Logic in CS Autumn 2024

Problem Sheet 2

S. Krishna

1. An adequate set of connectives is a set such that for every formula there is an equivalent formula with only connectives from that set. For example, $\{\neg, \vee\}$ is adequate for propositional logic since any occurrence of $\wedge$ and $\rightarrow$ can be removed using the equivalences

$$\varphi \rightarrow \psi \equiv \neg\varphi \vee \psi$$

$$\varphi \wedge \psi \equiv \neg(\neg\varphi \vee \neg\psi)$$

   (a) Show that $\{\neg, \wedge\}$, $\{\neg, \rightarrow\}$ and $\{\rightarrow, \perp\}$ are adequate sets of connectives. ($\perp$ treated as a nullary connective).

   (b) Show that if $C \subseteq \{\neg, \wedge, \vee, \rightarrow, \perp\}$ is adequate, then $\neg \in C$ or $\perp \in C$.

2. The binary connective `nand`, $F \downarrow G$, is defined by the truth table corresponding to $\neg(F \wedge G)$. Show that `nand` is complete - that is, it can express all binary Boolean connectives.

3. The binary connective `xor`, $F \oplus G$ is defined by the truth table corresponding to $(\neg F \wedge G) \vee (F \wedge \neg G)$. Show that `xor` is not complete - that is, it cannot express all binary Boolean connectives.

4. If a contradiction can be derived from a set of formulae, then the set of formulae is said to be inconsistent. Otherwise, the set of formulae is consistent. Let $\mathcal{F}$ be a set of formulae. Show that $\mathcal{F}$ is consistent iff it is satisfiable.

5. Suppose $\mathcal{F}$ is an inconsistent set of formulae. For each $G \in \mathcal{F}$, let $\mathcal{F}_G$ be the set obtained by removing G from $\mathcal{F}$.

   (a) Prove that for any $G \in \mathcal{F}$, $\mathcal{F}_G \vdash \neg G$, using the previous question.

   (b) Prove this using a formal proof.

6. Consider a formula φ which is of the form $C_1 \wedge C_2 \wedge \dots C_n$ where each clause C_i is of the form $(\top \rightarrow \alpha)$ or $(\alpha_1 \wedge \dots \alpha_n \rightarrow \beta)$ or $(\gamma \rightarrow \perp)$ where $\alpha, \alpha_i, \beta, \gamma$ are literals. A logician wishes to apply `HornSAT` to this formula φ by renaming negative literals (if any) with fresh positive literals. Thus, if any $\alpha, \alpha_i, \beta, \gamma$ was of the form $\neg p$, the logician will replace that $\neg p$ with a fresh variable p'. The logician claims that he can check satisfiability of φ correctly by applying `HornSAT` on the new formula (call it φ') in the following way: φ is satisfiable iff `HornSAT` concludes that φ' is satisfiable, and φ is unsatisfiable iff `HornSAT` concludes that φ' is unsatisfiable. Do you agree with the logician?

7. Using resolution, show that $P_1 \wedge P_2 \wedge P_3$ is a consequence of

$$F := (\neg P_1 \vee P_2) \wedge (\neg P_2 \vee P_3) \wedge (P_1 \vee \neg P_3) \wedge (P_1 \vee P_2 \vee P_3)\,.$$

8. Show that the satisfiability of any 2-CNF formula can be checked in polynomial time.

9. Call a set of formulae minimal unsatisfiable iff it is unsatisfiable, but every proper subset is satisfiable. Show that there exist minimal unsatisfiable sets of formulae of size n for each $n \geq 1$.

10. Consider a set $\Sigma = \{\varphi_1, \varphi_2, \dots\}$ of propositional logic formulae (note that Σ may be infinite). Show that Σ is unsatisfiable if and only if there exists a finite set $\Sigma' \subseteq \Sigma$ such that Σ' is unsatisfiable.