5 Mixed strategies

Chapter summary

Given a game in strategic form we extend the strategy set of a player to the set of all probability distributions over his strategies. The elements of the new set are called *mixed strategies*, while the elements of the original strategy set are called *pure strategies*. Thus, a mixed strategy is a probability distribution over pure strategies. For a strategic-form game with finitely many pure strategies for each player we define the *mixed extension* of the game, which is a game in strategic form in which the set of strategies of each player is his set of mixed strategies, and his payoff function is the multilinear extension of his payoff function in the original game.

The main result of the chapter is the Nash Theorem, which is one of the milestones of game theory. It states that the mixed extension always has a Nash equilibrium; that is, a Nash equilibrium in mixed strategies exists in every strategic-form game in which all players have finitely many pure strategies. We prove the theorem and provide ways to compute equilibria in special classes of games, although the problem of computing Nash equilibrium in general games is computationally hard.

We generalize the Nash Theorem to mixed extensions in which the set of strategies of each player is not the whole set of mixed strategies, but rather a polytope subset of this set.

We investigate the relation between utility theory discussed in Chapter 2 and mixed strategies, and define the *maxmin value* and the *minmax value* of a player (in mixed strategies), which measure respectively the amount that the player can guarantee to himself, and the lowest possible payoff that the other players can force on the player.

The concept of *evolutionary stable strategy*, which is the Nash equilibrium adapted to Darwin's Theory of Evolution, is presented in Section 5.8.

There are many examples of interactive situations (games) in which it is to a decision maker's advantage to be "unpredictable":

- If a baseball pitcher throws a waist-high fastball on every pitch, the other team's batters will have an easy time hitting the ball.
- If a tennis player always serves the ball to the same side of the court, his opponent will have an advantage in returning the serve.
- If a candidate for political office predictably issues announcements on particular dates, his opponents can adjust their campaign messages ahead of time to pre-empt him and gain valuable points at the polls.

- If a traffic police car is placed at the same junction at the same time every day, its effectiveness is reduced.

It is easy to add many more such examples, in a wide range of situations. How can we integrate this very natural consideration into our mathematical model?

Example 5.1 Consider the two-player zero-sum game depicted in Figure 5.1.

		Player II		
		L	*R*	$\min_{s_{\mathrm{II}}} u(s_{\mathrm{I}}, s_{\mathrm{II}})$
Player I	*T*	4	1	1
	B	2	3	2
$\max_{s_{\mathrm{I}}} u(s_{\mathrm{I}}, s_{\mathrm{II}})$		4	3	2, 3

Figure 5.1 A two-player zero-sum game; the security values of the players are circled

Player I's security level is 2; if he plays B he guarantees himself a payoff of at least 2. Player II's security level is 3; if he plays R he guarantees himself a payoff of at most 3.

This is written as

$$\underline{v} = \max_{s_{\mathrm{I}} \in \{T,B\}} \min_{s_{\mathrm{II}} \in \{L,R\}} u(s_{\mathrm{I}}, s_{\mathrm{II}}) = 2, \tag{5.1}$$

$$\overline{v} = \min_{s_{\mathrm{II}} \in \{L,R\}} \max_{s_{\mathrm{I}} \in \{T,B\}} u(s_{\mathrm{I}}, s_{\mathrm{II}}) = 3. \tag{5.2}$$

Since

$$\overline{v} = 3 > 2 = \underline{v}, \tag{5.3}$$

the game has no value.

Can one of the players, say Player I, guarantee a "better outcome" by playing "unpredictably"? Suppose that Player I tosses a coin with parameter $\frac{1}{4}$, that is, a coin that comes up heads with probability $\frac{1}{4}$ and tails with probability $\frac{3}{4}$. Suppose furthermore that Player I plays T if the result of the coin toss is heads and B if the result of the coin toss is tails. Such a strategy is called a mixed strategy.

What would that lead to? First of all, the payoffs would no longer be definite, but instead would be probabilistic payoffs. If Player II plays L the result is a lottery $[\frac{1}{4}(4), \frac{3}{4}(2)]$; that is, with probability $\frac{1}{4}$ Player II pays 4, and with probability $\frac{3}{4}$ pays 2. If these payoffs are the utilities of a player whose preference relation satisfies the von Neumann–Morgenstern axioms (see Chapter 2), then Player I's utility from this lottery is $\frac{1}{4} \times 4 + \frac{3}{4} \times 2 = 2\frac{1}{2}$. If, however, Player II plays R the result is the lottery $[\frac{1}{4}(1), \frac{3}{4}(3)]$. In this case, if the payoffs are utilities, Player I's utility from this lottery is $\frac{1}{4} \times 1 + \frac{3}{4} \times 3 = 2\frac{1}{2}$. ◀

5.1 The mixed extension of a strategic-form game

In the rest of this section, we will assume that the utilities of the players satisfy the von Neumann–Morgenstern axioms; hence their utility functions are linear (in probabilities).

In other words, the payoff (= utility) to a player from a lottery is the expected payoff of that lottery. With this definition of what a payoff is, Player I can guarantee that no matter what happens his expected payoff will be at least $2\frac{1}{2}$, in contrast to a security level of 2 if he does not base his strategy on the coin toss.

Definition 5.2 *Let $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a strategic-form game in which the set of strategies of each player is finite. A* mixed strategy *of player i is a probability distribution over his set of strategies S_i. Denote by*

$$\Sigma_i := \left\{ \sigma_i : S_i \to [0,1] \colon \sum_{s_i \in S_i} \sigma_i(s_i) = 1 \right\} \tag{5.4}$$

the set of mixed strategies of player i.

A mixed strategy of player i is, therefore, a probability distribution over S_i: $\sigma_i = (\sigma_i(s_i))_{s_i\in S_i}$. The number $\sigma_i(s_i)$ is the probability of playing the strategy s_i. To distinguish between the mixed strategies Σ_i and the strategies S_i, the latter are called *pure strategies*. Because all the results proved in previous chapters involved only pure strategies, the claims in them should be qualified accordingly. For example, Kuhn's Theorem (Theorem 4.49 on page 118) should be read as saying: In every finite game with perfect information, there is at least one equilibrium point in pure strategies.

We usually denote a mixed strategy using the notations for lotteries (see Chapter 2). For example, if Player I's set of pure strategies is $S_{\mathrm{I}} = \{A, B, C\}$, we denote the mixed strategy σ_{I} under which he chooses each pure strategy with probability $\frac{1}{3}$ by $\sigma_{\mathrm{I}} = [\frac{1}{3}(A), \frac{1}{3}(B), \frac{1}{3}(C)]$.

If $S_{\mathrm{I}} = \{H, T\}$, Player I's set of mixed strategies is

$$\Sigma_{\mathrm{I}} = \{[p_1(H), p_2(T)] \colon p_1 \geq 0, \quad p_2 \geq 0, \quad p_1 + p_2 = 1\}. \tag{5.5}$$

In this case, the set Σ_{I} is equivalent to the interval in $\mathbb{R}^2$ connecting $(1,0)$ to $(0,1)$. We can identify Σ_{I} with the interval $[0,1]$ by identifying every real number $x \in [0,1]$ with the probability distribution over $\{H, T\}$ that satisfies $p(H) = x$ and $p(T) = 1 - x$. If $S_{\mathrm{II}} = \{L, M, R\}$, Player II's set of mixed strategies is

$$\Sigma_{\mathrm{II}} = \{[p_1(L), p_2(M), p_2(R)] \colon p_1 \geq 0, \quad p_2 \geq 0, \quad p_3 \geq 0, \quad p_1 + p_2 + p_3 = 1\}. \tag{5.6}$$

In this case, the set Σ_{II} is equivalent to the triangle in $\mathbb{R}^3$ whose vertices are $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

For any finite set A, denote by $\Delta(A)$ the set of all probability distributions over A. That is,

$$\Delta(A) := \left\{ p : A \to [0,1] \colon \sum_{a\in A} p(a) = 1 \right\}. \tag{5.7}$$

The set $\Delta(A)$ is termed a *simplex* in $\mathbb{R}^{|A|}$. The dimension of the simplex $\Delta(A)$ is $|A| - 1$ (this follows from the constraint that $\sum_{a\in A} p(a) = 1$). We denote the number of pure strategies of player i by m_i, and we assume that his pure strategies have a particular ordering, with the denotation $S_i = \{s_i^1, s_i^2, \ldots, s_i^{m_i}\}$. It follows that the set of mixed strategies $\Sigma_i = \Delta(S_i)$ is a subset of $\mathbb{R}^{m_i}$ of dimension $m_i - 1$.

We identify a mixed strategy s_i with the pure strategy $\sigma_i = [1(s_i)]$, in which the pure strategy s_i is chosen with probability 1. This implies that every pure strategy can also be considered a mixed strategy.

We now define the mixed extension of a game.

Definition 5.3 *Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a strategic-form game in which for every player $i \in N$, the set of pure strategies S_i is nonempty and finite. Denote by $S := S_1 \times S_2 \times \cdots \times S_n$ the set of pure strategy vectors. The* mixed extension *of G is the game*

$$\Gamma = (N, (\Sigma_i)_{i \in N}, (U_i)_{i \in N}), \tag{5.8}$$

in which, for each $i \in N$, player i's set of strategies is $\Sigma_i = \Delta(S_i)$, and his payoff function is the function $U_i : \Sigma \to \mathbb{R}$, which associates each strategy vector $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma = \Sigma_1 \times \cdots \times \Sigma_n$ with the payoff

$$U_i(\sigma) = \mathbf{E}_\sigma[u_i(\sigma)] = \sum_{(s_1,\ldots,s_n) \in S} u_i(s_1, \ldots, s_n)\sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_n(s_n). \tag{5.9}$$

Remark 5.4 *Mixed strategies were defined above only for the case in which the sets of pure strategies are finite. It follows that the mixed extension of a game is only defined when the set of pure strategies of each player is finite. However, the concept of mixed strategy, and hence the mixed extension of a game, can be defined when the set of pure strategies of a player is a countable set (see Example 5.12 and Exercise 5.50). In that case the set $\Sigma_i = \Delta(S_i)$ is an infinite-dimensional set. It is possible to extend the definition of mixed strategy further to the case in which the set of strategies is any set in a measurable space, but that requires making use of concepts from measure theory that go beyond the background in mathematics assumed for this book.* ◆

Note that the fact that the mixed strategies of the players are statistically independent of each other plays a role in Equation (5.9), because the probability of drawing a particular vector of pure strategies $(s_1, s_2, \ldots, s_n)$ is the product $\sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_n(s_n)$. In other words, each player i conducts the lottery σ_i that chooses s_i independently of the lotteries conducted by the other players.

The mixed extension Γ of a strategic-form game G is itself a strategic-form game, in which the set of strategies of each player is of the cardinality of the continuum. It follows that all the concepts we defined in Chapter 4, such as dominant strategy, security level, and equilibrium, are also defined for Γ, and all the results we proved in Chapter 4 apply to mixed extensions of games.

Definition 5.5 *Let G be a game in strategic form, and Γ be its mixed extensions. Every equilibrium of Γ is called an* equilibrium in mixed strategies *of G. If G is a two-player zero-sum game, and if Γ has value v, then v is called the* value of G in mixed strategies.

Example 5.1 (*Continued*) Consider the two-player zero-sum game in Figure 5.2.

		Player II	
		L	R
Player I	T	4	1
	B	2	3

Figure 5.2 The game in strategic form

When Player I's strategy set contains two actions, T and B, we identify the mixed strategy $[x(T), (1-x)(B)]$ with the probability x of selecting the pure strategy T. Similarly, when Player II's strategy set contains two actions, L and R, we identify the mixed strategy $[y(L), (1-y)(R)]$ with the probability y of selecting the pure strategy L. For each pair of mixed strategies $x, y \in [0, 1]$ (with the identifications $x \approx [x(T), (1-x)(B)]$ and $y \approx [y(L), (1-y)(R)]$) the payoff is

$$U(x, y) = 4xy + 1x(1-y) + 2(1-x)y + 3(1-x)(1-y) \tag{5.10}$$

$$= 3 - 2x - y + 4xy. \tag{5.11}$$

This mixed extension is identical to the game over the unit square presented in Section 4.14.1. As we showed there, the game has the value $2\frac{1}{2}$, and its optimal strategies are $x = \frac{1}{4}$ and $y = \frac{1}{2}$. It follows that the value in mixed strategies of the game in Figure 5.2 is $2\frac{1}{2}$, and the optimal strategies of the players are $x^* = [\frac{1}{4}(T), \frac{3}{4}(B)]$ and $y^* = [\frac{1}{2}(L), \frac{1}{2}(R)]$. We conclude that this game has no value in pure strategies, but it does have a value in mixed strategies. ◀

The payoff function defined in Equation (5.10) is a linear function over x for each fixed y and, similarly, a linear function over y for each fixed x. Such a function is called a *bilinear function*. The analysis we conducted in Example 5.1 can be generalized to all two-player games where each player has two pure strategies. The extension to mixed strategies of such a game is a game on the unit square with bilinear payoff functions. In the converse direction, every zero-sum two-player game over the unit square with bilinear payoff functions is the extension to mixed strategies of a two-player zero-sum game in which each player has two pure strategies (Exercise 5.6).

The next theorem states that this property can be generalized to any number of players and any number of actions, as long as we properly generalize the concept of bilinearity to multilinearity.

Theorem 5.6 *Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a game in strategic form in which the set of strategies S_i of every player is finite, and let $\Gamma = (N, (\Sigma_i)_{i \in N}, (U_i)_{i \in N})$ be its mixed extension. Then for each player $i \in N$, the function U_i is a multilinear function in the n variables $(\sigma_i)_{i \in N}$, i.e., for every player i, for every $\sigma_i, \sigma_i' \in \Sigma_i$, and for every $\lambda \in [0, 1]$,*

$$U_i(\lambda\sigma_i + (1-\lambda)\sigma_i', \sigma_{-i}) = \lambda U_i(\sigma_i, \sigma_{-i}) + (1-\lambda)U_i(\sigma_i', \sigma_{-i}), \qquad \forall \sigma_{-i} \in \Sigma_{-i}.$$

Proof: Recall that

$$U_i(\sigma) = \sum_{(s_1,\ldots,s_n)\in S} u_i(s_1, \ldots, s_n)\sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_n(s_n). \tag{5.12}$$

The function U_i is a function of $\sum_{i=1}^{n} m_i$ variables:

$$\sigma_1\left(s_1^1\right), \sigma_1\left(s_1^2\right), \ldots, \sigma_1\left(s_1^{m_1}\right), \sigma_2\left(s_2^1\right), \ldots, \sigma_2\left(s_2^{m_2}\right), \ldots, \sigma_n(s_n^1), \ldots, \sigma_n(s_n^{m_n}). \quad (5.13)$$

For each $i \in N$, for all j, $1 \leq j \leq m_i$ and for each $s = (s_1, \ldots, s_n) \in S$, the function

$$\sigma_i\left(s_i^j\right) \mapsto u_i(s_1, \ldots, s_n)\sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_n(s_n) \quad (5.14)$$

is a constant function if $s_i \neq s_i^j$ and a linear function of $\sigma_i(s_i^j)$ with slope

$$u_i(s_1, \ldots, s_n)\sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_2(s_{i-1})\sigma_2(s_{i+1})\cdots\sigma_n(s_n) \quad (5.15)$$

if $s_i = s_i^j$. Thus, the function U_i, as the sum of linear functions in $\sigma_i(s_i^j)$, is also linear in $\sigma_i(s_i^j)$. It follows that for every $i \in N$, the function $U_i(\cdot, \sigma_{-i})$ is linear in each of the coordinates $\sigma_i(s_i^j)$ of σ_i, for all $\sigma_{-i} \in \Sigma_{-i}$:

$$U_i(\lambda\sigma_i + (1-\lambda)\sigma_i', \sigma_{-i}) = \lambda U_i(\sigma_i, \sigma_{-i}) + (1-\lambda)U_i(\sigma_i', \sigma_{-i}), \quad (5.16)$$

for every $\lambda \in [0, 1]$, and every $\sigma_i, \sigma_i' \in \Sigma_i$. □

Since a multilinear function over Σ is a continuous function (see Exercise 5.4), we have the following corollary of Theorem 5.6.

Corollary 5.7 *The payoff function U_i of player i is a continuous function in the extension to mixed strategies of every finite strategic-form game $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$.*

We can also derive a second corollary from Theorem 5.6, which can be used to determine whether a particular mixed strategy vector is an equilibrium.

Corollary 5.8 *Let $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a strategic-form game, and let Γ be its mixed extension. A mixed strategy vector σ^* is an equilibrium in mixed strategies of Γ if and only if for every player $i \in N$ and every pure strategy $s^i \in S^i$*

$$U_i(\sigma^*) \geq U_i(s_i, \sigma_{-i}^*). \quad (5.17)$$

Proof: If σ^* is an equilibrium in mixed strategies of Γ, then $U_i(\sigma^*) \geq U_i(\sigma_i, \sigma_{-i}^*)$ for every player $i \in N$ and every mixed strategy $\sigma_i \in \Sigma_i$. Since every pure strategy is in particular a mixed strategy, $U_i(\sigma^*) \geq U_i(s_i, \sigma_{-i}^*)$ for every player $i \in N$ and every pure strategy $s^i \in S^i$, and Equation (5.17) holds.

To show the converse implication, suppose that the mixed strategy vector σ^* satisfies Equation (5.17) for every player $i \in N$ and every pure strategy $s^i \in S^i$. Then for each mixed strategy σ_i of player i,

$$U_i(\sigma_i, \sigma_{-i}^*) = \sum_{s_i\in S_i} \sigma_i(s_i)U_i(s_i, \sigma_{-i}^*) \quad (5.18)$$

$$\leq \sum_{s_i\in S_i} \sigma_i(s_i)U_i(\sigma^*) \quad (5.19)$$

$$= U_i(\sigma^*)\sum_{s_i\in S_i} \sigma_i(s_i) = U_i(\sigma^*), \quad (5.20)$$

where Equation (5.18) follows from the fact that U_i is a multilinear function, and Equation (5.19) follows from Equation (5.17). In particular, σ^* is an equilibrium in mixed strategies of Γ. ☐

Example 5.9 **A mixed extension of a two-player game that is not zero-sum** Consider the two-player non-zero-sum game given by the payoff matrix shown in Figure 5.3.

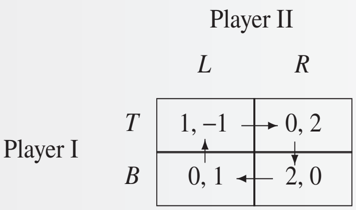

Figure 5.3 A two-player, non-zero-sum game without an equilibrium

As we now show, this game has no equilibrium in pure strategies (you can follow the arrows in Figure 5.3 to see why this is so).

- (T, L) is not an equilibrium, since Player II can gain by deviating to R.
- (T, R) is not an equilibrium, since Player I can gain by deviating to B.
- (B, L) is not an equilibrium, since Player I can gain by deviating to T.
- (B, R) is not an equilibrium, since Player II can gain by deviating to L.

Does this game have an equilibrium in mixed strategies? To answer this question, we first write out the mixed extension of the game:

- The set of players is the same as the set of players in the original game: $N = \{\mathrm{I}, \mathrm{II}\}$.
- Player I's set of strategies is $\Sigma_{\mathrm{I}} = \{[x(T), (1-x)(B)] \colon x \in [0,1]\}$, which can be identified with the interval $[0, 1]$.
- Player II's set of strategies is $\Sigma_{\mathrm{II}} = \{[y(L), (1-y)(R)] \colon y \in [0,1]\}$, which can be identified with the interval $[0, 1]$.
- Player I's payoff function is

$$U_{\mathrm{I}}(x, y) = xy + 2(1-x)(1-y) = 3xy - 2x - 2y + 2. \tag{5.21}$$

- Player II's payoff function is

$$U_{\mathrm{II}}(x, y) = -xy + 2x(1-y) + y(1-x) = -4xy + 2x + y. \tag{5.22}$$

This is the game on the unit square that we studied in Section 4.14.2 (page 123). We found a unique equilibrium for this game: $x^* = \frac{1}{4}$ and $y^* = \frac{2}{3}$. The unique equilibrium in mixed strategies of the given game is therefore

$$\left(\left[\tfrac{1}{4}(T), \tfrac{3}{4}(B)\right], \left[\tfrac{2}{3}(L), \tfrac{1}{3}(R)\right]\right). \tag{5.23}$$

◀

We have seen in this section two examples of two-player games, one a zero-sum game and the other a non-zero-sum game. Neither of them has an equilibrium in pure strategies, but they both have equilibria in mixed strategies. Do all games have equilibria in mixed strategies? John Nash, who defined the concept of equilibrium, answered this question affirmatively.

Theorem 5.10 (Nash [1950b, 1951]) *Every game in strategic form G, with a finite number of players and in which every player has a finite number of pure strategies, has an equilibrium in mixed strategies.*

The proof of Nash's Theorem will be presented later in this chapter. As a corollary, along with Theorem 4.45 on page 115, we have an analogous theorem for two-player zero-sum games. This special case was proven by von Neumann twenty-two years before Nash proved his theorem on the existence of the equilibrium that bears his name.

Theorem 5.11 (von Neumann's Minmax Theorem [1928]) *Every two-player zero-sum game in which every player has a finite number of pure strategies has a value in mixed strategies.*

In other words, in every two-player zero-sum game the minmax value in mixed strategies is equal to the maxmin value in mixed strategies. Nash regarded his result as a generalization of the Minmax Theorem to n players. This is, in fact, a generalization of the Minmax Theorem here to two-player games that may not be zero-sum, and to games with any finite number of players. On the other hand, as we noted on page 117, this is a generalization of only one aspect of the notion of the "value" of a game, namely, the aspect of stability. The other aspect of the value of a game – the security level – which characterizes the value in two-player zero-sum games, is not generalized by the Nash equilibrium.

Recall that the value in mixed strategies of a two-player zero-sum game, if it exists, is given by

$$v := \max_{\sigma_{\mathrm{I}} \in \Sigma_{\mathrm{I}}} \min_{\sigma_{\mathrm{II}} \in \Sigma_{\mathrm{II}}} U(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}) = \min_{\sigma_{\mathrm{II}} \in \Sigma_{\mathrm{II}}} \max_{\sigma_{\mathrm{I}} \in \Sigma_{\mathrm{I}}} U(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}). \tag{5.24}$$

Since the payoff function is multilinear, for every strategy σ_{I} of Player I, the function $\sigma_{\mathrm{II}} \mapsto U(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}})$ is linear.

A point x in a set $X \subseteq \mathbb{R}^n$ is called an *extreme point* if it is not the linear combination of two other points in the set (see Definition 23.2 on page 917). Every linear function defined over a compact set attains its maximum and minimum at extreme points. The set of extreme points of a collection of mixed strategies is the set of pure strategies (Exercise 5.5). It follows that for every strategy σ_{I} of Player I, it suffices to calculate the internal maximum in the middle term in Equation (5.24) over pure strategies. Similarly, for every strategy σ_{II} of Player II, it suffices to compute the internal maximum in the right-hand term in Equation (5.24) over pure strategies. That is, if v is the value in mixed strategies of the game, then

$$v = \max_{\sigma_{\mathrm{I}} \in \Sigma_{\mathrm{I}}} \min_{s_{\mathrm{II}} \in S_{\mathrm{II}}} U(\sigma_{\mathrm{I}}, s_{\mathrm{II}}) = \min_{\sigma_{\mathrm{II}} \in \Sigma_{\mathrm{II}}} \max_{s_{\mathrm{I}} \in S_{\mathrm{I}}} U(s_{\mathrm{I}}, \sigma_{\mathrm{II}}). \tag{5.25}$$

As the next example shows, when the number of pure strategies is infinite, Nash's Theorem and the Minmax Theorem do not hold.

Example 5.12 **Choosing the largest number** Consider the following two-player zero-sum game. Two players simultaneously and independently choose a positive integer. The player who chooses the smaller number pays a dollar to the player who chooses the largest number. If the two players choose the same integer, no exchange of money occurs. We will model this as a game in strategic form, and then show that it has no value in mixed strategies.

Both players have the same set of pure strategies:

$$S_{\mathrm{I}} = S_{\mathrm{II}} = \mathbb{N} = \{1, 2, 3, \dots\}. \tag{5.26}$$

This set is not finite; it is a countably infinite set. The payoff function is

$$u(s_{\mathrm{I}}, s_{\mathrm{II}}) = \begin{cases} 1 & \text{when } s_{\mathrm{I}} > s_{\mathrm{II}}, \\ 0 & \text{when } s_{\mathrm{I}} = s_{\mathrm{II}}, \\ -1 & \text{when } s_{\mathrm{I}} < s_{\mathrm{II}}. \end{cases} \tag{5.27}$$

A mixed strategy in this game is a probability distribution over the set of nonnegative integers:

$$\Sigma_{\mathrm{I}} = \Sigma_{\mathrm{II}} = \left\{ (x_1, x_2, \dots) \colon \sum_{k=1}^{\infty} x_k = 1, \quad x_k \geq 0 \quad \forall k \in \mathbb{N} \right\}. \tag{5.28}$$

We will show that

$$\sup_{\sigma_{\mathrm{I}} \in \Sigma_{\mathrm{I}}} \inf_{\sigma_{\mathrm{II}} \in \Sigma_{\mathrm{II}}} U(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}) = -1 \tag{5.29}$$

and

$$\inf_{\sigma_{\mathrm{II}} \in \Sigma_{\mathrm{II}}} \sup_{\sigma_{\mathrm{I}} \in \Sigma_{\mathrm{I}}} U(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}) = 1. \tag{5.30}$$

It will then follow from Equations (5.29) and (5.30) that the game has no value in mixed strategies. Let σ_{I} be the strategy of Player I, and let $\varepsilon \in (0, 1)$. Since σ_{I} is a distribution over $\mathbb{N}$, there exists a sufficiently large $k \in \mathbb{N}$ satisfying

$$\sigma_{\mathrm{I}}(\{1, 2, \dots, k\}) > 1 - \varepsilon. \tag{5.31}$$

In words, the probability that Player I will choose a number that is less than or equal to k is greater than $1 - \varepsilon$. But then, if Player II chooses the pure strategy $k + 1$ we will have

$$U(\sigma_{\mathrm{I}}, k + 1) < (1 - \varepsilon) \times (-1) + \varepsilon \times 1 = -1 + 2\varepsilon, \tag{5.32}$$

because with probability greater than $1 - \varepsilon$, Player I loses and the payoff is -1, and with probability less than ε, he wins and the payoff is 1. Since this is true for any $\varepsilon \in (0, 1)$, Equation (5.29) holds. Equation (5.30) is proved in a similar manner. ◀

We defined extensive-form games with the use of finite games; in particular, in every extensive-form game every player has a finite number of pure strategies. We therefore have the following corollary of Theorem 5.10.

Theorem 5.13 *Every extensive-form game has an equilibrium in mixed strategies.*

5.2 Computing equilibria in mixed strategies

Before we proceed to the proof of Nash's Theorem, we will consider the subject of computing equilibria in mixed strategies. When the number of players is large, and similarly when the number of strategies is large, finding an equilibrium, to say nothing of finding all the equilibria, is a very difficult problem, both theoretically and computationally. We will present only a few examples of computing equilibria in simple games.

5.2.1 The direct approach

The direct approach to finding equilibria is to write down the mixed extension of the strategic-form game and then to compute the equilibria in the mixed extension (assuming we can do that). In the case of a two-player game where each player has two pure strategies, the mixed extension is a game over the unit square with bilinear payoff functions, which can be solved as we did in Section 4.14 (page 121). Although this approach works well in two-player games where each player has two pure strategies, when there are more strategies, or more players, it becomes quite complicated.

We present here a few examples of this sort of computation. We start with two-player zero-sum games, where finding equilibria is equivalent to finding the value of the game, and equilibrium strategies are optimal strategies. Using Equation (5.25) we can find the value of the game by computing $\max_{\sigma_{\mathrm{I}}\in\Sigma_{\mathrm{I}}} \min_{s_{\mathrm{II}}\in S_{\mathrm{II}}} U(\sigma_{\mathrm{I}}, s_{\mathrm{II}})$ or $\min_{\sigma_{\mathrm{II}}\in\Sigma_{\mathrm{II}}} \max_{s_{\mathrm{I}}\in S_{\mathrm{I}}} U(s_{\mathrm{I}}, \sigma_{\mathrm{II}})$, which also enables us to find the optimal strategies of the players: every strategy σ_{I} at which the maximum of $\max_{\sigma_{\mathrm{I}}\in\Sigma_{\mathrm{I}}} \min_{s_{\mathrm{II}}\in S_{\mathrm{II}}} U(\sigma_{\mathrm{I}}, s_{\mathrm{II}})$ is obtained is an optimal strategy of Player I, and every strategy σ_{II} at which the minimum of $\min_{\sigma_{\mathrm{II}}\in\Sigma_{\mathrm{II}}} \max_{s_{\mathrm{I}}\in S_{\mathrm{I}}} U(s_{\mathrm{I}}, \sigma_{\mathrm{II}})$ is obtained is an optimal strategy of Player II.

The first game we consider is a game over the unit square. The computation presented here differs slightly from the computation in Section 4.14 (page 121).

Example 5.14 **A two-player zero-sum game, in which each player has two pure strategies** Consider the two-player zero-sum game in Figure 5.4.

		Player II	
		L	R
Player I	T	5	0
	B	3	4

Figure 5.4 A two-player zero-sum game

We begin by computing $\max_{\sigma_{\mathrm{I}}\in\Sigma_{\mathrm{I}}} \min_{s_{\mathrm{II}}\in S_{\mathrm{II}}} U(\sigma_{\mathrm{I}}, s_{\mathrm{II}})$ in this example. If Player I plays the mixed strategy $[x(T), (1-x)(B)]$, his payoff, as a function of x, depends on the strategy of Player II:

- If Player II plays L: $U(x, L) = 5x + 3(1-x) = 2x + 3$.
- If Player II plays R: $U(x, R) = 4(1-x) = -4x + 4$.

The graph in Figure 5.5 shows these two functions. The thick line plots the function representing the minimum payoff that Player I can receive if he plays x: $\min_{s_{\mathrm{II}}\in S_{\mathrm{II}}} U(x, s_{\mathrm{II}})$. This minimum is called the *lower envelope* of the payoffs of Player I.

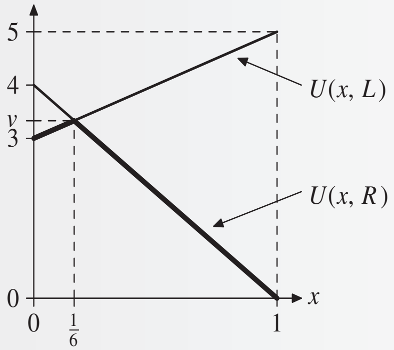

Figure 5.5 The payoff function of Player I and the lower envelope of those payoffs, in the game in Figure 5.4

The value of the game in mixed strategies equals $\max_{\sigma_{\mathrm{I}} \in \Sigma_{\mathrm{I}}} \min_{s_{\mathrm{II}} \in S_{\mathrm{II}}} U(\sigma_{\mathrm{I}}, s_{\mathrm{II}})$, which is the maximum of the lower envelope. This maximum is attained at the intersection point of the two corresponding lines appearing in Figure 5.5, i.e., at the point at which

$$2x + 3 = -4x + 4, \tag{5.33}$$

whose solution is $x = \frac{1}{6}$. It follows that Player I's optimal strategy is $x^* = [\frac{1}{6}(T), \frac{5}{6}(B)]$. The value of the game is the height of the intersection point, $v = 2 \times \frac{1}{6} + 3 = 3\frac{1}{3}$.

We conduct a similar calculation for finding Player II's optimal strategy, aimed at finding the strategy σ_{II} at which the minimum of $\min_{\sigma_{\mathrm{II}} \in \Sigma_{\mathrm{II}}} \max_{s_{\mathrm{I}} \in S_{\mathrm{I}}} U(s_{\mathrm{I}}, \sigma_{\mathrm{II}})$ is attained. For each one of the pure strategies T and B of Player I, we compute the payoff as a function of the mixed strategy y of Player II, and look at the upper envelope of these two lines (see Figure 5.6).

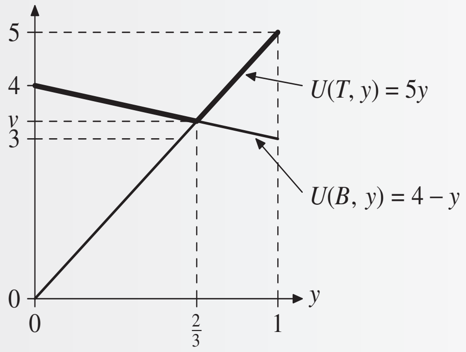

Figure 5.6 The payoff function of Player II and the upper envelope of those payoffs, in the game in Figure 5.4

The minimum of the upper envelope is attained at the point of intersection of these two lines. It is the solution of the equation $5y = 4 - y$, which is $y = \frac{2}{3}$. It follows that the optimal strategy of Player II is $y^* = [\frac{2}{3}(L), \frac{1}{3}(R)]$. The value of the game is the height of the intersection point,

$$U(B, y^*) = 4 - \tfrac{2}{3} = 3\tfrac{1}{3}. \tag{5.34}$$

This procedure can be used for finding the optimal strategies of every game in which the players each have two pure strategies. Note that the value v, as computed in Figure 5.6 (the minmax value), is identical to the value v computed in Figure 5.4 (the maxmin value): both are equal to $3\frac{1}{3}$. This equality follows from Theorem 5.11, which states that the game has a value in mixed strategies. ◀

The graphical procedure presented in Example 5.14 is very convenient. It can be extended to games in which one of the players has two pure strategies and the other player has any finite number of strategies. Suppose that Player I has two pure strategies. We can plot (as a straight line) the payoffs for each pure strategy of Player II as a function of x, the mixed strategy chosen by Player I. We can find the minimum of these lines (the lower envelope), and then find the maximum of the lower envelope. This maximum is the value of the game in mixed strategies.

Example 5.15 Consider the two-player zero-sum game in Figure 5.7.

		Player II		
		L	*M*	*R*
Player I	*T*	2	5	–1
	B	1	–2	5

Figure 5.7 The two-player zero-sum game in Example 5.15

If Player I plays the mixed strategy $[x(T), (1-x)(B)]$, his payoff, as a function of x, depends on the strategy chosen by Player II:

- If Player II plays L: $U(x, L) = 2x + (1-x) = 1 + x$.
- If Player II plays M: $U(x, M) = 5x - 2(1-x) = 7x - 2$.
- If Player II plays R: $U(x, R) = -x + 5(1-x) = -6x + 5$.

Figure 5.8 shows these three functions. As before, the thick line represents the function $\min_{y\in[0,1]} U(x, y)$. The maximum of the lower envelope is attained at the point x in the intersection of the lines $U(x, L)$ and $U(x, R)$, and it is therefore the solution of the equation $1 + x = -6x + 5$, which is $x = \frac{4}{7}$. It follows that the optimal strategy of Player I is $x^* = [\frac{4}{7}(T), \frac{3}{7}(B)]$. The maximum of the lower envelope is $U(\frac{4}{7}, L) = U(\frac{4}{7}, R) = 1\frac{4}{7}$; hence the value of the game in mixed strategies is $1\frac{4}{7}$.

How can we find optimal strategies for Player II? For each mixed strategy σ_{II} of Player II, the payoff $U(x, \sigma_{\mathrm{II}})$, as a function of x, is a linear function. In fact, it is the average of the functions $U(x, L)$, $U(x, M)$, and $U(x, R)$. If σ^*_{II} is an optimal strategy of Player II, then it guarantees that the payoff will be at most the value of the game, regardless of the mixed strategy x chosen by Player I. In other words, we must have

$$U(x, \sigma^*_{\mathrm{II}}) \leq 1\tfrac{4}{7}, \quad \forall x \in [0, 1]. \tag{5.35}$$

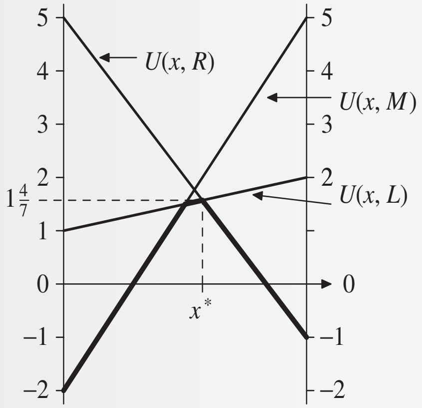

Figure 5.8 The graphs of the payoff functions of Player 1

Consider the graph in Figure 5.8. Since $U(\frac{4}{7}, \sigma_{\mathrm{II}}^*)$ is at most $1\frac{4}{7}$, but $U(\frac{4}{7}, L) = U(\frac{4}{7}, R) = 1\frac{4}{7}$ and $U(\frac{4}{7}, M) > 1\frac{4}{7}$, the only mixed strategies for which $U(\frac{4}{7}, \sigma_{\mathrm{II}}) \leq 1\frac{4}{7}$ are mixed strategies in which the probability of choosing the pure strategy M is 0, and in those mixed strategies $U(\frac{4}{7}, \sigma_{\mathrm{I}}) = 1\frac{4}{7}$.

Our task, therefore, is to find the appropriate weights for the pure strategies L and R that guarantee that the weighted average of $U(x, L)$ and $U(x, R)$ is the constant function $1\frac{4}{7}$. Since every weighted average of these functions equals $1\frac{4}{7}$ at the point $x = \frac{4}{7}$, it suffices to find weights that guarantee that the weighted average will be $1\frac{4}{7}$ at one additional point x, for example, at $x = 0$ (because a linear function that attains the same value at two distinct points is a constant function). This means we need to consider the equation

$$1\tfrac{4}{7} = qU(0, L) + (1 - q)U(0, R) = q + 5(1 - q) = 5 - 4q. \tag{5.36}$$

The solution to this equation is $q = \frac{6}{7}$, and therefore the unique optimal strategy of Player II is $\sigma_{\mathrm{II}}^* = [\frac{6}{7}(L), \frac{1}{7}(R)]$. ◀

The procedure used in the last example for finding an optimal strategy for Player II is a general one: after finding the value of the game and the optimal strategy of Player I, we need only look for pure strategies of Player II for which the intersection of the lines corresponding to their payoffs comprises the maximum of the lower envelope. In the above example, there were only two such pure strategies. In other cases, there may be more than two pure strategies comprising the maximum of the lower envelope. In such cases, we need only choose two such strategies: one for which the corresponding line is nonincreasing, and one for which the corresponding line is nondecreasing (see, for example, Figure 5.9(F)). After we have identified two such strategies, it remains to solve one linear equation and find a weighted average of the lines that yields a horizontal line.

Remark 5.16 *The above discussion shows that in every two-player zero-sum game in which Player I has two pure strategies and Player II has m_{II} pure strategies, Player II*

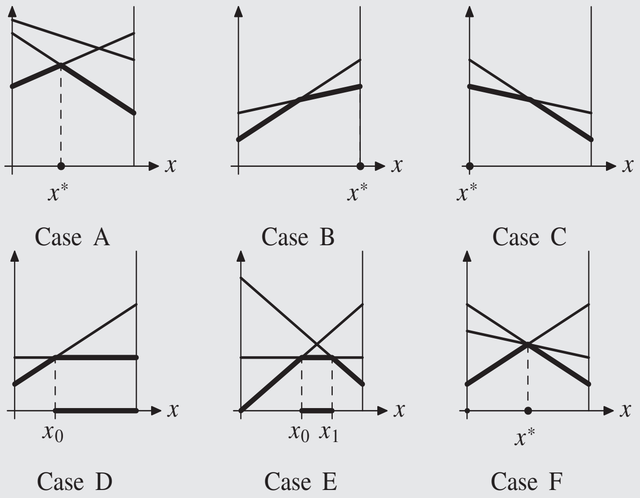

Figure 5.9 Possible graphs of payoffs as a function of x

has an optimal mixed strategy that chooses, with positive probability, at most two pure strategies. This is a special case of a more general result: in every two-player zero-sum game where Player I has m_{I} pure strategies and Player II has m_{II} pure strategies, if $m_{\mathrm{I}} < m_{\mathrm{II}}$ then Player II has an optimal mixed strategy that chooses, with positive probability, at most m_{I} pure strategies. ◆

To compute the value, we found the maximum of the lower envelope. In the example above, there was a unique maximum, which was attained in the line segment [0, 1]. In general there may not be a unique maximum, and the maximal value may be attained at one of the extreme points, $x = 0$ or $x = 1$. Figure 5.9 depicts six distinct possible graphs of payoff functions of $(U(x, s_{\mathrm{II}}))_{s_{\mathrm{II}} \in S_{\mathrm{II}}}$.

In cases A and F, the optimal strategy of Player I is attained at an internal point x^*. In case B, the maximum of the lower envelope is attained at $x^* = 1$, and in case C the maximum is attained at $x^* = 0$. In case D, the maximum is attained in the interval $[x_0, 1]$; hence every point in this interval is an optimal strategy of Player I. In case E, the maximum is attained in the interval $[x_0, x_1]$; hence every point in this interval is an optimal strategy of Player I.

As for Player II, his unique optimal strategy is at an internal point in case A (and therefore is not a pure strategy). His unique optimal strategy is a pure strategy in cases B, C, D, and E. In case F, Player II has a continuum of optimal strategies (see Exercise 5.11).

5.2.2 Computing equilibrium points

When dealing with a game that is not zero sum, the Nash equilibrium solution concept is not equivalent to the maxmin value. The computational procedure above will therefore not lead to Nash equilibrium points in that case, and we need other procedures.

The most straightforward and natural way to develop such a procedure is to build on the definition of the Nash equilibrium in terms of the "best reply." We have already seen such a procedure in Section 4.14.2 (page 123), when we looked at non-zero-sum games on the unit square. We present another example here, in which there is more than one equilibrium point.

Example 5.17 **Battle of the Sexes** The Battle of the Sexes game, which we saw in Example 4.21 (page 98), appears in Figure 5.10.

		Player II	
		F	*C*
Player I	*F*	2, 1	0, 0
	C	0, 0	1, 2

Figure 5.10 Battle of the Sexes

Recall that for each mixed strategy $[x(F), (1-x)(C)]$ of Player I (which we will refer to as x for short), we denoted the collection of best replies of Player II by:

$$\begin{aligned} \mathrm{br}_{\mathrm{II}}(x) &= \operatorname{argmax}_{y \in [0,1]} u_{\mathrm{II}}(x, y) && (5.37) \\ &= \{y \in [0,1] \colon u_{\mathrm{II}}(x, y) \geq u_{\mathrm{II}}(x, z) \quad \forall z \in [0,1]\}. && (5.38) \end{aligned}$$

Similarly, for each mixed strategy $[y(F), (1-y)(C)]$ of Player II (which we will refer to as y for short), we denoted the collection of best replies of Player I by:

$$\begin{aligned} \mathrm{br}_{\mathrm{I}}(y) &= \operatorname{argmax}_{x \in [0,1]} u_{\mathrm{I}}(x, y) && (5.39) \\ &= \{x \in [0,1] \colon u_{\mathrm{I}}(x, y) \geq u_{\mathrm{I}}(z, y) \quad \forall z \in [0,1]\}. && (5.40) \end{aligned}$$

In the Battle of the Sexes, these correspondences[1] are given by

$$\mathrm{br}_{\mathrm{II}}(x) = \begin{cases} 0 & \text{if } x < \frac{2}{3}, \\ [0,1] & \text{if } x = \frac{2}{3}, \\ 1 & \text{if } x > \frac{2}{3}. \end{cases} \qquad \mathrm{br}_{\mathrm{I}}(y) = \begin{cases} 0 & \text{if } y < \frac{1}{3}, \\ [0,1] & \text{if } y = \frac{1}{3}, \\ 1 & \text{if } y > \frac{1}{3}. \end{cases}$$

Figure 5.11 depicts the graphs of these two set-valued functions, br_{I} and $\mathrm{br}_{\mathrm{II}}$. The graph of $\mathrm{br}_{\mathrm{II}}$ is the lighter line, and the graph of br_{I} is the darker line. The two graphs are shown on the same set of axes, where the x-axis is the horizontal line, and the y-axis is the vertical line. For each $x \in [0,1]$, $\mathrm{br}_{\mathrm{II}}(x)$ is a point or a line located above x. For each $y \in [0,1]$, $\mathrm{br}_{\mathrm{I}}(y)$ is a point or a line located to the right of y.

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[1] A *set-valued function*, or a *correspondence*, is a multivalued function that associates every point in the domain with a set of values (as opposed to a single value, as is the case with an ordinary function).

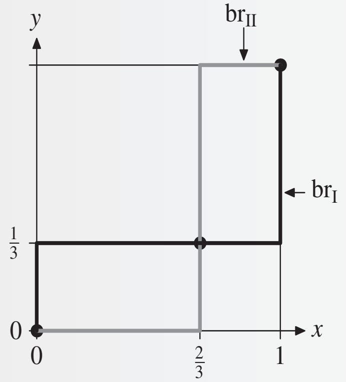


Figure 5.11 The graphs of br_{I} (black line) and of $\mathrm{br}_{\mathrm{II}}$ (grey line)

A point (x^*, y^*) is an equilibrium point if and only if $x^* \in \mathrm{br}_{\mathrm{I}}(y^*)$ and $y^* \in \mathrm{br}_{\mathrm{II}}(x^*)$. This is equivalent to (x^*, y^*) being a point at which the two graphs br_{I} and $\mathrm{br}_{\mathrm{II}}$ intersect (verify this for yourself). As Figure 5.11 shows, these graphs intersect in three points:

- $(x^*, y^*) = (0, 0)$: corresponding to the pure strategy equilibrium (C, C).
- $(x^*, y^*) = (1, 1)$: corresponding to the pure strategy equilibrium (F, F).
- $(x^*, y^*) = (\frac{2}{3}, \frac{1}{3})$: corresponding to the equilibrium in mixed strategies

$$x^* = \left[\tfrac{2}{3}(F), \tfrac{1}{3}(C)\right], \quad y^* = \left[\tfrac{1}{3}(F), \tfrac{2}{3}(C)\right]. \tag{5.41}$$

Note two interesting points:

- The payoff at the mixed strategy equilibrium is $(\frac{2}{3}, \frac{2}{3})$. For each player, this payoff is worse than the worst payoff he would receive if either of the pure strategy equilibria were chosen instead.
- The payoff $\frac{2}{3}$ is also the security level (maxmin value) of each of the two players (verify this), but the maxmin strategies guaranteeing this level are not equilibrium strategies; the maxmin strategy of Player I is $[\frac{1}{3}(F), \frac{2}{3}(C)]$, and the maxmin strategy of Player II is $[\frac{2}{3}(F), \frac{1}{3}(C)]$. ◀

This geometric procedure for computing equilibrium points, as intersection points of the graphs of the best replies of the players, is not applicable if there are more than two players or if each player has more than two pure strategies. But there are cases in which this procedure can be mimicked by finding solutions of algebraic equations corresponding to the intersections of best-response graphs.

5.2.3 The indifference principle

One effective tool for finding equilibria is the *indifference principle*. The indifference principle says that if a mixed equilibrium calls for a player to use two distinct pure strategies with positive probability, then the expected payoff to that player for using one of those pure strategies equals the expected payoff to him for using the other pure strategy, assuming that the other players are playing according to the equilibrium.

Theorem 5.18 *Let σ^* be an equilibrium in mixed strategies of a strategic-form game, and let s_i and $\widehat{s}_i$ be two pure strategies of player i. If $\sigma_i^*(s_i) > 0$ and $\sigma_i^*(\widehat{s}_i) > 0$, then*

$$U_i(s_i, \sigma_{-i}^*) = U_i(\widehat{s}_i, \sigma_{-i}^*). \tag{5.42}$$

The reason this theorem holds is simple: if the expected payoff to player i when he plays pure strategy s_i is higher than when he plays $\widehat{s}_i$, then he can improve his expected payoff by increasing the probability of playing s_i and decreasing the probability of playing $\widehat{s}_i$.

Proof: Suppose by contradiction that Equation (5.42) does not hold. Without loss of generality, suppose that

$$U_i(s_i, \sigma_{-i}^*) > U_i(\widehat{s}_i, \sigma_{-i}^*). \tag{5.43}$$

Let σ_i be the strategy of player i defined by

$$\sigma_i'(t_i) := \begin{cases} \sigma_i(t_i) & \text{if } t_i \notin \{s_i, \widehat{s}_i\}, \\ 0 & \text{if } t_i = \widehat{s}_i, \\ \sigma_i^*(s_i) + \sigma_i^*(\widehat{s}_i) & \text{if } t_i = s_i. \end{cases}$$

Then

$$\begin{aligned} U_i(\sigma_i, \sigma_{-i}^*) &= \sum_{t_i \in S_i} \sigma(t_i) U_i(t_i, \sigma_{-i}^*) & (5.44) \\ &= \sum_{t_i \notin \{s_i, \widehat{s}_i\}} \sigma^*(t_i) U_i(t_i, \sigma_{-i}^*) + (\sigma_i^*(s_i) + \sigma_i(\widehat{s}_i)) U_i(s_i, \sigma_{-i}^*) & (5.45) \\ &> \sum_{t_i \notin \{s_i, \widehat{s}_i\}} \sigma^*(t_i) U_i(t_i, \sigma_{-i}^*) + \sigma_i(s_i) U_i(s_i, \sigma_{-i}^*) + \sigma_i^*(\widehat{s}_i) U_i(\widehat{s}_i, \sigma_{-i}^*) & (5.46) \\ &= \sum_{t_i \in s_i} \sigma_i^*(t_i) U_i(t_i, \sigma_{-i}^*) & (5.47) \\ &= U_i(\sigma^*). & (5.48) \end{aligned}$$

The equalities in Equation (5.45) and Equation (5.47) follow from the definition of σ, and Equation (5.46) follows from Equation (5.43). But this contradicts the assumption that σ^* is an equilibrium, because player i can increase his payoff by deviating to strategy σ_i'. This contradiction shows that the assumption that Equation (5.42) does not hold was wrong, and the theorem therefore holds. □

We will show how the indifference principle can be used to find equilibria, by reconsidering the game in Example 5.9.

Example 5.9 (*Continued*) The payoff matrix in this game appears in Figure 5.12.

		Player II	
		L	R
Player I	T	1, –1	0, 2
	B	0, 1	2, 0

Figure 5.12 The payoff matrix in Example 5.9

As we have already seen, the only equilibrium point in this game is

$$\left(\left[\tfrac{1}{4}(T), \tfrac{3}{4}(B)\right], \left[\tfrac{2}{3}(L), \tfrac{1}{3}(R)\right]\right). \tag{5.49}$$

Definition 5.19 *A mixed strategy σ_i of player i is called a* completely mixed strategy *if $\sigma_i(s_i) > 0$ for every pure strategy $s_i \in S_i$. An equilibrium $\sigma^* = (\sigma_i^*)_{i \in N}$ is called a* completely mixed equilibrium *if for every player $i \in N$ the strategy σ_i^* is a completely mixed strategy.*

In words, a player's completely mixed strategy chooses each pure strategy with positive probability. It follows that at every completely mixed equilibrium, every pure strategy vector is chosen with positive probability.

We will now compute the equilibrium using the indifference principle. The first step is to ascertain, by direct inspection, that the game has no pure strategy equilibria. We can also ascertain that there is no Nash equilibrium of this game in which one of the two players plays a pure strategy. By Nash's Theorem (Theorem 5.10), the game has at least one equilibrium in mixed strategies, and it follows that at every equilibrium of the game both players play completely mixed strategies. For every pair of mixed strategies (x, y), we have that $U_{\mathrm{II}}(x, L) = 1 - 2x$, $U_{\mathrm{II}}(x, R) - 2x$, $U_{\mathrm{I}}(T, y) = y$, and $U_{\mathrm{I}}(B, y) = 2(1 - y)$. By the indifference principle, at equilibrium Player I is indifferent between playing T and playing B, and Player II is indifferent between L and R. In other words, if the equilibrium is (x^*, y^*), then:

- Player I is indifferent between T and B:

$$U_{\mathrm{I}}(T, y^*) = U_{\mathrm{I}}(B, y^*) \implies y^* = 2(1 - y^*) \implies y^* = \tfrac{2}{3}. \tag{5.50}$$

- Player II is indifferent between L and R:

$$U_{\mathrm{II}}(x^*, L) = U_{\mathrm{II}}(x^*, R) \implies 1 - 2x^* = 2x^* \implies x^* = \tfrac{1}{4}. \tag{5.51}$$

We have, indeed, found the same equilibrium that we found above, using a different procedure. Interestingly, in computing the mixed strategy equilibrium, each player's strategy is determined by the payoffs of the other player; each player plays in such a way that the other player is indifferent between his two pure strategies (and therefore the other player has no incentive to deviate). This is in marked contrast to the maxmin strategy of a player, which is determined solely by the player's own payoffs. This is yet another expression of the significant difference between the solution concepts of Nash equilibrium and maxmin strategy, in games that are not two-player zero-sum games. ◀

5.2.4 Dominance and equilibrium

The concept of strict dominance (Definition 4.6 on page 86) is a useful tool for computing equilibrium points. As we saw in Corollary 4.36 (page 109), in strategic-form games a strictly dominated strategy is chosen with probability 0 in each equilibrium. The next result, which is a generalization of that corollary, is useful for finding equilibria in mixed strategies.

Theorem 5.20 *Let $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a game in strategic form in which the sets $(S_i)_{i\in N}$ are all finite sets. If a pure strategy $s_i \in S_i$ of player i is strictly dominated by a mixed strategy $\sigma_i \in \Sigma_i$, then in every equilibrium of the game, the pure strategy s_i is chosen by player i with probability* 0.

Proof: Let s_i be a pure strategy of player i that is strictly dominated by a mixed strategy σ_i, and let $\widehat{\sigma} = (\widehat{\sigma}_i)_{i\in N}$ be a strategy vector in which player i chooses strategy s_i with positive probability: $\widehat{\sigma}_i(s_i) > 0$. We will show that $\widehat{\sigma}$ is not an equilibrium by showing that $\widehat{\sigma}_i$ is not a best reply of player i to $\widehat{\sigma}_{-i}$.

Define a mixed strategy $\sigma'_i \in \Sigma_i$ as follows:

$$\sigma'_i(t_i) = \begin{cases} \widehat{\sigma}_i(s_i)\cdot\sigma_i(s_i) & t_i = s_i, \\ \widehat{\sigma}_i(t_i) + \widehat{\sigma}_i(s_i)\cdot\sigma_i(t'_i) & t_i \neq s_i. \end{cases} \tag{5.52}$$

In words, player i, using strategy σ'_i, chooses his pure strategy in two stages: first he chooses a pure strategy using the probability distribution $\widehat{\sigma}_i$. If this choice leads to a pure strategy that differs from s_i, he plays that strategy. But if s_i is chosen, player i chooses another pure strategy using the distribution σ_i, and plays whichever pure strategy that leads to.

Finally, we show that σ'_i yields player i a payoff that is higher than $\widehat{\sigma}_i$, when played against $\widehat{\sigma}_{-i}$, and hence $\widehat{\sigma}$ cannot be an equilibrium. Since σ_i strictly dominates s_i, it follows that, in particular,

$$U_i(s_i, \widehat{\sigma}_{-i}) < U_i(\sigma_i, \widehat{\sigma}_{-i}), \tag{5.53}$$

and we have

$$\begin{aligned} U_i(\widehat{\sigma}_i, \widehat{\sigma}_{-i}) &= \sum_{t_i\in S_i} \widehat{\sigma}_i(t_i) U_i(t_i, \widehat{\sigma}_{-i}) && (5.54)\\ &= \sum_{t_i\neq s_i} \widehat{\sigma}_i(t_i) U_i(t_i, \widehat{\sigma}_{-i}) + \widehat{\sigma}_i(s_i) U_i(s_i, \widehat{\sigma}_{-i}) && (5.55)\\ &< \sum_{t_i\neq s_i} \widehat{\sigma}_i(t_i) U_i(t_i, \widehat{\sigma}_{-i}) + \widehat{\sigma}_i(s_i) U_i(\sigma_i, \widehat{\sigma}_{-i}) && (5.56)\\ &= \sum_{t_i\neq s_i} \sigma'_i(t_i) U_i(t_i, \widehat{\sigma}_{-i}) + \widehat{\sigma}_i(s_i) \sum_{t_i\in S_i} \sigma'_i(t_i) U_i(t_i, \widehat{\sigma}_{-i}) && (5.57)\\ &= \sum_{t_i\in S_i} \sigma'_i(t_i) U_i(t_i, \widehat{\sigma}_{-i}) && (5.58)\\ &= U_i(\sigma'_i, \widehat{\sigma}_{-i}). && (5.59) \end{aligned}$$

□

The next example shows how to use Theorem 5.20 to find equilibrium points in a game.

Example 5.21 Consider the following two-player game in which $N = \{\text{I}, \text{II}\}$ (Figure 5.13).

Player II

		L	C	R
	T	6, 2	0, 6	4, 4
Player I	M	2, 12	4, 3	2, 5
	B	0, 6	10, 0	2, 2

Figure 5.13 The strategic-form game in Example 5.21

In this game, no pure strategy is dominated by another pure strategy (verify this). However, strategy M of Player I is strictly dominated by the mixed strategy $[\frac{1}{2}(T), \frac{1}{2}(B)]$ (verify this). It follows from Theorem 5.20 that the deletion of strategy M has no effect on the set of equilibria in the game. Following the deletion of strategy M, we are left with the game shown in Figure 5.14.

Player II

		L	C	R
Player I	T	6, 2	0, 6	4, 4
	B	0, 6	10, 0	2, 2

Figure 5.14 The game after eliminating strategy M

In this game, strategy R of Player II is strictly dominated by the mixed strategy $[\frac{5}{12}(L), \frac{7}{12}(C)]$. We then delete R, which leaves us with the game shown in Figure 5.15.

Player II

		L	C
Player I	T	6, 2	0, 6
	B	0, 6	10, 0

Figure 5.15 The game after eliminating strategies M and R

The game shown in Figure 5.15 has no pure strategy equilibria (verify this). The only mixed equilibrium of this game, which can be computed using the indifference principle, is $([\frac{3}{5}(T), \frac{2}{5}(B)], [\frac{5}{8}(L), \frac{3}{8}(R)])$, which yields payoff $(\frac{15}{4}, \frac{18}{5})$ (verify this, too).

Since the strategies that were deleted were all strictly dominated strategies, the above equilibrium is also the only equilibrium of the original game. ◀

5.2.5 Two-player zero-sum games and linear programming

Computing the value of two-player zero-sum games, where each player has a finite number of strategies, and finding optimal strategies in such games, can be presented as a linear programming problem. It follows that these computations can be accomplished using known linear programming algorithms. In this section, we present linear programs that correspond to finding the value and optimal strategies of a two-player game. A brief survey of linear programming appears in Section 23.3 (page 945).

Let $(N, (S_i)_{i \in N}, u)$ be a two-player zero-sum game, in which the set of players is $N = \{\mathrm{I}, \mathrm{II}\}$. As usual, U denotes the multilinear extension of u.

Theorem 5.22 *Denote by Z_P the value of the following linear program in the variables $(x_{s_\mathrm{I}})_{s_\mathrm{I} \in S_\mathrm{I}}$.*

$$\begin{array}{ll} \textit{Compute:} & Z_P := \max z \\ \textit{subject to:} & \sum_{s_\mathrm{I} \in S_\mathrm{I}} x(s_\mathrm{I}) u(s_\mathrm{I}, s_\mathrm{II}) \geq z, \quad \forall s_\mathrm{II} \in S_\mathrm{II}; \\ & \sum_{s_\mathrm{I} \in S_\mathrm{I}} x(s_\mathrm{I}) = 1; \\ & x(s_\mathrm{I}) \geq 0, \quad \forall s_\mathrm{I} \in S_\mathrm{I}. \end{array}$$

Then Z_P is the value in mixed strategies of the game.

Proof: Denote by v the value of the game $(N, (S_i)_{i \in N}, u)$ in mixed strategies. We will show that $Z_P = v$ by showing that $Z_P \geq v$ and $Z_P \leq v$.

Step 1: $Z_P \geq v$.
If v is the value of the game, then Player I has an optimal strategy σ_I^* that guarantees a payoff of at least v, for every mixed strategy of Player II:

$$U(\sigma_\mathrm{I}^*, \sigma_\mathrm{II}) \geq v, \quad \forall \sigma_\mathrm{II} \in \Sigma_\mathrm{II}. \tag{5.60}$$

Since this inequality holds, in particular, for each pure strategy $s_\mathrm{II} \in S_\mathrm{II}$, the vector $(x, z) = (\sigma_\mathrm{I}^*, v)$ satisfies the constraints. Since Z_P is the largest real number z for which there exists a mixed strategy for Player I at which the constraints are satisfied, we have $Z_P \geq v$.

Step 2: $Z_P \leq v$.
We first show that $Z_P < \infty$. Suppose that (x, z) is a vector satisfying the constraints. Then $\sum_{s_\mathrm{I} \in S_\mathrm{I}} x(s_\mathrm{I}) u(s_\mathrm{I}, s_\mathrm{II}) \geq z$ and $\sum_{s_\mathrm{I} \in S_\mathrm{I}} x(s_\mathrm{I}) = 1$. This leads to

$$z \leq \sum_{s_\mathrm{I} \in S_\mathrm{I}} x(s_\mathrm{I}) u(s_\mathrm{I}, s_\mathrm{II}) \leq \max_{s_\mathrm{I} \in S_\mathrm{I}} \max_{s_\mathrm{II} \in S_\mathrm{II}} |u(s_\mathrm{I}, s_\mathrm{II})| \times \sum_{s_\mathrm{I} \in S_\mathrm{I}} x(s_\mathrm{I}) \tag{5.61}$$

$$= \max_{s_\mathrm{I} \in S_\mathrm{I}} \max_{s_\mathrm{II} \in S_\mathrm{II}} |u(s_\mathrm{I}, s_\mathrm{II})| < \infty. \tag{5.62}$$

The finiteness of the expression in (5.62) is due to the fact that each of the players has a finite number of pure strategies. Since Z_P is the value of the linear program, there exists a vector x such that (x, Z_P) satisfies the constraints. These constraints require x to be a mixed strategy of Player I. Similarly, the constraints imply that $u(x, s_\mathrm{II}) \geq Z_P$ for every pure strategy $s_\mathrm{II} \in S_\mathrm{II}$. The multilinearity of U implies that

$$U(x, \sigma_\mathrm{II}) \geq Z_P, \quad \forall \sigma_\mathrm{II} \in \Sigma_\mathrm{II}. \tag{5.63}$$

It follows that Player I has a mixed strategy guaranteeing a payoff of at least Z_P, and hence $v \geq Z_P$. □

The fact that the value of a game in mixed strategies can be found using linear programming is an expression of the strong connection that exists between the Minmax Theorem and the Duality Theorem. These two theorems are actually equivalent to each other. Exercise 5.65 presents a guided proof of the Minmax Theorem using the Duality Theorem. For the proof of the Duality Theorem using the Minmax Theorem, see Luce and Raifa [1957], Section A5.2.

5.2.6 Two-player games that are not zero sum

Computing the value of a two-player zero-sum game can be accomplished by solving a linear program. Similarly, computing equilibria in a two-player game that is not zero sum can be accomplished by solving a quadratic program. However, while there are efficient algorithms for solving linear programs, there are no known efficient algorithms for solving generic quadratic programs.

A straightforward method for finding equilibria in two-player games that are not zero sum is based on the following idea. Let $(\sigma_{\mathrm{I}}^*, \sigma_{\mathrm{II}}^*)$ be a Nash equilibrium in mixed strategies. Denote

$$\mathrm{supp}(\sigma_{\mathrm{I}}^*) := \{s_{\mathrm{I}} \in S_{\mathrm{I}} \colon \sigma_{\mathrm{I}}^*(s_{\mathrm{I}}) > 0\}, \tag{5.64}$$

$$\mathrm{supp}(\sigma_{\mathrm{II}}^*) := \{s_{\mathrm{II}} \in S_{\mathrm{II}} \colon \sigma_{\mathrm{II}}^*(s_{\mathrm{II}}) > 0\}. \tag{5.65}$$

The sets $\mathrm{supp}(\sigma_{\mathrm{I}}^*)$ and $\mathrm{supp}(\sigma_{\mathrm{II}}^*)$ are called the *support* of the mixed strategies σ_{I}^* and σ_{II}^* respectively, and they contain all the pure strategies that are chosen with positive probability under σ_{I}^* and σ_{II}^*, respectively. By the indifference principle (see Theorem 5.18 on page 160), at equilibrium any two pure strategies that are played by a particular player with positive probability yield the same payoff to that player. Choose $s_{\mathrm{I}}^0 \in \mathrm{supp}(\sigma_{\mathrm{I}})$ and $s_{\mathrm{II}}^0 \in \mathrm{supp}(\sigma_{\mathrm{II}}^*)$. Then $(\sigma_{\mathrm{I}}^*, \sigma_{\mathrm{II}}^*)$ satisfies the following constraints:

$$U_{\mathrm{I}}(s_{\mathrm{I}}^0, \sigma_{\mathrm{II}}^*) = U_{\mathrm{I}}(s_{\mathrm{I}}, \sigma_{\mathrm{II}}^*), \quad \forall s_{\mathrm{I}} \in \mathrm{supp}(\sigma_{\mathrm{I}}^*), \tag{5.66}$$

$$U_{\mathrm{II}}(\sigma_{\mathrm{I}}^*, s_{\mathrm{II}}^0) = U_{\mathrm{II}}(\sigma_{\mathrm{I}}^*, s_{\mathrm{II}}), \quad \forall s_{\mathrm{II}} \in \mathrm{supp}(\sigma_{\mathrm{II}}^*). \tag{5.67}$$

At equilibrium, neither player can profit from unilateral deviation; in particular,

$$U_{\mathrm{I}}(s_{\mathrm{I}}^0, \sigma_{\mathrm{II}}^*) \geq U_{\mathrm{I}}(s_{\mathrm{I}}, \sigma_{\mathrm{II}}^*), \quad \forall s_{\mathrm{I}} \in S_{\mathrm{I}} \setminus \mathrm{supp}(\sigma_{\mathrm{I}}), \tag{5.68}$$

$$U_{\mathrm{II}}(\sigma_{\mathrm{I}}^*, s_{\mathrm{II}}^0) \geq U_{\mathrm{II}}(\sigma_{\mathrm{I}}^*, s_{\mathrm{II}}), \quad \forall s_{\mathrm{II}} \in S_{\mathrm{II}} \setminus \mathrm{supp}(\sigma_{\mathrm{II}}). \tag{5.69}$$

Since U_{I} and U_{II} are multilinear functions, this is a system of equations that are linear in σ_{I}^* and σ_{II}^*. By taking into account the constraint that σ_{I}^* and σ_{II}^* are probability distributions, we conclude that $(\sigma_{\mathrm{I}}^*, \sigma_{\mathrm{II}}^*)$ is the solution of a system of linear equations. In addition, every pair of mixed strategies $(\sigma_{\mathrm{I}}^*, \sigma_{\mathrm{II}}^*)$ that solves Equations (5.66)–(5.69) is a Nash equilibrium.

This leads to the following direct algorithm for finding equilibria in a two-player game that is not zero sum: For every nonempty subset Y_{I} of S_{I} and every nonempty subset Y_{II} of S_{II}, determine whether there exists an equilibrium $(\sigma_{\mathrm{I}}^*, \sigma_{\mathrm{II}}^*)$ satisfying $Y_{\mathrm{I}} = \mathrm{supp}(\sigma_{\mathrm{I}}^*)$ and $Y_{\mathrm{II}} = \mathrm{supp}(\sigma_{\mathrm{II}}^*)$. The set of equilibria whose support is Y_{I} and Y_{II} is the set of solutions of the system of equations comprised of Equations (5.70)–(5.79), in which s_{I}^0 and s_{II}^0 are any two pure strategies in Y_{I} and Y_{II}, respectively (Exercise 5.54):

$$\sum_{s_{\mathrm{II}}\in S_{\mathrm{II}}} \sigma_{\mathrm{II}}(s_{\mathrm{II}})u_{\mathrm{I}}\left(s_{\mathrm{I}}^0, s_{\mathrm{II}}\right) = \sum_{s_{\mathrm{II}}\in S_{\mathrm{II}}} \sigma_{\mathrm{II}}(s_{\mathrm{II}})u_{\mathrm{I}}(s_{\mathrm{I}}, s_{\mathrm{II}}), \quad \forall s_{\mathrm{I}} \in Y_{\mathrm{I}}, \tag{5.70}$$

$$\sum_{s_{\mathrm{I}}\in S_{\mathrm{I}}} \sigma_{\mathrm{I}}(s_{\mathrm{I}})u_{\mathrm{I}}\left(s_{\mathrm{I}}, s_{\mathrm{II}}^0\right) = \sum_{s_{\mathrm{I}}\in S_{\mathrm{I}}} \sigma_{\mathrm{I}}(s_{\mathrm{I}})u_{\mathrm{I}}(s_{\mathrm{I}}, s_{\mathrm{II}}), \quad \forall s_{\mathrm{II}} \in Y_{\mathrm{II}}, \tag{5.71}$$

$$\sum_{s_{\mathrm{II}}\in S_{\mathrm{II}}} \sigma_{\mathrm{II}}(s_{\mathrm{II}})u_{\mathrm{I}}\left(s_{\mathrm{I}}^0, s_{\mathrm{II}}\right) \geq \sum_{s_{\mathrm{II}}\in S_{\mathrm{II}}} \sigma_{\mathrm{II}}(s_{\mathrm{II}})u_{\mathrm{I}}(s_{\mathrm{I}}, s_{\mathrm{II}}), \quad \forall s_{\mathrm{I}} \in S_{\mathrm{I}} \setminus Y_{\mathrm{I}}, \tag{5.72}$$

$$\sum_{s_{\mathrm{I}}\in S_{\mathrm{I}}} \sigma_{\mathrm{I}}(s_{\mathrm{I}})u_{\mathrm{I}}\left(s_{\mathrm{I}}, s_{\mathrm{II}}^0\right) \geq \sum_{s_{\mathrm{I}}\in S_{\mathrm{I}}} \sigma_{\mathrm{I}}(s_{\mathrm{I}})u_{\mathrm{I}}(s_{\mathrm{I}}, s_{\mathrm{II}}), \quad \forall s_{\mathrm{II}} \in S_{\mathrm{II}} \setminus Y_{\mathrm{II}}, \tag{5.73}$$

$$\sum_{s_{\mathrm{I}}\in S_{\mathrm{I}}} \sigma_{\mathrm{I}}(s_{\mathrm{I}}) = 1, \tag{5.74}$$

$$\sum_{s_{\mathrm{II}}\in S_{\mathrm{II}}} \sigma_{\mathrm{II}}(s_{\mathrm{II}}) = 1, \tag{5.75}$$

$$\sigma_{\mathrm{I}}(s_{\mathrm{I}}) > 0, \quad \forall s_{\mathrm{I}} \in Y_{\mathrm{I}}, \tag{5.76}$$

$$\sigma_{\mathrm{II}}(s_{\mathrm{II}}) > 0, \quad \forall s_{\mathrm{II}} \in Y_{\mathrm{II}}, \tag{5.77}$$

$$\sigma_{\mathrm{I}}(s_{\mathrm{I}}) = 0, \quad \forall s_{\mathrm{I}} \in S_{\mathrm{I}} \setminus Y_{\mathrm{I}}, \tag{5.78}$$

$$\sigma_{\mathrm{II}}(s_{\mathrm{II}}) = 0, \quad \forall s_{\mathrm{II}} \in S_{\mathrm{II}} \setminus Y_{\mathrm{II}}. \tag{5.79}$$

Determining whether this system of equations has a solution can be accomplished by solving a linear program. Because the number of nonempty subsets of S_{I} is $2^{m_{\mathrm{I}}} - 1$ and the number of empty subsets of S_{II} is $2^{m_{\mathrm{II}}} - 1$, the complexity of this algorithm is exponential in m_{I} and m_{II}, and hence this algorithm is computationally inefficient.

5.3 The proof of Nash's Theorem

This section is devoted to proving Nash's Theorem (Theorem 5.10), which states that every finite game has an equilibrium in mixed strategies. The proof of the theorem makes use of the following result.

Theorem 5.23 (Brouwer's Fixed Point Theorem) *Let X be a convex and compact set in a d-dimensional Euclidean space, and let $f : X \to X$ be a continuous function. Then there exists a point $x \in X$ such that $f(x) = x$. Such a point x is called a* fixed point *of f.*

Brouwer's Fixed Point Theorem states that every continuous function from a convex and compact set to itself has a fixed point, that is, a point that the function maps to itself.

In one dimension, Brouwer's Theorem takes an especially simple form. In one dimension, a convex and compact space is a closed line segment $[a, b]$. When $f : [a, b] \to [a, b]$ is a continuous function, one of the following three alternatives must obtain:

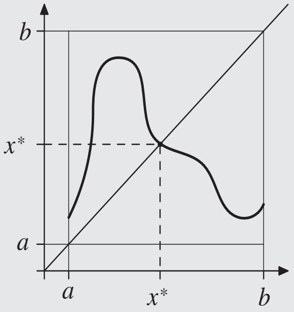


Figure 5.16 Brouwer's Theorem: a fixed point in the one-dimensional case

1. $f(a) = a$, hence a is a fixed point of f.
2. $f(b) = b$, hence b is a fixed point of f.
3. $f(a) > a$ and $f(b) < b$. Consider the function $g(x) = f(x) - x$, which is continuous, where $g(a) > 0$ and $g(b) < 0$. The Intermediate Value Theorem implies that there exists $x \in [a, b]$ satisfying $g(x) = 0$, that is to say, $f(x) = x$. Every such x is a fixed point of f.

The graphical expression of the proof of Brouwer's Fixed Point Theorem in one dimension is as follows: every continuous function on the segment $[a, b]$ must intersect the main diagonal in at least one point (see Figure 5.16).

If the dimension is two or greater, the proof of Brouwer's Fixed Point Theorem is not simple. It can be proved in several different ways, with a variety of mathematical tools. A proof of the theorem using Sperner's Lemma appears in Section 23.1.2 (page 935).

We will now prove Nash's Theorem using Brouwer's Fixed Point Theorem. The proofs of the following two claims are left to the reader (Exercises 5.1 and 5.2).

Theorem 5.24 *If player i's set of pure strategies S_i is finite, then his set of mixed strategies Σ_i is convex and compact.*

Theorem 5.25 *If $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ are compact sets, then the set $A \times B$ is a compact subset of $\mathbb{R}^{n+m}$. If A and B are convex sets, then $A \times B$ is a convex subset of $\mathbb{R}^{n+m}$.*

Theorems 5.24 and 5.25 imply that the set $\Sigma = \Sigma_1 \times \Sigma_2 \times \cdots \Sigma_n$ is a convex and compact subset of the Euclidean space $\mathbb{R}^{m_1+m_2+\cdots+m_n}$. The proof of Nash's Theorem then proceeds as follows. We will define a function $f : \Sigma \to \Sigma$, and prove that it satisfies the following two properties:

- f is a continuous function.
- Every fixed point of f is an equilibrium of the game.

Since Σ is convex and compact, and f is continuous, it follows from Brouwer's Fixed Point Theorem that f has at least one fixed point. The second property above then implies that the game has at least one equilibrium point.

The idea behind the definition of f is as follows. For each strategy vector σ, we define $f(\sigma) = (f_i(\sigma))_{i \in N}$ to be a vector of strategies, where $f_i(\sigma)$ is a strategy of player i. $f_i(\sigma)$

is defined in such a way that if σ_i is not a best reply to σ_{-i}, then $f_i(\sigma)$ is given by shifting σ_i in the direction of a "better reply" to σ_{-i}. It then follows that $f_i(\sigma) = \sigma_i$ if and only if σ_i is a best reply to σ_{-i}.

To define f, we first define an auxiliary function $g_i^j : \Sigma \to [0, \infty)$ for each player i and each index j, where $1 \leq j \leq m_i$. That is, for each vector of mixed strategies σ we define a nonnegative number $g_i^j(\sigma)$.

The payoff that player i receives under the vector of mixed strategies σ is $U_i(\sigma)$. The payoff he receives when he plays the pure strategy s_i^j, but all the other players play σ, is $U_i(s_i^j, \sigma_{-i})$. We define the function g_i^j as follows:

$$g_i^j(\sigma) := \max\left\{0, U_i\left(s_i^j, \sigma_{-i}\right) - U_i(\sigma)\right\}. \tag{5.80}$$

In words, $g_i^j(\sigma)$ equals 0 if player i cannot profit from deviating from σ_i to s_i^j. When $g_i^j(\sigma) > 0$, player i gains a higher payoff if he increases the probability of playing the pure strategy s_i^j. Because a player has a profitable deviation if and only if he has a profitable deviation to a pure strategy, we have the following result:

Claim 5.26 *The strategy vector σ is an equilibrium if and only if $g_i^j(\sigma) = 0$, for each player $i \in N$ and for all $j = 1, 2, \ldots, m_i$.*

To proceed with the proof, we need the following claim.

Claim 5.27 *For every player $i \in N$, and every $j = 1, 2, \ldots, m_i$, the function g_i^j is continuous.*

Proof: Let $i \in N$ be a player, and let $j \in \{1, 2, \ldots, m_i\}$. From Corollary 5.7 (page 149) the function U_i is continuous. The function $\sigma_{-i} \mapsto U_i(s_i^j, \sigma_{-i})$, as a function of σ_{-1} is therefore also continuous. In particular, the difference $U_i(s_i^j, \sigma_{-i}) - U_i(\sigma)$ is a continuous function. Since 0 is a continuous function, and since the maximum of continuous functions is a continuous function, we have that the function g_i^j is continuous. □

We can now define the function f. The function f has to satisfy the property that every one of its fixed points is an equilibrium of the game. It then follows that if σ is not an equilibrium, it must be the case that $\sigma \neq f(\sigma)$. How can we guarantee that? The main idea is to consider, for every player i, the indices j such that $g_i^j(\sigma) > 0$; these indices correspond to pure strategies at which $g_i^j(\sigma) > 0$, i.e., the strategies that will increase player i's payoff if he increases the probability that they will be played (and decreases the probability of playing pure strategies that do not satisfy this inequality). This idea leads to the following definition.

Because $f(\sigma)$ is an element in Σ, i.e., it is a vector of mixed strategies, $f_i^j(\sigma)$ is the probability that player i will play the pure strategy s_i^j. Define:

$$f_i^j(\sigma) := \frac{\sigma_i\left(s_i^j\right) + g_i^j(\sigma)}{1 + \sum_{k=1}^{m_i} g_i^k(\sigma)}. \tag{5.81}$$

In words, if s_i^j is a better reply than σ_i to σ_{-i}, we increase its probability by $g_i^j(\sigma)$, and then normalize the resulting numbers so that we obtain a probability distribution. We now turn our attention to the proof that f satisfies all its required properties.

Claim 5.28 *The range of f is Σ, i.e., $f(\Sigma) \subseteq \Sigma$.*

Proof: We need to show that $f(\sigma)$ is a vector of mixed strategies, for every $\sigma \in \Sigma$, i.e.,

1. $f_i^j(\sigma) \geq 0$ for all i, and for all $j \in \{1, 2, \ldots, m_i\}$.
2. $\sum_{j=1}^{m_i} f_i^j(\sigma) = 1$ for all players $i \in \mathbb{N}$.

The first condition holds because $g_i^j(\sigma)$ is nonnegative by definition, and hence the denominator in Equation (5.81) is at least 1, and the numerator is nonnegative.

As for the second condition, because $\sum_{j=1}^{m_i} \sigma_i(s_i^j) = 1$, it follows that

$$\sum_{j=1}^{m_i} f_i^j(\sigma) = \sum_{j=1}^{m_i} \frac{\sigma_i\left(s_i^j\right) + g_i^j(\sigma)}{1 + \sum_{k=1}^{m_i} g_i^k(\sigma)} \tag{5.82}$$

$$= \frac{\sum_{j=1}^{m_i}(\sigma_i\left(s_i^j\right) + g_i^j(\sigma))}{1 + \sum_{k=1}^{m_i} g_i^k(\sigma)} \tag{5.83}$$

$$= \frac{\sum_{j=1}^{m_i} \sigma_i\left(s_i^j\right) + \sum_{j=1}^{m_i} g_i^j(\sigma)}{1 + \sum_{j=1}^{m_i} g_i^j(\sigma)} = 1. \tag{5.84}$$

□

Claim 5.29 *f is a continuous function.*

Proof: Claim 5.27, implies that both the numerator and the denominator in the definition of f_i^j are continuous functions. As mentioned in the proof of Claim 5.28, the denominator in the definition of f_i^j is at least 1. Thus, f is the ratio of two continuous functions, where the denominator is always positive, and therefore it is a continuous function. □

To complete the proof of the theorem, we need to show that every fixed point of f is an equilibrium of the game. This is accomplished in several steps.

Claim 5.30 *Let σ be a fixed point of f. Then*

$$g_i^j(\sigma) = \sigma_i\left(s_i^j\right) \sum_{k=1}^{m_i} g_i^k(\sigma), \quad \forall i \in N, j \in \{1, 2, \ldots, m_i\}. \tag{5.85}$$

Proof: The strategy vector σ is a fixed point of f, and therefore $f(\sigma) = \sigma$. This is an equality between vectors; hence every coordinate in the vector on the left-hand side of the equation equals the corresponding coordinate in the vector on the right-hand side, i.e.,

$$f_i^j(\sigma) = \sigma_i\left(s_i^j\right), \quad \forall i \in N, j \in \{1, 2, \ldots, m_i\}. \tag{5.86}$$

From the definition of f

$$\frac{\sigma_i\left(s_i^j\right) + g_i^j(\sigma)}{1 + \sum_{k=1}^{m_i} g_i^k(\sigma)} = \sigma_i\left(s_i^j\right), \quad \forall i \in N, j \in \{1, 2, \ldots, m_i\}. \tag{5.87}$$

The denominator on the left-hand side is positive; multiplying both sides of the equations by the denominator yields

$$\sigma_i\left(s_i^j\right) + g_i^j(\sigma) = \sigma_i\left(s_i^j\right) + \sigma_i\left(s_i^j\right)\sum_{k=1}^{m_i} g_i^k(\sigma), \ \forall i \in N,\ j \in \{1, 2, \dots, m_i\}. \quad (5.88)$$

Cancelling the term $\sigma_i(s_i^j)$ from both sides of Equation (5.88) leads to Equation (5.85). ☐

We now turn to the proof of the last step.

Claim 5.31 *Let σ be a fixed point of f. Then σ is a Nash equilibrium.*

Proof: Suppose by contradiction that σ is not an equilibrium. Theorem 5.26 implies that there exists a player i, and $l \in \{1, 2, \dots, m_i\}$, such that $g_i^l(\sigma) > 0$. In particular, $\sum_{k=1}^{m_i} g_i^k(\sigma) > 0$; hence from Equation (5.85) we have

$$\sigma_i\left(s_i^j\right) > 0 \iff g_i^j(\sigma) > 0, \quad \forall j \in \{1, 2, \dots, m_i\}. \quad (5.89)$$

Because $g_i^l(\sigma) > 0$, one has in particular that $\sigma_i(s_i^l) > 0$. Since the function U_i is multilinear, $U_i(\sigma) = \sum_{j=1}^{m_i} \sigma_i(s_i^j) U_i(s_i^j, \sigma_{-i})$. This yields

$$0 = \sum_{j=1}^{m_i} \sigma_i\left(s_i^j\right)(U_i\left(s_i^j, \sigma_{-i}\right) - U_i(\sigma)) \quad (5.90)$$

$$= \sum_{\{j:\,\sigma_i\left(s_i^j\right)>0\}} \sigma_i\left(s_i^j\right)\left(U_i\left(s_i^j, \sigma_{-i}\right) - U_i(\sigma)\right) \quad (5.91)$$

$$= \sum_{\{j:\,\sigma_i\left(s_i^j\right)>0\}} \sigma_i\left(s_i^j\right) g_i^j(\sigma), \quad (5.92)$$

where the last equality holds because from Equation (5.89), if $\sigma_i(s_i^j) > 0$, then $g_i^j(\sigma) > 0$, and in this case $g_i^j(\sigma) = U_i(s_i^j, \sigma_{-i}) - U_i(\sigma)$. But the sum (Equation (5.92)) is positive: it contains at least one element ($j = l$), and by Equation (5.89) every summand in the sum is positive. This contradiction leads to the conclusion that σ must be a Nash equilibrium. ☐

5.4 Generalizing Nash's Theorem

There are situations in which, due to various constraints, a player cannot make use of some mixed strategies. For example, there may be situations in which player i cannot choose two pure strategies s_i and $\widehat{s_i}$ with different probability, and he is then forced to limit himself to mixed strategies σ_i in which $\sigma_i(s_i) = \sigma_i(\widehat{s_i})$. A player may find himself in a situation in which he must choose a particular pure strategy s_i with probability greater than or equal to some given number $p_i(s_i)$, and he is then forced to limit himself to mixed strategies σ_i in which $\sigma_i(s_i) \geq p_i(s_i)$. In both of these examples, the constraints can be translated into linear inequalities. A bounded set that is defined by the intersection of a finite number of half-spaces is called a *polytope*. The number of extreme points of every polytope S is finite, and every polytope is the convex hull of its extreme points: if $x^1, x^2, \dots, x^K$

are the extreme points of S, then S is the smallest convex set containing $x^1, x^2, \dots, x^K$ (see Definition 23.1 on page 917). In other words, for each $s \in S$ there exist nonnegative numbers $(\alpha^l)_{l=1}^K$ whose sum is 1, such that $s = \sum_{l=1}^K \alpha^l x^l$; conversely, for each vector of nonnegative numbers $(\alpha^l)_{l=1}^K$ whose sum is 1, the vector $\sum_{l=1}^K \alpha^l x^l$ is in S.

The space of mixed strategies Σ_i is a simplex, which is a polytope whose extreme points are unit vectors $e^1, e^2, \dots, e^{m_i}$, where $e^k = (0, \dots, 0, 1, 0, \dots, 0)$ is an m_i-dimensional vector whose k-th coordinate is 1, and all the other coordinates of e^k are 0. We will now show that Nash's Theorem still holds when the space of strategies of a player is a polytope, and not necessarily a simplex. We note that Nash's Theorem holds under even more generalized conditions, but we will not present those generalizations in this book.

Theorem 5.32 *Let $G = (N, (X_i)_{i\in N}, (U_i)_{i\in N})$ be a strategic-form game in which, for each player i,*

- *The set X_i is a polytope in $\mathbb{R}^{d_i}$.*
- *The function U_i is a multilinear function over the variables $(s_i)_{i\in N}$.*

Then G has an equilibrium.

Nash's Theorem (Theorem 5.10 on page 151) is a special case of Theorem 5.32, where $X_i = \Sigma_i$ for every player $i \in N$.

Proof: The set of strategies X_i of player i in the game G is a polytope. Denote the extreme points of this set by $\{x_i^1, x_i^2, \dots, x_i^{K_i}\}$. Define an auxiliary strategic-form game $\widehat{G}$ in which:

- The set of players is N.
- The set of pure strategies of player $i \in N$ is $L_i := \{1, 2, \dots, K_i\}$. Denote $L := \times_{i\in N} L_i$.
- For each vector of pure strategies $l = (l_1, l_2, \dots, l_n) \in L$, the payoff to player i is

$$v_i(l) := U_i\left(x_1^{l_1}, x_2^{l_2}, \dots, x_n^{l_n}\right). \tag{5.93}$$

It follows that in the auxiliary game every player i chooses an extreme point in his set of strategies X_i, and his payoff in the auxiliary game is given by U_i. For each $i \in N$, denote by V_i the multilinear extension of v_i. Since U_i is a multilinear function, player i's payoff function in the extension of $\widehat{G}$ to mixed strategies is

$$V_i(\alpha) = \sum_{l_1=1}^{k_1}\sum_{l_2=1}^{k_2}\cdots\sum_{l_n=1}^{k_n} \alpha_1^{l_1}\alpha_2^{l_2}\cdots\alpha_n^{l_n} v_i(l_1, l_2, \dots, l_n) \tag{5.94}$$

$$= \sum_{l_1=1}^{k_1}\sum_{l_2=1}^{k_2}\cdots\sum_{l_n=1}^{k_n} \alpha_1^{l_1}\alpha_2^{l_2}\cdots\alpha_n^{l_n} U_i\left(x_1^{l_1}, x_2^{l_2}, \dots, x_n^{l_n}\right) \tag{5.95}$$

$$= U_i\left(\sum_{l_1=1}^{k_1} \alpha_1^{l_1} x_1^{l_1}, \dots, \sum_{l_n=1}^{k_n} \alpha_n^{l_n} x_n^{l_n}\right). \tag{5.96}$$

The auxiliary game $\widehat{G}$ satisfies the conditions of Nash's Theorem (Theorem 5.10 on page 151), and it therefore has a Nash equilibrium in mixed strategies α^*. It follows that for every player i,

$$V_i(\alpha^*) \geq V_i(\alpha_i, \alpha^*_{-i}), \quad \forall i \in N, \forall \alpha_i \in \Delta(L_i). \tag{5.97}$$

Denote by $\alpha^*_i = (\alpha_i^{*,l_i})_{l_i=1}^{K_i}$ player i's strategy in the equilibrium α^*. Since X_i is a convex set, the weighted average

$$s^*_i := \sum_{l_i=1}^{K_i} \alpha_i^{*,l_i} x_i^{l_i} \tag{5.98}$$

is a point in X_i. We will now show that $s^* = (s^*_i)_{i \in N}$ is an equilibrium of the game G. Let $i \in N$ be a player, and let s_i be any strategy of player i. Since $\{x_i^1, x_i^2, \ldots, x_i^{K_i}\}$ are extreme points of S_i there exists a distribution $\alpha_i = (\alpha_i^{l_i})_{l_i=1}^{K_i}$ over L_i such that $s_i = \sum_{l_i=1}^{K_i} \alpha_i x_i^{l_i}$. Equations (5.98), (5.94), and (5.97) imply that, for each player $i \in N$,

$$U_i(s^*) = V_i({}^*\alpha) \geq V_i(\alpha_i, \alpha_{-i}) = U_i(s_i, s^*_{-i}). \tag{5.99}$$

That is, if player i deviates to s_i, he cannot profit. Since this is true for every player $i \in N$ and every strategy $s_i \in S_i$, the strategy vector s^* is an equilibrium of the game G. □

5.5 Utility theory and mixed strategies

In defining the mixed extension of a game, we defined the payoff that a vector of mixed strategies yields as the expected payoff when every player chooses a pure strategy according to the probability given by his mixed strategy. But how is this definition justified? In this section we will show that if the preferences of the players satisfy the von Neumann–Morgenstern axioms of utility theory (see Chapter 2), we can interpret the numerical values in each cell of the payoff matrix as the utility the players receive when the outcome of the game is that cell (see Figure 5.17).

Suppose that we are considering a two-player game, such as the game in Figure 5.17. In this game there are six possible outcomes, $O = \{A_1, A_2, \ldots, A_6\}$. Each player has a preference relation over the set of lotteries over O. Suppose that the two players have linear utility functions, u_1 and u_2 respectively, over the set of lotteries. Every pair of mixed strategies $x = (x_1, x_2)$ and $y = (y_1, y_2, y_3)$ induces a lottery over the possible outcomes. The probability of reaching each one of the possible outcomes is indicated in Figure 5.18.

In other words, every pair of mixed strategies (x, y) induces the following lottery $L_{x,y}$ over the outcomes:

$$L = L_{x,y} = [x_1y_1(A_1), x_1y_2(A_2), x_1y_3(A_3), x_2y_1(A_4), x_2y_2(A_5), x_2y_3(A_6)].$$

Since the utility function of the two players is linear, player i's utility from this lottery is

$$u_i(L_{x,y}) = x_1y_1u_i(A_1) + x_1y_2u_i(A_2) + x_1y_3u_i(A_3) + x_2y_1u_i(A_4) \\ + x_2y_2u_i(A_5) + x_2y_3u_i(A_6). \tag{5.100}$$

			Player II	
		y_1	y_2	y_3
Player I	x_1	A_1	A_2	A_3
	x_2	A_4	A_5	A_6

Figure 5.17 A two-player game in terms of outcomes

Outcome	A_1	A_2	A_3	A_4	A_5	A_6
Probability	x_1y_1	x_1y_2	x_1y_3	x_2y_1	x_2y_2	x_2y_3

Figure 5.18 The probability of reaching each of the outcomes

			Player II	
		y_1	y_2	y_3
Player I	x_1	$u_1(A_1), u_2(A_1)$	$u_1(A_2), u_2(A_2)$	$u_1(A_3), u_2(A_3)$
	x_2	$u_1(A_4), u_2(A_4)$	$u_1(A_5), u_2(A_5)$	$u_1(A_6), u_2(A_6)$

Figure 5.19 The game in Figure 5.17 in terms of utilities

Player i's utility from this lottery is therefore equal to his expected payoff in the strategic-form game in which in each cell of the payoff matrix we write the utilities of the players from the outcome obtained at that cell (Figure 5.19).

If, therefore, we assume that each player's goal is to maximize his utility, what we are seeking is the equilibria of the game in Figure 5.19. If (x, y) is an equilibrium of this game, then any player who unilaterally deviates from his equilibrium strategy cannot increase his utility.

Note that because, in general, the utility functions of the players differ from each other, the game in terms of utilities (Figure 5.19) is not a zero-sum game, even if the original game is a zero-sum game in which the outcome is a sum of money that Player II pays to Player I.

Recall that a player's utility function is determined up to a positive affine transformation (Corollary 2.23, on page 23). How does the presentation of a game change if a different choice of players' utility functions is made? Let v_1 and v_2 be two positive affine transformations of u_1 and u_2 respectively; i.e., u_i and v_i are equivalent representations of the utilities of player i that satisfy $v_i(L) = \alpha_i u_i(L) + \beta_i$ for every lottery L where $\alpha_i > 0$ and $\beta_i \in \mathbb{R}$ for $i = 1, 2$. The game in Figure 5.17 in terms of the utility functions v_1 and v_2 will be analogous to the matrix that appears in Figure 5.19, with u_1 and u_2 replaced by v_1 and v_2 respectively.

Example 5.33 Consider the two games depicted in Figure 5.20. Game B in Figure 5.20 is derived from Game A by adding a constant value of 6 to the payoff of Player II in every cell, whereby we have implemented a positive affine transformation (where $\alpha = 1$, $\beta = 6$) on the payoffs of Player II.

Game A (Player I: rows; Player II: columns)

Player I \ Player II	L	M
T	3, −3	−2, 2
B	5, −5	1, −1

Game B (Player I: rows; Player II: columns)

Player I \ Player II	L	M
T	3, 3	−2, 8
B	5, 1	1, 5

Figure 5.20 Adding a constant value to the payoffs of one of the players

While Game A is a zero-sum game, Game B is not a zero-sum game, because the sum of the utilities in each cell of the matrix is 6. Such a game is called a *constant-sum game*. Every constant-sum game can be transformed to a zero-sum game by adding a constant value to the payoffs of one of the players, whereby the concepts constant-sum game and zero-sum game are equivalent. As we will argue in Theorem 5.35 below, the equilibria of a game are unchanged by adding constant values to the payoffs. For example, in the two games in Figure 5.20, strategy B strictly dominates T for Player I, and strategy M strictly dominates L for Player II. It follows that in both of these games, the only equilibrium point is (B, M).

If we implement a positive affine transformation in which $\alpha \neq 1$ on the payoffs of the players, we will still end up with a game in which the only thing that has changed is the units in which we are measuring the utilities of the players. For example, the following game is derived from Game A in Figure 5.20 by implementing the affine transformation $x \mapsto 5x + 7$ on the payoffs of Player I (see Figure 5.21).

Player I \ Player II	L	M
T	22, −3	−3, 2
B	32, −5	12, −1

Figure 5.21 The utilities in Game A in Figure 5.20 after implementing the affine transformation $x \mapsto 5x + 7$ on the payoffs to Player I ◀

Games that differ only in the utility representations of the players are considered to be equivalent games.

Definition 5.34 *Two games in strategic form* $(N, (S_i)_{i\in N}, (u_i)_{i\in N})$ *and* $(N, (S_i)_{i\in N}, (v_i)_{i\in N})$ *with the same set of players and the same sets of pure strategies are* strategically equivalent *if for each player* $i \in N$ *the function* v_i *is a positive affine transformation* *of the function u_i. In other words, there exist $\alpha_i > 0$ and $\beta_i \in \mathbb{R}$ such that*

$$v_i(s) = \alpha_i u_i(s) + \beta_i, \quad \forall s \in S. \tag{5.101}$$

The name “strategic equivalence” comes from the next theorem, whose proof we leave as an exercise (Exercise 5.58).

Theorem 5.35 *Let G and $\widehat{G}$ be two strategically equivalent strategic-form games. Every equilibrium $\sigma = (\sigma_1, \ldots, \sigma_n)$ in mixed strategies of the game G is an equilibrium in mixed strategies of the game $\widehat{G}$.*

In other words, each equilibrium in the original game remains an equilibrium after changing the utility functions of the players by positive affine transformations. Note, however, that the equilibrium payoffs do change from one strategically equivalent game to another, in accordance with the positive affine transformation that has been implemented.

Corollary 5.36 *If the preferences of every player over lotteries over the outcomes of the game satisfy the von Neumann–Morgenstern axioms, then the set of equilibria of the game is independent of the particular utility functions used to represent the preferences.*

Given the payoff matrix in Figure 5.21 and asked whether or not this game is strategically equivalent to a zero-sum game, what should we do? If the game is strategically equivalent to a zero-sum game, then there exist two positive affine transformations f_1 and f_2 such that $f_2(u_2(s)) = -f_1(u_1(s))$ for every strategy vector $s \in S$. Since the inverse of a positive affine transformation is also a positive affine transformation (Exercise 2.19 on page 35), and the concatenation of two positive affine transformations is also a positive affine transformation (Exercise 2.20 on page 35), in this case the positive affine transformation $f_3 = -((f_1)^{-1} \circ (-f_2))$ satisfies the property that $f_3(u_2(s)) = -u_1(s)$ for every strategy vector $s \in S$. In other words, if the game is strategically equivalent to a zero-sum game, there exists a positive affine transformation that when applied to the utilities of Player II, yields the negative of the utilities of Player I. Denote such a transformation, assuming it exists, by $\alpha u + \beta$. Then we need to check whether there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that

$$-5\alpha + \beta = -32, \tag{5.102}$$

$$-3\alpha + \beta = -22, \tag{5.103}$$

$$-1\alpha + \beta = -12, \tag{5.104}$$

$$2\alpha + \beta = 3. \tag{5.105}$$

In order to ascertain whether this system of equations has a solution, we can find α and β that solve two of the above equations, and check whether they satisfy the rest of the equations. For example, if we solve Equations (5.102) and (5.103), we get $\alpha = 5$ and $\beta = -7$, and we can then check that these values do indeed also solve the Equations (5.104) and (5.105). Since we have found α and β solving the system of equations, we deduce that this game is strategically equivalent to a zero-sum game.

Remark 5.37 *Given the above, some people define a zero-sum game to be a game strategically equivalent to a game $(N, (S_i)_{i\in N}, (v_i)_{i\in N})$ in which $v_1 + v_2 = 0$.* ◆

The connection presented in this section between utility theory and game theory underscores the significance of utility theory. Representing the utilities of players by linear functions enables us to compute Nash equilibria with relative ease. Had we represented the players' preferences/indifferences by nonlinear utility functions, calculating equilibria would be far more complicated. This is similar to the way we select measurement scales in various fields. Many physical laws are expressed using the Celsius scale, because they can be given a simple expression. For example, consider the physical law that states that the change in the length of a metal rod is proportional to the change in its temperature. If temperature is measured in Fahrenheit, that law remains unchanged, since the Fahrenheit scale is a positive affine transformation of the Celsius scale. In contrast, if we were to measure temperature using, say, the log of the Celsius scale, many physical laws would have much more complicated formulations. Using linear utilities enables us to compute the utilities of simple lotteries using expected-value calculations, which simplifies the analysis of strategic-form games. This, of course, depends on the assumption that the preferences of the players can be represented by linear utility functions, i.e., that their preferences satisfy the von Neumann–Morgenstern axioms.

Another important point that has emerged from this discussion is that most daily situations do not correspond to two-player zero-sum games, even if the outcomes are in fact sums of money one person pays to another. This is because the utility of one player from receiving an amount of money x is usually not diametrically opposite to the utility of the other from paying this amount. That is, there are amounts $x \in \mathbb{R}$ for which $u_1(x) + u_2(-x) \neq 0$. On the other hand, as far as equilibria are concerned, the particular representation of the utilities of the players does not affect the set of equilibria of a game. If there exists a representation that leads to a zero-sum game, we are free to choose that representation, and if we do so, we can find equilibria by solving a linear program (see Section 5.2.5 on page 164).

One family of games that is always amenable to such a representation, which can be found easily, is the family of two-person games with two outcomes, where the preferences of the two players for the two alternative outcomes are diametrically opposed in these games. In such games we can always define the utilities of one of the players over the outcomes to be 1 or 0, and define the utilities of the other player over the outcomes to be -1 or 0. In contrast, zero-sum games are rare in the general family of two-player games. Nevertheless, two-player zero-sum games are very important in the study of game theory, as explained on pages 111–112.

5.6 The maxmin and the minmax in *n*-player games

In Section 4.10 (page 102), we defined the maxmin to be the best outcome that a player can guarantee for himself under his most pessimistic assumption regarding the behavior of the other players.

Definition 5.38 *The* maxmin in mixed strategies *of player i is defined as follows:*

$$\underline{v}_i := \max_{\sigma_i \in \Sigma_i} \min_{\sigma_{-i} \in \Sigma_{-i}} U_i(\sigma_i, \sigma_{-i}). \tag{5.106}$$

In two-player zero-sum games we also defined the concept of the minmax value, which is interpreted as the least payoff that the other players can guarantee that a player will get. In two-player zero-sum games, minimizing the payoff of one player is equivalent to maximizing the payoff of his opponent, and hence in two-player zero-sum games the maxmin of Player I is equal to the minmax of Player II. This is not true, however, in two-player games that are not zero-sum games, and in games with more than two players.

Analogously to the definition of the maxmin in Equation (5.106), the minmax value of a player is defined as follows.

Definition 5.39 *Let* $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ *be a strategic-form game. The* minmax value in mixed strategies *of player* i *is*

$$\overline{v}_i := \min_{\sigma_{-i}\in\Sigma_{-i}} \max_{\sigma_i\in\Sigma_i} U_i(\sigma_i, \sigma_{-i}). \tag{5.107}$$

$\overline{v}_i$ is the lowest possible payoff that the other players can force on player i.

A player's maxmin and minmax values depend solely on his payoff function, which is why different players in the same game may well have different maxmin and minmax values. One of the basic characteristics of these values is that a player's minmax value in mixed strategies is greater than or equal to his maxmin value in mixed strategies.

Theorem 5.40 *In every strategic-form game* $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$*, for each player* $i \in N$*,*

$$\overline{v}_i \geq \underline{v}_i. \tag{5.108}$$

Equation (5.108) is expected: if the other players can guarantee that player i will not receive more than $\overline{v}_i$, and player i can guarantee himself at least $\underline{v}_i$, then $\overline{v}_i \geq \underline{v}_i$.

Proof: Let $\widehat{\sigma}_{-i} \in \Sigma_{-i}$ be a strategy vector in which the minimum in Equation (5.107) is attained;, i.e.,

$$\overline{v}_i = \max_{\sigma_i\in\Sigma_i} U_i(\sigma_i, \widehat{\sigma}_{-i}) \leq \max_{\sigma_i\in\Sigma_i} U_i(\sigma_i, \sigma_{-i}), \qquad \forall \sigma_{-i} \in \Sigma_{-i}. \tag{5.109}$$

On the other hand,

$$U_i(\sigma_i, \widehat{\sigma}_{-i}) \geq \min_{\sigma_{-i}\in\Sigma_{-i}} U_i(\sigma_i, \sigma_{-i}), \quad \forall \sigma_i \in \Sigma_i. \tag{5.110}$$

Taking the maximum over all mixed strategies $\sigma_i \in \Sigma_i$ on both sides of the equation sign (5.110) yields

$$\overline{v}_i = \max_{\sigma_i\in\Sigma_i} U_i(\sigma_i, \widehat{\sigma}_{-i}) \geq \max_{\sigma_i\in\Sigma_i} \min_{\sigma_{-i}\in\Sigma_{-i}} U_i(\sigma_i, \sigma_{-i}) = \underline{v}_i. \tag{5.111}$$

We conclude that $\overline{v}_i \geq \underline{v}_i$, which is what we needed to show. □

In a two-player game $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ where $N = \{\mathrm{I}, \mathrm{II}\}$, the maxmin value in mixed strategies of each player is always equal to his minmax value in mixed strategies. For Player I, for example, these two values equal the value of the second two-player zero-sum game $G = (N, (S_i)_{i\in N}, (v_i)_{i\in N})$, in which $v_{\mathrm{I}} = u_{\mathrm{I}}$ and $v_{\mathrm{II}} = -u_{\mathrm{I}}$ (Exercise 5.64). As the next example shows, in a game with more than two players the maxmin value may be less than the minmax value.

Example 5.41 Consider the three-player game in which the set of players is $N = \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$, and every player has two pure strategies; Player I chooses a row (T or B), Player II chooses a column (L or R), and Player III chooses the matrix (W or E). The payoff function u_1 of Player I is shown in Figure 5.22.

	L	R
T	0	1
B	1	1

W

	L	R
T	1	1
B	1	0

E

Figure 5.22 Player I's payoff function in the game in Example 5.41

We compute the maxmin value in mixed strategies of player i. If Player I uses the mixed strategy $[x(T), (1-x)(B)]$, Player II uses the mixed strategy $[y(L), (1-y)(R)]$, and Player III uses the mixed strategy $[z(W), (1-z)(E)]$, then Player I's payoff is

$$U_{\mathrm{I}}(x, y, z) = 1 - xyz - (1-x)(1-y)(1-z). \tag{5.112}$$

We first find

$$\underline{v}_{\mathrm{I}} = \max_x \min_{y,z} U_{\mathrm{I}}(x, y, z) = \tfrac{1}{2}. \tag{5.113}$$

To see this, note that $U_{\mathrm{I}}(x, 1, 1) = x \leq \frac{1}{2}$ for every $x \leq \frac{1}{2}$, and $U_{\mathrm{I}}(x, 0, 0) = 1 - x \leq \frac{1}{2}$ for every $x \geq \frac{1}{2}$, and hence $\min_{y,z} U_{\mathrm{I}}(x, y, z) \leq \frac{1}{2}$ for every x. On the other hand, $U_{\mathrm{I}}(\frac{1}{2}, y, z) \geq \frac{1}{2}$ for each y and z and hence $\max_x \min_{y,z} U_{\mathrm{I}}(x, y, z) = \frac{1}{2}$, which is what we claimed.

We next turn to calculating the minmax value of Player I.

$$\begin{aligned}
\overline{v}_{\mathrm{I}} &= \min_{y,z} \max_x U_{\mathrm{I}}(x, y, z) && (5.114)\\
&= \min_{y,z} \max_x (1 - xyz - (1-x)(1-y)(1-z)) && (5.115)\\
&= \min_{y,z} \max_x (1 - (1-y)(1-z) + x(1-y-z)). && (5.116)
\end{aligned}$$

For every fixed y and z the function $x \mapsto (1 - (1-y)(1-z) + x(1-y-z))$ is linear; hence the maximum of Equation (5.116) is attained at the extreme point $x = 1$ if $1 - y - z \geq 0$, and at the extreme point $x = 0$ if $1 - y - z \leq 0$. This yields

$$\max_x (1 - (1-y)(1-z) + x(1-y-z)) = \begin{cases} 1 - (1-y)(1-z) & \text{if } y + z \geq 1, \\ 1 - yz & \text{if } y + z \leq 1. \end{cases}$$

The minimum of the function $1 - (1-y)(1-z)$ over the domain $y + z \geq 1$ is $\frac{3}{4}$, and is attained at $y = z = \frac{1}{2}$. The minimum of the function $1 - yz$ over the domain $y + z \leq 1$ is also $\frac{3}{4}$, and is attained at $y = z = \frac{1}{2}$. We therefore deduce that

$$\overline{v}_{\mathrm{I}} = \tfrac{3}{4}. \tag{5.117}$$

In other words, in this example

$$\underline{v}_{\mathrm{I}} = \tfrac{1}{2} < \tfrac{3}{4} = \overline{v}_{\mathrm{I}}. \tag{5.118}$$

◀

Why can the minmax value in mixed strategies of player i in a game with more than two players be greater than his maxmin value in mixed strategies? Note that since U_i is a function that is linear in σ_{-i}, the minimum in Equation (5.106) is attained at an extreme point of Σ_{-i}, i.e., at a point in S_{-i}. It follows that

$$\underline{v}_i = \max_{\sigma_i \in \Sigma_i} \min_{s_{-i} \in S_{-i}} U_i(\sigma_i, s_{-i}). \tag{5.119}$$

Consider the following two-player zero-sum auxiliary game $\widehat{G}$:

- The set of players is {I, II}.
- Player I's set of pure strategies is S_i.
- Player II's set of pure strategies is $S_{-i} = \times_{j \neq i} S_j$: Player II chooses a pure strategy for every player who is not player i.
- The payoff function is u_i, the payoff function of player i in the original game.

Player I's set of mixed strategies in the auxiliary game $\widehat{G}$ is $\Delta(S_i) = \Sigma_i$, which is player i's set of mixed strategies in the original game. Player II's set of mixed strategies in the auxiliary game $\widehat{G}$ is $\Delta(S_{-i})$; i.e., a mixed strategy of Player II in $\widehat{G}$ is a probability distribution over the set of pure strategy vectors of the players who are not player i. The Minmax Theorem (Theorem 5.11 on page 151) then implies that the game $\widehat{G}$ has a value in mixed strategies, which is equal to

$$\widehat{v} = \max_{\sigma_i \in \Sigma_i} \min_{\sigma_{-i} \in \Delta(S_{-i})} U_i(\sigma_i, \sigma_{-i}) = \min_{\sigma_{-i} \in \Delta(S_{-i})} \max_{\sigma_i \in \Sigma_i} U_i(\sigma_i, \sigma_{-i}). \tag{5.120}$$

Since for every mixed strategy σ_i of player i the function $\sigma_{-i} \mapsto U_i(\sigma_i, \sigma_{-i})$ is linear in the variables σ_{-i}, the minimum in the expression in the middle term of Equation (5.120) equals the minimum over the extreme points of $\Delta(S_{-i})$, which is the set S_{-i}. Therefore,

$$\widehat{v} = \max_{\sigma_i \in \Sigma_i} \min_{\sigma_{-i} \in \Delta(S_{-i})} U_i(\sigma_i, \sigma_{-i}) = \max_{\sigma_i \in \Sigma_i} \min_{s_{-i} \in S_{-i}} U_i(\sigma_i, s_{-i}) = \underline{v}_i, \tag{5.121}$$

where the last equality follows from Equation (5.119). Combining Equations (5.120) and (5.121) and substituting $S_{-i} = \times_{j \neq i} S_j$ yields

$$\underline{v}_i = \min_{\sigma_{-i} \in \Delta(\times_{j \neq i} S_j)} \max_{\sigma_i \in \Sigma_i} U_i(\sigma_i, \sigma_{-i}), \tag{5.122}$$

and using Equation (5.107), and substituting $\Sigma_{-i} = \times_{j \neq i} \Delta(S_j)$, yields

$$\overline{v}_i = \min_{\sigma_{-i} \in \times_{j \neq i} \Delta(S_j)} \max_{\sigma_i \in \Sigma_i} U_i(\sigma_i, \sigma_{-i}). \tag{5.123}$$

With the help of the last two equations, we can see that the difference between $\underline{v}_i$ (the value of the auxiliary game) and $\overline{v}_i$ is in the set over which the minimization is implemented, or more precisely, the order in which the Δ operator (the set of distributions over ...) and $\times_{j \neq i}$ (the Cartesian product over ...) are implemented. $\times_{j \neq i} \Delta(S_j)$ appears in the calculation of the minmax value $\overline{v}_i$, and $\Delta(\times_{j \neq i} S_j)$ appears in the calculation of the maxmin value $\underline{v}_i$. The relationship between these two sets is given by

$$\Delta(\times_{j \neq i} S_j) \supseteq \times_{j \neq i} \Delta(S_j). \tag{5.124}$$

The two sets in Equation (5.124) have the same extreme points, namely, the elements of the set S_{-i}. This fact was used in the derivation of Equation (5.122). Despite this, the inclusion in Equation (5.124) is a proper inclusion when the number of players is greater than 2. In this case $\underline{v}_i$ may be less than $\overline{v}_i$, since the minimization in Equation (5.122) is over a set larger than the set over which the minimization in Equation (5.123) is conducted.

In summary, in the auxiliary game, Player II represents all the players who are not i, and his mixed strategy is not necessarily the product distribution over the set $\times_{j \neq i} S_j$: for instance, in Example 5.41, for $i = 1$, the mixed strategy $[\frac{1}{2}(L, W), \frac{1}{2}(R, E)]$ is a possible strategy for Player II in the auxiliary game; i.e., it is in the set $\Delta(S_2 \times S_3)$, but it is not an element in $\Delta(S_2) \times \Delta(S_3)$. It follows that in the auxiliary game Player II can choose the vector of pure strategies for the players who are not i in a correlated manner, while in the original game, in which the players choose mixed strategies independently of each other, such correlation is impossible.

Theorem 4.29 (page 105) states that player i's payoff in any equilibrium is at least his maxmin value. As we now show, this payoff is also at least the player's minmax value.

Theorem 5.42 *For every Nash equilibrium σ^* in a strategic-form game and every player i we have $u_i(\sigma^*) \geq \overline{v}_i$.*

Proof: The result holds since

$$u_i(s^*) = \max_{s_i \in S_i} u_i(s_i, s^*_{-i}) \geq \min_{s_{-i} \in S_{-i}} \max_{s_i \in S_i} u_i(s_i, s_{-i}) = \overline{v}_i, \tag{5.125}$$

□

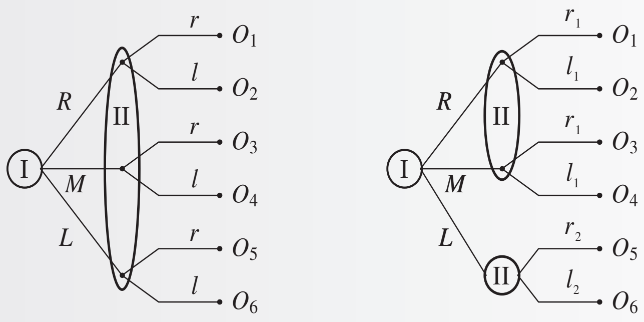
5.7 Imperfect information: the value of information

Recall that every extensive-form game is also a strategic-form game. In this section we study extensive-form games with information sets, and look into how their maxmin and minmax values in mixed strategies change when we add information for one of the players. Adding information to a player is expressed by splitting one or more of his information sets into subsets. Note that this gives the player pure strategies that were previously unavailable to him, while he does not lose any strategies that were available to him before he received the new information. The intuitive reason for this is that the player can always ignore the additional information he has received, and play the way he would have played before.

In this section only we denote the multilinear extension of player i's payoff function by u_i rather than U_i, which will denote an information set of player i.

Example 5.43 Consider Game A in Figure 5.23. In this game, every player has one information set. The set of pure strategies of Player I is $\{L, M, R\}$, and that of Player II is $\{l, r\}$. This game is equivalent to a strategic-form game, in which Player II, in choosing his action, does not know what action was chosen by Player I (Game A in Figure 5.24, where Player I is the row player and Player II is the column player).

Game B in Figure 5.23 is similar to Game A, except that we have split the information set of Player II into two information sets. In other words, Player II, in choosing his action, knows whether or not Player I has chosen L, and hence we have increased Player II's information.



Game A Game B

Figure 5.23 Adding information to Player II

With this additional information, the set of pure strategies of Player II is $\{l_1l_2, l_1r_2, r_1l_2, r_1r_2\}$, and the strategic description of this game is Game B in Figure 5.24.

Player II

Player I	l	r
L	O_6	O_5
M	O_4	O_3
R	O_2	O_1

Game A

Player II

Player I	l_1l_2	r_1r_2	l_1r_2	r_1l_2
L	O_6	O_5	O_5	O_6
M	O_4	O_3	O_4	O_3
R	O_2	O_1	O_2	O_1

Game B

Figure 5.24 Splitting an information set: the games in strategic form

Note that Game B in Figure 5.24, restricted only to the strategies l_1l_2 and r_1r_2, is equivalent, from the perspective of its outcomes, to Game A. In other words, the strategies l_1l_2 and r_1r_2 are identical to the strategies l and r, respectively, in Game A. In summary, adding information to a player enlarges his set of pure strategies. ◀

The phenomenon that we just saw in Example 5.43 can be generalized to every game in extensive form: if we split an information set of player i into two information sets, then every strategy in the original game is equivalent to a strategy in the new game, in which the player takes the same action in each of the two information sets split from the original information set. If we identify these two strategies, in the original game and the new game, we see that as a result of splitting an information set of player i, his set of pure strategies in the new game includes his set of strategies in the original game. Since every addition of information to a player is equivalent to splitting one of his information sets into a finite number of subsets, we can conclude that adding information to a player leads to a game with "more" pure strategies available to that player.

What effect does adding information to a player have on the outcome of a game? Does the additional information lead to a better or worse outcome for him? As we will now show, in a two-player zero-sum game, adding information to a player can never be to his detriment, and may well be to his advantage. In contrast, in a game that is not zero-sum, adding information to a player may sometimes be to his advantage, and sometimes to his detriment.

Theorem 5.44 *Let Γ be an extensive-form game, and let Γ' be the game derived from Γ by splitting several of player i's information sets. Then the maxmin value in mixed strategies of player i in the game Γ' is greater than or equal to his maxmin value in mixed strategies in Γ, and his minmax value in mixed strategies in Γ' is greater than or equal to his minmax value in mixed strategies in Γ.*

Proof: We will first prove the theorem's claim with respect to the maxmin value in mixed strategies. Denote by $\underline{v}_i$ the maxmin value of player i in Γ, and by $\underline{v}'_i$ the maxmin value of Γ'. For every player j, let S_j be the set of j's pure strategies in Γ, and S'_j be his set of pure strategies in Γ'. Denote by Σ_j player j's set of mixed strategies in Γ, and by Σ'_j his set of mixed strategies in Γ'. In going from Γ to Γ', the set of information sets of player j, for $j \neq i$, remains unchanged, and the set of pure strategies of each of these players also remains unchanged: $S_j = S'_j$, and therefore $\Sigma_j = \Sigma'_j$. In contrast, in going from Γ to Γ', some of player i's information sets have been split. In particular, every information set U'_i of player i in Γ' is contained in a single information set U_i of player i in Γ. It follows that every pure strategy in S_i can be regarded as a pure strategy in S'_i. Indeed, let $s_i \in S_i$ be a strategy of player i in Γ. Define a strategy $s'_i \in S'_i$ of player i in Γ' as follows: in each information set U'_i, the strategy s'_i chooses the action that s_i chooses in the unique information set U_i in Γ that contains U'_i. The strategies s_i and s'_i are effectively identical.

Consequently, every mixed strategy σ_i of player i in Γ can be regarded as a mixed strategy of player i in Γ': for every mixed strategy σ_i of player i in Γ there exists a mixed strategy σ'_i of player i in Γ' satisfying

$$u_i(\sigma_i, \sigma_{-i}) = u_i(\sigma'_i, \sigma_{-i}), \qquad \forall \sigma_{-i} \in \Sigma_{-i}. \tag{5.126}$$

Therefore,

$$\underline{v}_i = \max_{\sigma_i \in \Sigma_i} \min_{\sigma_{-i} \in \Sigma_{-i}} u_i(\sigma_i, \sigma_{-i}) \leq \max_{\sigma_i \in \Sigma'_i} \min_{\sigma_{-i} \in \Sigma_{-i}} u_i(\sigma_i, \sigma_{-i}) = \underline{v}'_i. \tag{5.127}$$

In words, player i's maxmin value in mixed strategies, as a result of having additional information, is no less than his maxmin value in mixed strategies without this information. The proof for the analogous claim for the minmax value follows similarly by the same argument. Denote by $\overline{v}_i$ player i's minmax value in Γ, and by $\overline{v}'_i$ his minmax value in Γ'. Since we can regard every mixed strategy of player i in Γ as a mixed strategy in Γ', then, for all $\sigma_{-i} \in \Sigma_{-i}$,

$$\max_{\sigma_i \in \Sigma_i} U_i(\sigma_i, \sigma_{-i}) \leq \max_{\sigma_i \in \Sigma'_i} U_i(\sigma_i, \sigma_{-i}). \tag{5.128}$$

Therefore, when we take the minimum over all mixed strategy vectors of the players in $N \setminus \{i\}$ we get

$$\overline{v}_i = \min_{\sigma_{-i} \in \Sigma_{-i}} \max_{\sigma_i \in \Sigma_i} U_i(\sigma_i, \sigma_{-i}) \leq \min_{\sigma_{-i} \in \Sigma_{-i}} \max_{\sigma_i \in \Sigma'_i} U_i(\sigma_i, \sigma_{-i}) = \overline{v}'_i. \tag{5.129}$$

Thus, the minmax value in mixed strategies of player i does not decrease when additional information is received. ☐

For two-player zero-sum games for which a value in mixed strategies always exists, we have the following corollary (see Exercise 5.61).

Theorem 5.45 *Let Γ be a two-player zero-sum game in extensive form and let Γ' be the game derived from Γ by splitting several information sets of Player I. Then the value of the game Γ' in mixed strategies is greater than or equal to the value of Γ in mixed strategies.*

The theorem is depicted in the following example.

Example 5.46 Consider the two-player zero-sum game comprised of the following two stages. In the first stage, one of the two matrices in Figure 5.25 is chosen by a coin toss (each of the matrices is chosen with probability $\frac{1}{2}$). The players are not informed which matrix has been chosen. In stage two, the two players play the game whose payoffs are given by the chosen matrix (the payoffs represent payments made by Player II to Player I).

Player II

		L	R
Player I	T	0	$\frac{1}{2}$
	B	0	1

Matrix G_1

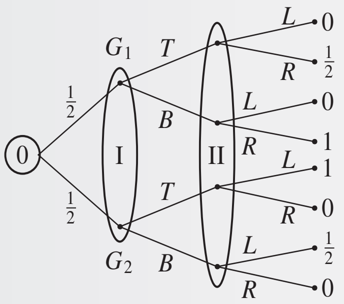
Player II

		L	R
Player I	T	1	0
	B	$\frac{1}{2}$	0

Matrix G_2

Figure 5.25 The matrices in Example 5.46

Figure 5.26 shows the game in extensive form and in strategic form.



Player II

		L	R
Player I	T	$\frac{1}{2}$	$\frac{1}{4}$
	B	$\frac{1}{4}$	$\frac{1}{2}$

Figure 5.26 The game in Example 5.46, shown in extensive form and in strategic form

Since the two players do not know which matrix has been chosen, each player has two pure strategies. Since the probability that each of the payoff matrices G_1 and G_2 will be chosen is $\frac{1}{2}$, the payoff matrix in this figure is the average of the payoff matrices G_1 and G_2: the payoff corresponding to each pure strategy vector is the average of the payoffs in the entries corresponding to that strategy vector in the two payoff matrices G_1 and G_2 (see page 79).

The value in mixed strategies of the game in Figure 5.26 is $\frac{3}{8}$. Player I's optimal strategy is $[\frac{1}{2}(T), \frac{1}{2}(B)]$, and Player II's optimal strategy is $[\frac{1}{2}(L), \frac{1}{2}(R)]$.

Consider now what happens if Player I is informed which matrix was chosen, but Player II remains ignorant of the choice. In that case, in the extensive form of the game Player I's information set in Figure 5.26 splits into two information sets, yielding the extensive-form game shown in Figure 5.27.

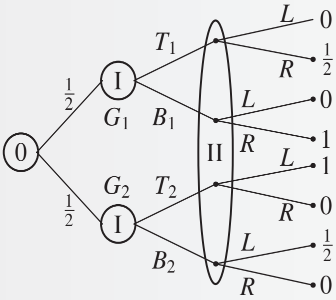


Figure 5.27 The game derived from the game in Figure 5.26 if Player I knows which matrix is chosen

In this game, Player I has four pure strategies (T_1T_2, T_1B_2, B_1T_2, B_1B_2), while Player II has two strategies (L and R). The corresponding strategic-form game appears in Figure 5.28.

Player II

Player I

	L	R
T_1T_2	$\frac{1}{2}$	$\frac{1}{4}$
T_1B_2	$\frac{1}{4}$	$\frac{1}{4}$
B_1T_2	$\frac{1}{2}$	$\frac{1}{2}$
B_1B_2	$\frac{1}{4}$	$\frac{1}{2}$

Figure 5.28 The game in Figure 5.27 in strategic form

The value in mixed strategies of this game is $\frac{1}{2}$, and B_1T_2 is Player I's optimal strategy. Since $\frac{1}{2} > \frac{3}{8}$, the added information is advantageous to Player I, in accordance with Theorem 5.45. ◀

In games that are not zero-sum, however, the situation is completely different. In the following example, Player I receives additional information, but this leads him to lose at the equilibrium point.

Example 5.47 **Detrimental addition of information** Consider Game A in Figure 5.29.

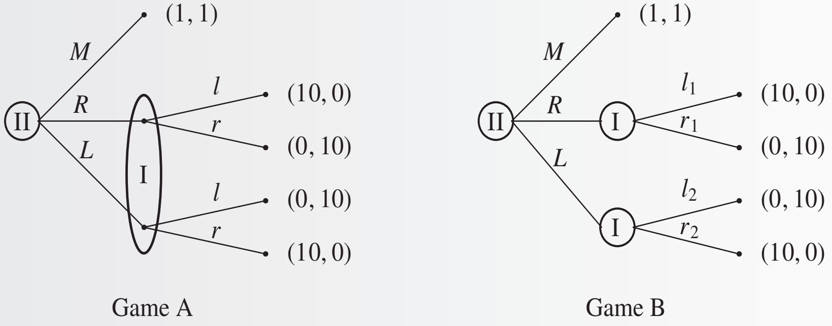


Figure 5.29 The games in Example 5.47

The only equilibrium point of this game is the following pair of mixed strategies:

- Player I plays $[\frac{1}{2}(l), \frac{1}{2}(r)]$.
- Player II plays $[\frac{1}{2}(L), \frac{1}{2}(R), 0(M)]$.

To see this, note that strategy M is strictly dominated by strategy $[\frac{1}{2}(L), \frac{1}{2}(R), 0(M)]$, and it follows from Theorem 4.35 (page 109) that it may be eliminated. After the elimination of this strategy, the resulting game is equivalent to Matching Pennies (Example 3.20 on page 52), whose sole equilibrium point is the mixed strategy under which both players choose each of their pure strategies with probability $\frac{1}{2}$ (verify that this is true). The equilibrium payoff is therefore (5, 5). When Player I receives additional information and can distinguish between the two vertices (see Game B in Figure 5.29), an equilibrium in the game is:

- Player I plays the pure strategy (l_1, r_2).
- Player II plays $[0(L), 0(R), 1(M)]$.

To see this, note that the pure strategy (l_1, r_2) of Player I is his best reply to Player II's strategy, and strategy $[0(L), 0(R), 1(M)]$ is Player II's best reply to (l_1, r_2). The equilibrium payoff is therefore (1, 1). This is not the only equilibrium of this game; there are more equilibria, but they all yield an equilibrium payoff of (1, 1) (Exercise 5.67). Adding information in this game is to Player I's detriment, because his payoff drops from 5 to 1. In this particular example, the additional information has also impacted Player II's payoff negatively, but this is not always the case (see Exercise 5.62).

The reason that additional information is to Player I's detriment is that he cannot ignore the new information: if the play of the game reaches one of the vertices that Player I controls, it is to his advantage to exploit the information he has, since ignoring it lowers his expected payoff. As a rational player, he must make use of the information. Player II knows this, and adapts his strategy to this new situation. It would be to Player I's advantage to commit to not using his additional information, but without such a binding commitment, Player II may not believe any "promise" that Player I makes to disregard his information. Careful consideration of this example brings out the source of this phenomenon: it is not the addition of information, *per se*, that is the cause of Player I's loss, but the fact that Player II knows that Player I has additional information, which leads Player II to change his behavior. ◀

A question naturally arises from the material in this section: why does the addition of information always (weakly) help a player in two-player zero-sum games, while in games that are not zero-sum, it may be advantageous or detrimental? The answer lies in the fact that there is a distinction between the concepts of the maxmin value and the equilibrium in games that are not zero sum, while in two-player zero-sum games the two concepts coincide. Additional information to a player (weakly) increases his maxmin value in every game, whether or not it is a two-player zero-sum game (Theorem 5.44). In a two-player zero-sum game, the unique equilibrium payoff is the value of the game (which is also the maxmin value), which is why adding information to a player always (weakly) increases his payoff. If the game is not a two-player zero-sum game, equilibrium payoffs, which can rise or fall with the addition of information, need not equal a player's maxmin value. The statement "adding information can be detrimental" specifically relates to situations in which a player's equilibrium payoff, after he gains information, can fall. But this is only relevant if we are expecting the outcome of a game to be an equilibrium, both before and after the addition of new information (we may perhaps expect this if the game has a unique equilibrium, or strictly dominant strategies, as in Exercise 5.63). In contrast, in situations in which we expect the players to play their maxmin strategies, adding information cannot be detrimental.

5.8 Evolutionarily stable strategies

Darwin's Theory of Evolution is based on the principle of the survival of the fittest, according to which new generations of the world's flora and fauna bear mutations.[2] An individual who has undergone a mutation will pass it on to his descendants. Only those individuals most adapted to their environment succeed in the struggle for survival.

It follows from this principle that, in the context of the process of evolution, every organism acts as if it were a rational creature, by which we mean a creature whose behavior is directed toward one goal: to maximize the expected number of its reproducing descendants. We say that it acts "as if" it were rational in order to stress that the individual organism is not a strategically planning creature. If an organism's inherited properties are not adapted to the struggle for survival, however, it will simply not have descendants.

For example, suppose that the expected number of surviving offspring per individual in a given population is three in every generation. If a mutation raising the number of expected offspring to four occurs in only one individual, eventually there will be essentially no individuals in the population carrying genes yielding the expected number of three offspring, because the ratio of individuals carrying the gene for an expected number of four descendants to individuals carrying the gene for an expected number of three descendants will grow exponentially over the generations.

If we relate to an organism's number of offspring as a payoff, we have described a process that is propelled by the maximization of payoffs. Since the concept of equilibrium in a game is also predicated on the idea that only strategies that maximize expected payoffs

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[2] A mutation is a change in a characteristic that an individual has that is brought on by a change in genetic material. In this section, we will use the term mutation to mean an individual in a population whose behavior has changed, and who passes on that change in behavior to his descendants.

(against the strategies used by the other players) will be chosen, we have a motivation for using ideas from game theory in order to explain evolutionary phenomena. Maynard Smith and Price [1973] showed that, in fact, it is possible to use the Nash equilibrium concept to shed new light on Darwin's theory. This section presents the basic ideas behind the application of game theory to the study of evolutionary biology. The interested reader can find descriptions of many phenomena in biology that are explicable using the theory developed in this section in Richard Dawkins's popular book, *The Selfish Gene* (Dawkins [1976]).

The next example, taken from Maynard Smith and Price [1973], introduces the main idea, and the general approach, used in this theory.

Example 5.48 Suppose that a particular animal can exhibit one of two possible behaviors: aggressive behavior or peaceful behavior. We will describe this by saying that there are two types of animals: hawks (who are aggressive), and doves (who are peaceful). The different types of behavior are expressed when an animal invades the territory of another animal of the same species. A hawk will aggressively repel the invader. A dove, in contrast, will yield to the aggressor and be driven out of its territory. If one of the two animals is a hawk and the other a dove, the outcome of this struggle is that the hawk ends up in the territory, while the dove is driven out, exposed to predators and other dangers. If both animals are doves, one of them will end up leaving the territory. Suppose that each of them leaves the territory in that situation with a probability of $\frac{1}{2}$. If both animals are hawks, a fight ensues, during which both of them are injured, perhaps fatally, and at most one of them will remain in the territory and produce offspring. Figure 5.30 presents an example of a matrix describing the expected number of offspring of each type of animal in this situation.

Note that the game in Figure 5.30 is symmetric; that is, both players have the same set of strategies $S_1 = S_2$, and their payoff functions satisfy $u_1(s_1, s_2) = u_2(s_2, s_1)$ for each $s_1, s_2 \in S$. This is an example of a "single-species" population, i.e., a population comprised of only one species of animal, where each individual can exhibit one of several possible behaviors.

		Invader	
		Dove	Hawk
Defender	Dove	4, 4	2, 8
	Hawk	8, 2	1, 1

Figure 5.30 The expected number of offspring following an encounter between two individuals in the population

Our focus here is on the dynamic process that develops under conditions of many random encounters between individuals in the population, along with the appearance of random mutations. A mutation is an individual in the population characterized by a particular behavior: it may be of type dove, or type hawk. More generally, a mutation can be of type x $(0 \leq x \leq 1)$; that is, the individual[3] will behave as a dove with probability x, and as a hawk with probability $1 - x$.

The expected number of offspring of an individual who randomly encounters another individual in the population depends on both its type and the type of the individual it has encountered; to be more precise, the expected number depends on the probability y that the encountered individual is a

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[3] For convenience, we will use the same symbol x to stand both for the real number between 0 and 1 specifying the probability of being of type "dove," and for the lottery $[x(\text{dove}), (1 - x)(\text{hawk})]$.

dove (or the probability $1 - y$ that the encountered individual is a hawk). This probability depends on the composition of the population, that is, on how many individuals there are of a given type in the population, and whether those types are "pure" doves or hawks, or mixed types x. Every population composition determines a unique real number y ($0 \leq y \leq 1$), which is the probability that a randomly chosen individual in the population will behave as a dove (in its next encounter).

Suppose now that a mutation, before it is born, can "decide" what type it will be (dove, hawk, or x between 0 and 1). This "decision" and the subsequent interactions the mutation will have with the population can be described by the matrix in Figure 5.31.

		Population	
		Dove y	Hawk $1-y$
Mutation	Dove	4, 4	2, 8
	Hawk	8, 2	1, 1

Figure 5.31 The Mutation–Population game

In this matrix, the columns and the rows represent behavioral types. If we treat the matrix as a game, the column player is the "population," which is implementing a fixed mixed strategy $[y(\text{Dove}), (1-y)(\text{Hawk})]$; i.e., with probability y the column player will behave as a dove and with probability $1 - y$ he will behave as a hawk. The row player, who is the mutation, in contrast chooses its type.

The expected payoff of a mutation from a random encounter is $4y + 2(1 - y)$ if it is a dove, $8y + (1 - y)$ if it is a hawk, and $x(4y + 2(1 - y)) + (1 - x)(8y + (1 - y))$ if it is of type x. For example, if the population is comprised of 80% doves ($y = 0.8$) and 20% hawks, and a new mutation is called upon to choose its type when it is born, what "should" the mutation choose? If the mutation chooses to be born a dove ($x = 1$), its expected number of offspring is $0.8 \times 4 + 0.2 \times 2 = 3.6$, while if it chooses to be born a hawk ($x = 0$), its expected number of offspring is $0.8 \times 8 + 0.2 \times 1 = 6.6$. It is therefore to the mutation's advantage to be born a hawk. No mutation, of course, has the capacity to decide whether it will be a hawk or a dove, because these characteristics are either inherited, or the result of a random change in genetic composition. What happens in practice is that individuals who have the characteristics of a hawk will reproduce more than individuals who have the characteristics of a dove. Over the generations, the number of hawks will rise, and the ratio of doves to hawks will not be 80% : 20% (because the percentage of hawks will be increasing). A population in which the ratio of doves to hawks is 80% : 20% is therefore evolutionarily unstable. Similarly, if the population is comprised of 10% doves ($y = 0.1$) and 90% hawks, we have an evolutionarily unstable situation (because the percentage of doves will increase). It can be shown that only if the population is comprised of 20% doves and 80% hawks will the expected number of offspring of each type be equal. When $y^* = 0.2$

$$0.2 \times 4 + 0.8 \times 2 = 2.4 = 0.2 \times 8 + 0.8 \times 1. \tag{5.130}$$

We therefore have $u_1(x, y^*) = 2.4$ for all $x \in [0, 1]$. Note that $y^* = 0.2$ is the symmetric equilibrium of the game in Figure 5.31, even when the player represented by the "population" can choose any mixed strategy. In other words, $u_1(x, y^*) \leq u_1(y^*, y^*)$ for each x, and $u_2(y^*, x) \leq u_2(y^*, y^*)$ for each x (in fact, the expressions on both sides of the inequality sign in all these cases is 2.4). Can we conclude that when the distribution of the population corresponds to the symmetric Nash equilibrium of the associated game, the population will be evolutionarily stable? The following example shows that to attain evolutionary stability, we need to impose a stronger condition that takes into account encounters between two mutations. ◀

Example 5.49 Consider the situation shown in Figure 5.32, in which the payoffs in each encounter are different from the ones above.

		Population	
		Dove y	Hawk $1 - y$
Mutation	Dove	4, 4	2, 2
	Hawk	2, 2	2, 2

Figure 5.32 The payoff matrix in a symmetric game

This game has two Nash equilibria, (Dove, Dove) and (Hawk, Hawk). The former corresponds to a population comprised solely of doves, and the second to a population comprised solely of hawks. When the population is entirely doves ($y = 1$), the expected number of offspring of type dove is 4, and the expected number of offspring of type hawk is 2, and therefore hawk mutations disappear over the generations, and the dove population remains stable. When the population is entirely hawks ($y = 0$), the expected number of offspring, of either type, is 2. But in this case, as long as the percentage of doves born is greater than 0, it is to a mutation's advantage to be born a dove. This is because the expected number of offspring of a mutation that is born a hawk is 2, but the expected number of offspring of a mutation that is born a dove is $2(1 - \varepsilon) + 4\varepsilon = 2 + 2\varepsilon$, where ε is the percentage of doves in the population (including mutations). In other words, when there are random changes in the composition of the population, it is to a mutation's advantage to be born a dove, because its expected number of offspring will be slightly higher than the expected number of offspring of a hawk. After a large number of generations have passed, the doves will form a majority of the population. This shows that a population comprised solely of hawks is not evolutionarily stable.

What happens if the population is comprised of doves, but many mutations occur, and the percentage of hawks in the population becomes ε? By a calculation similar to the one above, the expected number of offspring of a hawk is 2, while the expected number of offspring of a dove is $4 - 2\varepsilon$. As long as $\varepsilon < 1$, the expected number of offspring of a dove will be greater than that of a hawk, and hence the percentage of hawks in the population will decrease. This shows that a population comprised entirely of doves is evolutionarily stable.

If the population is comprised solely of hawks, where can a dove mutation come from? Such a mutation can arise randomly, as the result of a genetic change that occurs in an individual in the population. In general, even when a particular type is entirely absent from a population, in order to check whether the population is evolutionarily stable it is necessary to check what would happen if the absent type were to appear "ab initio." ◀

We will limit our focus in this section to two-player symmetric games. We will also assume that payoffs are nonnegative, since the payoffs in these games represent the expected number of offspring, which cannot be a negative number. Examples 5.48 and 5.49 lead to the following definition of evolutionarily stable strategy.

Definition 5.50 *A mixed strategy x^* in a two-player symmetric game is an* evolutionarily stable strategy (ESS) *if for every mixed strategy x that differs from x^* there exists $\varepsilon_0 = \varepsilon_0(x) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$,*

$$(1 - \varepsilon)u_1(x, x^*) + \varepsilon u_1(x, x) < (1 - \varepsilon)u_1(x^*, x^*) + \varepsilon u_1(x^*, x). \tag{5.131}$$

The biological interpretation of this definition is as follows. Since mutations occur in nature on a regular basis, we are dealing with populations mostly composed of "normal" individuals, with a minority of mutations. We will interpret x^* as the distribution of types among the normal individuals. Consider a mutation making use of strategy x, and assume that the proportion of this mutation in the population is ε. Every individual of type x will encounter a normal individual of type x^* with probability $1 - \varepsilon$, receiving in that case the payoff $u_1(x, x^*)$, and will encounter a mutation of type x with probability ε, receiving in that case the payoff $u_1(x, x)$. Equation (5.131) therefore says that in a population in which the proportion of mutations is ε, the expected payoff of a mutation (the left-hand side of the equal sign in Equation (5.131)) is smaller than the expected payoff of a normal individual (the right-hand side of the equal sign in Equation (5.131)), and hence the proportion of mutations will decrease and eventually disappear over time, with the composition of the population returning to being mostly x^*. An "evolutionarily stable equilibrium" is therefore a mixed strategy of the column player that corresponds to a population that is immune to being overtaken by mutations.

In Example 5.49, Equation (5.131) holds for the dove strategy ($x^* = 1$) for every $\varepsilon < 1$ and it is therefore an evolutionarily stable strategy. In contrast, Equation (5.131) does not hold for the hawk strategy ($x^* = 0$), and the hawk strategy is therefore not evolutionarily stable. As we saw in that example, for each $x \neq 0$ (where x denotes the proportion of doves in the population),

$$(1-\varepsilon)u_1(x, x^*) + \varepsilon u_1(x, x) = 2 + 2(1-x)^2 > 2 = (1-\varepsilon)u_1(x^*, x^*) + \varepsilon u_1(x^*, x).$$

By continuity, Equation (5.131) holds as a weak inequality for $\varepsilon = 0$. From this we deduce that every evolutionarily stable strategy defines a symmetric Nash equilibrium in the game. In particular, the concept of an evolutionarily stable equilibrium constitutes a refinement of the concept of Nash equilibrium.

Theorem 5.51 *If x^* is an evolutionarily stable strategy in a two-player symmetric game, then (x^*, x^*) is a symmetric Nash equilibrium in the game.*

As Example 5.49 shows, the opposite direction does not hold: if (x^*, x^*) is a symmetric Nash equilibrium, x^* is not necessarily an evolutionarily stable strategy. In this example, the strategy vector (Hawk, Hawk) is a symmetric Nash equilibrium, but the Hawk strategy is not an evolutionarily stable strategy.

The next theorem characterizes evolutionarily stable strategies.

Theorem 5.52 *A strategy x^* is evolutionarily stable if and only if for each $x \neq x^*$ only one of the following two conditions obtains:*

$$u_1(x, x^*) < u_1(x^*, x^*), \tag{5.132}$$

or

$$u_1(x, x^*) = u_1(x^*, x^*) \quad \textit{and} \quad u_1(x, x) < u_1(x^*, x). \tag{5.133}$$

The first condition states that if a mutation deviates from x^*, it will lose in its encounters with the normal population. The second condition says that if the payoff a mutation receives from encountering a normal individual is equal to that received by a normal individual encountering a normal individual, that mutation will receive a smaller payoff when it encounters the same mutation than a normal individual would in encountering the mutation. In both cases the population of normal individuals will increase faster than the population of mutations.

Proof: We will first prove that if x^* is an evolutionarily stable strategy then for each $x \neq x^*$ one of the conditions (5.132) or (5.133) holds. From Theorem 5.51, (x^*, x^*) is a Nash equilibrium, and therefore $u_1(x, x^*) \leq u_1(x^*, x^*)$ for each $x \neq x^*$. If for a particular x neither of the conditions (5.132) or (5.133) holds, then $u_1(x, x^*) = u_1(x^*, x^*)$ and $u_1(x, x) \geq u_1(x^*, x)$, but then Equation (5.131) does not hold for this x for any $\varepsilon > 0$, contradicting the fact that x^* is an evolutionarily stable strategy. It follows that for each x at least one of the two conditions (5.132) or (5.133) obtains.

Suppose next that for any mixed strategy $x \neq x^*$, at least one of the two conditions (5.132) or (5.133) obtains. We will prove that x^* is an evolutionarily stable strategy. If condition (5.132) obtains, then Equation (5.131) obtains for all $\varepsilon < \frac{u_1(x^*,x^*)-u_1(x,x^*)}{4M}$, where M is the upper bound of the payoffs: $M = \max_{s_1 \in S_1} \max_{s_2 \in S_2} u_1(s_1, s_2)$ (verify!). If condition (5.133) obtains then Equation (5.131) obtains for all $\varepsilon \in (0, 1]$. It follows that Equation (5.131) obtains in both cases, and therefore x^* is an evolutionarily stable strategy. □

If condition (5.132) obtains, then for each $x \neq x^*$, the equilibrium (x^*, x^*) is called a *strict equilibrium*. The next corollary follows from Theorem 5.52.

Corollary 5.53 *In a symmetric game, if* (x^*, x^*) *is a strict symmetric equilibrium then* x^* *is an evolutionarily stable equilibrium.*

Indeed, if (x^*, x^*) is a strict symmetric equilibrium, then condition (5.132) holds for every $x \neq x^*$. Theorem 5.52 and Corollary 5.53 yield a method for finding evolutionarily stable strategies: find all symmetric equilibria in the game, and for each one of them, determine whether or not it is a strict equilibrium. Every strict symmetric equilibrium defines an evolutionarily stable strategy. For every Nash equilibrium that is not strict, check whether condition (5.133) obtains for each x different from x^* for which condition (5.132) does not obtain (hence necessarily $u_1(x, x^*) = u_1(x^*, x^*)$).

Example 5.48 (*Continued*) Recall that the payoff function in this example is as shown in Figure 5.33.

		Population	
		Dove	Hawk
Mutation	Dove	4, 4	2, 8
	Hawk	8, 2	1, 1

Figure 5.33 The expected number of offspring from encounters between two individuals in Example 5.48

The symmetric mixed equilibrium is $([\frac{1}{5}(\text{Dove}), \frac{4}{5}(\text{Hawk})], [\frac{1}{5}(\text{Dove}), \frac{4}{5}(\text{Hawk})])$. The proportion of doves at equilibrium is $x^* = \frac{1}{5}$. Denote by x the proportion of doves in a mutation. Since the equilibrium is completely mixed, each of the two pure strategies yields the same expected payoff, and therefore $u_1(x, x^*) = u_1(x^*, x^*)$ for all $x \neq x^*$. To check whether $[\frac{1}{5}(\text{Dove}), \frac{4}{5}(\text{Hawk})]$ is an evolutionarily stable strategy, we need to check whether condition (5.133) obtains; that is, we need to check whether $u_1(x, x) < u_1(x^*, x)$ for every $x \neq x^*$.

This inequality can be written as

$$4x^2 + 2x(1-x) + 8(1-x)x + (1-x)^2 < \tfrac{1}{5}4x + \tfrac{1}{5}2(1-x) + \tfrac{4}{5}8x + \tfrac{4}{5}(1-x),$$

which can be simplified to

$$(5x-1)^2 > 0, \tag{5.134}$$

and this inequality obtains for each x different from $\frac{1}{5}$. We have thus proved that $[\frac{1}{5}(\text{Dove}), \frac{4}{5}(\text{Hawk})]$ is an evolutionarily stable strategy.

This game has two additional asymmetric Nash equilibria: (Dove, Hawk) and (Hawk, Dove). These equilibria do not contribute to the search for evolutionarily stable equilibria, since Theorem 5.51 relates evolutionarily stable equilibria solely to symmetric equilibria. ◀

Example 5.54 Consider another version of the Hawk–Dove game, in which the payoffs are as shown in Figure 5.34. This game has three symmetric equilibria: two pure equilibria, (Dove, Dove), (Hawk, Hawk), and one mixed, $([\frac{1}{2}(\text{Dove}), \frac{1}{2}(\text{Hawk})], [\frac{1}{2}(\text{Dove}), \frac{1}{2}(\text{Hawk})])$.

The pure equilibria (Dove, Dove) and (Hawk, Hawk) are strict equilibria, and hence the two pure strategies Dove and Hawk are evolutionarily stable strategies (Corollary 5.53) (see Figure 5.34).

		Population	
		Dove	Hawk
Mutation	Dove	4, 4	1, 3
	Hawk	3, 1	2, 2

Figure 5.34 The expected number of offspring in encounters between two individuals in Example 5.54

The strategy $x^* = [\frac{1}{2}(\text{Dove}), \frac{1}{2}(\text{Hawk})]$ is not evolutionarily stable. To see this, denote $x = [1(\text{Dove}), 0(\text{Hawk})]$. Then $u_1(x^*, x^*) = 2\frac{1}{2} = u_1(x, x^*)$, and $u_1(x, x) = 4 > 2\frac{1}{2} = u_1(x^*, x)$. From Theorem 5.52 it follows that the strategy $[\frac{1}{2}(\text{Dove}), \frac{1}{2}(\text{Hawk})]$ is not evolutionarily stable.

We can conclude from this that the population would be stable against mutations if the population were comprised entirely of doves or entirely of hawks. Any other composition of the population would not be stable against mutations. In addition, if the percentage of doves is greater than 50%, doves will reproduce faster than hawks and take over the population. On the other hand, if the percentage of doves is less than 50%, doves will reproduce more slowly than hawks, and eventually disappear from the population. If the percentage of doves is exactly 50%, as a result of mutations or random changes in the population stemming from variability in the number of offspring, the percentage of doves will differ from 50% in one of the subsequent generations, and then one of the two types will take over the population.

Although in this example a population composed entirely of doves reproduces at twice the rate of a population composed entirely of hawks, both populations are evolutionarily stable. ◀

Since Nash's Theorem (Theorem 5.10, page 151) guarantees the existence of a Nash equilibrium, an interesting question arises: does an evolutionarily stable strategy always exist? The answer is negative. It may well happen that an evolutionary process has no evolutionarily stable strategies. The next example, which is similar to Rock, Paper, Scissors, is taken from Maynard Smith [1982].

Example 5.55 Consider the symmetric game in which each player has the three pure strategies appearing in Figure 5.35.

		Player II		
		Rock	Paper	Scissors
	Rock	$\frac{2}{3}, \frac{2}{3}$	0, 1	1, 0
Player I	Paper	1, 0	$\frac{2}{3}, \frac{2}{3}$	0, 1
	Scissors	0, 1	1, 0	$\frac{2}{3}, \frac{2}{3}$

Figure 5.35 A game without an evolutionarily stable strategy

This game has only one Nash equilibrium (Exercise 5.70), which is symmetric, in which the players play the mixed strategy:

$$x^* = \left[\tfrac{1}{3}(\text{Rock}), \tfrac{1}{3}(\text{Paper}), \tfrac{1}{3}(\text{Scissors})\right]. \tag{5.135}$$

The corresponding equilibrium payoff is $u_1(x^*, x^*) = \frac{5}{9}$. We want to show that there is no evolutionarily stable strategy in this game. Since every evolutionarily stable strategy defines a symmetric Nash equilibrium, to ascertain that there is no evolutionarily stable strategy it suffices to check that the strategy x^* is not an evolutionarily stable strategy. The strategy x^* is completely mixed, and hence it leads to an identical payoff against any pure strategy: $u_1(x, x^*) = u_1(x^*, x^*)$ for all $x \neq x^*$.

Consider a mutation $x = [1(\text{Rock}), 0(\text{Paper}), 0(\text{Scissors})]$; condition (5.133) does not obtain for this mutation. To see this, note that $u_1(x, x) = \frac{2}{3}$, while $u_1(x^*, x) = \frac{2}{9} + \frac{1}{3} = \frac{5}{9}$, and hence $u_1(x, x) > u_1(x^*, x)$.

It is interesting to note that a biological system in which the number of offspring is given by the table in Figure 5.35 and the initial distribution of the population is $\left[\frac{1}{3}(\text{Rock}), \frac{1}{3}(\text{Paper}), \frac{1}{3}(\text{Scissors})\right]$ will never attain population stability, and instead will endlessly cycle through population configurations (see Hofbauer and Sigmund [2003] or Zeeman [1980]). If, for example, through mutation the proportion of Rocks in the population were to increase slightly, their relative numbers would keep rising, up to a certain point. At that point, the proportion of Papers would rise, until that process too stopped, with the proportion of Scissors then rising. But at a certain point the rise in the relative numbers of Scissors would stop, with Rocks then increasing, and the cycle would repeat endlessly. Analyzing the evolution of such systems is accomplished using tools from the theory of dynamic processes. The interested reader is directed to Hofbauer and Sigmund [2003]. ◀

5.9 Remarks

Exercise 5.13 is based on Alon, Brightwell, Kierstead, Kostochka, and Winkler [2006]. Exercise 5.27 is based on a discovery due to Lloyd Shapley, which indicates that the equilibrium concept has disadvantages (in addition to its advantages). A generalization of this result appears in Shapley [1994]. The Inspector Game in Exercise 5.28 is a special case of an example in Maschler [1966b], in which r on-site inspection teams may be sent, and there are n possible dates on which the Partial Test Ban Treaty can be abrogated. For a generalization of this model to the case in which there are several detectors, with varying probabilities of detecting what they are looking for, see Maschler [1967]. The interested reader can find a survey of several alternative models for the Inspector Game in Avenhaus, von Stengel, and Zamir [2002]. Exercise 5.29 is based on Biran and Tauman [2007]. Exercise 5.33 is based on an example in Diekmann [1985]. Exercise 5.34 is a variation of a popular lottery game conducted in Sweden by the Talpa Corporation. Exercise 5.44 is taken from Lehrer, Solan, and Viossat [2007]. Exercise 5.51 is from Peleg [1969]. Parts of Exercise 5.60 are taken from Altman and Solan [2006].

The authors thank Uzi Motro for reading and commenting on Section 5.8 and for suggesting Exercise 5.71. We also thank Avi Shmida, who provided us with the presentation of Exercise 5.72.

5.10 Exercises

5.1 Prove that if S is a finite set then $\Delta(S)$ is a convex and compact set.

5.2 Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be two compact sets. Prove that the product set $A \times B \subseteq \mathbb{R}^{n+m}$ is a compact set.

5.3 Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be two convex sets. Prove that the product set $A \times B \subseteq \mathbb{R}^{n+m}$ is a convex set.

5.4 Show that every multilinear function $f : \Sigma \to \mathbb{R}$ is continuous.

5.5 Prove that for every player i the set of extreme points of player i's collection of mixed strategies is his set of pure strategies.

5.6 Prove that every two-player zero-sum game over the unit square with bilinear payoff functions is the extension to mixed strategies of a two-player game in which each player has two pure strategies.

5.7 Show that for every vector σ_{-i} of mixed strategies of the other players, player i has a best reply that is a pure strategy.

5.8 Answer the following questions for each of the following games, which are all two-player zero-sum games. As is usually the case in this book, Player I is the row player and Player II is the column player.

(a) Write out the mixed extension of each game.

(b) Find the value in mixed strategies, and all the optimal mixed strategies of each of the two players.

	L	*R*
T	−1	−4
B	−3	3

Game A

	L	*R*
T	5	8
B	5	1

Game B

	L	*R*
T	5	4
B	2	3

Game C

	L	*R*
T	4	2
B	2	9

Game D

	L	*R*
T	5	4
B	5	6

Game E

	L	*R*
T	7	7
B	3	10

Game F

5.9 Find the value of the game in mixed strategies and all the optimal strategies of both players in each of the following two-player zero-sum games, where Player I is the row player and Player II is the column player.

	L	*R*
T	2	6
M	5	5
B	7	4

Game A

	L	*R*
d	15	−8
c	10	−4
b	5	−2
a	−3	8

Game B

	a	*b*	*c*	*d*
T	5	3	4	0
B	−3	2	−5	6

Game C

	L	*M*	*R*
T	6	4	3
B	3	7	9

Game D

	L	*M*	*R*
T	6	4	3
B	3	7	9

Game E

5.10 In each of the following items, find a two-player game in strategic form in which each player has two pure strategies, such that in the mixed extension of the game the payoff functions of the players are the specified functions. (Note that the games in parts (a) and (b) are zero-sum, but that the games in parts (c) and (d) are not zero-sum.)

(a) $U(x, y) = 5xy - 2x + 6y - 2$.

(b) $U(x, y) = -2xy + 4x - 7y$.

(c) $U_1(x, y) = 3xy - 4x + 5, \quad U_2(x, y) = 7xy + 7x - 8y + 12.$
(d) $U_1(x, y) = 3xy - 3x + 3y - 5, \quad U_2(x, y) = 7x - 8y + 12.$

5.11 For each of the graphs appearing in Figure 5.9 (page 157) find a two-player zero-sum game such that the graph of the functions $(U(x, s_{\mathrm{II}}))_{s_{\mathrm{II}} \in S_{\mathrm{II}}}$ is the same as the graph in the figure. For each of these games, compute the value in mixed strategies, and all the optimal strategies of Player I.

5.12 A (finite) square matrix $A = (a_{i,j})_{i,j}$ is called *anti-symmetric* if $a_{i,j} = -a_{j,i}$ for all i and j. Prove that if the payoff matrix of a two-player zero-sum game is anti-symmetric, then the value of the game in mixed strategies is 0. In addition, Player I's set of optimal strategies is identical to that of Player II, when we identify Player I's pure strategy given by row k with Player II's pure strategy given by column k.

5.13 Let $G = (V, E)$ be a directed graph, where V is a set of vertices, and E is a set of edges. A directed edge from vertex x to vertex y is denoted by (x, y). Suppose that the graph is complete, i.e., for every pair of edges $x, y \in V$, either $(x, y) \in E$ or $(y, x) \in E$, but not both. In particular, $(x, x) \in E$ for all $x \in E$. In this exercise, we will prove that there exists a distribution $q \in \Delta(V)$ satisfying

$$\sum_{\{y \in V \colon (y,x) \in E\}} q(y) \geq \tfrac{1}{2}, \quad \forall x \in V. \tag{5.136}$$

(a) Define a two-player zero-sum game in which the set of pure strategies of the two players is V, and the payoff function is defined as follows:

$$u(x, y) = \begin{cases} 0 & x = y, \\ 1 & x \neq y, (x, y) \in E, \\ -1 & x \neq y, (x, y) \notin E. \end{cases} \tag{5.137}$$

Prove that the payoff matrix of this game is an anti-symmetric matrix, and, using Exercise 5.12, deduce that its value in mixed strategies is 0.

(b) Show that every optimal strategy q of Player II in this game satisfies Equation (5.136).

5.14 A mixed strategy σ_i of player i is called *weakly dominated (by a mixed strategy)* if it is weakly dominated in the mixed extension of the game: there exists a mixed strategy $\widehat{\sigma}_i$ of player i satisfying

(a) For each strategy $s_{-i} \in S_{-i}$ of the other players:

$$U_i(\sigma_i, s_{-i}) \leq U_i(\widehat{\sigma}_i, s_{-i}). \tag{5.138}$$

(b) There exists a strategy $t_{-i} \in S_{-i}$ of the other players for which

$$U_i(\sigma_i, t_{-i}) < U_i(\widehat{\sigma}_i, t_{-i}). \tag{5.139}$$

Prove that the set of weakly dominated mixed strategies is a convex set.

5.15 Suppose that a mixed strategy σ_i of player i strictly dominates another of his mixed strategies, $\widehat{\sigma}_i$. Prove or disprove each of the following claims:

(a) Player i has a pure strategy $s_i \in S_i$ satisfying: (i) $\widehat{\sigma}_i(s_i) > 0$ and (ii) strategy s_i is not chosen by player i in any equilibrium.

(b) For each equilibrium $\sigma^* = (\sigma_i^*)_{i \in N}$ player i has a pure strategy $s_i \in S_i$ satisfying (a) $\widehat{\sigma}_i(s_i) > 0$ and (b) $\sigma_i^*(s_i) = 0$.

5.16 Suppose player i has a pure strategy s_i that is chosen with positive probability in each of his maxmin strategies. Prove that s_i is not weakly dominated by any other strategy (pure or mixed).

5.17 Suppose player i has a pure strategy s_i that is chosen with positive probability in one of his maxmin strategies. Is s_i chosen with positive probability in each of player i's maxmin strategies? Prove this claim, or provide a counterexample.

5.18 Suppose player i has a pure strategy s_i that is not weakly dominated by any of his other pure strategies. Is s_i chosen with positive probability in one of player i's maxmin strategies? Prove this claim, or provide a counterexample.

5.19 Let $(a_{i,j})_{1 \le i,j \le n}$ be nonnegative numbers satisfying $\sum_{j \neq i} a_{i,j} = a_{i,i}$ for all i. Julie and Sam are playing the following game. Julie writes down a natural number i, $1 \le i \le n$, on a slip of paper. Sam does not see the number that Julie has written. Sam then guesses what number Julie has chosen, and writes his guess, which is a natural number j, $1 \le j \le n$, on a slip of paper. The two players simultaneously show each other the numbers they have written down. If Sam has guessed correctly, Julie pays him $a_{i,i}$ dollars, where i is the number that Julie chose (and that Sam correctly guesses). If Sam was wrong in his guess ($i \neq j$), Sam pays Julie $a_{i,j}$ dollars.

Depict this game as a two-player zero-sum game in strategic form, and prove that the value in mixed strategies of the game is 0.

5.20 Consider the following two-player zero-sum game.

		Player II		
		L	*C*	*R*
	T	3	−3	0
Player I	*M*	2	6	4
	B	2	5	6

(a) Find a mixed strategy of Player I that guarantees him the same payoff against any pure strategy of Player II.

(b) Find a mixed strategy of Player II that guarantees him the same payoff against any pure strategy of Player I.

(c) Prove that the two strategies you found in (a) and (b) are the optimal strategies of the two players.

(d) Generalize this result: Suppose a two-player zero-sum game is represented by an $n \times m$ matrix.[4] Suppose each player has an *equalizing strategy*, meaning a strategy guaranteeing him the same payoff against any pure strategy his opponent may play. Prove that any equalizing strategy is an optimal strategy.

(e) Give an example of a two-player zero-sum game in which one of the players has an equalizing strategy that is not optimal. Why is this not a contradiction to (d)?

5.21 In the following payoff matrix of a two-person zero-sum game, no player has an optimal pure strategy.

		Player II	
		L	R
Player I	T	a	b
	B	c	d

What inequalities must the numbers a, b, c, d satisfy? Find the value in mixed strategies of this game.

5.22 Prove that in any n-person game, at Nash equilibrium, each player's payoff is greater than or equal to his maxmin value.

5.23 The goal of this exercise is to prove that in a two-player zero-sum game, each player's set of optimal strategies is a convex set. Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a two-player zero-sum game in which $N = \{\mathrm{I}, \mathrm{II}\}$. For each pair of mixed strategies $\sigma_{\mathrm{I}} = [p_{\mathrm{I}}^1(s_{\mathrm{I}}^1), \ldots, p_{\mathrm{I}}^{m_{\mathrm{I}}}(s_{\mathrm{I}}^{m_{\mathrm{I}}})]$ and $\widehat{\sigma}_{\mathrm{I}} = [\widehat{p}_{\mathrm{I}}^1(s_{\mathrm{I}}^1), \ldots, \widehat{p}_{\mathrm{I}}^{m_{\mathrm{I}}}(s_{\mathrm{I}}^{m_{\mathrm{I}}})]$, and each real number in the unit line interval $\alpha \in [0, 1]$, define a vector $q_{\mathrm{I}} = (q_{\mathrm{I}}^j)_{j=1}^{m_{\mathrm{I}}}$ as follows:

$$q_{\mathrm{I}}^j = \alpha p_{\mathrm{I}}^j + (1 - \alpha)\widehat{p}_{\mathrm{I}}^j. \tag{5.140}$$

(a) Prove that $q = (q_{\mathrm{I}}^j)_{j=1}^{m_{\mathrm{I}}}$ is a probability distribution.

(b) Define a mixed strategy τ_{I} of Player I as follows:

$$\tau_{\mathrm{I}} = \left[q_{\mathrm{I}}^1\left(s_{\mathrm{I}}^1\right), \ldots, q_{\mathrm{I}}^{m_{\mathrm{I}}}\left(s_{\mathrm{I}}^{m_{\mathrm{I}}}\right)\right]. \tag{5.141}$$

Prove that for every mixed strategy σ_{II} of Player II:

$$U(\tau_{\mathrm{I}}, \sigma_{\mathrm{II}}) = \alpha U(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}) + (1 - \alpha) U(\widehat{\sigma}_{\mathrm{I}}, \sigma_{\mathrm{II}}). \tag{5.142}$$

(c) We say that a strategy σ_{I} of Player I *guarantees* payoff v if $U(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}) \geq v$ for every strategy σ_{II} of Player II. Prove that if σ_{I} and $\widehat{\sigma}_{\mathrm{I}}$ guarantee Player I payoff v, then τ_{I} also guarantees Player I payoff v.

(d) Deduce that if σ_{I} and $\widehat{\sigma}_{\mathrm{I}}$ are optimal strategies of Player I, then τ_{I} is also an optimal strategy of Player I.

(e) Deduce that Player I's set of optimal strategies is a convex set in $\Delta(S_{\mathrm{I}})$.

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[4] This means that the matrix has n rows (pure strategies of Player I) and m columns (pure strategies of Player II).

5.24 The goal of this exercise is to prove that in a two-player zero-sum game, each player's set of optimal strategies, which we proved is a convex set in Exercise 5.23, is also a compact set. Let $(\sigma_{\mathrm{I}}^{k})_{k\in\mathbb{N}}$ be a sequence of optimal strategies of Player I, and for each $k \in \mathbb{N}$ denote $\sigma_{\mathrm{I}}^{k} = [p_{\mathrm{I}}^{k,1}(s_{\mathrm{I}}^{1}), \ldots, p_{\mathrm{I}}^{k,m_{\mathrm{I}}}(s_{\mathrm{I}}^{m_{\mathrm{I}}})]$. Suppose that for each $j = 1, 2, \ldots, m_{\mathrm{I}}$, the limit $p_{\mathrm{I}}^{*,j} = \lim_{k\to\infty} p_{\mathrm{I}}^{k,j}$ exists. Prove the following claims:

(a) The vector $(p_{\mathrm{I}}^{*,j})_{j=1}^{m_{\mathrm{I}}}$ is a probability distribution over S_{I}.
(b) Define a mixed strategy σ_{I}^{*} as follows:

$$\sigma_{\mathrm{I}}^{*} = \left[p_{\mathrm{I}}^{*,1}\left(s_{\mathrm{I}}^{1}\right), \ldots, p_{\mathrm{I}}^{*,m_{\mathrm{I}}}\left(s_{\mathrm{I}}^{m_{\mathrm{I}}}\right)\right]. \tag{5.143}$$

Prove that σ_{I}^{*} is also an optimal strategy of Player I.
(c) Deduce from this that Player I's set of optimal strategies is a compact subset of the set of mixed strategies $\Delta(S_{\mathrm{I}})$.

5.25 For each of the following games, where Player I is the row player and Player II is the column player:

(a) Write out the mixed extension of the game.
(b) Compute all the equilibria in mixed strategies.

	L	R
T	1, 1	4, 0
B	2, 10	3, 5

Game A

	L	R
T	1, 2	2, 2
B	0, 3	1, 1

Game B

	L	M	R
T	1, 1	0, 2	2, 0
B	0, 0	1, 0	−1, 3

Game C

5.26 For each of the following games, where Player I is the row player and Player II is the column player:

(a) Find all the equilibria in mixed strategies, and all the equilibrium payoffs.
(b) Find each player's maxmin strategy.
(c) What strategy would you advise each player to use in the game?

	L	R
T	5, 5	0, 8
B	8, 0	1, 1

Game A

	L	R
T	9, 5	10, 4
B	8, 4	15, 6

Game B

	L	R
T	5, 16	15, 8
B	16, 7	8, 15

Game C

	L	R
T	8, 3	10, 1
B	6, −6	3, 5

Game D

	L	R
T	4, 12	5, 10
B	3, 16	6, 22

Game E

	L	R
T	2, 2	3, 3
B	4, 0	2, −2

Game F

	L	R
T	15, 3	15, 10
B	15, 4	15, 7

Game G

5.27 Consider the two-player game in the figure below, in which each player has three pure strategies.

		Player II		
		L	*C*	*R*
Player I	*T*	0, 0	7, 6	6, 7
	M	6, 7	0, 0	7, 6
	B	7, 6	6, 7	0, 0

(a) Prove that $([\frac{1}{3}(T), \frac{1}{3}(M), \frac{1}{3}(B)]; [\frac{1}{3}(L), \frac{1}{3}(C), \frac{1}{3}(R)])$ is the game's unique equilibrium.

(b) Check that if Player I deviates to T, then Player II has a reply that leads both players to a higher payoff, relative to the equilibrium payoff. Why, then, will Player II not play that strategy?

5.28 **The Inspector Game** During the 1960s, within the framework of negotiations between the United States (US) and the Union of Soviet Socialist Republics (USSR) over nuclear arms limitations, a suggestion was raised that both countries commit to a moratorium on nuclear testing. One of the objections to this suggestion was the difficulty in supervising compliance with such a commitment. Detecting above-ground nuclear tests posed no problem, because it was easy to detect the radioactive fallout from a nuclear explosion conducted in the open. This was not true, however, with respect to underground tests, because it was difficult at the time to distinguish seismographically between an underground nuclear explosion and an earthquake. The US therefore suggested that in every case of suspicion that a nuclear test had been conducted, an inspection team be sent to perform on-site inspection. The USSR initially objected, regarding any inspection team sent by the US as a potential spy operation. At later stages in the negotiations, Soviet negotiators expressed readiness to accept three on-site inspections annually, while American negotiators demanded at least eight on-site inspections. The expected number of seismic events per year considered sufficiently strong to arouse suspicion was 300.

The model presented in this exercise assumes the following:

- The USSR can potentially conduct underground nuclear tests on one of two possible distinct dates, labeled A and B, where B is the later date.
- The USSR gains nothing from choosing one of these dates over the other for conducting an underground nuclear test, and the US loses nothing if one date is chosen over another.
- The USSR gains nothing from conducting nuclear tests on both of these dates over its utility from conducting a test on only one date, and the US loses nothing if tests are conducted on both dates over its utility from conducting a test on only one date.

- The US may send an inspection team on only one of the two dates, A or B, but not on both.
- The utilities of the two countries from the possible outcomes are:
  - If the Partial Test Ban Treaty (PTBT) is violated by the USSR and the US does not send an inspection team: the US receives 0 and the USSR receives 0.
  - If the PTBT is violated by the USSR and the US sends an inspection team: the US receives 1 and the USSR receives 1.
  - If the PTBT is not violated, the US receives α and the USSR receives β, where $\alpha > 1$ and $0 < \beta < 1$ (whether or not the US sends an inspection team).

Answer the following questions:

(a) Explain why the above conditions are imposed on the values of α and β.

(b) Plot, in the space of the utilities of the players, the convex hull of the points $(0, 1)$, $(1, 0)$, and (α, β). The convex hull includes all the results of all possible lotteries conducted on pairs of actions undertaken by the players.

(c) List the pure strategies available to each of the two countries.

(d) Write down the matrix of the game in which the pure strategies permitted to the US (the row player) are:
  - A: Send an inspection team on date A
  - B: Send an inspection team on date B

  and the pure strategies permitted to the USSR (the column player) are:
  - L: Conduct a nuclear test on date A
  - R: Do not conduct a nuclear test on date A. Conduct a nuclear test on date B, only if the US sent an inspection team on date A.

(e) Explain why the other pure strategies you wrote down in part (c) are either dominated by the strategies in paragraph (d), or equivalent to them.

(f) Show that the game you wrote down in paragraph (d) has only one equilibrium. Compute that equilibrium. Denote by $(v_{\mathrm{I}}^*, v_{\mathrm{II}}^*)$ the equilibrium payoff, and by $[x^*(A), (1-x^*)(B)]$ the equilibrium strategy of the US.

(g) Add to the graph you sketched in paragraph (b) the equilibrium payoff, and the payoff $U([x^*(A), (1-x^*)(B)], R)$ (where $U = (U_{\mathrm{I}}, U_{\mathrm{II}})$ is the vector of the utilities of the two players). Show that the point $U([x^*(A), (1-x^*)(B)], R)$ is located on the line segment connecting $(0, 1)$ with (α, β).

(h) Consider the following possible strategy of the US: play $[(x^* + \varepsilon)(A), (1 - x^* - \varepsilon)(B)]$, where $\varepsilon > 0$ is small, and commit to playing this mixed strategy.[5] Show that the best reply of the USSR to this mixed strategy is to play strategy R. What is the payoff to the two players from the strategy vector $([(x^* + \varepsilon)(A), (1 - x^* - \varepsilon)(B)], R)$? Which of the two countries gains from this, relative to the equilibrium payoff?

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[5] This model in effect extends the model of a strategic game by assuming that one of the players has the option to commit to implementing a particular strategy. One way of implementing this would be to conduct a public spin of a roulette wheel in the United Nations building, and to commit to letting the result of the roulette spin determine whether an inspection team will be sent: if the result indicates that the US should send an inspection team on date A, the USSR will be free to deny entry to a US inspection team on date B, without penalty.

(i) Prove that the USSR can guarantee itself a payoff of v_{II}^*, regardless of the mixed strategy used by the US, when it plays its maxmin strategy.

(j) Deduce from the last two paragraphs that, up to an order of ε, the US cannot expect to receive a payoff higher than the payoff it would receive from committing to play the strategy $[(x^* + \varepsilon)(A), (1 - x^* - \varepsilon)(B)]$, assuming that the USSR makes no errors in choosing its strategy.

5.29 Suppose Country A constructs facilities for the development of nuclear weapons. Country B sends a spy ring to Country A to ascertain whether it is developing nuclear weapons, and is considering bombing the new facilities. The spy ring sent by Country B is of quality α: if Country A is developing nuclear weapons, Country B's spy ring will correctly report this with probability α, and with probability $1 - \alpha$ it will report a false negative. If Country A is not developing nuclear weapons, Country B's spy ring will correctly report this with probability α, and with probability $1 - \alpha$ it will report a false positive. Country A must decide whether or not to develop nuclear weapons, and Country B, after receiving its spy reports, must decide whether or not to bomb Country A's new facilities. The payoffs to the two countries appear in the following table.

		Country B	
		Bomb	Don't Bomb
Country A	Don't Develop	$\frac{1}{2}, \frac{1}{2}$	$\frac{3}{4}$,1
	Develop	0, $\frac{3}{4}$	1,0

(a) Depict this situation as a strategic-form game. Are there any dominating strategies in the game?

(b) Verbally describe what it means to say that the quality of Country B's spy ring is $\alpha = \frac{1}{2}$. What if $\alpha = 1$?

(c) For each $\alpha \in [\frac{1}{2}, 1]$, find the game's set of equilibria.

(d) What is the set of equilibrium payoffs as a function of α? What is the α at which Country A's maximal equilibrium payoff is obtained? What is the α at which Country B's maximal equilibrium payoff is obtained?

(e) Assuming both countries play their equilibrium strategy, what is the probability that Country A will manage to develop nuclear weapons without being bombed?

5.30 Prove that in any two-player game,

$$\max_{\sigma_{\mathrm{I}} \in \Sigma_{\mathrm{I}}} \min_{\sigma_{\mathrm{II}} \in \Sigma_{\mathrm{II}}} u_{\mathrm{I}}(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}) = \max_{\sigma_{\mathrm{I}} \in \Sigma_{\mathrm{I}}} \min_{s_{\mathrm{II}} \in S_{\mathrm{II}}} u_{\mathrm{I}}(\sigma_{\mathrm{I}}, s_{\mathrm{II}}). \qquad (5.144)$$

That is, given a mixed strategy of Player I, Player II can guarantee that Player I will receive the minimal possible payoff by playing a pure strategy, without needing to resort to a mixed strategy.

5.31 Let σ_{-i} be a vector of mixed strategies of all players except for player i, in a strategic-form game. Let σ_i be a best reply of player i to σ_{-i}. The *support* of σ_i is the set of all pure strategies given positive probability in σ_i (see Equation (5.64) on page 165). Answer the following questions.

(a) Prove that for any pure strategy s_i of player i in the support of σ_i

$$U_i(s_i, \sigma_{-i}) = U(\sigma_i, \sigma_{-i}). \tag{5.145}$$

(b) Prove that for any mixed strategy $\widehat{\sigma}_i$ of player i whose support is contained in the support of σ_i

$$U_i(\widehat{\sigma}_i, \sigma_{-i}) = U(\sigma_i, \sigma_{-i}). \tag{5.146}$$

(c) Deduce that player i's set of best replies to every mixed strategy of the other players σ_{-i} is the convex hull of the pure strategies that give him a maximal payoff against σ_{-i}.

Recall that the *convex hull* of a set of points in a Euclidean space is the smallest convex set containing all of those points.

5.32 A game $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ is called *symmetric* if (a) each player has the same set of strategies: $S_i = S_j$ for each $i, j \in N$, and (b) the payoff functions satisfy

$$u_i(s_1, s_2, \ldots, s_n) = u_j(s_1, \ldots, s_{i-1}, s_j, s_{i+1}, \ldots, s_{j-1}, s_i, s_{j+1}, \ldots, s_n) \tag{5.147}$$

for any vector of pure strategies $s = (s_1, s_2, \ldots, s_n) \in S$ and for each pair of players i, j satisfying $i < j$.

Prove that in every symmetric game there exists a symmetric equilibrium in mixed strategies: an equilibrium $\sigma = (\sigma_i)_{i\in N}$ satisfying $\sigma_i = \sigma_j$ for each $i, j \in N$.

5.33 The Volunteer's Dilemma Ten people are arrested after committing a crime. The police lack sufficient resources to investigate the crime thoroughly. The chief investigator therefore presents the suspects with the following proposal: if at least one of them confesses, every suspect who has confessed will serve a one-year jail sentence, and all the rest will be released. If no one confesses to the crime, the police will continue their investigation, at the end of which each one of them will receive a ten-year jail sentence.

(a) Write down this situation as a strategic-form game, where the set of players is the set of people arrested, and the utility of each player (suspect) is 10 minus the number of years he spends in jail.

(b) Find all the equilibrium points in pure strategies. What is the intuitive meaning of such an equilibrium, and under what conditions is it reasonable for such an equilibrium to be attained?

(c) Find a symmetric equilibrium in mixed strategies. What is the probability that at this equilibrium no one volunteers to confess?

(d) Suppose the number of suspects is not 10, but n. Find a symmetric equilibrium in mixed strategies. What is the limit, as n goes to infinity, of the probability that in a symmetric equilibrium no one volunteers ? What can we conclude from this analysis for the topic of volunteering in large groups?

5.34 Consider the following lottery game, with n participants competing for a prize worth \$$M$ ($M > 1$). Every player may purchase as many numbers as he wishes in the range $\{1, 2, \ldots, K\}$, at a cost of \$1 per number. The set of all the numbers that have been purchased by only one of the players is then identified, and the winning number is the smallest number in that set. The (necessarily only) player who purchased that number is the lottery winner, receiving the full prize. If no number is purchased by only one player, no player receives a prize.

(a) Write down every player's set of pure strategies and payoff function.
(b) Show that a symmetric equilibrium exists, i.e., there exists an equilibrium in which every player uses the same mixed strategy.
(c) For $p_1 \in (0, 1)$, consider the following mixed strategy $\sigma_i(p_1)$ of player i: with probability p_1 purchase only the number 1, and with probability $1 - p_1$ do not purchase any number. What conditions must M, n, and p_1 satisfy for the strategy vector in which player i plays strategy $\sigma_i(p_1)$ to be a symmetric equilibrium?
(d) Show that if at equilibrium there is a positive probability that player i will not purchase any number, then his expected payoff is 0.
(e) Show that if $M < n$, meaning that the number of participants is greater than the value of the prize at equilibrium, there is a positive probability that no player purchases a number. Conclude from this that at every symmetric equilibrium the expected payoff of every player is 0. (Hint: Show that if with probability 1 every player purchases at least one number, the expected number of natural numbers purchased by all the players together is greater than the value of the prize M, and hence there is a player whose expected payoff is negative.)

5.35 The set of equilibria is a subset of the product space $\Delta(S_1) \times \Delta(S_2) \times \cdots \times \Delta(S_n)$. Prove that it is a compact set. Is it also a convex set? If you answer yes, provide a proof; if you answer no, provide a counterexample.

5.36 Let $M_{n,m}$ be the space of matrices of order $n \times m$ representing two-player zero-sum games in which Player I has n pure strategies and Player II has m pure strategies. Prove that the function that associates with every matrix $A = (a_{ij}) \in M_{n,m}$ the value in mixed strategies of the game that it represents is continuous in (a_{ij}).

Remark: The sequence of matrices $(A^k)_{k\in\mathbb{N}}$ in $M_{n,m}$, where $A^k = (a^k_{ij})$, converges to $A = (a_{ij})$, if

$$a_{ij} = \lim_{k\to\infty} a^k_{ij}, \quad \forall i, j. \tag{5.148}$$

5.37 Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times m$ matrices representing two-player zero-sum games in strategic form. Prove that the difference between the value of A and the value of B is less than or equal to

$$\max_{i=1}^{n} \max_{j=1}^{m} |a_{ij} - b_{ij}|. \tag{5.149}$$

5.38 Find matrices A and B of order $n \times m$ representing two-player zero-sum games, such that the value of the matrix $C := \frac{1}{2}A + \frac{1}{2}B$ is less than the value of A and less than the value of B.

5.39 Let $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ and $\widehat{\Gamma} = (N, (S_i)_{i\in N}, (\widehat{u}_i)_{i\in N})$ be two strategic-form games with the same sets of pure strategies. Denote the maximal difference between the payoff functions of the two games by

$$c = \max_{s\in S_1\times\cdots\times S_n} \max_{i\in N} |u_i(s) - \widehat{u}_i(s)|. \tag{5.150}$$

We say that the set of equilibria of G is *close to* the set of equilibria of $\widehat{G}$ if for every equilibrium x^* of G there is an equilibrium $\widehat{x}^*$ of $\widehat{G}$ such that

$$|u_i(x^*) - \widehat{u}_i(\widehat{x}^*)| \leq c, \quad \forall i \in N. \tag{5.151}$$

Find two games $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ and $\widehat{\Gamma} = (N, (S_i)_{i\in N}, (\widehat{u}_i)_{i\in N})$ such that the set of equilibria of G is not close to the set of equilibria of $\widehat{G}$. Can such a phenomenon exist in two-player zero-sum games? (See Exercise 5.37.)

5.40 Let $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ and $\widehat{\Gamma} = (N, (S_i)_{i\in N}, (\widehat{u}_i)_{i\in N})$ be two strategic-form games with the same sets of players, and the same sets of pure strategies such that $u_i(s) \geq \widehat{u}_i(s)$ for each strategy vector $s \in S$. Denote the multilinear extension of $\widehat{u}_i$ by $\widehat{U}_i$. Is it necessarily true that for each equilibrium σ of Γ there exists an equilibrium $\widehat{\sigma}$ of $\widehat{\Gamma}$ such that $U_i(\sigma) \geq \widehat{U}_i(\widehat{\sigma})$ for each player $i \in N$? In other words, when the payoffs increase, do the equilibrium payoffs also increase? Prove this claim, or find a counterexample.

5.41 Prove that in a two-player strategic-form game, the minmax value in mixed strategies of a player equals his maxmin value in mixed strategies.

5.42 Suppose that the following game has a unique equilibrium, given by a completely mixed strategy.

		Player II	
		L	R
Player I	T	a, b	e, f
	B	c, d	g, h

Answer the following questions:

(a) Prove that the payoff of each player at this equilibrium equals his maxmin value in mixed strategies.

(b) Compute the equilibria in mixed strategies and the maxmin strategies in mixed strategies of the two players. Did you find the same strategies in both cases?

5.43 Prove that the only equilibrium in the following three-player game, where Player I chooses a row (T or B), Player II chooses a column (L or R), and Player III chooses a matrix (W or E), is (T, L, W).

W

	L	R
T	$1, 1, 1$	$0, 1, 3$
B	$1, 3, 0$	$1, 0, 1$

E

	L	R
T	$3, 0, 1$	$1, 1, 0$
B	$0, 1, 1$	$0, 0, 0$

Guidance: First check whether there are equilibria in pure strategies. Then check whether there are equilibria in which two players play pure strategies, while the third plays a completely mixed strategy (meaning a strategy in which each one of his two pure strategies is chosen with positive probability). After that, check whether there are equilibria in which one player plays a pure strategy, and the other two play completely mixed strategies. Finally, check whether there are equilibria in which all the players play completely mixed strategies. Note the symmetry between the players; making use of the symmetry will reduce the amount of work you need to do.

5.44 In this exercise we will prove the following theorem:

Theorem 5.56 *A set $E \subseteq \mathbb{R}^2$ is the set of Nash equilibrium payoffs in a two-player game in strategic form if and only if E is the union of a finite number of rectangles of the form $[a, b] \times [c, d]$ (the rectangles are not necessarily disjoint from each other, and we do not rule out the possibility that in some of them $a = b$ and/or $c = d$).*

For every distribution x over a finite set S, the *support* of x, which is denoted $\operatorname{supp}(x)$, is the set of all elements of S that have positive probability under x:

$$\operatorname{supp}(x) := \{s \in S : x(s) > 0\}. \tag{5.152}$$

(a) Let (x_1, y_1) and (x_2, y_2) be two equilibria of a two-player strategic-form game with payoffs (a, c) and (b, d) satisfying $\operatorname{supp}(x_1) = \operatorname{supp}(x_2)$ and $\operatorname{supp}(y_1) = \operatorname{supp}(y_2)$. Prove that for every $0 \leq \alpha, \beta \leq 1$ the strategy vector $(\alpha x_1 + (1 - \alpha)x_2, \beta y_1 + (1 - \beta)y_2)$ is a Nash equilibrium with the same support, and with payoff $(\alpha a + (1 - \alpha)c, \beta b + (1 - \beta)d)$.

(b) Deduce that for any subset S'_{I} of Player I's pure strategies, and any subset S'_{II} of Player II's pure strategies, the set of Nash equilibria payoffs yielded by strategy vectors (x, y) satisfying $\operatorname{supp}(x) = S'_{\mathrm{I}}$ and $\operatorname{supp}(y) = S'_{\mathrm{II}}$ is a rectangle.

(c) Since the number of possible supports is finite, deduce that the set of equilibrium payoffs of every two-player game in strategic form is a union of a finite number of rectangles of the form $[a, b] \times [c, d]$.

(d) In this part, we will prove the converse of Theorem 5.56. Let K be a positive integer, and let $(a_k, b_k, c_k, d_k)_{k=1}^{K}$ be positive numbers satisfying $a_k \leq b_k$ and $c_k \leq d_k$ for all k. Define the set $A = \bigcup_{k=1}^{K}([a_k, b_k] \times [c_k, d_k])$, which is the union of a finite number of rectangles (if $a_k = b_k$ and/or $c_k = d_k$, the rectangle is degenerate). Prove that the set of equilibrium payoffs in the following game in strategic form in which each player has $2K$ actions is A.

a_1, b_1	c_1, b_1	$0, 0$	$0, 0$	$\cdots$	$0, b_1$	$0, b_1$
a_1, d_1	c_1, d_1	$0, 0$	$0, 0$	$\cdots$	$0, d_1$	$0, d_1$
$0, 0$	$0, 0$	a_2, b_2	c_2, b_2	$\cdots$	$0, b_2$	$0, b_2$
$0, 0$	$0, 0$	a_2, d_2	c_2, d_2	$\cdots$	$0, d_2$	$0, d_2$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$a_1, 0$	$c_1, 0$	$a_2, 0$	$c_2, 0$	$\cdots$	a_K, b_K	c_K, b_K
$a_1, 0$	$c_1, 0$	$a_2, 0$	$c_2, 0$	$\cdots$	a_K, d_K	c_K, d_K

5.45 In this exercise, we will show that Theorem 5.56 (page 206) only holds true in two-player games: when there are more than two players, the set of equilibrium payoffs is not necessarily a union of polytopes. Consider the following three-player game, in which Player I chooses a row (T or B), Player II chooses a column (L or R), and Player III chooses a matrix (W or E).

W

	L	R
T	$1, 0, 3$	$0, 0, 1$
B	$1, 1, 1$	$0, 1, 1$

E

	L	R
T	$0, 1, 4$	$0, 0, 0$
B	$1, 1, 0$	$1, 0, 0$

Show that the set of equilibria is

$$\left\{([x(T), (1-x)(B)], [y(L), (1-y)(R)], W) : 0 \leq x, y \leq 1, \ \ xy \leq \tfrac{1}{2}\right\}. \tag{5.153}$$

Deduce that the set of equilibrium payoffs is

$$\left\{(y, x, 1+2xy) \colon 0 \leq x, y \leq 1, \ \ xy \leq \tfrac{1}{2}\right\}. \tag{5.154}$$

and hence it is not the union of polytopes in $\mathbb{R}^3$.

Guidance: First show that at every equilibrium, Player III plays his pure strategy W with probability 1, by ascertaining what the best replies of Players I and II are if he does not do so, and what Player III's best reply is to these best replies.

5.46 Find all the equilibria in the following three-player game, in which Player I chooses a row (T or B), Player II chooses a column (L or R), and Player III chooses a matrix (W or E).

W

	L	*R*
T	0, 0, 0	1, 0, 0
B	0, 0, 1	0, 1, 0

E

	L	*R*
T	0, 1, 0	0, 0, 1
B	1, 0, 0	0, 0, 0

5.47 Tom, Dick, and Harry play the following game. At the first stage, Dick or Harry is chosen, each with probability $\frac{1}{2}$. If Dick has been chosen, he plays the Game A in Figure 5.36, with Tom as his opponent. If Harry has been chosen, he plays the Game B in Figure 5.36, with Tom as his opponent. Tom, however, does not know who his opponent is (and which of the two games is being played). The payoff to the player who is not chosen is 0.

Dick

		L	*R*
Tom	*T*	2, 5	0, 0
	B	0, 0	1, 1

Game A

Harry

		L	*R*
Tom	*T*	2, 5	0, 0
	B	0, 0	1, 1

Game B

Figure 5.36 The payoff matrices of the game in Exercise 5.47

Do the following:

(a) Draw the extensive form of the game.
(b) Write down strategic form of the game.
(c) Find two equilibria in pure strategies.
(d) Find an additional equilibrium in mixed strategies.

5.48 In this exercise, Tom, Dick, and Harry are in a situation similar to the one described in Exercise 5.47, but this time the payoff matrices are those shown in Figure 5.37.

Dick

		L	*R*
Tom	*T*	0, 0	3, −3
	B	1, −1	0, 0

Game A

Harry

		L	*R*
Tom	*T*	5, −5	0, 0
	B	0, 0	1, −1

Game B

Figure 5.37 The payoff matrices of the game in Exercise 5.48

(a) Depict the game in extensive form.
(b) Depict the game in strategic form.
(c) Find all the equilibria of this game.

5.49 Prove that in every two-player game on the unit square that is not zero sum, and in which the payoff functions of the two players are bilinear (see Section 4.14.2 on page 123), there exists an equilibrium in pure strategies.

5.50 In this exercise, we generalize Theorem 5.11 (page 151) to the case in which the set of pure strategies of one of the players is countable. Let Γ be a two-player zero-sum game in which Player I's set of pure strategies S_{I}, is finite, Player II's set of pure strategies $S_{\mathrm{II}} = \{1, 2, 3, \dots\}$ is a countable set, and the payoff function u is bounded.

Let Γ^n be a two-player zero-sum game in which Player I's set of pure strategies is S_{I}, Player II's set of pure strategies is $S^n_{\mathrm{II}} = \{1, 2, \dots, n\}$, and the payoff functions are identical to those of Γ. Let v^n be the value of the game Γ^n, and let $\sigma^n_{\mathrm{I}} \in \Delta(S_{\mathrm{I}})$ and $\sigma^n_{\mathrm{II}} \in \Delta(S_{\mathrm{II}})$ be the optimal strategies of the two players in this game, respectively.

(a) Prove that $(v^n)_{n\in\mathbb{N}}$ is a sequence of nonincreasing real numbers. Deduce that $v := \lim_{n\to\infty} v^n$ exists.

(b) Prove that each accumulation point σ_{I} of the sequence $(\sigma^n_{\mathrm{I}})_{n\in\mathbb{N}}$ satisfies[6]

$$\inf_{\sigma_{\mathrm{II}}\in\Delta(S_{\mathrm{II}})} U(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}) \geq v. \tag{5.155}$$

(c) Prove that for each $n \in \mathbb{N}$, the mixed strategy σ^n_{II} satisfies

$$\sup_{\sigma_{\mathrm{I}}\in\Delta(S_{\mathrm{I}})} U(\sigma_1, \sigma^n_{\mathrm{II}}) \leq v^n. \tag{5.156}$$

(d) Deduce that

$$\sup_{\sigma_{\mathrm{I}}\in\Delta(S_{\mathrm{I}})} \inf_{\sigma_{\mathrm{II}}\in\Delta(S_{\mathrm{II}})} U(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}) = \inf_{\sigma_{\mathrm{II}}\in\Delta(S_{\mathrm{II}})} \sup_{\sigma_{\mathrm{I}}\in\Delta(S_{\mathrm{I}})} U(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}) = v.$$

(e) Find an example of a game Γ in which the sequence $(\sigma^n_{\mathrm{II}})_{n\in\mathbb{N}}$ has no accumulation point.

(f) Show by a counterexample that (d) above does not necessarily hold when S_{I} is also countably infinite.

5.51 In this exercise we will present an example of a game with an infinite set of players that has no equilibrium in mixed strategies. Let $(N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a game in strategic form in which the set of players is the set of natural numbers $N = \{1, 2, 3, \dots\}$, each player $i \in N$ has two pure strategies $S_i = \{0, 1\}$, and player i's payoff function is

$$u_i(s_1, s_2, \dots) = \begin{cases} s_i & \text{if } \sum_{j\in N} s_j < \infty, \\ -s_i & \text{if } \sum_{j\in N} s_j = \infty. \end{cases} \tag{5.157}$$

(a) Prove that this game has no equilibrium in pure strategies.

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[6] Recall that σ_{I} is an *accumulation point* of a sequence $(\sigma^n_{\mathrm{I}})_{n\in\mathbb{N}}$ if there exists a subsequence $(\sigma^{n_k}_{\mathrm{I}})_{k\in\mathbb{N}}$ converging to σ^{I}.

(b) Using Kolmogorov's 0-1 Law,[7] prove that the game has no equilibrium in mixed strategies.

5.52 Let $f : \mathbb{R}^{n+m} \to \mathbb{R}$ be a homogeneous function, i.e., $f(cx, y) = cf(x, y) = f(x, cy)$, for every $x \in \mathbb{R}^n$, every $y \in \mathbb{R}^m$, and every $c \in \mathbb{R}$.

Prove that the value $\widehat{v}$ of the two-player zero-sum game $\widehat{G} = (\{\mathrm{I}, \mathrm{II}\}, \mathbb{R}^n_+, \mathbb{R}^m_+, f)$ exists;[8] that is, there exists $\widehat{v} \in \mathbb{R} \cup \{-\infty, \infty\}$ such that

$$\widehat{v} = \sup_{x \in \mathbb{R}^n_+} \inf_{y \in \mathbb{R}^m_+} f(x, y) = \inf_{y \in \mathbb{R}^m_+} \sup_{\mathbb{R}^n_+} f(x, y). \tag{5.158}$$

In addition, prove that the value of the game equals either 0, ∞, or $-\infty$.

Guidance: Consider first the game $G = (\{\mathrm{I}, \mathrm{II}\}, X, Y, f)$, where

$$X := \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i \leq 1, \quad x_i \geq 0 \quad \forall i = 1, 2, \ldots, n \right\}, \tag{5.159}$$

$$Y := \left\{ y \in \mathbb{R}^m : \sum_{j=1}^{m} x_j \leq 1, \quad y_j \geq 0 \quad \forall j = 1, 2, \ldots, m \right\}. \tag{5.160}$$

Show that the game G has a value v; then show that: if $v = 0$ then $\widehat{v} = 0$; if $v > 0$ then $\widehat{v} = \infty$; and if $v < 0$ then $\widehat{v} = -\infty$.

5.53 Show that every two-player constant-sum game is strategically equivalent to a zero-sum game. For the definition of strategic equivalence, see Definition 5.34 (page 174).

5.54 Prove that if $(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}})$ is the solution of the system of linear equations (5.70)–(5.79) (page 166), then $(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}})$ is a Nash equilibrium.

5.55 Suppose that the preferences of two players satisfy the von Neumann–Morgenstern axioms. Player I is indifferent between receiving \$600 with certainty and participating in a lottery in which he receives \$300 with probability $\frac{1}{4}$ and \$1,500 with probability $\frac{3}{4}$. He is also indifferent between receiving \$800 with certainty and participating in a lottery in which he receives \$600 with probability $\frac{1}{2}$ and \$1,500 with probability $\frac{1}{2}$.

Player II is indifferent between losing \$600 with certainty and participating in a lottery in which he loses \$300 with probability $\frac{1}{7}$ and \$800 with probability $\frac{6}{7}$. He is also indifferent between losing \$800 with certainty and participating in a lottery in which he loses \$300 with probability $\frac{1}{8}$ and \$1,500 with probability $\frac{7}{8}$. The players

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[7] Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of independent random numbers defined over a probability space $(\Omega, \mathcal{F}, p)$. An event A is called a *tail event* if it depends only on $(X_i)_{i \geq n}$, for each $n \in \mathbb{N}$. In other words, for any $n \in \mathbb{N}$, to ascertain whether $\omega \in A$ it suffices to know the values $(X_i(\omega))_{i \geq n}$, which means that we can ignore a finite number of the initial variables $X_1, X_2, \ldots, X_n$ (for any n). *Kolmogorov's* 0-1 *law* says that the probability of a tail event is either 0 or 1.

[8] For every natural number n the set $\mathbb{R}^n_+$ is the nonnegative quadrant of $\mathbb{R}^n$:

$$\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_i \geq 0, \quad \forall i = 1, 2, \ldots, n\}.$$

play the game whose payoff matrix is as follows, where the payoffs are dollars that Player II pays to Player I.

(a) Find linear utility functions for the two players representing the preference relations of the players over the possible outcomes.

The players play a game whose outcomes, in dollars paid by Player II to Player I, are given by the following matrix.

Player II

		L	M	R
Player I	T	$300	$800	$1,500
	B	$1,500	$600	$300

(b) Determine whether the game is zero sum.
(c) If you answered yes to the last question, find optimal strategies for each of the players. If not, find an equilibrium.

5.56 Which of the following games, where Player I is the row player and Player II is the column player, are strategically equivalent to two-player zero-sum games? For each game that is equivalent to a two-player zero-sum game, write explicitly the positive affine transformation that proves your answer.

	L	R
T	11, 2	5, 4
B	−7, 8	17, 0

Game A

	L	R
T	2, 7	4, 5
B	6, 3	−3, 12

Game B

	L	C	R
T	0, 12	5, 16	4, 22
B	8, 9	2, 10	7, 11

Game C

	L	C	R
T	11, 2	5, 4	−16, 11
B	−7, 8	17, 0	−1, 6

Game D

	L	C	R
T	9, 5	3, 7	−18, 14
B	−9, 11	15, 3	−4, 9

Game E

	L	C	R
T	9, 2	5, 6	12, −1
B	9, 8	1, 10	7, 4

Game F

5.57 (a) Find the value in mixed strategies and all the optimal strategies of each of the two players in the following two-player zero-sum game.

Player II

		L	R
Player I	T	12	8
	B	4	16

(b) Increase the utility of Player II by 18, to get

Player II

		L	R
Player I	T	$12, 6$	$8, 10$
	B	$4, 14$	$16, 2$

What are all the equilibria of this game? Justify your answer.

(c) Multiply the utility of the first player in the original game by 2, and add 3, to get the following game.

Player II

		L	R
Player I	T	$27, -12$	$19, -8$
	B	$11, -4$	$35, -16$

What are the equilibrium strategies and equilibrium payoffs in this game?

5.58 Prove Theorem 5.35 on page 175: let G and $\widehat{G}$ be two strategically equivalent strategic-form games. Every equilibrium in mixed strategies σ of G is an equilibrium in mixed strategies of $\widehat{G}$.

5.59 (a) Consider the following two-player game.

Player II

		L	R
Player I	T	$1, 0$	$-1, 1$
	B	$0, 1$	$0, 0$

Show that the only equilibrium in the game is $[\frac{1}{2}(T), \frac{1}{2}(B)], [\frac{1}{2}(L), \frac{1}{2}(R)]$.

(b) Consider next the two-player zero-sum game derived from the above game in which these payoffs are Player I's payoffs. Compute the value in mixed strategies of this game and all the optimal strategies of Player I.

(c) Suppose that Player I knows that Player II is implementing strategy $[\frac{1}{2}(L), \frac{1}{2}(R)]$, and he needs to decide whether to implement the mixed strategy $[\frac{1}{2}(T), \frac{1}{2}(B)]$, which is his part in the equilibrium, or whether to implement instead the pure strategy B, which guarantees him a payoff of 0. Explain in what sense the mixed strategy $[\frac{1}{2}(T), \frac{1}{2}(B)]$ is equivalent to the pure strategy B, from Player I's perspective.

5.60 A *strategic-form game with constraints* is a quintuple $(N, (S_i, u_i)_{i\in N}, c, \gamma)$ where N is the set of players, S_i is player i's set of pure strategies, $u_i : S \to \mathbb{R}$ is player i's payoff function, where $S = \times_{i\in N} S_i$, $c : S \to \mathbb{R}$ is a constraint function, and $\gamma \in \mathbb{R}$ is a bound. Extend c to mixed strategies in the following way:

$$C(\sigma) = \sum_{s\in S} \sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_n(s_n)c(s). \tag{5.161}$$

In a game with constraints, the vectors of mixed strategies that the players can play are limited to those vectors of mixed strategies satisfying the constraints. Formally, a vector of mixed strategies $\sigma = (\sigma_i)_{i\in N}$ is called *permissible* if $C(\sigma) \leq \gamma$. Games with constraints occur naturally when there is a resource whose use is limited.

(a) Consider the following two-player zero-sum game, with the payoffs and constraints appearing in the accompanying figure. The bound is $\gamma = 1$.

		Player II	
		L	R
Player I	T	0	1
	B	−1	0

Payoffs

		Player II	
		L	R
Player I	T	2	0
	B	0	0

Constraints

Compute

$$\max_{\sigma_{\mathrm{I}}\in\Sigma_{\mathrm{I}}} \min_{\{\sigma_{\mathrm{II}}\in\Sigma_{\mathrm{II}}\colon C(\sigma_{\mathrm{I}},\sigma_{\mathrm{II}})\leq\gamma\}} U(\sigma_{\mathrm{I}},\sigma_{\mathrm{II}})$$

and

$$\min_{\sigma_{\mathrm{II}}\in\Sigma_{\mathrm{II}}} \max_{\{\sigma_{\mathrm{I}}\in\Sigma_{\mathrm{I}}\colon C(\sigma_{\mathrm{I}},\sigma_{\mathrm{II}})\leq\gamma\}} U(\sigma_{\mathrm{I}},\sigma_{\mathrm{II}}).$$

(b) How many equilibria can you find in this game?

The following condition in games with constraints is called the Slater condition. For each player i and every vector of mixed strategies of the other players σ_{-i}

there exists a mixed strategy σ_i of player i such that $C(\sigma_i, \sigma_{-i}) < \gamma$ (note the strict inequality). The following items refer to games with constraints satisfying the Slater condition.

(c) Prove that in a two-player zero-sum game with constraints

$$\max_{\sigma_{\mathrm{I}} \in \Sigma_{\mathrm{I}}} \min_{\{\sigma_{\mathrm{II}} \in \Sigma_{\mathrm{II}} : C(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}) \leq \gamma\}} U(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}) \geq \min_{\sigma_{\mathrm{II}} \in \Sigma_{\mathrm{II}}} \max_{\{\sigma_{\mathrm{I}} \in \Sigma_{\mathrm{I}} : C(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}) \leq \gamma\}} U(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}).$$

Does this result contradict Theorem 5.40 on page 177? Explain.

(d) Go back to the n-player case. Using the compactness of Σ_i and Σ_{-i}, prove that for every player i,

$$\sup_{\sigma_{-i} \in \Sigma_{-i}} \inf_{\{\sigma_i \in \Sigma_i : C(\sigma_i, \sigma_{-i}) \leq \gamma\}} C(\sigma_i, \sigma_{-i}) < \gamma. \tag{5.162}$$

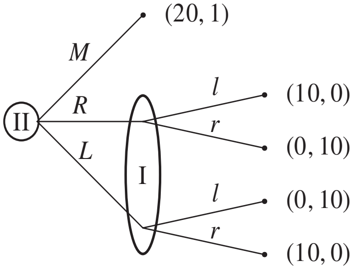
(e) Prove that for each strategy vector σ satisfying the constraints (i.e., $C(\sigma) \leq \gamma$), for each player i, and each sequence of strategy vectors $(\sigma_{-i}^k)_{k=1}^{\infty}$ converging to σ_{-i}, there exists a sequence $(\sigma_i^k)_{k=1}^{\infty}$ converging to σ_i such that $C(\sigma_i^k, \sigma_{-i}^k) \leq \gamma$ for every k.

(f) Using Kakutani's Fixed Point Theorem (Theorem 23.32 on page 939), show that in every strategic-form game with constraints there exists an equilibrium. In other words, show that there exists a permissible vector σ^* satisfying the condition that for each player $i \in N$ and each strategy σ_i^* of player i, if $(\sigma_i, \sigma_{-i}^*)$ is a permissible strategy vector, then $U_i(\sigma_i, \sigma_{-i}^*) \leq U_i(\sigma^*)$.

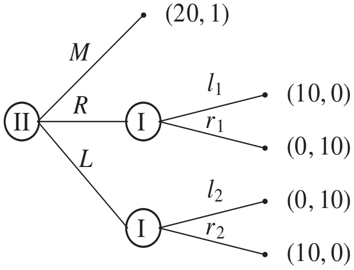
Hint: To prove part (c), denote by v the value of the game without constraints, and prove that the left-hand side of the inequality is greater than or equal to v, and the right-hand side of the inequality is less than or equal to v.

5.61 Prove Theorem 5.45 on page 183: if information is added to Player I in a two-player zero-sum game, the value of the game in mixed strategies does not decrease.

5.62 Compute the (unique) equilibrium payoff in each of the following two-player extensive-form games. Which player gains, and which player loses, from the addition of information to Player I, i.e., when moving from Game A to Game B? Is the result of adding information here identical to the result of adding information in Example 5.47 (page 185)? Why?



Game A



Game B

5.63 Consider the following two-player game composed of two stages. In the first stage, one of the two following matrices is chosen by a coin toss (with each matrix chosen with probability $\frac{1}{2}$). In the second stage, the two players play the strategic-form game whose payoff matrix is given by the matrix that has been chosen.

		Player II		
		L	C	R
Player I	T	$0, 0$	$1, -1$	$-1, 10$
	B	$-2, -2$	$-2, -2$	$-3, -12$

		Player II		
		L	C	R
Player I	T	$-1, 1$	$-2, -1$	$-2, -11$
	B	$1, \frac{1}{2}$	$-1, 0$	$-1, 10$

For each of the following cases, depict the game as an extensive-form game, and find the unique equilibrium:

(a) No player knows which matrix was chosen.
(b) Player I knows which matrix was chosen, but Player II does not know which matrix was chosen.

What effect does adding information to Player I have on the payoffs to the players at equilibrium?

5.64 Prove that in a two-player game, the maxmin value in mixed strategies of a player equals his minmax value in mixed strategies.

5.65 In this exercise we will prove von Neumann's Minmax Theorem (Exercise 5.11 on page 151), using the Duality Theorem from the theory of linear programming (see Section 23.3 on page 945 for a brief review of linear programming).

Let G be a two-player zero-sum game in which Player I has n pure strategies, Player II has m pure strategies, and the payoff matrix is A. Consider the following linear program, in the variables $y = (y_j)_{j=1}^m$, in which c is a real number, and $\vec{c}$ is an n-dimensional vector, all of whose coordinates equal c:

$$\begin{array}{ll} \text{Compute:} & Z_P := \min c, \\ \text{subject to:} & Ay^\top \leq \vec{c}, \\ & \sum_{j=1}^m y_j = 1, \\ & y \geq \vec{0}. \end{array}$$

(a) Write down the dual program.
(b) Show that the set of all y satisfying the constraints of the primal program is a compact set, and conclude that Z_P is finite.
(c) Show that the optimal solution to the primal program defines a mixed strategy for Player II that guarantees him an expected payoff of at most Z_P.
(d) Show that the optimal solution to the dual program defines a mixed strategy for Player I that guarantees an expected payoff of at least Z_D.
(e) Explain why the Duality Theorem is applicable here. Since the Duality Theorem implies that $Z_P = Z_D$, deduce that Z_P is the value of the game.

5.66 Prove the following claims for n-player extensive-form games:

(a) Adding information to one of the players does not increase the maxmin or the minmax value of the other players.
(b) Adding information to one of the players does not increase the minmax value of the other players.
(c) Adding information to one of the players may have no effect on his maxmin value.
(d) Adding information to one of the players may decrease the maxmin value of the other players.

5.67 Find all the equilibria of Game B in Figure 5.29 (page 185). What are the equilibria payoffs corresponding to these equilibria?

5.68 Find all the equilibrium points of the following games, and ascertain which of them defines an evolutionarily stable strategy.

		Population	
		Dove	Hawk
Mutation	Dove	2, 2	8, 3
	Hawk	3, 8	7, 7

Game A

		Population	
		Dove	Hawk
Mutation	Dove	2, 2	1, 3
	Hawk	3, 1	7, 7

Game B

		Population	
		Dove	Hawk
Mutation	Dove	2, 2	0, 1
	Hawk	1, 0	7, 7

Game C

		Population	
		Dove	Hawk
Mutation	Dove	1, 1	1, 1
	Hawk	1, 1	1, 1

Game D

		Population	
		Dove	Hawk
Mutation	Dove	2, 2	8, 8
	Hawk	8, 8	7, 7

Game E

		Population	
		Dove	Hawk
Mutation	Dove	1, 1	1, 1
	Hawk	1, 1	2, 2

Game F

5.69 Suppose that a symmetric two-player game, in which each player has two pure strategies and all payoffs are nonnegative, is given by the following figure.

		Player II	
		L	R
Player I	T	a, a	d, c
	B	c, d	b, b

What conditions on a, b, c, d guarantee the existence of an ESS?

5.70 Prove that the unique Nash equilibrium of Rock, Paper, Scissors (Example 4.3, on page 78) is

$$\left(\left[\tfrac{1}{3}(\text{Rock}), \tfrac{1}{3}(\text{Paper}), \tfrac{1}{3}(\text{Scissors})\right]; \left[\tfrac{1}{3}(\text{Rock}), \tfrac{1}{3}(\text{Paper}), \tfrac{1}{3}(\text{Scissors})\right]\right).$$

5.71 Suppose that the males and females of a particular animal species have two types of behavior: care for offspring, or abandonment of offspring. The expected number of offspring are presented in the following matrix.

		Mother	
		Care	Abandon
Father	Care	$V-c, V-c$	$\alpha V-c, \alpha V$
	Abandon	$\alpha V, \alpha V-c$	$0, 0$

Explanation: V is the expected number of surviving offspring if they are cared for by both parents. If only one parent cares for the offspring, the expected number of surviving offspring is reduced to αV, $0 < \alpha < 1$. In addition, a parent who cares for his or her offspring invests energy and time into that care, which reduces the number of surviving offspring he or she has by c (because he or she has fewer mating encounters with other animals).

Prove the following claims:

(a) If $V - c > \alpha V$ and $\alpha V - c > 0$ (which results in a relatively smaller investment, since $c < \alpha V$ and $c < (1 - \alpha)V$), then the only evolutionarily stable strategy is Care, meaning that both parents care for their offspring.
(b) If $V - c < \alpha V$ and $\alpha V - c < 0$ (which results in a high cost for caring for offspring), the only evolutionarily stable strategy is Abandon, and hence both parents abandon their offspring.
(c) If $\alpha < \frac{1}{2}$ (in this case $(1 - \alpha)V > \alpha V$, and investment in caring for offspring satisfies $(1 - \alpha)V > c > \alpha V$), there are two evolutionarily stable equilibria, Care and Abandon, showing that both Care and Abandon are evolutionarily stable strategies. Which equilibrium emerges in practice in the population depends on the initial conditions.

(d) If $\alpha > \frac{1}{2}$ (in this case $\alpha V > (1-\alpha)V$, and investment in caring for offspring satisfies $\alpha V > c > (1-\alpha)V$), the only evolutionarily stable equilibrium is the mixed strategy in which Care is chosen with probability $\frac{\alpha V - c}{(2\alpha-1)V}$.

Remark: The significance of $\alpha < \frac{1}{2}$ is that "two together are better than two separately." The significance of $\alpha > \frac{1}{2}$ is that "two together are worse than two separately."

5.72 A single male leopard can mate with all the female leopards on the savanna. Why, then, is every generation of leopards composed of 50% males and 50% females? Does this not constitute a waste of resources? Explain, using ideas presented in Section 5.8 (page 186), why the evolutionarily stable strategy is that at which the number of male leopards born equals the number of females born.[9]

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[9] In actual fact, the ratio of the males to females in most species is close to 50%, but not exactly 50%. We will not present here various explanations that have been suggested for this phenomenon.