CS215 Fall, 2024: Tutorial 1

Atharva Tambat

August 12, 2024

Table of Contents

Question 1

Using Chebyshev's Inequality:

$$P\{|X-\mu| \geq k\} \leq \frac{\sigma^2}{k^2}$$

Here, $\sigma^2 = 0$. Take $k = \frac{1}{n}$

$$P\left\{|X-\mu| \geq \frac{1}{n}\right\} \leq 0 \implies P\left\{|X-\mu| \geq \frac{1}{n}\right\} = 0$$

Take limit $n \to \infty$,

$$\lim_{n\to\infty} P\left\{|X-\mu| \geq \frac{1}{n}\right\} = P\left\{\lim_{n\to\infty}\left\{|X-\mu| \geq \frac{1}{n}\right\}\right\} = P\{X \neq \mu\} = 0$$

Table of Contents

1. Proof: Independence $\implies$ Uncorrelated
   For Independent variables X and Y, E[XY] = E[X]E[Y]
   Proof:

$$\begin{aligned}
E[XY] = \mu_{xy} &= \sum_i \sum_j x_i y_j p(x_i, y_j) \\
&= \sum_i \sum_j x_i y_j p(x_i) p(y_j) \\
&= \left(\sum_i x_i p(x_i) \right) \left(\sum_j y_j p(y_j) \right) \\
&= E[X]E[Y] = \mu_x \mu_y
\end{aligned}$$

Question 2 - Independence $\not\Leftrightarrow$ Uncorrelated

1. Proof: Uncorrelated $\not\Rightarrow$ Independence
   Proof: Counterexample: $X \in \{-1, 0, 1\}$

   $P(X = -1) = P(X = 0) = P(X = 1) = 1/3$

$$Y = \begin{cases} 1, & \text{if } X = 0 \\ 0, & \text{otherwise} \end{cases}$$

Table of Contents

Question 6 - Law of Total Probability

Applying the law of total probability:

$$P(E|E \cup F) = P(E|E \cup F, F)P(F) + P(E|E \cup F, \bar{F})P(\bar{F})$$

1. $P(A|C, D)$ - Probability of event A, given events C **and** D have occurred
2. $P(E|E \cup F, F) = \frac{P(E \cap (E \cup F) \cap F)}{P((E \cup F) \cap F)} = P(E \cap F)/P(F) = P(E|F)$
3. $P(E|E \cup F, \bar{F}) = 1.$ Why?

Question 6 - Law of Total Probability

Thus,

$$\begin{aligned} P(E|E \cup F) &= P(E|F)P(F) + (1 - P(F)) \\ &\geq P(E|F)P(F) + P(E|F)(1 - P(F)) \quad \textbf{Why?} \end{aligned}$$

So,

$$P(E|E \cup F) \geq P(E|F)(P(F) + 1 - P(F)) = P(E|F)$$

Table of Contents

Optimal Strategy

Ans: **Any strategy** has a winning probability of 1/52!!

Proof: Let us prove a stronger point - in a game with n cards, the winning probability is always 1/n regardless of the strategy.

1. $p \rightarrow$ probability that the strategy chooses the first card
2. $G \rightarrow$ Event that the first card is guessed

Two cases of winning:

- 1^{st} card is ace of spades - happens with probability 1/n
- 1^{st} card is not ace of spades: What is the probability of win, **given** we skip the first chance?

Optimal Strategy

H = first card is not ace of spades

$$P(H).P(\{win\}|H)$$

But $P(\{win\}|H)$ = probability of winning with n-1 cards = $\frac{1}{n-1}$ by induction hypothesis

$$P(H).P(\{win\}|H) = \frac{n-1}{n}\frac{1}{n-1} = \frac{1}{n}$$

Using the Law of total probability:

$$\begin{aligned} P(\{win\}) &= P(\{win\}|G)P(G) + P(\{win\}|\bar{G})(1 - P(G)) \\ &= \frac{1}{n}p + \frac{1}{n}(1-p) \\ &= \frac{1}{n} \end{aligned}$$

Table of Contents

Coin Flip

- $C_i \rightarrow$ Event that coin i is chosen
- $F_n \rightarrow$ Event that first n tosses are heads
- $H \rightarrow$ Event that the $(n+1)^{th}$ is a head

$$P(H|F_n) = ?$$

$$P(H|F_n) = \sum_{i=0}^{k} P(H|F_n C_i) P(C_i|F_n)$$

Coin Flip

$P(H|F_nC_i)$ - means given i^{th} coin is selected and first n tosses are heads, what is the probability $(n+1)^{th}$ toss is a head.

$$P(H|F_nC_i) = P(H|C_i) = \frac{i}{k} \ \ \textbf{Why?}$$

Also,

$$P(C_i|F_n) = \frac{P(C_iF_n)}{P(F_n)} = \frac{P(F_n|C_i)P(C_i)}{\sum_{j=0}^{k} P(F_n|C_j)P(C_j)} =$$

Coin Flip

$$P(C_i|F_n) = \frac{(i/k)^n[1/(k+1)]}{\sum_{j=0}^{n}(j/k)^n[1/(k+1)]}$$

Thus,

$$P(H|F_n) = \sum_{i=0}^{k} \frac{(i/k)^{n+1}}{\sum_{j=0}^{k}(j/k)^n} = \frac{\sum_{i=0}^{k}(i/k)^{n+1}}{\sum_{j=0}^{k}(j/k)^n}$$

Now, use the approximation to simplify the numerator and denominator