Logic in CS	Autumn 2024

Problem Sheet 2

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1. An adequate set of connectives is a set such that for every formula there is an equivalent formula with only connectives from that set. For example, $\{\neg, \vee\}$ is adequate for propositional logic since any occurrence of $\wedge$ and $\rightarrow$ can be removed using the equivalences

$$\varphi \rightarrow \psi \equiv \neg\varphi \vee \psi$$

$$\varphi \wedge \psi \equiv \neg(\neg\varphi \vee \neg\psi)$$

   (a) Show that $\{\neg, \wedge\}$, $\{\neg, \rightarrow\}$ and $\{\rightarrow, \perp\}$ are adequate sets of connectives. ($\perp$ treated as a nullary connective).

   (b) Show that if $C \subseteq \{\neg, \wedge, \vee, \rightarrow, \perp\}$ is adequate, then $\neg \in C$ or $\perp \in C$.

2. The binary connective `nand`, $F \downarrow G$, is defined by the truth table corresponding to $\neg(F \wedge G)$. Show that `nand` is complete - that is, it can express all binary Boolean connectives.

3. The binary connective `xor`, $F \oplus G$ is defined by the truth table corresponding to $(\neg F \wedge G) \vee (F \wedge \neg G)$. Show that `xor` is not complete - that is, it cannot express all binary Boolean connectives.

4. If a contradiction can be derived from a set of formulae, then the set of formulae is said to be inconsistent. Otherwise, the set of formulae is consistent. Let $\mathcal{F}$ be a set of formulae. Show that $\mathcal{F}$ is consistent iff it is satisfiable.

5. Suppose $\mathcal{F}$ is an inconsistent set of formulae. For each $G \in \mathcal{F}$, let $\mathcal{F}_G$ be the set obtained by removing G from $\mathcal{F}$.

   (a) Prove that for any $G \in \mathcal{F}$, $\mathcal{F}_G \vdash \neg G$, using the previous question.

   (b) Prove this using a formal proof.

6. Consider a formula φ which is of the form $C_1 \wedge C_2 \wedge \dots C_n$ where each clause C_i is of the form $(\top \rightarrow \alpha)$ or $(\alpha_1 \wedge \dots \alpha_n \rightarrow \beta)$ or $(\gamma \rightarrow \perp)$ where $\alpha, \alpha_i, \beta, \gamma$ are literals. A logician wishes to apply `HornSAT` to this formula φ by renaming negative literals (if any) with fresh positive literals. Thus, if any $\alpha, \alpha_i, \beta, \gamma$ was of the form $\neg p$, the logician will replace that $\neg p$ with a fresh variable p'. The logician claims that he can check satisfiability of φ correctly by applying `HornSAT` on the new formula (call it φ') in the following way: φ is satisfiable iff `HornSAT` concludes that φ' is satisfiable, and φ is unsatisfiable iff `HornSAT` concludes that φ' is unsatisfiable. Do you agree with the logician?

7. We have seen in class that `HornSAT` has a polynomial satisfiability, while general `SAT` is NP-complete. Here is a reduction called "Hornification" proposed by a student from `SAT` to `HornSAT`. Given a formula φ in CNF, "hornify" each non-horn clause as follows.

   - If we have a clause $C_1 = p \vee q \vee r$, then all occurrences of p, q are renamed to $\neg p'$ and $\neg q'$ where p', q' are fresh variables (In general, you could have chosen to rename all but one positive literal to Hornify). Clearly, this renaming can be done in polynomial time.
   - Additionally, to respect the relationship between the original variables and their renamed counterparts, add a new Horn clause $p' \wedge p \to \bot$ (and similarly for q') whenever you rename p as $\neg p'$. This ensures that the new variables p' and q' behave correctly with respect to their original negated forms.

   Call the new formula (on an expanded set of variables) as φ'. Since φ' is in `HornSAT`, we can check its satisfiability in polynomial time. Can we conclude that "Hornification" makes `SAT` to be in P?

8. Using resolution, show that $P_1 \wedge P_2 \wedge P_3$ is a consequence of

$$F := (\neg P_1 \vee P_2) \wedge (\neg P_2 \vee P_3) \wedge (P_1 \vee \neg P_3) \wedge (P_1 \vee P_2 \vee P_3)\,.$$

9. Show that the satisfiability of any 2-CNF formula can be checked in polynomial time.

10. Call a set of formulae minimal unsatisfiable iff it is unsatisfiable, but every proper subset is satisfiable. Show that there exist minimal unsatisfiable sets of formulae of size n for each $n \geq 1$.