IIT Bombay
CS 6001: GT&AMD
Endsem, 2023-24-I
Date: November 25, 2023

Roll No : 1234567890 **Dept.**:
e.g., 190040001 e.g., CSE

CS6001: Game Theory and Algorithmic Mechanism Design

Total: **10 × 4 = 40 points,** *Duration:* **2 hours, ATTEMPT ALL QUESTIONS**

Instructions:

1. This question-and-answersheet booklet contains a total of **6 sheets** of paper **(11 pages, pages 2 and 12 are blank)**. Please verify.
2. Write your roll number and department on **every side of every sheet** (except the blank sheet) of this booklet. Use only **black/blue ball-point pen**. The first 5 minutes of additional time is given exclusively for this activity.
3. Write final answers neatly with a pen **only in the given boxes**.
4. Use the rough sheets for scratch works / attempts to solution. **Write only the final solution (which may be a sequence of logical arguments) in a precise and succinct manner in the boxes provided**. Do not provide unnecessarily elaborate steps. The space within the boxes are sufficient for the correct and precise answers.
5. Submit your answerscripts to the teaching staff when you leave the exam hall or the time runs out (whichever is earlier). **Your exam will not be graded if you fail to return the paper**.
6. **This is a closed book, notes, internet exam. No communication device, e.g., cellphones, iPad, etc., is allowed**. Keep it switched off in your bag and keep the bag away from you. If anyone is found in possession of such devices during the exam, that answerscript may be disqualified for evaluation and DADAC may be invoked.
7. One A4 assistance sheet (text **on both sides**) is allowed for the exam.
8. **After you are done with your exam or the exam duration is over, please DO NOT rush to the desk for submitting your paper**. Please remain seated until we will collect all the papers, count them, and give you get a clear signal to leave your seat.

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Problem 1 (5 × 2 = 10 points). Consider a single-item auction problem with a single agent in the quasi-linear domain. The valuation of the agent for the item is denoted by v that lies in the closed interval $[a, b]$. Suppose there is a function q that is monotone non-decreasing in v.

(a) Consider a function $f(v)$ that satisfies

$$\exists v^* \in [a, b], \text{ s.t. } f(v) \leqslant 0, \text{ if } a \leqslant v \leqslant v^*; \ f(v) \geqslant 0, \text{ if } v^* < v \leqslant b, \text{ and } \int_a^b f(x)\, dx = 0,$$

i.e., the function f switches its sign at some valuation v^* and has an integral of zero. Prove that, $\int_a^b q(x) f(x)\, dx \geqslant 0$. Show only the steps of the derivation with the appropriate reasons at every step.

Consider $\int_a^b q(x) f(x) dx = \int_a^{v^*} q(x) f(x) dx + \int_{v^*}^b q(x) f(x) dx$ ... ①

note: $\forall x \in [a, v^*]$, $f(x) \leq 0$ and $q(v^*) \geq q(x)$

$\Rightarrow q(v^*) f(x) \leq q(x) f(x)$

Similarly, $\forall x \in (v^*, b]$, $f(x) \geq 0$ and $q(v^*) \leq q(x)$

$\Rightarrow q(v^*) f(x) \leq q(x) f(x)$

Hence the RHS of ① is $\geq q(v^*) \left[\int_a^{v^*} f(x) dx + \int_{v^*}^b f(x) dx \right]$

$= q(v^*) \int_a^b f(x) dx = 0$ □

(b) Now, consider two functions $f_1(v)$ and $f_2(v)$ with the following properties.

$$\exists v^* \in [a, b], \text{ s.t. } f_1(v) \leqslant f_2(v), \text{ if } a \leqslant v \leqslant v^*; \ f_1(v) \geqslant f_2(v), \text{ if } v^* < v \leqslant b,$$
$$\text{and } \int_a^b f_1(x)\, dx = \int_a^b f_2(x)\, dx,$$

i.e., the functions cross over at v^* and have the same integral. Use the result from the previous part of this question to show that $\int_a^b q(x) f_1(x)\, dx \geqslant \int_a^b q(x) f_2(x)\, dx$. Show only the steps of the derivation with the appropriate reasons at every step.

Let $f(v) := (f_1(v) - f_2(v)) \Rightarrow f(v) \leq 0 \ \forall a \leq v \leq v^*$
and $f(v) \geq 0 \ \forall v^* < v \leq b$.
and $\int_a^b f(v) dv$

using part (a), we get

$\int_a^b q(x) f(x) dx \geq 0 \Rightarrow \int_a^b q(x) f_1(x) dx \geq \int_a^b q(x) f_2(x) dx$ □

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(c) Suppose, we want to partially solve the revenue maximization for a slightly different setup, where there are two different allocation functions α and β for two separate types of the agent.[1] As a result, suppose we have the following optimization problem to solve (where q is the same monotone function described above).

$$\begin{aligned} \max_{\alpha,\beta} & \int_a^b q(x)\alpha(x)\ dx + \int_a^b q(x)\beta(x)\ dx \\ \text{s.t. } c & \geqslant \int_a^t (\alpha(x) - \beta(x))\ dx,\ \forall t \in [a, b]. \end{aligned} \tag{1}$$

The functions $\alpha(v)$ and $\beta(v)$ are both monotone non-decreasing in v and c is a constant. Also, $\alpha(v)$ is differentiable at every $v \in [a, b]$. Then prove the following claim constructively: *keeping β fixed, for every feasible solution α of Equation (1), there exists a function $\delta : [a, b] \to [0, 1]$ that*

(i) maintains the constraint, and

(ii) weakly improves the objective function.

In simple words, this question is *not* asking you to solve the optimization problem. Rather, it is aiming to obtain another feasible solution that weakly improves the objective function.

Construct such a δ that takes the minimum number of values, i.e., has the smallest *range*. Provide the most formal description of the function and minimize textual explanation. You do *not* need to explain why you chose such a δ in this part of the question.

The constructed function is as follows:

$\delta(v)$ is a step function s.t.

$$\int_a^b \delta(v)\, dv = \int_a^b \alpha(x)\, dx$$

note that such a δ always exists and is unique.

[1]Type of an agent may be more than just its valuation. For instance, you can imagine that $\alpha(v) = \mathbf{alloc}(v, \theta_1)$ and $\beta(v) = \mathbf{alloc}(v, \theta_2)$.

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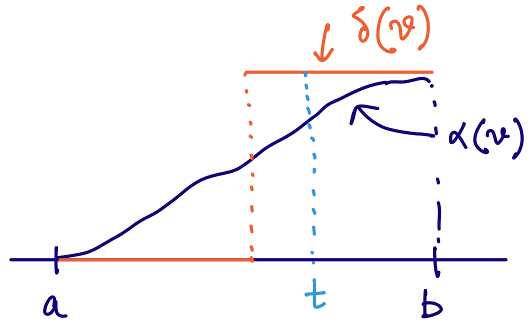
(d) Why does your construction satisfy condition (i) of the claim of part (c)? Please provide the most refined mathematical arguments as your answer.



[Since the integral is the same, any loss in the integral between $[t, b]$ is larger for the step function than that of α]

By construction

$$\int_a^b \delta(v)\, dv = \int_a^b \alpha(v)\, dv$$

Since δ is a step function

$\forall\, t \in [a, b]$

$$\int_a^t \delta(v)\, dv \leq \int_a^t \alpha(v)\, dv$$

hence the constraint is maintained
((i) is satisfied)

(e) Why does your construction satisfy condition (ii) of the claim of part (c)? Please provide the most refined mathematical arguments as your answer.

We saw in part (b) of this problem that if two functions cross over at some point then the function f_1 that stays above f_2 after the crossover point satisfies the following inequality

$$\int_a^b q(x) f_1(x)\, dx \geqslant \int_a^b q(x) f_2(x)\, dx$$

here $\delta(v)$ and $\alpha(v)$ satisfied the same conditions as f_1 and f_2 respectively. Hence

$$\int_a^b q(x)\, \delta(x)\, dx \geqslant \int_a^b q(x)\, \alpha(x)\, dx$$

hence (ii) is also satisfied.

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Problem 2 (10 points). Let f be a monotonic social choice function, let $a \in \textbf{range}(f)$, and let P be a strict preference profile. In this problem, we will prove the following in steps.

$$\text{if } aP_ib, \forall\, i \in N, \text{ then } f(P) \neq b.$$

Suppose, for contradiction, $f(P) = b$. Answer the following sub-questions using only first principles, i.e., do not use any derived results from the class.

(a) Use a property given in the question to assert existence of a profile. **2 points**

Since $a \in$ range(f)

$\exists$ a profile Q s.t. $f(Q) = a$

(b) Construct a new profile that helps making use of the monotonicity of f. **4 points**

Create a profile P' s.t.

$P_i'(1) = a$ and $P_i'(2) = b$ $\forall\, i \in N$, i.e.

P'

a	a	...	a
b	b		b
⋮	⋮		⋮

The first and the second preferred alternatives are a and b respectively for every agent in P'

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(c) Finish the proof by arguing about the contradiction. Write only the precise arguments. **4 points**

We know in P, $a P_i b$, $\forall i \in N$ and $f(P) = b$

In P', a remains above b and both alternatives move to positions 1 and 2 respectively. So, the 'if' condition of monotonicity $D(b, P_i) \subseteq D(b, P_i') \;\; \forall i \in N$ is satisfied. Hence, $f(P') = b$ (by monotonicity)

However, in Q, $f(Q) = a$ and in P', a moves to the top. Hence

$$D(a, Q_i) \subseteq D(a, P_i') \quad \forall i \in N$$

Hence, $f(P') = a$ (by monotonicity)

This is a contradiction. Hence $f(P) \neq b$.

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Problem 3 (10 points). Three friends, numbered 1, 2, and 3, are planning to choose a movie to watch together. The candidate movies are A, B, and C (exactly one of them can be chosen). The values (derived from their film interests) that the friends get from watching these movies are given in the table below.

	A	B	C
1	7	5	1
2	7	5	2
3	2	5	10

They agree that the decision should be as 'efficient' as possible, i.e., it should maximize the sum value of all the agents, but they are also concerned whether (i) this can be done truthfully, and (ii) using only internal transfers, i.e., keeping it budget balanced.

(a) If monetary transfer is allowed and that results in a quasi-linear utility to the agents (the friends), provide a payment rule that implements the efficient choice. **1 point**

Both VCG (pivotal payment) and Groves payment rules ensures efficient choice.

Important clarification: since the question missed mentioning dominant strategy truthfulness, the answers for dAGVA stands correct. However, for such a mechanism, which is prior-dependent, if the prior is not mentioned or explained, those parts will get no points.

[this solution assumes VCG payment, if you choose groves, then the actual h_i function should be mentioned for each profile]

(b) Which movie is decided? A **1 point**

(c) How much should each agent pay? **0.5 × 3 = 1.5 points**

Agent 1: $12 - 9 = 3$

Agent 2: $11 - 9 = 2$

Agent 3: $14 - 14 = 0$

(d) What is the net utility of each of the players? **0.5 × 3 = 1.5 points**

Agent 1: $7 - 3 = 4$

Agent 2: $7 - 2 = 5$

Agent 3: $2 - 0 = 2$

(e) Is this mechanism budget balanced? (yes/no) no **1 point**

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(f) Suppose they decide on a different mechanism instead. Agent 2's values are not considered in the movie decision process. The mechanism chosen in part (a) above is used for agents 1 and 3, the monetary surplus generated is transferred to agent 2. Does this mechanism satisfy properties (i) and (ii)? Precisely explain why. **1.5 points**

Yes. This mechanism satisfies both (i) and (ii)
It is truthful for agents 1 and 3 because of the property of VCG
It is truthful for agent 2 since the allocation and payment does not depend on her reported valuations.

Under this mechanism, the outcome will be movie C.

(g) What are the net utilities of the players? **0.5 $\times$ 3 = 1.5 points**

Agent 1: $1 - 0 = 1$

Agent 2: $10 - 6 = 4$

Agent 3: $2 - (-6) = 8$

(h) Is this mechanism efficient and why? **1 point**

This is not efficient since it chooses an inefficient outcome C which yields a social welfare of 13, while the efficient outcome is A having a welfare of 16.

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Problem 4 (10 points). The following table provides the allocation functions for auctioning a single indivisible item among two bidders. Each bidder has five *equally likely* valuations given by $\{1, 2, \ldots, 5\}$ and valuations of the agents are *independent*. The valuations are shown along the x and y axes for bidders 1 and 2 respectively in the first column of every row in the following table. The markers ■ and □ denote that the item is allocated to bidders 1 and 2 respectively.

(a) Answer if the following allocation functions are Bayesian or dominant strategy implementable or not. Wherever it is, provide the expected payment π_1 of bidder 1 that Bayesian implements the allocation. Note that $\pi_1(t_1) = \mathbb{E}(p_1(t_1, t_2) \mid t_1)$, where $p_1(t_1, t_2)$ is the payment of bidder 1 when the bid profile is (t_1, t_2). Show two steps (the formula and the values) as you arrive at the final value of the payment π_1. Write `NA` where payments are not possible. **9 × 1 = 9 points**

<table>
<tr><th>Allocation function</th><th>Bayesian Implementable? (Yes/No)</th><th>Dominant Strategy Implementable? (Yes/No)</th><th>Payment, $\pi_1(\cancel{t_1}\,4) - \pi_1(0)$</th></tr>
<tr><td>
<table>
<tr><td>5</td><td>□</td><td>□</td><td>□</td><td>□</td><td>■</td></tr>
<tr><td>4</td><td>□</td><td>□</td><td>□</td><td>■</td><td>□</td></tr>
<tr><td>3</td><td>□</td><td>□</td><td>□</td><td>□</td><td>■</td></tr>
<tr><td>2</td><td>□</td><td>□</td><td>□</td><td>■</td><td>□</td></tr>
<tr><td>1</td><td>□</td><td>□</td><td>■</td><td>□</td><td>■</td></tr>
<tr><td></td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr>
</table>
</td><td>Yes</td><td>No</td><td>$4 \cdot \alpha_1(4) - \int^{4} \alpha_1(x)\,dx$
$= 4 \times \frac{2}{5} - \left(0 + \overset{0}{0} + \frac{1}{5} + \frac{2}{5}\right)$
$= 1$</td></tr>
<tr><td>
<table>
<tr><td>5</td><td>□</td><td>□</td><td>□</td><td>□</td><td>□</td></tr>
<tr><td>4</td><td>□</td><td>□</td><td>□</td><td>□</td><td>■</td></tr>
<tr><td>3</td><td>□</td><td>□</td><td>□</td><td>■</td><td>■</td></tr>
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<tr><td></td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr>
</table>
</td><td>Yes</td><td>Yes</td><td>$4 \times \frac{3}{5} - \left(\frac{1}{5} + \frac{2}{5} + \frac{3}{5}\right)$
$= \frac{6}{5} = 1.2$</td></tr>
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<tr><td>5</td><td>□</td><td>□</td><td>□</td><td>□</td><td>■</td></tr>
<tr><td>4</td><td>□</td><td>□</td><td>□</td><td>■</td><td>□</td></tr>
<tr><td>3</td><td>□</td><td>□</td><td>■</td><td>□</td><td>■</td></tr>
<tr><td>2</td><td>□</td><td>□</td><td>□</td><td>■</td><td>□</td></tr>
<tr><td>1</td><td>□</td><td>□</td><td>□</td><td>□</td><td>■</td></tr>
<tr><td></td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr>
</table>
</td><td>No</td><td>No</td><td>NA</td></tr>
</table>

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(b) Explain how you arrived at the above set of answers. In particular, which principle(s) did you use to conclude to your answers above. **1 point**

The allocation rule is

Dominant strategy implementable if it is non-decreasing for every agent

AND

Bayesian implementable if it is non-decreasing in expectation for every agent.

The first allocation is NDE but not ND
second " is ND, hence is also NDE
third " is neither NDE nor ND

hence the answers.

Myerson payment formula is used to find $\pi_1(4) - \pi_1(0)$

END OF QUESTION PAPER. GOOD LUCK!