18 The Shapley value

Chapter summary

This chapter presents the Shapley value, which is one of the two most important single-valued solution concepts for coalitional games. It assigns to every coalitional game an imputation, which represents the payoff that each player can expect to obtain from participating in the game. The Shapley value is defined by an axiomatic approach: it is the unique solution concept that satisfies the efficiency, symmetry, null player, and additivity properties. An explicit formula is provided for the Shapley value of a coalitional game, as a linear function of the worths of the various coalitions. A second characterization, due to Peyton Young, involves a marginality property that replaces the additivity and null player properties.

The Shapley value of a convex game turns out to be an element of the core of the game, which implies in particular that the core of a convex game is nonempty. Similar to the core, the Shapley value is consistent: it satisfies a reduced game property, with respect to the Hart–Mas-Colell definition of the reduced game.

When applied to simple games, the Shapley value is known as the Shapley–Shubik power index and it is widely used in political science as a measure of the power distribution in committees.

This chapter studies the *Shapley value*, a single-valued solution concept for coalitional games first introduced in Shapley [1953]. Shapley's original goal was to answer the question "How much would a player be willing to pay for participating in a game?" Plainly, the answer to that question depends on how much a player expects to receive when he comes to play the game.

Assuming that the grand coalition N will be formed, how will the worth $v(N)$ be divided among the members of the coalition? If one may "expect" that player i will receive the sum φ_i, that sum will be called "player i's value in the game." We will also interpret φ_i as the sum that it is "reasonable" for player i to receive if the grand coalition N is formed.

Shapley proposed a solution concept that satisfies several properties, which have come to be known as the Shapley properties (or axioms). We leave it to the reader to judge to what extent the properties put forward by Shapley reflect his original goal. The properties may also be applicable to a fair judge who is hired to advise the players on how to divide their profit among themselves after forming the grand coalition.

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press

18.1 The Shapley properties

We start by presenting Shapley's properties, also sometimes called "axioms," as they form the basis of a theory developed from them.

Recall that a single-valued solution concept φ is a function associating every coalitional game $(N; v)$ and every coalitional structure $\mathcal{B}$ with an imputation $\varphi(N; v; \mathcal{B}) \in \mathbb{R}^N$. In this chapter, we will restrict our attention to a coalitional structure containing only one coalition, the grand coalition $\mathcal{B} = \{N\}$, and therefore omit mentioning the coalitional structure in the notation for solution concepts. As we have seen, every game with a set of players N is a vector[1] $z = (z_S)_{S \subseteq N} \in \mathbb{R}^{\mathcal{P}(N)}$ satisfying $z_\emptyset = 0$. It follows that for a fixed set of players, a single-valued solution concept is a function φ defined over the set $\{z \in \mathbb{R}^{\mathcal{P}(N)} \colon z_\emptyset = 0\}$, which is a $(2^n - 1)$-dimensional subspace of $\mathbb{R}^{\mathcal{P}(N)}$. The function φ associates every game in this subspace with an n-dimensional vector in $\mathbb{R}^N$. In other words, a single-valued solution concept is an infinite sequence of functions, one for each set of players N.

Definition 18.1 *Let φ be a single-valued solution concept, let $(N; v)$ be a coalitional game, and let $i \in N$ be a player. Then $\varphi_i(N; v)$ is called the* value of player i in $(N; v)$ *according to φ.*

18.1.1 Efficiency

The first property we present here is the efficiency property, which requires that the sum total that all the players expect to get equals $v(N)$, the worth of the grand coalition N.

Definition 18.2 *A solution concept φ satisfies* efficiency *if for every coalitional game $(N; v)$,*

$$\sum_{i \in N} \varphi_i(N; v) = v(N). \tag{18.1}$$

If we assume that the coalition that will form is the grand coalition N, then the sum total that the players expect to receive is $v(N)$, the total amount available to them, and it is reasonable to assume that rational players will divide the entire sum total, without "wasting" any part of it.

18.1.2 Symmetry

The next property is the symmetry property, which is essentially a "non-discrimination" property, because it states that two players with the same standing in the game (who differ only in their names) should expect the same amount.

Definition 18.3 *Let $(N; v)$ be a coalitional game, and let $i, j \in N$. Players i and j are* symmetric players *if for every coalition $S \subseteq N \setminus \{i, j\}$ (which contains neither i nor j as members),*

$$v(S \cup \{i\}) = v(S \cup \{j\}). \tag{18.2}$$

1 $\mathcal{P}(N)$ is the collection of all subsets of the set of players N.

Symmetric players give the same marginal contribution to every coalition that does not contain them, and they are therefore identical from a strategic perspective. Adding player i to a coalition is equivalent to adding player j to that coalition.

Definition 18.4 *A solution concept φ satisfies* symmetry[2] *if for every coalitional game $(N; v)$ and every pair of symmetric players i and j in the game:*

$$\varphi_i(N; v) = \varphi_j(N; v). \tag{18.3}$$

The symmetry property requires that the solution concept be independent of the names of the players if their contributions to every coalition are equal. Two such players ought to get the same share of $v(N)$, and therefore ought to be willing to pay the same sum for participating in the game. This property is a reasonable one to adopt when there are no other differences between the players, stemming from social standing, age, personality, and so on.

18.1.3 Covariance under strategic equivalence

Another reasonable property to apply is covariance under strategic equivalence (see Section 16.2 on page 668).

Definition 18.5 *A solution concept φ satisfies* covariance under strategic equivalence *if for every coalitional game $(N; v)$, every positive real number a, every vector $b \in \mathbb{R}^N$, and every player $i \in N$,*[3]

$$\varphi_i(N, av + b) = a\varphi_i(N; v) + b_i. \tag{18.4}$$

18.1.4 The null player property

The next property is the null player property, which states that if a player contributes nothing to any coalition he joins, then he should not expect to receive a positive amount for participating in the game.

Definition 18.6 *A player i is called a* null player *in a game $(N; v)$ if for every coalition $S \subseteq N$, including the empty coalition, one has*[4]

$$v(S) = v(S \cup \{i\}). \tag{18.5}$$

A null player contributes nothing to any coalition he chooses to join. In particular, if i is a null player, then $v(i) = 0$.

Definition 18.7 *A solution concept φ satisfies the* null player property *if for every coalitional game $(N; v)$ and every null player i in the game,*

$$\varphi_i(N; v) = 0. \tag{18.6}$$

2 This property is sometimes called the *equal treatment property*.

3 Recall that for every coalitional game $(N; v)$, for every $a > 0$, and for every $b \in \mathbb{R}^N$, the coalitional function $av + b$ is defined by

$$(av + b)(S) := av(S) + b(S).$$

4 If a coalition S contains player i then this equality holds trivially, because then $S \cup \{i\} = S$.

18.1.5 Additivity property

The next property looks at a pair of coalitional games with the same set of players, and connects the solution of those two games to the solution of their sum.[5]

Definition 18.8 *A solution concept φ satisfies* additivity *if for every pair of coalitional games $(N; v)$ and $(N; w)$,*

$$\varphi(N; v + w) = \varphi(N; v) + \varphi(N; w). \tag{18.7}$$

The additivity property is justified as follows: suppose that the same set of players participate in the two coalitional games $(N; v)$ and $(N; w)$. The amount that player i expects to receive in $(N; v)$ is $\varphi_i(N; v)$, and the amount that he expects to receive in $(N; w)$ is $\varphi_i(N; w)$. If the two games are independent, we may regard this situation as a single game in which each coalition S, if it forms, receives $v(S) + w(S)$. Additivity states that player i should expect to receive $\varphi_i(N; v) + \varphi_i(N; w)$ in the game $(N; v + w)$.

This justification depends on the answer to the question: "To what extent is the single game $(N; v + w)$ equivalent to playing both $(N; v)$ and $(N; w)$?" In other words, to what extent are the two games indeed independent of each other? If we accept the additivity property, then the sum of the expectations of the players in the game $(N; v + w)$ is the sum of their expectations in the games $(N; v)$ and $(N; w)$. But is the "strength" of a player in the game $(N; v + w)$ equal to the sum of his "strengths" in each of the games $(N; v)$ and $(N; w)$? It is possible, for example, that one player may be willing to give up a bit in the game $(N; v)$ in exchange for receiving much more in the game $(N; w)$.

Another interpretation that can be adduced in justification of the additivity property is that the players play only one of the games, $(N; v)$ or $(N; w)$, each with probability $\frac{1}{2}$. In this case, they will expect to receive in the combined game $\frac{1}{2}\varphi(N; v) + \frac{1}{2}\varphi(N; w)$. The worth of a coalition in the combined game is then $\frac{1}{2}v(S) + \frac{1}{2}w(S)$, yielding the requirement that $\varphi(N; \frac{1}{2}v + \frac{1}{2}w) = \frac{1}{2}\varphi(N; v) + \frac{1}{2}\varphi(N; w)$, which, together with the covariance under strategic equivalence, is equivalent to the definition of the additivity property. This justification assumes that the utility that the players receive from the lottery conducted to choose the game equals the expected utility under the lottery, but in practice this does not always hold. Since the additivity property reflects assumptions that do not hold in all situations, it has been criticized. There is therefore some interest in characterizing the Shapley value using a system of properties that does not include the additivity property. We will present such a characterization in Section 18.5.

18.2 Solutions satisfying some of the Shapley properties

In this section, we present several solutions, and check which of the Shapley properties they satisfy.

5 If $(N; v)$ and $(N; w)$ are two games, their sum is $(N; v + w)$, defined by $(v + w)(S) := v(S) + w(S)$ for every coalition $S \subseteq N$.

Example 18.9 Consider the solution concept ψ defined by

$$\psi_i(N; v) := v(i). \tag{18.8}$$

This solution concept satisfies additivity, symmetry, the null player property, and covariance under strategic equivalence. It does not, however, satisfy efficiency (Exercise 18.1). ◀

Example 18.10 A player i is called a *dummy player* if $v(S \cup \{i\}) = v(S) + v(i)$ for every coalition $S \subseteq N \setminus \{i\}$. Every null player is a dummy player. Denote by $d(v)$ the number of dummy players. Consider the solution concept ψ defined by

$$\psi_i(N; v) := \begin{cases} v(i) + \frac{v(N) - \sum_{j \in N} v(j)}{n - d(v)} & \text{if } i \text{ is not a dummy player,} \\ v(i) & \text{if } i \text{ is a dummy player.} \end{cases} \tag{18.9}$$

This solution concept satisfies efficiency, symmetry, covariance under strategic equivalence, and the null player property. It does not, however, satisfy additivity (Exercise 18.2).

Here is an example that shows that this solution concept does not satisfy additivity. Consider the following two three-player games:

$$v(1) = v(2) = v(3) = v(1,2) = v(1,3) = 0, \quad v(2,3) = v(1,2,3) = 1, \tag{18.10}$$

and

$$u(1) = u(2) = u(3) = u(1,3) = 0, \quad u(1,2) = u(2,3) = u(1,2,3) = 1. \tag{18.11}$$

In the game $(N; v)$, only Player 1 is a dummy player, and in the game $(N; u)$ there is no dummy player. Therefore,

$$\psi(N; v) = \left(0, \tfrac{1}{2}, \tfrac{1}{2}\right), \qquad \psi(N; u) = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right). \tag{18.12}$$

The game $(N; v + u)$ is the game

$$(v+u)(1) = (v+u)(2) = (v+u)(3) = u(1,3) = 0, \quad (v+u)(1,2) = 1,$$
$$(v+u)(2,3) = (v+u)(1,2,3) = 2.$$

There is no dummy player in this game, and therefore

$$\psi(N; v+u) = \left(\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{2}{3}\right) \neq \left(\tfrac{1}{3}, \tfrac{5}{6}, \tfrac{5}{6}\right) = \psi(N; v) + \psi(N; u). \tag{18.13}$$

In other words, φ does not satisfy additivity. ◀

Example 18.11 Consider the solution concept ψ defined by

$$\psi_i(N; v) := \max_{\{S \colon i \notin S\}} (v(S \cup \{i\}) - v(S)). \tag{18.14}$$

This is the maximal marginal contribution that player i can give to any coalition. This solution concept satisfies symmetry, the null player property, and covariance under strategic equivalence. It does not, however, satisfy efficiency and additivity (Exercise 18.3).

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press

We first show that ψ does not satisfy efficiency. To this end, define the following three-player game $(N; w)$:

$$w(1) = 0, \quad w(2) = w(3) = w(1,2) = w(1,3) = w(2,3) = w(1,2,3) = 1. \tag{18.15}$$

Applying Equation (18.14) yields

$$\psi(N; w) = (0, 1, 1). \tag{18.16}$$

Since $\sum_{i=1}^{3} \psi_i(N; w) = 2 \neq 1 = w(N)$, the solution concept ψ does not satisfy efficiency.

We next show that ψ does not satisfy additivity. Let $(N; v)$ be the game defined in Equation (18.10). Then

$$\psi(N; v) = (0, 1, 1). \tag{18.17}$$

The sum $v + w$ is the coalitional function given by

$$(v + w)(1) = 0, \tag{18.18}$$

$$(v + w)(2) = (v + w)(3) = (v + w)(1,2) = (v + w)(1,3) = 1, \tag{18.19}$$

$$(v + w)(2,3) = (v + w)(1,2,3) = 2, \tag{18.20}$$

and therefore

$$\psi(N; v + w) = (0, 1, 1) \neq (0, 1, 1) + (0, 1, 1) = \psi(N; v) + \psi(N; w), \tag{18.21}$$

from which we deduce that ψ does not satisfy additivity. ◀

Example 18.12 Consider the solution concept ψ defined by

$$\psi_i(N; v) := v(\{1, 2, \ldots, i\}) - v(\{1, 2, \ldots, i - 1\}). \tag{18.22}$$

This solution concept satisfies efficiency, additivity, the null player property, and covariance under strategic equivalence. It does not satisfy symmetry (Exercise 18.4). To show that this solution concept does not satisfy symmetry, note that for the game $(N; v)$ defined in Equation (18.10)

$$\psi(N; v) = (0, 0, 1), \tag{18.23}$$

even though Players 2 and 3 are symmetric players in this game. ◀

In the last example, we assumed the existence of a particular ordering of the players, namely, $1, 2, \ldots, n$. Clearly, this solution concept can be defined using any arbitrary ordering of the players. Denote by $\Pi(N)$ the set of all permutations of the set of players N (recall that the number of players is $n = |N|$). The set $\Pi(N)$ then contains $n!$ permutations. For every permutation $\pi \in \Pi(N)$, define

$$P_i(\pi) := \{j \in N : \pi(j) < \pi(i)\}. \tag{18.24}$$

This is the set of players ahead of player i when the players are ordered according to permutation π. Note that $P_i(\pi) = \emptyset$ if and only if $\pi(i) = 1$. Similarly, $P_i(\pi)$ contains only one element (which is $\pi^{-1}(1)$) if and only if $\pi(i) = 2$. Generally,

$$P_i(\pi) \cup \{i\} = P_k(\pi) \text{ if and only if } \pi(k) = \pi(i) + 1. \tag{18.25}$$

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press

For every permutation $\pi \in \Pi(N)$, define a solution concept ψ^π as follows.

$$\psi_i^\pi(N; v) := v(P_i(\pi) \cup \{i\}) - v(P_i(\pi)). \tag{18.26}$$

As in Example 18.12, this solution concept satisfies efficiency, additivity, the null player property, and covariance under strategic equivalence. It does not satisfy symmetry.

18.3 The definition and characterization of the Shapley value

Are there solution concepts that satisfy efficiency, additivity, the null player property, and symmetry? Shapley [1953] proved that this is indeed the case, and furthermore that there is a unique such solution concept.

Theorem 18.13 (Shapley [1953]) *There is a unique solution concept satisfying efficiency, additivity, the null player property, and symmetry.*

Shapley provided an explicit formula for computing this solution concept. In Example 18.12, we saw that for every permutation $\pi \in \Pi(N)$, the solution concept ψ^π satisfies all the Shapley properties except symmetry. To define a solution concept that in addition satisfies symmetry, we average the solution concepts ψ^π over all permutations of the set of players N.

Definition 18.14 *The* Shapley value *is the solution concept* Sh *defined as follows.*

$$\mathrm{Sh}_i(N; v) := \frac{1}{n!} \sum_{\pi \in \Pi(N)} (v(P_i(\pi) \cup \{i\}) - v(P_i(\pi))), \quad \forall i \in N. \tag{18.27}$$

Using the solution concept ψ^π defined in Equation (18.26), one may write Equation (18.27) as follows:

$$\mathrm{Sh}(N; v) := \frac{1}{n!} \sum_{\pi \in \Pi(N)} \psi^\pi(N; v). \tag{18.28}$$

Before proceeding to the proof that the Shapley value Sh satisfies the above-listed properties, we present a probabilistic interpretation of Equation (18.27). Suppose that the players enter a room one at a time. Each player, upon entry, receives his marginal contribution to the set composed of the players who entered the room before him; i.e., if S is the set of players already in the room as player i enters, player i receives $v(S \cup \{i\}) - v(S)$. If the ordering under which the players enter the room is chosen randomly, with every ordering given equal probability of being chosen, then the expected payoff of player i is given by Equation (18.27).

Theorem 18.15 *The Shapley value is the only single-valued solution concept satisfying efficiency, additivity, the null player property, and symmetry.*

Remark 18.16 *In Exercise 18.10, we show that there exist subfamilies of the class of coalitional games over which one may define a solution concept different from the Shapley value and that satisfies efficiency, additivity, the null player property, and symmetry.*

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press

Theorem 18.15, however, shows that the Shapley value is the only solution concept defined over all coalitional games that satisfies these four properties. ♦

We now present a formulation equivalent to Equation (18.27). Let i be a player, and S be an arbitrary coalition that does not include player i. What is the number of permutations π for which $P_i(\pi) = S$? For $P_i(\pi)$ to equal S, we must require that the players in S enter the room before player i under the permutation π, then player i, and after him the players in $N \setminus (S \cup \{i\})$. The number of different ways that the players in S can be ordered is $S!$, and the number of different ways that the players in $N \setminus (S \cup \{i\})$ can be ordered is $(n - |S| - 1)!$. It follows that the number of permutations π under which $P_i(\pi) = S$ is $|S|! \times (n - |S| - 1)!$. This shows that the Shapley value of player i can be computed as follows.

Theorem 18.17 *The Shapley value is given by the following equation:*

$$\mathrm{Sh}_i(N; v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|! \times (n - |S| - 1)!}{n!} (v(S \cup \{i\}) - v(S)). \tag{18.29}$$

We begin the proof of Theorem 18.15 by ascertaining that the Shapley value satisfies the four properties listed in the statement of the theorem.

Claim 18.18 *The Shapley value satisfies efficiency, additivity, the null player property, and symmetry. Furthermore, it satisfies covariance under strategic equivalence.*

Proof: Since

$$\mathrm{Sh}(N; v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} \psi^{\pi}(N; v), \tag{18.30}$$

and since for each permutation π, the solution concept ψ^{π} satisfies additivity, the null player property, and covariance under strategic equivalence, the Shapley value, as the average of these solution concepts, also satisfies the same list of properties (Exercise 18.6).

We now show that the Shapley value satisfies symmetry. Let i and j be two symmetric players. Define a function $f : \Pi(N) \to \Pi(N)$ that maps each permutation over the set of players to another permutation as follows. For every permutation π, the permutation $f(\pi)$ is identical to π except that it swaps player i with player j:

$$(f(\pi))(k) = \begin{cases} \pi(j) & \text{if } k = i, \\ \pi(i) & \text{if } k = j, \\ \pi(k) & \text{if } k \notin \{i, j\}. \end{cases} \tag{18.31}$$

Another way of saying this is that $f(\pi)$ is the composition of π with the permutation that swaps i with j and leaves all the other players in their place. The function f is bijective[6] (explain why).

We next show that if i and j are symmetric players, then the permutation π satisfies

$$v(P_i(\pi) \cup \{i\}) - v(P_i(\pi)) = v(P_j(f(\pi)) \cup \{j\}) - v(P_j(f(\pi))). \tag{18.32}$$

6 A function $f : A \to B$ is *bijective* if it is one-to-one and onto.

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press

$$\overbrace{\underbrace{k_1 \quad k_2 \quad k_3}_{P_i(\pi)} \quad i \quad k_4 \quad j \quad k_5}^{\pi} \qquad\qquad \overbrace{\underbrace{k_1 \quad k_2 \quad k_3}_{P_j(f(\pi))} \quad j \quad k_4 \quad i \quad k_5}^{f(\pi)}$$

Figure 18.1 The case in which player i appears before player j under permutation π

$$\overbrace{\underbrace{k_1 \quad k_2 \quad k_3 \quad j \quad k_4}_{P_i(\pi)} \quad i \quad k_5}^{\pi} \qquad\qquad \overbrace{\underbrace{k_1 \quad k_2 \quad k_3 \quad i \quad k_4}_{P_j(f(\pi))} \quad j \quad k_5}^{f(\pi)}$$

Figure 18.2 The case in which player i appears after player j under permutation π

Case 1: Player i appears before player j under permutation π, i.e., $j \notin P_i(\pi)$.
In this case, $P_i(\pi) = P_j(f(\pi))$ (see Figure 18.1), and in particular $v(P_i(\pi)) = v(P_j(f(\pi)))$. Since i and j are symmetric players, and since $P_i(\pi)$ contains neither i nor j, $v(P_i(\pi) \cup \{i\}) = v(P_j(f(\pi)) \cup \{j\})$. These two equalities together imply Equation (18.32).

Case 2: Player i appears after player j under permutation π, i.e., $j \in P_i(\pi)$.
In this case, $P_i(\pi) \cup \{i\} = P_j(f(\pi)) \cup \{j\}$ (see Figure 18.2), and in particular $v(P_i(\pi) \cup \{i\}) = v(P_j(f(\pi)) \cup \{j\})$. Similarly, $P_i(\pi) \setminus \{j\} = P_j(f(\pi)) \setminus \{i\}$. Since i and j are symmetric players, $v((P_i(\pi) \setminus \{j\}) \cup \{j\}) = v((P_j(f(\pi)) \setminus \{i\}) \cup \{i\})$, i.e., $v(P_i(\pi)) = v(P_j(f(\pi)))$. These two equalities together imply that Equation (18.32) holds in this case as well.

Since f is bijective,

$$\{f(\pi) \colon \pi \in \Pi(N)\} = \Pi(N). \tag{18.33}$$

Therefore,

$$\mathrm{Sh}_i(N; v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} (v(P_i(\pi) \cup \{i\}) - v(P_i(\pi))) \tag{18.34}$$

$$= \frac{1}{n!} \sum_{\pi \in \Pi(N)} \big(v(P_j(f(\pi)) \cup \{j\}) - v(P_j(f(\pi)))\big) \tag{18.35}$$

$$= \frac{1}{n!} \sum_{\mu \in \Pi(N)} \big(v(P_j(\mu) \cup \{j\}) - v(P_j(\mu))\big) \tag{18.36}$$

$$= \mathrm{Sh}_j(N; v), \tag{18.37}$$

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press

where Equation (18.35) follows from Equation (18.32) and Equation (18.36) follows from setting $\mu = f(\pi)$ and from using Equation (18.33). This shows that the Shapley value Sh satisfies symmetry, which is what we wanted to prove, completing the proof of Claim 18.18. □

We next prove the uniqueness claim in Theorem 18.15. Define, for every nonempty coalition T, a simple game called the *carrier game over* T. In the carrier game over T, a coalition is a winning coalition (with worth 1) if and only if it contains T.

Definition 18.19 *Let $T \subseteq N$ be a nonempty coalition. The* carrier game *over T is the simple game $(N; u_T)$ defined as follows. For each coalition $S \subseteq N$,*

$$u_T(S) := \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases} \tag{18.38}$$

Theorem 18.20 *Every game $(N; v)$ is a linear combination of carrier games.*

Proof: Recall that the space of the coalitional games over the set of players N is a vector space of dimension $2^n - 1$. The number of carrier games equals the number of nonempty coalitions in N, namely, $2^n - 1$. To prove the theorem, it suffices to show that the carrier games are linearly independent over $\mathbb{R}^{2^n-1}$; this will imply that they form a linear basis of the space of games. Indeed, every set of $2^n - 1$ independent vectors in a vector space of dimension $2^n - 1$ is a basis for that space, and therefore every element of the vector space can be written as a linear combination of the basis elements.

Suppose, by contradiction, that the carrier games are linearly dependent. Then there exists a linear combination of carrier games with non-zero coefficients that sums to the zero vector. In other words, there exist real numbers $(\alpha_T)_{\{T \subseteq N, T \neq \emptyset\}}$, not all zero, such that

$$\sum_{\{T \subseteq N, T \neq \emptyset\}} \alpha_T u_T(S) = 0, \qquad \forall S \subseteq N. \tag{18.39}$$

Let $\mathcal{T} = \{T \subseteq N : T \neq \emptyset, \alpha_T \neq 0\}$ be the set of all coalitions with non-zero coefficients in the linear combination in Equation (18.39). Since we assumed that not all coefficients are zero, the set $\mathcal{T}$ is nonempty. Let $S_0 \in \mathcal{T}$ be a minimal coalition in $\mathcal{T}$; i.e., there is no coalition in $\mathcal{T}$ strictly contained in S_0. We will show that $\sum_{\{T \subseteq N, T \neq \emptyset\}} \alpha_T u_T(S_0) \neq 0$, in contradiction to Equation (18.39). Note that

$$\sum_{\{T \subseteq N, T \neq \emptyset\}} \alpha_T u_T(S_0) = \sum_{\{T \subset S_0, T \neq \emptyset\}} \alpha_T u_T(S_0) + \alpha_{S_0} u_{S_0}(S_0) + \sum_{T \not\subseteq S_0} \alpha_T u_T(S_0). \tag{18.40}$$

$\alpha_T = 0$ for every T satisfying $T \subset S_0$, since S_0 is a minimal coalition in $\mathcal{T}$. For every T satisfying $T \not\subseteq S_0$, the definition of a carrier game implies that $u_T(S_0) = 0$. Therefore,

$$\sum_{\{T \subseteq N, T \neq \emptyset\}} \alpha_T u_T(S_0) = \alpha_{S_0} u_{S_0}(S_0) = \alpha_{S_0} \neq 0, \tag{18.41}$$

which is what we wanted to show. The contradiction implies that the assumption that the carrier games are linearly dependent is false; i.e., they are linearly independent. □

The next theorem states that every solution concept satisfying efficiency, symmetry, and the null player property is uniquely determined for every game that is the product of a carrier game by a scalar.

Theorem 18.21 *Let T be a nonempty coalition, and let α be a real number. Define a game $(N; u_{T,\alpha})$ as follows:*

$$u_{T,\alpha}(S) = \begin{cases} \alpha & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases} \tag{18.42}$$

If φ is a solution concept satisfying efficiency, symmetry, and the null player property, then

$$\varphi_i(N; u_{T,\alpha}) = \begin{cases} \frac{\alpha}{|T|} & \text{if } i \in T, \\ 0 & \text{if } i \notin T. \end{cases} \tag{18.43}$$

Proof: In the game $(N; u_{T,\alpha})$, every player $i \notin T$ is a null player, and every pair of players in T are symmetric. The claim of the theorem then follows from the assumption that φ satisfies efficiency, symmetry, and the null player property. □

Proof of Theorem 18.15: By Claim 18.18, the Shapley value Sh satisfies additivity, efficiency, symmetry, and the null player property. All that remains is to show that the Shapley value is the unique solution concept satisfying these properties. Let φ therefore be a solution concept satisfying these properties; we will show that $\varphi = \text{Sh}$. Let $(N; v)$ be a coalitional game. Theorem 18.20 implies that v is a sum of games of the form u_{T,α_T} for coalitions T in the collection $\{T \subseteq N, T \neq \emptyset\}$. In other words, there exist real numbers $(\alpha_T)_{\{T \subseteq N, T \neq \emptyset\}}$ such that

$$v(S) = \sum_{\{T \subseteq N, T \neq \emptyset\}} u_{T,\alpha_T}(S). \tag{18.44}$$

By Theorem 18.21, since both φ and Sh satisfy symmetry, efficiency, and the null player property,

$$\varphi(N; u_{T,\alpha_T}) = \text{Sh}(N; u_{T,\alpha_T}), \quad \forall T \subseteq N, T \neq \emptyset. \tag{18.45}$$

Since both φ and Sh satisfy additivity,

$$\varphi(N; v) = \sum_{\{T \subseteq N, T \neq \emptyset\}} \varphi(N; u_{T,\alpha_T}) = \sum_{\{T \subseteq N, T \neq \emptyset\}} \text{Sh}(N; u_{T,\alpha_T}) = \text{Sh}(N; v). \tag{18.46}$$

Because this is true for every game $(N; v)$, we conclude that $\varphi = \text{Sh}$. In other words, every solution satisfying additivity, efficiency, symmetry, and the null player property is identical to the Shapley value. This concludes the proof of Theorem 18.15. □

18.4 Examples

In this section we compute the Shapley value in several examples.

Example 18.22 **A two-player bargaining game** Let $(N;v)$ be a two-player game with the following coalitional function:

$$v(1) = v(2) = 0,\ v(1,2) = 1. \tag{18.47}$$

Since the two players are symmetric, $\mathrm{Sh}_1(N;v) = \mathrm{Sh}_2(N;v)$, the efficiency property implies that the Shapley value is

$$\mathrm{Sh}(N;v) = \left(\tfrac{1}{2}, \tfrac{1}{2}\right). \tag{18.48}$$

◀

Example 18.23 **A simple majority game with n players** Let $(N;v)$ be an n-player game with the following coalitional function:

$$v(S) = \begin{cases} 0 & \text{if } |S| \leq \frac{n}{2}, \\ 1 & \text{if } |S| > \frac{n}{2}. \end{cases} \tag{18.49}$$

Since all the players are symmetric, $\mathrm{Sh}_i(N;v) = \mathrm{Sh}_j(N;v)$ for every pair of players $i, j \in N$, and applying the efficiency property yields

$$\mathrm{Sh}(N;v) = \left(\tfrac{1}{n}, \tfrac{1}{n}, \ldots, \tfrac{1}{n}\right). \tag{18.50}$$

◀

Example 18.24 **The gloves game** Let $(N;v)$ be a three-player game with the following coalitional function:

$$v(1) = v(2) = v(3) = v(1,2) = 0, \quad v(1,3) = v(2,3) = v(1,2,3) = 1. \tag{18.51}$$

We previously saw this game in Example 17.5 on page 690, where we found that its core solely contains the imputation $(0,0,1)$. We next compute the Shapley value of the game. To this end, we use Equation (18.27). For every permutation of N (there are six such permutations) we list the marginal contribution of each player:

Permutation	Contribution of Player 1	Contribution of Player 2	Contribution of Player 3
(1, 2, 3)	$v(1) - v(\emptyset) = 0$	$v(1,2) - v(1) = 0$	$v(1,2,3) - v(1,2) = 1$
(1, 3, 2)	$v(1) - v(\emptyset) = 0$	$v(1,2,3) - v(1,3) = 0$	$v(1,3) - v(1) = 1$
(2, 1, 3)	$v(1,2) - v(2) = 0$	$v(2) - v(\emptyset) = 0$	$v(1,2,3) - v(1,2) = 1$
(2, 3, 1)	$v(1,2,3) - v(2,3) = 0$	$v(2) - v(\emptyset) = 0$	$v(2,3) - v(2) = 1$
(3, 1, 2)	$v(1,3) - v(3) = 1$	$v(1,2,3) - v(1,3) = 0$	$v(3) - v(\emptyset) = 0$
(3, 2, 1)	$v(1,2,3) - v(2,3) = 0$	$v(2,3) - v(3) = 1$	$v(3) - v(\emptyset) = 0$

Summing the contribution of each player, and dividing by the number of permutations, 6, yields the Shapley value:

$$\mathrm{Sh}(N;v) = \left(\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{2}{3}\right). \tag{18.52}$$

This imputation emphasizes the fact that although Player 3 is the strongest player here, holding the left glove, the Shapley value of the other players is not zero, in contrast to the core of the gloves game.

Note that in this computation, it sufficed to compute only one of the three columns, say the Shapley value of Player 1, and to use that to deduce the Shapley values of Players 2 and 3. (Explain why this is true. Which properties did you use in your explanation?)

The core of this game contains only one imputation, (0, 0, 1) (see Example 17.5 on page 690), and therefore even when the core is nonempty, the Shapley value may be outside the core. ◀

18.5 An alternative characterization of the Shapley value

The characterization of the Shapley value presented in the previous section relies on the additivity property. The motivation behind this property may be unpersuasive, and there are many cases in which it is unclear why additivity is a reasonable assumption. In this section we present a characterization of the Shapley value that does not use the additivity property. We first present several new properties, and then show that they can be used to characterize the Shapley value.

18.5.1 The marginality property

The property of monotonicity of marginal contributions requires that if a player contributes to each coalition in a game $(N; v)$ no less than he contributes to the same coalition in another game $(N; w)$ with the same set of players, then his value in $(N; v)$ is at least as great as his value in $(N; w)$.

Definition 18.25 *A solution concept φ satisfies* monotonicity of marginal contributions *if for every pair of games $(N; v)$ and $(N; w)$ with the same set of players, and for each player $i \in N$, if*

$$v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S), \quad \forall S \subseteq N \setminus \{i\}, \tag{18.53}$$

then

$$\varphi_i(N; v) \geq \varphi_i(N; w). \tag{18.54}$$

The property of monotonicity of marginal contributions implies the property of marginality, defined as follows.

Definition 18.26 *A solution concept φ satisfies* marginality *if for every pair of games $(N; v)$ and $(N; w)$ with the same set of players, and for every player i, if*

$$v(S \cup \{i\}) - v(S) = w(S \cup \{i\}) - w(S), \quad \forall S \subseteq N \setminus \{i\}, \tag{18.55}$$

then

$$\varphi_i(N; v) = \varphi_i(N; w). \tag{18.56}$$

This property imposes the property that the value of a player depends only on his marginal contribution to each coalition, and is independent of the marginal contributions of the other players to all possible coalitions.

Theorem 18.27 *The Shapley value satisfies monotonicity of marginal contributions, and it therefore also satisfies marginality.*

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press

The proof of Theorem 18.27 follows from the fact that by definition (see Definition 18.14) the Shapley value of player i is the weighted average of his marginal contributions to all possible coalitions. The next theorem connects several properties.

Theorem 18.28 *Every solution concept φ satisfying efficiency, symmetry, and marginality also satisfies the null player property.*

Proof: Let φ be a solution concept satisfying efficiency, symmetry, and marginality. We will show that φ also satisfies the null player property.

Let $(N; v)$ be a coalitional game, and let i be a null player in this game. Denote by z the zero game that is defined by $z(S) = 0$ for every coalition S. In $(N; z)$, all the players are symmetric, and the properties of efficiency and symmetry then imply that

$$\varphi_j(N; z) = 0 \quad \forall j \in N. \tag{18.57}$$

Since player i is a null player, one has

$$v(S \cup \{i\}) - v(S) = 0 = z(S \cup \{i\}) - z(S) \quad \forall S \subseteq N. \tag{18.58}$$

We deduce from this that the vector of marginal contributions of player i in $(N; v)$ equals his vector of marginal contributions in $(N; z)$. Marginality then implies that $\varphi_i(N; v) = \varphi_i(N; z) = 0$. $\square$

18.5.2 The second characterization of the Shapley value

The next theorem, which was proved in Young [1985], provides a second characterization of the Shapley value, replacing the properties of additivity and null player in the previous characterization with marginality.

Theorem 18.29 (Young [1985]) *The Shapley value is the unique single-valued solution concept satisfying efficiency, symmetry, and marginality.*

Proof: We have already seen (Claim 18.18 on page 755, and Theorem 18.27) that the Shapley value satisfies the three properties in the statement of the theorem. To prove the other direction of the claim of the theorem, we need to show that every solution concept φ satisfying efficiency, symmetry, and marginality is the Shapley value. The proof relies on the following definitions.

For every coalitional game $(N; v)$, denote

$$I(N; v) = \{S \subseteq N : \exists T \subseteq S, v(T) \neq 0\}. \tag{18.59}$$

A coalition S is not in $I(N; v)$ if and only if its worth, and the worth of all its subcoalitions, is 0. It follows that if S is a minimal coalition in $I(N; v)$ (i.e., none of its subcoalitions are in $I(N; v)$), then $v(S) \neq 0$, and the worth of all the subcoalitions of S is 0.

Given a game $(N; v)$, define for each coalition S a game $(N; v^S)$ as follows:

$$v^S(T) = v(S \cap T), \quad \forall T \subseteq N. \tag{18.60}$$

Every player $i \notin S$ is a null player in $(N; v^S)$. To see this, note that $S \cap (T \cup \{i\}) = S \cap T$, because $i \notin S$, and therefore

$$v^S(T \cup \{i\}) = v(S \cap (T \cup \{i\})) = v(S \cap T) = v^S(T). \tag{18.61}$$

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press

Claim 18.30 *Let S be a coalition in $I(N; v)$. Then $I(N; v - v^S) \subset I(N; v)$.*

Proof of Claim 18.30: We first show that $I(N; v - v^S) \subseteq I(N; v)$. If $T \in I(N; v - v^S)$, then there exists a subcoalition $R \subseteq T$ satisfying

$$0 \neq (v - v^S)(R) = v(R) - v^S(R) = v(R) - v(R \cap S). \tag{18.62}$$

Therefore, either $v(R) \neq 0$, or $v(R \cap S) \neq 0$ (or both inequalities hold). Since $R \subseteq T$, both R and $R \cap S$ are subsets of T. It follows that T has at least one subset whose worth is not 0, and therefore $T \in I(N; v)$, which is what we wanted to show.

To show that $I(N; v - v^S) \neq I(N; v)$, we will show that $S \notin I(N; v - v^S)$. Indeed, for every coalition $T \subseteq S$,

$$(v - v^S)(S) = v(S) - v(S \cap S) = v(S) - v(S) = 0. \tag{18.63}$$

From this we deduce that S is not in $I(N; v - v^S)$, which is what we wanted to show. □

To prove uniqueness in Theorem 18.29, let φ be a solution concept satisfying efficiency, symmetry, and marginality. We will show that $\varphi = \mathrm{Sh}$. By Theorem 18.28, the solution concept φ also satisfies the null player property.

The proof of uniqueness will be an inductive proof over $|I(N; v)|$, the number of elements in $I(N; v)$. If $|I(N; v)| = 0$, then $v(S) = 0$ for every coalition S, and then all the players are null players. Since both φ and Sh satisfy the null player property, $\varphi_i(N; v) = 0 = \mathrm{Sh}_i(N; v)$ for every player $i \in N$.

Assume by induction that $\varphi(N; v) = \mathrm{Sh}(N; v)$ for every game $(N; v)$ satisfying $|I(N; v)| < k$, and let $(N; v)$ be a game satisfying $|I(N; v)| = k$. Denote by $\widehat{S}$ the set formed by the intersection of all the coalitions in $I(N; v)$, i.e.,

$$\widehat{S} = \cap_{S \in I(N; v)} S. \tag{18.64}$$

Step 1: $\mathrm{Sh}_i(N; v) = \varphi_i(N; v)$ for each $i \notin \widehat{S}$.
Let $i \notin \widehat{S}$. Then there exists a coalition $S \in I(N; v)$ that does not contain i. By Claim 18.30,

$$|I(N; v - v^S)| < |I(N; v)| = k, \tag{18.65}$$

and the inductive hypothesis then implies that

$$\varphi_j(N; v - v^S) = \mathrm{Sh}_j(N; v - v^S) \quad \forall j \in N. \tag{18.66}$$

We next compute the marginal contribution of player i in $(N; v - v^S)$, and show that it equals the marginal contribution of that player in $(N; v)$. For every coalition $T \subseteq N \setminus \{i\}$,

$$\begin{aligned}
(v - v^S)(T \cup \{i\}) &= v(T \cup \{i\}) - v^S(T \cup \{i\}) && (18.67)\\
&= v(T \cup \{i\}) - v(S \cap (T \cup \{i\})) && (18.68)\\
&= v(T \cup \{i\}) - v(S \cap T) && (18.69)\\
&= v(T \cup \{i\}) - v^S(T), && (18.70)
\end{aligned}$$

and, therefore,

$$\begin{aligned}
(v - v^S)(T \cup \{i\}) - (v - v^S)(T) &= v(T \cup \{i\}) - v^S(T) - v(T) + v^S(T)\\
&= v(T \cup \{i\}) - v(T). && (18.71)
\end{aligned}$$

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press

By assumption, φ satisfies marginality. By Theorem 18.27, the Shapley value Sh also satisfies marginality. It follows that $\varphi_i(N;v) = \varphi_i(N;v - v^S)$ and $\mathrm{Sh}_i(N;v) = \mathrm{Sh}_i(N;v - v^S)$. Using Equation (18.66), we get

$$\varphi_i(N;v) = \mathrm{Sh}_i(N;v), \quad \forall i \notin \widehat{S}. \tag{18.72}$$

Step 2: $\mathrm{Sh}_i(N;v) = \varphi_i(N;v)$ for each $i \in \widehat{S}$.
If $\widehat{S} = \emptyset$, the conclusion follows vacuously.

If $|\widehat{S}| = 1$, the conclusion follows from Step 1 and the fact that both the Shapley value Sh and the solution concept φ are efficient. Suppose, therefore, that $\widehat{S}$ contains at least two players.

We first show that every coalition T that does not contain $\widehat{S}$ satisfies $v(T) = 0$. Indeed, by the definition of $I(N;v)$, if $v(T) \neq 0$ then $T \in I(N;v)$. In particular, T must then contain $\widehat{S}$, which is the intersection of all the coalitions in $I(N;v)$, contradicting the fact that T does not contain $\widehat{S}$.

We now show that any pair of players in $\widehat{S}$ are symmetric players. Let $i, j \in \widehat{S}$ be two different players. For every coalition T that contains neither player i nor player j, both the coalitions $T \cup \{i\}$ and $T \cup \{j\}$ do not contain $\widehat{S}$, and therefore $v(T \cup \{i\}) = 0 = v(T \cup \{j\})$. Since this equality holds for every coalition T that contains neither i nor j, these players are symmetric, which is what we wanted to show.

Since both φ and Sh satisfy symmetry,

$$\varphi_i(N;v) = \varphi_j(N;v), \qquad \mathrm{Sh}_i(N;v) = \mathrm{Sh}_j(N;v), \quad \forall i, j \in \widehat{S}. \tag{18.73}$$

Because we know from the previous step that $\varphi_k(N;v) = \mathrm{Sh}_k(N;v)$ for every player $k \notin \widehat{S}$, and since both φ and Sh satisfy efficiency,

$$\sum_{k \in \widehat{S}} \varphi_k(N;v) = v(N) - \sum_{k \notin \widehat{S}} \varphi_k(N;v) = v(N) - \sum_{k \notin \widehat{S}} \mathrm{Sh}_k(N;v) = \sum_{k \in \widehat{S}} \mathrm{Sh}_k(N;v). \tag{18.74}$$

Equations (18.73)–(18.74) imply that for every player $j \in \widehat{S}$,

$$\varphi_j(N;v) = \frac{1}{|\widehat{S}|} \sum_{k \in \widehat{S}} \varphi_k(N;v) = \frac{1}{|\widehat{S}|} \sum_{k \in \widehat{S}} \mathrm{Sh}_k(N;v) = \mathrm{Sh}_j(N;v). \tag{18.75}$$

This completes the proof of the theorem. □

18.6 Application: the Shapley–Shubik power index

The Shapley value can be used to measure the power of each member in a decision-making process. This application of the Shapley value was developed by Shapley and Shubik in 1954, and is called the Shapley–Shubik power index.

In this section, we will concentrate on simple monotonic games. Recall that simple games are coalitional games in which the worth of each coalition is 0 or 1. A simple game is monotonic if when the worth of a coalition is 1, the worth of every coalition containing it is also 1. These games can model decision-making in collectives containing several decision makers. $v(S) = 1$ if the members of the coalition S can impose a decision even

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press

when the other players are opposed to their decision. In this case, we say that the coalition S is a *winning* coalition. In contrast, $v(S) = 0$ if the members of S cannot impose their decisions on the other decision makers. In this case, we say that the coalition S is a *losing* coalition. In particular, in a simple monotonic game, a subcoalition of a losing coalition is also a losing coalition, and a coalition containing a winning coalition is also a winning coalition.

Here are three examples of simple monotonic games:

1. Simple majority games (see Example 18.23).
2. Unanimity games, in which a decision is accepted only if all the players agree to accept it:
$$v(S) = \begin{cases} 1 & \text{if } S = N, \\ 0 & \text{otherwise.} \end{cases} \tag{18.76}$$
3. Dictatorship games, in which a single player decides whether or not to accept or reject a decision; i.e., there exists a player i_0 such that
$$v(S) = \begin{cases} 1 & \text{if } i_0 \in S, \\ 0 & \text{if } i_0 \notin S. \end{cases} \tag{18.77}$$

Definition 18.31 *The* Shapley–Shubik power index *is the function associating each simple monotonic game with its Shapley value. The i-th coordinate of this vector is called the* power index of player i.

In a simple monotonic game, the Shapley value has a particularly simple form. In this case, $v(S \cup \{i\}) - v(S)$ either has the value 0 (if S and $S \cup \{i\}$ are both winning coalitions or both losing coalitions) or 1 (if S is a losing coalition, and $S \cup \{i\}$ is a winning coalition). By Theorem 18.17 on page 755, the Shapley value can therefore be written as follows:

$$\mathrm{Sh}_i(N; v) = \sum_{\{S \subseteq N:\ S\cup\{i\} \text{winning}, S \text{ losing}\}} \frac{|S|! \times (n - |S| - 1)!}{n!}. \tag{18.78}$$

In what sense can one say that the Shapley–Shubik power index measures the power of a player in a game? Suppose that a set of individuals N make decisions by voting on them, and the outcome of a vote is determined by a simple monotonic game. Then, for every ordering $(i_1, i_2, \ldots, i_n)$ of the players, there is precisely one player such that the set of all the players before him in the sequence are a losing coalition, and his joining the coalition changes it into a winning coalition. Such a player is called a *pivot player*.

Suppose that a proposed decision is being voted on. The proposal induces an ordering of the players, according to their opinions regarding the proposal; the first player in this ordering is the most enthusiastic supporter of the proposal, the last player in this ordering is the most vociferous opponent of the proposal, and the players in between these two are ordered by decreasing support. The most enthusiastic supporter and the most vociferous opponent vie for the support of the other players for their positions.

Assume that all the players are exposed to the same arguments for and against the proposed decision. If the pivot player supports and votes in favor of the proposal, since all the players before him in the ordering support it to an equal or greater extent than he, they too will vote for it, and the proposal will be accepted. If, in contrast, the pivot player votes against the proposal, since all the players after him in the ordering are less

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press

enthusiastic about it than he, they too will vote against it, and the proposal will be rejected. It follows that the arguments for and against the proposed decision will be directed at the pivot player, whose vote will be the deciding vote, and his power flows from this.

When the number of proposals to be decided upon is large, and they induce all possible orderings of the players with equal probabilities, the Shapley–Shubik power index of player i is the probability that player i will be a pivot player. In this sense, the index does measure the power of each player. If the proposals to be decided upon do not induce all possible orderings of the players with equal probabilities, the Shapley–Shubik power index lacks a clear justification, and variations of this index need to be defined.

18.6.1 The power index of the United Nations Security Council

The United Nations Security Council, the most important body in the international political system, was formed in the aftermath of the Second World War. At the time, it was composed of five permanent members[7] and six nonpermanent members. The council's original charter established that a resolution could be adopted only if it received supporting votes from at least seven members. In addition, every permanent member was granted veto power over any resolution. Ignoring the possibility of a council member abstaining from a vote, it followed that for a resolution to be passed by the Security Council, it had to be supported by all five permanent members and at least two nonpermanent members.

The veto power in the hands of the permanent members of the council was the target of criticism over the years by observers who objected to the "unbalanced power" it gives the permanent members relative to the nonpermanent members. The chorus of criticism led to a restructuring of the Security Council in 1965, giving the council the structure it maintains to this day. Under the new council structure, four nonpermanent members were added to the body, and the number of supporting members required for the adoption of a resolution was raised to nine, including, as before, all five permanent members. It was claimed that both the increase in the number of nonpermanent members, from six to ten, and the fact that adoption of a resolution now required at least 4 nonpermanent members in addition to the five permanent members, as opposed to a minimum of two nonpermanent members in the previous structure, significantly changed the balance of power in the council. Can this claim be sustained?

The Shapley–Shubik power index enables us to explore this question in a quantified manner. To do so, we compute the Shapley value of the members of the Security Council under both structures, prior to 1965 and after 1965, and then check what change, if any, occurred in the Shapley value as a result of the 1965 restructuring. The pre-1965 structure of the Security Council can be described by a coalitional game. If we denote by P the set of permanent members of the council, and by NP the set of nonpermanent members, the resulting game is a simple game in which the set of players is given by $N := P \cup NP$, and the coalitional function (ignoring the possibility of abstentions) is given by

$$v(S) = \begin{cases} 1 & \text{if } S \supset P \text{ and } |S| \geq 7, \\ 0 & \text{otherwise.} \end{cases} \tag{18.79}$$

7 The United States, the United Kingdom, the Union of the Soviet Socialist Republics, China, and France.

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press

In this coalitional function all the members of P are symmetric and all the members of NP are symmetric, and hence only two Shapley values need to be computed: the Shapley value of a member of P and the Shapley value of a member of NP. Let us first compute the Shapley value of a nonpermanent member $i \in NP$ using Equation (18.78). For every $i \in NP$ there are five coalitions S that are not winning coalitions but satisfy the property that $S \cup \{i\}$ is a winning coalition, namely, the coalitions containing the five permanent members along with one nonpermanent member other than i. The size of each such coalition is $|S| = 6$, and therefore Equation (18.78) enables us to compute the Shapley value of each nonpermanent member, which is

$$5 \times \frac{6! \times 4!}{11!} = \frac{1}{462} = 0.0021645. \tag{18.80}$$

Since every pair of nonpermanent members is symmetric, each nonpermanent member has the same Shapley value. It follows that the sum of the Shapley values of the six nonpermanent members is

$$6 \times \frac{1}{462} = \frac{6}{462} \approx 0.013. \tag{18.81}$$

Since the Shapley value satisfies efficiency, the sum of the Shapley values of the 5 permanent members is $1 - \frac{6}{462} = \frac{456}{462} \approx 0.987$. Furthermore, since every pair of permanent members is symmetric, the Shapley value of each permanent member is $\frac{1}{5} \cdot \frac{456}{462} = \frac{91.2}{462} = 0.1974$. Note the immense ratio $1 : \frac{456}{5} = 1 : 91.2$ between the power of a permanent member and the power of a nonpermanent member.

The simple game corresponding to the structure of the post-1965 Security Council is given by the following coalitional game, where the set R now contains ten players:

$$v(S) = \begin{cases} 1 & \text{if } S \supset P \text{ and } |S| \geq 9, \\ 0 & \text{otherwise.} \end{cases} \tag{18.82}$$

We now compute the Shapley value of a nonpermanent member i in this game. The number of coalitions S that are not winning coalitions but satisfy the property that $S \cup \{i\}$ is a winning coalition is $\binom{9}{3}$, because, in addition to the five permanent members, such a coalition must contain three out of the nine nonpermanent members different from i. Since such a coalition contains eight players, applying Equation (18.78) gives us the Shapley value of a nonpermanent member, which is

$$\binom{9}{3} \times \frac{8! \times 6!}{15!} = \frac{4}{2145} = 0.001865. \tag{18.83}$$

The power of all the nonpermanent members together is $\frac{40}{2145} \approx 0.0186$. This enables us to deduce that the Shapley value of each permanent member is

$$\frac{1}{5}\left(1 - 10 \times \frac{4}{2145}\right) = \frac{421}{2145} = 0.1963. \tag{18.84}$$

The change in the total power of the permanent members dropped from 0.987 to $\frac{2105}{2145} \approx 0.9814$ as a result of the restructuring of the Security Council, but the relative power ratio between pairs of members moved in a negative direction for the nonpermanent members: while the power of every permanent member fell relatively marginally (by about half a

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press

percent), the power of each nonpermanent member fell by 14%. The ratio between the power of a permanent member to a nonpermanent member rose to $1 : 105\frac{1}{4}$.

If one accepts the Shapley value as a reasonable index of power, the conclusion of this computation is that the restructuring of the Security Council in 1965 did not significantly change the basic balance of power on the council: almost all the power in the Council was and remained in the hands of the veto-wielding permanent members. Measuring power under a different index, the Banzhaf power index (see Exercise 18.27) results in the conclusion that the power of the permanent members fell to a greater extent after the restructuring, but even under the Banzhaf power index the power of the permanent members was still much higher than the power of the nonpermanent members. It is therefore not surprising that complaints about the imbalance of power on the Security Council continue to be heard, with suggested changes in the composition of the council and its voting rules regularly raised by members of the United Nations.

18.7 Convex games

Recall that a game $(N; v)$ is convex if for every pair of coalitions $S, T \subseteq N$,

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T). \tag{18.85}$$

In Theorem 17.55 (page 719) we proved that the core of such a game is never empty. In this section we will show that the Shapley value of a convex game is in the core of the game.

Theorem 18.32 *If $(N; v)$ is a convex game, then the Shapley value is in the core of the game.*

Proof: For every permutation $\pi \in \Pi(N)$, denote by w^π the vector in $\mathbb{R}^N$ such that for every $i \in \{1, 2, \dots, n\}$, its i-th coordinate is[8]

$$w_i^\pi = v(P_i(\pi) \cap \{i\}) - v(P_i(\pi)). \tag{18.86}$$

By Theorem 17.55 (page 719), for every $\pi \in \Pi(N)$, the imputation w^π is in the core of $(N; v)$. By Equation (18.27), the Shapley value is the average of the vectors $(w^\pi)_{\pi \in \Pi(N)}$:

$$\mathrm{Sh}(N; v) = \sum_{\pi \in \Pi(N)} \frac{1}{n!} w^\pi. \tag{18.87}$$

Since the core is a convex set (Theorem 17.3 on page 687), we conclude that the Shapley value is in the core. □

In Remark 17.59 (page 720) we noted that in a convex game the core equals the convex hull of the imputations defined by Equation (18.86). Since some of these vectors may be equal to each other, the Shapley value is not necessarily the core's center of mass.

8 Recall that $P_i(\pi)$ denotes the set of players preceding player i according to the permutation π.

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press

18.8 The consistency of the Shapley value

In the previous chapter on the core (see page 715), we defined the Davis–Maschler reduced game, and showed that the core satisfies the consistency property: if x is a point in the core of a game $(N; v)$, then for every nonempty coalition $S \subseteq N$, the vector $(x_i)_{i\in S}$ is a point in the core of the Davis–Maschler reduced game $(S; v_S)$. In this section, we prove that a similar property also holds for the Shapley value, but the definition of the reduced game is different in this case; it is the one introduced in Hart and Mas-Colell [1989]. Furthermore, we will prove that the reduced game property can be used to characterize the Shapley value axiomatically, similarly to the way that it can be used to characterize the nucleolus axiomatically (see Section 20.5 on page 816). Both of these single-valued solution concepts have the same system of characterizing axioms, but the definition of the reduced game differs between the two of them. It follows that if we have to choose between these two solution concepts, we can check which definition of a reduced game is more appropriate to the situation at hand.

Definition 18.33 *Let φ be a single-valued solution concept, let $(N; v)$ be a coalitional game, and let S be a nonempty coalition. The* Hart–Mas-Colell reduced game over S relative to φ *is the game $(S; \widetilde{v}_{S,\varphi})$, with the following coalitional function:*[9]

$$\widetilde{v}_{S,\varphi}(R) = v(R \cup S^c) - \sum_{i\in S^c} \varphi_i(R \cup S^c; v), \qquad \forall R \subseteq S,\ R \neq \emptyset, \tag{18.88}$$

$$\widetilde{v}_{S,\varphi}(\emptyset) = 0. \tag{18.89}$$

The idea behind the definition is that when a coalition R is formed in the reduced game over S, it adds as partners in the coalition all the players in S^c, computes how much each player should receive according to the solution concept φ in the reduced game, over the set of players $R \cup S^c$, and gives the players in S^c their shares in the solution. What remains after the members of S^c receive their share is the worth of the coalition R in the reduced game $\widetilde{v}_{S,\varphi}$.

Remark 18.34 *Recall that when S is a nonempty coalition and x is an individually rational vector in $\mathbb{R}^N$ the Davis–Maschler reduced game over S relative to x, denoted by $(S; w_S^x)$, is the game with the set of players S in which the coalitional function is*

$$w_S^x(R) = \begin{cases} \max_{Q\subseteq N\setminus S}(v(R\cup Q) - x(Q)) & \emptyset \neq R \subset S, \\ x(S) & R = S, \\ 0 & R = \emptyset. \end{cases} \tag{18.90}$$

There are two differences between the Davis–Maschler reduced game and the Hart–Mas-Colell reduced game:

1. *The Hart–Mas-Colell reduced game is appropriate only for single-valued solution concepts, while the Davis–Maschler reduced game is appropriate for single-valued and set-valued solution concepts because it is defined in relation to each imputation in the solution.*

9 We alternatively use the notation S^c and $N \setminus S$ to denote the complement coalition of S.

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press

2. *In the definition of the Hart–Mas-Colell reduced game, the members of R must add all the players who are not members of S as partners, while in the Davis–Maschler reduced game they may choose as partners the most beneficial subset, from their perspective, of the players in S^c.* ♦

Example 18.35 **The reduced game of the carrier game** Recall that for every nonempty coalition T the carrier game over T is the game $(N; u_T)$ (see Definition 18.19) in which

$$u_T(R) = \begin{cases} 1 & \text{if } R \supseteq T, \\ 0 & \text{if } R \not\supseteq T. \end{cases} \tag{18.91}$$

By Theorem 18.21 (page 758) the Shapley value of the carrier game u_T is

$$\mathrm{Sh}_i(N; u_T) = \begin{cases} \frac{1}{|T|} & \text{if } i \in T, \\ 0 & \text{if } i \notin T. \end{cases} \tag{18.92}$$

Let $S \subseteq N$ be a nonempty coalition. We will compute the Hart–Mas-Colell reduced game over S relative to the Shapley value Sh.

Case 1: $S \cap T = \emptyset$.
Let $R \subseteq S$. Then $R \subseteq N \setminus T$, and therefore every player in R is a null player. Since $R \cup S^c \supseteq S^c \supseteq T$,

$$u_T(R \cup S^c) = 1 = \sum_{i \in T} \mathrm{Sh}_i(R \cup S^c; u_T) = \sum_{i \in S^c} \mathrm{Sh}_i(R \cup S^c; u_T). \tag{18.93}$$

Therefore

$$\widetilde{u}_{S,\mathrm{Sh}}(R) = v(R \cup S^c) - \sum_{i \in S^c} \mathrm{Sh}_i(R \cup S^c; u_T) = 0, \tag{18.94}$$

which implies that the reduced game over S of the carrier game T is the zero game.

Case 2: $S \cap T \neq \emptyset$.
Let $R \subseteq S$ be a nonempty coalition. If $R \supseteq S \cap T$ then $R \cup S^c \supseteq T$ (see Figure 9.3), and therefore $u_T(R \cup S^c) = 1$. The calculation of the Shapley value of the carrier game of T yields that $\sum_{i \notin S} \mathrm{Sh}_i(R \cup S^c; u_T) = \frac{|T \setminus S|}{|T|}$.

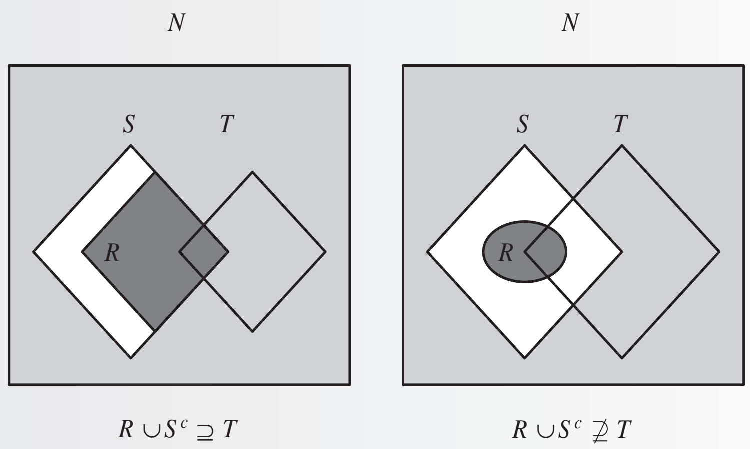


Figure 18.3

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press

If $R \not\supseteq S \cap T$, then $R \cup S^c \not\supseteq T$ (see Figure 18.3), and therefore the game $(R \cup S^c; u_T)$ is the zero game.

We deduce from this that $\widetilde{u}_{T,S,\mathrm{Sh}}$, the Hart–Mas-Colell reduced game of u_T, is given by

$$\widetilde{u}_{T,S,\mathrm{Sh}}(R) = \begin{cases} 1 - \frac{|T \setminus S|}{|T|} & R \supseteq S \cap T, \\ 0 & R \not\supseteq S \cap T. \end{cases} \tag{18.95}$$

Note that in this case, all the players in $S \cap T$ are symmetric, and all the players not in $S \cap T$ are null players. ◀

A solution concept φ is called *linear* if for every pair of coalitional games $(N; v)$ and $(N; u)$ with the same set of players and every pair of real numbers α and β,

$$\varphi_i(N; \alpha u + \beta v) = \alpha \varphi_i(N; u) + \beta \varphi_i(N; v), \quad \forall i \in N. \tag{18.96}$$

The Shapley value is a linear solution concept (Exercise 18.8).

The following result follows from the definitions and is left to the reader as an exercise (Exercise 18.33). It states that the correspondence between a coalitional game and its reduced game relative to a linear solution concept φ is a linear function.

Theorem 18.36 *Let φ be a linear solution concept. Then for every pair of coalitional games $(N; v)$ and $(N; u)$ and for every pair of real numbers α and β,*

$$\widetilde{w}_{S,\varphi} = \alpha \widetilde{u}_{S,\varphi} + \beta \widetilde{v}_{S,\varphi}, \tag{18.97}$$

for every nonempty coalition $S \subseteq N$, where $w = \alpha u + \beta v$.

Definition 18.37 *A single-valued solution concept φ is said to be* consistent relative to the Hart–Mas-Colell reduced game *if for every game $(N; v)$, every nonempty coalition S, and every player $i \in S$,*

$$\varphi_i(N; v) = \varphi_i(S; \widetilde{v}_{S,\varphi}). \tag{18.98}$$

Theorem 18.38 *The Shapley value is consistent relative to the Hart–Mas-Colell reduced game: for every game $(N; v)$, and every nonempty coalition S,*

$$\mathrm{Sh}_i(N; v) = \mathrm{Sh}_i(S; \widetilde{v}_{S,\mathrm{Sh}}), \quad \forall i \in S. \tag{18.99}$$

Proof: Denote by $\widetilde{G}$ the set of games satisfying Equation (18.99). We need to show that $\widetilde{G}$ contains the set of all games. This is accomplished in two steps. In the first step, we prove that Equation (18.99) holds for the set of carrier games, which forms a basis for the vector space of all games. In the second step we show, using Theorem 18.36, that a linear combination of games satisfying Equation (18.99) also satisfies that equation.

Step 1: $\widetilde{G}$ contains the set of all carrier games.
Let T be a nonempty coalition. We will show that $(N; u_T)$ is in $\widetilde{G}$. Let $S \subseteq N$ be a nonempty coalition. If $S \cap T = \emptyset$, the left-hand side of Equation (18.99) equals 0 because the players in S are null players in the game $(N; u_T)$. We showed in Example 18.35 that the reduced game $\widetilde{u}_{T,S,\mathrm{Sh}}$ is the zero game, and therefore the Shapley value of all the players in S is 0. Thus, Equation (18.99) holds, with both sides being 0.

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press

Suppose now that $S \cap T \neq \emptyset$. As we showed in Example 18.35, in the reduced game $\widetilde{u}_{T,S,\mathrm{Sh}}$, players who are not in $S \cap T$ are null players. and all the players in $S \cap T$ are symmetric. Therefore, the Shapley value of the reduced game $\widetilde{u}_{T,S,\mathrm{Sh}}$ is

$$\mathrm{Sh}_i(S;\widetilde{u}_{T,S,\mathrm{Sh}}) = \begin{cases} \frac{1}{|T|} & \text{if } i \in S \cap T, \\ 0 & \text{if } i \in S \setminus T. \end{cases} \tag{18.100}$$

By Theorem 18.21 we deduce that in this case Equation (18.99) also holds.

Step 2: $\widetilde{G}$ contains the space of all games.
Let $(N; v)$ be a coalitional game, and let $S \subseteq N$ be a nonempty coalition. We will show that Equation (18.99) is satisfied for $(N; v)$ and $S \subseteq N$. By Theorem 18.20 (page 757) every game is equal to a linear combination of carrier games. There therefore exist real numbers $(\alpha_T)_{\{T\subseteq N, T\neq\emptyset\}}$ such that

$$v = \sum_{\{T\subseteq N, T\neq\emptyset\}} \alpha_T u_T. \tag{18.101}$$

Since the Shapley value is a linear solution concept, it follows in particular that

$$\mathrm{Sh}_i(N; v) = \sum_{\{T\subseteq N, T\neq\emptyset\}} \alpha_T \mathrm{Sh}_i(N; u_T), \quad \forall i \in S. \tag{18.102}$$

Since Equation (18.99) holds for all carrier games, we deduce that

$$\mathrm{Sh}_i(N; v) = \sum_{\{T\subseteq N, T\neq\emptyset\}} \alpha_T \mathrm{Sh}_i(S; \widetilde{u}_{T,S,\mathrm{Sh}}), \quad \forall i \in S. \tag{18.103}$$

Using once again the fact that the Shapley value is a linear solution concept yields

$$\mathrm{Sh}_i(N; v) = \mathrm{Sh}_i\left(S; \sum_{\{T\subseteq N, T\neq\emptyset\}} \alpha_T \widetilde{u}_{T,S,\mathrm{Sh}}\right), \quad \forall i \in S. \tag{18.104}$$

To complete this proof we need to show that

$$\mathrm{Sh}_i\left(S; \sum_{\{T\subseteq N, T\neq\emptyset\}} \alpha_T \widetilde{u}_{T,S,\mathrm{Sh}}\right) = \mathrm{Sh}_i(S; \widetilde{v}_{S,\mathrm{Sh}}), \quad \forall i \in S, \tag{18.105}$$

and for this purpose, it suffices to show (again using the linearity of Sh) that

$$\sum_{\{T\subseteq N, T\neq\emptyset\}} \alpha_T \widetilde{u}_{T,S,\mathrm{Sh}} = \widetilde{v}_{S,\mathrm{Sh}}. \tag{18.106}$$

From Theorem 18.36, the function associating each game with its reduced game is linear, and this equality therefore holds. □

Theorem 18.39 *The Shapley value is the unique single-valued solution concept satisfying efficiency, symmetry, and covariance under strategic equivalence, and is also consistent relative to the Hart–Mas-Colell reduced game.*

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press

In comparison with Theorem 18.15 (page 754), the properties of additivity and the null player are replaced in Theorem 18.39 by the properties of covariance under strategic equivalence and consistent relative to the Hart–Mas-Colell reduced game.

Proof: Let φ be a single-valued solution concept satisfying efficiency, symmetry, and covariance under strategic equivalence, which is also consistent relative to the Hart–Mas-Colell reduced game. We will show that φ coincides with the Shapley value Sh. The proof is accomplished by induction over the number of players n.

Step 1: $n = 1$.
When $N = \{1\}$, since both the Shapley value and φ satisfy efficiency,

$$\varphi_1(N; v) = v(1) = \mathrm{Sh}_1(N; v). \tag{18.107}$$

Step 2: $n = 2$.
For two-player games, every single-valued solution concept φ satisfying efficiency, covariance under strategic equivalence, and symmetry is given by (see Exercise 18.5)

$$\varphi_1(N; v) = v(1) + \frac{v(1,2) - v(1) - v(2)}{2}, \tag{18.108}$$

$$\varphi_2(N; v) = v(2) + \frac{v(1,2) - v(1) - v(2)}{2}. \tag{18.109}$$

Since both the Shapley value and φ satisfy these properties, we deduce that

$$\varphi_i(N; v) = \mathrm{Sh}_i(N; v), \quad i = 1, 2. \tag{18.110}$$

Note that by Equations (18.108)–(18.109)

$$\varphi_1(N; v) - \varphi_2(N; v) = v(1) - v(2) = \mathrm{Sh}_1(N; v) - \mathrm{Sh}_2(N; v). \tag{18.111}$$

Step 3: $n > 2$.
Assume by induction that for every k-player game $(K; u)$, such that $2 \le k < n$, $\varphi(K; u) = \mathrm{Sh}(K; u)$. Let $(N; v)$ be an n-player game. We will prove that $\varphi(N; v) = \mathrm{Sh}(N; v)$. Let i and j be two players, and let $S := \{i, j\}$. Since $\{i\} \cup S^c = N \setminus \{j\}$ and $\{j\} \cup S^c = N \setminus \{i\}$, the Hart–Mas-Colell reduced game over S relative to φ is the game $(\{i, j\}; \widetilde{v}_{S,\varphi})$ given by the following coalitional function:

$$\widetilde{v}_{S,\varphi}(i) = v(N \setminus \{j\}) - \sum_{k \neq i,j} \varphi_k(N \setminus \{j\}; v), \tag{18.112}$$

$$\widetilde{v}_{S,\varphi}(j) = v(N \setminus \{i\}) - \sum_{k \neq i,j} \varphi_k(N \setminus \{i\}; v). \tag{18.113}$$

Similarly, the Hart–Mas-Colell reduced game over S relative to Sh is the game $(\{i, j\}; \widetilde{v}_{S,\mathrm{Sh}})$ with the following coalitional function:

$$\widetilde{v}_{S,\mathrm{Sh}}(i) = v(N \setminus \{j\}) - \sum_{k \neq i,j} \mathrm{Sh}_k(N \setminus \{j\}; v), \tag{18.114}$$

$$\widetilde{v}_{S,\mathrm{Sh}}(j) = v(N \setminus \{i\}) - \sum_{k \neq i,j} \mathrm{Sh}_k(N \setminus \{i\}; v). \tag{18.115}$$

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press

The inductive hypothesis, applied to the $(n-1)$-player games $(N \setminus \{i\}; v)$ and $(N \setminus \{j\}; v)$, and Equations (18.112)–(18.115), together yield

$$\varphi_k(N \setminus \{i\}; v) = \mathrm{Sh}_k(N \setminus \{i\}; v), \quad \forall k \neq i, \tag{18.116}$$

$$\varphi_k(N \setminus \{j\}; v) = \mathrm{Sh}_k(N \setminus \{j\}; v), \quad \forall k \neq j. \tag{18.117}$$

By Equations (18.112)–(18.113):

$$\widetilde{v}_{S,\varphi}(i) = \widetilde{v}_{S,\mathrm{Sh}}(i), \tag{18.118}$$

$$\widetilde{v}_{S,\varphi}(j) = \widetilde{v}_{S,\mathrm{Sh}}(j). \tag{18.119}$$

Applying Equation (18.111) to the reduced game over S, one has

$$\varphi_i(S, \widetilde{v}_{S,\varphi}) - \varphi_j(S, \widetilde{v}_{S,\varphi}) = \widetilde{v}_{S,\varphi}(i) - \widetilde{v}_{S,\varphi}(j), \tag{18.120}$$

$$\mathrm{Sh}_i(S, \widetilde{v}_{S,\mathrm{Sh}}) - \mathrm{Sh}_j(S, \widetilde{v}_{S,\mathrm{Sh}}) = \widetilde{v}_{S,\mathrm{Sh}}(i) - \widetilde{v}_{S,\mathrm{Sh}}(j). \tag{18.121}$$

By Equations (18.118)–(18.119), the right-hand side of Equation (18.120) equals the right-hand side of Equation (18.121), and therefore

$$\varphi_i(S, \widetilde{v}_{S,\varphi}) - \varphi_j(S, \widetilde{v}_{S,\varphi}) = \mathrm{Sh}_i(S, \widetilde{v}_{S,\mathrm{Sh}}) - \mathrm{Sh}_j(S, \widetilde{v}_{S,\mathrm{Sh}}). \tag{18.122}$$

By Theorem 18.38, the Shapley value is consistent relative to the Hart–Mas-Colell reduced game, and therefore

$$\mathrm{Sh}_k(S, \widetilde{v}_{S,\mathrm{Sh}}) = \mathrm{Sh}_k(N; v), \qquad \forall k \in \{i, j\}. \tag{18.123}$$

Since φ is consistent relative to the Hart–Mas-Colell reduced game, one has

$$\varphi_k(S, \widetilde{v}_{S,\varphi}) = \varphi_k(N; v), \qquad \forall k \in \{i, j\}. \tag{18.124}$$

Inserting this into Equation (18.122) yields

$$\varphi_i(N; v) - \varphi_j(N; v) = \mathrm{Sh}_i(N; v) - \mathrm{Sh}_j(N; v). \tag{18.125}$$

Since i and j are arbitrary players, Equation (18.125) holds for any pair of players $i, j \in N$. Since both the Shapley value and φ satisfy efficiency,

$$\sum_{k \in N} \varphi_k(N; v) = v(N) = \sum_{k \in N} \mathrm{Sh}_k(N; v), \tag{18.126}$$

Summing Equation (18.125) over $j \in N$, and using Equation (18.126), yields

$$n\varphi_i(N; v) - v(N) = n\mathrm{Sh}_i(N; v) - v(N). \tag{18.127}$$

This further implies that for every player $i \in N$,

$$\varphi_i(N; v) = \mathrm{Sh}_i(N; v). \tag{18.128}$$

This completes the inductive step, and the proof that $\varphi = \mathrm{Sh}$. □

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press

18.9 Remarks

The proof of Theorem 18.29 (page 761) presented in this section is from Neyman [1989]. The concept of consistency presented in this chapter first appeared in Hart and Mas-Colell [1989], who proved that the Shapley value is consistent.

The potential function in Exercise 18.26 was introduced in Hart and Mas-Colell [1989]. The interested reader is directed to Felsenthal and Machover [1998] for an insightful discussion of the Shapley–Shubik power index and the Banzhaf power index (see Exercise 18.27). The interested reader is similarly directed to Aumann and Shapley [1974] for a thorough exposition of the Shapley value in games with a continuum of players.

18.10 Exercises

18.1 Prove that the solution concept defined in Example 18.9 (page 752) satisfies additivity, symmetry, the null player property, and covariance under strategic equivalence, but does not satisfy efficiency.

18.2 Prove that the solution concept defined in Example 18.10 (page 752) satisfies efficiency, symmetry, the null player property, and covariance under strategic equivalence.

18.3 Prove that the solution concept defined in Example 18.11 (page 752) satisfies the null player property, symmetry, and covariance under strategic equivalence.

18.4 Prove that the solution concept defined in Example 18.12 (page 753) satisfies efficiency, additivity, the null player property, and covariance under strategic equivalence.

18.5 Prove that when restricted to two-player games, every single-valued solution concept φ satisfying efficiency, covariance under strategic equivalence, and symmetry is given by the following equations:

$$\varphi_1(N;v) = v(1) + \frac{v(1,2) - v(1) - v(2)}{2} = \frac{v(1,2) + v(1) - v(2)}{2}, \tag{18.129}$$

$$\varphi_2(N;v) = v(2) + \frac{v(1,2) - v(1) - v(2)}{2} = \frac{v(1,2) - v(1) + v(2)}{2}. \tag{18.130}$$

18.6 Let ψ^1 and ψ^2 be two solution concepts both satisfying symmetry, efficiency, the null player property, and covariance under strategic equivalence, and let $\lambda \in [0, 1]$. Define a solution concept ψ by $\psi := \lambda\psi^1 + (1 - \lambda)\psi^2$. Prove that ψ also satisfies symmetry, efficiency, the null player property, and covariance under strategic equivalence.

18.7 Decompose the following game $(N; v)$ as a linear combination of carrier games. The set of players is $N = \{1, 2, 3, 4\}$, and the coalitional function is

$$v(1) = 6, \quad v(2) = 12, \quad v(3) = 0, \quad v(4) = 18,$$
$$v(1, 2) = 24, \quad v(1, 3) = 48, \quad v(1, 4) = 60, \quad v(2, 3) = 12,$$
$$v(2, 4) = 32, \quad v(3, 4) = 38$$
$$v(1, 2, 3) = 120, \quad v(1, 2, 4) = 89, \quad v(1, 3, 4) = 150,$$
$$v(2, 3, 4) = 179, \quad v(1, 2, 3, 4) = 240.$$

18.8 Prove that the Shapley value is a linear solution concept: for every list of K games $((N; v_k))_{k=1}^K$, and every list of K real numbers $(\alpha_k)_{k=1}^K$:

$$\mathrm{Sh}\left(N; \sum_{k=1}^K \alpha_k v_k\right) = \sum_{k=1}^K \alpha_k \mathrm{Sh}(N; v_k). \tag{18.131}$$

18.9 In Theorem 18.15 (page 754), can any one of the properties (efficiency, symmetry, null player, or additivity) be replaced by covariance under strategic equivalence, while maintaining the conclusion of the theorem? Prove your answer.

18.10 A single-valued solution concept is a function associating an imputation with every coalitional game. One may also define solution concepts that are only defined for a subset of the class of coalitional games. Let $\mathcal{F}$ be a subset of the class of coalitional games. A *single-valued solution concept for* $\mathcal{F}$ is a function associating an imputation with each coalitional game in $\mathcal{F}$. The Shapley value is the only solution concept satisfying the four properties proposed by Shapley for the class of all coalitional games. In this exercise, we show that there exist families of coalitional games over which solution concepts different from the Shapley value that nevertheless satisfy Shapley's four properties can be defined.

A family $\mathcal{F}$ of coalitional games is called *additively closed* if for every pair of coalitional games $(N; v)$ and $(N; u)$ in $\mathcal{F}$, the game $(N; v + u)$ is also in $\mathcal{F}$.

Find a family of coalitional games that is additively closed and a single-valued solution concept defined over that family that satisfies the four Shapley properties but is not the Shapley value.

Explain why this exercise does not contradict Theorem 18.15 on page 754.

18.11 A coalitional game $(N; v)$ is called *additive* if every coalition S satisfies $v(S) = \sum_{i \in S} v(i)$. What is the Shapley value of each player i in an additive game?

18.12 Let $a \in \mathbb{R}^N$ be a vector. Compute the Shapley value of the coalitional game $(N; v)$ defined as follows:

$$v(S) := \left(\sum_{i \in S} a_i\right)^2, \quad \emptyset \neq S \subseteq N. \tag{18.132}$$

18.13 In this exercise, we present an algorithm for computing a solution concept. Given a coalitional game $(N; v)$:

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press

(a) Choose a coalition whose worth is not 0, and divide this worth equally among the members of the coalition (this is called the *dividend* given to the members of the coalition).
(b) Subtract the worth of this coalition from the worth of every coalition containing it, or equal to it. This defines a new coalitional function (where subtracting a negative number is understood to be equivalent to adding the absolute value of that number).
(c) Repeat this process until there are no more coalitions whose worth is not 0.

For example, consider the game $(N; v)$ defined by the set of players $N = \{1, 2, 3\}$, and the coalitional function

$$v(1) = 6, \quad v(2) = 12, \quad v(3) = 18, \quad v(1,2) = 30, \quad v(1,3) = 60,$$
$$v(2,3) = 90, \quad v(1,2,3) = 120.$$

The following table summarizes a stage of the algorithm in each row, and includes the coalitional function at the beginning of that stage, the chosen coalition (whose worth is not 0), and the payoff given to each player at that stage. The last line presents the sum total of all payoffs received by each player.

Stage	1	2	3	1, 2	1, 3	2, 3	1, 2, 3	Coalition	1	2	3
1	6	12	18	30	60	90	120	1, 2	15	15	0
2	6	12	18	0	60	90	90	2	0	12	0
3	6	0	18	−12	60	78	78	1, 3	30	0	30
4	6	0	18	−12	0	78	18	1	6	0	0
5	0	0	18	−18	−6	78	12	3	0	0	18
6	0	0	0	−18	−24	60	−6	1, 2, 3	−2	−2	−2
7	0	0	0	−18	−24	60	0	1, 2	−9	−9	0
8	0	0	0	0	−24	60	18	1, 3	−12	0	−12
9	0	0	0	0	0	60	42	2, 3	0	30	30
10	0	0	0	0	0	0	−18	1, 2, 3	−6	−6	−6
11	0	0	0	0	0	0	0				
									22	40	58

Prove the following claims:

(a) This process always terminates.
(b) The total payoffs received by the players are the Shapley value of the game (and are therefore independent of the order in which the coalitions are chosen).

Remark 18.40 *The algorithm terminates in the least number of steps if we first choose the coalitions containing only one player, then the coalitions containing two players each, and so on. This process was first presented by John Harsanyi.* ♦

18.14 Compute the Shapley value of the game in Exercise 18.7, using the algorithm described in Exercise 18.13.

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press

18.15 For every game $(N; v)$, define the *dual game* $(N; v^*)$ as follows:

$$v^*(S) = v(N) - v(N \setminus S). \tag{18.133}$$

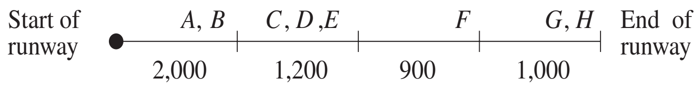
Prove the following claims:

(a) If $(N; v^*)$ is the dual game to $(N; v)$, then $(N; v)$ is the dual game to $(N; v^*)$.
(b) $\mathrm{Sh}(N; v) = \mathrm{Sh}(N; v^*)$.

18.16 The maintenance costs of airport runways are usually charged to the airlines landing planes at that airport. But light planes require shorter runways than heavy planes, and this raises the question of how to determine a fair allocation of maintenance costs among airlines with different types of planes.

Define a cost game $(N; c)$, where N is the set of all planes landing at an airport, and $c(S)$, for each coalition S, is the maintenance cost of the shortest runway that can accommodate all the planes in the coalition.

The following figure depicts an example in which eight planes, labeled A, B, C, D, E, F, G, and H land at an airport on a daily basis. Each plane requires the entire length of the runway up to (and including) the interval on which it is located in the figure. For example, plane F needs the first three segments of the runway. The weekly maintenance costs of each runway segment appear at the bottom of the figure. For example, $c(A, D, E) = 3{,}200$, $c(A) = 2{,}000$, and $c(C, F, G) = 5{,}100$.



Prove that if the Shapley value of this game is used to determine the allocation of costs, then the maintenance cost of each runway segment is borne equally by the planes using that segment.

For example, in the above figure,

$$\mathrm{Sh}_A(N; c) = \frac{2{,}000}{8} = 250, \tag{18.134}$$

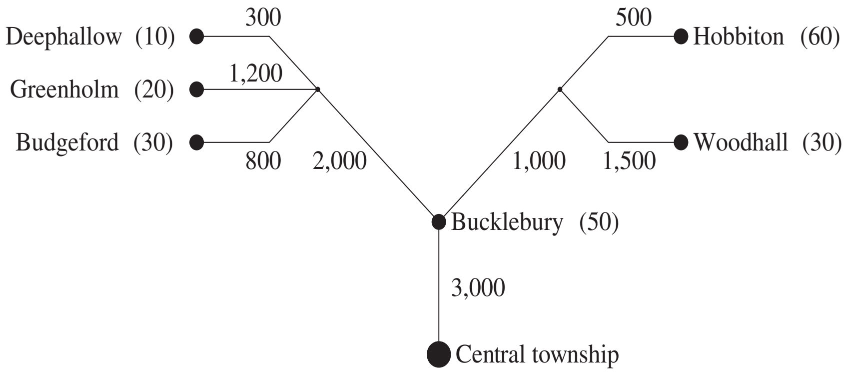
$$\mathrm{Sh}_F(N; c) = \frac{2{,}000}{8} + \frac{1{,}200}{6} + \frac{900}{3} = 750. \tag{18.135}$$

18.17 This exercise considers maintenance costs associated with a road network connecting villages to a central township. The network is depicted as a tree, with the central township at the root of the tree. Each village is associated with a node of the tree, and there are additional nodes of the tree that represent road intersections. The villages vary in their numbers of inhabitants. An example appears in the following figure, which depicts six villages and two intersections; the number of inhabitants in each village appears in the figure, near that village's name, and each segment of road connecting two intersections, or connecting the township to an intersection, is labeled with that segment's maintenance cost.

A cost game $(N; c)$ is derived from the network, where N is the set of residents in all the villages (in this example $|N| = 200$), and for each coalition $S \subseteq N$, $c(S)$

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press

is the maintenance cost of the minimal subtree required to maintain the network of roads connecting all the members in S to the central township.



Prove that if the Shapley value of such a game is used to determine the allocation of costs, then the maintenance cost of each road segment is borne equally by all the people using that segment. For example, in the figure above, the Shapley value of every resident of Hobbiton is

$$\frac{500}{60} + \frac{1{,}000}{90} + \frac{3{,}000}{200} = 34\tfrac{4}{9}. \tag{18.136}$$

18.18 For every pair of games over the same set of players $(N; v)$ and $(N; w)$, define the maximum game $(N; v \vee w)$ as follows:

$$(v \vee w)(S) = \max\{v(S), w(S)\}, \quad \forall S \subseteq N, \tag{18.137}$$

and the minimum game $(N; v \wedge w)$ by

$$(v \wedge w)(S) = \min\{v(S), w(S)\}, \quad \forall S \subseteq N. \tag{18.138}$$

Suppose that $(N; v)$ and $(N; w)$ are two weighted majority games (see Section 16.1.4). Are the games $(N; v \vee w)$ and $(N; v \wedge w)$ also weighted majority games?

18.19 Let $(N; v)$ and $(N; w)$ be two coalitional games over the same set of players. Does $\mathrm{Sh}_i(N; v \vee w) \geq \mathrm{Sh}_i(N; v)$ hold for every player $i \in N$? Either prove this statement or provide a counterexample. For the definition of the game $(N; v \vee w)$, see Exercise 18.18.

18.20 Let Σ_N be the set of simple monotonic games over a set of players N.

(a) Prove that if $(N; v), (N; w) \in \Sigma_N$ then $(N; v \vee w)$ and $(N; v \wedge w)$ are also games in Σ_N.

Since a sum of games in Σ_N is not necessarily in Σ_N, the additivity property is not applicable to this family of games. This motivated Dubey [1975] to define the following *valuation property*. A solution concept φ over Σ_N satisfies the *valuation*

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press

property if for every pair[10] of games $(N; v)$ and $(N; w)$ in Σ_N,

$$\varphi(N; v) + \varphi(N; w) = \varphi(N; v \wedge w) + \varphi(N; v \vee w). \tag{18.139}$$

(b) Prove that the Shapley value is the unique solution concept over Σ_N satisfying efficiency, symmetry, the null player property, and the valuation property.

18.21 Let $(a_i)_{i \in N}$ be real numbers. Let v be the following coalitional function:

$$v(S) = \begin{cases} 0 & \text{if } |S| \leq k, \\ \sum_{i \in S} a_i & \text{if } |S| > k. \end{cases} \tag{18.140}$$

Compute the Shapley value of the game $(N; v)$ for every $k = 0, 1, \ldots, n$.

18.22 Consider a weighted majority game with four players, with quota $q = \frac{1}{2}$, and weights $(0.1, 0.2, 0.3, 0.4)$ (see Section 16.4, page 671, for a discussion on weighted majority games).

(a) Write down the coalitional game corresponding to this weighted majority game.
(b) Compute the Shapley value of the game.
(c) Compute the core of the game.

18.23 Compute the Shapley value of the weighted majority game with $N + 1$ players, weights $(\frac{N}{3}, 1, 1, 1, \ldots, 1)$, and quota $q = \frac{N}{2}$. Suppose that N is divisible by 6. What is the limit of the Shapley value of the player with weight $\frac{N}{3}$ as N goes to infinity?

18.24 Compute the Shapley value of the weighted majority game with $N + 2$ players, weights $(\frac{N}{3}, \frac{N}{3}, 1, 1, 1, \ldots, 1)$, and quota $q = \frac{5N}{6}$. Suppose that N is divisible by 6. What is the limit of the Shapley value of each player with weight $\frac{N}{3}$ as N goes to infinity?

18.25 Compute the Shapley value of the weighted majority game with $N + 2$ players, weights $(\frac{N}{3}, \frac{N}{3}, 1, 1, 1, \ldots, 1)$, and quota $q = N$. Suppose that N is divisible by 6. What is the limit of the Shapley value of each player with weight $\frac{N}{3}$ as N goes to infinity?

18.26 Let U be a nonempty set of players. Denote by Γ_U^* the family of coalitional games $(N; v)$ such that $N \subseteq U$, i.e., those games in which the set of players is taken from U.

Let $P : \Gamma_U^* \to \mathbb{R}$ be a function associating each game in Γ_U^* with a real number.

Definition 18.41 *For every game* $(N; v) \in \Gamma_U^*$*, the* marginal contribution *of player* i *in* N *to the game* $(N; v)$ *relative to* P *is*

$$D_i P(N; v) := \begin{cases} P(N; v) & \textit{if } |N| = 1, \\ P(N; v) - P(N \setminus \{i\}; v) & \textit{if } |N| \geq 2. \end{cases} \tag{18.141}$$

In Equation (18.141), $(N \setminus \{i\}; v)$ is the game in which the set of players is $N \setminus \{i\}$, and the coalitional function is the function v restricted to this set of players.

10 For the definition of the games $v \wedge w$ and $v \vee w$, see Exercise 18.18.

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press

Definition 18.42 *A function $P : \Gamma^*_U \to \mathbb{R}$ is called a* potential function *over Γ^*_U if for every $(N; v) \in \Gamma^*_U$ the sum of the marginal contributions equals $v(N)$:*

$$\sum_{i \in N} D_i P(N; v) = v(N). \tag{18.142}$$

Prove the following claims:

(a) For every nonempty set of players U, there is a unique potential function $P : \Gamma^*_U \to \mathbb{R}$.
(b) If P is a potential function, then for every game $(N; v) \in \Gamma^*_U$, and every $i \in N$,

$$D_i P(N; v) = \mathrm{Sh}_i(N; v). \tag{18.143}$$

18.27 Let $(N; v)$ be a simple monotonic game satisfying $v(N) = 1$. For each player i, define $B_i(N; v)$ to be the number[11] of coalitions S satisfying $v(S) = 0$ and $v(S \cup \{i\}) = 1$. The *Banzhaf value* of player i is defined to be

$$\mathrm{BZ}_i(N; v) := \frac{B_i(N; v)}{\sum_{j \in N} B_j(N; v)}. \tag{18.144}$$

Similarly to the Shapley–Shubik power index, the Banzhaf value also constitutes a power index, measuring the relative power of each player.

(a) Which of the following properties are satisfied by the Banzhaf value: efficiency, the null player property, additivity, marginality, symmetry?
(b) Compute the Banzhaf value of the game in Exercise 18.22.
(c) Find a formula for the Banzhaf value of the games in Exercises 18.23–18.24.
(d) Compute the Banzhaf value of the members of the United Nations Security Council, both in its pre-1965 structure and in its post-1965 structure (see Section 18.6.1 on page 765).

18.28 A cost game $(N; c)$ is called *convex* if

$$c(S) + c(T) \geq c(S \cup T) + c(S \cap T), \quad \forall S, T \subseteq N. \tag{18.145}$$

Prove that this property is equivalent to the following property:

$$c(S \cup \{i\}) - c(S) \geq c(T \cup \{i\}) - c(T), \quad \forall i \in N, \forall S \subseteq T \subseteq N \setminus \{i\}. \tag{18.146}$$

Prove that the airport game in Exercise 18.16 is a convex cost game.

18.29 Construct a road network game, as in Exercise 18.17, for which the corresponding cost game is not convex.
Hint: Construct a road network in which the villages and the central township are geographically situated on a circle.

18.30 Let i be a null player in a coalitional game $(N; v)$. Compute the Hart–Mas-Colell reduced game over $N \setminus \{i\}$ relative to the Shapley value Sh.

---

11 Such a coalition S is called a *swing* for player i.

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press

18.31 Let $(N; v)$ be an additive game (for the definition of an additive game, see Exercise 18.11) and let S be a nonempty coalition. Compute the Hart–Mas-Colell reduced game over S relative to the Shapley value Sh.

18.32 Let $(N; v)$ be a coalitional game, and let $(N; v^*)$ be its dual game (for the definition of a dual game, see Exercise 18.15). For a nonempty coalition S, write down the coalition function of the Hart–Mas-Colell reduced game of $(N; v^*)$ over S relative to the Shapley value Sh, in terms of v and $\widetilde{v}_{S,\mathrm{Sh}}$.

18.33 Prove that for every linear solution concept φ and for every nonempty coalition $S \subseteq N$, the function that assigns to each coalition game its Hart–Mas-Colell reduced game over S relative to φ is a linear function. That is, for every pair of coalitional games $(N; v)$ and $(N; u)$ and for every pair of real numbers α and β,

$$\widetilde{w}_{S,\varphi} = \alpha \widetilde{u}_{S,\varphi} + \beta \widetilde{v}_{S,\varphi}. \tag{18.147}$$

https://doi.org/10.1017/CBO9780511794216.019 Published online by Cambridge University Press