3 Extensive-form games

Chapter summary

In this chapter we introduce a graphic way of describing a game, the description in *extensive form*, which depicts the rules of the game, the order in which the players make their moves, the information available to players when they are called to take an action, the termination rules, and the outcome at any terminal point. A game in extensive form is given by a *game tree*, which consists of a directed graph in which the set of vertices represents positions in the game, and a distinguished vertex, called the *root*, represents the starting position of the game. A vertex with no outgoing edges represents a terminal position in which play ends. To each terminal vertex corresponds an outcome that is realized when the play terminates at that vertex. Any nonterminal vertex represents either a chance move (e.g., a toss of a die or a shuffle of a deck of cards) or a move of one of the players. To any chance-move vertex corresponds a probability distribution over the edges emanating from that vertex, which correspond to the possible outcomes of the chance move.

To describe games with imperfect information, in which players do not necessarily know the full board position (like poker), we introduce the notion of *information sets*. An information set of a player is a set of decision vertices of the player that are indistinguishable by him given his information at that stage of the game. A game of *perfect information* is a game in which all information sets consist of a single vertex. In such a game whenever a player is called to take an action, he knows the exact history of actions and chance moves that led to that position.

A *strategy* of a player is a function that assigns to each of his information sets an action available to him at that information set. A path from the root to a terminal vertex is called a *play* of the game. When the game has no chance moves, any vector of strategies (one for each player) determines the play of the game, and hence the outcome. In a game with chance moves, any vector of strategies determines a probability distribution over the possible outcomes of the game.

This chapter presents the theory of games in extensive form. It will be shown that many familiar games, including the game of chess studied in Chapter 1, can be described formally as extensive-form games, and that Theorem 1.4 can be generalized to every finite extensive-form game.

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3.1 An example

How does one describe a game? Every description of a game must include the following elements:

- A set of players (decision makers).
- The possible actions available to each player.
- Rules determining the order in which players make their moves.
- A rule determining when the game ends.
- A rule determining the outcome of every possible game ending.

A natural way to depict a game is graphically, where every player's action is depicted as a transition from one vertex to another vertex in a graph (as we saw in Figure 1.1 for the game of chess).

Example 3.1 Consider the simple game shown in Figure 3.1. We start with a table with four squares, labeled 1, 2, 3, and 4.

2	4
1	3

Figure 3.1 The game board in Example 3.1

Two players, labeled Players I and II, participate in the game. Player I has the opening move, in which he "captures" one of the squares. By alternate turns, each player captures one of the squares, subject to the following conditions:

1. A square may be captured by a player only if it has not been previously captured by either player.
2. Square 4 may not be captured if Square 2 or Square 3 has been previously captured.
3. The game ends when Square 1 is captured. The player who captures Square 1 is the losing player.

A graphic depiction of this game appears in Figure 3.2.

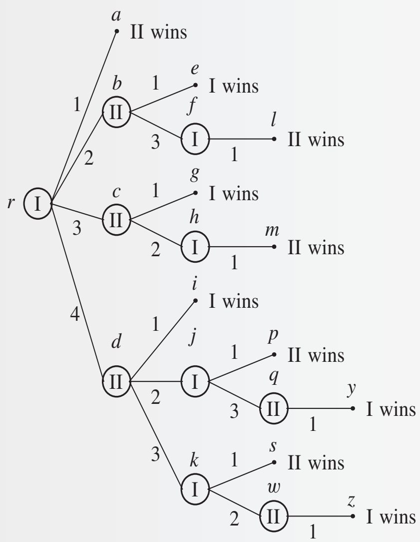

Figure 3.2 The game tree in Example 3.1

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Every circled vertex in Figure 3.2 represents a decision by a player, and is labeled with the number of that player. The terminal vertices of the game are indicated by dark dots. The edges of the graph depict game actions. The number that appears next to each edge corresponds to the square that is captured. Next to every terminal vertex, the corresponding game outcome is indicated. A game depicted by such a graph is called a **game in extensive form**, or **extensive-form game**. ◄

As the example illustrates, a graph that describes a game has a special structure, and is sometimes called a **game tree**. To provide a formal definition of a game tree, we first define a tree.

3.2 Graphs and trees

Definition 3.2 *A (finite)* directed graph *is a pair $G = (V, E)$, where:*

- *V is a finite set, whose elements are called* vertices.
- *$E \subseteq V \times V$ is a finite set of pairs of vertices, whose elements are called* edges. *Each directed edge is composed of two vertices: the two ends of the edge (it is possible for both ends of a single edge to be the same vertex).*

A convenient way of depicting a graph geometrically is by representing each vertex by a dot and each edge by an arrow (a straight line, an arc, or a circle) connecting two vertices. Illustrative examples of geometric depictions of graphs are presented in Figure 3.3.

Remark 3.3 *Most of the games that are described in this book are finite games, and can therefore be represented by finite graphs. But there are infinite games, whose representation requires infinite graphs.* ♦

Definition 3.4 *Let x^1 and x^{K+1} be two vertices in a graph G. A* path *from x^1 to x^{K+1} is a finite sequence of vertices and edges of the form*

$$x^1, e^1, x^2, e^2, \ldots, e^K, x^{K+1} \tag{3.1}$$

in which the vertices are distinct: $e^k \neq e^l$ for every $k \neq l$ and for $1 \leq k \leq K$, the edge e^k connected vertex x^k with vertex x^{k+1}. The number K is called the path length. *A path is called* cyclic *if $K \geq 1$ and $x^1 = x^{K+1}$.*

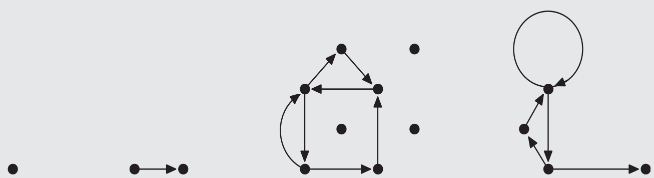

Figure 3.3 Examples of graphs

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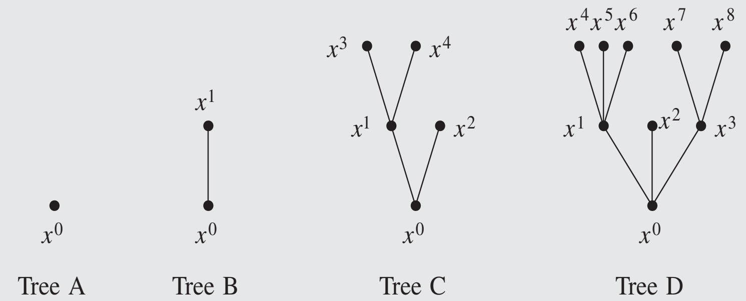

Figure 3.4 Examples of trees

Definition 3.5 *A* tree *is a triple* $G = (V, E, x^0)$, *where* (V, E) *is a directed graph,* $x^0 \in V$ *is a vertex called the* root *of the tree, and for every vertex* $x \in V$ *there is a unique path in the graph from* x^0 *to* x.

The definition of a tree implies that a graph containing only one vertex is a tree: the triple $(\{x^0\}, \emptyset, x^0)$ is a tree. The requirement that for each vertex $x \in V$ there exists a unique path from the root to x guarantees that if there is an edge from a vertex $\widehat{x}$ to a vertex x then $\widehat{x}$ is "closer" to the root than x: the path leading from the root to x passes through $\widehat{x}$ (while the path from the root to $\widehat{x}$ does not pass through x). It follows that there is no need to state explicitly the directions of the edges in the tree. Figure 3.4 shows several trees. Tree A contains only one vertex, the root x^0. Tree B contains two vertices and one edge, from x^0 to x^1. Tree C contains four edges, from x^0 to x^1, from x^0 to x^2, from x^1 to x^3, and from x^1 to x^4.

A vertex x is called a *child* of a vertex $\widehat{x}$ if there is a directed edge from $\widehat{x}$ to x. For example, in the tree in Figure 3.2, g and h are children of c, and s and w are children of k. A vertex x is a *leaf* (or a *terminal point*) if it has no children, meaning that there are no directed edges emanating from x.

3.3 Game trees

Various games can be represented by trees. When a tree represents a game, the root of the tree corresponds to the *initial position* of the game, and every *game position* is represented by a vertex of the tree. The children of each vertex v are the vertices corresponding to the game positions that can be arrived at from v via one action. In other words, the number of children of a vertex is equal to the number of possible actions in the game position corresponding to that vertex.

Figure 3.2 indicates that in addition to the game tree, we need two further components in order to describe a game fully:

- For every vertex that is not a leaf, we need to specify the player who is to take an action at that vertex.
- At each leaf, we need to describe the outcome of the game.

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Definition 3.6 *Let B be a nonempty set. A* partition *of B is a collection $B_1, B_2, \ldots, B_K$ of pairwise disjoint and nonempty subsets of B whose union is B.*

We are now ready for the first definition of a game in extensive form. We will later add more elements to this definition.

Definition 3.7 *A* game in extensive form *(or* extensive-form game*) is an ordered vector*[1]

$$\Gamma = (N, V, E, x^0, (V_i)_{i \in N}, O, u), \tag{3.2}$$

where:

- *N is a finite set of players.*
- *(V, E, x^0) is a tree called the* game tree.
- *$(V_i)_{i \in N}$ is a partition of the set of vertices that are not leaves.*
- *O is the set of possible game outcomes.*
- *u is a function associating every leaf of the tree with a game outcome in the set O.*

By "possible outcome" we mean a detailed description of what happens as a result of the actions undertaken by the players. Some examples of outcomes include:

1. Player I is declared the winner of the game, and Player II the loser.
2. Player I receives \$2, Player II receives \$3, and Player III receives \$5.
3. Player I gets to go out to the cinema with Player II, while Player III is left at home.
4. If the game describes bargaining between two parties, the outcome is the detailed description of the points agreed upon in the bargaining.
5. In most of the following chapters, an outcome $u(x)$ at a leaf x will be a vector of real numbers representing the utility[2] of each player when a play reaches leaf x.

For each player $i \in N$, the set V_i is player i's set of decision vertices, and for each leaf x, the outcome at that leaf is $u(x)$.

Note that the partition $(V_i)_{i \in N}$ may contain empty sets. We accept the possibility of empty sets in $(V_i)_{i \in N}$ in order to be able to treat games in which a player may not be required to make any moves, but is still a game participant who is affected by the outcome of the game.

In the example in Figure 3.2,

$$\begin{aligned} N &= \{\mathrm{I}, \mathrm{II}\}, \\ V &= \{r, a, b, c, d, e, f, g, h, i, j, k, l, m, p, q, s, w, y, z\}, \\ x^0 &= r, \\ V_{\mathrm{I}} &= \{r, f, h, j, k\}, \\ V_{\mathrm{II}} &= \{b, c, d, q, w\}. \end{aligned}$$

The set of possible outcomes is

$$O = \{\text{I wins, II wins}\}, \tag{3.3}$$

1 The word "ordered" indicates the convention that the elements of the game in extensive form appear in a specific order: the first element is the set of players, the second is the set of vertices, etc.

2 The subject of utility theory is discussed in Chapter 2.

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and the function u is given by

$$u(a) = u(l) = u(m) = u(p) = u(s) = \text{II wins},$$
$$u(e) = u(g) = u(i) = u(y) = u(z) = \text{I wins}.$$

The requirement that $(V_i)_{i \in N}$ be a partition of the set of vertices that are not leaves stems from the fact that at each game situation there is one and only one player who is called upon to take an action. For each vertex x that is not a leaf, there is a single player $i \in N$ for whom $x \in V_i$. That player is called the *decision maker at vertex* x, and denoted by $J(x)$. In the example in Figure 3.2,

$$J(r) = J(f) = J(h) = J(j) = J(k) = \text{I},$$
$$J(b) = J(c) = J(d) = J(q) = J(w) = \text{II}.$$

Denote by $C(x)$ the set of all children of a non-leaf vertex x. Every edge that leads from x to one of its children is called a possible *action* at x. We will associate every action with the child to which it is connected, and denote by $A(x)$ the set of all actions that are possible at the vertex x. Later, we will define more complicated games, in which such a mapping between the possible actions at x and the children of x does not exist.

An extensive-form game proceeds in the following manner:

- Player $J(x^0)$ initiates the game by choosing a possible action in $A(x^0)$. Equivalently, he chooses an element x^1 in the set $C(x^0)$.
- If x^1 is not a leaf, Player $J(x^1)$ chooses a possible action in $A(x^1)$ (equivalently, an element $x^2 \in C(x^1)$).
- The game continues in this manner, until a leaf vertex x is reached, and then the game ends with outcome $u(x)$.

By definition, the collection of the vertices of the graph is a finite set, so that the game necessarily ends at a leaf, yielding a sequence of vertices $(x^0, x^1, \ldots, x^k)$, where x^0 is the root of the tree, x^k is a leaf, and $x^{l+1} \in C(x^l)$ for $l = 0, 1, \ldots, k-1$. This sequence is called a *play*.[3] Every play ends at a particular leaf x^k with outcome $u(x^k)$. Similarly, every leaf x^k determines a unique play, which corresponds to the unique path connecting the root x^0 with x^k.

It follows from the above description that every player who is to take an action knows the current state of the game, meaning that he knows all the actions in the game that led to the current point in the play. This implicit assumption is called *perfect information*, an important concept to be studied in detail when we discuss the broader family of games with imperfect information. Definition 3.7 therefore defines extensive-form games with perfect information.

Remark 3.8 *An extensive-form game, as defined here, is a finite game: the number of vertices V is finite. It is possible to define extensive-form games in which the game tree*

3 Note carefully the words that are used here: a *game* is a general description of rules, whereas a *play* is a sequence of actions conducted in a particular instance of playing the game. For example, chess is a game; the sequence of actions in a particular chess match between two players is a play.

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(V, E, x^0) is infinite. When the game tree is infinite, there are two possibilities to be considered. It is possible that the depth of the tree is bounded, i.e., that there exists a natural number L such that the length of every path in the tree is less than or equal to L. This corresponds to a game that ends after at most L actions have been played, and there is at least one player who has an infinite number of actions available at an information set. The other possibility is that the depth of the vertices of the tree is not bounded; that is, there exists an infinite path in the game tree. This corresponds to a game that might never end. The definition of extensive-form game can be generalized to the case in which the game tree is infinite. Accomplishing this requires implementing mathematical tools from measure theory that go beyond the scope of this book. With the exception of a few examples in this chapter, we will assume here that extensive-form games are finite. ♦

We are now ready to present one of the central concepts of game theory: the concept of strategy. A strategy is a prescription for how to play a game. The definition is as follows.

Definition 3.9 *A* strategy *for player i is a function s_i mapping each vertex $x \in V_i$ to an element in $A(x)$(equivalently, to an element in $C(x)$).*

According to this definition, a strategy includes instructions on how to behave at each vertex in the game tree, including vertices that previous actions by the player preclude from being reached. For example, in the game of chess, even if White's strategy calls for opening by moving a pawn from c2 to c3, the strategy must include instructions on how White should play in his second move if in his first move he instead moved a pawn from c2 to c4, and Black then took his action.

The main reason this definition is used is its simplicity: it does not require us to provide details regarding which vertices need to be dealt with in the strategy and which can be ignored. We will later see that this definition is also needed for further developments of the theory, which take into account the possibility of errors on the part of players, leading to situations that were unintended.

Definition 3.10 *A* strategy vector *is a list of strategies $s = (s_i)_{i \in N}$, one for each player.*

Player i's set of strategies is denoted by S_i, and the set of all strategy vectors is denoted $S = S_1 \times S_2 \times \ldots S_n$. Every strategy vector $s = (s_i)_{i \in N}$ determines a unique play (path from the root to a leaf). The play that is determined by a strategy vector $s = (s_i)_{i \in N}$ is $(x^0, x^1, x^2, \ldots, x^k)$, where x^1 is the choice of player $J(x^0)$, based on his strategy, x^2 is the choice of player $J(x^1)$ based on his strategy, and so on, and x^k is a leaf. The play corresponds to the terminal point x^k (with outcome $u(x^k)$), which we also denote by $u(s)$.

We next proceed to define the concept of subgame:

Definition 3.11 *Let $\Gamma = (N, V, E, x^0, (V_i)_{i \in N}, O, u)$ be an extensive-form game (with perfect information), and let $x \in V$ be a vertex in the game tree. The* subgame *starting at x, denoted by $\Gamma(x)$, is the extensive-form game $\Gamma(x) = (N, V(x), E(x), x, (V_i(x))_{i \in N}, O, u)$, where:*

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- *The set of players N is as in the game Γ.*
- *The set of vertices $V(x)$ includes x, and all the vertices that are descendants of x in the game tree (V, E, x^0), that is, the children of x, their children, the children of these children, and so on.*
- *The set of edges $E(x)$ includes all the edges in E that connect the vertices in $V(x)$.*
- *The set of vertices at which player i is a decision maker is $V_i(x) = V_i \cap V(x)$.*
- *The set of possible outcomes is the set of possible outcomes in the game Γ.*
- *The function mapping leaves to outcomes is the function u, restricted to the set of leaves in the game tree $(V(x), E(x), x)$.*

The original game Γ is itself a subgame: $\Gamma(x^0) = \Gamma$. In addition, every leaf x defines a subgame in which no player can make a choice. We next focus on games with two players, I and II, whose set of outcomes is $O = \{\text{I wins, II wins, Draw}\}$. We will define the concepts of a winning strategy and a strategy guaranteeing at least a draw for such games.

Definition 3.12 *Let Γ be an extensive-form game with Players I and II, whose set of outcomes is $O = \{I\ wins, II\ wins, Draw\}$. A strategy s_{I} of Player I is called a* winning strategy *if*

$$u(s_{\mathrm{I}}, s_{\mathrm{II}}) = I\ wins, \quad \forall s_{\mathrm{II}} \in S_{\mathrm{II}}. \tag{3.4}$$

A strategy s_{I} of Player I is called a strategy guaranteeing at least a draw *if*

$$u(s_{\mathrm{I}}, s_{\mathrm{II}}) \in \{I\ wins, Draw\}, \quad \forall s_{\mathrm{II}} \in S_{\mathrm{II}}. \tag{3.5}$$

A winning strategy for Player II, and a strategy guaranteeing at least a draw for Player II, are defined similarly.

Theorem 3.13 (Von Neumann [1928]) *In every two-player game (with perfect information) in which the set of outcomes is $O = \{I\ wins, II\ wins, Draw\}$, one and only one of the following three alternatives holds:*

1. *Player I has a winning strategy.*
2. *Player II has a winning strategy.*
3. *Each of the two players has a strategy guaranteeing at least a draw.*

The proof of Theorem 3.13 is similar to the proof of Theorem 1.4 for the game of chess (page 3), and it is left to the reader as an exercise (Exercise 3.7). As we saw above, in proving Theorem 1.4 we did not, in fact, make use of any of the rules specific to the game of chess; the proof is valid for any game that satisfies the three properties (C1)–(C3) specified on page 6.

Examples of additional games to which Theorem 3.13 applies include, for example, checkers, the game Nim (see Exercise 3.14), and the game Hex (see Exercise 3.19).

Remark 3.14 *In our definition of a game in extensive form, we assumed that the game tree is finite. The proof of Theorem 3.13 shows that the theorem also holds when the game tree is infinite, but the depth of the vertices of the tree is bounded: there exists a natural*

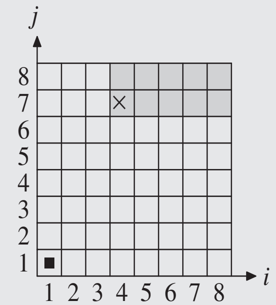

Figure 3.5 Gale's game for $n = m = 8$

number L such that the depth of each vertex in the tree is less than or equal to L. It turns out that the theorem is not true when the depth of the vertices of the tree is unbounded. See Mycielski [1992], Claim 3.1. ♦

We now consider another game that satisfies the conditions of Theorem 3.13. This game is interesting because we can prove which of the three possibilities of the theorem holds in this game, but we do not know how to calculate the appropriate strategy, in contrast to the game of chess, in which we do not even know which of the three alternatives holds.

3.4 Chomp: David Gale's game

The game described in this section is known by the name of Chomp, and was invented by David Gale (see Gale [1974]). It is played on an $n \times m$ board of squares. Each square is denoted by a pair of coordinates (i, j), $1 \leq i \leq n$ and $1 \leq j \leq m$: i is the horizontal coordinate, and j is the vertical coordinate. Figure 3.5 depicts the game board for $n = m = 8$.

Every player in turn captures a square, subject to the following rules: if at a certain stage the square (i_0, j_0) has been captured, no square that is located north-east of (i_0, j_0) can be captured in subsequent moves. This means that after (i_0, j_0) has been captured, all the squares (i, j) satisfying $i \geq i_0$ and $j \geq j_0$ can be regarded as if they have been removed from the board. In Figure 3.5, for example, square (4, 7) has been captured, so that all the darkened squares in the figure are regarded as having been removed.

Player I has the opening move. The player who captures square (1, 1) (which is marked in Figure 3.5 with a black inner square) is declared the loser. We note that the game in Example 3.1 is David Gale's game for $n = m = 2$.

Theorem 3.15 *In David Gale's game on an $n \times n$ board, the following strategy is a winning strategy for Player I: in the opening move capture square* $(2, 2)$*, thus leaving only the squares in row $j = 1$ and column $i = 1$ (see Figure 3.6). From that point on, play symmetrically to Player II's actions. That is, if Player II captures square* (i, j)*, Player I captures square* (j, i) *in the following move.*

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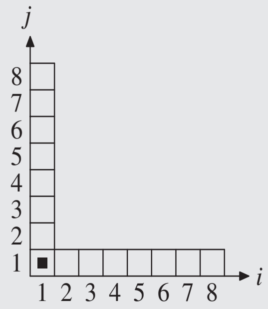

Figure 3.6 The game board after Player I has captured square (2, 2)

The above strategy is well defined. That is, if Player II captures square (i, j) (and $(i, j) \neq (1, 1)$), square (j, i) has not yet been removed from the board (verify this!). This strategy is also a winning strategy when the board is infinite, $\infty \times \infty$.

What happens if the board is rectangular but not square? Which player then has a winning strategy? As the next theorem states, the opening player always has a winning strategy.

Theorem 3.16 *For every finite $n \times m$ board (with $n > 1$ or $m > 1$), Player I, who has the opening move, has a winning strategy.*

Proof: The game satisfies the conditions of von Neumann's Theorem (Theorem 3.13), and therefore one of the three possibilities of the theorem must hold. Since the game cannot end in a draw, there are only two remaining possibilities:

1. Player I has a winning strategy.
2. Player II has a winning strategy.

Theorem 3.16 will be proved once the following claim is proved.

Claim 3.17 *For every finite $n \times m$ board (with $n > 1$ or $m > 1$), if Player II has a winning strategy, then Player I also has a winning strategy.*

Since it is impossible for both players to have winning strategies, it follows that Player II cannot have a winning strategy, and therefore the only remaining possibility is that Player I has a winning strategy.

Proof of Claim 3.17: Suppose that Player II has a winning strategy s_{II}. This strategy guarantees Player II victory over any strategy used by Player I. In particular, the strategy grants Player II victory even if Player I captures square (n, m) (the top-rightmost square) in the opening move. Suppose that Player II's next action, as called for by strategy s_{II}, is to capture square (i_0, j_0) (see Figure 3.7(a)).

From this point on, a new game is effectively being played, as depicted in Figure 3.7(b). In this game Player I has the opening move, and Player II, using strategy s_{II}, guarantees

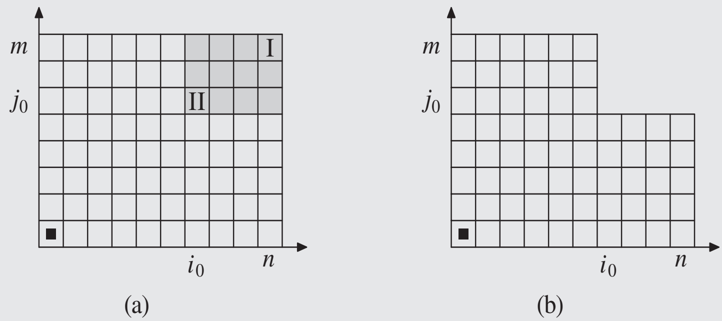

Figure 3.7 The board after the first action (a) and the board after the second action (b)

himself victory. In other words, the player who implements the opening move in this game is the losing player. But Player I can guarantee himself the situation depicted in Figure 3.7(b) when Player II opens, by choosing the square (i_0, j_0) on his first move. In conclusion, a winning strategy in the original game for Player I is to open with (i_0, j_0) and then continue according to strategy s_{II}, thus completing the proof of the claim. □

It follows from Claim 3.17 that Player II has no winning strategy, so that Player I must have a winning strategy. □

The conclusion of the theorem is particularly striking, given the fact that for $n \neq m$ we do not know how to find the winning strategy for Player I, even with the aid of computers, on relatively small boards of n and m between 30 to 40.

3.5 Games with chance moves

In the games we have seen so far, the transition from one state to another is always accomplished by actions undertaken by the players. Such a model is appropriate for games such as chess and checkers, but not for card games or dice games (such as poker or backgammon) in which the transition from one state to another may depend on a chance process: in card games, the shuffle of the deck, and in backgammon the toss of the dice. It is possible to come up with situations in which transitions from state to state depend on other chance factors, such as the weather, earthquakes, or the stock market. These sorts of state transitions are called *chance moves*. To accommodate this feature, our model is expanded by labeling some of the vertices in the game tree (V, E, x^0) as chance moves. The edges emanating from vertices corresponding to chance moves represent the possible outcomes of a lottery, and next to each such edge is listed the probability that the outcome it represents will be the result of the lottery.

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Example 3.18 **A game with chance moves** Consider the two-player game depicted in Figure 3.8.

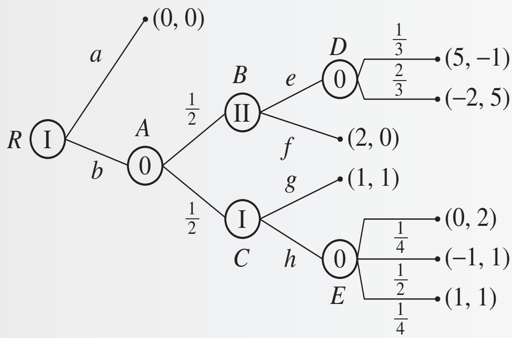

Figure 3.8 An example of a game with chance moves

The outcomes of the game are noted by pairs of numbers $(z_{\mathrm{I}}, z_{\mathrm{II}})$, where z_{I} is the monetary payoff to Player I, and z_{II} is the monetary payoff to Player II.

The verbal description of this game is as follows. At the root of the game (vertex R) Player I has the choice of selecting between action a, which leads to the termination of the game with payoff (0, 0), and action b, which leads to a chance move at vertex A. The chance move is a lottery (or a flip of a coin) leading with probability $\frac{1}{2}$ to state B, which is a decision vertex of Player II, and with probability $\frac{1}{2}$ to state C, which is a decision vertex of Player I. At state B, Player II chooses between action f, leading to a termination of the game with payoff (2, 0), and action e leading to state D which is a chance move; at this chance move, with probability $\frac{1}{3}$ the game ends with payoff $(5, -1)$, and with probability $\frac{2}{3}$ the game ends with payoff $(-2, 5)$. At state C, Player I chooses between action g, leading to the termination of the game with payoff (1, 1), and action h, leading to a chance move at vertex E. At this chance move the game ends, with payoff (0, 2), or $(-1, 1)$, or (1, 1), with respective probabilities $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$. ◀

Formally, the addition of chance moves to the model proceeds as follows. We add a new player, who is called "Nature," and denoted by 0. The set of players is thus expanded to $N \cup \{0\}$. For every vertex x at which a chance move is implemented, we denote by p_x the probability vector over the possible outcomes of a lottery that is implemented at vertex x. This leads to the following definition of a game in extensive form.

Definition 3.19 *A game in extensive form (with perfect information and chance moves) is a vector*

$$\Gamma = (N, V, E, x^0, (V_i)_{i \in N \cup \{0\}}, (p_x)_{x \in V_0}, O, u), \tag{3.6}$$

where:

- *N is a finite set of players.*
- *(V, E, x^0) is the game tree.*
- *$(V_i)_{i \in N \cup \{0\}}$ is a partition of the set of vertices that are not leaves.*
- *For every vertex $x \in V_0$, p_x is a probability distribution over the edges emanating from x.*

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- *O is the set of possible outcomes.*
- *u is a function mapping each leaf of the game tree to an outcome in O.*

The notation used in the extension of the model is the same as the previous notation, with the following changes:

- The partition of the set of vertices is now $(V_i)_{i \in N \cup \{0\}}$. We have, therefore, added the set V_0 to the partition, where V_0 is the set of vertices at which a chance move is implemented.
- For each vertex $x \in V_0$, a vector p_x, which is a probability distribution over the edges emanating from x, has been added to the model.

Games with chance moves are played similarly to games without chance moves, the only difference being that at vertices with chance moves a lottery is implemented, to determine the action to be undertaken at that vertex. We can regard a vertex x with a chance move as a roulette wheel, with the area of the pockets of the roulette wheel proportional to the values p_x. When the game is at a chance vertex, the wheel is spun, and the pocket at which the wheel settles specifies the new state of the game.

Note that in this description we have included a hidden assumption, namely, that the probabilities of the chance moves are known to all the players, even when the game includes moves that involve the probability of rain, or an earthquake, or a stock market crash, and so forth. In such situations, we presume that the probability assessments of these occurrences are known by all the players. More advanced models take into account the possibility that the players do not all necessarily share the same assessments of the probabilities of chance moves. Such models are considered in Chapters 9, 10, and 11.

In a game without chance moves, a strategy vector determines a unique play of the game (and therefore also a unique game outcome). When a game includes chance moves, a strategy vector determines a probability distribution over the possible game outcomes.

Example 3.18 (*Continued*) (See Figure 3.8.) Suppose that Player I uses strategy s_{I}, defined as

$$s_{\mathrm{I}}(R) = b, s_{\mathrm{I}}(C) = h, \tag{3.7}$$

and that Player II uses strategy s_{II}, defined as

$$s_{\mathrm{II}}(B) = f. \tag{3.8}$$

Then:

- the play $R \to A \to B \to (2, 0)$ occurs with probability $1/2$, leading to outcome $(2, 0)$;
- the play $R \to A \to C \to E \to (0, 2)$ occurs with probability $1/8$, leading to outcome $(0, 2)$;
- the play $R \to A \to C \to E \to (-1, 1)$ occurs with probability $1/4$, leading to outcome $(-1, 1)$;
- the play $R \to A \to C \to E \to (1, 1)$ occurs with probability $1/8$, leading to outcome $(1, 1)$.

◀

Using this model of games with chance moves, we can represent games such as backgammon, Monopoly, Chutes and Ladders, and dice games (but not card games such as poker and bridge, which are not games with perfect information, because players do

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not know what cards the other players are holding). Note that von Neumann's Theorem (Theorem 3.13) does not hold in games with chance moves. In dice games, such as backgammon, a player who benefits from favorable rolls of the dice can win regardless of whether or not he has the first move, and regardless of the strategy adopted by his opponent.

3.6 Games with imperfect information

One of the distinguishing properties of the games we have seen so far is that at every stage of the game each of the players has perfect knowledge of all the developments in the game prior to that stage: he knows exactly which actions were taken by all the other players, and if there were chance moves, he knows what the results of the chance moves were. In other words, every player, when it is his turn to take an action, knows precisely at which vertex in the game tree the game is currently at. A game satisfying this condition is called a *game with perfect information*.

The assumption of perfect information is clearly a very restrictive assumption, limiting the potential scope of analysis. Players often do not know all the actions taken by the other players and/or the results of chance moves (for example, in many card games the hand of cards each player holds is not known to the other players). The following game is perhaps the simplest example of a game with imperfect information.

Example 3.20 **Matching Pennies** The game Matching Pennies is a two-player game in which each player chooses one of the sides of a coin, H (for heads) or T (for tails) in the following way: each player inserts into an envelope a slip of paper on which his choice is written. The envelopes are sealed and submitted to a referee. If both players have selected the same side of the coin, Player II pays one dollar to Player I. If they have selected opposite sides of the coin, Player I pays one dollar to Player II. The depiction of Matching Pennies as an extensive-form game appears in Figure 3.9. In Figure 3.9, Player I's actions are denoted by upper case letters, and Player II's actions are depicted by lower case letters.

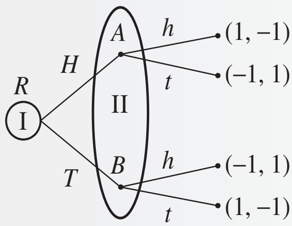

Figure 3.9 The game Matching Pennies as a game in extensive form

Figure 3.9 introduces a new element to the depictions of extensive-form games: the two vertices A and B of Player II are surrounded by an ellipse. This visual element represents the fact that when Player II is in the position of selecting between h and t, he does not know whether the game state is currently at vertex A or vertex B, because he does not know whether Player I has selected H or T. These two vertices together form an *information set* of Player II. ◀

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Remark 3.21 *The verbal description of Matching Pennies is symmetric between the two players, but in Figure 3.9 the players are not symmetric. The figure depicts Player I making his choice before Player II's choice, with Player II not knowing which choice Player I made; this is done in order to depict the game conveniently as a tree. We could alternatively have drawn the tree with Player II making his choice before Player I, with Player I not knowing which choice Player II made. Both trees are equivalent, and they are equivalent to the verbal description of the game in which the two players make their choices simultaneously.* ♦

In general, a player's information set consists of a set of vertices that satisfy the property that when play reaches one of these vertices, the player knows that play has reached one of these vertices, but he does not know which vertex has been reached. The next example illustrates this concept.

Example 3.22 Consider the following situation. David Beckham, star mid-fielder for Manchester United, is interested in leaving the team and signing up instead with either Real Madrid, Bayern Munich, or AC Milan. Both Bayern Munich and AC Milan have told Beckham they want to hire him, and even announced their interest in the star to the media.

Beckham has yet to hear anything on the matter from Real Madrid. With the season fast approaching, Beckham has only a week to determine which club he will be playing for. Real Madrid announces that it will entertain proposals of interest from players only up to midnight tonight, because its Board of Directors will be meeting tomorrow to discuss to which players the club will be making offers (Real Madrid's Board of Directors does not rule out making offers to players who have not approached it with proposals of interest). Only after the meeting will Real Madrid make offers to the players it wishes to add to its roster for the next season.

Beckham needs to decide whether to approach Real Madrid today with an expression of interest, or wait until tomorrow, hoping that the club will make him an offer on its own initiative. Real Madrid's Board of Directors will be called upon to consider two alternatives: hiring an outside expert to assess Beckham's potential contribution to the team, or dropping all considerations of hiring Beckham, without even asking for an expert's opinion. If Real Madrid hires an outside expert, the club will make an offer to Beckham if the outside expert's assessment is positive, and decline to make an offer to Beckham if the assessment is negative. The outside expert, if hired by Real Madrid, will not be informed whether or not Beckham has approached Real Madrid. If Beckham fails to receive an offer from Real Madrid, he will not know whether that is because the expert determined his contribution to the team unworthy of a contract, or because the team did not even ask for an expert's opinion. After a week, whether or not he receives an offer from Real Madrid, Beckham must decide which club he will be playing for next season, Bayern Munich, AC Milan, or Real Madrid, assuming the latter has sent him an offer. This situation can be described as a three-player game (see Figure 3.10) (verify this).

There are three information sets in this game that contain more than one vertex. The expert does not know whether or not Beckham has approached Real Madrid with an expression of interest. If Beckham has not received an offer from Real Madrid, he does not know whether that is because the expert determined his contribution to the team unworthy of a contract, or because the team did not even ask for an expert's opinion.

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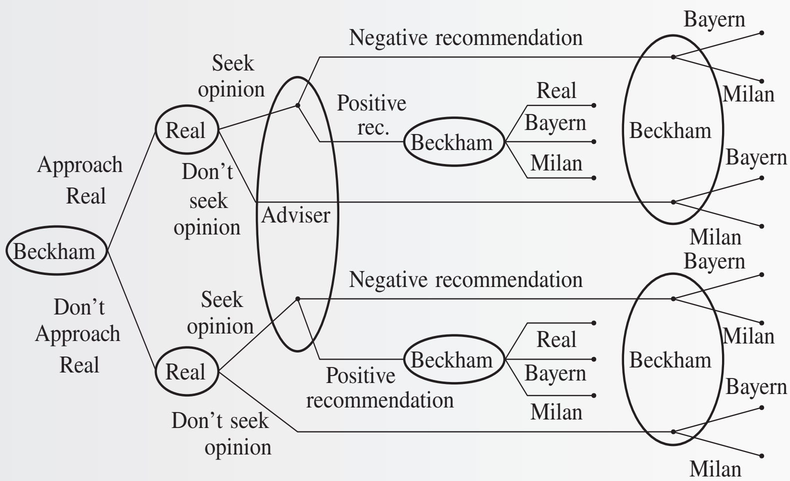

Figure 3.10 The game in Example 3.22 in extensive form

The addition of information sets to our model leads to the following definition.

Definition 3.23 *Let* $\Gamma = (N, V, E, x^0, (V_i)_{i \in N \cup \{0\}}, (p_x)_{x \in V_0}, O, u)$ *be a game in extensive form. An* information set *of player* i *is a pair* $(U_i, A(U_i))$ *such that*

- $U_i = \{x_i^1, x_i^2, \dots, x_i^j\}$ *is a subset of* V_i *that satisfies the property that at each vertex in* U_i *player* i *has the same number of actions* $l_i = l_i(U_i)$*, i.e.,*

$$|A(x_i^j)| = l_i, \qquad \forall j = 1, 2, \dots, m. \tag{3.9}$$

- $A(U_i)$ *is a partition of the* ml_i *edges* $\bigcup_{j=1}^m A(x_i^j)$ *to* l_i *disjoint sets, each of which contains one element from the sets* $(A(x_i^j))_{j=1}^m$*. We denote the elements of the partition by* $a_i^1, a_i^2, \dots, a_i^{l_i}$*. The partition* $A(U_i)$ *is called the* action set *of player* i *in the information set* U_i*.*

We now explain the significance of the definition. When the play of the game arrives at vertex x in information set U_i, all that player i knows is that the play has arrived at one of the vertices in this information set. The player therefore cannot choose a particular edge emanating from x. Each element of the partition a_i^l contains m edges, one edge for each vertex in the information set. The partition elements $a_i^1, a_i^2, \dots, a_i^{l_i}$ are the "actions" from which the player can choose; if player i chooses one of the elements from the partition a_i^l, the play continues along the unique edge in the intersection $a_i^l \cap A(x)$. For this reason, when we depict games with information sets, we denote edges located in the same partition elements by the same letter.

Definition 3.24 *A* game in extensive form *(with chance moves and with imperfect information) is a vector*

$$\Gamma = (N, V, E, x^0, (V_i)_{i \in N \cup \{0\}}, (p_x)_{x \in V_0}, (U_i^j)_{i \in N}^{j=1,\dots,k_i}, O, u), \tag{3.10}$$

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where:

- N *is a finite set of players.*
- (V, E, x^0) *is a game tree.*
- $(V_i)_{i \in N \cup \{0\}}$ *is a partition of the set of vertices that are not leaves.*
- *For each vertex* $x \in V_0$, p_x *is a probability distribution over the set of edges emanating from* x.
- *For each player* $i \in N$, $(U_i^j)^{j=1,\ldots,k_i}$ *is a partition of* V_i.
- *For each player* $i \in N$ *and every* $j \in \{1, 2, \ldots, k_i\}$, *the pair* $(U_i^j, A(U_i^j))$ *is an information set of player* i.
- O *is a set of possible outcomes.*
- u *is a function mapping each leaf of the game tree to a possible outcome in* O.

We have added information sets to the previous definition of a game in extensive form (Definition 3.19): $(U_i^j)^{j=1,\ldots,k_i}$ is a partition of V_i. Every element U_i^j in this partition is an information set of player i. Note that the information sets are defined only for players $i \in N$, because, as noted above, Nature has no information sets.

In a game with imperfect information, each player i, when choosing an action, does not know at which vertex x the play is located. He only knows the information set U_i^j that contains x. The player then chooses one of the equivalence classes of actions available to him in U_i^j, i.e., an element in $A(U_i^j)$.

The game proceeds as described on pages 44 and 51, with one difference: when the play is x, the decision maker at that state, player $J(x)$, knows only the information set $U_{J(x)}^j$ that contains x, and he chooses an element a in $A(U_{J(x)}^j)$.

We can now describe many more games as games in extensive form: various card games such as poker and bridge, games of strategy such as Stratego, and many real-life situations, such as bargaining between two parties.

Definition 3.25 *An extensive-form game is called a* game with perfect information *for* player i *if each information set of player* i *contains only one vertex. An extensive-form game is called a* game with perfect information *if it is a game with perfect information for all of the players.*

In Definition 3.11 (page 45), we defined a subgame starting at a vertex x, to be the game defined by restriction to the subtree starting at x. A natural question arises as to how this definition can be adapted to games in which players have information sets that contain several vertices, because player i may have an information set $(U_i, A(U_i))$ where u_i contains both vertices that are in the subtree starting at x, and vertices that are outside this subtree. We will say that $\Gamma(x)$ is a *subgame* only if for every player i and each of this information sets $(U_i, A(U_i))$, the set U_i is either contained entirely inside the subtree starting at x, or disjoint from this subtree. For simplicity we will often refer to U_i as an information set, and omit the set partition $A(U_i)$.

Example 3.26 Consider the two-player game with chance moves and with imperfect information that is described in Figure 3.11. The outcomes, the names of the actions, and the probabilities assigned to the chance moves are not specified in this game (as they are not needed for our discussion).

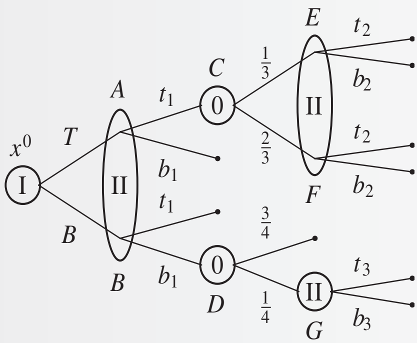

Figure 3.11 The game in Example 3.26 in extensive form

The game in Figure 3.11 has four subgames: $\Gamma(R)$, $\Gamma(C)$, $\Gamma(D)$, and $\Gamma(G)$. The subtree starting at A (or at B) cannot represent a subgame, because the information set $\{A, B\}$ of Player II is neither contained in, nor disjoint from, the subtree. It would therefore be incorrect to write $\Gamma(A)$ (or $\Gamma(B)$). Similarly, the subtrees that start at E and F cannot represent subgames, because the information set $\{E, F\}$ of Player II is neither contained in, nor disjoint from, each of these subtrees. ◀

3.6.1 Strategies in games with imperfect information

Recall that a player's strategy is a set of instructions telling the player which action to choose, every time he is called upon to play. When we dealt with games with perfect information, in which each player, when coming to make a decision, knows the current vertex x, a strategy was defined as a function $s_i : V_i \to V$, where $s_i(x) \in C(x)$ for every $x \in V_i$. In a game with imperfect information, when choosing an action, the player knows the information set that contains the current vertex. Therefore a strategy is a function that assigns an action to each information set.

Definition 3.27 *A* strategy *of player i is a function from each of his information sets to the set of actions available at that information set, i.e.,*

$$s_i : \mathcal{U}_i \to \bigcup_{j=1}^{k_i} A(U_i^j), \tag{3.11}$$

where $\mathcal{U}_i = \{U_i^1, \ldots, U_i^{k_i}\}$ is the collection of player i's information sets, and for each information set $U_i^j \in \mathcal{U}_i$,

$$s_i(U_i^j) \in A(U_i^j). \tag{3.12}$$

Just as in games with chance moves and perfect information, a strategy vector determines a distribution over the outcomes of a game. For example, in Example 3.22, suppose that the players implement the following strategies:

- David Beckham approaches Real Madrid, and then chooses to play for Real Madrid if Real Madrid then makes him an offer; otherwise, he chooses to play for Bayern Munich.
- Real Madrid hires an outside expert if Beckham approaches it, and does not hire an outside expert if Beckham does not approach the club. Real Madrid makes an offer to

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Beckham only if Beckham first approaches the club, and if the outside expert gives a positive recommendation.

- The outside expert recommends that Real Madrid not make an offer to Beckham.

There are no chance moves in this game, so that the strategy vector determines a unique play of the game, and therefore also a unique outcome: Beckham ends up playing for Bayern Munich, after he approaches Real Madrid, Real Madrid in turn hires an outside expert to provide a recommendation, the expert returns with a negative recommendation, Real Madrid does not make an offer to Beckham, and Beckham then decides to play for Bayern Munich.

3.7 Exercises

3.1 Describe the following situation as an extensive-form game. Three piles of matches are on a table. One pile contains a single match, a second pile contains two matches, and the third pile contains three matches. Two players alternately remove matches from the table. In each move, the player whose turn it is to act at that move may remove matches from one and only one pile, and must remove at least one match. The player who removes the last match loses the game.

By drawing arrows on the game tree, identify a way that one of the players can guarantee victory.

3.2 **Candidate choice** Depict the following situation as a game in extensive form. Eric, Larry, and Sergey are senior partners in a law firm. The three are considering candidates for joining their firm. Three candidates are under consideration: Lee, Rebecca, and John. The choice procedure, which reflects the seniority relations between the three law firm partners, is as follows:

- Eric makes the initial proposal of one of the candidates.
- Larry follows by proposing a candidate of his own (who may be the same candidate that Eric proposed).
- Sergey then proposes a candidate (who may be one of the previously proposed candidates).
- A candidate who receives the support of two of the partners is accepted into the firm. If no candidate has the support of two partners, all three candidates are rejected.

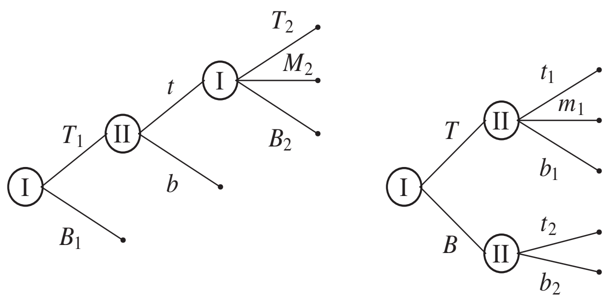
3.3 **Does aggressiveness pay off?** Depict the following situation as a game in extensive form. A bird is defending its territory. When another bird attempts to invade this territory, the first bird is faced with two alternatives: to stand and fight for its territory, or to flee and seek another place for its home. The payoff to each bird is defined to be the expected number of descendants it will leave for posterity, and these are calculated as follows:

- If the invading bird yields to the defending bird and instead flies to another territory, the payoff is: 6 descendants for the defending bird, 4 descendants for the invading bird.

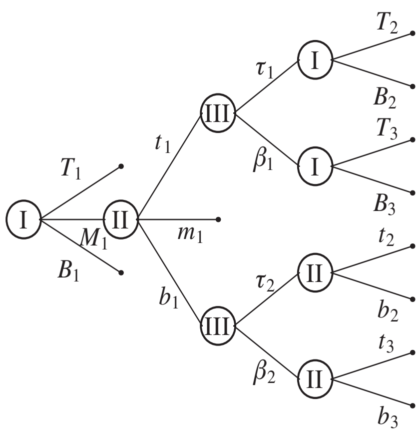
- If the invading bird presses an attack and the defending bird flies to another territory, the payoff is: 4 descendants for the defending bird, 6 descendants for the invading bird.
- If the invading bird presses an attack and the defending bird stands its ground and fights, the payoff is: 2 descendants for the defending bird, 2 descendants for the invading bird.

3.4 Depict the following situation as a game in extensive form. Peter and his three children, Andrew, James, and John, manage a communications corporation. Andrew is the eldest child, James the second-born, and John the youngest of the children. Two candidates have submitted offers for the position of corporate accountant at the communications corporation. The choice of a new accountant is conducted as follows: Peter first chooses two of his three children. The two selected children conduct a meeting to discuss the strengths and weaknesses of each of the two candidates. The elder of the two children then proposes a candidate. The younger of the two children expresses either agreement or disagreement to the proposed candidate. A candidate is accepted to the position only if two children support his candidacy. If neither candidate enjoys the support of two children, both candidates are rejected.

3.5 (a) How many strategies has each player got in each of the following three games (the outcomes of the games are not specified in the figures).



Game A　　Game B



Game C

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	O	X
X		

Figure 3.12 The board of the game Tic-Tac-Toe, after three moves

(b) Write out in full all the strategies of each player in each of the three games.
(c) How many different plays are possible in each of the games?

3.6 In a single-player game in which at each vertex x that is not the root the player has m_x actions, how many strategies has the player got?

3.7 Prove von Neumann's Theorem (Theorem 3.13 on page 46): in every two-player finite game with perfect information in which the set of outcomes is $O = \{\text{I wins, II wins, Draw}\}$, one and only one of the following three alternatives holds:

(a) Player I has a winning strategy.
(b) Player II has a winning strategy.
(c) Each of the two players has a strategy guaranteeing at least a draw.

Where does your proof make use of the assumption that the game is finite?

3.8 **Tic-Tac-Toe** How many strategies has Player I got in Tic-Tac-Toe, in which two players play on a 3×3 board, as depicted in Figure 3.12? Player I makes the first move, and each player in turn chooses a square that has not previously been selected. Player I places an X in every square that he chooses, and Player II places an O in every square that he chooses. The game ends when every square has been selected. The first player who has managed to place his mark in three adjoining squares, where those three squares form either a column, a row, or a diagonal, is the winner.[4] (Do not attempt to draw a full game tree. Despite the fact that the rules of the game are quite simple, the game tree is exceedingly large. Despite the size of the game tree, with a little experience players quickly learn how to ensure at least a draw in every play of the game.)

3.9 By definition, a player's strategy prescribes his selected action at each vertex in the game tree. Consider the following game.

Player I has four strategies, T_1T_2, T_1B_2, B_1T_2, B_1B_2. Two of these strategies, B_1T_2 and B_1B_2, regardless of the strategy used by Player II, yield the same play of the game, because if Player I has selected action B_1 at the root vertex, he will never get to his second decision vertex. We can therefore eliminate one of these two strategies and define a *reduced strategy* B_1, which only stipulates that Player I chooses B_1 at the root of the game. In the game appearing in the above

[4] The game, of course, can effectively be ended when one of the players has clearly ensured victory for himself, but calculating the number of strategies in that case is more complicated.

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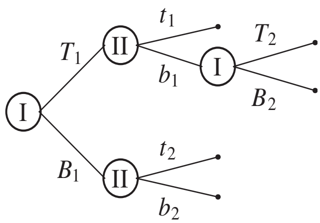
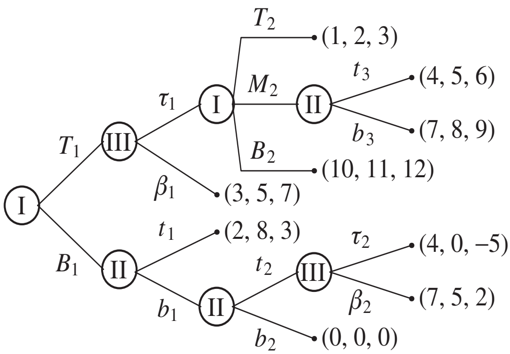


figure, the reduced strategies of Player I are T_1T_2, T_1B_2, and B_1. The reduced strategies of Player II are the same as his regular strategies, t_1t_2, t_1b_2, b_1t_2, and b_1b_2, because Player II does not know to which vertex Player I's choice will lead. Formally, a *reduced strategy* τ_i of player i is a function from a subcollection $\widehat{\mathcal{U}}_i$ of player i's collection of information sets to actions, satisfying the following two conditions:

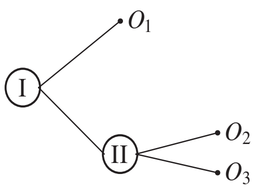
(i) For any strategy vector of the remaining players σ_{-i}, given the vector (τ_i, σ_{-i}), the game will definitely not get to an information set of player i that is not in the collection $\widehat{\mathcal{U}}_i$.
(ii) There is no strict subcollection of $\widehat{\mathcal{U}}_i$ satisfying condition (i).

(a) List the reduced strategies of each of the players in the game depicted in the following figure.



(b) What outcome of the game will obtain if the three players make use of the reduced strategies $\{(B_1), (t_1, t_3), (\beta_1, \tau_2)\}$?
(c) Can any player increase his payoff by unilaterally making use of a different strategy (assuming that the other two players continue to play according to the strategies of part (b))?

3.10 Consider the game in the following figure.



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The outcomes O_1, O_2, and O_3 are distinct and taken from the set {I wins, II wins, Draw}.

(a) Is there a choice of O_1, O_2, and O_3 such that Player I can guarantee victory for himself? Justify your answer.
(b) Is there a choice of O_1, O_2, and O_3 such that Player II can guarantee victory for himself? Justify your answer.
(c) Is there a choice of O_1, O_2, and O_3 such that both players can guarantee for themselves at least a draw? Justify your answer.

3.11 **The Battle of the Sexes** The game in this exercise, called Battle of the Sexes, is an oft-quoted example in game theory (see also Example 4.21 on page 98). The name given to the game comes from the story often attached to it: a couple is trying to determine how to spend their time next Friday night. The available activities in their town are attending a concert (C), or watching a football match (F). The man prefers football, while the woman prefers the concert, but both of them prefer being together to spending time apart.

The pleasure each member of the couple receives from the available alternatives is quantified as follows:

- From watching the football match together: 2 for the man, 1 for the woman.
- From attending the concert together: 1 for the man, 2 for the woman.
- From spending time apart: 0 for the man, 0 for the woman.

The couple do not communicate well with each other, so each one chooses where he or she will go on Friday night before discovering what the other selected, and refuses to change his or her mind (alternatively, we can imagine each one going directly to his or her choice directly from work, without informing the other). Depict this situation as a game in extensive form.

3.12 **The Centipede game**[5] The game tree appearing in Figure 3.13 depicts a two-player game in extensive form (note that the tree is shortened; there are another 94 choice vertices and another 94 leaves that do not appear in the figure). The payoffs appear as pairs (x, y), where x is the payoff to Player I (in thousands of dollars) and y is the payoff to Player II (in thousands of dollars). The players make moves in alternating turns, with Player I making the first move.

Every player has a till into which money is added throughout the play of the game. At the root of the game, Player I's till contains \$1,000, and Player II's till is empty. Every player in turn, at her move, can elect either to stop the game (S), in which case every player receives as payoff the amount of money in her till, or to continue to play. Each time a player elects to continue the game, she removes \$1,000 from his till and places them in the other player's till, while simultaneously the game-master adds another \$2,000 to the other player's till. If no player has stopped the game after 100 turns have passed, the game ends, and each player receives the amount of money in her till at that point.

How would you play this game in the role of Player I? Justify your answer!

5 The Centipede game was invented by Robert Rosenthal (see Rosenthal [1981]).

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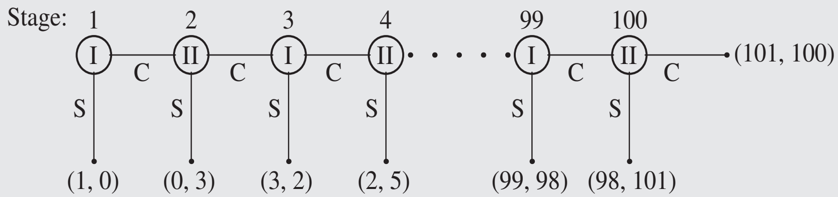

Figure 3.13 The Centipede game (outcomes are in payoffs of thousands of dollars)

3.13 Consider the following game. Two players, each in turn, place a quarter on a round table, in such a way that the coins are never stacked one over another (although the coins may touch each other); every quarter must be placed fully on the table. The first player who cannot place an additional quarter on the table at his turn, without stacking it on an already placed quarter, loses the game (and the other player is the winner). Prove that the opening player has a winning strategy.

3.14 Nim[6] Nim is a two-player game, in which piles of matches are placed before the players (the number of piles in the game is finite, and each pile contains a finite number of matches). Each player in turn chooses a pile, and removes any number of matches from the pile he has selected (he must remove at least one match). The player who removes the last match wins the game.

(a) Does von Neumann's Theorem (Theorem 3.13 on page 46) imply that one of the players must have a winning strategy? Justify your answer!

We present here a series of guided exercises for constructing a winning strategy in the game of Nim.

At the beginning of play, list, in a column, the number of matches in each pile, expressed in base 2. For example, if there are 4 piles containing, respectively, 2, 12, 13, and 21 matches, list:

$$\begin{array}{r} 10 \\ 1100 \\ 1101 \\ 10101 \end{array}$$

Next, check whether the number of 1s in each column is odd or even. In the above example, counting from the right, in the first and fourth columns the number of 1s is even, while in the second, third, and fifth columns the number of 1s is odd.

A position in the game will be called a "winning position" if the number of 1s in each column is even. The game state depicted above is not a winning position.

(b) Prove that, starting from any position that is not a winning position, it is possible to get to a winning position in one move (that is, by removing matches from a

6 Nim is an ancient game, probably originating in China. There are accounts of the game being played in Europe as early as the 15th century. The proof presented in this exercise is due to Bouton [1901].

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single pile). In our example, if 18 matches are removed from the largest pile, the remaining four piles will have 2, 12, 13, and 3 matches, respectively, which in base 2 are represented as

$$\begin{array}{r} 10 \\ 1100 \\ 1101 \\ 11 \end{array}$$

which is a winning position, as every column has an even number of 1s.

(c) Prove that at a winning position, every legal action leads to a non-winning position.
(d) Explain why at the end of every play of the game, the position of the game will be a winning position.
(e) Explain how we can identify which player can guarantee victory for himself (given the initial number of piles of matches and the number of matches in each pile), and describe that player's winning strategy.

3.15 The game considered in this exercise is exactly like the game of Nim of the previous exercise, except that here the player who removes the last match loses the game. (The game described in Exercise 3.1 is an example of such a game.)

(a) Is it possible for one of the players in this game to guarantee victory? Justify your answer.
(b) Explain how we can identify which player can guarantee victory for himself in this game (given the initial number of piles of matches and the number of matches in each pile), and describe that player's winning strategy.

3.16 Answer the following questions relating to David Gale's game of Chomp (see Section 3.4 on page 47).

(a) Which of the two players has a winning strategy in a game of Chomp played on a $2 \times \infty$ board? Justify your answer. Describe the winning strategy.
(b) Which of the two players has a winning strategy in a game of Chomp played on an $m \times \infty$ board, where m is any finite integer? Justify your answer. Describe the winning strategy.
(c) Find two winning strategies for Player I in a game of Chomp played on an $\infty \times \infty$ board.

3.17 Show that the conclusion of von Neumann's Theorem (Theorem 3.13, page 46) does not hold for the Matching Pennies game (Example 3.20, page 52), where we interpret the payoff $(1, -1)$ as victory for Player I and the payoff $(-1, 1)$ as victory for Player II.

Which condition in the statement of the theorem fails to obtain in Matching Pennies?

3.18 Prove that von Neumann's Theorem (Theorem 3.13, page 46) holds in games in extensive form with perfect information and without chance moves, in which the game tree has a countable number of vertices, but the depth of every vertex is

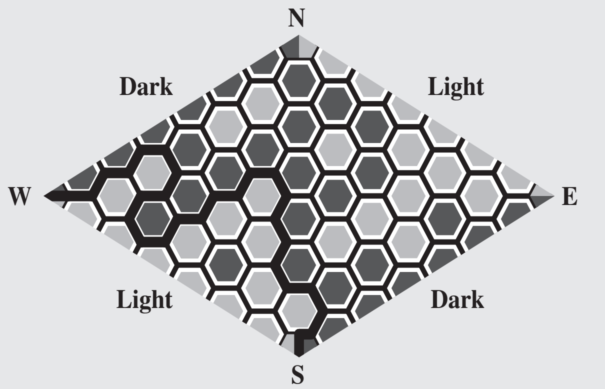

Figure 3.14 The Hex game board for $n = 6$ (in the play depicted here, dark is the winner)

bounded; i.e., there exists a positive integer K that is greater than the length of every path in the game tree.

3.19 **Hex** Hex[7] is a two-player game conducted on a rhombus containing n^2 hexagonal cells, as depicted in Figure 3.14 for $n = 6$.

The players control opposite sides of the rhombus (in the accompanying figure, the names of the players are "Light" and "Dark"). Light controls the south-west (SW) and north-east (NE) sides, while Dark controls the north-west (NW) and south-east sides (SE). The game proceeds as follows. Dark has the opening move. Every player in turn chooses an unoccupied hexagon, and occupies it with a colored game piece. A player who manages to connect the two sides he controls with a continuous path[8] of hexagons occupied by his pieces is declared a winner. If neither player can do so, a draw is called. We will show that a play of the game can never end in a draw. In Figure 3.14, we depict a play of the game won by Dark. Note that, by the rules, the players can keep placing game pieces until the entire board has been filled, so that a priori it might seem as if it might be possible for both players to win, but it turns out to be impossible, as we will prove. There is, in fact, an intuitive argument for why a draw cannot occur: imagine that one player's game pieces are bodies of water, and the other player's game pieces are dry land. If the water player is a winner, it means that he has managed to create a water channel connecting his sides, through which no land-bridge constructed by the opposing player can cross. We will see that turning this intuitive argument into a formal proof is not at all easy.[9]

7 Hex was invented in 1942 by a student named Piet Hein, who called it Polygon. It was reinvented, independently, by John Nash in 1948. The name Hex was given to the game by Parker Bros., who sold a commercial version of it. The proof that the game cannot end in a draw, and that there cannot be two winners, is due to David Gale [1979]. The presentation in this exercise is due to Jack van Rijswijck (see http://www.cs.ualberta.ca/~javhar/). The authors thank Taco Hoekwater for assisting them in preparing the figure of the game board.

8 A *continuous path* is a chain of adjacent hexagons, where two hexagons are called "adjacent" if they share a common edge.

9 This argument is equivalent to Jordan's Theorem, which states that a closed, continuous curve divides a plane into two parts, in such a way that every continuous curve that connects a point in one of the two disconnected parts with a point in the other part must necessarily pass through the original curve.

For simplicity, assume that the edges of the board, as in Figure 3.14, are also composed of (half) hexagons. The hexagons composing each edge will be assumed to be colored with the color of the player who controls that respective edge of the board. Given a fully covered board, we construct a broken line (which begins at the corner labeled W). Every leg of the broken line separates a game piece of one color from a game piece of the other color (see Figure 3.14).

(a) Prove that within the board, with the exception of the corners, the line can always be continued in a unique manner.
(b) Prove that the broken line will never return to a vertex through which it previously passed (hint: use induction).
(c) From the first two claims, and the fact that the board is finite, conclude that the broken line must end at a corner of the board (not the corner from which it starts). Keep in mind that one side of the broken line always touches hexagons of one color (including the hexagons comprising the edges of the rhombus), and the other side of the line always touches hexagons of the other color.
(d) Prove that if the broken line ends at corner S, the sides controlled by Dark are connected by dark-colored hexagons, so that Dark has won (as in Figure 3.14). Similarly, if the broken line ends at corner N, Light has won.
(e) Prove that it is impossible for the broken line to end at corner E.
(f) Conclude that a draw is impossible.
(g) Conclude that it is impossible for both players to win.
(h) Prove that the player with the opening move has a winning strategy.

Guidance for the last part: Based on von Neumann's Theorem (Theorem 3.13, page 46), and previous claims, one (and only one) of the players has a winning strategy. Call the player with the opening move Player I, and the other player, Player II. Suppose that Player II has a winning strategy. We will prove then that Player I has a winning strategy too, contradicting von Neumann's Theorem. The winning strategy for Player I is as follows: in the opening move, place a game piece on any hexagon on the board. Call that game piece the "special piece." In subsequent moves, play as if (i) you are Player II (and use his winning strategy), (ii) the special piece has not been placed, and (iii) your opponent is Player I. If the strategy requires placing a game piece where the special game piece has already been placed, put a piece on any empty hexagon, and from there on call that game piece the "special piece."

3.20 *And-Or* is a two-player game played on a full binary tree with a root, of depth n (see Figure 3.15). Every player in turn chooses a leaf of the tree that has not previously been selected, and assigns it the value 1 or 0. After all the leaves have been assigned a value, a value for the entire tree is calculated as in the figure. The first step involves calculating the value of the vertices at one level above the level of the leaves: the value of each such vertex is calculated using the logic "or" function, operating on the values assigned to its children. Next, a value is calculated for each vertex one level up, with that value calculated using the logic "and" function, operating on the

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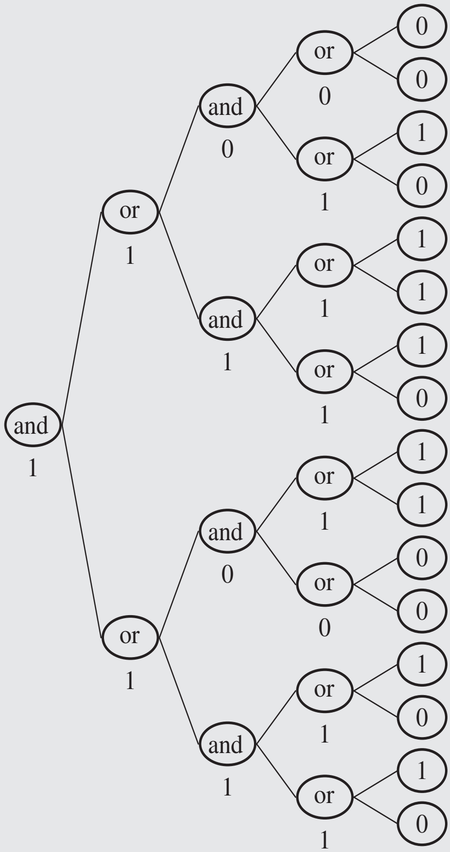

Figure 3.15 A depiction of the game And-Or of depth $n = 4$ as an extensive-form game

values previously calculated for their respective children. The truth tables of the "and" and "or" functions are:[10]

x	y	x and y	x or y
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

Equivalently, x and $y = \min\{x, y\}$ and x or $y = \max\{x, y\}$. The values of all the vertices of the tree are alternately calculated in this manner recursively, with the value of each vertex calculated using either the "and" or "or" functions, operating

10 Equivalently, "x or y" $= x \vee y = \max\{x, y\}$, and "$x$ and y" $= x \wedge y = \min\{x, y\}$.

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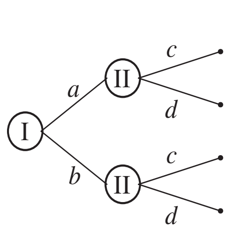
on values calculated for their respective children. Player I wins if the value of the root vertex is 1, and loses if the value of the root vertex is 0. Figure 3.15 shows the end of a play of this game, and the calculations of vertex values by use of the "and" and "or" functions. In this figure, Player I is the winner.

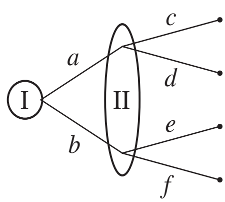
Answer the following questions:

(a) Which player has a winning strategy in a game played on a tree of depth two?
(b) Which player has a winning strategy in a game played on a tree of depth $2k$, where k is any positive integer?
Guidance: To find the winning strategy in a game played on a tree of depth $2k$, keep in mind that you can first calculate inductively the winning strategy for a game played on a tree of depth $2k - 2$.

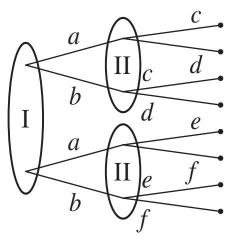
3.21 Each one of the following figures cannot depict a game in extensive form. For each one, explain why.



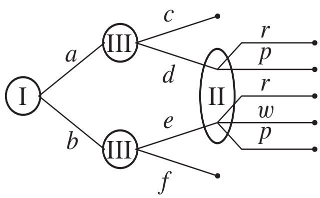
Part A



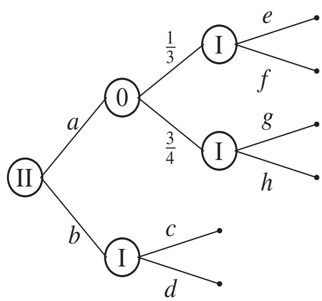
Part B



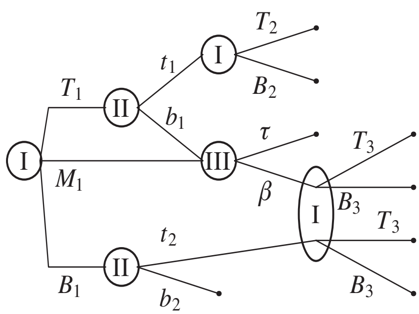
Part C



Part D



Part E



Part F

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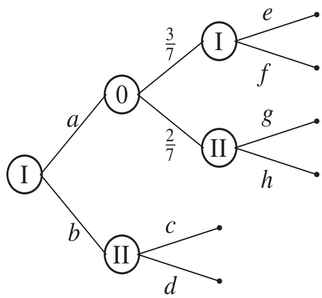
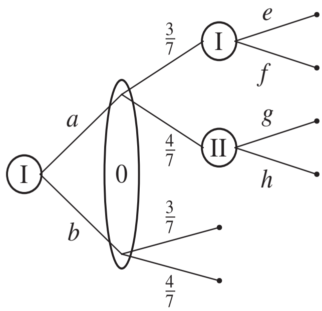
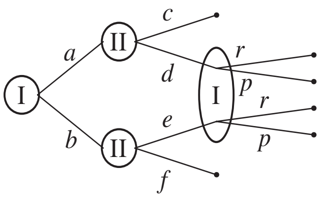


Part G

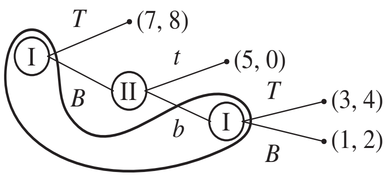


Part H

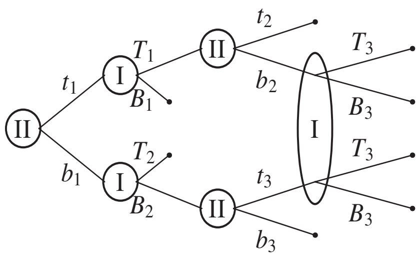
3.22 In each of the following games, Player I has an information set containing more than one vertex. What exactly has Player I "forgotten" (or could "forget") during the play of each game?



Game A

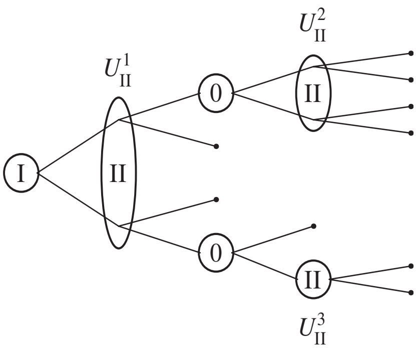


Game B

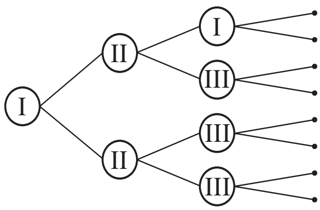


Game C

3.23 In which information sets for the following game does Player II know the action taken by Player I?



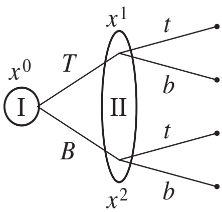
3.24 Sketch the information sets in the following game tree in each of the situations described in this exercise.



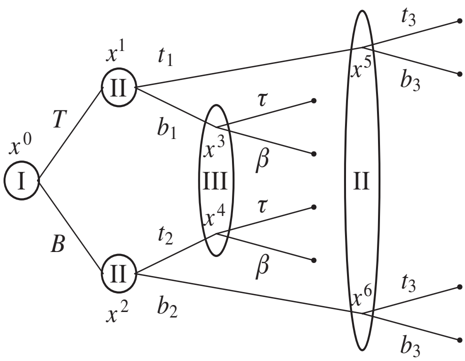
(a) Player II does not know what Player I selected, while Player III knows what Player I selected, but if Player I moved down, Player III does not know what Player II selected.
(b) Player II does not know what Player I selected, and Player III does not know the selections of either Player I or Player II.
(c) At every one of his decision points, Player I cannot remember whether or not he has previously made any moves.

3.25 For each of the following games:

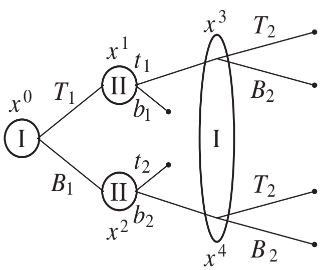
(a) List all of the subgames.
(b) For each information set, note what the player to whom the information set belongs knows, and what he does not know, at that information set.



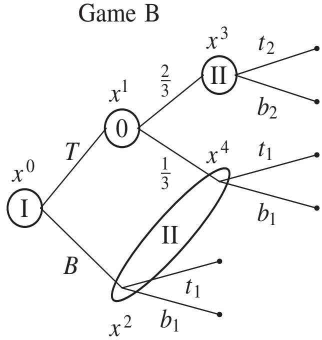
Game A



Game B



Game C



Game D

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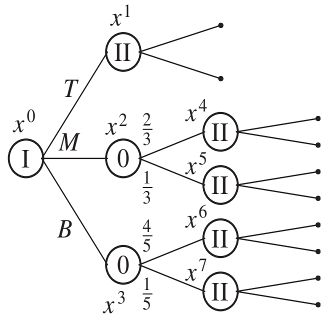
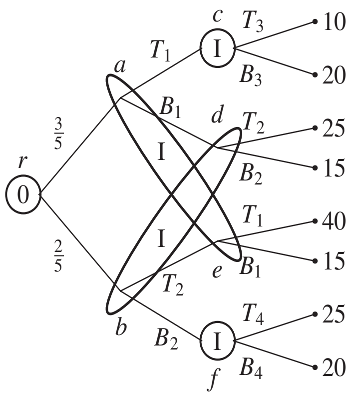
3.26 Only a partial depiction of a game in extensive form is presented in the accompanying figure of this exercise. Sketch the information sets describing each of the following situations.

(a) Player II, at his decision points, knows what Player I selected, but does not know the result of the chance move.
(b) Player II, at his decision points, knows the result of the chance move (where relevant). If Player I has selected T, Player II knows that this is the case, but if Player I selected either B or M, Player II does not know which of these two actions was selected.
(c) Player II, at his decision points, knows both the result of the chance move and any choice made by Player I.



3.27 (a) What does Player I know, and what does he not know, at each information set in the following game:



(b) How many strategies has Player I got?
(c) The outcome of the game is the payment to Player I. What do you recommend Player I should play in this game?

3.28 How many strategies has Player II got in the game in the figure in this exercise, in each of the described situations? Justify your answers.

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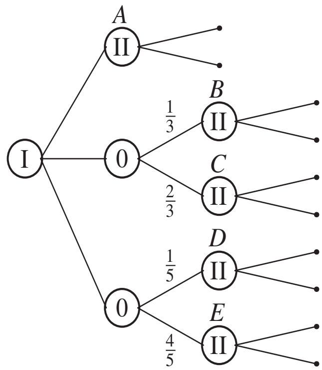
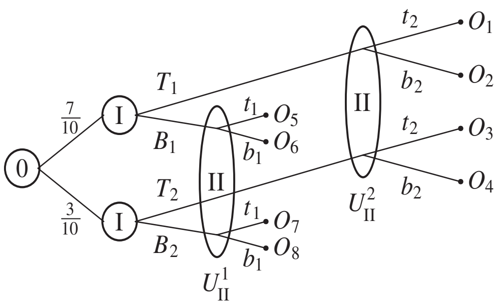


(a) The information sets of Player II are: $\{A\}$, $\{B, C\}$, $\{D, E\}$.
(b) The information sets of Player II are: $\{A, B\}$, $\{C\}$, $\{D, E\}$.
(c) The information sets of Player II are: $\{A, B, C\}$, $\{D, E\}$.
(d) The information sets of Player II are: $\{A, B, D\}$, $\{C\}$, $\{E\}$.

3.29 Consider the following two-player game.



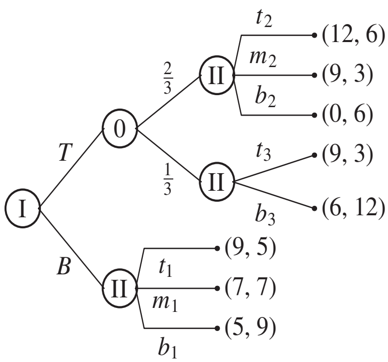
(a) What does Player II know, and what does he not know, at each of his information sets?
(b) Depict the same game as a game in extensive form in which Player II makes his move prior to the chance move, and Player I makes his move after the chance move.
(c) Depict the same game as a game in extensive form in which Player I makes his move prior to the chance move, and Player II makes his move after the chance move.

3.30 Depict the following situation as a game in extensive form. Two corporations manufacturing nearly identical chocolate bars are independently considering whether or not to increase their advertising budgets by \$500,000. The sales experts of both corporations are of the opinion that if both corporations increase their advertising budgets, they will each get an equal share of the market, and the same result will ensue if neither corporation increases its advertising budget. In contrast, if one corporation increases its advertising budget while the other maintains the same level of advertising, the corporation that increases its advertising budget will grab an 80% market share, and the other will be left with a 20% market share.

The decisions of the chief executives of the two corporations are made simultaneously; neither one of the chief executives knows what the decision of the other chief executive is at the time he makes his decision.

3.31 **Investments** Depict the following situation as a game in extensive form. Jack has \$100,000 at his disposal, which he would like to invest. His options include investing in gold for one year; if he does so, the expectation is that there is a probability of 30% that the price of gold will rise, yielding Jack a profit of \$20,000, and a probability of 70% that the price of gold will drop, causing Jack to lose \$10,000. Jack can alternatively invest his money in shares of the Future Energies corporation; if he does so, the expectation is that there is a probability of 60% that the price of the shares will rise, yielding Jack a profit of \$50,000, and a probability of 40% that the price of the shares will drop to such an extent that Jack will lose his entire investment. Another option open to Jack is placing the money in a safe index-linked money market account yielding a 5% return.

3.32 In the game depicted in Figure 3.16 , if Player I chooses T, there is an ensuing chance move, after which Player II has a turn, but if Player I chooses B, there is no chance move, and Player II has an immediately ensuing turn (without a chance move). The outcome of the game is a pair of numbers (x, y) in which x is the payoff for Player I and y is the payoff for Player II.



(a) What are all the strategies available to Player I?
(b) How many strategies has Player II got? List all of them.
(c) What is the expected payoff to each player if Player I plays B and Player II plays (t_1, b_2, t_3)?
(d) What is the expected payoff to each player if Player I plays T and Player II plays (t_1, b_2, t_3)?

3.33 The following questions relate to Figure 3.16. The outcome of the game is a triple (x, y, z) representing the payoff to each player, with x denoting the payoff to Player I, y the payoff to Player II and z the payoff to Player III.

The outcome of the game is a pair of numbers, representing a payment to each player.

(a) Depict, by drawing arrows, strategies (a, c, e), (h, j, l), and (m, p, q) of the three players.

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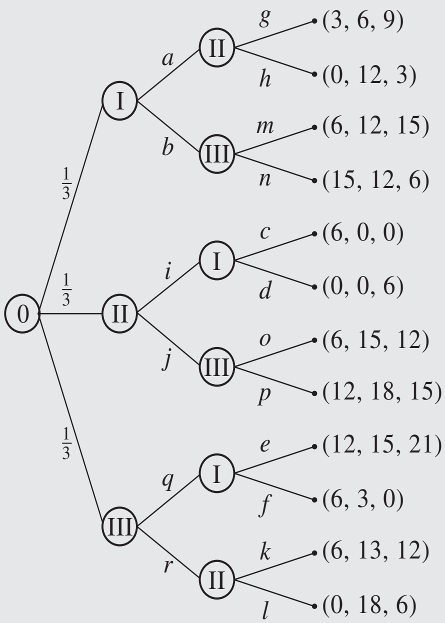


Figure 3.16

(b) Calculate the expected payoff if the players make use of the strategies in part (a).

(c) How would you play this game, if you were Player I? Assume that each player is striving to maximize his expected payoff.

3.34 Bill asks Al to choose heads or tails. After Al has made his choice (without disclosing it to Bill), Bill flips a coin. If the coin falls on Al's choice, Al wins. Otherwise, Bill wins. Depict this situation as a game in extensive form.

3.35 A pack of three cards, labeled 1, 2, and 3, is shuffled. William, Mary, and Anne each take a card from the pack. Each of the two players holding a card with low values (1 or 2) pays the amount of money appearing on the card he or she is holding to the player holding the high-valued card (namely, 3). Depict this situation as a game in extensive form.

3.36 Depict the game trees of the following variants of the candidate game appearing in Exercise 3.2:

(a) Eric does not announce which candidate he prefers until the end of the game. He instead writes down the name of his candidate on a slip of paper, and shows that slip of paper to the others only after Larry and Sergey have announced their preferred candidate.

(b) A secret ballot is conducted: no player announces his preferred candidate until the end of the game.

(c) Eric and Sergey keep their candidate preference a secret until the end of the game, but Larry announces his candidate preference as soon as he has made his choice.

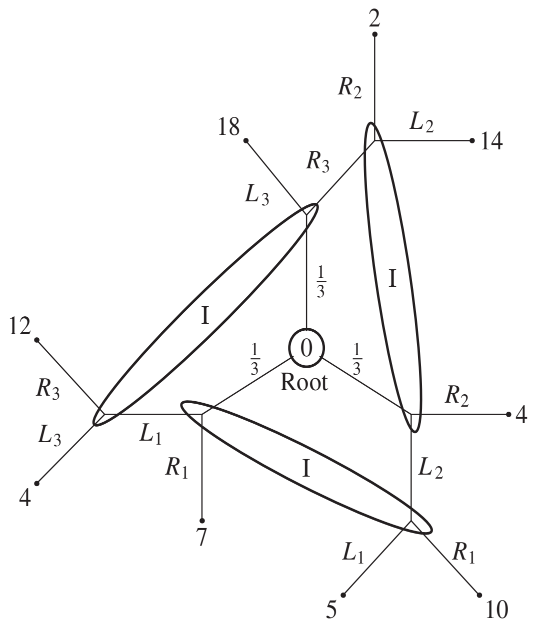
3.37 Describe the game Rock, Paper, Scissors as an extensive-form game (if you are unfamiliar with this game, see page 78 for a description).

3.38 Consider the following game. Player I has the opening move, in which he chooses an action in the set $\{L, R\}$. A lottery is then conducted, with either λ or ρ selected, both with probability $\frac{1}{2}$. Finally, Player II chooses either l or r. The outcomes of the game are not specified. Depict the game tree associated with the extensive-form game in each of the following situations:

(a) Player II, at his turn, knows Player I's choice, but does not know the outcome of the lottery.
(b) Player II, at his turn, knows the outcome of the lottery, but does not know Player I's choice.
(c) Player II, at his turn, knows the outcome of the lottery only if Player I has selected L.
(d) Player II, at his turn, knows Player I's choice if the outcome of the lottery is λ, but does not know Player I's choice if the outcome of the lottery is ρ.
(e) Player II, at his turn, does not know Player I's choice, and also does not know the outcome of the lottery.

3.39 In the following game, the root is a chance move, Player I has three information sets, and the outcome is the amount of money that Player I receives.

(a) What does Player I know in each of his information sets, and what does he not know?
(b) What would you recommend Player I to play, assuming that he wants to maximize his expected payoff? Justify your answer.



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