INDIAN INSTITUTE OF TECHNOLOGY, BOMBAY

Department of Mathematics

SI 427 (Probability Theory)

Tutorial Sheet-I

1. Let $\Omega \neq \emptyset$ be a finite set and $\mathcal{F}$ is a field of subsets of Ω. Show that $\mathcal{F}$ is a σ-field.

2. Write down the expression in set notation corresponding to each of the following events.
(i) The event which occurs if exactly one of the events A, B occurs.
(ii) The event which occurs if none of the events A, B, C occurs.
(iii) The event which occurs if exactly two of the events A, B, C occurs.

3. Let $(\Omega, \mathcal{F}, P)$ be a probability space and A, B, C be events in $\mathcal{F}$ such that
$$P(A) = 0.7,\ P(B) = 0.6, P(C) = 0.5, P(AB) = 0.4,$$
$$P(AC) = 0.3, P(BC) = 0.2 \text{ and } P(ABC) = 0.1\,.$$
Find (i) $P(A \cup B \cup C)$ (ii) $P(A^cC)$ (iii) $P(A^cB^cC^c)$.

4. Does there exists a probability measure P such that the events A, B, C satisfies
$$P(A) = 0.6,\ P(B) = 0.8, P(C) = 0.7, P(AB) = 0.5,$$
$$P(AC) = 0.4, P(BC) = 0.5 \text{ and } P(ABC) = 0.1\,?$$

5. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $A_1, A_2 \dots A_n$ be events in $\mathcal{F}$. Show that
$$P(A_1 \dots A_n) \ \geq \ \sum_{i=1}^{n} P(A_i) \ - \ n + 1\,.$$

6. Define $P : \mathcal{P}(\mathbb{N}) \to [0,\, 1]$ as follows.
$$P(A) \ = \ \begin{cases} 0 & \text{if} \quad A \text{ is finite} \\ 1 & \text{if} \quad A \text{ is infinite} \end{cases}$$

Is P a probability measure? Justify your answer.

7. Let Ω be a nonempty set and $\mathcal{F}$ is a σ-field of subsets of Ω. Let $P : \mathcal{F} \to [0,\ 1]$ be such that

   (i) $P(\Omega) = 1$

   (ii)For $A_1, A_2 \in \mathcal{F}$ disjoint

   $$P(A_1 \cup A_2) \;=\; P(A_1) + P(A_2)$$

   (iii) Whenever $\{A_n\}$ is a decreasing sequence of events from $\mathcal{F}$, $\lim_{n\to\infty} P(A_n) = P(\cap_{i=1}^{\infty} A_i)$.

   Then show that P is a probability measure.

8. Let $(\Omega,\ \mathcal{F}, P)$ be a probability space and $A_1, A_2 \cdots \in \mathcal{F}$ be such that $P(A_iA_j) = 0,\ i \neq j$. Show that

   $$P\Big(\cup_{n=1}^{\infty} A_n\Big) \;=\; \sum_{n=1}^{\infty} P(A_n)\,.$$

9. Let Ω be an uncountable set and

   $$\mathcal{F} \;=\; \{A \subseteq \Omega \,|\, \text{either } A \text{ is countable or } A^c \text{ is countable }\}\,.$$

   Show that $\mathcal{F}$ is a σ-field.

10. Let $\mathcal{D}_1,\ \mathcal{D}_2$ be two nonempty family of subsets of a nonempty set Ω such that $\mathcal{D}_1 \subset \mathcal{D}_2$. Show that $\sigma(\mathcal{D}_1) \subseteq \sigma(\mathcal{D}_2)$ Is the inclusion always strict?

11. In a group of n 'unrelated' individuals, none born on a leap year, show that the probability that atleast two share a birth day exceeds half if $n \geq 23$.

12. (Maxwell-Boltzmann Statistics) A configuration of n balls placed in r urns is an arrangement $(k_1, \cdots, k_r)$ where k_i denotes the number of balls in the ith urn and $k_1 + \cdots + k_r = n$. If the balls and urns are distinguishable and the balls are distributed at random into the urns, find the probability of a particular configuration.

13. Let $\pi = (\pi_1, \pi_2, \cdots, \pi_{52})$ denote a permutation of numbers from 1 to 52 done at random (You may think it as a perfectly shuffled deck of cards). By coincidence at a location k, we mean $\pi_k = k$. What is the probability that there is atleast one coincidence.

14. Does there exists a σ-field with number of elements 4098? Justify your answer.