CS 228 : Logic in Computer Science

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Moving On : Temporal Logics

- Starting Linear Temporal Logic (LTL)

Transition Systems

A Transition System is a tuple $(S, Act, \rightarrow, I, AP, L)$ where

- S is a set of states
- Act is a set of actions
- $s \xrightarrow{\alpha} s'$ in $S \times Act \times S$ is the transition relation
- $I \subseteq S$ is the set of initial states
- AP is the set of atomic propositions
- $L : S \rightarrow 2^{AP}$ is the labeling function

Traces of Transition Systems

- Labels of the locations represent values of all observable propositions $\in AP$
- Captures system state
- Focus on sequences $L(s_0)L(s_1)\ldots$ of labels of locations
- Such sequences are called traces
- Assuming transition systems have no terminal states,
  - Traces are infinite words over 2^{AP}
  - Traces $\in (2^{AP})^\omega$

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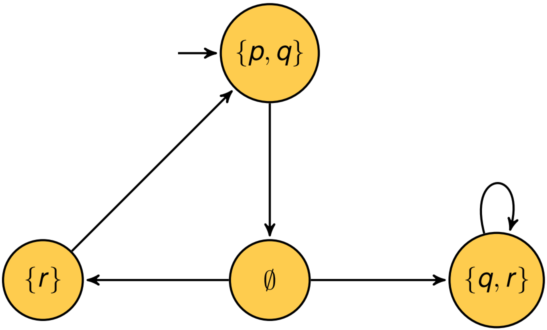
- $Traces(TS) = \bigcup_{s \in I} Traces(s)$

Example Traces



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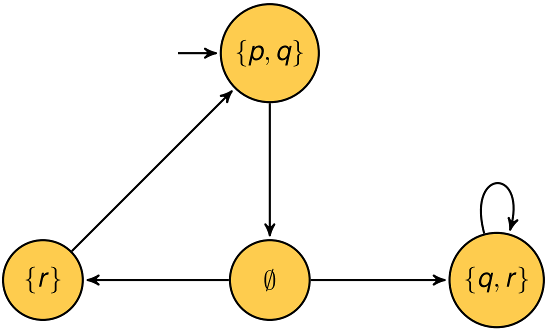

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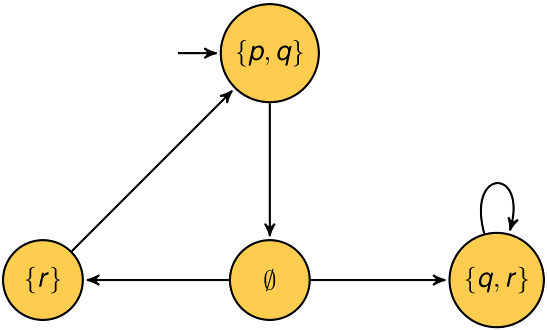

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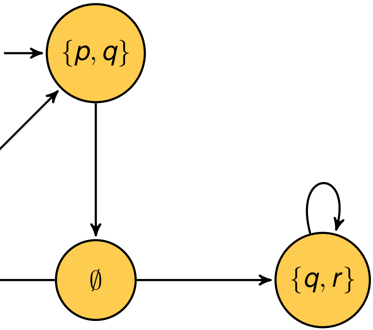

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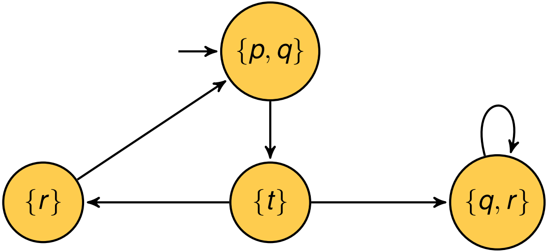
Linear Time Properties

- Linear-time properties specify traces that a TS must have
- A LT property P over AP is a subset of $(2^{AP})^{\omega}$
- TS over AP satisfies a LT property P over AP

$$TS \models P \text{ iff } Traces(TS) \subseteq P$$

- $s \in S$ satisfies LT property P (denoted $s \models P$) iff $Traces(s) \subseteq P$

Specifying Traces



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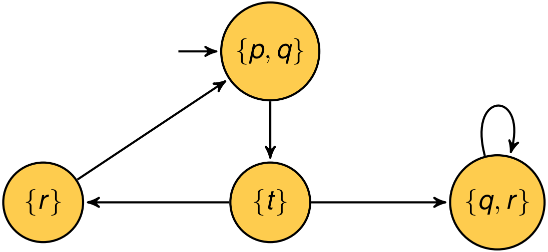

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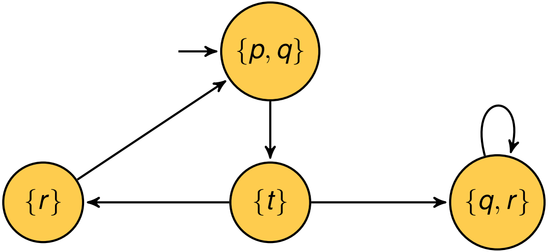

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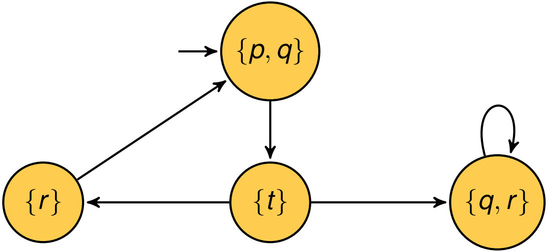

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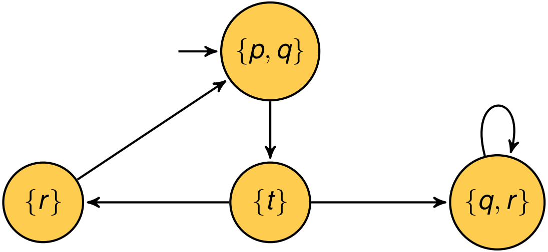

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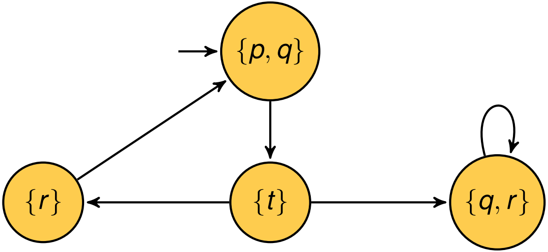

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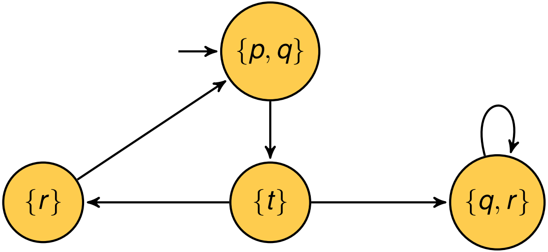

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- Propositional logic formulae over *AP*
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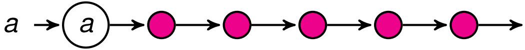
  - $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$
- Temporal Operators
  - $\bigcirc\varphi$ (Next φ)
  - $\varphi \,\mathsf{U} \psi$ (φ holds until a ψ-state is reached)
- LTL : Logic for describing LT properties

Semantics

LTL formulae over AP interpreted over words over Σ^ω, $\Sigma = 2^{AP}$

This is not an automaton, but a word over Σ



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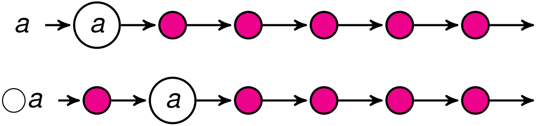

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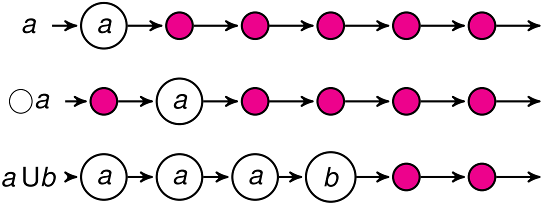

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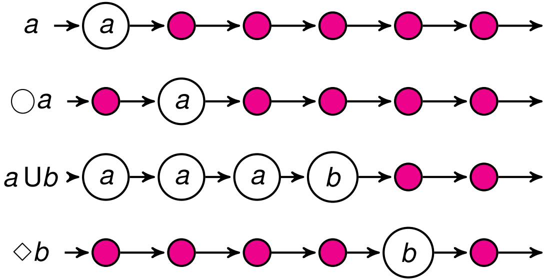

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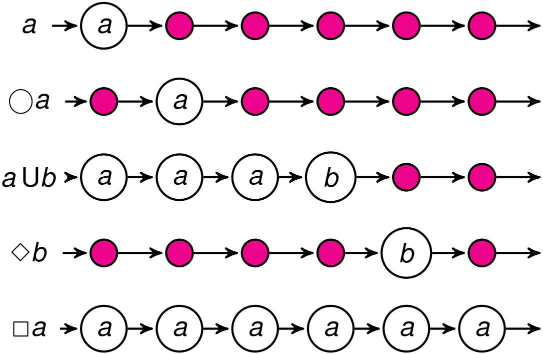

Derived Operators

- $\mathit{true} = \varphi \vee \neg\varphi$
- $\mathit{false} = \neg\mathit{true}$
- $\Diamond\varphi = \mathit{true}\,\mathsf{U}\varphi$ (Eventually φ)
- $\Box\varphi = \neg\Diamond\neg\varphi$ (Forever φ)

Precedence

- Unary Operators bind stronger than Binary
- $\bigcirc$ and $\neg$ equally strong
- U takes precedence over $\wedge, \vee, \rightarrow$
  - $a \vee b\,\mathsf{U}c \equiv a \vee (b\,\mathsf{U}c)$
  - $\bigcirc a\,\mathsf{U}\neg b \equiv (\bigcirc a)\,\mathsf{U}(\neg b)$

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Given LTL formula φ over AP,

$$L(\varphi) = \{\sigma \in (2^{AP})^\omega \mid \sigma \models \varphi\}$$

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- $\sigma \models \varphi \, \mathsf{U} \psi$ iff
  $\exists j \geqslant 0$ such that $A_j A_{j+1} \ldots \models \psi \wedge \forall 0 \leqslant i < j, A_i A_{i+1} \ldots \models \varphi$

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If $\sigma = A_0 A_1 A_2 \ldots$, $\sigma \models \varphi$ is also written as $\sigma, 0 \models \varphi$. This simply means $A_0 A_1 A_2 \ldots \models \varphi$. One can also define $\sigma, i \models \varphi$ to mean $A_i A_{i+1} A_{i+2} \ldots \models \varphi$ to talk about a suffix of the word σ satisfying a property.