_n)\,. \end{aligned}$$

(4) Finite sub-additivity:

$$P(A \cup B) \;\leq\; P(A) \;+\; P(B)\,.$$

(5) Continuity property:
(i) For $A_1 \;\subseteq\; A_2 \subseteq\; \ldots$

$$P(\cup_{n=1}^{\infty} A_n) \;=\; \lim_{n \to \infty} P(A_n)\,.$$

(ii) For $A_1 \;\supseteq\; A_2 \;\supseteq\; \ldots$,

$$P(\cap_{n=1}^{\infty} A_n) \;=\; \lim_{n \to \infty} P(A_n)\,.$$

(6) Boole's inequality (Countable sub-additivity):

$$P(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} P(A_n)\,.$$

Proof. Since $\Omega = A \cup A^c$, using finite additivity, we have

$$1 = P(A) + P(A^c).$$

This implies (1).

Consider

$$A \subseteq B \implies B = A \cup B \setminus A\,.$$

Therefore

$$P(B) = P(A) + P(B \setminus A) \Rightarrow P(B) \geq P(A)\,,$$

since $P(B \setminus A) \geq 0$. This proves (2).

We prove (3) by induction. For $n = 2$

$$A_1 \cup A_2 = A_1 \cup (A_2 \setminus A_1)\,.$$

and

$$A_2 = (A_2 \setminus A_1) \cup (A_1 A_2)\,.$$

(Here $A_1 A_2 = A_1 \cap A_2$). Hence we have

$$P(A_1 \cup A_2) = P(A_1) + P(A_2 \setminus A_1)$$

and

$$P(A_2) = P(A_2 \setminus A_1) + P(A_1 A_2)\,.$$

Combining the above, we have

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 A_2)\,.$$

Assume that equality holds for $n \leq m$. Consider

$$
\begin{aligned}
P(\cup_{k=1}^{m+1} A_k) &= P(\cup_{k=1}^{m} A_k \cup A_{m+1}) \\
&= P(\cup_{k=1}^{m} A_k) + P(A_{m+1}) - P([\cup_{k=1}^{m} A_k] \cap A_{m+1}) \\
&= P(\cup_{k=1}^{m} A_k) + P(A_{m+1}) - P(\cup_{k=1}^{m} A_k A_{m+1}) \\
&= \sum_{k=1}^{m} P(A_k) - \sum_{1 \leq i < j \leq m} P(A_i A_j) \\
&\quad + \cdots + (-1)^{m+1} P(A_1 \cdots A_m) + P(A_{m+1}) \\
&\quad - \Big[\sum_{k=1}^{m} P(A_k A_{m+1}) - \sum_{1 \leq i < j \leq m} P(A_i A_{m+1} A_j A_{m+1}) \\
&\quad + \cdots + (-1)^{m+1} P(A_1 A_{m+1} A_2 A_{m+1} \cdots A_m A_{m+1}) \Big] \\
&= \sum_{k=1}^{m+1} P(A_k) - \Big[\sum_{1 \leq i < j \leq m} P(A_i A_j) + \sum_{k=1}^{m} P(A_k A_{m+1}) \Big] \\
&\quad + \Big[\sum_{1 \leq i < j < k \leq m} P(A_i A_j A_k) + \sum_{1 \leq i < j \leq m} P(A_i A_j A_{m+1}) \Big] \\
&\quad + \cdots + (-1)^{m+2} P(A_1 A_2 \cdots A_m A_{m+1}) \\
&= \sum_{k=1}^{m+1} P(A_k) - \sum_{1 \leq i < j \leq m+1} P(A_i A_j) \\
&\quad + \sum_{1 \leq i < j < k \leq m+1} P(A_i A_j A_k) \cdots + (-1)^{m+2} P(A_1 \cdots A_{m+1}) .
\end{aligned}
$$

Therefore the result true for $n = m + 1$. Hence by induction property (3) follows.

From property (3), we have

$$P(A \cup B) = P(A) + P(B) - P(AB) .$$

Hence

$$P(A \cup B) \leq P(A) + P(B), .$$

Thus we have (4).

Now we prove (5)(i). Set

$$B_1 \;=\; A_1,\; B_n \;=\; A_n \;\setminus\; A_{n-1},\; n = 2, 3, \cdots .$$

Then

$$B_n \in \mathcal{F} \;\forall\; n = 1, 2, \cdots ,$$

$B_n' s$ are disjoint and

$$A_n \;=\; \cup_{k=1}^{n} B_k,\, n \geq 1\,. \tag{0.1}$$

Also

$$\cup_{n=1}^{\infty} B_n \;=\; \cup_{n=1}^{\infty} A_n\,. \tag{0.2}$$

To see the above,

$$\begin{aligned} \omega \in \cup_{n=1}^{\infty} A_n \;&\Rightarrow\; \omega \in A_n = \cup_{k=1}^{n} B_k \text{ for some } n \geq 1 \\ &\Rightarrow\; \omega \in B_m \text{ for some } m \leq n \\ &\Rightarrow\; \omega \in \cup_{n=1}^{\infty} B_n. \end{aligned}$$

Reverse inclusion is easier (left it as an exercise).
Using (0.1), we get

$$P(A_n) \;=\; \sum_{k=1}^{n} P(B_k)\,.$$

Hence, from the definition of series convergence, we get

$$\lim_{n\to\infty} P(A_n) \;=\; \lim_{n\to\infty} \sum_{k=1}^{n} P(B_k) \;=\; \sum_{n=1}^{\infty} P(B_n). \tag{0.3}$$

Similarly, using (0.2), we get

$$P(\cup_{n=1}^{\infty} A_n) \;=\; P(\cup_{n=1}^{\infty} B_n) \;=\; \sum_{n=1}^{\infty} P(B_n)\,.$$

Therefore from (0.3), we have

$$P(\cup_{n=1}^{\infty} A_n) \;=\; \lim_{n\to\infty} P(A_n).$$

Proof of (5)(ii) is as follows.
Note that

$$A_1^c \subseteq A_2^c \;\cdots .$$

Now using (5)(i) we have

$$\lim_{n\to\infty} P(A_n^c) \;=\; P(\cup_{n=1}^{\infty} A_n^c)\,.$$

i.e.,

$$1 - \lim_{n\to\infty} P(A_n) \;=\; P[(\cap_{n=1}^{\infty} A_n)^c] \;=\; 1 - P(\cap_{n=1}^{\infty} A_n)\,.$$

Hence

$$\lim_{n\to\infty} P(A_n) \;=\; P(\cap_{n=1}^{\infty} A_n)\,.$$

From property (4), it follows that

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) \le P(A_1) \;+\; \cdots \;+\; P(A_n) \;\forall\; n \ge 1\,.$$

i.e.,

$$P(\cup_{k=1}^{n} A_k) \;\le\; \sum_{k=1}^{n} P(A_k) \;\forall\; n \ge 1\,.$$

Therefore

$$P(\cup_{k=1}^{n} A_k) \;\le\; \sum_{k=1}^{\infty} P(A_k) \;\forall\; n \ge 1\,. \tag{0.4}$$

Set

$$B_n \;=\; \cup_{k=1}^{n} A_k\,.$$

Then $B_1 \subseteq B_2 \subseteq \cdots$ and are in $\mathcal{F}$. Also

$$\cup_{n=1}^{\infty} A_n \;=\; \cup_{n=1}^{\infty} B_n\,.$$

Hence

$$P(\cup_{n=1}^{\infty} A_n) \;=\; P(\cup_{n=1}^{\infty} B_n) \;=\; \lim_{n\to\infty} P(B_n) \;=\; \lim_{n\to\infty} P(\cup_{k=1}^{n} A_k)\,. \tag{0.5}$$

Here the second equality follows from the continuity property 5(i). Using (0.5), letting $n \to \infty$ in (0.4), we have

$$P(\cup_{n=1}^{\infty} A_n) \;\le\; \sum_{k=1}^{\infty} P(A_k)\,.$$

Result 1: Let $\mathcal{F}$ be a field with finitely many elements. Then $\mathcal{F}$ is a σ-field.
Proof:

I will give a proof, using the set theory we know. From given, there exists n such that $\#\mathcal{F} = n$. Hence $\mathcal{F} = \{B_1, B_2, \cdots, B_n\}$ for some $B_i \in P(\Omega), i = 1, 2, \cdots, n$. Now since $A_n \in \mathcal{F}$ for all n, we have $\{A_1, A_2, \cdots\} \subseteq \{B_1, B_2, \cdots, B_n\}$ (definition of subset). Hence there exists $\{i_1, i_1, \cdots, i_k\} \subseteq \{1, 2, \cdots, n\}$ such that $\{A_1, A_2, \cdots\} = \{B_{i_1}, B_{i_1}, \cdots, B_{i_k}\}$. This is what one meant by A_n's repeat after some time (no ambiguity here). Hence

$$\cup_{n=1}^{\infty} A_n = \cup_{l=1}^{k} B_{i_l}.$$

i.e. the proof of countable union is indeed a finite union is complete. Now $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$, since $\mathcal{F}$ is a field.

Result 2: Let $\{A_1, A_2, \cdots, A_n\}$ be a partition of Ω. Then $\sigma(\{A_1, A_2, \cdots, A_n\})$ is the set of all finite union of A_i's.

Proof: Set

$$\mathcal{D} = \text{set of all finite union of } A_i's.$$

Our task to show that $\sigma(\{A_1, A_2, \cdots, A_n\}) = \mathcal{D}$.

By looking at the definition of $\sigma(\{A_1, A_2, \cdots, A_n\})$, we need to show that

1. $\mathcal{D}$ is a σ-field,
2. $A_1, A_2, \cdots, A_n \in \mathcal{D}$,
3. $\mathcal{D}$ is the smallest σ-field satisfying 2.

For checking 1., we need to verify the definition of σ-field for $\mathcal{D}$.
Note $\Omega = A_1 \cup \cdots \cup A_n$, a finite union of A_i's, and hence in $\mathcal{D}$.
For $A \in \mathcal{D}$, since A is a finite union of A_i's, $A = A_{i_1} \cup \cdots \cup A_{i_k}$ for some $i_1, \cdots, i_k$.
Now,

$$A^c = \bigcup_{i \in \{1,2,\cdots,n\} \setminus \{i_1,\cdots,i_k\}} A_i \quad \in \mathcal{D}.$$

Here keep in mind the following. When $A = \Omega$, then $\{i_1, i_2, \cdots, i_k\} = \{1, 2, \cdots, n\}$. Then

$$A^c = \bigcup_{i \in \{1,2,\cdots,n\} \setminus \{i_1,\cdots,i_k\}} A_i$$

is the empty union of sets which is $\emptyset$.

Now for $A = A_{i_1} \cup \cdots \cup A_{i_k}, B \ = A_{j_1} \cup \cdots \cup A_{j_l} \in \mathcal{D}$,

$$A \cup B \ = \ A_{i_1} \cup \cdots \cup A_{i_k} \cup A_{j_1} \cup \cdots \cup A_{j_l} \in \mathcal{D}.$$

Hence $\mathcal{D}$ is a field and hence a σ-field by using Problem 1.

For checking 2., note that each A_i is a 'one' union of A_i and hence are in $\mathcal{D}$. This prove 2.

Note that proving 3. is same as proving the statement 'if $\mathcal{G}$ is a σ-field containing $\{A_1, \cdots, A_n\}$, then $\mathcal{D} \subseteq \mathcal{G}$'.

Let $\mathcal{G}$ is a σ-field containing $\{A_1, \cdots, A_n\}$, then from the definition of σ-field, it contains all the finite union of A_i's. Hence $\mathcal{D} \subseteq \mathcal{G}$. That's all.

For additional reading-not a part of the syllabus

Recall that all the examples of probability spaces we had seen till now are with sample space finite or countable and the σ-field as the power set of the sample space. Now let us look at a random experiment with uncountable sample space and the σ-field as a proper subset of the power set.

Consider the random experiment in Example 1.0.5, i.e, pick a point 'at random' from the interval $(0,\ 1]$. Since point is picked 'at random', the probability measure should satisfy the following.

$$P[a, b] \ = \ P(a, b] \ = \ P[a, b) \ = \ P(a, b) \ = \ b - a\,. \tag{0.6}$$

The σ-field we are using to define P is $\mathcal{B}_{(0,1]}$, the σ-field generated by all intervals in $(0,\ 1]$. $\mathcal{B}_{(0,1]}$ is called the Borel σ-field of subsets of $(0,\ 1]$.

Our aim is to define P for all elements of $\mathcal{B}_{(0,\ 1]}$, preserving (0.6). Set

$\mathcal{B}_0 :=$ all finite union of intervals in $(0,\ 1]$ of the form $(a,\ b]$, $0 \le a \le b \le 1$.

Clearly $\Omega = (0,\ 1] \in \mathcal{B}_0$.

Let $A \in \mathcal{B}_0$. then A can be represented as

$$A \ = \ \cup_{i=1}^{n} I_i\,,$$

where $I_i \ = \ (a_i,\ b_i]$,

$$0 \le a_1 < b_1 \le a_2 < b_2 \le a_3 < b_3 \cdots \le a_m < b_m\,.$$

Then

$$A^c \ = \ J_1 \cup J_2 \cup \cdots \cup J_m \cup J_{m+1}\,,$$

where

$$J_1 = [0, a_1],\ J_i = (b_{i-1}, a_i], i = 2, 3, \cdots, m, J_{m+1} = (b_m, 1].$$

Therefore $A^c \in \mathcal{B}_0$.
For $A, B \in \mathcal{B}_0$, it follows from the definition of $\mathcal{B}_0$ that $A \cup B \in \mathcal{B}_0$.

Hence $\mathcal{B}_0$ is a field.

Define P on $\mathcal{B}_0$ as follows.

$$P(A) = \sum_{i=1}^{m} P(J_i), \tag{0.7}$$

where

$$A = \cup_{i=1}^{n} J_i\,,$$

J_i's are pair wise disjoint intervals of the form $(a,\ b]$.

Extension of P from $\mathcal{B}_0$ to $\mathcal{B}$ follows from the extension theorem by Caratheodary. To understand the statement of the extension theorem, we need the following definition.

Definition 1.8 (Probability measure on a field) Let Ω be a nonempty set and $\mathcal{F}$ be a field. Then $P : \mathcal{F} \to [0, 1]$ is said to be a probability measure on $\mathcal{F}$ if
(i) $P(\Omega) = 1$
(ii) if $A_1, A_2, \cdots \in \mathcal{F}$ be such that $A_i's$ are pairwise disjoint and $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$, then

$$P(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)\,.$$

Example 0.3 *The set function P given by (0.7) is a probability measure on the field $\mathcal{B}_0$.*

Theorem 0.2 *(Extension Theorem) A probability measure defined on a field $\mathcal{F}$ has a unique extension to $\sigma(\mathcal{F})$.*

Using Theorem 1.0.2, one can extend P defined by (0.7) to $\sigma(\mathcal{B}_0)$. Since

$$\sigma(\mathcal{B}_0) = \mathcal{B}_{(0,\,1]}\,,$$

there exists a unique probability measure P on $\mathcal{B}_{(0,\,1]}$ preserving (0.6).

Chapter 2: Random Variables-General Facts

Key words: Random variable, Borel σ-field of subsets of $\mathbb{R}$, σ-field generated by random variable.

This chapter explains some general ideas related to random variables. In many situations, one is interested only in some aspects of a random experiment/phenomenon. For example, consider the experiment of tossing 3 unbiased coins and we are only interested in number of 'Heads' turned up. The probability space corresponding to the experiment of tossing 3 coins is given by

$$\Omega \;=\; \{(\omega_1, \omega_2, \omega_3) \,|\, \omega_i \in \{H, T\}\} \,, \mathcal{F} \;=\; \mathcal{P}(\Omega)$$

and P is described by

$$P(\{(\omega_1, \omega_2, \omega_3)\}) \;=\; \frac{1}{8}\,.$$

Our interest is in knowing no. of 'Heads', i.e., in the map $(\omega_1, \omega_2, \omega_3) \to I_{\{\omega_1=H\}} + I_{\{\omega_2=H\}} + I_{\{\omega_3=H\}}$. Here I_A denotes the 'indicator' function of A defined by

$$I_A(\omega) \;=\; \begin{cases} 1 & \text{if} \quad \omega \in A \\ 0 & \text{if} \quad \omega \in A^c. \end{cases}$$

In general we are interested in functions of sample space. Also in many cases, one never observe directly the underlying random phenomenon but observe 'measurements' coming from it and these measurements can be modeled as functions of the sample space. So in short, mainly our interest lies in functions of sample space and its analysis to make 'conclusions' about the random phenomenon. But to study a function associated with a random phenomenon, one should be able to assign probabilities to 'reasonably large class of events associated with the function. But in general we can't do this for all functions defined on the sample space. So one need to restrict ourself to certain class of functions of the sample space for which we can do this and we call them random variables. In short, random variables are nothing but 'measurable observations' from a random phenomenon.

Definition 2.1 Let Ω be a sample space and $\mathcal{F}$ a σ-field of subsets of Ω. A function $X : \Omega \;\to \mathbb{R}$ is said to be a random variable (with respect to $\mathcal{F}$) if

$$\{\omega \in \Omega \,|\, X(\omega) \leq x\} \in \mathcal{F} \text{ for all } x \in \mathbb{R}\,.$$

Now on, we denote $\{\omega \in \Omega \,|\, X(\omega) \leq x\}$ by $\{X \leq x\}$. Also we supress Ω and $\mathcal{F}$ from the description of X when Ω and $\mathcal{F}$ are understood from the context, i.e. we just say that 'X is a random variable'

Remark 0.1 *(1) In the definition of random variable, the probability measure P is not in the picture.*

(2) Also it is possible that a function which is not be a random variable with respect to a σ-field becomes a random variable with respect to another σ-field.

Example 0.4 *Let $\Omega = \{H, T\}$, $\mathcal{F} = \mathcal{P}(\Omega)$. Define $X : \Omega \to \mathbb{R}$ as follows:*

$$X(H) = 1 \,, \; X(T) = 0 \,.$$

Then X is a random variable.

Note that any map $Z : \Omega \to \mathbb{R}$ is a random variable for the probability space given in Example 0.4.

Now, we give an example, where this is not the case.

Example 0.5 *Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\mathcal{F} = \sigma(\{\{2\}, \{4\}, \{6\}, \{1, 3, 5\}\})$. Then $X(1) = 1, X(\omega) = 0$ otherwise is not a random variable with respect to $\mathcal{F}$. Note that X is a random variable with respect to the σ-field $\mathcal{P}(\Omega)$.*

X is a random variable imply that the 'basic events' $X^{-1}\Big((-\infty, \; x]\Big)$ associated with X are in $\mathcal{F}$. Does this imply a larger class of events associated with X are in $\mathcal{F}$ (i.e. can be assigned probabilities) ? To examine this, one need the following σ-field.

Definition 2.2 The σ-field generated by the collection of all open sets in $\mathbb{R}$ is called the **Borel σ-field** of subsets of $\mathbb{R}$ and is denoted by $\mathcal{B}_{\mathbb{R}}$. Also if $B \in \mathcal{B}_{\mathbb{R}}$, then B is said to be a **Borel set**.

Lemma 0.1 *Let $\mathcal{I}_1 = \{(-\infty, \, x] \,|\, x \in \mathbb{R}\}$. Then $\sigma(\mathcal{I}_1) = \mathcal{B}_{\mathbb{R}}$*

Proof. For $x \in \mathbb{R}$,

$$(-\infty, \; x] = (x, \; \infty)^c \in \mathcal{B}_{\mathbb{R}} \,.$$

Therefore

$$\mathcal{I}_1 \subseteq \mathcal{B}_{\mathbb{R}} \,.$$

Hence

$$\sigma(\mathcal{I}_1) \subseteq \mathcal{B}_{\mathbb{R}} \,.$$

For $x \in \mathbb{R}$,

$$(-\infty,\ x)\ =\ \cup_{n=1}^{\infty}\Big(-\infty,\ x-\frac{1}{n}\Big].$$

Therefore

$$\{(-\infty,\ x)\,|\,x \in \mathbb{R}\} \subseteq \sigma(\mathcal{I}_1)\,.$$

Also, for $a,\ b \in \mathbb{R},\ a < b$,

$$(a,\ b)\ =\ (-\infty,\ b) \setminus (-\infty,\ a] \in \sigma(\mathcal{I}_1)\,.$$

Therefore, $\sigma(\mathcal{I}_1)$ contains all open intervals for the form (a, b). Since any open set in $\mathbb{R}$ can be written as a (countable) union of open intervals with rational end points,[1] it follows that

$$\mathcal{O} \subseteq \sigma(\mathcal{I}_1)\,,$$

where $\mathcal{O}$ denote the set of all open sets in $\mathbb{R}$. Thus,

$$\mathcal{B}_{\mathbb{R}} \subseteq \sigma(\mathcal{I}_1)\,.$$

This completes the proof.

[1]Recall that from the definition of open set, given any element of the open set there is an open interval containg the element contained in the open set. Now using the property that ' between any two reals there is a rational' we can choose another open interval with rational end points containing the element and contained in the open set.