7 Equilibrium refinements

Chapter summary

The most important solution concept in noncooperative game theory is the Nash equilibrium. When games possess many Nash equilibria, we sometimes want to know which equilibria are more reasonable than others. In this chapter we present and study some refinements of the concept of Nash equilibrium.

In Section 7.1 we study *subgame perfect equilibrium*, which is a solution concept for extensive-form games. The idea behind this refinement is to rule out noncredible threats, that is, "irrational" behavior off the equilibrium path whose goal is to deter deviations. In games with perfect information, a subgame perfect equilibrium always exists, and it can be found using the process of *backward induction*.

The second refinement, presented in Section 7.3, is the *perfect equilibrium*, which is based on the idea that players might make mistakes when choosing their strategies. In extensive-form games there are two types of perfect equilibria corresponding to the two types of mistakes that players may make: one, called *strategic-form perfect equilibrium*, assumes that players may make a mistake at the outset of the game, when they choose the pure strategy they will implement throughout the game. The other, called *extensive-form perfect equilibrium*, assumes that players may make mistakes in choosing an action in each information set. We show by examples that these two concepts are different and prove that every extensive-form game possesses perfect equilibria of both types, and that every extensive-form perfect equilibrium is a subgame perfect equilibrium.

The last concept in this chapter, presented in Section 7.4, is the *sequential equilibrium* in extensive-form games. It is proved that every finite extensive-form game with perfect recall has a sequential equilibrium. Finally, we study the relationship between the sequential equilibrium and the extensive-form perfect equilibrium.

When a game has more than one equilibrium, we may wish to choose some equilibria over others based on "reasonable" criteria. Such a choice is termed a "refinement" of the equilibrium concept. Refinements can be derived in both extensive-form games and strategic-form games. We will consider several equilibrium refinements in this chapter, namely, perfect equilibrium, subgame perfect equilibrium, and sequential equilibrium.

Throughout this chapter, when we analyze extensive-form games, we will assume that if the game has chance vertices, every possible move at every chance vertex is chosen with positive probability. If there is a move at a chance vertex that is chosen with probability 0, it, and all the vertices following it in the tree, may be omitted, and we may consider instead the resulting smaller tree.

7.1 Subgame perfect equilibrium

The concept of subgame perfect equilibrium, which is a refinement of equilibrium in extensive-form games, is presented in this section. In an extensive-form game, each strategy vector σ defines a path from the root to one of the leaves of the game tree, namely, the path that is obtained when each player implements the strategy vector σ. When the strategy vector σ is a Nash equilibrium, the path that is thus obtained is called the *equilibrium path*. If x is a vertex along the equilibrium path, and if $\Gamma(x)$ is a subgame, then the strategy vector σ restricted to the subgame $\Gamma(x)$ is also a Nash equilibrium because each profitable deviation for a player in the subgame $\Gamma(x)$ is also a profitable deviation in the original game (explain why). In contrast, if the vertex x is not located along the equilibrium path (in which case it is said to be *off the equilibrium path*) then the strategy vector σ restricted to the subgame $\Gamma(x)$ is not necessarily a Nash equilibrium of the subgame. The following example illustrates this point.

Example 7.1 Consider the two-player extensive-form game shown in Figure 7.1.

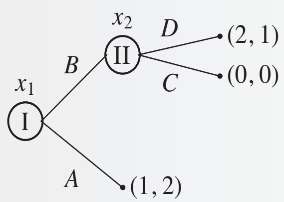

Figure 7.1 The extensive-form game in Example 7.1

Figure 7.2 shows the corresponding strategic form of this game.

Player I \ Player II	C	D
A	1, 2	1, 2
B	0, 0	2, 1

Figure 7.2 The strategic-form game, and two pure-strategy equilibria of the game

This game has two pure-strategy equilibria, (B, D) and (A, C). Player I clearly prefers (B, D), while Player II prefers (A, C). In addition, the game has a continuum of mixed-strategy equilibria: $(A, [y(C), (1 - y)(D)])$ for $y \geq \frac{1}{2}$, with payoff $(1, 2)$, which is identical to the payoff of (A, C). Which equilibrium is more likely to be played?

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The extensive form of the game is depicted again twice in Figure 7.3, with the thick lines corresponding to the two pure-strategy equilibria.

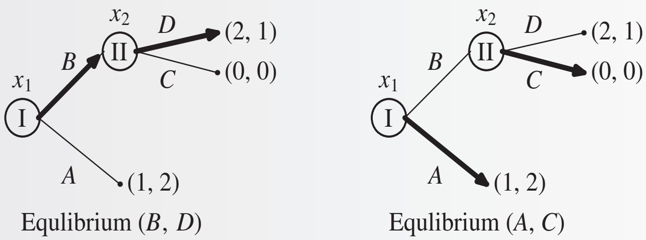

Figure 7.3 The pure-strategy equilibria of the game in extensive form

This game has one proper subgame, $\Gamma(x_2)$, in which only Player II is active. The two equilibria (A, C) and (B, D) induce different plays in this subgame. While the restriction of the equilibrium (B, D) to $\Gamma(x_2)$, namely D, is an equilibrium of the subgame $\Gamma(x_2)$, the restriction of the equilibrium (A, C) to $\Gamma(x_2)$, namely C, is not an equilibrium in this subgame, since a deviation to D is profitable for Player II. The vertex x_2 is on the equilibrium path of (B, D), which is $x_1 \to x_2 \to (2, 1)$, and it is not on the equilibrium path of (A, C), which is $x_1 \to (1, 2)$. This is necessarily so, since if x_2 were on the equilibrium path of (A, C) and Player II did not play an equilibrium in the subgame $\Gamma(x_2)$, then (A, C) could not be an equilibrium (why ?).

Player II's strategy at vertex x_2 seems irrational: the equilibrium strategy calls on him to choose C, which yields him a payoff of 0, instead of D, which yields him a payoff of 1. This choice, in fact, is never actually made when this equilibrium is played, because Player I chooses A at vertex x_1, but the equilibrium, as constructed, says that "if Player I were to choose B, then Player II would choose C." This may be regarded as a threat directed by Player II to Player I: if you "dare" choose B, I will choose C, and then you will get 0 instead of 1, which you would get by choosing A. This threat is intended by Player II to persuade Player I to choose the action that leads to payoff $(1, 2)$, which Player II prefers to $(2, 1)$. Is this a *credible* threat?

Whether or not a threat is credible depends on many factors, which are not expressed in our model of the game: previous interaction between the players, reputation, behavioral norms, and so on. Consideration of these factors, which may be important and interesting, is beyond the scope of this book. We will, however, consider what happens if we invalidate such threats, on the grounds that they are not "rational."

Another way of saying the same thing is: the restriction of the equilibrium (A, C) to the subgame $\Gamma(x_2)$, which begins at vertex x_2 (the subgame in which only Player II has a move), yields the strategy C, which is not an equilibrium of that subgame. This observation led to the concept of subgame perfect equilibrium that we develop in this section. ◀

Reinhard Selten [1965, 1973] suggested that the equilibria that should be chosen in extensive-form games are those equilibria that are also equilibria when restricted to each subgame. In other words, Selten suggested choosing those equilibria at which the actions of the players are still in equilibrium even when they are off the equilibrium path.

By definition, a strategy σ_i tells player i which action to choose at each of his information sets, even at information sets that will not be arrived at during the play of the game that results from implementing σ_i (whether due to moves chosen by player i, or moves chosen by the other players). It follows that, for every strategy vector σ, it is possible to compute the payoff of each player if the play of the game is at vertex x (even if the play has not arrived at x when the players implement σ). Denote by $u_i(\sigma \mid x)$ player i's payoff in the subgame $\Gamma(x)$ when the players implement the strategy vector σ, if the play of the game is at vertex x. For example, in the game in Example 7.1, $u_1((A, C) \mid x_1) = 1$ and $u_1((A, C) \mid x_2) = 0$ (note that x_2 is not reached when (A,C) is played).

The payoff $u_i(\sigma \mid x)$ depends only on the restriction of the strategy vector σ to the subgame $\Gamma(x)$. We will therefore use the same notation to denote the payoff when σ is a strategy vector in $\Gamma(x)$ and when in the strategy vector σ some of the strategies are in Γ and some in $\Gamma(x)$.

Definition 7.2 *A strategy vector σ^* (in mixed strategies or behavior strategies) in an extensive-form game Γ is called a* subgame perfect equilibrium *if for every subgame, the restriction of the strategy vector σ^* to the subgame is a Nash equilibrium of that subgame: for every player $i \in N$, every strategy σ_i, and every subgame $\Gamma(x)$,*

$$u_i(\sigma^* \mid x) \geq u_i(\sigma_i, \sigma^*_{-i} \mid x). \tag{7.1}$$

As we saw in Example 7.1, the equilibrium (A, C) is not a subgame perfect equilibrium. In contrast, the equilibrium (B, D) is a subgame perfect equilibrium: the choice D is an equilibrium of the subgame starting at x_2. For each $y \in [\frac{1}{2}, 1]$, the equilibrium in mixed strategies $(A, [y(C), (1 - y)(D)])$ is not a subgame perfect equilibrium, because the choice of C with positive probability is not an equilibrium of the subgame starting at x_2.

Note that in the strategic form of the game, Player II's strategy C is (weakly) dominated by the strategy D, and hence the elimination of dominated strategies in this game eliminates the equilibrium (A, C) (and the mixed-strategy equilibria for $y \in [\frac{1}{2}, 1]$), leaving only the subgame perfect equilibrium (B, D). A solution concept based on the elimination of weakly dominated strategies, and its relation to the concept of subgame perfect equilibrium, will be studied in Section 7.3.

Remark 7.3 *Since every game is a subgame of itself, by definition, every subgame perfect equilibrium is a Nash equilibrium. The concept of subgame perfect equilibrium is therefore a refinement of the concept of Nash equilibrium.* ♦

As previously stated, each leaf x in a game tree defines a sub-tree $\Gamma(x)$ in which effectively no player participates. An extensive-form game that does not include any subgame other than itself and the subgames defined by the leaves is called a *game without nontrivial subgames*. For such games, the condition appearing in Definition 7.2 holds vacuously, and we therefore deduce the following corollary.

Theorem 7.4 *In an extensive-form game without nontrivial subgames, every Nash equilibrium (in mixed strategies or behavior strategies) is a subgame perfect equilibrium.*

For each strategy vector σ, and each vertex x in the game tree, denote by $\mathbf{P}_\sigma(x)$ the probability that the play of the game will visit vertex x when the players implement the strategy vector σ.

Theorem 7.5 *Let σ^* be a Nash equilibrium (in mixed strategies or behavior strategies) of an extensive-form game Γ, and let $\Gamma(x)$ be a subgame of Γ. If $\mathbf{P}_{\sigma^*}(x) > 0$, then the strategy*

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vector σ^* restricted to the subgame $\Gamma(x)$ is a Nash equilibrium (in mixed strategies or behavior strategies) of $\Gamma(x)$.

This theorem underscores the fact that the extra conditions that make a Nash equilibrium a subgame perfect equilibrium apply to subgames $\Gamma(x)$ for which $\mathbf{P}_\sigma(x) = 0$, such as for example the subgame $\Gamma(x_2)$ in Example 7.1, under the equilibrium (A, C).

Proof: The idea behind the proof is as follows. If in the subgame $\Gamma(x)$ the strategy vector σ^* restricted to the subgame were not a Nash equilibrium, then there would exist a player i who could profit in that subgame by deviating from σ_i^* to a different strategy, say σ_i', in the subgame. Since the play of the game visits the subgame $\Gamma(x)$ with positive probability, the player can profit in Γ by deviating from σ_i^*, by implementing σ_i' if the game gets to x.

We now proceed to the formal proof. Let $\Gamma(x)$ be the subgame of Γ starting at vertex x and let σ^* be a Nash equilibrium of Γ satisfying $\mathbf{P}_{\sigma^*}(x) > 0$. Let σ_i' be a strategy of player i in the subgame $\Gamma(x)$. Denote by σ_i the strategy[1] of player i that coincides with σ^* except in the subgame $\Gamma(x)$, where it coincides with σ_i'.

Since σ^* and $(\sigma_i, \sigma_{-i}^*)$ coincide at all vertices that are not in $\Gamma(x)$,

$$\mathbf{P}_{\sigma^*}(x) = \mathbf{P}_{(\sigma_i, \sigma_{-i}^*)}(x). \tag{7.2}$$

Denote by $\widehat{u}_i$ the expected payoff of player i, conditional on the play of the game not arriving at the subgame $\Gamma(x)$ when the players implement the strategy vector σ^*. Then

$$u_i(\sigma^*) = \mathbf{P}_{\sigma^*}(x) u_i(\sigma^* \mid x) + (1 - \mathbf{P}_{\sigma^*}(x))\widehat{u}_i. \tag{7.3}$$

Writing out the analogous equation for the strategy vector $(\sigma_i, \sigma_{-i}^*)$ and using Equation (7.2) yields

$$\begin{aligned} u_i(\sigma_i, \sigma_{-i}^*) &= \mathbf{P}_{(\sigma_i, \sigma_{-i}^*)}(x) u_i((\sigma_i', \sigma_{-i}^*) \mid x) + (1 - \mathbf{P}_{(\sigma_i, \sigma_{-i}^*)}(x))\widehat{u}_i && (7.4)\\ &= \mathbf{P}_{\sigma^*}(x) u_i((\sigma_i', \sigma_{-i}^*) \mid x) + (1 - \mathbf{P}_{\sigma^*}(x))\widehat{u}_i. && (7.5) \end{aligned}$$

Since σ^* is an equilibrium,

$$\begin{aligned} \mathbf{P}_{\sigma^*}(x) u_i(\sigma^* \mid x) + (1 - \mathbf{P}_{\sigma^*}(x))\widehat{u}_i &= u_i(\sigma^*) && (7.6)\\ &\geq u_i(\sigma_i, \sigma_{-i}^*) && (7.7)\\ &= \mathbf{P}_{\sigma^*}(x) u_i((\sigma_i', \sigma_{-i}^*) \mid x) + (1 - \mathbf{P}_{\sigma^*}(x))\widehat{u}_i. && (7.8) \end{aligned}$$

Since $\mathbf{P}_{\sigma^*}(x) > 0$, one has

$$u_i(\sigma^* \mid x) \geq u_i((\sigma_i', \sigma_{-i}^*) \mid x). \tag{7.9}$$

1 When σ_i^* and σ_i' are behavior strategies, the strategy σ_i coincides with σ_i^* in player i's information sets that are not in $\Gamma(x)$, and with σ_i' in player i's information sets that are in the subgame of $\Gamma(x)$.

When σ_i^* and σ_i' are mixed strategies, the strategy σ_i is defined as follows: every pure strategy s_i of player i is composed of the pair (s_i^1, s_i^2), in which s_i^1 associates a move with each of player i's information sets in the subgame $\Gamma(x)$, and s_i^2 associates a move with each of player i's information sets that are not in the subgame $\Gamma(x)$. Then $\sigma_i(s_i^1, s_i^2) := \sigma_i'(s_i^1) \sum_{\{\widehat{s}_i : \widehat{s}_i^2 = s_i^2\}} \sigma^*(\widehat{s}_i)$.

Since $\Gamma(x)$ is a subgame, every information set that is in $\Gamma(x)$ does not contain vertices that are not in that subgame; hence the strategy σ_i is well defined in both cases.

Since this inequality is satisfied for each player i, and each strategy σ_i, in the subgame $\Gamma(x)$, the strategy vector σ^* restricted to the subgame $\Gamma(x)$ is a Nash equilibrium of $\Gamma(x)$. □

Recall that, given a mixed strategy σ_i of player i, we denote by $\sigma_i(s_i)$ the probability that the pure strategy s_i will be chosen, and given a behavior strategy σ_i, we denote by $\sigma_i(U_i; a_i)$ the probability that the action a_i will be chosen in the information set U_i of player i.

Definition 7.6 *A mixed strategy σ_i of player i is called* completely mixed *if $\sigma_i(s_i) > 0$ for each $s_i \in S_i$. A behavior strategy σ_i of player i is called* completely mixed *if $\sigma_i(U_i; a_i) > 0$ for each information set U_i of player i, and each action $a_i \in A(U_i)$.*

A mixed strategy is completely mixed if under the strategy a player chooses each of his pure strategies with positive probability, and a behavior strategy is completely mixed if at each of his information sets, the player chooses with positive probability each of his possible actions at that information set.

Since at each chance vertex every action is chosen with positive probability, if each player i uses a completely mixed strategy σ_i, then $\mathbf{P}_\sigma(x) > 0$ for each vertex x in the game tree. This leads to the following corollary of Theorem 7.5.

Corollary 7.7 *Let Γ be an extensive-form game. Then every Nash equilibrium in completely mixed strategies (behavior strategies or mixed strategies) is a subgame perfect equilibrium.*

As Theorem 7.4 states, in games whose only subgames are the game itself, every Nash equilibrium is a subgame perfect equilibrium; in such cases, subgame perfection imposes no further conditions beyond the conditions defining the Nash equilibrium. In contrast, when a game has a large number of subgames, the concept of subgame perfection becomes significant, because a Nash equilibrium must meet a large number of conditions to be a subgame perfect equilibrium. The most extreme case of such a game is a game with perfect information. Recall that a game with perfect information is a game in which every information set is composed of only one vertex. In such a game, every vertex is the root of a subgame.

Example 7.8 Figure 7.4 depicts a two-player game with perfect information.

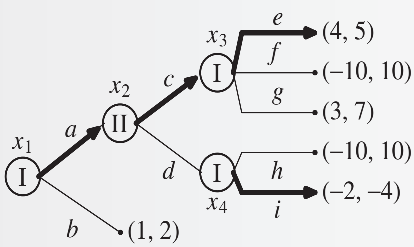

Figure 7.4 A subgame perfect equilibrium in a game with perfect information

To find a subgame perfect equilibrium, start with the smallest subgames: those whose roots are vertices adjacent to the leaf vertices, in this case the subgames $\Gamma(x_3)$ and $\Gamma(x_4)$. The only equilibrium in the subgame $\Gamma(x_3)$ is the one in which Player I chooses e, because this action leads

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to the result (4, 5), which includes the best payoff Player I can receive in the subgame. Similar reasoning shows that the only equilibrium in the subgame $\Gamma(x_4)$ is the one in which Player I chooses i, leading to the result $(-2, -4)$. We can now replace the subgame $\Gamma(x_3)$ with the result of its equilibrium, (4, 5), and the subgame $\Gamma(x_4)$ with the result of its equilibrium, $(-2, -4)$. This yields the game depicted in Figure 7.5 (this procedure is called "folding" the game).

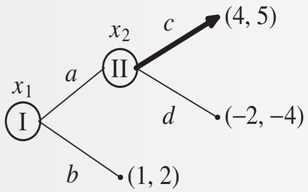

Figure 7.5 The folded game

Now, in the subgame starting at x_2, at equilibrium, Player II chooses c, leading to the result (4, 5). Folding this subgame leads to the game depicted in Figure 7.6.

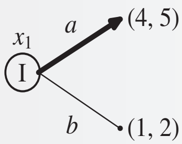

Figure 7.6 The game after further folding

In this game, at equilibrium Player I chooses a, leading to the result (4, 5). Recapitulating all the stages just described gives us the subgame perfect equilibrium shown in Figure 7.4. ◀

This process is called *backward induction* (see Remark 4.50 on page 121). The process leads to the equilibrium $((a, e, i), c)$, which by construction is a subgame perfect equilibrium. Backward induction leads, in a similar way, to a subgame perfect equilibrium in pure strategies in every (finite) game with perfect information. We thus have the following theorem.

Theorem 7.9 *Every finite extensive-form game with perfect information has a subgame perfect equilibrium in pure strategies.*

The proof of the theorem is accomplished by backward induction on the subgames, from the smallest (starting from the vertices adjacent to the leaves) to the largest (starting from the root of the tree). The formal proof is left to the reader (Exercise 7.8). We will later show (Theorem 7.37 on page 271) that every extensive-form game with perfect recall has a subgame perfect equilibrium in behavior strategies.

With regard to games with incomplete information, we can reuse the idea of "folding" a game to prove the following theorem.

Theorem 7.10 *Every extensive-form game with perfect recall has a subgame perfect equilibrium in mixed strategies.*

The proof of the theorem is left to the reader as an exercise (Exercise 7.15).

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Remark 7.11 *In the last two theorems we used the fact that an extensive-form game is finite by definition: the number of decision vertices is finite, and the number of actions at every decision vertex is finite. To prove Theorem 7.9, it is not necessary to assume that the game tree is finite; it suffices to assume that there exists a natural number L such that the length of each path emanating from the root is no greater than L. Without this assumption, the process of backward induction cannot begin, and these two theorems are not valid. These theorems do not hold in games that are not finite. There are examples of infinite two-player games that have no equilibria (see, for example, Mycielski [1992], Claim 3.1). Such examples are beyond the scope of this book. Exercise 7.16 presents an example of a game with imperfect information in which one of the players has a continuum of pure strategies, but the game has no subgame perfect equilibria.* ♦

Remark 7.12 *In games with perfect information, when the backward induction process reaches a vertex at which a player has more than one action that maximizes his payoff, any one of them can be chosen in order to continue the process. Each choice leads to a different equilibrium, and therefore the backward induction process can identify several equilibria (all of which are subgame perfect equilibria).* ♦

Remark 7.13 *The process of backward induction is in effect the game-theory version of the dynamic programming principle widely used in operations research. This is a very natural and useful approach to multistage optimization: start with optimizing the action chosen at the last stage, stage n, for every state of the system at stage $n-1$. Continue by optimizing the action chosen at stage $n-1$ for every state of the system at stage $n-2$, and so on.* ♦

Backward induction is a very convincing logical method. However, its use in game theory sometimes raises questions stemming from the fact that unlike dynamic optimization problems with a single decision maker, games involve several interacting decision makers. We will consider several examples illustrating the limits of backward induction in games. We first construct an example of a game that has an equilibrium that is not subgame perfect, but is preferred by both players to all the subgame perfect equilibria of the game.

Example 7.14 A two-player extensive-form game with two equilibria is depicted in Figure 7.7.

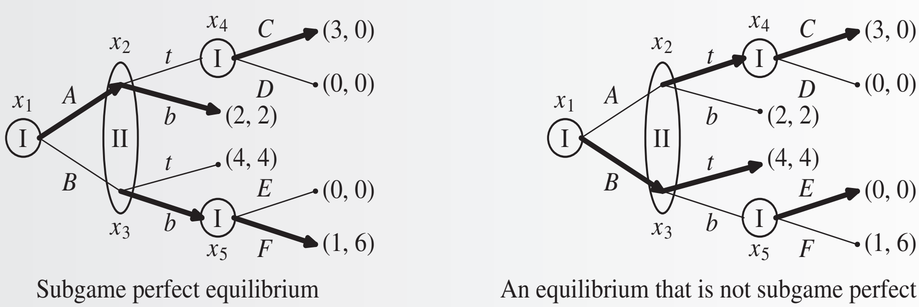

Figure 7.7 A game with two equilibria

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Using backward induction, we find that the only subgame perfect equilibrium in the game is $((A, C, F,), b)$, leading to the payoff (2, 2). The equilibrium $((B, C, E), t)$ leads to the payoff (4, 4) (verify that this is indeed an equilibrium). This equilibrium is not a subgame perfect equilibrium, since it calls on Player I to choose E in the subgame $\Gamma(x_4)$, which is not an equilibrium. This choice by Player I may be regarded as a threat to Player II: "if you choose b (in an attempt to get 6) instead of t, I will choose E and you will get 0." What is interesting in this example is that both players have an "interest" in maintaining this threat, because it serves both of them: it enables them to receive the payoff (4, 4), which is preferred by both of them to the payoff (2, 2) that they would receive under the game's only subgame perfect equilibrium. ◀

Example 7.15 **The repeated Prisoner's Dilemma** Consider the Prisoner's Dilemma game with the payoff shown in Figure 7.8.

Player I \ Player II	D	C
D	1, 1	4, 0
C	0, 4	3, 3

Figure 7.8 Prisoner's Dilemma

Suppose two players play the Prisoner's Dilemma game 100 times, with each player at each stage informed of the action chosen by the other player (and therefore also the payoff at each stage).

We can analyze this game using backward induction: at equilibrium, at the 100th (i.e., the last) repetition, each of the players chooses D (which strictly dominates C), independently of the actions undertaken in the previous stages: for every other choice, a player choosing C can profit by deviating to D. This means that in the game played at the 99th stage, what the players choose has no effect on what will happen at the 100th stage, so that at equilibrium each player chooses D at stage 99, and so forth. Backward induction leads to the result that the only subgame perfect equilibrium is the strategy vector under which both players choose D at every stage.[2] In Exercise 7.9, the reader is asked to turn this proof idea into a formal proof.

In fact, it can be shown that in every equilibrium (not necessarily subgame perfect equilibrium) in this 100-stage game the players play (D, D) in every stage (see Chapter 13). This does not seem reasonable: one would most likely expect rational players to find a way to obtain the payoff (3, 3), at least in the initial stages of the game, and not play (D, D), which yields only (1, 1), in every stage. A large number of empirical studies confirm that in fact players do usually cooperate during many stages when playing the repeated Prisoner's Dilemma, in order to obtain a higher payoff than that indicated by the equilibrium strategy. ◀

Example 7.16 **The Centipede game** The Centipede game that we saw in Exercise 3.12 on page 61 is also a two-player game with 100 stages, but unlike the repeated Prisoner's Dilemma, in the Centipede game the actions of the players are implemented sequentially, rather than simultaneously: in the odd stages, $t = 1, 3, \ldots, 99$, Player I has a turn, and he decides whether to stop the game (S) or to

2 This is a verbal description of the process of backward induction in a game tree with 100 stages. Writing out the formal backward induction process in full when the game tree is this large is, of course, not practical.

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continue (C). If he stops the game at stage t, the payoff is $(t, t-1)$ (hence Player I receives t, and Player II receives $t-1$), and if he instead chooses C, the game continues on to the next stage. In the even stages, $t = 2, 4, \ldots, 100$, Player II has a turn, and he also chooses between stopping the game (S) and continuing (C). If he stops the game at stage t, the payoff is $(t-2, t+1)$. If neither player chooses to stop in the first 99 stages, the game ends after 100 stages, with the payoff of 101 to Player I and 100 to Player II. The visual depiction of the game in extensive form explains why it is called the Centipede game (see Figure 7.9).

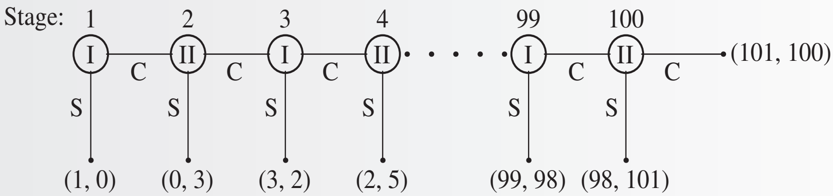

Figure 7.9 The Centipede game

What does backward induction lead to in this game? At stage 100, Player II should choose to stop the game: if he stops the game, he leaves the table with \$101, while if the game continues he will only get \$100. Since that is the case, at stage 99, Player I should stop the game: he knows that if he chooses to continue the game, Player II will stop the game at the next stage, and Player I will end up with \$98, while if he stops, he walks away with \$99. Subgame perfection requires him to stop at stage 99. A similar analysis obtains at every stage; hence the only subgame perfect equilibrium in the game is the strategy at which each player stops the game at every one of his turns. In particular, at this equilibrium, Player I stops the game at the first stage, and the payoff is $(1, 0)$. This result is unreasonable: shrewd players will not stop the game and be satisfied with the payoff $(1, 0)$ when they can both do much better by continuing for several stages. Empirical studies reveal that many people do indeed "climb the centipede" up to a certain level, and then one of them stops the game.

It can be shown that at every Nash equilibrium of the game (not necessarily subgame perfect equilibrium), Player I chooses S at the first stage (Exercise 4.19 on page 134). ◀

7.2 Rationality, backward induction, and forward induction

The last two examples indicate that backward induction alone is insufficient to describe rational behavior. Kohlberg and Mertens [1986] argued that backward induction requires that at each stage every player looks only at the continuation of the game from that stage forwards, and ignores the fact that the game has reached that stage. But if the game has reached a particular vertex in the game tree, that fact itself gives information about the behavior of the other players, and this should be taken into account. For example, if I am playing the repeated Prisoner's Dilemma, and at the third stage it transpires that the other player played C in the previous two stages, then I need to take this into account, beyond regarding it as "irrational." Perhaps the other player is signaling that we should both play (C, C)? Similarly, if the Centipede game reaches the second stage, then Player

I must have deviated from equilibrium, and not have stopped the game at the first stage. It seems reasonable to conjecture that if Player II chooses not to stop the game at that point, then Player I will not stop at stage 3. Backward induction implies that Player I should stop at stage 3, but it also implies that he should stop at stage 1. If he did not stop then, why should he stop now? The approach that grants significance to the history of the game is called *forward induction*. We will not present a formal description of the forward induction concept, and instead only give an example of it.

Example 7.17 Consider the two-player extensive-form game depicted in Figure 7.10.

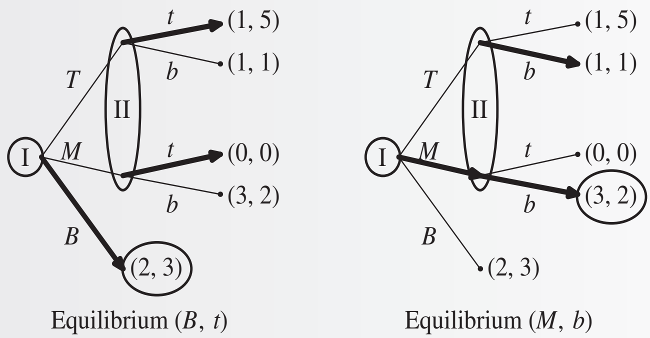

Figure 7.10 An extensive-form game with two subgame perfect equilibria

This game has two equilibria in pure strategies:

- (B, t), with payoff $(2, 3)$.
- (M, b), with payoff $(3, 2)$.

Since the game has no nontrivial subgames, both equilibria are subgame perfect equilibria (Theorem 7.4). Is (B, t) a reasonable equilibrium? Is it reasonable for Player II to choose t? If he is called on to choose an action, that means that Player I has not chosen B, which would guarantee him a payment of 2. It is unreasonable for him to have chosen T, which guarantees him only 1, and he therefore must have chosen M, which gives him the chance to obtain 3. In other words, although Player II cannot distinguish between the two vertices in his information set, from the very fact that the game has arrived at the information set and that he is being called upon to choose an action, he can deduce, assuming that Player I is rational, that Player I has played M and not T. This analysis leads to the conclusion that Player II should prefer to play b, if called upon to choose an action, and (M, b) is therefore a more reasonable equilibrium. This convincing choice between the two equilibria was arrived at through forward induction. ◀

Inductive reasoning, and the inductive use of the concept of rationality, has the potential of raising questions regarding the consistency of rationality itself. Consider the game depicted in Figure 7.11.

The only subgame perfect equilibrium of this game is $((r, c), a)$, which yields the payoff $(2, 1)$. Why does Player II choose a at x_2? Because if Player I is rational, he will then choose c, leading to the payoff $(1, 2)$, which Player II prefers to the payoff $(1, 1)$ that would result if he were to choose b. But is Player I really rational? Consider the fact that

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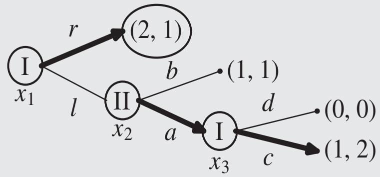

Figure 7.11 A game with only one subgame perfect equilibrium

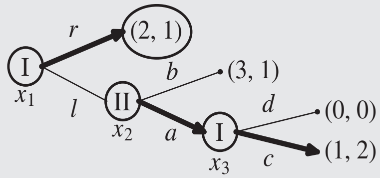

Figure 7.12 A game with only one subgame perfect equilibrium

if the game has arrived at x_2, and Player II is called upon to play, then Player I must be irrational: Player I must have chosen l, which yields him at most 1, instead of choosing r, which guarantees him 2. Then why should Player II assume that Player I will be rational at x_3? Perhaps it would be more rational for Player II to choose b, and guarantee himself a payoff of 1, instead of running the risk that Player I may again be irrational and choose d, which will yield the payoff (0, 0).

The game depicted in Figure 7.12, which is just like the previous game except that the payoff (1, 1) is replaced by (3, 1), is even more problematic.

This game also has only one subgame perfect equilibrium, $((r, c), a)$, yielding payoff (2, 1). Again, by backward induction, Player I will not choose l, which leads to the payoff (1, 2). Player II, at x_2, must therefore conclude that Player I is irrational (because Player I must have chosen l at x_1, which by backward induction leads to him getting 1, instead of r, which guarantees him a payoff of 2). And if Player I is irrational, then Player II may need to fear that if he chooses a, Player I will then choose d and the end result will be (0, 0). It is therefore possible that at x_2, Player II will choose b, in order to guarantee himself a payoff of 1. But, if that is the case, Player I is better off choosing l at x_1, because then he will receive 3, instead of 2, which is what choosing r gets him. So is Player I really irrational if he chooses l? Perhaps Player I's choice of l is a calculated choice, aimed at making Player II think that he is irrational, and therefore leading Player II to choose b? Then which one of Player I's choices at vertex x_1 is rational, and which is irrational?

7.3 Perfect equilibrium

This section presents the concept of perfect equilibrium. While subgame perfect equilibrium is a refinement of the concept of Nash equilibrium applicable only to extensive-form

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games, perfect equilibrium is a refinement of the concept of Nash equilibrium that is applicable to extensive-form games and strategic-form games.

After introducing the concept of subgame perfect equilibrium in 1965, Selten revisited it in 1975, using the following example.

Example 7.18 Consider the three-player game depicted in Figure 7.13.

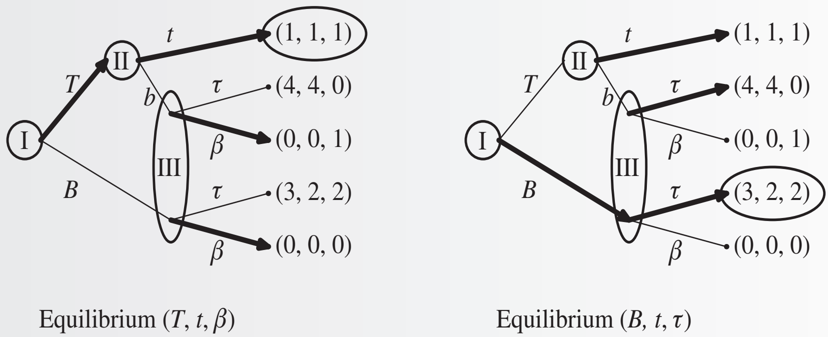

Figure 7.13 A game in extensive-form, along with two equilibria

Since this game has no nontrivial subgames, every equilibrium of the game is a subgame perfect equilibrium. There are two equilibria in pure strategies:

- (T, t, β), with payoff (1, 1, 1).
- (B, t, τ), with payoff (3, 2, 2).

(Check that each of these two strategy vectors does indeed form a Nash equilibrium.)

Selten argued that Player II's behavior in the equilibrium (B, t, τ) is irrational. The reasoning is as follows: if Player II is called upon to play, that means that Player I misplayed, playing T instead of B, because at equilibrium he is supposed to play B. Since Player III is supposed to play τ at that equilibrium, if Player II deviates and plays b, he will get 4 instead of 1. ◀

Selten introduced the concept of the "trembling hand," which requires rational players to take into account the possibility that mistakes may occur, even if they occur with small probability. The equilibrium concept corresponding to this type of rationality is called "perfect equilibrium." In an extensive-form game, a mistake can occur in two ways. A player may, at the beginning of the play of a game, with small probability mistakenly choose a pure strategy that differs from the one he intends to choose; such a mistake can cause deviations at every information set that is arrived at in the ensuing play. A second possibility is that the mistakes in different information sets are independent of each other; at each information set there is a small probability that a mistake will be made in choosing the action. As we will see later in this chapter, these two ways in which mistakes can occur lead to alternative perfect equilibrium concepts.

The analysis of this solution concept therefore requires careful attention to these details. We will first present the concept of perfect equilibrium for strategic-form games, in Section 7.3.1, and present its extensive-form game version in Section 7.3.2.

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7.3.1 Perfect equilibrium in strategic-form games

Definition 7.19 *Let $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a game in strategic form in which the set of pure strategies of each player is finite. A* perturbation vector of player i *is a vector $\varepsilon_i = (\varepsilon_i(s_i))_{s_i\in S_i}$ satisfying $\varepsilon_i(s_i) > 0$ for each $s_i \in S_i$, and*

$$\sum_{s_i\in S_i} \varepsilon_i(s_i) \leq 1, \quad \forall i \in N. \tag{7.10}$$

A perturbation vector *is a vector $\varepsilon = (\varepsilon_i)_{i\in N}$, where ε_i is a perturbation vector of player i for each $i \in N$.*

For every perturbation vector ε, the ε-perturbed game *is the game $\Gamma(\varepsilon) = (N, (\Sigma_i(\varepsilon_i))_{i\in N}, (u_i)_{i\in N})$ where player i's strategy set is*

$$\Sigma_i(\varepsilon_i) := \{\sigma_i \in \Sigma_i : \sigma_i(s_i) \geq \varepsilon_i(s_i), \quad \forall s_i \in S_i\}. \tag{7.11}$$

In words, in the ε-perturbed game $\Gamma(\varepsilon)$, every pure strategy s_i is chosen with probability greater than or equal to $\varepsilon_i(s_i)$. The condition in Equation (7.10) guarantees that the strategy set $\Sigma_i(\varepsilon_i)$ is not empty. Furthermore, $\Sigma_i(\varepsilon_i)$ is a compact and convex set (Exercise 7.17). The following theorem therefore follows from Theorem 5.32 (page 171).

Theorem 7.20 *Every (finite) ε-perturbed game has an equilibrium; i.e., there exists a mixed-strategy vector $\sigma^* = (\sigma_i^*)_{i\in N}$ satisfying $\sigma_i^* \in \Sigma_i(\varepsilon_i)$ for each player $i \in N$, and*

$$U_i(\sigma^*) \geq U_i(\sigma_i, \sigma^*_{-i}), \quad \forall i \in N, \ \forall \sigma_i \in \Sigma_i(\varepsilon_i). \tag{7.12}$$

Given a perturbation vector ε, denote by

$$M(\varepsilon) := \max_{i\in N, s_i\in S_i} \varepsilon_i(s_i) \tag{7.13}$$

the maximal perturbation in $\Gamma(\varepsilon)$, and by

$$m(\varepsilon) := \min_{i\in N, s_i\in S_i} \varepsilon_i(s_i) \tag{7.14}$$

the minimal perturbation. Note that $m(\varepsilon) > 0$.

Example 7.21 Consider the two-player game depicted in Figure 7.14.

Player II

Player I		L	R
	T	1, 1	0, 0
	B	0, 0	0, 0

Figure 7.14 The strategic-form game of Example 7.21

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This game has two pure strategy equilibria, (T, L) and (B, R). Consider now the ε-perturbed game, where the perturbation vector $\varepsilon = (\varepsilon_1, \varepsilon_2)$ is as follows.

$$\begin{array}{ll} \varepsilon_1(T) = \eta, & \varepsilon_2(L) = \eta, \\ \varepsilon_1(B) = \eta^2, & \varepsilon_2(R) = 2\eta, \end{array}$$

where $\eta \in (0, \frac{1}{3}]$. Then

$$M(\varepsilon) = 2\eta, \quad m(\varepsilon) = \eta^2. \tag{7.15}$$

Since $m(\varepsilon) > 0$, all the strategies in $\Sigma_1(\varepsilon_1)$ and $\Sigma_2(\varepsilon_2)$ are completely mixed strategies. In particular, in the perturbed game, Player I's payoff under T is always greater than his payoff under B: if Player I plays B, he receives 0, while if he plays T his expected payoff is positive. It follows that Player I's best reply to every strategy in $\Sigma_2(\varepsilon_2)$ is to play T with the maximally allowed probability; this means that the best reply is $[(1 - \eta^2)(T), \eta^2(B)]$.

Similarly, we can calculate that Player II's expected payoff is greatest when he plays L, and his best reply to every strategy in $\Sigma_1(\varepsilon_1)$ is $[(1 - 2\eta)(L), 2\eta(R)]$. It follows that the only equilibrium in this ε-perturbed game is

$$([(1 - \eta^2)(T), \eta^2(B)], [(1 - 2\eta)(L), 2\eta(R)]). \tag{7.16}$$

◀

In Example 7.21 Player I's pure strategy T weakly dominates his pure strategy B. In this case, when Player II is restricted to playing mixed strategies, the strategy T always leads to a higher payoff than the strategy B, and therefore at equilibrium Player I plays the pure strategy B with the minimal possible probability. This line of reasoning is generalized to the following theorem, whose proof is left to the reader.

Theorem 7.22 *If s_i is a weakly dominated strategy, then at every equilibrium σ of the ε-perturbed game,*

$$\sigma_i(s_i) = \varepsilon_i(s_i). \tag{7.17}$$

Let $(\varepsilon^k)_{k\in\mathbb{N}}$ be a sequence of perturbation vectors satisfying $\lim_{k\to\infty} M(\varepsilon^k) = 0$: the maximal constraint converges to 0. Then for every completely mixed strategy σ_i of player i, there exists $k_0 \in \mathbb{N}$ such that $\sigma_i \in \Sigma_i(\varepsilon_i^k)$ for every $k_0 \geq k$. Indeed, denote $c := \min_{s_i \in S_i} \sigma_i(s_i) > 0$ and choose $k_0 \in \mathbb{N}$, where $M(\varepsilon_i^k) \leq c$ for all $k \geq k_0$. Then $\sigma_i \in \Sigma_i(\varepsilon_i^k)$ for every $k \geq k_0$. Since every mixed strategy in Σ_i can be approximated by a completely mixed strategy (Exercise 7.18), we deduce the following theorem.

Theorem 7.23 *Let $(\varepsilon^k)_{k\in\mathbb{N}}$ be a sequence of perturbation vectors satisfying $\lim_{k\to\infty} M(\varepsilon^k) = 0$. For every mixed strategy $\sigma_i \in \Sigma_i$ of player i, there exists a sequence $(\sigma_i^k)_{k\in\mathbb{N}}$ of mixed strategies of player i satisfying the following two properties:*

- *$\sigma_i^k \in \Sigma_i(\varepsilon_i^k)$ for each $k \in \mathbb{N}$.*
- *$\lim_{k\to\infty} \sigma_i^k$ exists and equals σ_i.*

The following theorem, which is a corollary of Theorem 7.23, states that the limit of equilibria in an ε-perturbed game, where the perturbation vectors $(\varepsilon^k)_{k\in\mathbb{N}}$ are positive and converge to zero, is necessarily a Nash equilibrium of the original unperturbed game.

Theorem 7.24 *Let $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a strategic-form game. For each $k \in \mathbb{N}$, let ε^k be a perturbation vector, and let σ^k be an equilibrium of the ε^k-perturbed game $\Gamma(\varepsilon^k)$. If*

1. $\lim_{k\to\infty} M(\varepsilon^k) = 0$,
2. $\lim_{k\to\infty} \sigma^k$ *exists and equals the mixed strategy vector* σ,

then σ is a Nash equilibrium of the original game Γ.

Proof: To show that σ is a Nash equilibrium, we need to show that no player can profit from a unilateral deviation. Let σ_i' be a strategy of player i. By Theorem 7.23, there exists a sequence of strategies $(\sigma_i'^k)_{k\in\mathbb{N}}$ converging to σ_i', and satisfying $\sigma_i'^k \in \Sigma_i(\varepsilon_i^k)$ for each $k \in \mathbb{N}$.

Since σ^k is an equilibrium in the ε^k-perturbed game $\Gamma(\varepsilon^k)$,

$$u_i(\sigma^k) \geq u_i\left(\sigma_i'^k, \sigma_{-i}^k\right). \tag{7.18}$$

By the continuity of the payoff function u_i,

$$u_i(\sigma) = \lim_{k\to\infty} u_i(\sigma^k) \geq \lim_{k\to\infty} u_i\left(\sigma_i'^k, \sigma_{-i}^k\right) = u_i(\sigma_i', \sigma_{-i}). \tag{7.19}$$

Since this inequality obtains for every player $i \in N$ and every mixed strategy $\sigma_i' \in \Sigma_i$, it follows that σ is a Nash equilibrium. □

A mixed strategy vector that is the limit of equilibria in perturbed games, where the perturbation vectors are all positive, and converge to zero, is called a *perfect equilibrium*.

Definition 7.25 *A mixed strategy vector σ in a strategy-form game $(N, (S_i)_{i\in N}, (u_i)_{i\in N})$ is a* perfect equilibrium *if there exists a sequence of perturbation vectors $(\varepsilon^k)_{k\in\mathbb{N}}$ satisfying $\lim_{k\to\infty} M(\varepsilon^k) = 0$, and for each $k \in \mathbb{N}$ there exists an equilibrium σ^k of $\Gamma(\varepsilon^k)$ such that*

$$\lim_{k\to\infty} \sigma^k = \sigma. \tag{7.20}$$

The following corollary of Theorem 7.24 states that the concept of perfect equilibrium is a refinement of the concept of Nash equilibrium.

Corollary 7.26 *Every perfect equilibrium of a finite strategic-form game is a Nash equilibrium.*

The game in Example 7.21 (page 264) has two equilibria, (T, L) and (B, R). The equilibrium (T, L) is a perfect equilibrium: (T, L) is the limit of the equilibria given by Equation (7.16), as η converges to 0. We will later show that (B, R) is not a perfect equilibrium. In Example 7.18 (page 263), the equilibrium (T, t, β) is a perfect equilibrium, but the equilibrium (B, t, τ) is not a perfect equilibrium (Exercise 7.30). The next theorem states that every finite game has at least one perfect equilibrium.

Theorem 7.27 *Every finite strategic-form game has at least one perfect equilibrium.*

Proof: Let Γ be a finite strategic form game, and let $(\varepsilon^k)_{k\in\mathbb{N}}$ be a sequence of perturbation vectors satisfying $\lim_{k\to\infty} M(\varepsilon^k) = 0$. For example, $\varepsilon_i^k(s_i) = \frac{1}{k|S_i|}$ for each player

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$i \in \mathbb{N}$ and for each $s_i \in S_i$. Theorem 7.20 implies that for each $k \in \mathbb{N}$ the game $\Gamma(\varepsilon^k)$ has an equilibrium in mixed strategies σ^k. Since the space of mixed strategy vectors Σ is compact (see Exercise 5.1 on page 194), the sequence $(\sigma^k)_{k\in\mathbb{N}}$ has a convergent subsequence $(\sigma^{k_j})_{j\in\mathbb{N}}$. Denote the limit of this subsequence by σ. Applying Theorem 7.24 to the sequence of perturbation vectors $(\varepsilon^{k_j})_{j\in\mathbb{N}}$, and to the sequence of equilibria $(\sigma^{k_j})_{j\in\mathbb{N}}$, leads to the conclusion that σ is a perfect equilibrium of the original game. □

As a corollary of Theorem 7.22, and from the definition of perfect equilibrium, we can deduce the following theorem (Exercise 7.22).

Theorem 7.28 *In every perfect equilibrium, every (weakly) dominated strategy is chosen with probability zero.*

In other words, no weakly dominated strategy can be a part of a perfect equilibrium. This means that, for example, in Example 7.21, the strategy vector (B, R) is not a perfect equilibrium, since B is a dominated strategy of Player I (and R is a dominated strategy of Player II).

As Exercise 7.28 shows, the converse of this theorem is not true: it is possible for a Nash equilibrium to choose every dominated strategy with probability zero, but not to be a perfect equilibrium. The following theorem states that a completely mixed equilibrium must be a perfect equilibrium.

Theorem 7.29 *Every equilibrium in completely mixed strategies in a strategic-form game is a perfect equilibrium.*

Proof: Let σ^* be a completely mixed equilibrium of a strategic-form game Γ. Then $c := \min_{i\in N} \min_{s_i\in S_i} \sigma_i^*(s_i) > 0$.

Let $(\varepsilon^k)_{k\in\mathbb{N}}$ be a sequence of perturbation vectors satisfying $\lim_{k\to\infty} M(\varepsilon^k) = 0$. Since $\lim_{k\to\infty} M(\varepsilon^k) = 0$, it must be the case that $M(\varepsilon^k) < c$ for sufficiently large k. Hence for each such k, we may conclude that $\sigma_i^* \in \Sigma_i(\varepsilon_i^k)$ for every player i; i.e., σ^* is a possible strategy vector in the game $\Gamma(\varepsilon^k)$. Let $K_0 \in \mathbb{N}$ be sufficiently large so that for each $k \geq K_0$, one has $\sigma_i^* \in \Sigma_i(\varepsilon_i^k)$ for every player i. Since $\Gamma(\varepsilon^k)$ has fewer strategies than Γ, Theorem 4.31 (page 107) implies that σ^* is an equilibrium of $\Gamma(\varepsilon^k)$.

We may therefore apply Theorem 7.24 to the sequences $(\varepsilon^k)_{k\geq K_0}$ and the constant sequence $(\sigma^*)_{n=K_0}^{\infty}$, to conclude that σ^* is a perfect equilibrium, which is what we needed to show. □

7.3.2 Perfect equilibrium in extensive-form games

Since every extensive-form game can be presented as a strategic-form game, the concept of perfect equilibrium, as defined in Definition 7.25, also applies to extensive-form games. This definition of perfect equilibrium for extensive-form games is called *strategic-form perfect equilibrium*.

Theorem 7.27 implies the following corollary (Exercise 7.32).

Theorem 7.30 *Every extensive-form game has a strategic-form perfect equilibrium.*

In this section, we will study the concept of extensive-form perfect equilibrium, where the mistakes that each player makes in different information sets are independent of each other. We will limit our focus to extensive-form games with perfect recall. By Kuhn's Theorem, in such games each behavior strategy has an equivalent mixed strategy, and the converse also holds. Let Γ be an extensive-form game with perfect recall. Denote by $\mathcal{U}_i$ player i's set of information sets. Recall that we denote player i's set of possible actions at information set U_i by $A(U_i)$.

When we are dealing with behavior strategies, a perturbation vector δ_i of player i is a vector associating a positive real number with each action $a_i \in \bigcup_{U_i \in \mathcal{U}_i} A(U_i)$ of player i, such that $\sum_{a_i \in A(U_i)} \delta_i(a_i) \leq 1$ for each information set $U_i \in \mathcal{U}_i$. Let $\delta = (\delta_i)_{i \in N}$ be a set of perturbation vectors, one for each player. Denote the maximal perturbation in $\Gamma(\varepsilon)$ by

$$M(\delta) := \max_{\{i \in N, a_i \in \bigcup_{U_i \in \mathcal{U}_i} A(U_i)\}} \delta_i(a_i), \tag{7.21}$$

and the minimal perturbation by

$$m(\delta) := \min_{\{i \in N, a_i \in \bigcup_{U_i \in \mathcal{U}_i} A(U_i)\}} \delta_i(a_i) > 0. \tag{7.22}$$

The game $\Gamma(\delta)$ is the extensive-form game such that player i's set of strategies, denoted by $\mathcal{B}_i(\delta_i)$, is the set of behavior strategies in which every action a_i is chosen with probability greater than or equal to $\delta_i(a_i)$, that is,

$$\mathcal{B}_i(\delta_i) := \left\{\sigma_i \in \bigtimes_{U_i \in \mathcal{U}_i} \Delta(A(U_i)) \colon \sigma_i(U_i; a_i) \geq \delta_i(a_i), \quad \forall i \in N, \forall U_i \in \mathcal{U}_i, \forall a_i \in A(U_i)\right\}. \tag{7.23}$$

Since every possible action at every chance vertex is chosen with positive probability, and since $m(\delta) > 0$, it follows that $\mathbf{P}_\sigma(x) > 0$ for every vertex x, and every behavior strategy vector $\sigma = (\sigma_i)_{i \in N}$ in $\Gamma(\delta)$: the play of the game arrives at every vertex x with positive probability. For each vertex x such that $\Gamma(x)$ is a subgame, denote by $\Gamma(x; \delta)$ the subgame of $\Gamma(\delta)$ starting at the vertex x. Similarly to Theorem 7.5, we have the following result, whose proof is left to the reader (Exercise 7.33).

Theorem 7.31 *Let Γ be an extensive-form game, and let $\Gamma(x)$ be a subgame of Γ. Let δ be a perturbation vector, and let σ^* be a Nash equilibrium (in behavior strategies) of the game $\Gamma(\delta)$. Then the strategy vector σ^*, restricted to the subgame $\Gamma(x)$, is a Nash equilibrium of $\Gamma(x; \delta)$.*

Similar to Definition 7.25, which is based on mixed strategies, the next definition bases the concept of perfect equilibrium on behavior strategies.

Definition 7.32 *A behavior strategy vector σ in an extensive-form game Γ is called an* extensive-form perfect equilibrium *if there exists a sequence of perturbation vectors*

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$(\delta^k)_{k\in\mathbb{N}}$ satisfying $\lim_{k\to\infty} M(\delta^k)=0$, and for each $k\in\mathbb{N}$ there exists an equilibrium σ^k of $\Gamma(\delta^k)$, such that $\lim_{k\to\infty}\sigma^k=\sigma$ is satisfied.

These concepts, strategic-form perfect equilibrium and extensive-form perfect equilibrium, differ from each other: a strategic-form perfect equilibrium is a vector of mixed strategies, while an extensive-form perfect equilibrium is a vector of behavior strategies. Despite the fact that in games with perfect recall there is an equivalence between mixed strategies and behavior strategies (see Chapter 6), an extensive-form perfect equilibrium may fail to be a strategic-form perfect equilibrium. In other words, a vector of mixed strategies, each equivalent to a behavior strategy in an extensive-form perfect equilibrium, may fail to be a strategic-form perfect equilibrium (Exercise 7.36). Conversely, a strategic-form perfect equilibrium may not necessarily be an extensive-form equilibrium (Exercise 7.37). The conceptual difference between these two concepts is similar to the difference between mixed strategies and behavior strategies: in a mixed strategy, a player randomly chooses a pure strategy at the start of a game, while in a behavior strategy he randomly chooses an action at each of his information sets. Underlying the concept of strategic-form perfect equilibrium is the assumption that a player may mistakenly choose, at the start of the game, a pure strategy different from the one he intended to choose. In contrast, underlying the concept of extensive-form perfect equilibrium is the assumption that a player may mistakenly choose an action different from the one he intended at any of his information sets. In extensive-form games where each player has a single information set, these two concepts are identical, because in that case the set of mixed strategies of each player is identical with his set of behavior strategies.

As stated above, Selten defined the concept of perfect equilibrium in order to further "refine" the concept of subgame perfect equilibrium in extensive-form games. We will now show that this is indeed a refinement: every extensive-form perfect equilibrium is a subgame perfect equilibrium in behavior strategies. (This result can also be proved directly.) Since every subgame perfect equilibrium is a Nash equilibrium (Remark 7.3 on page 254), we will then conclude that every extensive-form perfect equilibrium is a Nash equilibrium in behavior strategies. This result can also be proved directly; see Exercise 7.31.

Theorem 7.33 *Let Γ be an extensive-form game. Every extensive-form perfect equilibrium of Γ is a subgame perfect equilibrium.*

The analogous theorem for strategic-form perfect equilibrium does not obtain (see Exercise 7.37). Before we proceed to the proof of the theorem, we present a technical result analogous to Theorem 7.23, which states that every behavior strategy may be approximated by a sequence of behavior strategies in perturbed games, where the perturbations converge to zero. The proof of this theorem is left to the reader (Exercise 7.34).

Theorem 7.34 *Let $(\delta^k)_{k\in\mathbb{N}}$ be a sequence of perturbation vectors satisfying $\lim_{k\to\infty} M(\delta^k)=0$. For each behavior strategy $\sigma_i\in\mathcal{B}_i$ of player i, there exists a sequence $(\sigma_i^k)_{k\in\mathbb{N}}$ of behavior strategies satisfying the following two properties:*

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- $\sigma_i^k \in \mathcal{B}_i(\delta_i^k)$ *for each* $k \in \mathbb{N}$.
- $\lim_{k\to\infty} \sigma_i^k$ *exists and equals* σ_i.

Proof of Theorem 7.33: Let $\sigma^* = (\sigma_i^*)_{i\in N}$ be an extensive-form perfect equilibrium, and let $\Gamma(x)$ be a subgame (starting at vertex x). We will show that the restriction of σ^* to this subgame is a subgame perfect equilibrium.

By the definition of extensive-form perfect equilibrium, for each $k \in \mathbb{N}$ there exists a perturbation vector δ^k, and an equilibrium σ^k in the δ^k-perturbed game satisfying $\lim_{k\to\infty} M(\delta^k) = 0$, and $\lim_{k\to\infty} \sigma^k = \sigma^*$. Theorem 7.31 implies that the strategy vector σ^k is a Nash equilibrium in behavior strategies of the game $\Gamma(x;\delta^k)$. Let σ_i' be a behavior strategy of player i. We will show that

$$u_i(\sigma^* \mid x) \geq u_i((\sigma_i', \sigma_{-i}^*) \mid x). \tag{7.24}$$

Theorem 7.34 implies that there exists a sequence $(\sigma_i'^k)_{k\in\mathbb{N}}$ of behavior strategies converging to σ_i' and satisfying $\sigma_i'^k \in \mathcal{B}_i(\delta_i^k)$ for each $k \in \mathbb{N}$. Since σ^k is an equilibrium of the subgame $\Gamma(x;\delta^k)$,

$$u_i(\sigma^k \mid x) \geq u_i\left(\left(\sigma_i'^k, \sigma_{-i}^k\right) \mid x\right). \tag{7.25}$$

Equation (7.24) is now derived from Equation (7.25) by using the continuity of the payoff function u_i and passing to the limit as $k \to \infty$. □

The next example shows that the converse of Theorem 7.33 does not obtain; a subgame perfect equilibrium need not be an extensive-form perfect equilibrium.

Example 7.35 Consider the two-player extensive-form game depicted in Figure 7.15. This game has two pure-strategy equilibria, (A, L) and (B, R). Each of these equilibria is a subgame perfect equilibrium, since the game has no nontrivial subgames (see Theorem 7.4).

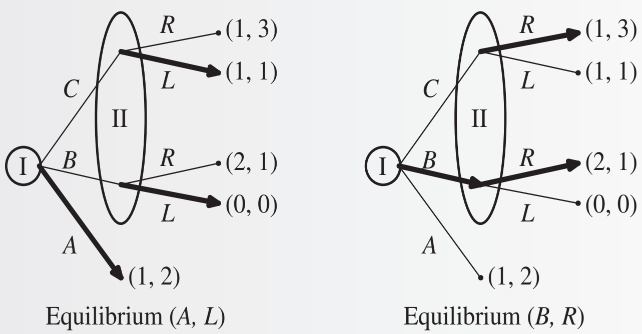

Figure 7.15 The game in Example 7.35, along with two of its equilibria

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The equilibrium (A, L) is not an extensive-form perfect equilibrium. Indeed, since each player has a single information set, if (A, L) were an extensive-form perfect equilibrium it would also be a strategic-form perfect equilibrium (Exercise 7.39). But the strategy L is a weakly dominated strategy, and therefore Theorem 7.28 implies that it cannot form part of a strategic-form perfect equilibrium.

Showing that (B, R) is an extensive-form perfect equilibrium is left to the reader (Exercise 7.47). ◀

Together with Theorem 7.33, the last example proves that the concept of extensive-form perfect equilibrium is a refinement of the concept of subgame perfect equilibrium. Note that in this example, a subgame perfect equilibrium that is not an extensive-form perfect equilibrium is given in pure strategies, and therefore the inclusion of the set of extensive-form perfect equilibria in the set of subgame perfect equilibria is a proper inclusion, even when only pure strategy equilibria are involved.

Theorem 7.33 states that every extensive-form perfect equilibrium is a subgame perfect equilibrium, and therefore also a Nash equilibrium. It follows that if a game has no Nash equilibria in behavior strategies, then it has no extensive-form perfect equilibria. By Theorem 6.16 (page 235) this can happen only if the game does not have perfect recall. Example 6.17 (page 236) describes such a game.

As we now show, a finite extensive-form game with perfect recall always has an extensive-form perfect equilibrium.

Theorem 7.36 *Every finite extensive-form game with perfect recall has an extensive-form perfect equilibrium.*

Proof: Let Γ be a finite extensive-form game with perfect recall, and let $(\delta^k)_{k\in\mathbb{N}}$ be a sequence of perturbation vectors satisfying $\lim_{k\to\infty} M(\delta^k) = 0$. Since all the players have perfect recall, Theorem 6.16 (page 235) shows that $\Gamma(\delta^k)$ has an equilibrium σ^k in behavior strategies. Since the space of behavior strategy vectors $\times_{i\in N} \mathcal{B}_i$ is compact, the sequence $(\sigma^k)_{k\in\mathbb{N}}$ has a convergent subsequence $(\sigma^{k_j})_{j\in\mathbb{N}}$, converging to a limit σ^*. Then σ^* is an extensive-form perfect equilibrium. □

Theorems 7.36 and 7.33 lead to the following result.

Theorem 7.37 *Every finite extensive-form game with perfect recall has a subgame perfect equilibrium in behavior strategies.*

7.4 Sequential equilibrium

This section presents another equilibrium concept for extensive-form games, which differs from the three concepts we have studied so far in this chapter, subgame perfect equilibrium, strategic-form perfect equilibrium, and extensive-form perfect equilibrium. The subgame perfect equilibrium concept assumes that players analyze each game from the leaves to the root, with every player, at each of his information sets, choosing an action under the assumption that in each future subgame, all the players will implement equilibrium

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strategies. The two perfect equilibrium concepts assume that each player has a positive, albeit small, probability of making a mistake, and that the other players take this into account when they choose their actions. The sequential equilibrium concept is based on the principle that at each stage of a game, the player whose turn it is to choose an action has a belief, i.e., a probability distribution, about which vertex in his information set is the true vertex at which the game is currently located, and a belief about how the play of the game will proceed given any action that he may choose. These beliefs are based on the information structure of the game (the information sets) and the strategies of the players. Given these beliefs, at each of his information sets, each player chooses the action that gives him the highest expected payoff.

In this section we will deal only with games with perfect recall. We will later, in Example 7.60 (on page 283), remark on why it is unclear how the concept of sequential equilibrium can be generalized to games without perfect recall. Recall that a player's behavior strategy in an extensive-form game is a function associating each of that player's information sets with a probability distribution over the set of possible actions at that information set. Such a probability distribution is called a *mixed action*.

Before we begin the formal presentation, we will look at an example that illustrates the concept of sequential equilibrium and the ideas behind it.

Example 7.38 Consider the two-player extensive-form game depicted in Figure 7.16.

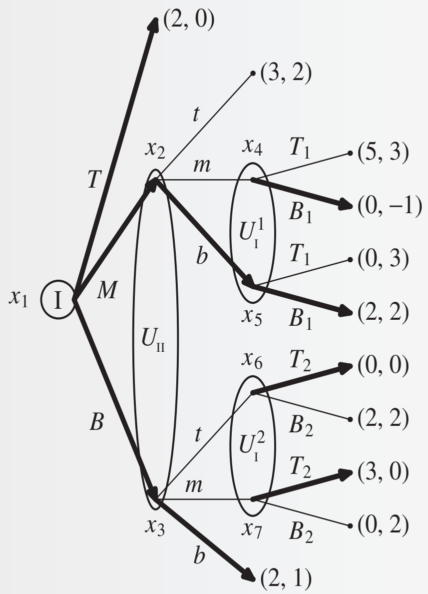

Figure 7.16 The game in Example 7.38 and a strategy vector

The following pair of behavior strategies $\sigma = (\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}})$ is a Nash equilibrium of the game (see Figure 7.16, where the actions taken in this equilibrium appear as bold lines):

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- Player I's strategy σ_{I}:
  - At information set U_{I}^1, choose the mixed action $[\frac{3}{12}(T), \frac{4}{12}(M), \frac{5}{12}(B)]$.
  - At information set U_{I}^2, choose B_1.
  - At information set U_{I}^3, choose T_2.
- Player II's strategy σ_{II}: Choose b.

(Check that this is indeed an equilibrium.) The strategy vector σ determines, for each vertex x, the probability $\mathbf{P}_\sigma(x)$ that a play of the game will visit this vertex:

$$\begin{array}{lll} \mathbf{P}_\sigma(x_2) = \frac{4}{12}, & \mathbf{P}_\sigma(x_3) = \frac{5}{12}, & \mathbf{P}_\sigma(x_4) = 0, \\ \mathbf{P}_\sigma(x_5) = \frac{4}{12}, & \mathbf{P}_\sigma(x_6) = 0, & \mathbf{P}_\sigma(x_7) = 0. \end{array}$$

When Player II is called upon to play, he knows that the play of the game is located at information set $U_{\mathrm{II}} = \{x_2, x_3\}$. Knowing Player I's strategy, he can calculate the conditional probability of each one of the vertices in this information set to be the actual position of the play,

$$\mathbf{P}_\sigma(x_2 \mid U_{\mathrm{II}}) = \frac{\mathbf{P}_\sigma(x_2)}{\mathbf{P}_\sigma(x_2) + \mathbf{P}_\sigma(x_3)} = \frac{\frac{4}{12}}{\frac{4}{12} + \frac{5}{12}} = \frac{4}{9}, \tag{7.26}$$

and similarly, $\mathbf{P}_\sigma(x_3 \mid U_{\mathrm{II}}) = \frac{5}{9}$. The conditional probability $\mathbf{P}_\sigma(\cdot \mid U_{\mathrm{II}})$ is Player II's belief at information set U_{II}.

Similarly, when Player I is called to play at information set U_{I}^1, he cannot distinguish between x_4 and x_5, but knowing Player II's strategy σ_{II}, Player I can ascribe probability 1 to the play of the game being at vertex x_5. Formally, this is the following conditional distribution,

$$\mathbf{P}_\sigma\left(x_5 \mid U_{\mathrm{I}}^1\right) = \frac{\mathbf{P}_\sigma(x_5)}{\mathbf{P}_\sigma(x_4) + \mathbf{P}_\sigma(x_5)} = \frac{\frac{4}{12}}{0 + \frac{4}{12}} = 1, \tag{7.27}$$

with a similar calculation yielding $\mathbf{P}_\sigma(x_4 \mid U_{\mathrm{I}}^1) = 0$. The conditional distribution $\mathbf{P}_\sigma(\cdot \mid U_{\mathrm{I}}^1)$ is Player I's belief at information set U_{I}^1. Does Player I also have a belief at information set U_{I}^2? The answer to this question is negative, because if the players implement the strategy vector σ, the play of the game will not visit this information set. Formally, the conditional distribution $\mathbf{P}_\sigma(\cdot \mid U_{\mathrm{I}}^2)$ is undefined, because the probability $\mathbf{P}_\sigma(U_{\mathrm{I}}^2) = \mathbf{P}_\sigma(x_6) + \mathbf{P}_\sigma(x_7)$, which represents the probability that the play of the game will arrive at information set U_{I}^2, equals 0. ◀

For each strategy vector σ, and for each player i and each of his information sets U_i, denote $\mathbf{P}_\sigma(U_i) := \sum_{x \in U_i} \mathbf{P}_\sigma(x)$. Since the game has perfect recall, every path from the root passes through every information set at most once, and hence $\mathbf{P}_\sigma(U_i)$ is the probability that a play of the game will visit information set U_i when the players implement the strategy vector σ. As we saw, the strategy vector σ determines a belief $\mathbf{P}_\sigma(\cdot \mid U_i)$ over the vertices of information set U_i, for each information set satisfying $\mathbf{P}_\sigma(U_i) > 0$: when player i is called upon to play in information set U_i, his belief about the vertex at which the play of the game is located is given by the conditional distribution $\mathbf{P}_\sigma(\cdot \mid U_i)$.

Beliefs, calculated this way from the strategy vector σ, satisfy the property of *consistency*. In other words, the beliefs are consistent with the distribution $\mathbf{P}_\sigma$ over the vertices of the game tree, and with Bayes' formula for calculating conditional probability. We say that the strategy vector σ determines a *partial belief system*. The word partial denotes the

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fact that the beliefs are defined only for some of the information sets; they are not defined for information sets that the strategy vector σ leads to with probability 0.

Example 7.38 (*Continued*) We will now explore the connection between an action chosen by a player at a given information set, and his belief at that information set. Player I, at information set U_I^1, ascribes probability 1 to the play of the game being located at vertex x_5. He therefore regards action B_1 as being the optimal action for him: B_1 leads to a payoff 2, while T_1 leads to a payoff 0. Given his belief at U_I^1, Player I is rational in choosing B_1. If, in contrast, according to his belief at U_I^1 he had ascribed high probability to the play of the game being located at vertex x_4 (a probability greater than or equal to $\frac{2}{7}$) it would have been rational for him to choose T_1. This property, in which a player's strategy calls on him to choose an action maximizing his expected payoff at each information set, given his belief at that information set, is termed *sequential rationality*.

We will now check whether sequential rationality obtains at Player II's information set U_{II} in the equilibrium we previously presented in this example. As we computed above, Player II's belief about the vertex at which the play of the game is located, given that it has arrived at information set U_{II}, is

$$\mathbf{P}_\sigma(x_2 \mid U_{II}) = \tfrac{4}{9}, \qquad \mathbf{P}_\sigma(x_3 \mid U_{II}) = \tfrac{5}{9}. \tag{7.28}$$

Given this belief, and the strategy vector σ, if Player II chooses action b, he receives a payoff of 2 with probability $\frac{4}{9}$ (if the play of the game is at vertex x_2) and a payoff of 1 with probability $\frac{5}{9}$ (if the play of the game is at vertex x_3). His expected payoff is therefore

$$\tfrac{4}{9} \times 2 + \tfrac{5}{9} \times 1 = \tfrac{13}{9}. \tag{7.29}$$

A similar calculation shows that if he chooses action m, his expected payoff is

$$\tfrac{4}{9} \times (-1) + \tfrac{5}{9} \times 0 = -\tfrac{4}{9}, \tag{7.30}$$

and if he chooses action t his expected payoff is

$$\tfrac{4}{9} \times 2 + \tfrac{5}{9} \times 0 = \tfrac{8}{9}. \tag{7.31}$$

The strategy σ_{II} calls on Player II to choose action b at information set U_{II}, which does indeed maximize his expected payoff, relative to his belief. In other words, Player II's strategy is sequentially rational.

We next ascertain that Player I's strategy is sequentially rational at information set U_I^1, containing a single vertex, x_1. When the play of the game arrives at information set U_I^1, Player I knows that x_1 must be the vertex at which the play of the game is located, because the information set contains only one vertex. The mixed strategy $[\frac{3}{12}(T), \frac{4}{12}(M), \frac{5}{12}(B)]$ maximizes Player I's expected payoff if and only if all three actions yield him the same expected payoff. This is due to the fact that the payoff is a linear function of the probabilities in which the various actions are implemented by Player I at information set U_I^1. We encountered a similar argument at the indifference principle (Theorem 5.18, page 160). The reader is asked to verify that each of these three actions yield the payoff 2, and therefore any mixed action implemented by Player I at information set U_I^1 satisfies sequential rationality, in particular the mixed action $[\frac{3}{12}(T), \frac{4}{12}(M), \frac{5}{12}(B)]$ implemented in σ_I. ◀

In Example 7.38, we saw that the strategy vector σ induces a partial belief system over the players' information sets, and that each player i's strategy is sequentially rational at

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each information set U_i for which the belief $\mathbf{P}_\sigma(\cdot \mid U_i)$ is defined, i.e., at each information set at which the play of the game arrives with positive probability under the strategy vector σ.

The main idea behind the concept of sequential equilibrium is that the property of sequential rationality should be satisfied at every information set, including those information sets that are visited with probability 0 under the strategy vector σ. This requirement is similar to the requirement that the subgame perfect equilibrium be an equilibrium both on the equilibrium path, and off the equilibrium path. A sequential equilibrium therefore requires specifying players' beliefs at all information sets. A sequential equilibrium is thus a pair (σ, μ), where $\sigma = (\sigma_i)_{i\in N}$ is a vector of behavior strategies, and μ is a complete belief system; i.e., with every player and every information set of that player, μ associates a belief: a distribution over the vertices of that information set. The pair (σ, μ) must satisfy two properties: the beliefs μ must be consistent with Bayes' formula and with the strategy vector σ, and σ must be sequentially rational given the beliefs μ.

The main stage in the development of the concept of sequential equilibrium is defining the concept of consistency of beliefs μ with respect to a given strategy vector σ. Doing this requires extending the partial belief system $\mathbf{P}_\sigma$ to every information set U_i for which $\mathbf{P}_\sigma(U_i) = 0$. This extension is based on Selten's trembling hand principle, which was discussed in the section defining perfect equilibrium. Denote by $\mathcal{U}$ the set of the information sets of all the players.

Definition 7.39 *A* complete belief system μ *is a vector* $\mu = (\mu_U)_{U\in\mathcal{U}}$ *associating each information set* $U \in \mathcal{U}$ *with a distribution over the vertices in* U.

Definition 7.40 *Let* $\mathcal{U}' \subseteq \mathcal{U}$ *be a partial collection of information sets. A* partial belief system μ *(with respect to the* $\mathcal{U}'$*) is a vector* $\mu = (\mu_U)_{U\in\mathcal{U}'}$ *associating each information set* $U \in \mathcal{U}'$ *with a distribution over the vertices in* U.

If μ is a partial belief system, denote by $\mathcal{U}_\mu$ the collection of information sets at which μ is defined.

Although the definition of a belief system is independent of the strategy vector implemented by the players, we are interested in belief systems that are closely related to the strategy vector σ. The partial belief system that is induced by σ plays a central role in the definition of sequential equilibrium.

Definition 7.41 *Let* σ *be a strategy vector. Let* $\mathcal{U}_\sigma = \{U \in \mathcal{U} : \mathbf{P}_\sigma(U) > 0\}$ *be the collection of all information sets that the play of the game visits with positive probability when the players implement the strategy vector* σ*. The* partial belief system induced by the strategy vector σ *is the collection of distributions* $\mu_\sigma = (\mu_{\sigma,U})_{U\in\mathcal{U}_\sigma}$*, satisfying, for each* $U \in \mathcal{U}_\sigma$*,*

$$\mu_{\sigma,U}(x) := \mathbf{P}_\sigma(x \mid U) = \frac{\mathbf{P}_\sigma(x)}{\mathbf{P}_\sigma(U)}, \quad \forall x \in U. \tag{7.32}$$

Note that $\mathcal{U}_\sigma = \mathcal{U}_{\mu_\sigma}$. To avoid using both denotations, we will henceforth use only the denotation $\mathcal{U}_{\mu_\sigma}$.

Remark 7.42 *Since we have assumed that at each chance vertex, every action is chosen with positive probability, it follows that if all the strategies in the strategy vector σ are completely mixed, i.e., at each information set every action is chosen with positive probability, then $\mathbf{P}_\sigma(U_i) > 0$ for each player i and each of his information sets U_i; hence in this case the belief system μ_σ is a complete belief system: it defines a belief at each information set in the game ($\mathcal{U}_{\mu_\sigma} = \mathcal{U}$).* ♦

Recall that $u_i(\sigma \mid x)$ is the expected payoff of player i when the players implement the strategy σ, given that the play of the game is at vertex x. It follows that player i's expected payoff when the players implement the strategy vector σ, given that the game arrives at information set U_i and given his belief, is

$$u_i(\sigma \mid U_i, \mu) := \sum_{x \in U_i} \mu_{\sigma, U_i}(x) u_i(\sigma \mid x). \tag{7.33}$$

Definition 7.43 *Let σ be a vector of behavior strategies, μ be a partial belief system, and $U_i \in \mathcal{U}_\mu$ be an information set of player i. The strategy vector σ is called* rational *at information set U_i, relative to μ, if for each behavior strategy σ_i' of player i*

$$u_i(\sigma \mid U_i, \mu) \geq u_i((\sigma_i', \sigma_{-i}) \mid U_i, \mu). \tag{7.34}$$

The pair (σ, μ) is called sequentially rational *if for each player i and each information set $U_i \in \mathcal{U}_\mu$, the strategy vector σ is rational at U_i relative to μ.*

As the following theorems show, there exists a close connection between the concepts of sequential rationality and those of Nash equilibrium and subgame perfect equilibrium.

Theorem 7.44 *In an extensive-form game with perfect recall, if the pair (σ, μ_σ) is sequentially rational, then the strategy vector σ is a Nash equilibrium in behavior strategies.*

Proof: Let $i \in N$ be a player, and let σ_i' be any behavior strategy of player i. We will prove that $u_i(\sigma) \geq u_i(\sigma_i', \sigma_{-i})$.

We say that an information set U_i of player i is *highest* if every path from the root to a vertex in U_i does not pass through any other information set of player i. Denote by $\widehat{\mathcal{U}}_i$ the set of player i's highest information sets: any path from the root to a leaf that passes through an information set of player i necessarily passes through an information set in $\widehat{\mathcal{U}}_i$. Denote by $p^*_{\sigma,i}$ the probability that, when the strategy vector σ is played, a play of the game will not pass through any of player i's information sets, and denote by $u^*_{\sigma,i}$ player i's expected payoff, given that the play of the game according to the strategy vector does not pass through any of player i's information sets.

Note that $p^*_{\sigma,i}$ and $u^*_{\sigma,i}$ are independent of player i's strategy, since these values depend on plays of the game in which player i does not participate. Similarly, for every

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information set $U_i \in \widehat{\mathcal{U}}_i$, the probability $\mathbf{P}_\sigma(U_i)$ is independent of player i's strategy, since these probabilities depend on actions chosen at vertices that are not under player i's control.

Using this notation, we have that

$$p_\sigma(U_i) = p_{(\sigma_i',\sigma_{-i})}(U_i), \quad \forall U_i \in \widehat{\mathcal{U}}_i, \tag{7.35}$$

$$u_i(\sigma) = \sum_{U_i \in \widehat{\mathcal{U}}_i} p_\sigma(U_i) u_i(\sigma \mid U_i, \mu_\sigma) + p^*_{\sigma,i} u^*_{\sigma,i}, \tag{7.36}$$

$$u_i(\sigma_i', \sigma_{-i}) = \sum_{U_i \in \widehat{\mathcal{U}}_i} p_{(\sigma_i',\sigma_{-i})}(U_i) u_i((\sigma_i', \sigma_{-i}) \mid U_i, \mu_\sigma) + p^*_{\sigma,i} u^*_{\sigma,i} \tag{7.37}$$

$$= \sum_{U_i \in \widehat{\mathcal{U}}_i} p_\sigma(U_i) u_i((\sigma_i', \sigma_{-i}) \mid U_i, \mu_\sigma) + p^*_{\sigma,i} u^*_{\sigma,i}, \tag{7.38}$$

where Equation (7.38) follows from Equation (7.35). Since for every $U_i \in \mathcal{U}_{\mu_\sigma}$, the pair (σ, μ_σ) is sequentially rational at U_i,

$$u_i(\sigma \mid U_i, \mu_\sigma) \geq u_i((\sigma_i', \sigma_{-i}) \mid U_i, \mu_\sigma). \tag{7.39}$$

Equations (7.36)–(7.39) imply that

$$u_i(\sigma) \geq u_i(\sigma_i', \sigma_{-i}), \tag{7.40}$$

which is what we wanted to prove. □

The following theorem, whose proof is left to the reader (Exercise 7.41), is the converse of Theorem 7.44.

Theorem 7.45 *If σ^* is a Nash equilibrium in behavior strategies, then the pair $(\sigma^*, \mu_{\sigma^*})$ is sequentially rational at every information set in $\mathcal{U}_{\mu_{\sigma^*}}$.*

In a game with perfect information, every information set contains only one vertex, and therefore when called on to make a move, a player knows at which vertex the play of the game is located. In this case, we denote by $\widehat{\mu}$ the complete belief system in which $\widehat{\mu}_U = [1(x)]$, for every information set $U = \{x\}$. The next theorem, whose proof is left to the reader (Exercise 7.42), characterizes subgame perfect equilibria using sequential rationality.

Theorem 7.46 *In a game with perfect information, a behavior strategy vector σ is a subgame perfect equilibrium if and only if the pair $(\sigma, \widehat{\mu})$ is sequentially rational at each vertex in the game.*

As previously stated, the main idea behind the sequential equilibrium refinement is to expand the definition of rationality to information sets U_i at which $\mathbf{P}_\sigma(U_i) = 0$. This is accomplished by the trembling hand principle: player i may find himself in an information set U_i for which $\mathbf{P}_\sigma(U_i) = 0$, due to a mistake (tremble) on the part of one of the players, and we require that even if this should happen, the player ought to behave rationally relative to beliefs that are "consistent" with such mistakes. In other words, we extend the partial belief system μ_σ to a complete belief system μ that is consistent with the trembling hand principle, and we require that σ be sequentially rational not only with respect to μ_σ, but also with respect to μ.

Remark 7.47 *A belief at an information set U_i is a probability distribution over the vertices in U_i, i.e., an element of the compact set $\Delta(U_i)$. A complete belief system is a vector of beliefs, one belief per information set, and therefore a vector in the compact set $\times_{U\in\mathcal{U}} \Delta(U)$. Since this set is compact, every sequence of complete belief systems has a convergent subsequence.* ♦

Definition 7.48 *An* assessment *is a pair (σ, μ) in which $\sigma = (\sigma_i)_{i\in N}$ is a vector of behavior strategies, and $\mu = (\mu_U)_{U\in\mathcal{U}}$ is a complete belief system.*

Definition 7.49 *An assessment (σ, μ) is called* consistent *if there exists a sequence of completely mixed behavior strategy vectors $(\sigma^k)_{k\in\mathbb{N}}$ satisfying the following conditions:*

(i) The strategies $(\sigma^k)_{k\in\mathbb{N}}$ converge to σ, i.e., $\lim_{k\to\infty} \sigma^k = \sigma$.
(ii) The sequence of beliefs $(\mu_{\sigma^k})_{k\in\mathbb{N}}$ induced by $(\sigma^k)_{k\in\mathbb{N}}$ converges to the belief system μ,

$$\mu_\sigma(U) = \lim_{k\to\infty} \mu_{\sigma^k}(U), \quad \forall U \in \mathcal{U}. \tag{7.41}$$

Remark 7.50 *If σ is a completely mixed behavior strategy vector, then μ_σ is a complete belief system (Remark 7.42). In this case, (σ, μ_σ) is a consistent system. This follows directly from Definition 7.49, using the sequence $(\sigma^k)_{k\in\mathbb{N}}$ defined by $\sigma^k = \sigma$ for all $k \in \mathbb{N}$.* ♦

Remark 7.51 *Since the strategies σ^k in Definition 7.49 are completely mixed strategies, for every $k \in \mathbb{N}$ the belief system μ_{σ^k} is a complete belief system (Remark 7.42), and hence the limit μ is also a complete belief system (Remark 7.47).* ♦

Definition 7.52 *An assessment (σ, μ) is called a* sequential equilibrium *if it is consistent and sequentially rational.*

Remark 7.53 *By definition, if an assessment (σ, μ) is sequentially rational then it is rational at each information set at which the belief μ is defined. Since the belief system of an assessment is a complete belief system, it follows that a sequentially rational assessment (σ, μ) is rational at each information set.* ♦

The following result, which is a corollary of Theorem 7.44, shows that the concept of sequential equilibrium is a refinement of the concept of Nash equilibrium.

Theorem 7.54 *In an extensive-form game with perfect recall, if the assessment (σ, μ) is a sequential equilibrium, then the strategy vector σ is a Nash equilibrium in behavior strategies.*

All of the above leads to:

Theorem 7.55 *In an extensive-form game with perfect recall, if σ is a Nash equilibrium in completely mixed behavior strategies, then (σ, μ_σ) is a sequential equilibrium.*

Proof: Remark 7.50 implies that the pair (σ, μ_σ) is a consistent assessment. Theorem 7.45 implies that this assessment is sequentially rational. □

Example 7.56 Consider the two-player extensive-form game depicted in Figure 7.17.

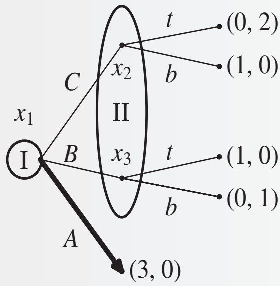

Figure 7.17 The game in Example 7.56, and a strictly dominant strategy of Player I

The strategy A strictly dominates Player I's two other strategies, and hence at every Nash equilibrium, Player I chooses A. This means that at any Nash equilibrium Player II's strategy has no effect at all on the play of the game, and hence in this game there is a continuum of equilibria in mixed strategies, $(A, [y(t), (1-y)(b)])$, for $0 \leq y \leq 1$.

We now compute all the sequential equilibria of the game. Since every sequential equilibrium is a Nash equilibrium (Theorem 7.54), every sequential equilibrium (σ, μ) satisfies $\sigma_{\mathrm{I}} = A$.

Player II's belief at her sole information set is therefore $\mu_{U_{\mathrm{II}}} = (\mu_{U_{\mathrm{II}}}(x_2), \mu_{U_{\mathrm{II}}}(x_3))$. Is every belief $\mu_{U_{\mathrm{II}}}$ part of a consistent complete belief system? The answer is positive: the only condition that the assessment (σ, μ) needs to satisfy in order to be a consistent assessment is $\sigma_{\mathrm{I}} = A$. This follows directly from Definition 7.49, using the sequence

$$\sigma_{\mathrm{I}}^k = \left[\frac{k-1}{k}(A), \frac{\mu_{U_{\mathrm{II}}}(x_2)}{k}(B), \frac{\mu_{U_{\mathrm{II}}}(x_3)}{k}(C)\right]$$

for each $k \in \mathbb{N}$.

We next check which beliefs of Player II at information set U_{II} are rational at U_{II}. If the play of the game is at information set U_{II}, action b yields Player II the expected payoff $2\mu_{U_{\mathrm{II}}}(x_2)$, and action t yields the expected payoff $\mu_{U_{\mathrm{II}}}(x_3)$. Since $\mu_{U_{\mathrm{II}}}(x_2) + \mu_{U_{\mathrm{II}}}(x_3) = 1$, we deduce the following:

- If $\mu_{U_{\mathrm{II}}}(x_2) > \frac{1}{3}$, then the only action that is rational for Player II at information set U_{II} is t.
- If $\mu_{U_{\mathrm{II}}}(x_2) < \frac{1}{3}$, then the only action that is rational for Player II at information set U_{II} is b.
- If $\mu_{U_{\mathrm{II}}}(x_2) = \frac{1}{3}$, then every mixed action of Player II is rational at information set U_{II}.

In other words, the set of sequentially rational equilibria consists of the following assessments:

- $\sigma_{\mathrm{I}} = A$, $\sigma_{\mathrm{II}} = t$, $\mu_{U_{\mathrm{II}}} = [y(x_2), (1-y)(x_3)]$ for $y > \frac{1}{3}$.
- $\sigma_{\mathrm{I}} = A$, $\sigma_{\mathrm{II}} = b$, $\mu_{U_{\mathrm{II}}} = [y(x_2), (1-y)(x_3)]$ for $y < \frac{1}{3}$.
- $\sigma_{\mathrm{I}} = A$, $\sigma_{\mathrm{II}} = [z(t), (1-z)(b)]$ for $z \in [0, 1]$, $\mu_{U_{\mathrm{II}}} = [\frac{1}{3}(x_2), \frac{2}{3}(x_3)]$. ◄

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Example 7.38 (*Continued*) We have seen that the following pair (σ, μ_σ) satisfies the properties of partial consistency, and sequential rationality, at every information set U for which $\mathbf{P}_\sigma(U) > 0$:

- Player I:
  - plays the mixed action $[\frac{3}{12}(T), \frac{4}{12}(M), \frac{5}{12}(B)]$ at U_{I}^1.
  - chooses B_1 at information set U_{I}^1.
  - chooses T_2 at information set U_{I}^2.
- Player II chooses b.
- Player II's belief at information set U_{II} is $[\frac{4}{9}(x_2), \frac{5}{9}(x_3)]$.
- Player I's belief at information set U_{I}^2 is $[1(x_5)]$.

We now show that (σ, μ_σ) can be extended to a sequential equilibrium. To do so, we need to specify what Player I's belief is at information set U_{I}^3. Denote this belief by $\mu_{U_{\mathrm{I}}^3} = (\mu_{U_{\mathrm{I}}^3}(x_6), \mu_{U_{\mathrm{I}}^3}(x_7))$. Note that for each $\mu_{U_{\mathrm{I}}^3}$, the assessment $(\sigma, \mu_\sigma, \mu_{U_{\mathrm{I}}^3})$ is consistent. This is achieved by defining

$$\sigma_{\mathrm{I}}^k = \sigma_{\mathrm{I}}, \qquad \sigma_{\mathrm{II}}^k = \left[\frac{\mu_{U_{\mathrm{I}}^3}(x_6)}{k}(t), \frac{\mu_{U_{\mathrm{I}}^3}(x_7)}{k}(m), \frac{k-1}{k}(b)\right], \tag{7.42}$$

and using Definition 7.49. Finally, the action T_2 yields Player I the expected payoff $3\mu_{U_{\mathrm{I}}^3}(x_7)$, and action B_1 yields him the expected payoff $2\mu_{U_{\mathrm{I}}^3}(x_6)$. It follows that action T_2 is rational if $\mu_{U_{\mathrm{I}}^3}(x_6) \le \frac{3}{5}$. We deduce that the assessment (σ, μ_σ) can be expanded to a sequential equilibrium (σ, μ) if we add:

- the belief of Player I at information set U_{I}^3 is $[p(x_6), (1-p)(x_7)]$, where $p \in [0, \frac{3}{5}]$. ◀

Sequential equilibrium, and extensive-form perfect equilibrium, are similar but not identical concepts. The following theorem states that every extensive-form perfect equilibrium can be completed to a sequential equilibrium. Example 7.59, which is presented after the proof of the theorem, shows that the converse does not obtain, and therefore the concept of extensive-form perfect equilibrium is a refinement of the concept of sequential equilibrium.

Theorem 7.57 *Let σ be an extensive-form perfect equilibrium in an extensive-form game with perfect recall Γ. Then σ can be completed to a sequential equilibrium: there exists a complete belief system $\mu = (\mu_U)_{U \in \mathcal{U}}$ satisfying the condition that the pair (σ, μ) is a sequential equilibrium.*

Since by Theorem 7.30 (page 268) every finite extensive-form game with perfect recall has an extensive-form perfect equilibrium, we immediately deduce the following corollary of Theorem 7.57.

Corollary 7.58 *Every finite extensive-form game with perfect recall has a sequential equilibrium.*

Proof of Theorem 7.57: Since σ is an extensive-form perfect equilibrium, there exists a sequence $(\delta^k)_{k \in \mathbb{N}}$ of perturbation vectors satisfying $\lim_{k\to\infty} M(\delta_k) = 0$, and for each $k \in \mathbb{N}$, there exists an equilibrium σ^k of the δ^k-perturbed game $\Gamma(\delta^k)$, satisfying

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$\lim_{k\to\infty} \sigma^k = \sigma$. Theorem 7.55 implies that for each $k \in \mathbb{N}$, the assessment $(\sigma^k, \mu_{\sigma^k})$ is a sequential equilibrium in the game $\Gamma(\delta^k)$.

By Remark 7.47, there exists an increasing sequence $(k_j)_{j\in\mathbb{N}}$ of natural numbers satisfying the condition that the sequence $(\mu_{\sigma^{k_j}})_{j\in\mathbb{N}}$ converges to a complete belief system μ. We deduce from this that (σ, μ) is a consistent assessment.

We now prove that (σ, μ) is a sequentially rational assessment. Let $i \in N$ be a player, U_i be an information set of player i, and σ'_i be a behavior strategy of player i. By Theorem 7.34, there exists a sequence $(\sigma_i'^k)_{k\in\mathbb{N}}$ of behavior strategies of player i converging to σ'_i and satisfying the condition that for each $k \in \mathbb{N}$, the strategy $\sigma_i'^k$ is a possible strategy for player i in the game $\Gamma(\delta^k)$. Since the assessment $(\sigma^k, \mu_{\sigma^k})$ is a sequential equilibrium in the game $\Gamma(\delta^k)$, one has

$$u_i(\sigma^k \mid U_i, \mu_{\sigma^k}) \geq u_i\left(\left(\sigma_i'^k, \sigma_{-i}\right) \mid U_i, \mu_{\sigma^k}\right). \tag{7.43}$$

From the continuity of the payoff function, and consideration of the subsequence $(k_j)_{j\in\mathbb{N}}$, we conclude that

$$u_i(\sigma \mid U_i, \mu) \geq u_i((\sigma'_i, \sigma_{-i}) \mid U_i, \mu). \tag{7.44}$$

This completes the proof that the pair (σ, μ) is sequentially rational, and hence a sequential equilibrium. □

We will now show that the converse of Theorem 7.57 does not hold: there exist games that have a sequential equilibrium of the form (σ, μ), where the strategy vector σ is not an extensive-form perfect equilibrium.

Example 7.59 Consider the two-player extensive-form game depicted in Figure 7.18. In this game there are two Nash equilibria in pure strategies: (T, t) and (B, b). Since every player has a single information set, the set of strategic-form perfect equilibria equals the set of extensive-form perfect equilibria. Since strategy T dominates strategy B (and strategy t dominates strategy b), only (T, t) is a strategic-form perfect equilibrium (see Theorem 7.28 on page 267). However, as we will now show, both (T, t) and (B, b) form elements of sequential equilibrium.

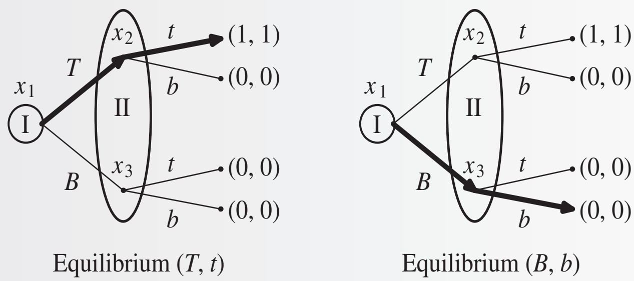

Figure 7.18 The game in Example 7.59, along with two sequential equilibria

Under both equilibria, the play of the game visits every information set, and therefore the beliefs of the players in these equilibria are as follows:

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- At the equilibrium (T, t), the beliefs of the players are $[1(x_1)]$ and $[1(x_2)]$ respectively.
- At the equilibrium (B, b), the beliefs of the players are $[1(x_1)]$ and $[1(x_3)]$ respectively.

We first show that the pair $((T, t), [1(x_1)], [1(x_1)])$ is a sequential equilibrium. To show that this pair is consistent define

$$\sigma^k = \left(\left[\frac{k-1}{k}(T), \frac{1}{k}(B)\right], \left[\frac{k-1}{k}(t), \frac{1}{k}(b)\right]\right), \quad \forall k \in \mathbb{N}. \tag{7.45}$$

Then $\mu_{\sigma^k}(U_{\mathrm{II}}) = \left[\frac{k-1}{k}(x_2), \frac{1}{k}(x_3)\right]$ for all $k \in \mathbb{N}$, $\lim_{k\to\infty} \sigma^k = (T, t)$, and $\lim_{k\to\infty} \mu_{\sigma^k}(U_{\mathrm{II}}) = [1(x_2)]$. This pair is sequentially rational because the payoff to each of the players is 1, which is the maximal payoff in the game.

We next show that the pair $((B, b), [1(x_1)], [1(x_3)])$ is also a sequential equilibrium. To show that this pair is consistent define

$$\sigma^k = \left(\left[\frac{1}{k}(T), \frac{k-1}{k}(B)\right], \left[\frac{1}{k}(t), \frac{k-1}{k}(b)\right]\right), \quad \forall k \in \mathbb{N}. \tag{7.46}$$

Then $\mu_{\sigma^k}(U_{\mathrm{II}}) = \left[\frac{1}{k}(x_2), \frac{k-1}{k}(x_3)\right]$ for all $k \in \mathbb{N}$, $\lim_{k\to\infty} \sigma^k = (B, b)$, and $\lim_{k\to\infty} \mu_{\sigma^k}(U_{\mathrm{II}}) = [1(x_3)]$. This pair is sequentially rational because Player I receives 0 whether he plays T or B, and given his belief at information set $\{x_2, x_3\}$, Player II receives 0 whether he plays t or plays b. ◀

In summary, the main differences between the three refinements of Nash equilibrium in extensive-form games are as follows:

- A mixed strategy vector σ is a strategic-form perfect equilibrium if it is the limit of equilibria in completely mixed strategies $(\sigma^k)_{k\in\mathbb{N}}$ of a sequence of perturbed games, where the perturbations converge to zero.
- A mixed strategy vector σ is an extensive-form perfect equilibrium if it is the limit of equilibria in completely mixed behavior strategies $(\sigma^k)_{k\in\mathbb{N}}$ of a sequence of perturbed games, where the perturbations converge to zero.
- An assessment (σ, μ) is a sequential equilibrium if μ is the limit of a sequence of beliefs $(\mu_{\sigma^k})_{k\in\mathbb{N}}$ induced by a sequence of strategies $(\sigma^k)_{k\in\mathbb{N}}$ converging to σ in a sequence of games with perturbations converging to zero (the consistency property), and for each player i, at each of his information sets, σ_i is the best reply to σ_{-i} according to μ (the sequential rationality property).

As we saw in Example 7.59, if (σ, μ) is a sequential equilibrium then the strategy vector σ is not necessarily an extensive-form perfect equilibrium. This is due to the fact that the definition of extensive-form perfect equilibrium contains a condition that is not contained in the definition of sequential equilibrium: for σ to be an extensive-form perfect equilibrium, σ^k must be an equilibrium of the corresponding perturbed game for every $k \in \mathbb{N}$; i.e., the sequential rationality property must obtain for every element of the sequence $(\sigma^k)_{k\in\mathbb{N}}$, while for (σ, μ) to be a sequential equilibrium, the sequential rationality property must hold only in the limit, σ.

The next example illustrates why it is not clear how to extend the definition of sequential equilibrium to games with imperfect recall.

Example 7.60 **The Absent-Minded Driver** Consider the Absent-Minded Driver game depicted in Figure 7.19, which we previously encountered in Example 6.9 (page 225). The game contains a single player, who cannot distinguish between the two vertices in the game tree, and hence, at any vertex cannot recall whether or not he has played in the past.

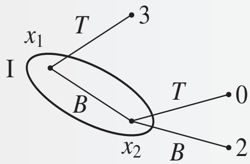

Figure 7.19 The Absent-Minded Driver game

The only Nash equilibrium in this game is T, because this strategy yields a payoff of 3, which is the game's highest payoff.

We now check whether the concept of sequential equilibrium can be adapted to this example. We first need to contend with the fact that because there are paths that visit the same information set several times, we need to reconsider what a belief at a vertex means. Suppose that the player implements strategy $\sigma = [1(B)]$, in which he plays action B. The play of the game will visit the vertex x_1, and the vertex x_2, hence $p_\sigma(x_1) = p_\sigma(x_2) = 1$, and $p_\sigma(U) = 1$ holds for the information set $U = \{x_1, x_2\}$. It follows that Equation (7.32) does not define a belief system, because $p_\sigma(U) \neq p_\sigma(x_1) + p_\sigma(x_2)$. We therefore need to define the player's belief system as follows:

$$\mu_U(x_1) = \frac{p_\sigma(x_1)}{p_\sigma(x_1) + p_\sigma(x_2)} = \tfrac{1}{2}, \quad \mu_U(x_2) = \frac{p_\sigma(x_2)}{p_\sigma(x_1)+p_\sigma(x_2)} = \tfrac{1}{2}. \qquad (7.47)$$

In words, if the player implements strategy B, at his information set he ascribes equal probability to the play of the game being at either of the vertices x_1 and x_2.

Is the concept of sequential equilibrium applicable in this game? We will show that the assessment $(B, [\frac{1}{2}(x_1), \frac{1}{2}(x_2)])$ is sequentially rational, and therefore is a sequential equilibrium according to Definition 7.52, despite the fact that the strategy B is not a Nash equilibrium. If Player I implements strategy B at his information set, his expected payoff is 2, because he believes the play of the game is located at either x_1, or x_2, with equal probability, and in either case, if he implements strategy B, his expected payoff is 2. If, however, Player I implements strategy T at this information set, his expected payoff is $\frac{3}{2}$: the player ascribes probability $\frac{1}{2}$ to the play of the game being located at vertex x_1, which yields a payoff of 3 if he implements strategy T, and he ascribes probability $\frac{1}{2}$ to the play of the game being located at vertex x_2, in which case he receives a payoff of 0 if he implements strategy T. It follows that $(B, [\frac{1}{2}(x_1), \frac{1}{2}(x_2)])$ is a sequentially rational assessment, despite the fact that B is not an equilibrium.

The reason that Theorem 7.54 does not hold in games with imperfect recall is due to the fact that if there exists a path from the root that passes through two different vertices in the same information set U of a player, then when the player changes the action that he implements at U, he may also change his belief at U. This possibility is not taken into account in the definition of sequential equilibrium. ◀

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7.5 Remarks

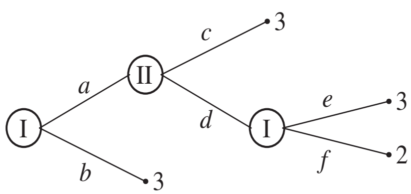
Sections 7.1 and 7.3 are based on the research conducted by Reinhardt Selten, who was awarded the Nobel Memorial Prize in Economics in 1994 for his contributions to refinements of the Nash equilibrium. The concept of sequential equilibrium first appeared in Kreps and Wilson [1982]. The interested reader may find a wealth of material on the concepts of subgame perfect equilibrium, and perfect equilibrium, in van Damme [1987]. Exercises 7.6 and 7.7 are based on Glazer and Ma [1989]. Exercise 7.13 is based on Selten [1978]. Exercise 7.14 is a variation of an example appearing in Rubinstein [1982]. Exercise 7.16 is based on an example appearing in Harris, Reny, and Robson [1995]. Exercise 7.26 is based on an example appearing in van Damme [1987, page 28]. Exercise 7.37 is based on an example appearing in Selten [1975]. The game in Exercise 7.46 is taken from Selten [1975]. The game in Exercise 7.48 is based on a game appearing in Kreps and Ramey [1987]. Exercise 7.49 is taken from Kohlberg and Mertens [1986]. Exercise 7.52 is based on an example appearing in Banks and Sobel [1987]. Exercise 7.53 is based on an example appearing in Cho and Kreps [1987]. Exercise 7.54 is based on an example appearing in Camerer and Weigelt [1988].

7.6 Exercises

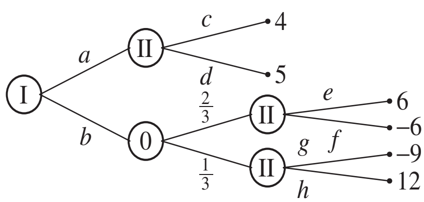
7.1 (a) What is the number of subgames in a game with perfect information whose game tree has eight vertices?
(b) What is the number of subgames in a game whose game tree has eight vertices and one information set, which contains two vertices (with all other information sets containing only one vertex)?
(c) What is the number of subgames in a game whose game tree has eight vertices, three of which are chance vertices?

7.2 Answer the following questions, for each of the following two-player zero-sum extensive-form games:

(a) Find all the equilibria obtained by backward induction.
(b) Describe the corresponding strategic-form game.
(c) Check whether there exist additional equilibria.

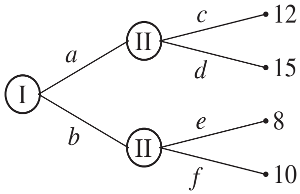


Game A



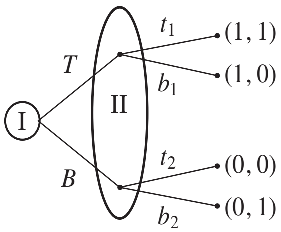
Game B

7.3 Find all the equilibria of the following two-player zero-sum game.

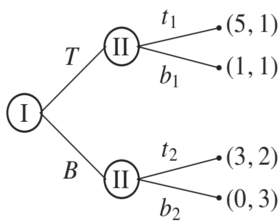


Explain why one cannot obtain all the equilibria of the game by implementing backward induction.

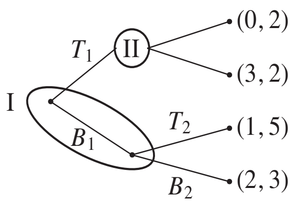
7.4 Find all the subgame perfect equilibria of the following games.



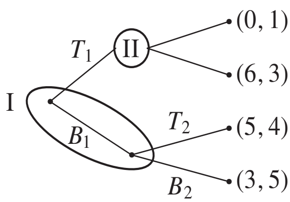
Game A



Game B



Game C



Game D

7.5 **The Ultimatum game** Allen and Rick need to divide \$100 between them as follows: first Allen suggests an integer x between 0 and 100 (which is the amount of money he wants for himself). Rick, on hearing the suggested amount, decides whether to accept or reject. If Rick accepts, the payoff of the game is $(x, 1 - x)$: Allen receives x dollars, and Rick receives $100 - x$ dollars. If Rick chooses to reject, neither player receives any money.

(a) Describe this situation as an extensive-form game.
(b) What is the set of pure strategies each player has?
(c) Show that any result $(a, 100 - a)$, $a \in \{0, 1, \dots, 100\}$, is a Nash equilibrium payoff. What are the corresponding equilibrium strategies?
(d) Find all the subgame perfect equilibria of this game.

7.6 **The Judgment of Solomon** Elizabeth and Mary appear before King Solomon at his palace, along with an infant. Each woman claims that the infant is her child. The

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child is “worth” 100 dinars to his true mother, but he is only “worth” 50 dinars to the woman who is not his mother. The king knows that one of these two women is the true mother of the child, and he knows the “value” that the true mother ascribes to the child, and the “value” that the impostor ascribes to the child, but he does not know which woman is the true mother, and which the impostor.

To determine which of the two women is the true mother, the king explains to Elizabeth and Mary that he will implement the following steps:

(i) He will ask Elizabeth whether the child is hers. If she answers negatively, the child will be given to Mary. If she answers affirmatively, the king will continue to the next step.
(ii) He will ask Mary if the child is hers. If she answers negatively, the child will be given to Elizabeth. If she answers affirmatively, Mary will pay the king 75 dinars, and receive the child, and Elizabeth will pay the king 10 dinars.

Answer the following questions:

(a) Describe the mechanism implemented by the king using two extensive-form games: in one extensive-form game Elizabeth is the true mother of the child, and in the second extensive-form game Mary is the true mother of the child.
(b) Prove that the mechanism implemented by the king guarantees that despite the fact that he does not know which of the above extensive-form games is being played, in each game the only subgame perfect equilibrium is the one under which the true mother gets her child and neither woman pays anything at all.
(c) Find another equilibrium of each game, which is not the subgame perfect equilibrium.

7.7 The following is a generalization of the “Judgment of Solomon,” discussed in Exercise 7.6.

Emperor Caligula wishes to grant a prize-winning horse as a gift to one of his friends, Claudius or Marcus. The value that Claudius ascribes to the horse is in the set $\{u_1, u_2, \ldots, u_n\}$, and the value that Marcus ascribes to the horse is in the set $\{v_1, v_2, \ldots, v_m\}$. Each one of the emperor’s friends knows the precise value that he ascribes to the horse, and he also knows the precise value that the other friend ascribes to the horse, but the only thing that the emperor knows is that the value that each of his friends ascribes to the horse is taken from the appropriate set of possible values. The emperor wishes to give the horse to the friend who values the horse most highly, but does not want to take money from his friends.

The emperor implements the following steps:

(i) Let $\varepsilon > 0$ be a positive number satisfying the condition that for each i and j, if $u_i \neq v_j$ then $|u_i - v_j| > \varepsilon$.
(ii) The emperor will ask Claudius if he values the horse at least as much as Marcus does. If Claudius answers negatively, the horse will be given to Marcus. If Claudius answers affirmatively, the emperor will continue to the next stage.
(iii) The emperor will ask Marcus if he values the horse more than Claudius does. If Marcus answers negatively, the horse will be given to Claudius. If Marcus

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answers affirmatively, the two friends will each pay the emperor $\varepsilon/4$, and the emperor will continue to the next step.

(iv) Claudius will be called upon to suggest a value $u \in \{u_1, u_2, \dots, u_n\}$.

(v) Knowing Claudius' suggested value, Marcus will be called upon to suggest a value $v \in \{v_1, v_2, \dots, v_m\}$.

(vi) The individual who suggested the higher value receives the horse, with the emperor keeping the horse in case of a draw. The winner pays $\max\{u, v\} - \frac{\varepsilon}{2}$ for the horse. The loser pays nothing.

Answer the following questions:

(a) Describe the sequence of steps implemented by the emperor as an extensive-form game. Assume that at the start of the game the following move of chance is implemented, which determines the private value of the horse for each of the two friends: the private value of the horse for Claudius is chosen from the set $\{u_1, u_2, \dots, u_n\}$ using the uniform distribution, and the private value of the horse for Marcus is chosen from the set $\{v_1, v_2, \dots, v_m\}$ using the uniform distribution.

(b) Prove that the only subgame perfect equilibrium of the game leads to the friend who values the horse the most receiving the horse (in case both friends equally value the horse, Claudius receives the horse).

7.8 Prove Theorem 7.9 on page 257: every (finite) extensive-form game with perfect information has a subgame perfect equilibrium in pure strategies.

7.9 Prove that in the 100-times-repeated Prisoner's Dilemma game (see Example 7.15 on page 259), the only subgame perfect equilibrium is the one where both players choose D in all stages of the game (after every history of previous actions).

7.10 (a) Find all the equilibria of the following two-player game.

		Player II	
		L	R
Player I	T	3, 0	1, 2
	B	2, 0	1, 5

(b) Suppose the players play the game twice; after the first time they have played the game, they know the actions chosen by both of them, and hence each player may condition his action in the second stage on the actions that were chosen in the first stage.

Describe this two-stage game as an extensive-form game.

(c) What are all the subgames of the two-stage game?

(d) Find all the subgame perfect equilibria of the two-stage game.

7.11 **The one-stage deviation principle for subgame perfect equilibria** Recall that $u_i(\sigma \mid x)$ is the payoff to player i when the players implement the strategy vector σ, given that the play of the game has arrived at the vertex x.

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Prove that a strategy vector $\sigma^* = (\sigma_i^*)_{i \in \mathbb{N}}$ in an extensive-form game with perfect information is a subgame perfect equilibrium if and only if for each player $i \in N$, every decision vertex x, and every strategy $\widehat{\sigma}_i$ of player i that is identical to σ_i^* at every one of his decision vertices except for x,

$$u_i(\sigma^* \mid x) \geq u_i((\widehat{\sigma}_i, \sigma_{-i}^*) \mid x). \tag{7.48}$$

Guidance: To prove that σ^* is a subgame perfect equilibrium if the condition above obtains, one needs to prove that the condition $u_i(\sigma^* \mid x) \geq u_i((\sigma_i, \sigma_{-i}^*) \mid x)$ holds for every vertex x, every player i, and every strategy σ_i. This can be accomplished by induction on the number of vertices in the game tree as follows. Suppose that this condition does not hold. Among all the triples (x, i, σ_i) for which it does not hold, choose a triple such that the number of vertices where σ_i differs from σ_i^* is minimal. Denote by $\mathcal{X}$ the set of all vertices such that σ_i differs from σ_i^*. By assumption, $|\mathcal{X}| \geq 1$. From the vertices in $\mathcal{X}$, choose a "highest" vertex, i.e., a vertex such that every path from the root to it does not pass through any other vertex in $\mathcal{X}$. Apply the inductive hypothesis to all the subgames beginning at the other vertices in $\mathcal{X}$.

7.12 Principal-Agent game Hillary manages a technology development company. A company customer asks Hillary to implement a particular project. Because it is unclear whether or not the project is feasible, the customer offers to pay Hillary \$2 million at the start of work on the project, and an additional \$4 million upon its completion (if the project is never completed, the customer pays nothing beyond the initial \$2 million payment). Hillary seeks to hire Bill to implement the project. The success of the project depends on the amount of effort Bill invests in his work: if he fails to invest effort, the project will fail; if he does invest effort, the project will succeed with probability p, and will fail with probability $1 - p$. Bill assesses the cost of investing effort in the project (i.e., the amount of time he will need to devote to work at the expense of the time he would otherwise give to his family, friends, and hobbies) as equivalent to \$1 million. Bill has received another job offer that will pay him \$1 million without requiring him to invest a great deal of time and effort. In order to incentivize Bill to take the job she is offering, Hillary offers him a bonus, to be paid upon the successful completion of the project, beyond the salary of \$1 million.

Answer the following questions:

(a) Depict this situation as an extensive-form game, where Hillary first determines the salary and bonus that she will offer Bill, and Bill afterwards decides whether or not to take the job offered by Hillary. If Bill takes the job offered by Hillary, Bill then needs to decide whether or not to invest effort in working on the project. Finally, if Bill decides to invest effort on the project, a chance move determines whether the project is a success or a failure. Note that the salary and bonus that Hillary can offer Bill need not be expressed in integers.

(b) Find all the subgame perfect equilibria of this game, assuming that both Hillary and Bill are risk-neutral, i.e., each of them seeks to maximize the expected payoff he or she receives.

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(c) What does Hillary need to persuade Bill of during their job interview, in order to increase her expected payoff at equilibrium?

7.13 The Chainstore game A national chain of electronics stores has franchises in shopping centers in ten different cities. In each shopping center, the chainstore's franchise is the only electronics store. Ten local competitors, one in each city, are each contemplating opening a rival electronics store in the local shopping center, in the following sequence. The first competitor decides whether or not to open a rival electronics store in his city. The second competitor checks whether or not the first competitor has opened an electronics store, and takes into account the national chainstore's response to this development, before deciding whether or not he will open a rival electronics store in his city. The third competitor checks whether or not the first and second competitors have opened electronics stores, and takes into account the national chainstore's response to these developments, before deciding whether or not he will open a rival electronics store in his city, and so on. If a competitor decides not to open a rival electronics store, the competitor's payoff is 0, and the national chain store's payoff is 5. If a competitor does decide to open a rival electronics store, his payoff depends on the response of the national chainstore. If the national chainstore responds by undercutting prices in that city, the competitor and the chainstore lose 1 each. If the national chainstore does not respond by undercutting prices in that city, the competitor and the national chainstore each receive a payoff of 3.

(a) Describe this situation as an extensive-form game.
(b) Find all the subgame perfect equilibria.
(c) Find a Nash equilibrium that is not a subgame perfect equilibrium, and explain why it fails to be a subgame perfect equilibrium.

7.14 Alternating Offers game Debby and Barack are jointly conducting a project that will pay them a total payoff of \$100. Every delay in implementing the project reduces payment for completing the project. How should they divide this money between them? The two decide to implement the following procedure: Debby starts by offering a division $(x_D, 100 - x_D)$, where x_D is a number in $[0, 100]$ representing the amount of money that Debby receives under the terms of this offer, while Barack receives $100 - x_D$. Barack may accept or reject Debby's offer. If he rejects the offer, he may propose a counteroffer $(y_D, 99 - y_D)$ where y_D is a number in $[0, 99]$ representing the amount of money that Debby receives under the terms of this offer, while Barack receives $99 - y_D$. Barack's offer can only divide \$99 between the two players, because the delay caused by his rejection of Debby's offer has reduced the payment for completing the project by \$1. Debby may accept or reject Barack's offer. If she rejects the offer, she may then propose yet another counteroffer, and so on. Each additional round of offers, however, reduces the amount of money available by \$1: if the two players come to an agreement on a division after the kth offer has been passed between them, then they can divide only $(101 - k)$ dollars between them. If the two players cannot come to any agreement, after 100 rounds of alternating offers, they drop plans to conduct the project jointly, and each receives 0.

Describe this situation as an extensive-form game, and find all of its subgame perfect equilibria.

7.15 Prove Theorem 7.10: every extensive-form game with perfect recall has a subgame perfect equilibrium in mixed strategies.

7.16 **A game without a subgame perfect equilibrium** Consider the four-player extensive-form game in Figure 7.20. In this game, Player I's set of pure strategies is the interval $[-1, 1]$; i.e., Player I chooses a number a in this interval. The other players, Players II, III, and IV, each have two available actions, at each of their information sets. Figure 7.20 depicts only one subgame tree, after Player I has chosen an action. All the other possible subtrees are identical to the one shown here.

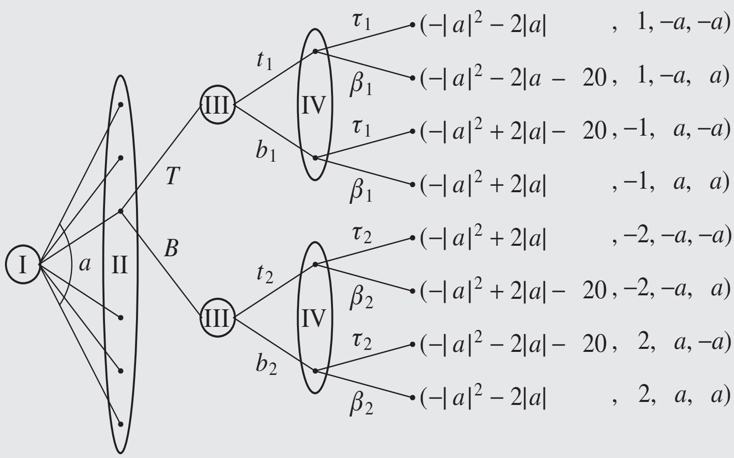

Figure 7.20 A game without subgame perfect equilibria

This game may be regarded as a two-stage game: in the first stage, Players I and II choose their actions simultaneously (where Player I chooses $a \in [-1, 1]$, and Player II chooses T or B), and in the second stage, Players III and IV, after learning which actions were chosen by Players I and II, choose their actions simultaneously.

Suppose that the game has a subgame perfect equilibrium, denoted by $\sigma = (\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}, \sigma_{\mathrm{III}}, \sigma_{\mathrm{IV}})$. Answer the following questions:

(a) What are all the subgames of this game?
(b) What will Players III and IV play under σ, when $a \neq 0$?
(c) What are the payoffs of Players III and IV, when $a = 0$?
(d) Denote by β the probability that Player II plays the pure strategy B, under strategy σ_{II}. Explain why there does not exist a subgame perfect equilibrium such that Player I plays $a = 0$, Player II plays $\beta = \frac{1}{2}$, and if $a = 0$, Player

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III chooses t_1 with probability $\frac{1}{4}$, and chooses t_2 with probability $\frac{1}{8}$, while Player IV chooses τ_1 and τ_2 with probability 1.

(e) Depict the expected payoff of Player I as a function of a and β, in the case where $a \neq 0$.

(f) Find the upper bound of the possible payoffs Player I can receive, in the case where $a = 0$.

(g) What is Player I's best reply when $\beta < \frac{1}{2}$? What are the best replies of Players III and IV, given Player I's strategy? What is Player II's best reply to these strategies of Players I, III, and IV?

(h) What is Player I's best reply when $\beta > \frac{1}{2}$? What are the best replies of Players III and IV, given Player I's strategy? What is Player II's best reply to these strategies of Players I, III, and IV?

(i) Suppose that $\beta = \frac{1}{2}$. What is the optimal payoff that Player I can receive? Deduce that under σ, Player I necessarily plays $a = 0$, and his payoff is then 0. What does this say about the strategies of Players III and IV? What is Player II's best reply to these strategies of Players I, III, and IV?

(j) Conclude that this game has no subgame perfect equilibrium.

(k) Find a Nash equilibrium of this game.

This exercise does not contradict Theorem 7.37, which states that every finite extensive-form game with perfect recall has a subgame perfect equilibrium in behavior strategies, because this game is infinite: Player I has a continuum of pure strategies.

7.17 Prove that for each player i, and every vector of perturbations ε_i, the set of strategies $\Sigma_i(\varepsilon_i)$ (see Equation (7.11)) is compact and convex.

7.18 Prove that every mixed strategy $\sigma_i \in \Sigma_i$ can be approximated by a completely mixed strategy; that is, for every $\delta > 0$ there is a completely mixed strategy σ_i' of player i that satisfies $\max_{s_i \in S_i} |\sigma_i(s_i) - \sigma_i'(s_i)| < \delta$.

7.19 Prove that the set of perfect equilibria of a strategic-form game is a closed subset of $\times_{i \in N} \Sigma_i$.

7.20 Find all the perfect equilibria in each of the following games, in which Player I is the row player and Player II is the column player.

	L	M
T	1, 1	1, 0
B	1, 0	0, 1

Game A

	L	C	R
T	1, 1	0, 0	−1, −2
M	0, 0	0, 0	0, −2
B	−2, 1	−2, 0	−2, −2

Game B

7.21 Consider the following two-player strategic-form game:

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	L	C	R
T	1, 2	3, 0	0, 3
M	1, 1	2, 2	2, 0
B	1, 2	0, 3	3, 0

(a) Prove that $([x_1(T), x_2(M), (1 - x_1 - x_2)(B)], L)$ is a Nash equilibrium of this game if and only if $\frac{1}{3} \leq x_1 \leq \frac{2}{3}$, $0 \leq x_2 \leq 2 - 3x_1$, and $x_1 + x_2 \leq 1$.

(b) Prove that the equilibria identified in part (a) are all the Nash equilibria of the game.

(c) Prove that if $([x_1(T), x_2(M), (1 - x_1 - x_2)(B)], L)$ is a perfect equilibrium, then $1 - x_1 - x_2 > 0$.

(d) Prove that for every $x_1 \in (\frac{1}{3}, \frac{1}{2})$ the strategy vector $([x_1(T), (1 - x_1)(M)], L)$ is a perfect equilibrium.

(e) Using Exercise 7.19 determine the set of perfect equilibria of this game.

7.22 Prove Theorem 7.28 (page 267): in a perfect equilibrium, every weakly dominated strategy is chosen with probability 0.

7.23 Let σ_1 and σ_2 be optimal strategies (in pure or mixed strategies) of two players in a two-player zero-sum game. Is (σ_1, σ_2) necessarily a perfect equilibrium? If so, prove it. If not, provide a counterexample.

7.24 A pure strategy s_i of player i is said to be *weakly dominated by a mixed strategy* if player i has a mixed strategy σ_i satisfying:

(a) For each strategy $s_{-i} \in S_{-i}$ of the other players,

$$u_i(s_i, s_{-i}) \leq U_i(\sigma_i, s_{-i}). \tag{7.49}$$

(b) There exists a strategy $t_{-i} \in S_{-i}$ of the other players satisfying

$$u_i(s_i, t_{-i}) < U_i(\sigma_i, t_{-i}). \tag{7.50}$$

Prove that in a perfect equilibrium, every pure strategy that is weakly dominated by a mixed strategy is chosen with probability 0.

7.25 (a) Prove that (T, L) is the only perfect equilibrium in pure strategies of the following game.

Player II

Player I		L	M
	T	6, 6	0, 0
	B	0, 4	4, 4

(b) Prove that in the following game, which is obtained from the game in part (a) by adding a dominated pure strategy to each player, (B, M) is a perfect equilibrium.

		Player II L	M	R
Player I	T	6, 6	0, 0	2, 0
	B	0, 4	4, 4	2, 0
	I	0, 0	0, 2	2, 2

7.26 In this exercise, we will show that in a three-player game a vector of strategies that makes use solely of strategies that are not dominated is not necessarily a perfect equilibrium. To do so, consider the following three-player game, where Player I chooses a row (T or B), Player II chooses a column (L or R), and Player III chooses a matrix (W or E).

W

	L	R
T	1, 1, 1	1, 0, 1
B	1, 1, 1	0, 0, 1

E

	L	R
T	1, 1, 0	0, 0, 0
B	0, 1, 0	1, 0, 0

(a) Find all the dominated strategies.
(b) Find all the Nash equilibria of this game.
(c) Find all the perfect equilibria of this game.

7.27 Prove that the following definition of perfect equilibrium is equivalent to Definition 7.25 (page 266).

Definition 7.61 *A strategy vector σ is called a* perfect equilibrium *if there exists a sequence $(\sigma^k)_{k\in\mathbb{N}}$ of vectors of completely mixed strategies satisfying:*

- *For each player $i \in N$, the limit $\lim_{k\to\infty} \sigma_i^k$ exists and equals σ_i.*
- *σ is a best reply to σ_{-i}^k, for each $k \in \mathbb{N}$, and each player $i \in N$.*

7.28 Prove that, in the following game, (B, L) is a Nash equilibrium, but not a perfect equilibrium.

		Player II L	M	R
Player I	T	1, 1	3, 3	0, 0
	C	1, 1	0, 0	3, 3
	B	1, 1	1, 1	1, 1

7.29 Show that in the game in Example 7.35 (page 270) the equilibrium (B, R) is an extensive-form perfect equilibrium. Does this game have additional Nash equilibria? If so, which of them is also an extensive-form perfect equilibrium? Justify your answer.

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7.30 Prove that, in the game in Example 7.18 (page 263), the equilibrium (T, t, β) is an extensive-form perfect equilibrium, but the equilibrium (B, t, τ) is not an extensive-form perfect equilibrium.

7.31 Prove directly the following theorem which is analogous to Corollary 7.26 (page 266) for extensive-form perfect equilibria: every extensive-form perfect equilibrium is a Nash equilibrium in behavior strategies. To prove this, first prove the analog result to Theorem 7.24 for a sequence of equilibria in perturbed games $(\Gamma(\delta^k))_{k\in\mathbb{N}}$.

7.32 Prove Theorem 7.30 (page 268): every finite extensive-form game has a strategic-form perfect equilibrium.

7.33 Prove Theorem 7.31 (page 268): let δ be a perturbation vector, let σ^* be a Nash equilibrium (in behavior strategies) in the game $\Gamma(\delta)$, and let $\Gamma(x)$ be a subgame of Γ. Then the strategy vector σ^*, restricted to the subgame $\Gamma(x)$, is a Nash equilibrium (in behavior strategies) of $\Gamma(x;\delta)$.

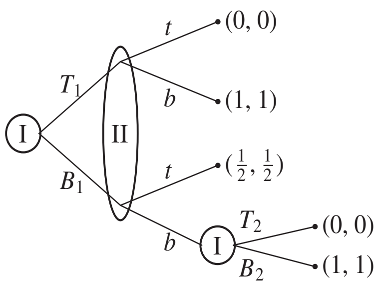
7.34 Prove Theorem 7.34 (page 269): let $(\delta^k)_{k\in\mathbb{N}}$ be a sequence of perturbation vectors satisfying $\lim_{k\to\infty} M(\delta^k) = 0$. Then for every behavior strategy $\sigma_i \in \mathcal{B}_i$ of player i there exists a sequence $(\sigma_i^k)_{k\in\mathbb{N}}$ of behavior strategies satisfying the following two properties:

- $\sigma_i^k \in \mathcal{B}_i(\delta_i^k)$ for each $k \in \mathbb{N}$.
- $\lim_{k\to\infty} \sigma_i^k$ exists and equals σ_i.

7.35 Prove Theorem 7.36 (page 271): every finite extensive-form game with perfect recall has an extensive-form perfect equilibrium.

7.36 This exercise shows that an extensive-form perfect equilibrium is not necessarily a strategic-form perfect equilibrium.

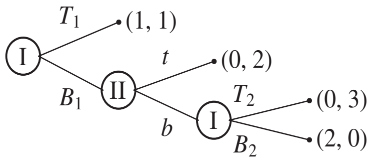
In the following game, find an extensive-form perfect equilibrium that is not a strategic-form perfect equilibrium.



Does the game have another Nash equilibrium? Does it have another subgame perfect equilibrium?

7.37 This exercise proves the converse to what we showed in Exercise 7.36: a strategic-form perfect equilibrium is not necessarily an extensive-form perfect equilibrium.

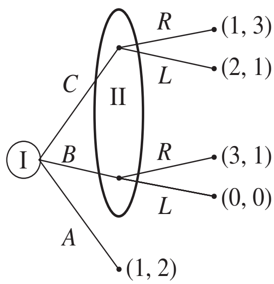
(a) Prove that the following game has a unique extensive-form perfect equilibrium.



(b) Show that this game has another equilibrium, which is a strategic-form perfect equilibrium. To do so, construct the corresponding strategic-form game, and show that it has more than one perfect equilibrium.

(c) Does this game have a strategic-form perfect equilibrium that is not a subgame perfect equilibrium?

7.38 Show that the following game has a unique Nash equilibrium, and in particular a unique extensive-form perfect equilibrium and a unique strategic-form perfect equilibrium.



7.39 Prove that in an extensive-form game in which every player has a single information set, every strategic-form perfect equilibrium is equivalent to an extensive-form perfect equilibrium, and that the converse also holds.

7.40 Consider the extensive-form game shown in Figure 7.21.

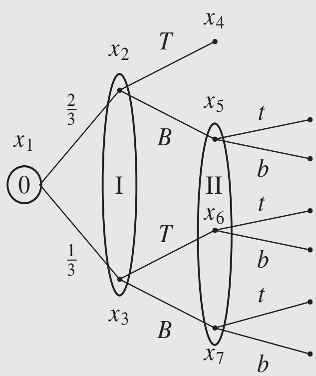


Figure 7.21 A game in extensive form

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For each of the following pairs, explain why it is not a consistent assessment of the game:

(a) $([\frac{1}{2}(T), \frac{1}{2}(B)], t, [\frac{1}{2}(x_2), \frac{1}{2}(x_3)], [\frac{1}{4}(x_4), \frac{1}{4}(x_5), \frac{1}{2}(x_6)])$.
(b) $([\frac{1}{2}(T), \frac{1}{2}(B)], b, [\frac{2}{3}(x_2), \frac{1}{3}(x_3)], [\frac{1}{3}(x_4), \frac{1}{3}(x_5), \frac{1}{3}(x_6)])$.
(c) $(T, t, [\frac{2}{3}(x_2), \frac{1}{3}(x_3)], [\frac{2}{3}(x_4), \frac{1}{3}(x_6)])$.
(d) $(T, t, [\frac{2}{3}(x_2), \frac{1}{3}(x_3)], [\frac{1}{2}(x_5), \frac{1}{2}(x_6)])$.

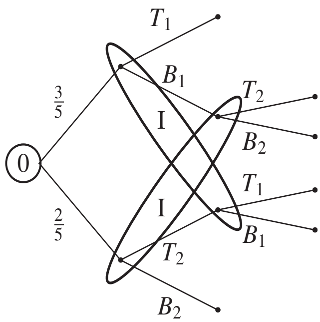
7.41 Prove Theorem 7.45 (page 277): if σ^* is a Nash equilibrium in behavior strategies, then the pair $(\sigma^*, \mu_{\sigma^*})$ is sequentially rational in every information set U satisfying $\mathbf{P}_{\sigma^*}(U) > 0$.

7.42 Prove Theorem 7.46 (page 277): in a game with perfect information, a vector of behavior strategies σ is a subgame perfect equilibrium if and only if the pair $(\sigma, \widehat{\mu})$ is sequentially rational at every information set of the game, where $\widehat{\mu}$ is a complete belief system such that $\widehat{\mu}_U = [1(x)]$ for every information set $U = \{x\}$.

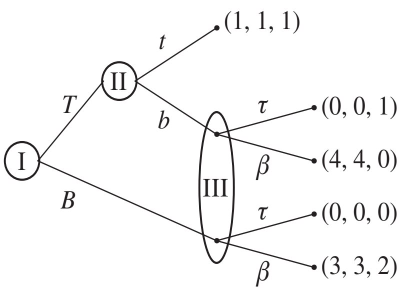
7.43 List all the consistent assessments of the extensive-form game in Exercise 7.40 (Figure 7.21).

7.44 List all the consistent assessments of the extensive-form game in Example 7.17 (page 261).

7.45 List all the consistent assessments of the following extensive-form game.



7.46 List all the consistent assessments, and all the sequentially rational assessments of the following game.



7.47 Find all the sequential equilibria of the game in Example 7.35 (page 270).

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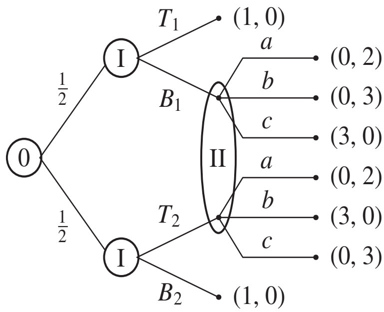
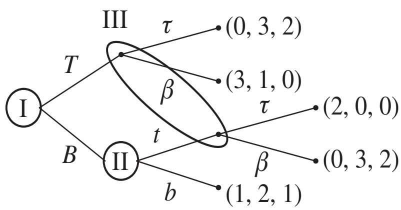
7.48 Consider the following extensive-form game.



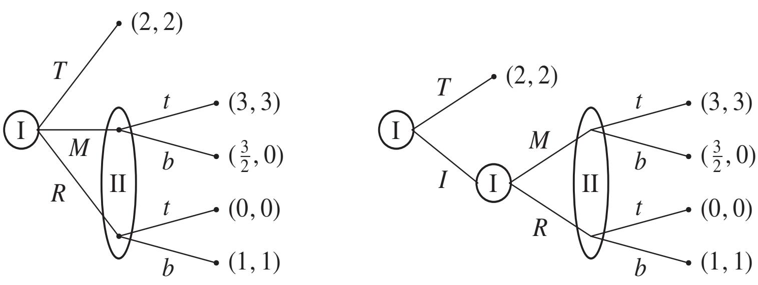
(a) Prove that in this game at every Nash equilibrium Player I plays (T_1, B_2).
(b) List all the Nash equilibria of the game.
(c) Which of these Nash equilibria can be completed to a sequential equilibrium, and for each such sequential equilibrium, what is the corresponding belief of Player II at his information sets? Justify your answer.

7.49 Find all the sequential equilibria of the following game.



7.50 The following example shows that the set of sequential equilibria is sensitive to the way in which a player makes decisions: it makes a difference whether the player, when called upon to choose an action from among a set of three possible actions, eliminates the actions he will not choose one by one, or simultaneously.

Consider the two extensive-form games below. Show that (2, 2) is a sequential equilibrium payoff in Game A, but not a sequential equilibrium payoff in Game B.



Game A Game B

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7.51 In an extensive-form game with perfect recall, is every Nash equilibrium part of a sequential equilibrium? That is, for every Nash equilibrium σ^* does there exist a belief system μ that satisfies the property that (σ^*, μ) is a sequential equilibrium? If yes, prove it. If not, construct a counterexample.

7.52 **Pre-trial settlement** A contractor is being sued for damages by a municipality that hired him to construct a bridge, because the bridge has collapsed. The contractor knows whether or not the collapse of the bridge is due to negligence on his part, or due to an act of nature beyond his control, but the municipality does not know which of these two alternatives is the true one. Both sides know that if the matter is settled by a court trial, the truth will eventually be uncovered.

The contractor can try to arrive at a pre-trial settlement with the municipality. He has two alternatives: to make a low settlement offer, under which he pays the municipality \$300,000, or a high offer, under which he pays the municipality \$500,000. After the contractor has submitted a settlement offer, the municipality must decide whether or not to accept it. Both parties know that if the suit goes to trial, the contractor will pay lawyer fees of \$600,000, and that, in addition to this expense, if the court finds him guilty of negligence, he will be required to pay the municipality \$500,000 in damages. Assume that the municipality has no lawyer fees to pay.

Answer the following questions:

(a) Describe this situation as an extensive-form game, where the root of the game is a chance move that determines with equal probability whether the contractor was negligent or not.
(b) Explain the significance of the above assumption, that a chance move determines with equal probability whether the contractor was negligent or not.
(c) Find all the Nash equilibria of this game.
(d) Find all the sequential equilibria of this game.
(e) Repeat items (c) and (d) when the chance move selects whether the contractor was negligent or not with probabilities p and $1 - p$ respectively.

7.53 **Signaling game** Caesar is at a cafe, trying to choose what to drink with breakfast: beer or orange juice. Brutus, sitting at a nearby table, is pondering whether or not to challenge Caesar to a duel after breakfast. Brutus does not know whether Caesar is brave or cowardly, and he will only dare to challenge Caesar if Caesar is cowardly. If he fights a cowardly opponent, he receives one unit of utility, and he receives the same single unit of utility if he avoids fighting a brave opponent. In contrast, he loses one unit of utility if he fights a brave opponent, and similarly loses one unit of utility if he dishonors himself by failing to fight a cowardly opponent. Brutus ascribes probability 0.9 to Caesar being brave, and probability 0.1 to Caesar being a coward. Caesar has no interest in fighting Brutus: he loses 2 units of utility if he fights Brutus, but loses nothing if there is no fight. Caesar knows whether he is brave or cowardly. He can use the drink he orders for breakfast to signal his type, because it is commonly known that brave types receive one unit of utility if they drink beer (and receive nothing if they drink orange juice), while cowards receive one unit of utility if they drink orange juice (and receive nothing if they drink beer). Assume that Caesar's utility is additive; for example, he receives three units of utility if he is brave, drinks beer, and avoids fighting Brutus. Answer the following questions:

(a) Describe this situation as an extensive-form game, where the root of the game tree is a chance move that determines whether Caesar is brave (with probability 0.9) or cowardly (with probability 0.1).
(b) Find all the Nash equilibria of the game.
(c) Find all the sequential equilibria of the game.

7.54 Henry seeks a loan to form a new company, and submits a request for a loan to Rockefeller. Rockefeller knows that p percent of people asking him for loans are conscientious, who feel guilty if they default on their loans, and $1 - p$ percent of people asking him for loans have no compunction about defaulting on their loans, but he does not know whether or not Henry is a conscientious borrower. Rockefeller is free to grant Henry a loan, or to refuse to give him a loan. If Henry receives the loan, he can decide to repay the loan, or to default. If Rockefeller refuses to loan money to Henry, both sides receive 10 units. If Rockefeller loans Henry the money he needs to form a company, and Henry repays the loan, Rockefeller receives 40 units, while Henry receives 60 units. If Rockefeller loans Henry the money he needs to form a company, but Henry defaults on the loan, Rockefeller loses x units, and Henry's payoff depends on his type: if he is a conscientious borrower, he receives 0, but if he has no compunction about defaulting, he gains 150 units. Answer the following questions:

(a) Describe this situation as an extensive-form game, where the root of the game tree is a chance move that determines Henry's type.
(b) Find all the Nash equilibria, and the sequential equilibria, of this game, in the following three cases:
  (i) $p = \frac{1}{3}$, and $x = 100$.
  (ii) $p = 0.1$, and $x = 50$.
  (iii) $p = 0$, and $x = 75$.

7.55 **The one-stage deviation principle for sequential equilibria** Let (σ, μ) be a consistent assessment in an extensive-form game Γ with perfect recall. Prove that the assessment (σ, μ) is a sequential equilibrium if and only if for each player $i \in N$, and every information set U_i

$$u_i(\sigma \mid U_i, \mu) \geq u_i(\widehat{\sigma}_i, \sigma_{-i} \mid U_i, \mu), \tag{7.51}$$

under every strategy $\widehat{\sigma}_i$ that differs from σ_i only at the information set U_i.
Guidance: To prove that if the condition holds then (σ, μ) is a sequential equilibrium, consider a player i and any information set U_i of his, along with any strategy σ'_i. Show that $u_i(\sigma \mid U_i, \mu) \geq u_i((\sigma'_i, \sigma_{-i}) \mid U_i, \mu)$. The proof of this inequality can be accomplished by induction on the number of information sets of player i over which σ'_i differs from σ_i.