12 Auctions

Chapter summary

In this chapter we present the theory of auctions, which is considered to be one of the most successful applications of game theory, and in particular of games with incomplete information. We mainly study symmetric auctions with independent private values and risk-neutral buyers. An auction is presented as a game with incomplete information and the main interest is in the (Bayesian) equilibrium of this game, that is, in the *bidding strategies* of the buyers and in the expected revenue of the seller. A hallmark of this theory is the *Revenue Equivalence Theorem*, which states that in any equilibrium of an auction method in which (a) the winner is the buyer with the highest valuation for the auctioned item, and (b) any buyer who assigns private value 0 to the auctioned item pays nothing, the expected revenue of the seller is independent of the auction method. This theorem implies that a wide range of auction methods yield the seller the same expected revenue. We also prove that the expected revenue to the seller increases if all buyers are risk averse, and it decreases if all buyers are risk seeking.

The theory is then extended to selling mechanisms. These are abstract mechanisms to sell items to buyers that include, e.g., post-auction bargaining between the seller and the buyers who placed the highest bids. We prove the *revelation principle* for selling mechanisms, which allows us to consider only a simple class of mechanisms, called *incentive-compatible direct selling mechanisms*. We then prove the Revenue Equivalence Theorem for selling mechanisms, and identify the selling mechanism that yields the seller the highest expected profit. This turns out to be a sealed-bid second-price auction with a reserve price.

Auctions and tenders are mechanisms for the buying and selling of objects by way of bids submitted by potential buyers, with the auctioned object sold to the highest bidder. Auctions have been known since antiquity. The earliest mention of auctions appears in the fifth century BCE, in the writings of Herodotus (Book One, Clio 194):

> When they arrive at Babylon in their voyage and have disposed of their cargo, they sell by auction the ribs of the boat and all the straw.

Herodotus also tells of a Babylonian custom of selling young women by public auction to men seeking wives (Book One, Clio 196). In 193 CE, the Roman emperor Pertinax was assassinated by his Praetorian Guard. In an attempt to win the support of the guard and be crowned the next emperor, Titus Flavius Sulpicianus offered to pay 20,000 sesterces to each member of the guard. Upon hearing of Sulpicianus' offer, Marcus Severus Didius

Julianus countered with an offer of 25,000 sesterces to each member of the Praetorian guard, and ascended to the throne; in effect, the Roman Empire had been auctioned to the highest bidder. Julianus did not live long to enjoy the prize he had won; within three months, three other generals laid claim to the crown, and Julianus was beheaded.

Auctions and tenders are ubiquitous nowadays. A very partial list of examples of objects sold in this way includes Treasury bills, mining rights, objects of fine art, bottles of wine, and repossessed houses. A major milestone in the history of auctions was achieved in the 1995 auctioning of the rights to radio-spectrum frequencies in the United States, which resulted in the federal government pocketing an unprecedented profit of 8.7 billion dollars.

The main reasons for preferring auctions and tenders to other sales mechanisms are the speed with which deals are concluded, the revelation of information achieved by these mechanisms, and the prevention of improper conduct on the part of sales agents and purchasers (an especially important reason when the seller is a public body).

As we will show in this chapter, an auction is a special case of a game with incomplete information. Many of the games we encounter in daily life are highly complex. Even when the theory assures us that these games have equilibria, in most cases the equilibria are hard to compute, and it is therefore difficult to predict what buyers will do, or to advise them on the way to play. This is also true, in general, with respect to auctions. However, as we will show, under certain assumptions, it is possible to compute the equilibrium strategies in auctions, to describe how the equilibria will change if the parameters of the game are changed (e.g., the utility functions of the buyers), and to compare the expected outcomes (for both buyers and seller) when the rules of the game (or the auction method) are changed. The theory developed in this chapter provides insights useful for participating in auctions and designing auctions.

The theory of auctions is one of the most successful application of game theory, and in particular of games with incomplete information. The theory is not simple, but it is very elegant. The combination of mathematical challenge with clear applicability makes the theory of auctions a central element of modern economic theory.

In the literature auctions are classified in several ways:

- **Open-bid or sealed-bid auction.** In an open-bid auction, the buyers hear or see each other in real time, as bids are made, and are able immediately to offer counter bids. In a sealed-bid auction, all the buyers submit their bids simultaneously, and no buyer knows the offers made by the other buyers.

  Art objects are usually sold in open-bid auctions. Most public auctions conducted on the Web, and many large state-run auctions, such as the auctioning of radio-spectrum frequencies, are also open-bid auctions. In contrast, tenders for government contracts, and auctions for the sale of assets taken into receivership after a corporate bankruptcy, are usually conducted in sealed-bid auctions.
- **Private value or common value.** A buyer's assessment of the worth of an object offered for sale is called a *value*. The literature on auction theory distinguishes between *private values* and *common values*. When the value of an object for a buyer is a private value, it is independent of the value as assessed by the other buyers. A private value is always known ahead of time to the buyer, with no uncertainty.

When the value is a common value, it is identical for all the buyers, but is unknown. This occurs, for example, in tenders for oil-drilling rights, where there is uncertainty regarding the amount of oil that can be extracted from the oil field, and in tenders for the real-estate development rights, where there is uncertainty regarding the potential demand for apartments, and the final price at which the apartments will be sold.

Most auctions share both characteristics to a certain extent: the value of any object, whether it is a valuable work of art or a drilling project, is never known with certainty. This unknown value is common among the buyers when measured in dollars and cents, but there is also a private component, determined by personal taste, financial resources and the future plans of the buyer. When the object offered for sale is, for example, a real-estate development or oil-drilling rights, the expected financial revenue that the project will yield is the common value component. When the object is a Treasury bond or shares in a company, the difference between the sale price and the purchase price is the common value component. This component is common to all the buyers, but is unknown to them, and each buyer may have different information (or different assessments) regarding this value. The financial abilities of the buyer, his future plans, and other possibilities available to him, should he fail to win the auction, also affect the value of an object to the buyer. These factors differ from one buyer to another, and what influences one buyer usually has no effect on another. This is the private component of the value of an object for a buyer.

The literature includes general auction models that use general valuation functions, and take into account the possibility that the private information of the buyers regarding the common but unknown value of an object may be interdependent (see Milgrom and Weber, [1982]).

- **Selling a single object or several objects.** Auctions differ with respect to the number of objects offered. Sometimes only one object is offered, such as a Chagall painting, a license to operate a television station for five years, or a letter from Marilyn Monroe to Elvis Presley. Sometimes, several copies of the same object are offered, such as batches of Treasury bonds, or shares in a company listed on the stock exchange. There are also cases in which several objects with different characteristics are offered at once. For example, in recent years some countries have conducted auctions of regional communication licenses (covering mobile telephone rights, broadcast radio rights, and so on), with licenses for different regions offered simultaneously.

In this chapter, we will focus on the case in which the buyers in an auction have independent private values, and only one object is offered for sale. This is the simplest case from the mathematical perspective. It is also historically the first case that was studied in the literature. Despite the simplicity of this model, the mathematical analysis is not trivial and the results are both elegant and applicable.

We close this introduction to the chapter with a remark on terminology. In previous chapters, the term "payoff" meant the expected payoff of a buyer. In this chapter, we will use the term "payment" to refer to the amount of money a buyer pays to the seller, and the term "profit" to denote the expected profit of the buyer, which is defined as the difference between the buyer's expected utility from receiving the object (the probability that he will win the auction times the utility he receives from winning the object) and the expected payment the buyer pays the seller.

12.1 Notation

The participants in the auctions will be called *buyers*. For every random variable X, denote its cumulative distribution function by F_X. That is,

$$F_X(c) = \mathbf{P}(X \leq c), \quad \forall c \in \mathbb{R}. \tag{12.1}$$

If X is a continuous random variable, denote its density function by f_X. In this case,

$$F_X(c) = \int_{-\infty}^{c} f_X(x)\mathrm{d}x, \quad \forall c \in \mathbb{R}. \tag{12.2}$$

12.2 Common auction methods

The following list details the most common auction methods:

1. **Open-bid ascending auction (English auction).** This is the most common public auction. It is characterized by an auctioneer who publicly declares the price of the object offered for sale. The opening price is low, and as long as there are at least two buyers willing to pay the declared price, the auctioneer raises the price (either in discrete jumps, or in a continuous manner using a clock). Each buyer raises a hand as long as he is willing to pay the last price that the auctioneer has declared. The auction ends when all hands except one have been lowered, and the object is sold to the last buyer whose hand is still raised, at the last price declared by the auctioneer. If the auction ends in a draw (i.e., the last two or more buyers whose hands were raised drop out of the auction at the same time), a previously agreed rule (such as tossing a coin) is employed to determine who wins the object, which is then sold to the winner at the price that was current when they lowered their hands.

   Web-based auctions and auctions of works of art (such as those conducted at Sotheby's and Christie's), typically use this method.
2. **Open-bid descending auction (Dutch auction).** A Dutch auction operates in the reverse direction of the English auction. In this method, the auctioneer begins by declaring a very high price, higher than any buyer could be expected to pay. As long as no buyer is willing to pay the last declared price, the auctioneer lowers the declared price (either in discrete jumps or in a continuous manner using a clock), up to the point at which at least one buyer is willing to pay the declared price and indicates his readiness by raising his hand or pressing a button to stop the clock. If the price drops below a previously declared minimum, the auction is stopped, and the object on offer is not sold. Similarly to the English auction, a previously agreed rule is employed to determine who wins the auction if two or more buyers stop the clock at the same time.

   The flower auction at the Aalsmeer Flower Exchange, near Amsterdam, is conducted using this method.

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3. **Sealed-bid first-price auction.** In this method, every buyer in the auction submits a sealed envelope containing the price he is willing to pay for the offered object. After all buyers have submitted their offers, the auctioneer opens the envelopes and reads the offers they contain. The buyer who has submitted the highest bid wins the offered object, and pays the price that he has bid. A previously agreed rule determines how to resolve draws.
4. **Sealed-bid second-price auction (Vickery auction).** The sealed-bid second-price auction method is similar to the first-price sealed-bid auction method, except that the winner of the auction, i.e., the buyer who submitted the highest bid, pays the *second*-highest price among the bid prices for the offered object. A previously agreed-upon rule determines the winner in case of a draw, with the winner in this case paying what he bid (which is, in the case of a draw, also the second-highest bid).

We mention here in passing several sealed-bid auction methods that, despite being important, will not be studied in detail in this book. In each of these methods, the winner of the auction is the buyer who has submitted the highest bid (if several buyers have submitted the same highest bid, the winner is determined by a previously agreed-upon rule).

1. A **sealed-bid auction with a reserve price** is a sealed-bid auction in which every bid that is lower than a minimal price, as determined by the seller, is disqualified. In a sealed-bid first-price auction with a reserve price, the winner of the auction pays the highest bid for the object; in a sealed-bid second-price auction with a reserve price, the winner pays either the second-highest bid for the object or the reserve price, whichever is higher.
2. An **auction with an entry fee** is a sealed-bid auction in which every buyer must pay an entry fee for participating in the auction, whether or not he wins the auction. The winner of the auction also pays for the object he has won, in addition to the entry fee. In a sealed-bid first-price auction with an entry fee, the winner of the auction pays the highest bid for the object; in a sealed-bid second-price auction with an entry fee, the winner pays the second-highest bid for the object. In an auction with an entry fee, a buyer's strategy is composed of two components: whether or not to participate in the auction (and pay the entry fee), and if so, how high a bid to submit.
3. An **all-pay auction** is a sealed- or open-bid auction in which every buyer pays the amount of money he has bid, whether or not he has won the object for sale. All-pay auctions are appropriate models for competitions, such as arms races between countries, or research and development competitions between companies racing to be the first to market with a new innovation. In these cases, all the buyers in the race, or competition, end up paying the full amounts of their investments, whether or not they win.

12.3 Definition of a sealed-bid auction with private values

In a sealed-bid auction, every buyer submits a bid, and the rules of the auction determine who wins the object for sale, and the amounts of money that the buyers (the winner, and perhaps also the other buyers) must pay. The winner is usually the highest bidder, but it is possible to define auctions in which the winner is not necessarily the highest bidder.

Definition 12.1 *A* sealed-bid auction *(with independent private values) is a vector* $(N, (\mathbb{V}_i, F_i)_{i \in N}, p, C)$*, where:*

- $N = \{1, 2, \dots, n\}$ *is the set of buyers.*
- $\mathbb{V}_i \subseteq \mathbb{R}$ *is the set of possible private values of buyer* i*, for each* $i \in N$*. Denote by* $\mathbb{V}^N := \mathbb{V}_1 \times \mathbb{V}_2 \times \cdots \times \mathbb{V}_n$ *the set of vectors of private value.*
- *For each buyer* $i \in N$ *there is a cumulative distribution function* F_i *over his set of private values* $\mathbb{V}_i$*.*
- $p : [0, \infty)^N \to \Delta(N)$ *is a function associating each vector of bids* $b \in [0, \infty)^N$ *with a distribution according to which the buyer who wins the auctioned object is identified.*[1]
- $C : N \times [0, \infty)^N \to \mathbb{R}^N$ *is a function determining the payment each buyer pays, for each vector of bids* $b \in [0, \infty)^N$*, depending on which buyer* $i_* \in N$ *is the winner.*

A sealed-bid auction is conducted as follows:

- The private value v_i of each buyer i is chosen randomly from the set $\mathbb{V}_i$, according to the cumulative distribution function F_i.
- Every buyer i learns his private value v_i, but not the private values of the other buyers.
- Every buyer i submits a bid $b_i \in [0, \infty)$ (depending on his private value v_i).
- The buyer who wins the auctioned object, i_*, is chosen according to the distribution $p(b_1, b_2, \dots, b_n)$; the probability that buyer i wins the object is $p_i(b_1, b_2, \dots, b_n)$.
- Every buyer i pays the sum $C_i(i_*; b_1, b_2, \dots, b_n)$.

For simplicity we will sometimes denote an auction by (p, C) instead of $(N, (F_i)_{i \in N}, p, C)$. Note several points relating to this definition:

- The private values of the buyers are independent, and therefore the vector of private values $(v_1, v_2, \dots, v_n)$ is drawn according to a product distribution, whose cumulative distribution function is $F^N := F_1 \times F_2 \times \cdots \times F_n$. A more general model would take into account the possibility of general joint distributions, thereby enabling the modeling of situations of interdependency between the private values of different buyers.
- In most of the auctions with which we are familiar, the winner of the auction is the highest bidder. In other words, if there is a buyer i such that $b_i > \max_{j \neq i} b_j$, then $p(b_1, b_2, \dots, b_n)$ is a degenerate distribution ascribing probability 1 to buyer i. If two (or more) buyers submit the same highest bid, a previously agreed-upon rule is implemented to determine the winner. That rule may be deterministic (for example, among the buyers who have submitted the highest bid, the winner is the buyer who submitted his bid first), or probabilistic (for example, the winner may be determined by the toss of a fair coin).
- In the most familiar payment functions, the winner pays either the highest, or the second-highest bid for the auctioned object. The payment function in the definition of a sealed-bid auction is more general, and enables the modeling of entry-fee favoritism (e.g., incentives for certain sectors), and all-pay auctions. It also enables the modeling

1 Recall that $\Delta(N) := \{x \in [0, 1]^N : \sum_{i \in N} x_i = 1\}$ is the set of all probability distributions over the set of buyers $N = \{1, 2, \dots, n\}$.

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of auctions with less-familiar rules, such as third-price auctions, in which the winner pays the third-highest bid.

The private value of buyer i is a random variable whose cumulative distribution function is F_i. This random value is denoted by V_i. A sealed-bid auction can be presented as a Harsanyi game with incomplete information (see Section 9.4) in the following way:

- The set of players is the set of buyers $N = \{1, 2, \ldots, n\}$.
- Player i's set of types is $\mathbb{V}_i$.
- The distribution over the set of type vectors is a product distribution with cumulative distribution function $F^N = F_1 \times F_2 \times \cdots \times F_n$.
- For each type vector $v \in \mathbb{V}^N$, the state of the world is the state game s_v, where buyer i's set of actions is $[0, \infty)$ and for every action vector $x \in [0, \infty)^N$, buyer i's profit is

$$p_i(x)v_i - \sum_{i_* \in N} p_{i_*}(x) C_i(i_*; x). \tag{12.3}$$

In words, if buyer i is the winner, he receives v_i (his private value for the object), and he pays $C_i(i_*; x)$ in any event (whether or not he is the winner), where i_* is the winning buyer.

A formal definition of an open-bid auction depends on the specific method used in conducting the auction, and may be very complex. For example, in the most common open-bid auction method, the English auction, a buyer's decision on whether or not to stop bidding at a certain moment depends on the identities of the other buyers, both those who have already quit the auction, and those who are still bidding, and the prices at which those who have already quit chose to stop bidding. We will not present a formal definition of an open-bid auction in this book.

Example 12.2 **Sealed-bid second-price auction** In a sealed-bid second-price auction, the winner is the highest bidder. If several buyers have submitted the highest bid then each of them has the same probability of winning: denote by $N(x) = \{i \in N : x_i = \max_{j \in N} x_j\}$ the set of buyers who have submitted the highest bid, and by i_* the buyer who wins the auctioned object. Then:

$$p_i(x) = \begin{cases} 0 & i \notin N(x), \\ \frac{1}{|N(x)|} & i \in N(x), \end{cases} \tag{12.4}$$

$$C_i(i_*; x) = \begin{cases} 0 & i \neq i_*, \\ \max_{j \neq i} x_j & i = i_*. \end{cases} \tag{12.5}$$

Note that if at least two buyers have submitted the same highest bid, the auctioned object is sold at this highest bid; that is, if $|N(x)| \geq 2$, then $C_{i_*}(i_*; x) = \max_{j \in N} x_j$. ◀

A *pure strategy* of buyer i in a sealed-bid auction is a measurable function[2]

$$\beta_i : [0, \infty) \to [0, \infty). \tag{12.6}$$

2 Recall that for every subset $X \subseteq \mathbb{R}$, a real-valued function $f : X \to [0, \infty)$ is *measurable* if for each number $y \in [0, \infty)$, the set $f^{-1}([0, y]) = \{x \in X : f(x) \leq y\}$ is a measurable set.

If buyer i uses pure strategy β_i, then when his type is v_i he bids $\beta_i(v_i)$. If the buyers use the strategy vector $\beta = (\beta_i)_{i \in N}$, buyer i's expected profit is

$$u_i(\beta) = \int_{\mathbb{V}^N} \left(p_i(\beta_1(x_1), \ldots, \beta_n(x_n)) v_i - \sum_{i_* \in N} p_{i_*}(x) C_i(i_*; \beta_1(x_1), \ldots, \beta_n(x_n)) \right) \mathrm{d}F^N(x). \tag{12.7}$$

The next theorem points out a connection between two of the auction methods described above.

Theorem 12.3 *The open-bid descending auction method is equivalent to the sealed-bid first-price auction method: both methods describe the same strategic-form game, with the same strategy sets and the same payoff functions.*

Proof: The set of (pure) strategies in a sealed-bid first-price auction, for each buyer i, is the set of all measurable functions $\beta_i : \mathbb{V}_i \to [0, \infty)$. This set is also the set of buyer i's strategies in an open descending auction. Indeed, a strategy of buyer i is a function detailing how he should play at each of his information sets. An open descending auction ends when the clock is stopped. Hence his only information consists of the current price. A strategy of buyer i then only needs to determine, for each of his possible private values, the announced price at which he will stop the clock (if no other buyer has stopped the clock before that price has been announced). In other words, every strategy of buyer i is a measurable function $\beta_i : [0, \infty) \to [0, \infty)$.

In both auctions, every strategy vector $\beta = (\beta_i)_{i \in N}$ leads to the same outcome in both auctions: in a sealed-bid first-price auction, the winning buyer is the one who submits the highest bid, $\max_{i \in N} \beta_i(v_i)$, and the price he pays for the auctioned object is his bid. In an open descending auction, the winning buyer is the one who stops the clock at the price $\max_{i \in N} \beta_i(v_i)$, and the price he pays for the auctioned object is that price. It follows that both types of auction correspond to the same strategic-form game. □

Remark 12.4 *Note that this equivalence obtains without any assumption on the information that each buyer has regarding the other buyers, their preferences, and their identities, or even the number of other buyers. Similarly, it does not depend on the assumption that the private values of the buyers are independent.* ◆

We next present additional relations between auction methods based on the concept of equilibrium.

12.4 Equilibrium

Having defined games, buyers, strategies, and payoffs, we next introduce the concept of equilibrium. In actual fact, since an auction is a game with incomplete information (because the type of each buyer, which is his private value, is known to him but not to the other buyers), the concept of equilibrium introduced here is that of Bayesian equilibrium

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(see Definition 9.49 on page 354), corresponding to the interim stage, when each buyer knows his private value.

Let $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ be a strategy vector. Denote by

$$\beta_{-i}(x_{-i}) := (\beta_j(x_j))_{j \neq i} \tag{12.8}$$

the vector of bids of the buyers other than i, given their types and their strategies. Denote by $u_i(\beta; v_i)$ buyer i's expected profit under the strategy vector β, when his private value is v_i,

$$\begin{aligned} u_i(\beta; v_i) :=& \int_{\mathbb{V}_{-i}} \Big(p_i(\beta_i(v_i), \beta_{-i}(x_{-i}))v_i \\ &- \sum_{i_* \in N} p_{i_*}(\beta_i(v_i), \beta_{-i}(x_{-i}))C_i(i_*; \beta_i(v_i), \beta_{-i}(x_{-i}))\Big)\, \mathrm{d}F_{-i}(x_{-i}). \end{aligned} \tag{12.9}$$

Here, $\mathbb{V}_{-i} := \times_{j \neq i} \mathbb{V}_j$ is the space of the vectors of the private values of all the buyers except for buyer i, and $F_{-i} := \times_{j \neq i} F_j$ is the cumulative distribution function of the multidimensional random variable $V_{-i} = (V_j)_{j \neq i}$.

Note that the expected profit $u_i(\beta; v_i)$ depends on β_i, buyer i's strategy, only via $\beta_i(v_i)$, the bid of buyer i with private value v_i. We denote by $u_i(b_i, \beta_{-i}; v_i)$ the expected profit of buyer i with private value v_i when he submits bid b_i and the other buyers use strategy vector β_{-i}.

Definition 12.5 *A strategy vector β^* is an* equilibrium *(or a* Bayesian equilibrium*) if for every buyer $i \in N$ and every private value $v_i \in \mathbb{V}_i$*

$$u_i(\beta^*; v_i) \geq u_i(b_i, \beta^*_{-i}; v_i), \quad \forall b_i \in [0, \infty). \tag{12.10}$$

In other words, β^* is an equilibrium if no buyer i with private value v_i can profit by deviating from his equilibrium bid $\beta^*_i(v_i)$ to another bid b_i.

Remark 12.6 *Analyzing auctions using mixed strategies is beyond the scope of this book. In the auctions covered in this chapter, the distribution of the private value of each buyer is continuous, and we will show that under appropriate assumptions, equilibria in pure strategies exist in these auctions. Note that if β^* is an equilibrium in pure strategies, then no buyer can increase his payoff by deviating to a mixed strategy. To see this, recall that a mixed strategy is a distribution over pure strategies. If the buyer could increase his payoff by deviating to a mixed strategy, then he could do the same by deviating to one of the pure strategies in the support of the mixed strategy. This shows that every equilibrium in pure strategies is also an equilibrium in mixed strategies. In general, however, it is possible for all the equilibria in an auction to be equilibria in completely mixed strategies (see for example Vickrey [1961]).* ♦

In Section 4.6 (page 91) we considered sealed-bid second-price auctions, and proved the following result (see Theorem 4.15 on page 92).

Theorem 12.7 *In a sealed-bid second-price auction, the strategy of buyer i in which he bids his private value weakly dominates all his other strategies.*

Remark 12.8 *As noted in Remark 12.4, Theorem 12.7 obtains under very few assumptions: we assume nothing regarding the behavior of the other buyers, the number of other buyers, or their identities. In other words, in a sealed-bid second-price auction, revealing your private value is a (weakly) dominant strategy. This is a great advantage that the sealed-bid second-price auction has over other auction methods: it incentivizes every buyer to reveal his true preferences, i.e., how much he truly is willing to pay for the object. From the seller's perspective, this is an advantage, because he need not be concerned that the buyers will conceal their preferences and act as if they value the object less than they really do. Another, secondary advantage for the seller is that if a buyer who submitted a high bid does not win the auction, the seller, knowing his true preferences, might be able to offer him a similar object.* ♦

An important consequence of Theorem 12.7 is:

Theorem 12.9 *In a sealed-bid second-price auction, the strategy vector in which every buyer's bid equals his private value is an equilibrium.*

Proof: As stated in Theorem 12.7, in a second-price auction, bidding the true value is a dominant strategy. Corollary 4.27 (page 105; see also Exercise 10.52 on page 433) states that a vector of dominant strategies is an equilibrium, and therefore this strategy vector is a Bayesian equilibrium. □

Although we have not defined a game corresponding to an open-bid ascending auctions, and in particular not defined a strategy in such an auction, it is possible to regard a behavior under which the buyer lowers his hand and no longer participates in the auction when the declared price reaches his private value as a "strategy" in this type of auctions. We will show that this is a dominant strategy for such a buyer. Since we have not presented the necessary definitions, the proof here is not a formal proof.

Theorem 12.10 *In an open ascending auction (English auction), the strategy of buyer i that calls on him to lower his hand when the declared price reaches his private value, weakly dominates all his other strategies.*

Proof: As long as the declared price is lower than buyer i's private value, he receives 0 with certainty if he quits the auction. On the other hand, if he continues to bid, he stands to receive a positive profit (and certainly cannot lose). When the declared price equals buyer i's private value, if he quits he receives 0 with certainty, but if he continues to bid he may win the auction and end up paying more for the object than he values it for. Here we are relying on the fact that buyer i knows his private value, and that this value is independent of the values of the other buyers, so that the information given by the timing that the other buyers choose for quitting the auction is irrelevant to his strategic considerations. □

Similarly to the proof of Theorem 12.9, and referring to Theorem 12.10, we can prove the following theorem.

Theorem 12.11 *In an open-bid ascending auction, the strategy vector in which every buyer lowers his hand when the declared price equals his private value is an equilibrium.*

Remark 12.12 *Note that in the dominant strategy equilibrium of the English auction established in Theorem 12.11, the winner of the object is the buyer with the highest private value and the selling price is the second highest private value. This is the same allocation and the same payment as in the dominant strategy equilibrium of the sealed-bid second-price auction established in Theorem 12.7.* ♦

Remark 12.13 *There are other equilibria in sealed-bid second-price auctions, in addition to the equilibrium in which every buyer's bid equals his private value. For example, if the private values of two buyers are independent and uniformly distributed over the interval* $[0, 1]$*, the strategy vector in which buyer* 1*'s bid is* $b_1 = 1$ *(for every private value* v_1*), and buyer* 2*'s bid is* $b_2 = 0$ *(for every private value* v_2*), is an equilibrium (Exercise 12.4)* ♦

12.5 The symmetric model with independent private values

In this section we will study models of sealed-bid auctions that satisfy the following assumptions:

(A1) Single object for sale: There is only one object offered for sale in the auction, and it is indivisible.
(A2) The seller is willing to sell the object at any nonnegative price.
(A3) There are n buyers, denoted by $1, 2, \ldots, n$.
(A4) Private values: All buyers have the same set of possible private values $\mathbb{V}$. This set can be a closed bounded interval $[0, \overline{v}]$ or the set of nonnegative numbers $[0, \infty)$. Every buyer knows his private value of the object. The random values $V_1, V_2, \ldots, V_n$ of the private values of the buyers are independent and identically distributed. Denote by F the common cumulative distribution function of the random variables V_i, $i = 1, 2, \ldots, n$. The support of this distribution is $\mathbb{V}$.
(A5) Continuity: For each i, the random variable V_i is continuous, and its density function, which we denote by f, is continuous and positive (this is the density function of the cumulative distribution function F of (A4)).
(A6) Risk neutrality: All the buyers are risk neutral, and therefore seek to maximize their expected profits.

We further assume that Assumptions (A1)–(A6) are common knowledge among the buyers (see Definition 9.17 on page 331). An auction model satisfying Assumptions (A1)–(A6) is called a *symmetric auction with independent private values*. This is the model studied in this section.

Since every buyer knows his own private value, any additional information, and in particular information regarding the private values of the other buyers, has no effect on his private value. That means that when buyer i's private value is v_i, then if he wins the auctioned object at price p, his profit is $v_i - p$, whether or not he knows the private values of the other buyers. In more general models in which buyers do not know with certainty the value of the auctioned object, the information a buyer has regarding the private values of the other buyers may be important to him, because it may be relevant to updating his own private value. Note that even if, after the auction is completed, the winner knows only that he has won, and not the details of the private values of the other buyers, he still obtains information about the other buyers' private values: he knows that the private values of the other buyers were sufficiently low for them not to submit bids higher than his bid.

The assumption that $V_1, V_2, \ldots, V_n$ are identically distributed is equivalent to the statement that prior to the random selection of the private values, the buyers are symmetric; each buyer, in his strategic considerations, assumes that all of the other buyers are similar to each other and to him.

12.5.1 Analyzing auctions: an example

Definition 12.14 *In a symmetric auction with independent private values, an equilibrium* $(\beta_1^*, \beta_2^*, \ldots, \beta_n^*)$ *is called a* symmetric equilibrium $\beta_i^* = \beta_j^*$ *for all* $1 \leq i, j \leq n$*; that is, all buyers implement the same strategy.*

When $\beta^* = (\beta_i^*)_{i \in N}$ is a symmetric equilibrium, we abuse notations and denote the common strategy also by β^*, that is, $\beta^* = \beta_i^*$ for every $i \in N$. Such a strategy is called a **symmetric equilibrium strategy**. We will denote by β_{-i}^* the vector of strategies in which all buyers except buyer i implement strategy β^*. We will sometimes denote the symmetric equilibrium strategy also by β_i^* when we want to focus on the strategy implemented by buyer i.

Example 12.15 **Two buyers with uniformly distributed private values**[3] Suppose that there are two buyers, and that V_i has uniform distribution over $[0, 1]$ for $i = 1, 2$ (and by Assumption (A4) V_1 and V_2 are independent). We will show that in a sealed-bid first-price auction the following strategy is a symmetric equilibrium:

$$\beta_i^*(v_i) = \frac{v_i}{2}, \qquad i = 1, 2. \tag{12.11}$$

This equilibrium calls on each buyer to submit a bid that is half of his private value. Suppose that buyer 2 implements this strategy. Then if buyer 1's private value is v_1, and her submitted bid is b_1, her expected profit is

$$u_1(b_1, \beta_2^*; v_1) = u_1\left(b_1, \frac{V_2}{2}; v_1\right) \tag{12.12}$$

$$= \mathbf{P}\left(b_1 > \frac{V_2}{2}\right)(v_1 - b_1) \tag{12.13}$$

$$= \mathbf{P}(2b_1 > V_2)(v_1 - b_1) \tag{12.14}$$

$$= \min\{2b_1, 1\}(v_1 - b_1). \tag{12.15}$$

This function is quadratic over the interval $b_1 \in [0, \frac{1}{2}]$ (attaining its maximum at $b_1 = \frac{v_1}{2}$), and linear, with a negative slope, when $b_1 \geq \frac{1}{2}$. The graph of the function $b_1 \mapsto u_1(b_1, \beta_2^*; v_1)$ is shown in Figure 12.1 for the case $v_1 \leq \frac{1}{2}$ and the case $v_1 > \frac{1}{2}$.

3 This example also appears on page 412 in Chapter 10.

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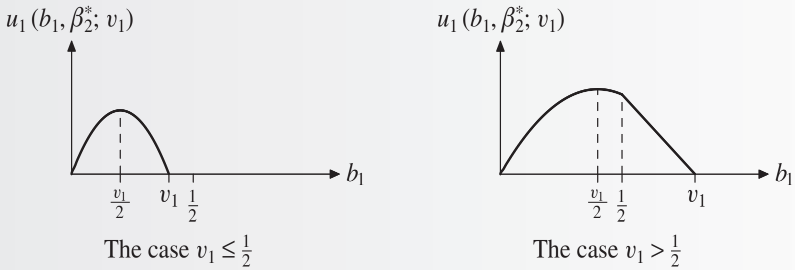

Figure 12.1 The payoff to buyer 1, as a function of b_1, when buyer 2 implements β_2^*

In both cases, the function attains its maximum at the point $b_1 = \frac{v_1}{2}$. This implies that $b_1^*(v_1) = \frac{v_1}{2}$ is the best response to β_2^*, which in turn means that the strategy vector $\beta^* = (\beta_1^*, \beta_2^*)$ is a symmetric equilibrium.

We note that from our results so far we can observe that different auction methods have different equilibria:

- In the sealed-bid first-price auction in Example 12.15, a symmetric equilibrium is given by $\beta_i^*(v_i) = \frac{v_i}{2}$.
- In a sealed-bid second-price auction, a symmetric equilibrium is given by $\beta_i^*(v_i) = v_i$ (Theorem 12.9, and Exercise 12.3).

Which auction method is preferable from the perspective of the seller? To answer this question, we need to calculate the seller's expected revenue in each of the two auction methods. The seller's expected revenue equals the expected sale price. At the equilibrium that we have calculated, the expected sale price is

$$\mathbf{E}\left[\max\left\{\frac{V_1}{2}, \frac{V_2}{2}\right\}\right] = \frac{1}{2}\mathbf{E}[\max\{V_1, V_2\}]. \tag{12.16}$$

Denote $Z := \max\{V_1, V_2\}$. Since V_1 and V_2 are independent, and have uniform distribution over $[0, 1]$, the cumulative distribution function of Z is

$$F_Z(z) = \mathbf{P}(Z \leq z) = \mathbf{P}(\max\{V_1, V_2\} \leq z) = \mathbf{P}(V_1 \leq z) \times \mathbf{P}(V_2 \leq z) = z^2. \tag{12.17}$$

It follows that the density function of Z is

$$f_Z(z) = \begin{cases} 2z & \text{if } 0 \leq z \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{12.18}$$

We deduce from this that the expected revenue is

$$\tfrac{1}{2}\mathbf{E}[Z] = \tfrac{1}{2}\int_0^1 z f_Z(z)\mathrm{d}z = \int_0^1 z^2 \mathrm{d}z = \tfrac{1}{3}. \tag{12.19}$$

The seller's expected revenue in a sealed-bid second-price auction is given by $\mathbf{E}[\min\{V_1, V_2\}]$. Note that

$$\min\{V_1, V_2\} + \max\{V_1, V_2\} = V_1 + V_2, \tag{12.20}$$

and hence

$$\mathbf{E}[\min\{V_1, V_2\}] + \mathbf{E}[\max\{V_1, V_2\}] = \mathbf{E}[V_1] + \mathbf{E}[V_2] = \tfrac{1}{2} + \tfrac{1}{2} = 1. \tag{12.21}$$

We have already calculated that $\mathbf{E}[\max\{V_1, V_2\}] = \mathbf{E}[Z] = \frac{2}{3}$, so $\mathbf{E}[\min\{V_1, V_2\}] = \frac{1}{3}$. In other words, the seller's expected revenue in a sealed-bid second-price auction is $\frac{1}{3}$. ◀

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Corollary 12.16 *In Example 12.15, in equilibrium, the expected revenue of the seller is the same, whether the auction method used is a sealed-bid first-price auction or second-price auction.*

This result is surprising at first sight, because one "would expect" that the seller would be better off selling the object at the price of the highest bid submitted, rather than the second-highest bid. However, buyers in a sealed-bid first-price auction will submit bids that are lower than those they would submit in a sealed-bid second-price auction, because in a sealed-bid first-price auction the winner pays what he bids, while in a sealed-bid second-price auction the winner pays less than his bid. The fact that these two opposing elements (on one hand, the sale price in a sealed-bid first-price auction is the highest bid, while on the other hand, bids are lower in a sealed-bid first-price auction) cancel each other out and lead to the same expected revenue, is a mathematical result that is far from self-evident.

The equivalence between sealed-bid first-price auctions and open-bid descending auctions (Theorem 12.3), and the equivalence between the equilibrium payments in sealed-bid second-price auctions and open-bid ascending auctions (Remark 12.12), lead to the following corollary.

Corollary 12.17 *In Example 12.15, all four auction methods presented, the sealed-bid first price auction, sealed-bid second price auction, open-bid ascending auction and open-bid descending auction yield the seller the same expected revenue in equilibrium.*

As we will see later, the equivalence of the expected profit in these four auction methods follows from a more general result (Theorem 12.23), called the Revenue Equivalence Theorem.

12.5.2 Equilibrium strategies

In this section, we will compute the equilibria of several auction methods.

Definition 12.18 *A symmetric equilibrium strategy β^* is* monotonically increasing *if the higher the private value, the higher the buyer's bid:*

$$v < v' \implies \beta^*(v) < \beta^*(v'), \quad \forall v, v' \in \mathbb{V}. \tag{12.22}$$

If β is a monotonically increasing symmetric equilibrium, the winner of the auction is the buyer with the highest private value. Since the distribution of V is continuous, the probability that two buyers have the same private value is 0. We proceed now to find monotonically increasing symmetric equilibria. Define

$$Y = \max\{V_2, V_3, \ldots, V_n\}. \tag{12.23}$$

This is a random variable, whose value equals the highest private value of buyers $2, \ldots, n$. From buyer 1's perspective, this is the highest private value of his competitors. In a monotonically increasing symmetric equilibrium, buyer 1 wins the auction if and only if $Y < V_1$. (As we previously stated, the event $Y = V_1$ has probability 0, so we ignore it, as it has no effect on the expected profit.)

The following theorem identifies a specific symmetric equilibrium, in symmetric auctions with independent private values.

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Theorem 12.19 *In a symmetric auction with independent private values the following strategy defines a symmetric equilibrium:*

$$\beta(v) := \mathbf{E}[Y \mid Y \leq v], \quad \forall v \in \mathbb{V} \setminus \{0\}, \tag{12.24}$$

and $\beta(0) := 0$.

Proof:
Step 1: β is a monotonically increasing function.
Recall that for every random variable X, and every pair of disjoint events A and B:

$$\mathbf{E}[X \mid A \cup B] = \mathbf{P}(A \mid A \cup B)\mathbf{E}[X \mid A] + \mathbf{P}(B \mid A \cup B)\mathbf{E}[X \mid B]. \tag{12.25}$$

Note that by the assumption that the density function f is positive (Assumption (A5)), it follows that Y is a continuous random variable with positive density function f_Y. Let v be an interior point of $\mathbb{V}_1$; that is, $v > 0$ and if $\mathbb{V} = [0, \overline{v}]$ is a bounded interval, then $v < \overline{v}$. For every $\delta > 0$ satisfying $v + \delta \in \mathbb{V}$,

$$\begin{aligned} \beta(v+\delta) &= \mathbf{E}[Y \mid Y \leq v+\delta] \\ &= \mathbf{P}(Y \leq v \mid Y \leq v+\delta) \times \mathbf{E}[Y \mid Y \leq v] \\ &\quad + \mathbf{P}(v < Y \leq v+\delta \mid Y \leq v+\delta) \times \mathbf{E}[Y \mid v < Y \leq v+\delta] \\ &= \frac{\mathbf{P}(Y \leq v)}{\mathbf{P}(Y \leq v+\delta)}\mathbf{E}[Y \mid Y \leq v] \\ &\quad + \frac{\mathbf{P}(v < Y \leq v+\delta)}{\mathbf{P}(Y \leq v+\delta)}\mathbf{E}[Y \mid v < Y \leq v+\delta]. \end{aligned} \tag{12.26}$$

Since the density function f_Y is positive, and since v is an interior point of $\mathbb{V}_1$,

$$\mathbf{E}[Y \mid Y \leq v] \leq v < \mathbf{E}[Y \mid v < Y \leq v+\delta]. \tag{12.27}$$

From Equations (12.26)–(12.26), we deduce that the $\beta(v+\delta)$ is the weighted average of two numbers, which, by Equation (12.27), satisfy the property that one is strictly greater than the other. Since the density function f_Y is positive, and since $v > 0$, the weights of both terms are positive. It follows that $\beta(v+\delta)$ is strictly greater than the minimal number among the two, which is, according to Equation (12.27), $\mathbf{E}[Y \mid Y \leq v]$, that is,

$$\beta(v+\delta) > \mathbf{E}[Y \mid Y \leq v] = \beta(v). \tag{12.28}$$

Therefore, β is an increasing function.

Step 2: β is a continuous function.
We will first show that β is continuous at $v = 0$. This obtains because for each $v \in \mathbb{V}$, $v > 0$,

$$0 \leq \beta(v) = \mathbf{E}[Y \mid Y \leq v] \leq v, \tag{12.29}$$

leading to $\lim_{v \to 0} \beta(v) = 0 = \beta(0)$. We next show that β is continuous at each interior point v of $\mathbb{V}$. By the definition of conditional expectation,

$$\beta(v) = \mathbf{E}[Y \mid Y \leq v] = \frac{\int_0^v y f_Y(y)\mathrm{d}y}{\mathbf{P}(Y \leq v)} = \frac{1}{F_Y(v)} \int_0^v y f_Y(y)\mathrm{d}y. \tag{12.30}$$

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Since the random variable Y is continuous, the cumulative distribution function F_Y is a continuous function. Since the density function f_Y is positive for all $v > 0$, $F_Y(v) > 0$; the denominator is not zero. It follows that β is the quotient of two continuous functions of v in which the function in the denominator is non-zero for $v > 0$, and hence it is a continuous function.

Note that from Equation (12.30) we can deduce, by integrating by parts, that

$$F_Y(v)\mathbf{E}[Y \mid Y \leq v] = \int_0^v y f_Y(y)\mathrm{d}y = vF_Y(v) - \int_0^v F_Y(y)\mathrm{d}y. \tag{12.31}$$

This equation will be useful later in the proof.

Step 3: β is a symmetric equilibrium strategy.
Suppose that buyers $2, 3, \ldots, n$ all implement strategy β. We will show that in that case, buyer 1's best reply is the same strategy β. Let v_1 be buyer 1's private value. If $\mathbb{V} = [0, \overline{v}]$ is a bounded interval, suppose that $v < \overline{v}$, which occurs with probability 1 (why?). Since the function β is monotonically strictly increasing (Step 1) and continuous (Step 2), it has a continuous inverse β^{-1}.

Buyer 1's expected profit, when he bids b_1, is

$$u_1(b_1, \beta_{-1}; v_1) = \mathbf{P}(\beta(Y) < b_1)(v_1 - b_1). \tag{12.32}$$

If buyer 1 bids $b_1 = 0$, with probability 1 he does not win the auction: since the density of Y is positive in the interval $\mathbb{V}$, the probability that $Y > 0$ is 1, and since β is monotonically increasing, with probability 1 another buyer bids more than 0. We deduce that $u_1(0, \beta_{-1}; v_1) = 0$.

If buyer 1 bids b_1 greater than or equal to his private value v_1, by Equation (12.32) his expected payoff is nonpositive:

$$u_1(b_1, \beta_{-1}; v_1) \leq 0, \quad \forall b_1 \geq v_1. \tag{12.33}$$

Since the density of V_i is positive in the set $\mathbb{V}$, the probability $\mathbf{P}(\beta(Y) < b_1)$ is positive for every $b_1 > 0$. It follows that in the domain $\mathbb{V} \setminus \{0\}$ the function $b_1 \mapsto u_1(b_1, \beta_{-1}; v_1)$ is the product of two positive functions and it is therefore a positive function. To summarize, we proved that $u_1(b_1, \beta_{-1}; v_1)$ is positive for $b_1 \in (0, v_1)$ and nonpositive for $b_1 \notin (0, v_1)$, and therefore the function $b_1 \mapsto u_1(b_1, \beta_{-1}; v_1)$, attains its maximum at an interior point of $[0, v_1]$. We next present the expected profit $u_1(b_1, \beta_{-1}; v_1)$ in a more useful form:

$$u_1(b_1, \beta_{-1}; v_1) = \mathbf{P}(\beta(Y) < b_1)(v_1 - b_1) \tag{12.34}$$
$$= \mathbf{P}(Y < \beta^{-1}(b_1))(v_1 - b_1) \tag{12.35}$$
$$= F_Y(\beta^{-1}(b_1)) \times (v_1 - \beta(\beta^{-1}(b_1))) \tag{12.36}$$
$$= F_Y(\beta^{-1}(b_1)) \times (v_1 - \mathbf{E}[Y \mid Y \leq \beta^{-1}(b_1)]). \tag{12.37}$$

Let b_1 be an interior point of the interval $\beta(\mathbb{V})$, which is the image of β, and denote $z_1 := \beta^{-1}(b_1)$. Then $b_1 = \beta(z_1)$, and hence

$$u_1(\beta(z_1), \beta_{-1}; v_1) = F_Y(z_1) \times (v_1 - \mathbf{E}[Y \mid Y \leq z_1]). \tag{12.38}$$

Denote the right-hand side of Equation (12.38) by $h(z_1)$:

$$h(z_1) := F_Y(z_1) \times (v_1 - \mathbf{E}[Y \mid Y \leq z_1]) \tag{12.39}$$
$$= F_Y(z_1)(v_1 - z_1) + F_Y(z_1)z_1 - F_Y(z_1)\mathbf{E}[Y \mid Y \leq z_1] \tag{12.40}$$
$$= F_Y(z_1)(v_1 - z_1) + \int_0^{z_1} F_Y(y)\mathrm{d}y, \tag{12.41}$$

where Equation (12.41) follows from Equation (12.31).

To find the point $b_1 \in \mathbb{V}$ at which the maximum of $u_1(b_1, \beta_{-1}; v_1)$ is attained, it suffices to find the point $z_1 \in \mathbb{V}$ at which the maximum of $h(z_1)$ is attained. To do so, differentiate h; the function h is differentiable over the interval $(0, v_1)$, and its derivative is

$$h'(z_1) = f_Y(z_1)(v_1 - z_1) - F_Y(z_1) + F_Y(z_1) = f_Y(z_1)(v_1 - z_1). \tag{12.42}$$

The derivative h' equals zero at a single point, $z_1 = v_1$, which is therefore the maximum of h. In other words, buyer 1's best reply, when his private value is v_1 and the other buyers implement strategy β, is $\beta^{-1}(b_1) = z_1 = v_1$, i.e., $b_1 = \beta(v_1)$. □

In summary, our results on equilibrium strategies in sealed-bid first-price and second-price auctions are as follows:

Corollary 12.20 *In a symmetric sealed-bid auction with independent private values:*

- $\beta(v) = \mathbf{E}[Y \mid Y \leq v]$ *is a symmetric equilibrium strategy in the sealed-bid first-price auction.*
- $\beta(v) = v$ *is a symmetric equilibrium strategy in the sealed-bid second-price auction.*

Example 12.15 (*Continued*) When there are two buyers, with private values uniformly distributed over $[0, 1]$, $\beta(v) = \mathbf{E}[Y \mid Y \leq v] = \frac{v}{2}$. This is the symmetric equilibrium strategy we found on page 472. ◀

We next compute the expected profits of the buyers and the seller in these two auction methods.

Theorem 12.21 *In the symmetric equilibria given by Corollary 12.20, the expected payment that a buyer with private value v makes for the object is $F_Y(v) \times \mathbf{E}[Y \mid Y \leq v]$, in both sealed-bid first-price and second-price auctions.*

Proof: At equilibrium in a sealed-bid second-price auction, a buyer with private value v submits a bid of v. He wins the auction with probability $F_Y(v)$, and the expected amount he pays is $\mathbf{E}[Y \mid Y \leq v]$. His expected payment is therefore $F_Y(v) \times \mathbf{E}[Y \mid Y \leq v]$, as claimed above.

At equilibrium in a sealed-bid first-price auction, a buyer with private value v submits a bid of $\mathbf{E}[Y \mid Y \leq v]$. He wins the auction with probability $F_Y(v)$, and pays what he bid. His expected payment for the object is therefore also $F_Y(v) \times \mathbf{E}[Y \mid Y \leq v]$. □

Corollary 12.22 *In a symmetric sealed-bid auction with independent private values, at the symmetric equilibrium, the expected revenue of a seller in both sealed-bid first-price* *and second-price auctions, is*

$$\pi = n \int_{\mathbb{V}} F_Y(v)\mathbf{E}[Y \mid Y \leq v] f(v)\mathrm{d}v. \tag{12.43}$$

Proof: The expected payment of a buyer with private value v is $F_Y(v)\mathbf{E}[Y \mid Y \leq v]$. It follows that the expected payment made by each buyer is $\int_{\mathbb{V}} F_Y(v)\mathbf{E}[Y \mid Y \leq v] f(v)\mathrm{d}v$. Since the seller's expected revenue is the sum of the expected payments of the n buyers, the result follows. □

12.5.3 The Revenue Equivalence Theorem

In the previous section, we saw that the symmetric and monotonically increasing equilibrium that we found in sealed-bid first-price and second-price auctions always yields the seller the same expected revenue. Is this coincidental, or is there a more general result implying this? As we shall see in the sequel, the Revenue Equivalence Theorem shows that there is indeed a more general result, ascertaining that the expected revenue of the seller is constant over a broad family of auction methods.

Recall that we denote by (p, C) a sealed-bid auction in which the winner is determined by the function p, and each buyer's payment is determined by the function C. Let $\beta : \mathbb{V} \to [0, \infty)$ be a monotonically increasing strategy. Denote by $e_i(v_i) = e(p, C, \beta; v_i)$ the expected payment that buyer i with private value v_i pays in auction method (p, C), when all the buyers implement strategy β:

$$\begin{aligned} &e_i(v_i) \\ &:= \int_{\mathbb{V}_{-i}} \sum_{i_* \in N} p_{i_*}(\beta(v_1), \beta(v_2), \dots, \beta(v_n)) C_i(i_*; \beta(v_1), \beta(v_2), \dots, \beta(v_n)) dF_{-i}(v_{-i}). \end{aligned} \tag{12.44}$$

Theorem 12.23 *Let β be a symmetric and monotonically increasing equilibrium in a sealed-bid symmetric auction with independent private values satisfying the following properties: (a) the winner of the auction is the buyer with the highest private value, and (b) the expected payment made by a buyer with private value* 0 *is* 0*. Then*

$$e_i(v_i) = F_Y(v_i)\mathbf{E}[Y \mid Y \leq v_i]. \tag{12.45}$$

Property (a) of the equilibrium is known as the "efficiency condition": an efficient auction is one in which, at equilibrium, the auctioned object is allocated to the buyer who most highly values it. Since the seller is willing to sell the auctioned object at any nonnegative price, and since the private values of the buyers are nonnegative, in an efficient auction the object is sold with probability 1. Under a symmetric and monotonically increasing equilibrium, the object is sold to the highest bidder.

The expression on the right-hand side of Equation (12.45) is independent of the auction methods, and depends solely on the distribution of Y, which is determined by the distribution of the private values of the buyers. Theorem 12.23 states therefore that the expected payment that a buyer with private value v makes is independent of the auction method, and depends only on the distribution of the private values of the buyers. It follows that if n risk-neutral buyers are asked whether they prefer to participate in a sealed-bid first-price auction, or a sealed-bid second-price auction, they have no reason to prefer one to the other.

By integrating Equation (12.45) over buyer types, we deduce that the seller's expected revenue is independent of the auction method:

Corollary 12.24 (The Revenue Equivalence Theorem) *In a symmetric sealed-bid auction with independent private values, let β be a symmetric and monotonically increasing equilibrium satisfying the following properties: (a) the winner is the buyer with the highest private value, and (b) the expected payment of each buyer with private value* 0 *is* 0*. Then the seller's expected revenue is*

$$\pi = n \int_{\mathbb{V}} e_i(v) f(v) \mathrm{d}v, \tag{12.46}$$

where

$$e_i(v) = F_Y(v) \mathbf{E}[Y \mid Y \leq v]. \tag{12.47}$$

Proof of Theorem 12.23: Since the private values are independent and identically distributed, the cumulative distribution function of Y is

$$F_Y = F^{n-1}. \tag{12.48}$$

Let β be a symmetric and monotonically increasing equilibrium strategy in a sealed-bid auction (p, C). Let $v_1 \in \mathbb{V}$ be a private value of buyer 1 (which is not 0 and is not $\overline{v}$ if $\mathbb{V} = [0, \overline{v}]$ is a bounded interval). If buyer 1 with private value v_1 deviates from the strategy and plays as if his private value is z_1, he wins only if z_1 is higher than the private values of the other buyers, and the probability of that occurring is $F_Y(z_1)$. His profit in this case is

$$u_1(\beta z_1, \beta_{-1}; v_1) = v_1 F_Y(z_1) - e_1(z_1). \tag{12.49}$$

Since β is an equilibrium, buyer 1's best reply is $z_1 = v_1$. In other words, the function $z_1 \mapsto u_1(\beta z_1, \beta_{-1}; v_1)$ attains its maximum at $z_1 = v_1$, which is an interior point of $\mathbb{V}$.

We next prove that the function e_1 is differentiable, and compute its derivative. Since the function $z_1 \mapsto u_1(\beta z_1, \beta_{-1}; v_1)$ attains its maximum at $z_1 = v_1$, for any pair of interior points v_1, z_1 in the interval $\mathbb{V}$ one has

$$v_1 F_Y(z_1) - e_1(z_1) = u_1(\beta z_1, \beta_{-1}; v_1) \leq u_1(\beta v_1, \beta_{-1}; v_1) = v_1 F_Y(v_1) - e_1(v_1). \tag{12.50}$$

By exchanging the roles of z_1 and v_1, we deduce that for every pair of interior points v_1, z_1 in the interval $\mathbb{V}$ one has

$$z_1 F_Y(v_1) - e_1(v_1) = u_1(\beta v_1, \beta_{-1}; z_1) \leq u_1(\beta z_1, \beta_{-1}; z_1) = z_1 F_Y(z_1) - e_1(z_1). \tag{12.51}$$

From Equations (12.50) and (12.51), by rearrangement, we have:

$$e_1(v_1) - e_1(z_1) \leq (F_Y(v_1) - F_Y(z_1)) v_1, \tag{12.52}$$

$$e_1(v_1) - e_1(z_1) \geq (F_Y(v_1) - F_Y(z_1)) z_1. \tag{12.53}$$

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For $z_1 \neq v_1$, dividing Equations (12.52) and (12.53) by $v_1 - z_1$, and taking the limit as z_1 goes to v_1, we get

$$\lim_{z_1 \to v_1} \frac{e_1(v_1) - e_1(z_1)}{v_1 - z_1} = v_1 f_Y(v_1), \quad \forall v_1 \in \mathbb{V}, v_1 \notin \{0, \overline{v}\}. \tag{12.54}$$

In particular, e_1 is a differentiable function and its derivative is $e_1'(v_1) = v_1 f_Y(v_1)$ for every $v_1 \in \mathbb{V}_1$. Note that the derivative e_1' is independent of the auction method. Since $e_1(0) = 0$, by integration, for every $v_1 \in \mathbb{V}$ (including the extreme points) we get

$$e_1(v_1) = e_1(0) + \int_0^{v_1} e_1'(y)\mathrm{d}y = \int_0^{v_1} y f_Y(y)\mathrm{d}y = F_Y(v_1)\mathbf{E}[Y \mid Y \leq v_1], \tag{12.55}$$

which is what we wanted to prove. □

We now show how to use the Revenue Equivalence Theorem to find symmetric equilibrium strategies in various auctions.

Theorem 12.25 *Let β be a symmetric, monotonically increasing equilibrium strategy, satisfying $\beta(0) = 0$ in a symmetric sealed-bid first-price auction with independent private values. Then*

$$\beta(v) = \mathbf{E}[Y \mid Y \leq v]. \tag{12.56}$$

This theorem complements Theorem 12.19, where we proved that $\beta(v) = \mathbf{E}[Y \mid Y \leq v]$ is a symmetric equilibrium strategy that is monotonically increasing and satisfies $\beta(0) = 0$. Theorem 12.25 shows that this is the unique such symmetric equilibrium in sealed-bid first-price auctions.

Proof: Since the function β is monotonic, a buyer with private value v wins the auction if and only if his private value is higher than the private values of all the other buyers. It follows that the probability that a buyer with value v wins the auction is $F_Y(v)$. If he wins, he pays his bid, meaning that he pays $\beta(v)$. The expected payment that the buyer makes is therefore

$$e(v) = F_Y(v)\beta(v). \tag{12.57}$$

Since β satisfies the conditions of Theorem 12.23 (note that the condition that $\beta(0) = 0$ guarantees that at this equilibrium, $e(0) = 0$), Theorem 12.23 implies that

$$e(v) = F_Y(v)\mathbf{E}[Y \mid Y \leq v]. \tag{12.58}$$

Since $F_Y(v) > 0$ for every $v > 0$, from Equations (12.57)–(12.58) we get

$$\beta(v) = \mathbf{E}[Y \mid Y \leq v], \tag{12.59}$$

which is what we wanted to show. □

The following theorem exhibits the equilibrium of an all-pay auction in which every buyer pays the amount of his bid, whether or not he wins the auctioned object (see page 465).

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Theorem 12.26 *Let β be a symmetric, monotonically increasing equilibrium strategy, satisfying $\beta(0) = 0$, in a symmetric sealed-bid all-pay auction with independent private values. Then*

$$\beta(v) = F_Y(v)\mathbf{E}[Y \mid Y \leq v]. \tag{12.60}$$

Proof: In a sealed-bid all-pay auction, every buyer pays his bid, in any event, and it follows that the payment that a buyer with private value v makes is $e(v) = \beta(v)$. Since the conditions of Theorem 12.23 are guaranteed by the monotonicity of β and the condition $\beta(0) = 0$, we deduce that $e(v) = F_Y(v)\mathbf{E}[Y \mid Y \leq v]$. It follows that $\beta(v) = F_Y(v)\mathbf{E}[Y \mid Y \leq v]$, which is what we needed to prove. □

Example 12.15 (*Continued*) **A sealed-bid first-price auction with two buyers** Consider a sealed-bid first-price auction with two buyers, where the private values of the buyers are independent and uniformly distributed over [0, 1]. We will compute the following:

- $e(v)$, the expected payment of a buyer with a private value v.
- e, the buyer's expected payment, before he knows his private value.
- $E = ne$, the seller's expected revenue.

For each $v \in [0, 1]$, one has $F_Y(v) = v$ and $f_Y(v) = 1$, and we have seen that $\beta(v) = \mathbf{E}[Y \mid Y \leq v] = \frac{v}{2}$. Therefore,

$$e(v) = F_Y(v)\mathbf{E}[Y \mid Y \leq v] = \frac{v^2}{2}, \tag{12.61}$$

$$e = \int_0^1 \frac{v^2}{2} \mathrm{d}v = \left.\frac{v^3}{6}\right]_0^1 = \tfrac{1}{6}, \tag{12.62}$$

$$\pi = 2\left(\tfrac{1}{6}\right) = \tfrac{1}{3}. \tag{12.63}$$

The seller's expected revenue π is $\frac{1}{3}$, as we computed directly on page 473. ◀

Example 12.27 **A sealed-bid first-price auction with an arbitrary number of buyers** Consider a symmetric sealed-bid first-price auction with $n \geq 2$ buyers. The private values of the buyers are independent and uniformly distributed over [0, 1]. Then $F_Y(v) = v^{n-1}$ and $f_Y(v) = (n-1)v^{n-2}$, for each $v \in [0, 1]$. By Theorem 12.26, the symmetric equilibrium strategy is

$$\beta(v) = \mathbf{E}[Y \mid Y \leq v] = \frac{\int_0^v x f_Y(x)\mathrm{d}x}{F_Y(v)} = \frac{\int_0^v (n-1)x^{n-1}\mathrm{d}x}{v^{n-1}} = \frac{n-1}{n}v. \tag{12.64}$$

It follows that the expected payment of a buyer with private value v is

$$e(v) = F_Y(v)\mathbf{E}[Y \mid Y \leq v] = \frac{n-1}{n}v^n. \tag{12.65}$$

The buyer's expected payment, before he knows his private value, is

$$e = \frac{n-1}{n}\int_0^1 v^n \mathrm{d}v = \frac{n-1}{n}\frac{1}{n+1}, \tag{12.66}$$

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and the seller's expected revenue is

$$\pi = ne = \frac{n-1}{n+1}. \tag{12.67}$$

This value converges to 1 as n increases to infinity. Since the seller's revenue equals the sale price, we deduce that the sale price converges to 1 as the number of buyers approaches infinity (explain intuitively why this should be expected). ◀

12.5.4 Entry fees

We have assumed, up to now, that participation in an auction is free, and that buyers therefore lose nothing in submitting bids. In this section, we explore, via examples, how adding entry fees for auctions may affect the strategies of buyers, and the seller's expected revenue.

Example 12.28 **Sealed-bid second-price auction with entry fee** Consider a sealed-bid second-price auction with entry fee $\lambda \in [0, 1]$. In such an auction, a buyer may decide not to participate; for example, he may decline to participate if his private value is lower than the entry free. If there is only one buyer submitting a bid, that buyer can win the auction by bidding 0.

As in second-price auctions without entry fees, when a buyer decides to participate in a second-price auction with an entry fee, his bid will be his private value of the auctioned object. To formulate this claim precisely, denote the set of actions of each buyer by $A = \mathbb{R}_+ \cup \{\text{“no”}\}$, where “no” means “don't participate in the auction” and $x \in \mathbb{R}_+$ means “participate in the auction, pay the entry fee λ and bid the price x”. A (pure) strategy of buyer i is a measurable function $\beta_i : \mathbb{V}_i \to A$. That is, when buyer i's private value is v_i he implements action $\beta_i(v_i)$.

Theorem 12.29 *In a sealed-bid second-price auction with entry fee, for every strategy β_i of buyer i the following strategy $\widehat{\beta}_i$ weakly dominates β_i, if $\widehat{\beta}_i \neq \beta_i$,*

$$\widehat{\beta}_i(v) = \begin{cases} \text{“no”} & \beta_i(v) = \text{“no”}, \\ v & \beta_i(v) = x. \end{cases} \tag{12.68}$$

The proof of the theorem is similar to the proof of Theorem 4.15 (page 92); the proof is left to the reader (Exercise 12.22). Theorem 12.29 implies that to find an equilibrium in a sealed-bid second-price auction with entry fees, we have to find for each buyer the set of private values for which he will participate in the auction.

Suppose that there are two buyers, and that the private values V_1 and V_2 are independent and uniformly distributed over [0, 1]. Since the buyer knows his own private value before he submits his bid, if his private value is low, he will not participate in the auction. There must therefore exist a threshold value v_0 such that no buyer with a private value below v_0 will participate in the auction.

Suppose that buyer 1's private value equals the threshold $V_1 = v_0$. If the equilibrium is monotonic, this buyer will win the auction if and only if buyer 2 does not participate in the auction, since if buyer 2 participates, with probability 1 his private value V_2 is greater than v_0, and therefore buyer 2's bid is greater than buyer 1's private value v_0. It follows that $\mathbf{P}(\text{winning the auction} \mid v_0) = v_0$. On the other hand, when the private value of buyer 1 equals the threshold value v_0, he is indifferent between participating and not participating. The buyer's expected profit if he participates is

$$\mathbf{P}(\text{winning the auction} \mid v_0) \times v_0 - \lambda = (v_0)^2 - \lambda, \tag{12.69}$$

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and his profit if he does not participate is 0, we deduce that $(v_0)^2 - \lambda = 0$, or $v_0 = \sqrt{\lambda}$. An equilibrium strategy in this game is therefore

$$\beta(v) = \begin{cases} \text{“Don't participate”} & \text{if } v < \sqrt{\lambda}, \\ v & \text{if } v \geq \sqrt{\lambda}. \end{cases} \tag{12.70}$$

The probability that each buyer will participate in the auction is $1 - \sqrt{\lambda}$.

To compute the seller's expected revenue, denote $V_{\max} = \max\{V_1, V_2\}$, and $V_{\min} = \min\{V_1, V_2\}$.

- If $V_{\min} \geq v_0$, both buyers participate in the auction, the seller receives 2λ as entry fee, and the sale price of the auctioned object is $V_{\min}$.
- If $V_{\max} < v_0$, no buyer will participate, and the seller's revenue is 0.
- If $V_{\min} < v_0 \leq V_{\max}$, only one buyer participates in the auction, the seller receives λ as entry fee, and the sale price of the auctioned object will be 0.

The seller's expected revenue, as a function of the entry fee λ, is therefore

$$\begin{aligned} \pi(\lambda) = {} & \mathbf{P}(V_{\min} \geq v_0)(2\lambda + \mathbf{E}[V_{\min} \mid V_{\min} \geq v_0]) \\ & + \mathbf{P}(V_{\min} < v_0 \leq V_{\max}) \times \lambda. \end{aligned} \tag{12.71}$$

Now,

$$F_{V_{\min}}(z) = \mathbf{P}(V_{\min} \leq z) = z + (1 - z)z = z(2 - z), \tag{12.72}$$

$$f_{V_{\min}}(z) = F'_{V_{\min}}(z) = 2(1 - z), \tag{12.73}$$

$$\mathbf{P}(V_{\min} \geq v_0) = (1 - v_0)^2, \tag{12.74}$$

$$\mathbf{P}(V_{\min} < v_0 \leq V_{\max}) = 2\mathbf{P}(V_1 < v_0 \leq V_2) = 2v_0(1 - v_0), \tag{12.75}$$

$$\begin{aligned} \mathbf{E}[V_{\min} \mid V_{\min} \geq v_0] &= \frac{1}{\mathbf{P}(V_{\min} \geq v_0)} \int_{v_0}^{1} v f_{V_{\min}}(v) \mathrm{d}v \\ &= \frac{1}{(1 - v_0)^2} \int_{v_0}^{1} 2v(1 - v) \mathrm{d}v = \frac{2v_0 + 1}{3}. \end{aligned} \tag{12.76}$$

By inserting the values of Equations (12.74)–(12.76) in (12.71), and using the fact that $v_0 = \sqrt{\lambda}$, we get

$$\pi(\lambda) = (1 - \sqrt{\lambda})^2 \left(2\lambda + \frac{2\sqrt{\lambda} + 1}{3}\right) + 2\sqrt{\lambda}(1 - \sqrt{\lambda}) \times \lambda \tag{12.77}$$

$$= \frac{(1 - \sqrt{\lambda})}{3}\left((1 - \sqrt{\lambda})(1 + 2\sqrt{\lambda} + 6\lambda) + (6\lambda\sqrt{\lambda})\right) \tag{12.78}$$

$$= \frac{(1 - \sqrt{\lambda})(4\lambda + \sqrt{\lambda} + 1)}{3} \tag{12.79}$$

This is a concave function of λ, satisfying $\pi(0) = \frac{1}{3}$ and $\pi(1) = 0$. Differentiating the function π, we have:

$$\pi'(\lambda) = [(1 - \sqrt{\lambda})(4\lambda + \sqrt{\lambda} + 1)]' = [1 + 3\lambda - 4\lambda^{\frac{3}{2}}]' = 3 - 6\sqrt{\lambda}. \tag{12.80}$$

The derivative π' vanishes at $\lambda^* = \frac{1}{4}$, where

$$\pi\left(\tfrac{1}{4}\right) = \tfrac{1}{3}\left(1 - \tfrac{1}{2}\right)\left(4 \cdot \tfrac{1}{4} + \tfrac{1}{2} + 1\right) = \tfrac{1}{3} \cdot \tfrac{1}{2} \cdot \tfrac{5}{2} = \tfrac{5}{12} > \tfrac{1}{3}. \tag{12.81}$$

Because $\pi(\frac{1}{4})$ is greater than $\pi(0)$ and greater than $\pi(1)$, the function π attains its maximum at the point $\lambda^* = \frac{1}{4}$, and hence the entry fee maximizing the seller's expected revenue is $\lambda^* = \frac{1}{4}$. We conclude that in this case, a sealed-bid second-price auction with entry fee $\frac{1}{4}$ yields the seller an expected revenue that is greater than what he can receive from a sealed-bid second-price auction without entry fees. ◀

Remark 12.30 *The fact that the seller's expected revenue from a sealed-bid second-price auction with entry fee is greater than his expected revenue from a sealed-bid second-price auction without entry fee does not contradict the Revenue Equivalence Theorem (Theorem 12.23), because the auction with entry fees does not satisfy the efficiency property: if both buyers have private values lower than the entry fee, the auctioned object is not sold, despite the fact that there is a buyer willing to pay for it.* ◆

12.6 The Envelope Theorem

Recall that, given the strategy of the other buyers β_{-i}, the profit of buyer i with private value v_i who submits a bid b_i, is $u_i(b_i, \beta_{-i}; v_i)$. The expected profit of the buyer is the difference between the product of the probability he will win the auction and his private value of the auctioned object, and his expected payment to the seller:

$$\begin{aligned} u_i(b_i, \beta_{-i}; v_i) = \mathbf{P}(&\text{buyer } i \text{ wins the auction } \mid b_i, \beta_{-i}) \times v_i \\ &- \mathbf{E}[\text{buyer } i\text{'s payment to the seller } \mid b_i, \beta_{-i}]. \end{aligned} \tag{12.82}$$

Both the probability of winning the auction and buyer i's payment depend on the bid b_i that he submits (and the strategies of the other buyers), but not on his private value. The

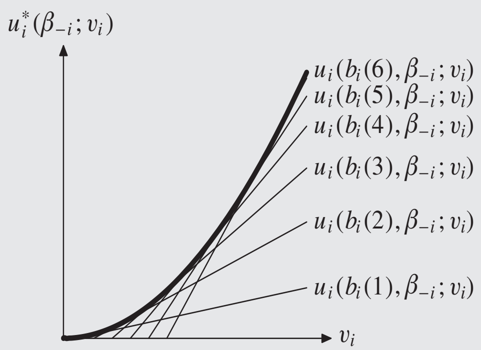

Figure 12.2 The function $u_i(b, \beta_{-i}; v_i)$, for different values of b_i and the upper envelope $u_i^*(\beta_{-i}; v_i)$ (the bold curve)

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function $u_i(b_i, \beta_{-i}; v_i)$ is therefore linear in v_i. At equilibrium, the buyer's bid maximizes his expected profit. The buyer's profit at equilibrium is therefore

$$u_i^*(\beta_{-i}; v_i) := \max_{b_i \geq 0} u_i(b_i, \beta_{-i}; v_i). \tag{12.83}$$

This function is called the *upper envelope*, because if we draw the function $v_i \mapsto u_i(b_i, \beta_{-i}; v_i)$ for every b_i, then $u_i^*(\beta_{-i}; v_i)$ is the upper envelope of this family of linear functions. Figure 12.2 shows some of these linear functions $v_i \mapsto u_i(b_i, \beta_{-i}; v_i)$ for various values of b_i, along with the upper envelope. Denote by $b_i^*(v_i)$ a value of b_i at which the maximum of $u_i(b_i, \beta_{-i}; v_i)$ for a given v_i is obtained:

$$u_i^*(\beta_{-i}; v_i) = u_i(b_i^*(v_i), \beta_{-i}; v_i). \tag{12.84}$$

That is, b_i^* is a best reply of the buyer to the strategy vector β_{-i}. Assuming that the function u_i is differentiable[4] and the function $v_i \mapsto b_i^*(v_i)$ is also differentiable, the function $v_i \mapsto u_i^*(\beta_{-i}; v_i)$ is differentiable, and its derivative is

$$\begin{aligned}\frac{\partial u_i^*}{\partial v_i}&(\beta_{-i}; v_i)\\ &= \frac{\partial u_i}{\partial v_i}(b_i, \beta_{-i}; v_i)_{|b_i = b_i^*(v_i)} + \frac{\partial u_i}{\partial b_i}(b_i, \beta_{-i}; v_i)_{|b_i = b_i^*(v_i)} \cdot \frac{\mathrm{d}b_i^*(v_i)}{\mathrm{d}v_i}.\end{aligned} \tag{12.85}$$

If for each private value v_i, $\frac{\partial u_i}{\partial b_i}(b_i, \beta_{-i}; v_i)_{|b_i = b_i^*(v_i)} = 0$ at the maximal point $b_i^*(v_i)$, then the second term is zero, leading to the following conclusion, called the Envelope Theorem, which has many applications in economics.

Theorem 12.31 (The Envelope Theorem) *Let b_i^* be a best response of buyer i to the strategy vector β_{-i} of the other buyers; i.e., b_i^* satisfies Equation (12.84). If the function $u_i(b_1, \beta_{-i}; v_i)$ is differentiable, and the function $v_i \mapsto b_i^*(v_i)$ is differentiable and satisfies $\frac{\partial u_i}{\partial b_i}(b_i, \beta_{-i}; v_i)_{|b_i = b_i^*(v_i)} = 0$ for every private value v_i, then*

$$\frac{\partial u_i^*}{\partial v_i}(\beta_{-i}; v_i) = \frac{\partial u_i}{\partial v_i}(b_i, \beta_{-i}; v_i)_{|b_i = b_i^*(v_i)}. \tag{12.86}$$

Remark 12.32 *The condition $\frac{\partial u_i}{\partial b_i}(b_i, \beta_{-i}; v_i)_{|b_i = b_i^*(v_i)} = 0$ holds if $0 < b_i^*(v_i) < \infty$ because this is the first-order condition for a local maximum. If follows that it is necessary to check that it holds only if the maximum is at an extreme point, i.e., only if $b_i^*(v_i) = 0$.* ♦

To apply the chain rule, the function $v_i \mapsto b_i^*(v_i)$ has to be differentiable. That means that the equilibrium strategy must be differentiable. A symmetric equilibrium β^* is called a *differentiable symmetric equilibrium* if the function β^* is differentiable.

4 A real-valued multi-variable function is *differentiable* if it is continuously differentiable (i.e., its derivative is continuous) with respect to each variable. This is equivalent to it being differentiable in every direction in the space of variables.

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Example 12.33 **Sealed-bid first-price auction** Consider a sealed-bid first-price auction with n buyers. Suppose that the private values of the buyers are independent of each other, that they are all in the unit interval $[0, 1]$, and that they share the same cumulative distribution function F. Assuming that there exists a monotonically increasing and differentiable symmetric equilibrium strategy β^*, we can compute it using the Envelope Theorem. Recall that β^*_{-i} is the strategy vector in which all the buyers, except for buyer i, use the strategy β^*. The expected profit of buyer i with private value v_i who submits a bid b_i is

$$u_i(b_i, \beta^*_{-i}; v_i) = \mathbf{P}(\text{the buyer wins the auction} \mid b_i, \beta^*_{-i}) \times (v_i - b_i) \tag{12.87}$$

$$= (F((\beta^*)^{-1}(b_i)))^{n-1} \times (v_i - b_i). \tag{12.88}$$

Since β^* is an equilibrium, $\beta^*_i(v_i)$ is the best response of buyer i with private value v_i to β^*_{-i}, and therefore $b^*_i(v_i) = \beta^*_i(v_i)$. If β is differentiable, then β^{-1} is also differentiable, and then $u_i(b_i, \beta^*_{-i}; v_i)$ is differentiable. Since the strategy β^* is monotonically increasing, and because $\beta^*(0) \geq 0$ and $\beta^*(1) \leq 1$, it follows that $0 \leq \beta^*(v_i) \leq 1$ for all $v_i \in (0, 1)$, and therefore $\frac{\partial u_i}{\partial b_i}(b_i, \beta^*_{-i}; v_i)_{|b_i=\beta^*_i(v_i)} = 0$. The Envelope Theorem implies that

$$\frac{\partial u^*_i}{\partial v_i}(\beta^*_{-i}; v_i) = \frac{\partial u_i}{\partial v_i}(b_i, \beta^*_{-i}; v_i)_{|b_i=\beta^*_i(v_i)} = (F(v_i))^{n-1}. \tag{12.89}$$

Note that $u^*_i(\beta^*_{-i}; 0) = 0$, i.e., the profit of a buyer with private value 0 is 0. By integrating, we get

$$u^*_i(\beta^*_{-i}; v_i) = \int_0^{v_i} (F(x_i))^{n-1} \mathrm{d}x_i. \tag{12.90}$$

From this equation, along with Equation (12.88), for $b_i = b^*_i(v_i) = \beta^*(v_i)$, we get

$$(F(v_i))^{n-1} \left(v_i - \beta^*(v_i)\right) = \int_0^{v_i} (F(x_i))^{n-1} \mathrm{d}x_i. \tag{12.91}$$

After moving terms from one side of the equals sign to the other, we have:

$$\beta^*(v_i) = v_i - \frac{\int_0^{v_i} (F(x_i))^{n-1} \mathrm{d}x_i}{(F(v_i))^{n-1}}. \tag{12.92}$$

In other words, if a monotonically increasing, differentiable, and symmetric equilibrium exists, it is necessarily given by Equation (12.92).

Recall that, according to Theorem 12.19, in a symmetric sealed-bid first-price auction with independent private values, the symmetric equilibrium is $\beta^*(v) = \mathbf{E}[Y \mid Y \leq v]$. It follows that in the case before us, in which the distribution of the private values of the buyers is the uniform distribution over $[0, 1]$, this expression must be equal to the expression given by Equation (12.92). The reader is asked to check directly that these two expressions indeed equal each other in Exercise 12.24. ◀

Example 12.34 **Sealed-bid first-price auction with a reserve price** An auction with a reserve price ρ is an auction in which every bid below ρ is invalid. Consider a sealed-bid first-price auction with a reserve price $\rho \in [0, 1]$ and two buyers whose private values are independent and uniformly distributed over $[0, 1]$. What is the symmetric equilibrium strategy β^*? A buyer with a private value lower than or equal to ρ cannot profit no matter what bid he makes. Using the Envelope Theorem we can find a symmetric, monotonically increasing and differentiable equilibrium strategy satisfying

$\beta^*(v_1) = v_1$ for all $v_1 \in [0, \rho]$. This choice is arbitrary and it guarantees that a bid by a buyer whose private value is less than ρ is invalid.[5]

Step 1: $\rho \le \beta^*(v_1) < v_1$ for all $v_1 \in (\rho, 1]$.
$u_1(b_1, \beta^*; v_1) = 0$ for all $b_1 < \rho$, and $u_1(b_1, \beta^*; v_1) \le 0$ for all $b_1 \ge v_1$. For all $b_1 \in (\rho, v_1)$,

$$u_1(b_1, \beta^*; v_1) \ge \mathbf{P}(V_2 < \rho)(b_1 - \rho) > 0. \tag{12.93}$$

It follows that the maximum of the function $b_1 \mapsto u_1(b_1, \beta^*; v_1)$ is attained at a point in the interval $[\rho, v_1)$.

Step 2: $\rho < \beta^*(v_1) < v_1$ for all $v_1 \in (\rho, 1]$.
Suppose by contradiction that there exists $v_1 \in (\rho, 1]$ such that $\beta^*(v_1) = \rho$ and let $\widehat{v}_1 \in (\rho, v_1)$. Since β^* is a monotonic strategy,

$$\beta^*(\widehat{v}_1) < \beta^*(v_1) = \rho, \tag{12.94}$$

in contradiction to Step 1.

Step 3: Computing β^*.
For $v_1 \in (\rho, 1]$ the maximum of the function $b_1 \mapsto u_1(b_1, \beta^*; v_1)$ is attained at a point in the interval (ρ, v_1), this is a local maximum, and since β^* is a differentiable function, $\frac{\partial u_1}{\partial b_1}(b_1, \beta^*; v_1)_{|b_1=\beta^*(v_1)} = 0$.

By the Envelope Theorem:

$$\frac{\partial u_1^*}{\partial v_1}(\beta^*; v_1) = \frac{\partial u_1}{\partial v_1}(b_1, \beta^*; v_1)_{|b_1=\beta^*(v_1)}. \tag{12.95}$$

Since the distribution of V_2 is the uniform distribution over $[0, 1]$, and since β^* is monotonically increasing,

$$u_1(b_1, \beta^*; v_1) = \mathbf{P}(\beta^*(V_2) < b_1) \times (v_1 - b_1) = (\beta^*)^{-1}(b_1) \times (v_1 - b_1). \tag{12.96}$$

Therefore,

$$\frac{\partial u_1}{\partial v_1}(b_1, \beta^*; v_1) = (\beta^*)^{-1}(b_1), \tag{12.97}$$

and Equations (12.82) and (12.95)–(12.97) imply that

$$\frac{\partial u_1^*}{\partial v_1}(\beta^*; v_1) = (\beta^*)^{-1}(\beta^*(v_1)) = v_1. \tag{12.98}$$

For $v_1 \le \rho$, the profit is zero: $u_1^*(\beta^*_{-1}; v_1) = 0$. By integration, we get

$$u_1^*(\beta^*; v_1) = \int_\rho^{v_1} \frac{\partial u_1^*}{\partial t_1}(\beta^*; t_1)\mathrm{d}t_1 = \int_\rho^{v_1} t_1 \mathrm{d}t_1 = \left.\frac{(t_1)^2}{2}\right|_\rho^{v_1} = \frac{(v_1)^2}{2} - \frac{\rho^2}{2}. \tag{12.99}$$

On the other hand, the buyer's profit $u_1^*(\beta^*; v_1)$ can be computed directly: in a symmetric, monotonically increasing equilibrium, buyer 1's profit is the probability that the private value of buyer 2 is lower than v_1 times the profit $(v_1 - \beta^*(v_1))$ if he wins:

$$u_1^*(\beta^*; v_1) = u_1(\beta^*(v_1), \beta^*; v_1) = v_1(v_1 - \beta^*(v_1)), \qquad \forall v_1 \in (\rho, 1]. \tag{12.100}$$

5 In this example we have two buyers, and therefore the vector β^*_{-1} of the strategies played in the symmetric equilibrium by all players except Player 1 is β^*. We therefore write β^* instead of β^*_{-1}.

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From Equations (12.99)–(12.100) we conclude that

$$\beta^*(v_1) = \frac{v_1}{2} + \frac{\rho^2}{2v_1}, \qquad \forall v_1 \in (\rho, 1]. \tag{12.101}$$

Step 4: Computing the seller's expected revenue.
We have shown that if there exists a monotonically increasing and differentiable symmetric equilibrium strategy in the interval $(\rho, 1)$ then that strategy is defined by Equation (12.101). The strategy that we found is indeed differentiable. To see that it is monotonically increasing, we look at its derivative:

$$(\beta^*)'(v_1) = \tfrac{1}{2} - \tfrac{\rho^2}{2(v_1)^2}, \quad \forall v_1 \in (\rho, 1]. \tag{12.102}$$

We see that for $v_1 \in (\rho, 1]$ it is indeed the case that $(\beta^*)'(v_1) > 0$.

Note that for $\rho = 0$ (an auction without a minimum price), by Equation (12.101), $\beta^*(v_1) = \frac{v_1}{2}$, which is the solution that we found for sealed-bid first-price auctions without a reserve price (Example 12.15 on page 472).

What is the seller's expected revenue? Computing this requires first computing each buyer's expected payment. Buyer 1's payment is 0 when $v_1 \le \rho$. If $v_1 > \rho$, he wins only if $v_1 > v_2$ (an event that occurs with probability v_1), and then he pays $\beta^*(v_1)$ (we are ignoring the possibility that $v_1 = v_2$, which occurs with probability 0). The expected payment of buyer 1 is, therefore,

$$\begin{aligned} e &= \int_\rho^1 v_1\beta^*(v_1)\mathrm{d}v_1 = \int_\rho^1 v_1\left(\frac{v_1}{2} + \frac{\rho^2}{2v_1}\right)\mathrm{d}v_1 = \tfrac{1}{2}\left(\tfrac{(v_1)^3}{3} + \rho^2 v_1\right)\Big|_\rho^1 \\ &= \tfrac{1}{6} + \tfrac{\rho^2}{2} - \tfrac{2}{3}\rho^3. \end{aligned} \tag{12.103}$$

Since there are two buyers, the seller's expected revenue is

$$\pi(\rho) = 2e = \tfrac{1}{3} + \rho^2 - \tfrac{4}{3}\rho^3. \tag{12.104}$$

Note that $\pi(0) = \frac{1}{3}$: in a sealed-bid first-price auction without a reserve price, the seller's expected revenue is $\frac{1}{3}$. Similarly, $\pi(1) = 0$: when the reserve price is 1, with probability 1 no buyer wins the object and the seller's expected revenue is 0. What is the reserve price that maximizes the seller's expected payoff? To compute that, differentiate the function π, and set the derivative to 0:

$$0 = \pi'(\rho) = 2\rho - 4\rho^2. \tag{12.105}$$

It follows that the reserve price that maximizes the seller's expected revenue is $\rho = \frac{1}{2}$, at which the seller's expected revenue is $\pi(\frac{1}{2}) = \frac{5}{12}$. Since $\frac{5}{12} \ge \frac{1}{3}$, introducing a reserve price is beneficial for the seller. Note that $\frac{5}{12}$ is also the seller's expected revenue in a sealed-bid second-price auction with entry fee $\frac{1}{4}$ (see Example 12.28). ◀

12.7 Risk aversion

One of the underlying assumptions of our analysis so far has been that the buyers participating in auctions are risk neutral, and therefore their goal is to maximize their expected profits. What happens if we drop this assumption? In this section, we will see how risk-averse buyers behave in sealed-bid first-price and second-price auctions. We will consider auction models satisfying Assumptions (A1)–(A5), thus omitting the risk neutrality Assumption (A6). For simplicity we maintain the term "symmetric auction with independent private values" for this model.

Suppose that the buyers satisfy the von Neumann–Morgenstern axioms with respect to their utility for money (for a review of utility theory, see Chapter 2). In addition, suppose that each buyer has the same monotonically increasing utility function for money, $U : \mathbb{R} \to \mathbb{R}$, satisfying $U(0) = 0$.

If buyer i's private value is v_i, and his bid is b_i, the buyer's profit is:

- 0, if he does not win the auction.
- $v_i - b_i$, if he wins.

If we denote by α_{b_i} the probability that the buyer wins the auction if he bids b_i, then when he bids b_i, he is effectively facing the lottery:

$$[\alpha_{b_i}(v_i - b_i), (1 - \alpha_{b_i})0]. \tag{12.106}$$

Since the buyer's preference relation satisfies the von Neumann–Morgenstern axioms, and since $U(0) = 0$, his utility from this lottery is

$$U[\alpha_{b_i}(v_i - b_i), (1 - \alpha_{b_i})0] = \alpha_{b_i} U(v_i - b_i). \tag{12.107}$$

Recall (see Section 2.7 on page 23), that a buyer is *risk-averse* if his utility function for money U is concave, is *risk-neutral* if his utility function U is linear, and is *risk-seeking* if his utility function U is convex.

In a sealed-bid second-price auction, the strategy

$$\beta(v) = v \tag{12.108}$$

still weakly dominates all other strategies, even if the buyers are risk-averse (or risk-seeking). The reasoning behind this is the same reasoning behind the similar conclusion we presented in the case of risk-neutral buyers, using the fact that U is monotonic (see Exercise 12.25). The situation is quite different in a sealed-bid first-price auction.

Theorem 12.35 *Consider a symmetric sealed-bid first-price auction with independent private values. Suppose that each buyer has the same utility function for money U that is monotonically increasing, differentiable, and strictly concave. Let γ be a monotonically increasing, differentiable, and symmetric equilibrium strategy satisfying $\gamma(0) = 0$. Then γ is the solution of the differential equation*

$$\gamma'(v) = (n-1) \times \frac{f(v)}{F(v)} \times \frac{U(v - \gamma(v))}{U'(v - \gamma(v))}, \quad \forall v > 0 \tag{12.109}$$

with initial condition $\gamma(0) = 0$.

Proof: Since γ is monotonically increasing, if buyer i bids b_i, and the other buyers implement strategy γ, the probability that buyer i wins the auction is $(F(\gamma^{-1}(b_i)))^{n-1}$. Since $U(0) = 0$, the buyer's utility is

$$u_i(b_i, \gamma_{-i}; v_i) = (F(\gamma^{-1}(b_i)))^{n-1} \times U(v_i - b_i). \tag{12.110}$$

We will first check that for $v_i > 0$, the maximum of this function is attained at a value b_i, which is in the interval $(0, v_i)$. This is accomplished by showing that $u_i(0, \gamma_{-i}; v_i) = u_i(\gamma_i, \gamma_{-i}; v_i) = 0$ and $u_i(b_i, \gamma_{-i}; v_i) < 0$ for each $b_i > v_i$, while $u_i(b_i, \gamma_{-i}; v_i) > 0$ for each $b_i \in (0, v_i)$.

- If buyer i's bid is $b_i = v_i$, his utility from winning is 0, and therefore $u_i(\gamma_i, \gamma_{-i}; v_i) = 0$.
- Since $\gamma(0) = 0$ it follows that $F(\gamma^{-1}(0)) = 0$, and therefore $u_i(0, \gamma_{-i}; v_i) = 0$.
- Since U is monotonically increasing and $U(0) = 0$, $v_i - b_i < 0$ for $b_i > v_i$, and therefore $U(v_i - b_i) < 0$. By Assumptions (A4) and (A5), $F(\gamma^{-1}(b_i)) > 0$ for every $b_i > v_i$, and therefore $u_i(b_i, \gamma_{-i}; v_i) < 0$ for every $b_i > v_i$.
- Finally, we show that $u_i(b_i, \gamma_{-i}; v_i) > 0$ for every $b_i \in (0, v_i)$. For every $b_i > 0$, since γ is monotonically increasing, $\gamma^{-1}(b_i) > 0$, and Assumptions (A4) and (A5) imply that $F(\gamma^{-1}(b_i)) > 0$. Since the utility function is monotonically increasing, $U(v_i - b_i) > 0$, and therefore $u_i(b_i, \gamma_{-i}; v_i) > 0$.

We deduce that the maximum of the function $b_i \mapsto u_i(b_i, \gamma_{-i}; v_i)$ is indeed attained at a point in the open interval $(0, v_i)$.

We next differentiate the function u_i in Equation (12.110) (which is differentiable because both U and γ are differentiable), yielding

$$\begin{aligned}\frac{\partial u_i}{\partial b_i}(b_i, \gamma_{-i}; v_i) = (n-1)\frac{f(\gamma^{-1}(b_i))}{\gamma'(\gamma^{-1}(b_i))}(F(\gamma^{-1}(b_i)))^{n-2}U(v_i - b_i)\\ - (F(\gamma^{-1}(b_i)))^{n-1}U'(v_i - b_i). \qquad (12.111)\end{aligned}$$

Since the strategy γ is a symmetric equilibrium strategy, the maximum of this function is attained at $b_i = \gamma(v_i)$; and at that point, the derivative vanishes. Thus, by substituting $b_i = \gamma(v_i)$ in Equation (12.111) one has

$$\begin{aligned}0 &= \frac{\partial u_i}{\partial b_i}(b_i, \gamma_{-i}; v_i)_{|b_i=\gamma(v_i)}\\ &= (n-1)\frac{f(v_i)}{\gamma'(v_i)}(F(v_i))^{n-2}U(v_i - \gamma(v_i)) - (F(v_i))^{n-1}U'(v_i - \gamma(v_i)). \qquad (12.112)\end{aligned}$$

Because $v_i > 0$, and by Assumption (A5), $F(v_i) > 0$. Reducing the factor $(F(v_i))^{n-2}$ in Equation (12.112), and rearranging the remaining equation, yields Equation (12.109). □

We now prove that when all the buyers are risk averse and have the same utility function, the submitted bids in equilibrium are higher than the bids that would be submitted by risk-neutral buyers. The intuition behind this result is that risk-averse buyers are more concerned about not winning the auction, and therefore they submit higher bids than risk-neutral buyers.

Theorem 12.36 *Suppose that in a symmetric sealed-bid first-price auction with independent private values, each buyer's utility function U is monotonically increasing, differentiable, strictly concave, and satisfies $U(0) = 0$. Let γ be a monotonically increasing, differentiable, symmetric equilibrium strategy satisfying $\gamma(0) = 0$, and let β be monotonically increasing, differentiable, symmetric equilibrium strategy satisfying $\beta(0) = 0$ in the auction when the buyers are risk-neutral. Then $\gamma(v) > \beta(v)$ for each $v > 0$.*

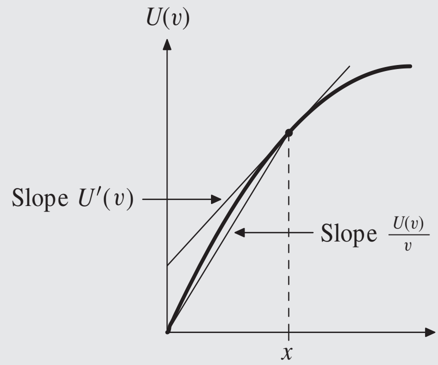

Figure 12.3 $\frac{U(v)}{v} > U'(v)$ for a strictly concave function U

Proof: Theorem 12.35 implies that

$$\gamma'(v) = (n-1)\frac{f(v)}{F(v)} \times \frac{U(x-\gamma(v))}{U'(v-\gamma(v))}, \quad \forall v > 0 \tag{12.113}$$

Since the strategy β also satisfies the conditions of Theorem 12.35, this strategy, for risk-neutral buyers, satisfies Equation (12.113) with utility function $U(v) = v$

$$\beta'(v) = (n-1)\frac{f(v)}{F(v)} \times (v - \beta(v)), \quad \forall v > 0. \tag{12.114}$$

Since U is a strictly concave function, and $U(0) = 0$, it follows that $U'(v) < \frac{U(v)}{v}$ (see Figure 12.3), or equivalently, $\frac{U(v)}{U'(v)} > v$.

It follows that

$$\gamma'(v) = (n-1)\frac{f(v)}{F(v)} \times \frac{U(v-\gamma(v))}{U'(v-\gamma(v))} > (n-1)\frac{f(v)}{F(v)} \times (v-\gamma(v)). \tag{12.115}$$

To show that $\gamma(v) > \beta(v)$ for each $v > 0$, note that if $v_0 > 0$ satisfies $\gamma(v_0) \le \beta(v_0)$, then

$$\gamma'(v_0) > (n-1)\frac{f(v_0)}{F(v_0)} \times (v_0 - \gamma(v_0)) \tag{12.116}$$

$$\ge (n-1)\frac{f(v_0)}{F(v_0)} \times (v_0 - \beta(v_0)) = \beta'(v_0) > 0. \tag{12.117}$$

Define $\delta(v) := \gamma(v) - \beta(v)$ for all $v \in \mathbb{V}$. Equations (12.116)–(12.117) show that $\delta'(v) > 0$ for each $v > 0$ such that $\delta(v) \le 0$. It follows that if there exists $v_0 > 0$ such that $\delta(v_0) \le 0$, then $\delta(v) < 0$ for each $v \in [0, v_0)$. Since $\delta(0) = \gamma(0) - \beta(0) = 0$, there does not exist $v_0 > 0$ such that $\delta(v_0) \le 0$. In other words, $\gamma(v) > \beta(v)$ for each $v > 0$. □

In the model used in this section, in which all the buyers have the same utility function for money, the bids submitted by the buyers in the symmetric equilibrium of the sealed-bid first-price auctions are higher if all buyers are risk averse, which implies that the seller's expected revenue is higher. In contrast, in sealed-bid second-price auctions, the bids submitted by buyers in the symmetric equilibrium equal their private values, whether

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they are risk-averse or risk-neutral, and hence the seller's expected revenue is equal in either case. This leads to the following corollary.

Corollary 12.37 *In a symmetric sealed-bid auction with independent private values, when buyers are risk averse, and they all have the same monotonically increasing, differentiable, and strictly concave utility function, the seller's expected revenue in the symmetric equilibrium is higher in a sealed-bid first-price auction than in a sealed-bid second-price auction. In particular, this proves that the Revenue Equivalence Theorem does not apply when the buyers are risk averse.*

The converse corollary can similarly be proved for risk-seeking buyers (Exercise 12.26): In a symmetric sealed-bid auction with independent private values, when the buyers are risk seeking and they all have the same monotonically increasing, differentiable, and strictly convex utility functions the seller's expected revenue in the symmetric equilibrium is lower in a first-price auction than in a second-price auction.

12.8 Mechanism design

We have presented up to now several auction methods, which we analyzed by computing an equilibrium in each auction, and studying its properties. The advantages and disadvantages of a particular auction method were then judged by the properties of its equilibrium. A natural question that arises is whether one can plan an auction method, or more generally a selling mechanism, that can be expected to yield a "desired outcome." In other words, what we seek is a selling mechanism whose equilibrium has "desired properties," such as efficiency and maximizing the revenue of the seller. In this section we will study mechanism design, a subject that focuses on these sorts of questions.

Definition 12.38 *A* selling problem *is a vector* $(N; (\mathbb{V}_i, f_i)_{i\in N})$ *such that:*

- $N = \{1, 2, \dots, n\}$ *is a set of buyers.*
- $\mathbb{V}_i$ *is a bounded interval* $[0, \overline{v}]$ *or an infinite interval* $[0, \infty)$.
- $f_i : \mathbb{V}_i \to [0, \infty)$ *is a density function, i.e.,* $\int_{\mathbb{V}_i} f_i(v)\mathrm{d}v = 1$.

A selling problem serves as a model for the following situation:

- A seller wishes to sell an indivisible object, whose value for the seller is normalized to be 0.
- The set of buyers is $N = \{1, 2, \dots, n\}$. The buyers are all risk-neutral, and each seeks to maximize the expected value of his profit.
- The private value of buyer i is a random variable V_i with values in the interval $\mathbb{V}_i$. This is a continuous random variable whose density function is f_i. The random variables $(V_i)_{i\in N}$ are independent random variables that do not necessarily have the same distribution; we do not rule out the possibility that $f_i \neq f_j$ for different buyers i and j.
- Each buyer knows his private value and does not know the private values of the other buyers, but he knows the distributions of the random variables $(V_j)_{j\neq i}$.

Denote by $\mathbb{V}^N := \mathbb{V}_1 \times \mathbb{V}_2 \times \cdots \times \mathbb{V}_n$ the space of all possible vectors of private values. Since the private value of the buyers are independent, the joint density function of the vector $V = (V_1, V_2, \ldots, V_n)$ is

$$f_V(v) = \prod_{i \in N} f_i(v_i). \tag{12.118}$$

Denote:

$$f_{-i}(v_{-i}) := \prod_{j \neq i} f_j(v_j). \tag{12.119}$$

f_{-i} is the joint density function of $V_{-i} = (V_j)_{j \neq i}$. Since the private values are independent, this is also the marginal density of V over $\mathbb{V}_{-i}$, conditioned on V_i, where $\mathbb{V}_{-i} := \times_{j \neq i} \mathbb{V}_j$ is the set of all possible private value vectors of all the buyers except for buyer i.

Definition 12.39 *A* selling mechanism *for a selling problem $(N, (\mathbb{V}_i, f_i)_{i \in N})$ is a vector $((\Theta_i, \widehat{q}_i, \widehat{\mu}_i)_{i \in N})$ where, for each buyer $i \in N$:*

1. *Θ_i is a measurable space[6] of messages that buyer i can send to the seller. The space of the message vectors is $\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n$.*
2. *$\widehat{q}_i : \Theta \to [0, 1]$ is a function mapping each vector of messages to the probability that buyer i wins the object. (Necessarily, $\sum_{i \in N} \widehat{q}_i(\theta) \leq 1$, for every $\theta \in \Theta$.)*
3. *$\widehat{\mu}_i : \Theta \to \mathbb{R}$ is a function mapping every vector of messages to the payment that buyer i makes to the seller (whether or not he wins the object).*

If $\sum_{i \in N} \widehat{q}_i(\theta) < 1$ for a particular $\theta \in \Theta$, then when the message vector received by the seller is θ there is a positive probability that the object will not be sold (and will therefore remain in the possession of the seller).

Given a selling mechanism, we define the following game:

- The set of players (buyers) is N.
- Each buyer $i \in N$ chooses a message $\theta_i \in \Theta_i$. Denote $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$.
- Buyer i wins the object with probability $\widehat{q}_i(\theta)$. The object remains in the possession of the seller with probability $1 - \sum_{i \in N} \widehat{q}_i(\theta)$.
- Every buyer i pays to the seller the amount $\widehat{\mu}_i(\theta)$.

The space of messages in a selling mechanism may be very complex: if the selling mechanism includes negotiations, the message space may include the buyer's first offer, his second offer to every counteroffer of the seller, and so on.

From now on we study selling mechanisms for a given selling problem.

6 Recall that Θ_i is a *measurable space* if it has an associated σ-algebra, i.e., a collection of subsets of Θ_i containing the empty set that is closed under countable unions and set complementation.

Example 12.40 **Sealed-bid first-price auction** A sealed-bid first-price auction with risk-neutral buyers can be presented as a selling mechanism as follows:

- $\Theta_i = [0, \infty)$: Buyer i's message is a nonnegative number; this is buyer i's bid.
- Denote by

$$N(\theta) := |\{i \in N : \theta_i = \max_{j \in N} \theta_j\}| \tag{12.120}$$

the number of buyers who submit the highest bid.[7]

$$\widehat{q}_i(\theta) = \begin{cases} \frac{1}{N(\theta)} & \text{if } \theta_i = \max_{j\in N} \theta_j, \\ 0 & \text{if } \theta_i < \max_{j\in N} \theta_j. \end{cases} \tag{12.121}$$

- The payment that buyer i makes is

$$\widehat{\mu}_i(\theta) = \begin{cases} \frac{\theta_i}{N(\theta)} & \text{if } \theta_i = \max_{j\in N} \theta_j, \\ 0 & \text{if } \theta_i < \max_{j\in N} \theta_j. \end{cases} \tag{12.122}$$

This description differs from the auction descriptions we previously presented only in the payment that the buyer who submits the highest bid makes, and only in the case that several buyers submit the same bid. In the description of an auction as a selling mechanism, all the buyers who submitted the highest bid equally share the cost of that bid whether or not they finally get the object, while in the description on page 465, only the winner of the object pays its full price, which equals his bid. But this difference does not change the strategic considerations of the buyers: when there are several buyers submitting the same highest bid and the winner of the object is chosen from among them according to the uniform distribution, in both cases the expected payment that the buyer who wins the auction makes is the amount that he bid, divided by $N(\theta)$, and his probability of winning is $\frac{1}{N(\theta)}$. Since the buyers are risk-neutral and the goal of each buyer is to maximize his expected profit, the strategic considerations of the buyers, under both definitions, are unchanged. ◀

Example 12.41 **Sealed-bid second-price auction** Similar to a sealed-bid first-price auction, a sealed-bid second-price auction with risk-neutral buyers can also be presented as a selling mechanism. The only difference is in the payment function, $\widehat{\mu}$, which is given as follows:

$$\widehat{\mu}_i(\theta) = \begin{cases} \frac{\max_{j\neq i} \theta_j}{N(\theta)} & \text{if } \theta_i = \max_{j\in N} \theta_j, \\ 0 & \text{if } \theta_i < \max_{j\in N} \theta_j. \end{cases} \tag{12.123}$$

Again, $N(\theta)$ is the number of buyers who submitted the highest bid. ◀

As the following theorem states, every sealed-bid auction is an example of a selling mechanism. The proof of the theorem is left to the reader (Exercise 12.28).

Theorem 12.42 *Every sealed-bid auction with risk-neutral buyers can be presented as a selling mechanism.*

The game that corresponds to a selling mechanism is a Harsanyi game with incomplete information (see Definition 9.39 on page 347).

7 Recall that for every finite set A, the number of elements in A is denoted by $|A|$.

- The set of players is the set of buyers $N = \{1, 2, \dots, n\}$.
- Player i's set of types is $\mathbb{V}_i$. Denote $\mathbb{V}^N := \times_{i \in N} \mathbb{V}_i$.
- The distribution of the set of type vectors is a product distribution, whose density is f_V.
- For each type vector $v \in \mathbb{V}^N$, the state of nature s_v is the state game defined by
  - player i's set of actions is Θ_i;
  - for each vector of actions $\theta \in \Theta$, buyer i's utility is

$$u_i(v; \theta) = \widehat{q}(\theta) v_i - \widehat{\mu}_i(\theta). \tag{12.124}$$

  In other words, the buyer pays $\widehat{\mu}_i(\theta)$ in any event, and if he wins the auctioned object, he receives v_i, his value of the object.

A pure strategy for player i is a measurable function $\beta_i : \mathbb{V}_i \to \Theta_i$. For each strategy vector $\beta_{-i} = (\beta_j)_{j \neq i}$ of the other buyers, denote by $u_i(\theta_i, \beta_{-i}; v_i)$ buyer i's expected profit, when his private value is v_i, and he sends message θ_i:

$$u_i(\theta_i, \beta_{-i}; v_i)$$
$$= \int_{\mathbb{V}_{-i}} u_i(v_i; \beta_1(v_1), \dots, \beta_{i-1}(v_{i-1}), \theta_i, \beta_{i+1}(v_{i+1}), \dots, \beta_n(v_n)) f_{-i}(v) \mathrm{d}v_{-i}. \tag{12.125}$$

The definition of a Bayesian equilibrium of the game with incomplete information that corresponds to a selling mechanism is as follows (see Definition 9.49 on page 354).

Definition 12.43 *A vector $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ of strategies is a (Bayesian)* equilibrium *if for each buyer $i \in N$, and every private value $v_i \in \mathbb{V}_i$,*

$$u_i(\beta_i(v_i), \beta_{-i}; v_i) \geq u_i(\theta_i, \beta_{-i}; v_i), \quad \forall \theta_i \in \Theta_i. \tag{12.126}$$

A simple set of mechanisms is the set of direct selling mechanisms, in which the set of messages of each buyer is his set of private values.

Definition 12.44 *A selling mechanism $(\Theta_i, \widehat{q}_i, \widehat{\mu}_i)_{i \in N}$ is called* direct *if $\Theta_i = \mathbb{V}_i$ for each buyer $i \in N$.*

A direct selling mechanism is a mechanism in which every buyer is required to report a private value; he may report his true private value or make up any other value to report. We will denote a direct selling mechanism by $(\widehat{q}, \widehat{\mu})$ for short, where $\widehat{q} = (\widehat{q}_i)_{i \in N}$ and $\widehat{\mu} = (\widehat{\mu}_i)_{i \in N}$.

When a mechanism is direct, a possible strategy that a buyer may use is to report his true private value:

$$\beta_i^*(v_i) = v_i, \quad \forall v_i \in \mathbb{V}_i. \tag{12.127}$$

We refer to this as the "truth-telling" strategy.

Definition 12.45 *A direct selling mechanism $(\widehat{q}, \widehat{\mu})$ is* incentive compatible *if the vector $\beta^* = (\beta_i^*)_{i \in N}$ of truth-telling strategies $\beta_i^*(v_i) = v_i$ is an equilibrium.*

The reason we use the term "incentive compatible" is because if β^* is an equilibrium, then each buyer has an incentive to report his private value truthfully: he cannot profit by lying in reporting his private value. This property is analogous to the nonmanipulability property that we will discuss in Chapter 21, on social choice theory.

The direct selling mechanism depicted in Example 12.41 for a sealed-bid second-price auction is an incentive compatible mechanism, because in a sealed-bid second-price auction, the strategy vector β^* in which every buyer's bid equals his private value is an equilibrium. In contrast, the direct selling mechanism depicted in Example 12.40, corresponding to a sealed-bid first-price auction, is not incentive compatible, because in sealed-bid first-price auction the strategy vector β^* is not an equilibrium. As the next example shows, when the buyers are symmetric, it is nevertheless possible to describe sealed-bid first-price auctions as incentive-compatible direct selling mechanisms.

Example 12.46 **Sealed-bid first-price auction: another representation as a selling mechanism** Consider a symmetric sealed-bid first-price auction that satisfies Assumptions (A4)–(A6). Let $\beta = (\beta_i)_{i \in N}$ be an equilibrium of the auction. Consider the following direct selling mechanism:

- $\Theta_i = [0, \infty)$: buyer i's message is a nonnegative number.
- The probability that buyer i wins the auctioned object is

$$\widehat{q}_i(\theta) = \begin{cases} \frac{1}{N(\theta)} & \text{if } \theta_i = \max_{j \in N} \theta_j, \\ 0 & \text{if } \theta_i < \max_{j \in N} \theta_j. \end{cases} \tag{12.128}$$

- The expected payment that buyer i makes is

$$\widehat{\mu}_i(\theta) = \begin{cases} \beta_i(\theta_i) & \text{if } \theta_i = \max_{j \in N} \theta_j, \\ 0 & \text{if } \theta_i < \max_{j \in N} \theta_j. \end{cases} \tag{12.129}$$

In words, a buyer submitting the highest bid pays the expected sum that he would pay under equilibrium β when the private values are $(\theta_i)_{i \in N}$. Since β is a symmetric equilibrium strategy, the strategy vector β^*, under which each buyer reports his private value, is an equilibrium in this selling mechanism, and therefore in particular this selling mechanism is incentive compatible. ◄

12.8.1 The revelation principle

The idea in Example 12.46 can be generalized to any selling mechanism: let $(\Theta_i, \widehat{q}_i, \widehat{\mu}_i)_{i \in N}$ be a selling mechanism, and let $\widehat{\beta}$ be an equilibrium of this mechanism. We can then define a direct selling mechanism (q, μ), as follows: if the buyers report the private values vector $v = (v_i)_{i \in N}$, the mechanism computes what message $\widehat{\beta}_i(v_i)$ the buyer would have sent in the original mechanism, and then proceeds exactly as that mechanism would have done under those messages: for each buyer $i \in N$,

$$q_i(v) := \widehat{q}_i(\widehat{\beta}_1(v_1), \ldots, \widehat{\beta}_n(v_n)), \tag{12.130}$$

$$\mu_i(v) := \widehat{\mu}_i(\widehat{\beta}_1(v_1), \ldots, \widehat{\beta}_n(v_n)). \tag{12.131}$$

The mechanism (q, μ) is schematically described in Figure 12.4. Since the strategy vector $\widehat{\beta}$ is an equilibrium of the mechanism $(\Theta_i, \widehat{q}_i, \widehat{\mu}_i)_{i \in N}$, the strategy vector β^* according to which every buyer reports his true private value is an equilibrium of the mechanism (q, μ). This leads to the following theorem, which is called the revelation principle.

Theorem 12.47 **(Myerson [1979])** *Let $(\Theta_i, \widehat{q}_i, \widehat{\mu}_i)_{i \in N}$ be a selling mechanism, and let $\widehat{\beta}$ be an equilibrium of this mechanism. There exists an incentive-compatible direct selling*

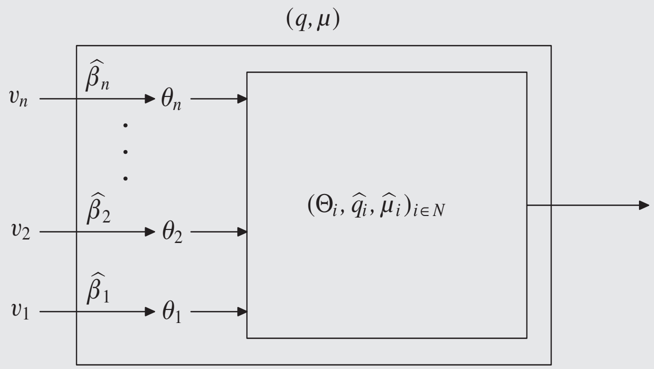

Figure 12.4 A selling mechanism, along with a θ incentive-compatible direct selling mechanism that is equivalent to it

mechanism (q, μ) satisfying that the outcome of the original mechanism under $\widehat{\beta}$ is identical to the outcome of (q, μ) under β^* (which is the truth-telling equilibrium):

$$\widehat{q}\left(\widehat{\beta}_1(v_1), \widehat{\beta}_2(v_2), \ldots, \widehat{\beta}_n(v_n)\right) = q(v_1, v_2, \ldots, v_n), \quad \forall (v_1, \ldots, v_n) \in \mathbb{V}, \tag{12.132}$$

$$\widehat{\mu}\left(\widehat{\beta}_1(v_1), \widehat{\beta}_2(v_2), \ldots, \widehat{\beta}_n(v_n)\right) = \mu(v_1, v_2, \ldots, v_n), \quad \forall (v_1, \ldots, v_n) \in \mathbb{V}. \tag{12.133}$$

The proof of the theorem is left to the reader as an exercise (Exercise 12.30). The theorem's importance stems from the fact that incentive-compatible direct selling mechanisms are simple and easy to work with, resulting in simpler mathematical analysis. The space of messages in a generic selling mechanism may be quite large, and the revelation principle simplifies the effort required to analyze selling mechanisms. It implies that it suffices to consider only incentive-compatible direct selling mechanisms, because every general selling mechanism has an incentive-compatible, direct mechanism that is equivalent to it in the sense of Theorem 12.47.

12.8.2 The Revenue Equivalence Theorem

In this section we prove a general Revenue Equivalence Theorem for selling mechanisms, which includes the Revenue Equivalence Theorem for auctions as a special case (Theorem 12.23 on page 478). To that end, we first introduce some new notation, and prove intermediate results.

Consider a direct selling mechanism (q, μ). When buyer i reports that his private value is x_i, and the other buyers report their true private values, the probability that buyer i wins the object is

$$Q_i(x_i) = \int_{\mathbb{V}_{-i}} q_i(x_i, v_{-i}) f_{-i}(v_{-i}) \mathrm{d}v_{-i}, \tag{12.134}$$

and the expected payment he makes is

$$M_i(x_i) = \int_{\mathbb{V}_{-i}} \mu_i(x_i, v_{-i}) f_{-i}(v_{-i}) \mathrm{d}v_{-i}. \tag{12.135}$$

Because the private value of the buyers are independent, these two quantities are independent of buyer i's true private value, and depend solely on the message x_i that he reports.

Buyer i's expected profit, when his true private value is v_i and he reports x_i, is

$$u_i(x_i, \beta^*_{-i}; v_i) = Q_i(x_i)v_i - M_i(x_i). \tag{12.136}$$

In other words, the buyer's expected profit is the probability that he wins the auction, times his private value, less the expected payment that he makes. Given this, the following equation is obtained:

$$u_i(x_i, \beta^*_{-i}; v_i) = Q_i(x_i)v_i - M_i(x_i) \tag{12.137}$$
$$= Q_i(x_i)x_i - M_i(x_i) + Q_i(x_i)(v_i - x_i) \tag{12.138}$$
$$= u_i(x_i, \beta^*_{-i}; x_i) + Q_i(x_i)(v_i - x_i). \tag{12.139}$$

Denote buyer i's expected profit when he reports his true private value by

$$W_i(v_i) = u_i(v_i, \beta^*_{-i}; v_i). \tag{12.140}$$

Inserting $x_i = v_i$ into Equation (12.136), one has

$$W_i(v_i) = Q_i(v_i)v_i - M_i(v_i). \tag{12.141}$$

Theorem 12.48 *A direct selling mechanism (q, μ) is incentive compatible if and only if*

$$W_i(v_i) \geq W_i(x_i) + Q_i(x_i)(v_i - x_i), \quad \forall i \in N, \forall v_i \in \mathbb{V}_i, \forall x_i \in \mathbb{V}_i. \tag{12.142}$$

Proof: A direct selling mechanism (q, μ) is incentive compatible if and only if equilibrium is attained when all buyers report their true private values. This means that for each buyer i, each private value $v_i \in \mathbb{V}_i$, and each possible report $x_i \in \mathbb{V}_i$,

$$u_i(v_i, \beta^*_{-i}; v_i) \geq u_i(x_i, \beta^*_{-i}; v_i). \tag{12.143}$$

Since $u_i(v_i, \beta^*_{-i}; v_i) = W_i(v_i)$ (Equation (12.140)), Equations (12.137)–(12.139) imply that Equation (12.143) is equivalent to

$$W_i(v_i) \geq u_i(x_i, \beta^*_{-i}; x_i) + Q_i(x_i)(v_i - x_i) = W_i(x_i) + Q_i(x_i)(v_i - x_i). \tag{12.144}$$

In other words, (q, μ) is incentive compatible if and only if Equation (12.144) obtains, which is what we needed to prove. □

The following theorem yields an explicit formula for computing a buyer's expected profit in an incentive-compatible direct selling mechanism.

Theorem 12.49 *Let (q, μ) be an incentive-compatible direct selling mechanism. Then for each $v_i \in \mathbb{V}_i$,*

$$W_i(v_i) = W_i(0) + \int_0^{v_i} Q_i(t_i)\mathrm{d}t_i \tag{12.145}$$

$$= -M_i(0) + \int_0^{v_i} Q_i(t_i)\mathrm{d}t_i. \tag{12.146}$$

We see from Equation (12.146) that buyer i's expected profit depends on the payment he makes for the object, M_i, only through $M_i(0)$ – the sum that he pays when the private value that he reports is 0. Inserting Equation (12.141) into Equation (12.146), one gets

$$M_i(v_i) = M_i(0) + Q_i(v_i)v_i - \int_0^{v_i} Q_i(t_i)\mathrm{d}t_i. \tag{12.147}$$

This equation is an explicit formula for a buyer's expected payment under the truth-telling equilibrium β^* as a function of $M_i(0)$, and of his probability of winning.

Proof of Theorem 12.49: Note that if Equation (12.145) is satisfied, then Equation (12.146) also holds, since

$$W_i(0) = u_i(0, \beta^*_{-i}; 0) = Q_i(0) \times 0 - M_i(0) = -M_i(0). \tag{12.148}$$

To prove Equation (12.145), we first prove that the function Q_i is monotonically nondecreasing. Since (q, μ) is an incentive-compatible mechanism, Theorem 12.48 implies that

$$W_i(v_i) - W_i(x_i) \geq Q_i(x_i)(v_i - x_i), \quad \forall i \in N, \forall v_i \in \mathbb{V}_i, \forall x_i \in \mathbb{V}_i. \tag{12.149}$$

In particular:

$$W_i(v_i) - W_i(x_i) \geq Q_i(x_i)(v_i - x_i), \quad \forall v_i \geq x_i. \tag{12.150}$$

Reversing the roles of x_i and v_i in Equation (12.149), and re-inserting $v_i \geq x_i$, one gets

$$W_i(x_i) - W_i(v_i) \geq Q_i(v_i)(x_i - v_i), \quad \forall v_i \geq x_i. \tag{12.151}$$

Multiplying both sides of the inequality sign in this equation by -1 yields:

$$W_i(v_i) - W_i(x_i) \leq Q_i(v_i)(v_i - x_i), \quad \forall v_i \geq x_i. \tag{12.152}$$

Note the resemblance between the inequalities in Equations (12.150) and (12.152): the only difference is the argument of Q_i, and the direction of the inequality sign. Equations (12.150) and (12.152) imply that for each x_i and v_i,

$$Q_i(x_i)(v_i - x_i) \leq W_i(v_i) - W_i(x_i) \leq Q_i(v_i)(v_i - x_i), \quad \forall v_i \geq x_i. \tag{12.153}$$

For $v_i > x_i$, we can divide Equation (12.153) by $v_i - x_i$, which yields

$$\text{If } v_i > x_i \text{ then } Q_i(v_i) \geq Q_i(x_i). \tag{12.154}$$

That is, the function Q_i is a monotonically nondecreasing function, and is therefore in particular integrable.

We next turn to the proof that Equation (12.145) is satisfied. Let $v_i \in \mathbb{V}_i$, $v_i > 0$, and consider the integral $\int_0^{v_i} Q_i(t_i)\mathrm{d}t_i$ as the limit of Riemann sums of the function Q_i. Divide the interval $[0, v_i]$ into L intervals of length $\delta = \frac{v_i}{L}$; denote by $z^k = (k+1)\delta$ the rightmost (upper) end of the k-th interval, and by $x^k = k\delta$ its leftmost (lower) end. Inserting $v_i = z^k$ and $x_i = x^k$ in Equation (12.153) and summing over $k = 0, 1, \ldots, L-1$ yields

$$\sum_{k=0}^{L-1} Q_i(x^k)(z^k - x^k) \leq \sum_{k=0}^{L-1}(W_i(z^k) - W_i(x^k)) \leq \sum_{k=0}^{L-1} Q_i(z^k)(z^k - x^k). \tag{12.155}$$

The middle series is a telescopic series that sums to $W_i(v_i) - W_i(0)$. The left series is a Riemann sum, where the value of the function is taken to be its value at the leftmost end of each interval, and the right series is a Riemann sum, where the value of the function is taken to be its value at the rightmost end of the interval. By increasing L (letting δ approach 0), both the right series and the left series converge to $\int_0^{v_i} Q_i(t_i)\mathrm{d}t_i$, which yields

$$\int_0^{v_i} Q_i(t_i)\mathrm{d}t_i = W_i(v_i) - W_i(0), \tag{12.156}$$

and hence Equation (12.145) is satisfied. □

Corollary 12.50 *Let (q, μ) and $(\widetilde{q}, \widetilde{\mu})$ be two incentive-compatible direct selling mechanisms defined over the same selling problem $(N, (\mathbb{V}_i, f_i)_{i\in N})$ and satisfying:*

- *$q = \widetilde{q}$: the rule determining the winner is identical in both mechanisms.*
- *$\mu_i(0, v_{-i}) = \widetilde{\mu}_i(0, v_{-i})$: a buyer who reports that his private value is 0 pays the same sum in both mechanisms.*

Then at the truth-telling equilibrium β^* the expected profit of each buyer is the same in both mechanisms:

$$u_i(v_i, \beta^*_{-i}; v_i) = \widetilde{u}_i(v_i, \beta^*_{-i}; v_i), \tag{12.157}$$

where $\widetilde{u}_i(v_i, \beta^*_{-i}; v_i)$ is seller i's expected revenue under equilibrium β^* using the selling mechanism $(\widetilde{q}, \widetilde{\mu})$.

Proof: Since both mechanisms apply identical rules for determining the winner, the probability that a buyer with private value v_i wins is equal in both mechanisms: the second term in Equation (12.146) is therefore the same in both mechanisms. Since in both cases a buyer who reports 0 pays the same amount, the first term in Equation (12.146) is also the same in both mechanisms. Equation (12.146) therefore implies that the expected profit of each buyer is equal in both mechanisms. □

Since every auction is, in particular, a selling mechanism, Corollary 12.50 implies the Revenue Equivalence Theorem (Corollary 12.24, page 479) as the reader is asked to prove in Exercise 12.31.

12.9 Individually rational mechanisms

A direct selling mechanism is called individually rational if at the truth-telling equilibrium β^*, the expected profit of each buyer is nonnegative.

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Definition 12.51 *A direct selling mechanism is called* individually rational *if* $W_i(v_i) \geq 0$ *for each buyer* i *and for every* $v_i \in \mathbb{V}_i$.

If $W_i(v_i) < 0$, buyer i with private value v_i will not want to participate, because by doing so he is liable to lose. Therefore, assuming that the equilibrium β^* is attained, a buyer cannot lose by participating in a sale by way of a direct and individually rational selling mechanism.

Theorem 12.52 *An incentive-compatible direct selling mechanism is individually rational if and only if* $M_i(0) \leq 0$, *for each buyer* $i \in N$.

Proof: From Equations (12.145)–(12.146):

$$W_i(v_i) = -M_i(0) + \int_0^{v_i} Q_i(t_i)\mathrm{d}t. \tag{12.158}$$

Since the function Q_i is nonnegative, the right-hand side of the equation is minimal when $v_i = 0$. That is, $W_i(v_i)$ is minimal at $v_i = 0$. Therefore, the mechanism is individually rational if and only if $0 \leq W_i(0) = -M_i(0)$, that is, if and only if $M_i(0) \leq 0$, for each buyer $i \in N$, which is what we needed to prove. □

12.10 Finding the optimal mechanism

In this section, we will find the incentive-compatible and individually rational mechanism that maximizes the seller's expected revenue. We will assume the following condition, which is similar to Assumption (A5):

(B) For every buyer i, the density function f_i of buyer i's private values is positive over the interval $\mathbb{V}_i$.

Remark 12.53 *Changing a density function at a finite (or countable) number of points does not affect the distribution. Therefore, if a density function is zero at a finite number of points, it can be changed to satisfy Assumption (B).* ♦

Define, for each buyer i, a function $c_i : \mathbb{V}_i \to \mathbb{R}$ as follows:

$$c_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}. \tag{12.159}$$

This function depends only on the distribution of the buyer's private values. From Assumption (B), the function c_i is well defined over the interval $\mathbb{V}_i$.

Example 12.54 **The private value is uniformly distributed over** $[0, 1]$ Since $f_i(v_i) = 1$ and $F_i(v_i) = v_i$, for each $v_i \in [0, 1]$:

$$c_i(v_i) = v_i - \frac{1 - v_i}{1} = 2v_i - 1. \tag{12.160}$$

◀

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Example 12.55 **The distribution of the private values over** $[0, 1]$ **is given by the cumulative distribution function** $F_i(v_i) = v_i(2 - v_i)$ In this case, $f_i(v_i) = 2(1 - v_i)$, and therefore

$$c_i(v_i) = v_i - \frac{1 - v_i(2 - v_i)}{2(1 - v_i)} = \frac{3v_i - 1}{2}. \tag{12.161}$$

◀

We will first prove the following claim:

Theorem 12.56 $\mathbf{E}[c_i(V_i)] = 0$ *for each buyer* $i \in N$.

Proof:

$$\begin{aligned}\mathbf{E}[c_i(V_i)] &= \int_{\mathbb{V}_i} \left(v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right) f_i(v_i)\mathrm{d}v_i \\ &= \mathbf{E}[V_i] - \int_{\mathbb{V}_i} (1 - F_i(v_i))\mathrm{d}v_i = 0, \end{aligned} \tag{12.162}$$

where the last equality obtains because for every nonnegative random variable,

$$\mathbf{E}[X] = \int_0^\infty (1 - F_X(x))\mathrm{d}x. \tag{12.163}$$

□

Using the functions $(c_i)_{i \in N}$, we define a direct selling mechanism (q^*, μ^*).

Definition 12.57 *Define the direct mechanism* (q^*, μ^*) *as follows:*

$$q_i^*(v) = \begin{cases} 0 & c_i(v_i) \leq 0, \\ 0 & c_i(v_i) < \max_{j \in N} c_j(v_j), \\ \frac{1}{|\{l : c_l(v_l) = \max_{j \in N} c_j(v_j)\}|} & c_i(v_i) = \max_{j \in N} c_j(v_j) > 0. \end{cases} \tag{12.164}$$

$$\mu_i^*(v) = v_i q_i^*(v) - \int_0^{v_i} q_i^*(t_i, v_{-i})\mathrm{d}t_i. \tag{12.165}$$

In other words, buyer i wins the object (with positive probability) only if $c_i(v_i)$ is positive and maximal.

First, we will show that if the function c_i is nondecreasing, then the mechanism (q^*, μ^*) is incentive compatible and individually rational. Then we will show that, if c_i is monotonically nondecreasing, (q^*, μ^*) maximize the seller's expected revenue among the incentive-compatible and individually rational direct selling mechanisms.

Theorem 12.58 *If for each buyer* $i \in N$ *the function* c_i *is nondecreasing, then the direct mechanism* (q^*, μ^*) *is incentive compatible and individually rational.*

Proof: In the direct mechanism (q^*, μ^*), when each buyer reports his true private value, for each buyer i with private value is v_i denote by $Q_i^*(v_i)$ buyer i's probability of winning the object, by $M_i^*(v_i)$ the expected payment that buyer i makes, and by $U_i^*(v_i)$ his expected profit in this case. Also denote by $u_i^*(x_i, \beta_{-i}^*; v_i)$ the expected profit of buyer i with private value v_i if he reports x_i while all other buyers truthfully report their private value.

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Step 1: The function Q_i^* is nondecreasing: $Q_i^*(t_i) \geq Q_i^*(x_i)$ for all $t_i \geq x_i$.
Since the function c_i is nondecreasing, the definition of q^* (Equation (12.164)) implies that the greater the buyer's private value, the greater his probability of winning, i.e., for each buyer i, for any $t_i \geq x_i$ and for any $v_{-i} \in \mathbb{V}_{-i}$,

$$q_i^*(t_i, v_{-i}) \geq q_i^*(x_i, v_{-i}). \tag{12.166}$$

Integrating over $\mathbb{V}_{-i}$ yields

$$\int_{\mathbb{V}_{-i}} q_i^*(t_i, v_{-i}) f_{-i}(v_{-i}) \mathrm{d}v_{-i} \geq \int_{\mathbb{V}_{-i}} q_i^*(x_i, v_{-i}) f_{-i}(v_{-i}) \mathrm{d}v_{-i}. \tag{12.167}$$

By the definition of Q_i^* (see Equation (12.134)),

$$Q_i^*(t_i) \geq Q_i^*(x_i), \quad \forall i \in N, \forall t_i \geq x_i. \tag{12.168}$$

Step 2: $u_i^*(x_i, \beta_{-i}^*; v_i) = Q_i^*(x_i)(v_i - x_i) + \int_0^{x_i} Q_i^*(t_i)\mathrm{d}t_i$ for each $i \in N$ and every $x_i \in \mathbb{V}_i$.
By definition (see Equation (12.136)),

$$u_i^*(x_i, \beta_{-i}^*; v_i) = Q_i^*(x_i) v_i - M_i^*(x_i), \tag{12.169}$$

and

$$\begin{aligned} M_i^*(x_i) &= \int_{\mathbb{V}_{-i}} \mu_i^*(x_i, v_{-i}) f_{-i}(v_{-i}) \mathrm{d}v_{-i} \\ &= \int_{\mathbb{V}_{-i}} \left(x_i q_i^*(x_i, v_{-i}) - \int_0^{x_i} q_i^*(t_i, v_{-i}) \mathrm{d}t_i \right) f_{-i}(v_{-i}) \mathrm{d}v_{-i} \\ &= x_i Q_i^*(x_i) - \int_0^{x_i} Q_i^*(t_i) \mathrm{d}t_i, \end{aligned} \tag{12.170}$$

where the first equality follows from Equation (12.135), the second equality follows the definition of μ^* (Equation (12.165)), and the third equality follows from the definition of Q_i^*, and changing the order of integration. Inserting this equation into Equation (12.169), one has

$$u_i^*(x_i, \beta_{-i}^*; v_i) = Q_i^*(x_i)(v_i - x_i) + \int_0^{x_i} Q_i^*(t_i) \mathrm{d}t_i, \tag{12.171}$$

as claimed. This equation obtains for every $x_i \in \mathbb{V}_i$, and in particular, for $x_i = v_i$, it becomes $u_i^*(v_i, \beta_{-i}^*; v_i) = \int_0^{v_i} Q_i^*(t_i)\mathrm{d}t_i$.

Step 3: The mechanism (q^*, μ^*) is a incentive-compatible direct selling mechanism.
The mechanism is incentive compatible if and only if reporting the truth is an equilibrium:

$$u_i^*(v_i, \beta_{-i}^*; v_i) \geq u_i^*(x_i, \beta_{-i}^*; v_i), \quad \forall v_i, x_i \in \mathbb{V}_i. \tag{12.172}$$

By substituting the expression for $u_i^*(x_i, \beta_{-i}^*; v_i)$ obtained in Equation (12.171) into both the left-hand side and the right-hand side of Inequality (12.172) we deduce that the mechanism is incentive compatible if and only if

$$Q_i^*(x_i)(v_i - x_i) + \int_0^{x_i} Q_i^*(t_i) \mathrm{d}t_i \leq \int_0^{v_i} Q_i^*(t_i) \mathrm{d}t_i, \quad \forall v_i, x_i \in \mathbb{V}_i. \tag{12.173}$$

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The last equation holds if and only if

$$Q_i^*(x_i)(v_i - x_i) \leq \int_{x_i}^{v_i} Q_i^*(t_i)\mathrm{d}t_i, \quad \forall v_i, x_i \in \mathbb{V}_i. \tag{12.174}$$

By Step 1 the function Q_i^* is nondecreasing. Thus, if $x_i \leq v_i$ then $Q_i^*(x_i) \leq Q_i^*(t)$ for all $t \in [x_i, v_i]$, and therefore

$$Q_i^*(x_i)(v_i - x_i) = \int_{x_i}^{v_i} Q_i^*(x_i)\mathrm{d}t_i \leq \int_{x_i}^{v_i} Q_i^*(t_i)\mathrm{d}t_i \tag{12.175}$$

and Equation (12.174) holds. If $x_i > v_i$,

$$Q_i^*(x_i)(x_i - v_i) = \int_{v_i}^{x_i} Q_i^*(x_i)\mathrm{d}t_i \geq \int_{v_i}^{x_i} Q_i^*(t_i)\mathrm{d}t_i, \tag{12.176}$$

and therefore

$$Q_i^*(x_i)(v_i - x_i) = -Q_i^*(x_i)(x_i - v_i) \leq -\int_{v_i}^{x_i} Q_i^*(t_i)\mathrm{d}t_i = \int_{x_i}^{v_i} Q_i^*(t_i)\mathrm{d}t_i,$$

and Equation (12.174) also holds.

Step 4: The mechanism (q^*, μ^*) is individually rational.
Since the mechanism is direct and incentive compatible, by Theorem 12.52 it suffices to show that $M_i^*(0) \leq 0$ for every buyer $i \in N$. By Equation (12.165), for every $v_{-i} \in \mathbb{V}_{-i}$,

$$\mu_i^*(0, v_{-i}) = 0 \cdot q_i^*(0, v_{-i}) + \int_0^0 q_i^*(t_i, v_{-i})\mathrm{d}t_i = 0. \tag{12.177}$$

It follows that

$$M_i^*(0) = \int_{\mathbb{V}_{-i}} \mu_i^*(0, v_{-i}) f_{-i}(v_{-i})\mathrm{d}v_{-i} = 0, \tag{12.178}$$

and therefore in particular $M_i^*(0) \leq 0$. □

The next theorem shows that if the functions $(c_i)_{i \in N}$ are monotonically nondecreasing, then the mechanism (q^*, μ^*) is optimal from the seller's perspective.

Theorem 12.59 *Consider the selling problem $(N, (\mathbb{V}_i, f_i)_{i \in N})$ and suppose that the functions $(c_i)_{i \in N}$ defined by Equation (12.159) are monotonically nondecreasing. Then the mechanism (q^*, μ^*) defined by Equations (12.164)–(12.165) maximizes the seller's expected revenue within all incentive-compatible, individually rational direct selling mechanisms.*

Proof: The seller's revenue is the sum of the payments made by the buyers. His expected revenue, which we will denote by π, is therefore

$$\pi = \sum_{i \in N} \mathbf{E}[M_i(V_i)]. \tag{12.179}$$

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From[8] Equation (12.147),

$$\mathbf{E}[M_i(V_i)] = \int_0^{\overline{v}_i} M_i(v_i) f_i(v_i) \mathrm{d}v_i$$
$$= M_i(0) + \int_0^{\overline{v}_i} Q_i(v_i) v_i f_i(v_i) \mathrm{d}v_i$$
$$- \int_0^{\overline{v}_i} \left(\int_0^{v_i} Q_i(t_i) \mathrm{d}t_i \right) f_i(v_i) \mathrm{d}v_i. \quad (12.180)$$

Changing the order of integration in the last term, and using the fact that $F_i(\overline{v}_i) = 1$ yields:

$$\int_0^{\overline{v}_i} \left(\int_0^{v_i} Q_i(t_i) \mathrm{d}t_i \right) f_i(v_i) \mathrm{d}v_i = \int_0^{\overline{v}_i} \left(\int_{t_i}^{\overline{v}_i} Q_i(t_i) f_i(v_i) \mathrm{d}v_i \right) \mathrm{d}t_i$$
$$= \int_0^{\overline{v}_i} Q_i(t_i)(1 - F_i(t_i)) \mathrm{d}t_i. \quad (12.181)$$

It follows that

$$\mathbf{E}[M_i(V_i)] = M_i(0) + \int_0^{\overline{v}_i} Q_i(v_i) v_i f_i(v_i) \mathrm{d}v_i - \int_0^{\overline{v}_i} Q_i(t_i)(1 - F_i(t_i)) \mathrm{d}t_i \quad (12.182)$$
$$= M_i(0) + \int_0^{\overline{v}_i} Q_i(v_i) f_i(v_i) \left(v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right) \mathrm{d}v_i \quad (12.183)$$
$$= M_i(0) + \int_0^{\overline{v}_i} Q_i(v_i) c_i(v_i) f_i(v_i) \mathrm{d}v_i \quad (12.184)$$
$$= M_i(0) + \int_{\mathbb{V}^N} q_i(v) c_i(v_i) f_V(v) \mathrm{d}v. \quad (12.185)$$

Equation (12.184) follows from the definition of c_i, and Equation (12.185) holds by Equation (12.134) (page 497) together with the fact that the private values are independent. By summing over $i \in N$, we deduce that the seller should maximize the quantity

$$\pi = \sum_{i \in N} M_i(0) + \int_{\mathbb{V}^N} \left(\sum_{i \in N} q_i(v) c_i(v_i) \right) f_V(v) \mathrm{d}v. \quad (12.186)$$

The first term depends only on $(M_i(0))_{i \in N}$, i.e., only on $(\mu_i)_{i \in N}$, and the second term depends only on $(q_i)_{i \in N}$. To maximize π it therefore suffices to maximize each term separately.

Start with the second term. For each $v \in \mathbb{V}$, consider $\sum_{i \in N} q_i(v) c_i(v_i)$. What are the coefficients $(q_i(v))_{i \in N}$ that maximize this sum?

- If $c_i(v_i) < 0$ for every $i \in N$, the sum is maximized when $q_i(v) = 0$ for each $i \in N$.
- If $\max_{i \in N} c_i(v_i) \geq 0$, the maximum of the sum (under the constraint $\sum_{i \in N} q_i(v) \leq 1$) is $\max_{i \in N} c_i(v_i)$: give positive weights (summing to 1) only to those buyers for whom $c_i(v_i)$ is maximal.

8 When $\mathbb{V}_i$ is not bounded we denote $\overline{v}_i = \infty$ and $F_i(\overline{v}_i) := \lim_{v_i \to \infty} F_i(v_i)$.

Since $(q_i^*)_{i \in N}$ have been defined to satisfy these conditions (see Equation (12.164)), it follows that the second term in Equation (12.186) is maximal for $q = q^*$. We next turn to the first term in Equation (12.186). By Theorem 12.52, $M_i(0) \leq 0$ for every individually rational direct selling mechanism. Therefore, the first term is not greater than 0. As we proved in Step 4 above (Equation (12.178)), $M_i^*(0) = 0$ for all $i \in N$, and therefore μ^* maximizes the first term in Equation (12.186). The definition of $(\mu_i^*)_{i \in N}$ implies that $M_i^*(0) = 0$. It follows that there is no incentive-compatible direct selling mechanism that yields the seller an expected revenue greater than his expected revenue yielded by (q^*, μ^*). □

Example 12.60 **Private values distributed over different intervals** Suppose that there are two buyers, where buyer 1's private value is uniformly distributed over $[0, 1]$, and buyer 2's private value is uniformly distributed over $[0, 2]$. As we saw in Example 12.54, $c_1(v_1) = 2v_1 - 1$, and therefore $c_1(v_1) < 0$ if and only if $v_1 < \frac{1}{2}$. For buyer 2, $f_2(v_2) = \frac{1}{2}$ and $F_2(v_2) = \frac{v_2}{2}$, and therefore $c_2(v_2) = v_2 - \frac{1 - v_2/2}{1/2} = 2v_2 - 2$. It follows that $c_2(v_2) < 0$ if and only if $v_2 < 1$. Since

$$c_1(v_1) > c_2(v_2) \iff 2v_1 - 1 > 2v_2 - 2 \iff v_1 > v_2 - \tfrac{1}{2}, \qquad (12.187)$$

the optimal allocation rule q^* is defined as follows:

- If $v_1 < \frac{1}{2}$ and $v_2 < 1$, the object is not sold ($q_i^*(v_i) = 0$ for $i = 1, 2$).
- If $v_1 \geq \frac{1}{2}$ and $v_1 > v_2 - \frac{1}{2}$, buyer 1 wins the object, ($q_1^*(v_1) = 1$ and $q_2^*(v_2) = 0$).
- If $v_2 > 1$ and $v_2 > v_1 + \frac{1}{2}$, buyer 2 wins the object, ($q_2^*(v_2) = 1$ and $q_1^*(v_1) = 0$).
- If $v_1 \geq \frac{1}{2}$, $v_2 \geq 1$ and $v_1 = v_2 - \frac{1}{2}$, each buyer wins the object with probability $\frac{1}{2}$, ($q_1^*(v_1) = q_2^*(v_2) = \frac{1}{2}$).

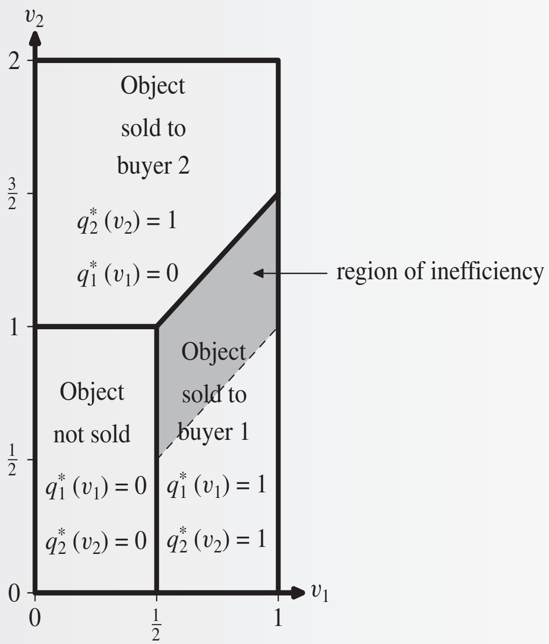

Figure 12.5 The function q^* in Example 12.60

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To compute the payment function we first compute

$$\int_0^{v_1} q_1^*(t_1, v_2)\mathrm{d}t_1 = \begin{cases} 0 & v_1 < \frac{1}{2}, \\ 0 & v_1 < v_2 - \frac{1}{2}, \\ v_1 - \max\left\{\frac{1}{2}, v_2 - \frac{1}{2}\right\} & v_1 \geq \frac{1}{2} \text{ and } v_1 \geq v_2 - \frac{1}{2}. \end{cases} \tag{12.188}$$

$$\int_0^{v_2} q_2^*(v_1, t_2)\mathrm{d}t_2 = \begin{cases} 0 & v_2 < 1, \\ 0 & v_2 < v_1 + \frac{1}{2}, \\ v_2 - \max\left\{1, v_1 + \frac{1}{2}\right\} & v_1 \geq 1 \text{ and } v_2 \geq v_1 + \frac{1}{2}. \end{cases} \tag{12.189}$$

Equation (12.165) yields

$$\mu_1^*(v_1, v_2) = \begin{cases} 0 & \text{Buyer 1 does not win,} \\ \max\left\{\frac{1}{2}, v_2 - \frac{1}{2}\right\} & \text{Buyer 1 wins.} \end{cases} \tag{12.190}$$

$$\mu_2^*(v_1, v_2) = \begin{cases} 0 & \text{Buyer 2 does not win,} \\ \max\left\{1, v_1 + \frac{1}{2}\right\} & \text{Buyer 2 wins.} \end{cases} \tag{12.191}$$

Note that in this case, each buyer has a different minimum price: $\frac{1}{2}$ for buyer 1 and 1 for buyer 2. Note also that when $\frac{1}{2} < v_1 < v_2 < v_1 + \frac{1}{2}$ (the shaded area in Figure 12.5), buyer 1 wins the object, despite the fact that his private value is less than buyer 2's private value. In other words, the selling mechanism that maximizes the seller's expected revenue is not efficient: the winner is not necessarily the buyer with the highest private value. ◀

When private values are independent and identically distributed, Theorem 12.59 leads to the following corollary.

Corollary 12.61 *If the private values of the buyers are independent and identically distributed, and if the functions $(c_i)_{i \in N}$ are monotonically nondecreasing, the incentive-compatible direct selling mechanism that maximizes the seller's expected revenue is a sealed-bid second-price auction with a reserve price.*

Proof: Since the private values of the buyers are identically distributed, $c_i = c_j =: c$ for every pair of buyers i, j. The function c is monotonically nondecreasing, and therefore a buyer for whom $c(v_i)$ is maximal is one whose private value v_i is maximal. Denote

$$\rho^* = \inf\{t_i \in \mathbb{V}_i : c_i(t_i) > 0\}, \tag{12.192}$$

which is independent of i, since $\mathbb{V}_i = \mathbb{V}_j$ for all $i, j \in N$.

To simplify the analysis, suppose that c_i is a continuous function, and therefore $c_i(\rho^*) = 0$. By the definition of q^* (Equation (12.164)), the buyer who wins the object is the one who submits the highest bid, as long as that bid is greater than ρ^*:

$$q_i^*(v_i) = \begin{cases} 0 & \text{if } v_i < \rho^* \text{ or } v_i \leq \max_{j \in N} v_j, \\ \frac{1}{|\{l : v_l = \max_{j \in N} v_j\}|} & \text{if } v_i > \rho^* \text{ and } v_i = \max_{j \in N} v_j. \end{cases} \tag{12.193}$$

We next calculate $\mu^*(v_i) = v_i q_i^*(v_i) - \int_0^{v_i} q_i^*(t_i, v_{-i})\mathrm{d}t_i$ (see Equation (12.165) on page 502):

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- If $v_i \leq \rho^*$ or $v_i < \max_{j \neq i} v_j$, then $q_i^*(v_i) = 0$, and $q_i^*(t_i) = 0$ for each $t_i \in [0, v_i)$, and hence in this case

$$\mu_i^*(v_i) = v_i q_i^*(v_i) - \int_0^{v_i} q_i^*(t_i, v_{-i}) \mathrm{d}t_i = 0. \tag{12.194}$$

  In other words, a buyer whose bid is lower than the highest bid or less than or equal to ρ^* pays nothing.
- If $v_i > \rho^*$ and $v_i \geq \max_{j \neq i} v_j$ then $q_i^*(v_i) = \frac{1}{|\{i : v_i = \max_{j \in N} v_j\}|}$, and $q_i^*(t_i) = 0$ for all $t_i \in [0, v_i)$. Hence, in this case,

$$\mu_i^*(v_i) = v_i q_i^*(v_i) - \int_0^{v_i} q_i^*(t_i, v_{-i}) \mathrm{d}t_i = \frac{v_i}{|\{l : v_l = \max_{j \in N} v_j\}|}. \tag{12.195}$$

  In words, all the buyers who bid the maximum bid, provided it is at least ρ^*, equally share the payment to the seller.

In summary,

$$\begin{aligned} \mu^*(v_i) &= v_i q_i^*(v_i) - \int_0^{v_i} q_i^*(t_i, v_{-i}) \mathrm{d}t_i \\ &= \begin{cases} 0 & v_i < \rho^* \text{ or } v_i < \max_{j \in N} v_j, \\ \frac{\max_{j \neq i} v_j}{|\{i : c_i(v_i) = \max_{j \in N} c_j(v_j)\}|} & v_i \geq \rho^* \text{ and } v_i = \max_{j \in N} v_j. \end{cases} \end{aligned} \tag{12.196}$$

In other words, (q^*, μ^*) is a sealed-bid second-price auction with a reserve price ρ^*. □

12.11 Remarks

The first use of game theory to study auctions was accomplished by economist William Vickrey [1961, 1962]. Vickrey, 1914–96, was awarded the Nobel Memorial Prize in Economics in 1996 for his contributions to the study of incentives when buyers have different information, and the implications incentives have on auction theory. The results in Section 12.7 are based on Holt [1980].

In this chapter we studied symmetric sealed-bid auction with independent private values. The theory of asymmetric sealed-bid auctions and sealed-bid auctions in which the private values of the buyers are not independent is mathematically complex. The interested reader is directed to Milgrom and Weber [1982], Lebrun [1999], Maskin and Riley [2000], Fibich, Gavious, and Sela [2004], Reny and Zamir [2004], and Kaplan and Zamir [2011, 2012].

The significance of mechanism design in economic theory was recognized by the Nobel Prize Committee in 2007, when it awarded the prize to three researchers who were instrumental in developing mechanism design, Leonid Hurwicz, Eric Maskin, and Roger Myerson. The revelation principle was proved by Myerson [1979], and the structure of the optimal mechanism was proved by Myerson [1981]. The reader interested in further deepening his understanding of auction theory and mechanism design is directed to Krishna [2002], Milgrom [2004], or Klemperer [2004].

Exercise 12.1 is based on Wolfstetter [1996]. Exercises 12.19, 12.45 and 12.46 are based on examples that appear in Krishna [2002]. Exercises 12.20 and 12.21 are based on Kaplan and Zamir [2012].

The authors wish to thank Vijay Krishna for answering questions during the composition of this chapter. Many thanks are due to the students in the Topics in Game Theory course that was conducted at Tel Aviv University in 2005, for their many comments on this chapter, with special thanks going to Ronen Eldan and Ayala Mashiah-Yaakovi.

12.12 Exercises

In each of the exercises in this chapter, assume that buyers are risk-neutral, unless it is explicitly noted otherwise.

12.1 **Selling an object at the monopoly price** Andrew is interested in selling a rare car (whose value in his eyes we will normalize to 0). Assume there are n buyers and that buyer i's private value of the car, V_i, is uniformly distributed over $[0, 1]$. The private values of the buyers are independent.

Instead of conducting an auction, Andrew intends on setting a price for the car, publicizing this price, and selling the car only to buyers who are willing to pay this price; if no buyer is willing to pay the price, the car will not be sold, and if more than one buyer is willing to pay the price, the car will be sold to one of them based on a fair lottery that gives each of them equal probability of winning.

Answer the following questions:

(a) Find Andrew's expected revenue as a function of the price x that he sets.
(b) Find the price x^* that maximizes Andrew's expected revenue.
(c) What is the maximal expected revenue that Andrew can obtain, as a function of n?
(d) Compare Andrew's maximal revenue with the revenue he would gain if he sells the car by way of a sealed-bid first-price auction. For which values of n does a sealed-bid first-price auction yield a higher revenue?

12.2 (a) Explain what a buyer in an open-bid decreasing auction knows when the current announced price is x that he did not know prior to the auction.
(b) Explain what a buyer in an open-bid increasing auction knows when the current announced price is x that he did not know prior to the auction.

12.3 Prove that in a symmetric sealed-bid second-price auction with independent private values the only monotonically increasing, symmetric equilibrium is the equilibrium in which every buyer submits a bid equal to his private value.

12.4 Suppose that $\mathbb{V} = [0, \overline{v}]$ is a bounded interval. Show that in a symmetric sealed-bid second-price auction with independent private values the strategy vector under which buyer 1 bids $\overline{v}$ and all the other buyers bid 0 is an (asymmetric) equilibrium. Is it also an equilibrium in a sealed-bid first-price auction? Justify your answer. Is there an equilibrium in an open-bid ascending auction that is analogous to this equilibrium?

12.5 Consider a sealed-bid second-price auction where the private values of the buyers are independent and identically distributed with the uniform distribution over $[0, 1]$. Show that the strategy under which a buyer bids his private value does not strictly dominate all his other strategies.

12.6 Brent and Stuart are the only buyers in a sealed-bid first-price auction of medical devices. Brent knows that Stuart's private value is uniformly distributed over $[0, 2]$, and that Stuart's strategy is $\beta(v) = \frac{v^2}{3} + \frac{v}{3}$.

(a) What is Brent's optimal strategy?
(b) What is Brent's expected payment if he implements this optimal strategy (as a function of his own private value)?

12.7 (a) Suppose that the private values of two buyers in a sealed-bid first-price auction are independent and uniformly distributed over the set $\{0, 1, 2\}$. In other words, each buyer has three possible private values. The bids in the auction must be nonnegative integers. Find all the equilibria.
(b) Find all the equilibria, under the same assumptions, when the auction is a sealed-bid second-price auction.
(c) Compare the seller's expected revenue under both auction methods. What have you discovered?

12.8 Denote by E^I the seller's revenue in a sealed-bid first-price auction, and by E^{II} the seller's revenue in a sealed-bid second-price auction. Find the variance of E^I and of E^{II} when there are n buyers, whose private values are independent and uniformly distributed over $[0, 1]$. Which of the two is the lesser?

12.9 Consider a sealed-bid second-price auction with n buyers whose private values are independent and uniformly distributed over $[0, 1]$. Find an asymmetric equilibrium at which every buyer wins with probability $\frac{1}{n}$. Can you find an equilibrium at which every buyer i wins with probability α_i, for any collection of nonnegative numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ whose sum is 1?

12.10 Consider a sealed-bid first-price auction with three buyers, where the private values of the buyers are independent and uniformly distributed over $[0, 2]$.

(a) Find a symmetric equilibrium of the game.
(b) What is the seller's expected revenue?

12.11 Prove that in a symmetric sealed-bid auction with independent private values the random variable $Y = \max\{V_2, V_3, \ldots, V_n\}$ is a continuous random variable and its density function f_Y is a positive function.

12.12 Which of the following auction methods satisfy the conditions of the Revenue Equivalence Theorem (Corollary 12.24 on page 479): sealed-bid first-price auctions (see Example 12.15 on page 472), sealed-bid second-price auctions, sealed-bid first-price auctions with a reserve price (see Example 12.34 on page 486), sealed-bid second-price auctions with a reserve price, sealed-bid second-price auctions with entry fees (see Example 12.28 on page 482). When only some of the conditions are satisfied, specify which conditions are not satisfied, and justify your answer.

12.13 Compute the seller's expected revenue in a sealed-bid second-price auction with a reserve price ρ, in which there are two buyers whose private values are independent and uniformly distributed over $[0, 1]$. Compare the results here with the results we computed for sealed-bid first-price auctions with a reserve price ρ (Equation (12.104) on page 488).

12.14 (a) Compute the seller's expected revenue in a sealed-bid first-price auction with a reserve price ρ, with n buyers whose private values are independent and uniformly distributed over $[0, 1]$.
(b) What is the reserve price ρ^* that maximizes the seller's expected revenue?
(c) Repeat items (a) and (b) for a sealed-bid second-price auction with a reserve price ρ.
(d) Did you obtain the same results in both cases? Explain why.
(e) Compare the expected revenue computed here with the expected revenue of a seller who is selling the object by setting the monopoly price (see Exercise 12.1). Which expected revenue is higher?
(f) What does the optimal reserve price in items (b) and (c) above converge to when the number of buyers increases to infinity?

12.15 Consider a symmetric sealed-bid first-price auction with independent private values with n buyers whose cumulative distribution function F is given by $F(v) = v^2$. Find the symmetric equilibrium, compute $e_i(v_i)$ (the expected payment of buyer i if his private value is v_i), compute e_i (the payment that buyer i makes), and compute π (the seller's expected revenue).

12.16 Repeat Exercise 12.15 when the cumulative distribution function of each buyer i's private value is $F_i(v) = v^3$.

12.17 Consider a sealed-bid second-price auction with entry fee λ and n buyers, whose private values are independent and uniformly distributed over $[0, 1]$.

(a) Find a symmetric equilibrium.
(b) What is the seller's expected revenue?
(c) Which entry fee maximizes the seller's expected revenue?
(d) What value does the optimal entry fee approach as the number of buyers increases to ∞?

12.18 Repeat Exercise 12.17 when the auction method conducted is a sealed-bid first-price auction with entry fee λ.

12.19 In this exercise, using Theorem 12.23 (page 478), compute a symmetric equilibrium β in a sealed-bid third-price auction, with n buyers whose private values are independent; each V_i has uniform distribution over $[0, 1]$. The winner of this auction is the buyer submitting the highest bid, and he pays the third-highest bid.[9] If several buyers have submitted the highest bid, the winner is chosen from among them by a fair lottery granting each equal probability of winning.

Denote the highest bid from among $V_2, V_3, \ldots, V_n$ by Y, and the second-highest bid from among $V_2, V_3, \ldots, V_n$ by W. Denote by F_i the cumulative distribution function of V_i, and by f_i its density function.

(a) Prove that for every $v_1 \in (0, 1]$, the conditional cumulative distribution function of W, given $Y \leq v_1$, is

$$F_{(W|Y\leq v_1)}(w) = (F_i(w))^{n-2} \times \frac{(n-1)F_i(v_1) - (n-2)F_i(w)}{(F_i(v_1))^{n-1}}, \quad \forall w \in [0, v_1]. \quad (12.197)$$

(b) Compute the conditional density function $f_{(W|Y\leq v_1)}$.

(c) Denote

$$h(y) = (n-2)(F_1(y))^{n-3} f_1(y), \quad \forall y \in [0, 1]. \quad (12.198)$$

The Revenue Equivalence Theorem implies that the expected payment of buyer 1 with private value $v_1 \in (0, 1]$ is given by $F_Y(v_1)\mathbf{E}[\beta(W) \mid Y \leq v_1]$. Conclude from this that

$$\int_0^{v_1} \beta(y)(n-1)h(y)(F_1(v_1) - F_1(y))\mathrm{d}y = \int_0^{v_1} y f_Y(y)\mathrm{d}y, \quad \forall v_1 \in (0, 1]. \quad (12.199)$$

(d) Differentiate Equation (12.199) by v_1, and show that

$$\int_0^{v_1} \beta(y)h(y)\mathrm{d}y = v_1(F_1(v_1))^{n-2}, \quad \forall v_1 \in (0, 1). \quad (12.200)$$

(e) Differentiate Equation (12.200) by v_1, and show that the solution to this equation is

$$\beta(v_1) = v_1 + \frac{F_1(v_1)}{(n-2)f_1(v_1)}, \quad \forall v_1 \in (0, 1). \quad (12.201)$$

(f) Under what conditions is β a symmetric equilibrium?

12.20 Consider a sealed-bid first-price auction with two buyers whose private values are independent; the private value of buyer 1 has uniform distribution over the interval $[0, 3]$, and the private value of buyer 2 has uniform distribution over the interval $[3, 4]$. Answer the following questions:

(a) Prove that the following pair of strategies form an equilibrium

$$\beta_1(v_1) = 1 + \tfrac{v_1}{2}, \quad (12.202)$$

$$\beta_2(v_2) = \tfrac{1}{2} + \tfrac{v_2}{2}. \quad (12.203)$$

9 If only two buyers have submitted the highest bid, the next-highest bid is the sum of money that the winner pays for the auctioned object. If three buyers have submitted the highest bid, that bid is the amount of money the winner pays for the auctioned object.

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(b) Is the probability that buyer 2 wins the auction equal to 1?

(c) Compute the seller's expected revenue if the buyers implement the strategies (β_1, β_2).

(d) Compute the seller's expected revenue in a sealed-bid second-price auction. Is that the same expected revenue as in part (c) above?

12.21 Consider a sealed-bid second-price auction with two buyers whose private values are independent; the private value of buyer 1 has uniform distribution over the interval $[0, m+z]$, and the private value of buyer 2 has uniform distribution over the interval $[\frac{3m}{2}, \frac{3m}{2}+z]$, where $m, z > 0$. Show that the following pair of strategies form an equilibrium.

$$\beta_1(v_1) = \tfrac{v_1}{2} + \tfrac{m}{2}, \tag{12.204}$$

$$\beta_2(v_2) = \tfrac{v_2}{2} + \tfrac{m}{4}. \tag{12.205}$$

Note that these equilibrium strategies are independent of z. This is a generalization of Exercise 12.20 (which is the special case in which $m = 2, z = 1$).

12.22 Prove Theorem 12.29 (page 482): in a sealed-bid second-price auction with a reserve price, for each buyer strategy β the following strategy $\widehat{\beta}$ weakly dominates β, if $\widehat{\beta} \neq \beta$,

$$\widehat{\beta}(v) = \begin{cases} \text{"no"} & \beta(v) = \text{"no"}, \\ v & \beta(v) = x. \end{cases} \tag{12.206}$$

12.23 Consider a sealed-bid second-price auction with two buyers, whose private values are independent; buyer 1's private value is uniformly distributed over [0, 1], and buyer 2's private value is uniformly distributed over [0, 2].

(a) For each buyer, find all weakly dominant strategies.

(b) Consider the equilibrium in which every buyer bids his private value. What is the probability that buyer 1 wins the auction, under this equilibrium? What is the seller's expected revenue in this case?

(c) Prove that at any equilibrium $\beta = (\beta_1, \beta_2)$ satisfying $\beta_1(v) = \beta_2(v)$ for all $v \in [0, 1]$, one has $\beta_1(v) = \beta_2(v) = v$ for all $v \in [0, 1]$.

(d) Are there equilibria at which a buyer, whose private value v_i is less than 1, does not submit the bid $\beta_i(v_i) = v_i$?

12.24 (a) Prove that if F is a cumulative distribution function over [0, 1], and if Y is the maximum of $n-1$ independent random variables with cumulative distribution function F, then

$$\mathbf{E}[Y \mid Y \leq v] = v - \frac{\int_0^v (F(x))^{n-1} \mathrm{d}x}{(F(v))^{n-1}}. \tag{12.207}$$

Hint: Differentiate the function $x(F(x))^{n-1}$.

(b) Use Equation (12.207) to write explicitly the symmetric equilibrium β^* in a symmetric sealed-bid first-price auction with independent private

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values with n buyers when $\mathbb{V} = [0, 1]$ and (a) $F(x) = x$, (b) $F(x) = x^2$, and (c) $F(x) = x(2 - x)$.

12.25 Prove that in a sealed-bid second-price auction, the strategy under which a buyer submits a bid equal to his private value weakly dominates all his other strategies, also when the buyer is risk-averse or risk-seeking.

12.26 Prove that in a symmetric sealed-bid auction with independent private values in which all the buyers are risk-seeking, and have the same strictly convex, differentiable, and monotonically increasing utility function, the seller's expected revenue is lower in a sealed-bid first-price auction than in a sealed-bid second-price auction.

12.27 Suppose that in a sealed-bid first-price auction there are n buyers whose private values are independent and uniformly distributed over $[0, 1]$. Suppose further that the utility function of all buyers is $U(x) = x^c$.

(a) Find the values c for which the buyers are risk-averse, risk-neutral, and risk-seeking.
(b) Prove that a symmetric equilibrium γ must satisfy the following differential equation:

$$\gamma'(x) = \frac{n-1}{c}\frac{f(x)}{F(x)}(x - \gamma(x)). \tag{12.208}$$

(c) Show that the following strategy is a symmetric equilibrium:

$$\gamma(v) = v - \frac{\int_0^v F^{\frac{n-1}{c}}(x)\mathrm{d}x}{F^{\frac{n-1}{c}}(v)}. \tag{12.209}$$

(d) Compute the symmetric equilibrium γ for the case that F is the uniform distribution over $[0, 1]$.
(e) Compare the strategy that you found for arbitrary c, with the symmetric equilibrium in the case in which the buyers are risk-neutral (see Exercise 12.24). Ascertain that risk-averse buyers submit higher bids than risk-neutral buyers, and that risk-seeking buyers submit lower bids than risk-neutral buyers.
(f) What is the seller's expected revenue as a function of c? Is this an increasing function?

This exercise shows that a symmetric equilibrium in a sealed-bid first-price auction with n buyers, where the utility function of each buyer is $U(x) = x^c$, is also a symmetric equilibrium in the same auction with $\frac{n-1}{c} + 1$ risk-neutral buyers.[10] In other words, a risk-averse buyer behaves like a risk-neutral buyer in an auction with more buyers, and therefore increases his bid.

12.28 Prove Theorem 12.42 on page 494: every sealed-bid auction with risk-neutral buyers can be presented as a selling mechanism.

12.29 Can there be more than one equilibrium for an incentive-compatible direct selling mechanism? If your answer is yes, present an example. If not, justify your answer.

10 Under the assumption that $\frac{n-1}{c}$ is an integer.

If your answer is yes, do all these equilibria yield the same expected revenue for the seller? Justify your answer.

12.30 Prove the revelation principle (Theorem 12.47 on page 496): let $(\Theta_i, \widehat{q}_i, \widehat{\mu}_i)_{i \in N}$ be a selling mechanism, and $\widehat{\beta}$ be an equilibrium of this mechanism. Then the outcome of this mechanism under $\widehat{\beta}$ is identical to the outcome under β^* (the equilibrium when all buyers reveal their true values) in mechanism (q, μ), defined by Equations (12.132)–(12.133).

12.31 Let (p, C) and $(\widetilde{p}, \widetilde{C})$ be two symmetric sealed-bid auctions with independent private values defined over the same set N of risk-neutral buyers. Suppose that in both auctions the winner of the auction is the buyer who submits the highest bid. and a buyer who submits a bid of 0 pays nothing. Let β and $\widetilde{\beta}$ be symmetric and monotonically increasing equilibrium strategies in (p, C) and in $(\widetilde{p}, \widetilde{C})$, respectively (for the same distributions of private values). Using Corollary 12.50 (page 500) prove that the seller's expected revenue is the same under both equilibria, and the buyer's expected profit given his private value is also identical in both auctions.

12.32 Consider the following cumulative distribution function, where $k \in \mathbb{N}$:

$$F_i(v_i) = (v_i)^k, \quad 0 \le v \le 1. \tag{12.210}$$

Compute the function c_i (see Equation (12.159) on page 501). For which values k is the function c_i monotonically increasing?

12.33 Suppose that V_i is distributed according to the exponential distribution with parameter λ (i.e, $\mathbb{V}_i = [0, \infty)$ and $f_i(v_i) = \lambda e^{-\lambda v_i}$ for each $v_i \ge 0$). Compute the function c_i. Is c_i monotonically increasing?

12.34 For each of the following auctions and their respective equilibria β, construct an incentive-compatible direct selling mechanism whose truth-telling equilibrium β^* is equivalent to the equilibrium β.

(a) Auction method: sealed-bid second-price auction; equilibrium $\beta = (\beta_i)_{i \in N}$, where $\beta_i(v_i) = v_i$ for each buyer i and each $v_i \in \mathbb{V}_i$.
(b) Auction method: a sealed-bid second-price auction in which $\mathbb{V}_i = [0, 1]$ for every buyer i, equilibrium $\beta = (\beta_i)_{i \in N}$ where $\beta_1(v_1) = 1$ for all $v_1 \in [0, 1]$ and $\beta_i(v_i) = 0$ for each buyer $i \neq 1$ and all $v_1 \in [0, 1]$.
(c) Auction method: sealed-bid first-price auction with $n = 2$, and the private values of the buyers are independent and uniformly distributed over $[0, 1]$; equilibrium $\beta = (\beta_i)_{i=1,2}$, given by $\beta_i(v_i) = \frac{v_i}{2}$.
(d) Auction method: sealed-bid first-price auction with a reserve price ρ and two buyers whose private values are independent and uniformly distributed over the interval $[0, 1]$; equilibrium given in Example 12.34 (page 486).

12.35 Suppose that there are n buyers whose private values are independent and uniformly distributed over $[0, 1]$. Answer the following questions:

(a) What is the individually rational, incentive-compatible direct selling mechanism that maximizes the seller's expected revenue?

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(b) In this mechanism, what is each buyer's probability of winning the object, assuming that each buyer reports his true private value?
(c) What is the seller's expected revenue in this case?

12.36 Repeat Exercise 12.35 for the case in which there are two buyers, and their private values V_1 and V_2 are independent; V_1 is uniformly distributed over $[0, 2]$ and V_2 is uniformly distributed over $[0, 3]$.

12.37 Repeat Exercise 12.35 for the case in which there are two buyers, and their private values V_1 and V_2 are independent, V_1 is uniformly distributed over $[0, 1]$ and V_2 is uniformly distributed over $[0, 1]$ given by the cumulative distribution function $F_2(v) = v^2$.

12.38 What is the seller's expected revenue in a sealed-bid first-price auction in which there are two buyers, and their private values V_1 and V_2 are independent; V_1 is uniformly distributed over $[0, 2]$ and V_2 is uniformly distributed over $[0, 3]$?
Hint: Use the Revenue Equivalence Theorem.

12.39 Repeat Exercise 12.38 for the case in which there are two buyers, and their private values V_1 and V_2 are independent; V_1 is uniformly distributed over $[0, 1]$ and V_2 is uniformly distributed over $[0, 3]$.

12.40 Partition the following list of auction methods into groups, each of which contains methods whose symmetric equilibrium yields the same expected revenue for the seller. In all these methods consider the symmetric case with independent private values and a symmetric and monotonic increasing equilibrium.

(a) Sealed-bid first-price auction.
(b) Sealed-bid first-price auction with a reserve price ρ.
(c) Sealed-bid first-price auction with an entry fee λ.
(d) Sealed-bid second-price auction.
(e) Sealed-bid second-price auction with a reserve price ρ.
(f) Sealed-bid second-price auction with an entry fee λ.
(g) Sealed-bid third-price auction (in which the winning buyer is the one who submits the highest bid, and the price he pays for the object is the third-highest price bid. See Exercise 12.19).
(h) Sealed-bid all-pay auction.
(i) Sealed-bid all-pay auction with a reserve price ρ.

12.41 Among the auction methods listed in Exercise 12.40, do there exist methods in which a buyer's submitted bid in an equilibrium may be greater than his private value? Justify your answer.

12.42 Suppose that there are n buyers participating in an auction, where the private values $V_1, V_2, \ldots, V_n$ are independent and identically distributed, with the cumulative distribution function $F_i(v_i) = (v_i)^2$ for $v \in [0, 1]$. Which auction maximizes the seller's expected revenue? What is the seller's expected revenue in that auction?

12.43 Suppose there are n buyers participating in an auction, where the private values $V_1, V_2, \dots, V_n$ are independent and identically distributed, with the cumulative distribution function

$$F_i(v_i) = \tfrac{1}{2}v_i + \tfrac{1}{2}(v_i)^2, \quad v_i \in [0,1]. \tag{12.211}$$

Answer the following questions:

(a) Which auction maximizes the seller's expected revenue?
(b) What is the seller's expected revenue in this auction? (To answer this, it suffices to write the formula for the seller's expected revenue, and specify the values of the variables. There is no need to compute the formula explicitly.)
(c) Is the seller's expected revenue monotonically increasing as the number of buyers in the auction increases? Justify your answer.

12.44 Suppose there are n buyers participating in an auction, where the private values $V_1, V_2, \dots, V_n$ are independent and for each $i \in \{1, 2, \dots, n\}$, v_i is uniformly distributed over $[0, \overline{v}_i]$. Suppose further that $\overline{v}_1 < \overline{v}_2 < \cdots < \overline{v}_n$. Answer the following questions:

(a) Which selling mechanism maximizes the seller's expected revenue?
(b) What is the seller's expected revenue under this mechanism?
(c) What is the probability that buyer n wins the object under this mechanism?

In the last two items, it suffices to write down the appropriate formula, with no need to solve it explicitly.

12.45 Suppose there are two buyers with independent private values uniformly distributed over $[0, 1]$. Buyer 2 faces budget limitations, and the maximal sum that he can bid is $\frac{1}{4}$. Buyer 1 is free of budget limitations. Answer the following questions:

(a) Find an equilibrium if the buyers are participating in a sealed-bid second-price auction.
(b) Compute the seller's expected revenue, given the equilibrium you found.

Consider what happens if the buyers participate instead in a sealed-bid first-price auction. To avoid a situation in which buyer 1 bids a price $\frac{1}{4} + \varepsilon$, where $\varepsilon > 0$ is very small, define the function p according to which if both buyers bid $\frac{1}{4}$ buyer 1 is declared the winner (and if both buyers submit an identical bid that is lower than $\frac{1}{4}$, each of them is chosen the winner with equal probability $\frac{1}{2}$). Answer the following questions, and justify your answers:

(c) Is the following strategy vector (β_1, β_2) an equilibrium?

$$\beta_1(v_1) = \tfrac{v_1}{2}, \qquad \beta_2(v_2) = \min\left\{\tfrac{v_2}{2}, \tfrac{1}{4}\right\}. \tag{12.212}$$

(d) If your answer to item (c) is negative, find a nondecreasing equilibrium.
(e) Let (β_1^*, β_2^*) be a nondecreasing equilibrium. What are $\beta_1^*(0)$ and $\beta_2^*(0)$? Is $\beta_1^*(1) = \beta_2^*(1)$?

(f) Does Corollary 12.50 (page 500) enable you to deduce that the seller's expected revenue under the nondecreasing equilibrium in this case equals the expected revenue that you found in item (b)?

(g) Does Theorem 12.59 (page 504) enable you to deduce that the individually rational, incentive-compatible direct selling mechanism that maximizes the seller's expected revenue is a sealed-bid second-price auction with a reserve price?

12.46 This exercise explores the case in which the number of buyers in an auction is unknown.

Suppose that there are N potential buyers whose private values $V_1, V_2, \ldots, V_N$ are independent and have identical continuous distribution over $[0, 1]$. Denote by F the common cumulative distribution function.

The number of buyers participating in this auction is unknown to any of the participating buyers. Each buyer ascribes probability p_n to the event that there are n participating buyers, in addition to himself, where $\sum_{n=0}^{N-1} p_n = 1$. Note that each buyer has the same belief about the distribution of the number of buyers in the auction.

(a) Find a symmetric equilibrium of this situation when the selling takes place by way of a sealed-bid second-price auction. Explain why this is an equilibrium.

(b) Denote $G^{(n)}(z) = (F(z))^n$. Prove that the expected profit of a participating buyer, whose private value is v, is

$$\sum_{n=0}^{N-1} p_n G^{(n)}(v) \mathbf{E}\big[Y_1^{(n)} \mid Y_1^{(n)} < v\big], \tag{12.213}$$

where $Y_1^{(n)}$ is the maximum of n independent random variables sharing the same cumulative distribution function F.

(c) Prove the Revenue Equivalence Theorem in this case: Consider a symmetric sealed-bid auction with independent private values, and let β be a monotonically increasing symmetric equilibrium satisfying the assumptions that (a) the winner is the buyer submitting the highest bid, and (b) the expected payment of a buyer whose private value is 0, is 0. Then the expected payment of a buyer whose private value is v is given by Equation (12.213).

(d) Compute a symmetric equilibrium strategy when the selling takes place by way of a sealed-bid first-price auction.

(e) Explain how Theorem 12.59 (page 504) can be used to show that the optimal selling mechanism in this case is a sealed-bid second-price auction with a reserve price.