CS 405/6001: Game Theory and Algorithmic Mechanism Design

Problem Set 1
Course Homepage

[Week 1]

1. "The outcome of every play of the game of chess is either a victory for White, a victory for Black, or a draw." Is this statement equivalent to the result of von Neumann? Justify your answer.
   Note: the result of von Neumann states that,
   "In chess, one and only one of the following statements is true
   - W has a winning strategy
   - B has a winning strategy
   - Each player has a draw guaranteeing strategy"

2. Consider a simplified version of chess, where the game starts with only two kings on the board: a white king (W) and a black king (B). The rules of the game are as follows:
   - Both kings can move one square in any direction (standard chess king moves).
   - If one king can move to the square occupied by the other king, the game ends, and the player whose king made the move wins.
   - If neither player can move their king without being captured in the next move, the game ends in a draw.

   Now in this game, answer the following:

   (a) Can White guarantee a draw or a win in this game regardless of Black's moves? If so, what is the strategy?

   (b) Can Black guarantee a draw or a win in this game regardless of White's moves? If so, what is the strategy?

3. In a game of chess, two players (White and Black) are rational and intelligent. White always moves first. Let's assume the game is in a specific situation where both players know that a sequence of exactly 4 moves will lead to checkmate by White. However, this sequence is not obvious, and Black is still trying to defend.

   White can either:
   - Make a safe move that preserves the game state but delays checkmate, or
   - Attempt the checkmate sequence, which requires White to predict all of Black's responses perfectly.

   Black can either:
   - Try to prevent checkmate by playing the best defensive moves, or

- Resign, acknowledging that checkmate is inevitable.

Based on this game situation, answer the following:

(a) If both players are rational and intelligent, what should White do?

(b) What should Black do?

(c) What is the likely outcome of the game?

4. Five individuals (A, B, C, D, and E) are deciding whether to adopt a new policy. The policy will be adopted if and only if all five individuals vote in favor. Each person knows their own preference but does not know the preferences of others. They all know, however, that at least three people want the policy to be adopted. This information is common knowledge.

   Each individual gains 3 points if the policy is adopted and they voted for it. They gain 0 points if the policy is adopted and they voted against it. If the policy is not adopted, they gain 1 point regardless of how they voted.

   For this voting, answer the following:

   (a) What will each rational and intelligent individual do?

   (b) Will the policy be adopted?

5. Five pirates of varying ages have a treasure of 100 gold coins. On their ship, they decide to divide the coins according to this method:

   The oldest pirate suggests a way to distribute the coins, and all pirates (including the oldest) cast a vote either in favor or against the proposal. If 50% or more of the pirates vote in favor, the coins are divided as proposed. If not, the proposing pirate is thrown overboard, and the process begins again with the remaining pirates.

   Given that pirates are a ruthless bunch, if a pirate would receive the same number of coins whether they voted for or against a proposal, they would vote against it to ensure the proposing pirate is thrown overboard.

   Assuming all five pirates are intelligent, rational, greedy, and want to stay alive (and surprisingly good at math for pirates), what will the outcome be?

6. Cannibals ambush a safari in the jungle and capture three men, giving them a single chance to escape without being eaten.

   The captives are lined up in order of height and tied to stakes. The man at the back can see the hats on the heads of the two men in front of him, the middle man can only see the hat of the man in front, and the man at the front cannot see anyone's hat. The cannibals show the men five hats—three black and two white.

   Afterward, blindfolds are placed over each man's eyes, and a hat is put on each of their heads, with the two remaining hats hidden. The blindfolds are then removed, and the cannibals tell the men that if one of them can correctly guess the color of the hat he is wearing, they will all be set free.

The man at the rear, who can see both of his companions' hats but not his own, says, "I don't know." The man in the middle, who can see only the hat of the man in front, also says, "I don't know." But the man at the front, who cannot see anyone's hat, confidently says, "I know!"

How did he figure out the color of his hat, and what color was it? Your response should explain the logical conclusions that led each cannibal to react the way they did.

[Week 2]

7. Find a game that has at least one equilibrium, but in which iterative elimination of dominated strategies yields a game with no equilibria.

8. In a first-price auction, participants submit their bids in sealed envelopes. The bidder with the highest bid wins the auction and is required to pay the amount they bid. If there are ties, meaning multiple bidders have the highest bid, a random draw determines which of these top bidders wins, and the winning bidder pays their bid amount.

   (a) In this situation, does the strategy β_i^* of buyer i, in which he bids his private value for the item, weakly dominate all his other strategies?

   (b) Find a strategy of buyer i that weakly dominates strategy β_i^*.

   Does the strategy of bidding one's private value weakly dominate all other strategies? Provide an explanation for your answer.

9. In the three-player game described, where Player I selects a row (A or B), Player II chooses a column (a or b), and Player III picks a matrix (α, β, or γ), determine all the equilibria of the game.

	a	b
A	0, 0, 5	0, 0, 0
B	2, 0, 0	0, 0, 0

α

	a	b
A	1, 2, 3	0, 0, 0
B	0, 0, 0	1, 2, 3

β

	a	b
A	0, 0, 0	0, 0, 0
B	0, 5, 0	0, 0, 4

γ

10. Provide an example of a game $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ in strategic form where the game $\hat{G}$, which is obtained by eliminating one strategy from one player's strategy set, has an equilibrium that does not exist in the original game G.

11. Let A and B be two finite sets, and let $u : A \times B \to \mathbb{R}$ be an arbitrary function. Prove that

$$\max_{a \in A} \min_{b \in B} u(a, b) \leq \min_{b \in B} \max_{a \in A} u(a, b)$$

12. In a game involving fifty players, each participant writes down an integer from the set $\{0, 1, \ldots, 100\}$ on a separate slip of paper and submits it. The game master then calculates

the average x of all the numbers submitted. The winner is the player (or players) whose number is closest to $\frac{2}{3}x$. The prize of \$1,000 is shared equally among the winners.

Describe this as a strategic-form game and determine all the Nash equilibria. What would your strategy be in this game, and why?

[Week 3]

13. Show that in any n-person game, the payoff for each player at a Nash equilibrium is always at least as high as their maxmin value.

14. For each of the following games, where Player I is the row player and Player II is the column player:

    (a) Find all the equilibria in mixed strategies, and all the equilibrium payoffs.

    (b) Find each player's maxmin strategy.

    (c) What strategy would you advise each player to use in the game?

	L	R
T	5, 5	0, 8
B	8, 0	1, 1

Game A

	L	R
T	9, 5	10, 4
B	8, 4	15, 6

Game B

	L	R
T	5, 16	15, 8
B	16, 7	8, 15

Game C

	L	R
T	8, 3	10, 1
B	6, −6	3, 5

Game D

	L	R
T	4, 12	5, 10
B	3, 16	6, 22

Game E

	L	R
T	2, 2	3, 3
B	4, 0	2, −2

Game F

	L	R
T	15, 3	15, 10
B	15, 4	15, 7

Game G

15. Assume Country A builds facilities for nuclear weapons development. Country B sends a spy ring with quality α to determine whether Country A is indeed developing nuclear weapons and is considering bombing the facilities. The spy ring from Country B will accurately report nuclear development with probability α if it is occurring, and will incorrectly report it with probability $1-\alpha$. Conversely, if no nuclear development is happening, the spy ring will correctly report this with probability α and falsely indicate development with probability $1-\alpha$. Country A must decide on nuclear development, while Country B, based on the spy report, must choose whether to bomb Country A's facilities. The payoffs for both countries are detailed in the following table.

		Country B	
		Bomb	Don't Bomb
Country A	Don't Develop	$\frac{1}{2}, \frac{1}{2}$	$\frac{3}{4}, 1$
	Develop	$0, \frac{3}{4}$	$1, 0$

(a) Depict this situation as a strategic-form game. Are there any dominating strate- gies in the game?

(b) Describe what it means to say that the quality of Country B's spy ring is $\alpha = \frac{1}{2}$. What if $\alpha = 1$?

(c) For each $\alpha \in [\frac{1}{2}, 1]$, find the game's set of equilibria.

(d) What is the set of equilibrium payoffs as a function of α? What is the α at which Country A's maximal equilibrium payoff is obtained? What is the α at which Country B's maximal equilibrium payoff is obtained?

(e) Assuming both countries play their equilibrium strategy, what is the probability that Country A will manage to develop nuclear weapons without being bombed?

16. Ten individuals are arrested for a crime, but the police do not have enough resources to conduct a thorough investigation. Consequently, the chief investigator offers the following deal: if at least one suspect confesses, each confessing suspect will receive a one-year jail term, while those who do not confess will be released. If no one confesses, the police will extend their investigation, and if they still fail to identify the culprit, all suspects will face a ten-year prison sentence.

    (a) Represent this scenario as a strategic-form game where the players are the arrested individuals. Each player's utility is calculated as 10 minus the number of years they spend in jail.

    (b) Identify all the equilibrium points in pure strategies. What does such an equilibrium signify intuitively, and under what conditions is it plausible for this equilibrium to be achieved?

    (c) Determine a symmetric equilibrium in mixed strategies. What is the probability that, at this equilibrium, no one chooses to confess?

    (d) Assume there are n suspects instead of 10. Find a symmetric equilibrium in mixed strategies. As n approaches infinity, what is the limit of the probability that no one volunteers to confess in this symmetric equilibrium? What implications can we draw from this analysis regarding volunteering in large groups?

    Note : Symmetric equilibrium in mixed strategies: an equilibrium $\sigma = (\sigma_i)_{i \in N}$ satisfying $\sigma_i = \sigma_j$ for each $i, j \in N$

17. Consider the following two-player game composed of two stages. In the first stage, one of the two following matrices is chosen by a coin toss (with each matrix chosen with probability $\frac{1}{2}$). In the second stage, the two players play the strategic-form game whose payoff matrix is given by the matrix that has been chosen.

		Karan		
		L	C	R
Isha	T	$0, 0$	$1, -1$	$-1, 10$
	B	$-2, -2$	$-2, -2$	$-3, -12$

		Karan		
		L	C	R
Isha	T	$-1, 1$	$-2, -1$	$-2, -11$
	B	$1, \frac{1}{2}$	$-1, 0$	$-1, 10$

For each of the following cases, find the unique equilibrium:

(a) No player knows which matrix was chosen.

(b) Isha knows which matrix was chosen, but Karan does not know which matrix was chosen.

What effect does adding information to Isha have on the payoffs to the players at equilibrium?

18. Consider a lottery game with n participants competing for a prize of $\$M$ (where $M > 1$). Each player can buy as many numbers as they want from the set $1, 2, \ldots, K$, at a cost of \$1 per number. After all numbers are purchased, the game identifies the numbers that were bought by exactly one player. The winning number is the smallest among these unique numbers. The player who purchased this winning number claims the entire prize. If no number is bought by only one player, no prize is awarded.

    (a) Write down every player's set of pure strategies and payoff function.

    (b) Prove that a symmetric equilibrium is present, meaning there is an equilibrium where all players employ the same mixed strategy.

    (c) Given $p_1 \in (0, 1)$, examine the mixed strategy $\sigma_i(p_1)$ for player i: with probability p_1, the player selects only the number 1, while with probability $1 - p_1$, the player abstains from purchasing any numbers. What specific conditions must be met by M, n, and p_1 to ensure that the strategy profile where each player adopts $\sigma_i(p_1)$ constitutes a symmetric equilibrium?

    (d) Show that if at equilibrium there is a positive probability that player i will not purchase any number, then his expected payoff is 0.

    (e) Demonstrate that if $M < n$, indicating that the number of participants exceeds the prize value at equilibrium, there is a non-zero probability that no player buys a number. From this, conclude that in every symmetric equilibrium, the expected payoff for each player is zero. (Hint: Prove that if every player buys at least one number with probability 1, the expected total number of numbers purchased by all players exceeds the prize value M, leading to at least one player having a negative expected payoff.)

[Week 4]

19. Prove that in every correlated equilibrium, the payoff to each player i is at least his maxmin value in mixed strategies.
$$\underline{v}_i = \max_{\sigma_i \in \Delta S_i} \min_{\sigma_{-i} \in \Delta S_{-i}} U_i(\sigma_i, \sigma_{-i}).$$

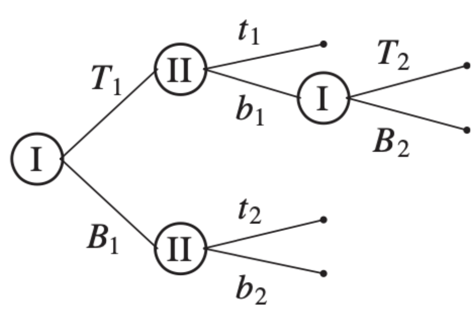
20. Show that there exists a unique correlated equilibrium in the following game, in which $a, b, c, d \in (-\frac{1}{4}, \frac{1}{4})$. Find this correlated equilibrium. What is the limit of the correlated equilibrium payoff as a, b, c, and d approach 0?

		Player II L	R
Player I	T	$1, 0$	$c, 1+d$
	B	$0, 1$	$1+a, b$

21. Represent the following scenario as a game in extensive form. Ehan, Lalit, and Sushant are senior partners at a law firm evaluating candidates to join their team. The candidates under consideration are Krish, Roshni, and Jamal. The decision-making process, reflecting the hierarchy among the three partners, is as follows:

    - Ehan makes the initial proposal of one of the candidates.
    - Lalit follows by proposing a candidate of his own (who may be the same candidate that Ehan proposed).
    - Sushant then proposes a candidate (who may be one of the previously proposed candidates).
    - A candidate who receives the support of two of the partners is accepted into the firm. If no candidate has the support of two partners, all three candidates are rejected.

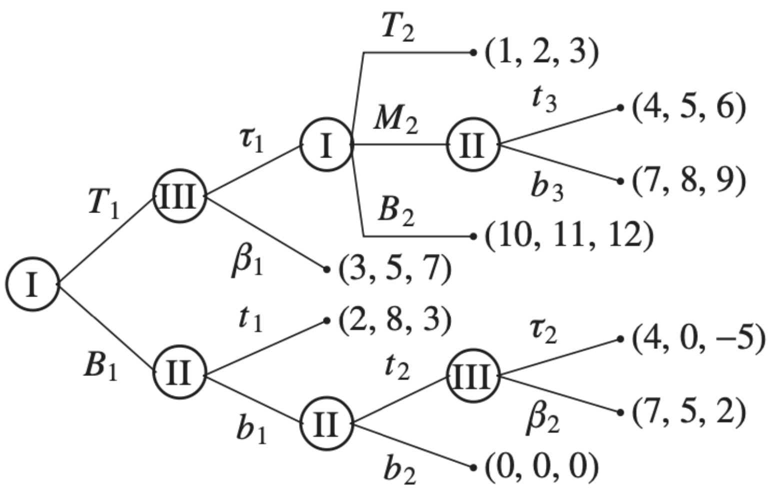
22. By definition, a player's strategy prescribes his selected action at each vertex in the game tree. Consider the following game. Player I has four strategies, $T_1T_2, T_1B_2, B_1T_2, B_1B_2$. Two of these strategies, B_1T_2 and B_1B_2, regardless of the strategy used by Player II, yield the same play of the game, because if Player I has selected action B_1 at the root vertex, he will never get to his second decision vertex. We can therefore eliminate one of these two strategies and define a *reduced strategy* B_1, which only stipulates that Player I chooses B_1 at the root of the game.

    In the game appearing in the above figure, the reduced strategies of Player I are T_1T_2, T_1B_2, and B_1. The reduced strategies of Player II are the same as his regular strategies, t_1t_2, t_1b_2, b_1t_2, and b_1b_2, because Player II does not know to which vertex Player I's choice will lead. Formally, a *reduced strategy* τ of player i is a function from a subcollection $\hat{U}_i$ of player i's collection of information sets to actions, satisfying the following two conditions:

(i) For any strategy vector of the remaining players σ_{-i}, given the vector (τ_i, σ_{-i}), the game will definitely not get to an information set of player i that is not in the collection $\hat{U}_i$.

(ii) There is no strict subcollection of $\hat{U}_i$ satisfying condition (i).



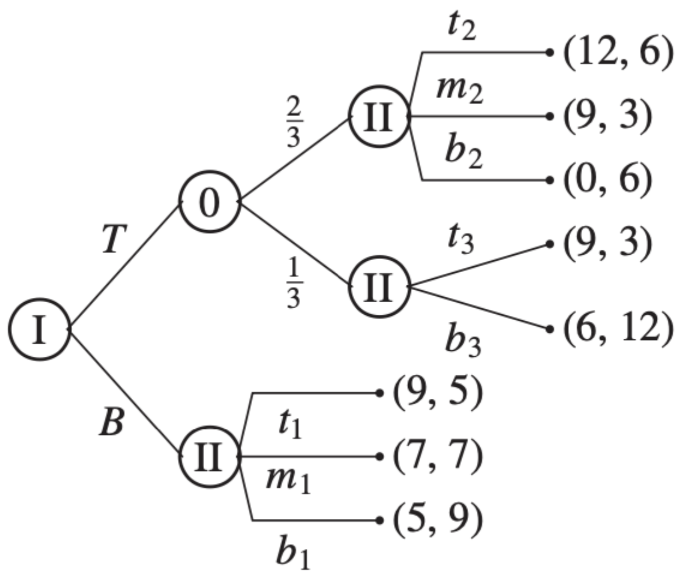
(a) List the reduced strategies of each of the players in the game depicted in the above figure.

(b) What outcome of the game will obtain if the three players make use of the reduced strategies $(B_1), (t_1, t_3), (\beta_1, \tau_2)$?

(c) Can any player increase his payoff by unilaterally making use of a different strategy (assuming that the other two players continue to play according to the strategies of part (b))?

23. Consider the following game: Two players take turns placing quarters on a round table. The coins must not be stacked on top of each other, although they can touch. Each quarter must be fully placed on the table. The player who cannot place a quarter on the table on their turn, without stacking it on an already placed coin, loses the game, and the other player wins. Demonstrate that the first player has a winning strategy.

24. In the game depicted below , if Player I chooses T , there is an ensuing chance move, after which Player II has a turn, but if Player I chooses B, there is no chance move, and Player II has an immediately ensuing turn (without a chance move). The outcome of the game is a pair of numbers (x, y) in which x is the payoff for Player I and y is the payoff for Player II.



(a) What are all the strategies available to Player I?

(b) How many strategies has Player II got? List all of them.

(c) What is the expected payoff to each player if Player I plays B and Player II plays (t_1, b_2, t_3)?

(d) What is the expected payoff to each player if Player I plays T and Player II plays (t_1, b_2, t_3)?