Roll No : 123456789 **Dept.**: CSE
e.g., 190040001 e.g., CSE

IIT Bombay
CS 6001: GT&AMD
Quiz 2, 2023-24-I
Date: October 27, 2023

CS6001: Game Theory and Algorithmic Mechanism Design

Total: **10 $\times$ 3 = 30 points,** *Duration:* **1 hour, ATTEMPT ALL QUESTIONS**

Instructions:

1. This question-and-answersheet booklet contains a total of **4 sheets** of paper **(8 pages, page 2 is blank)**. Please verify.
2. Write your roll number and department on **every side of every sheet** (except the blank sheet) of this booklet. Use only **black/blue ball-point pen**. The first 5 minutes of additional time is given exclusively for this activity.
3. Write final answers neatly with a pen **only in the given boxes**.
4. Use the rough sheets for scratch works / attempts to solution. **Write only the final solution (which may be a sequence of logical arguments) in a precise and succinct manner in the boxes provided**. Do not provide unnecessarily elaborate steps. The space within the boxes are sufficient for the correct and precise answers.
5. Submit your answerscripts to the teaching staff when you leave the exam hall or the time runs out (whichever is earlier). **Your exam will not be graded if you fail to return the paper**.
6. **This is a closed book, notes, internet exam. No communication device, e.g., cellphones, iPad, etc., is allowed**. Keep it switched off in your bag and keep the bag away from you. If anyone is found in possession of such devices during the exam, that answerscript may be disqualified for evaluation and DADAC may be invoked.
7. One A4 assistance sheet (text **only on one side**) is allowed for the exam.
8. **After you are done with your exam or the exam duration is over, please DO NOT rush to the desk for submitting your paper**. Please remain seated until we will collect all the papers, count them, and give you get a clear signal to leave your seat.

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Problem 1 (1 × 10 points). A security agency pays its employees on an hourly basis in **two** shifts of work. During the 12 hours of day-time, they pay $w_d =$ ₹100 per hour, and during the 12 hours of night-time, they pay $w_n =$ ₹200 per hour. However, each employee has a different cost factor for day and night as given by the utility expression below. The *private* cost factors for day and night for employee i is given by d_i and n_i respectively. The utilities for an allocation $x = (x(d), x(n))$ are as follows, where $x_i(d)$ and $x_i(n)$ are the allocated hours to employee i during day and night respectively (hour allocations can be any real number).

$$u_i^{(d)}(x(d), d_i) = w_d \cdot x_i(d) - d_i \cdot \frac{x_i^2(d)}{2}$$
$$u_i^{(n)}(x(n), n_i) = w_n \cdot x_i(n) - n_i \cdot \frac{x_i^2(n)}{2}$$
$$u_i(x, (d_i, n_i)) = u_i^{(d)}(x(d), d_i) + u_i^{(n)}(x(n), n_i)$$

Consider 5 employees that are to be scheduled during the 24-hour period. The job requires **only one employee** to be at the security post at any given time. If the uniform rule (Sprumont [1991], as discussed in class) is applied for the allocation of both day and night duty hours *separately*, what will be the allocated duty hours of each of the employees? The cost factors for the employees are given in Figure 1. Fill your answers in the empty boxes of Figure 2.

$i \rightarrow$	1	2	3	4	5
d_i	50	100	20	12.5	25
n_i	200	200	100	$200/3$	$200/3$

Figure 1: Cost factor (₹/hour^2)

$i \longrightarrow$	1	2	3	4	5
$x_i(d)$	2	1	3	3	3
$x_i(n)$	2	2	2	3	3

Figure 2: Allocation (hours), **write your answers in the boxes above**

Here, every player (employee) has a single-peaked preference over the share of the task. Since the task of day hours is disjoint from the night hours, these two times are handled separately. The day time and night time peaks of each agent are $\frac{w_d}{d_i}$ and $\frac{w_n}{n_i}$ respectively. The day peaks are 2, 1, 5, 8, 4 and night peaks are 1, 1, 2, 3, 3. Using the uniform rule on these (two different cases), we get the above numbers.

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Problem 2 (10 points). Consider **single peaked** preferences for n players. Recall that this type of preferences has a **common order** over the alternatives and the preferences have a *single peak* w.r.t. that common order for every player. Define a social choice function (SCF) $f : \mathcal{S}^n \to A$ that takes a single-peaked preference profile $P \in \mathcal{S}^n$ and outputs an alternative in A.

(a) Since each player is identified with a unique peak in this domain, one can define the SCF in terms of their peaks. Consider the SCF f^{left} that picks the leftmost peak among the player peaks. Is this SCF strategyproof? (Yes/No – reasons are not needed as part of this question) **1 point**

Yes

(b) Explain your answer to the previous part of this question. If your answer was **yes**, explain the reason in about 3 sentences. If you answer was **no**, provide a counterexample (you can draw the example and explain) with 5 agents. **2 points**

Since the SCF chooses the leftmost peak, the agent that has this peak has no reason to misreport. All other agents' peaks are either at this point or on the right of it. The only way they can change the outcome is by reporting a peak even left to it, but that is less preferred than the current outcome since the preferences are single peaked.

(c) Is the SCF f^{left} **Condorcet consistent**? (Yes/No – reasons are not needed as part of this question)

Recall: A *Condorcet winner* is an alternative that defeats all other alternatives in pairwise election. An SCF is *Condorcet consistent* if it selects *the* Condorcet winner whenever one exists. **1 point**

No

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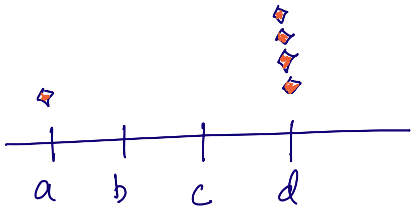
(d) Explain your answer to the previous part of this question. If your answer was **yes**, explain the reason in about 3 sentences. If you answer was **no**, provide a counterexample (you can draw the example and explain) with 5 agents. **2 points**



Consider the above example where there are 4 alternatives a, b, c, d. All agents except one have their peaks at d and that one agent has her peak at a. Any Condorcet consistent SCF will give d as the outcome while f^{left} gives a. Hence the answer.

(e) Find an SCF (different from the above one) in this setting that is **Condorcet consistent**. Explain the SCF in at most 2 sentences. is Condorcet consistent **2 ~~1~~ points**

why

An SCF that picks the median of all the reported peaks is CC. $f^{med}(P) = \text{median}(p_1, \dots, p_n)$, where p_i's are agent peaks.

If the median of the peaks is picked as the outcome, then for any profile a majority of the agents prefer that outcome over any other alternative. Hence, that outcome will beat every other alternative in pairwise election (or will not lose in a pairwise election). This suffices for a Condorcet winner.

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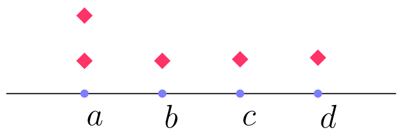
(f) What will be the outcome of this new SCF on the example profile below? There are 4 alternatives: a through d with the common ordering as shown in the figure. The diamonds show a player's peak at that alternative. 1 ~~**2**~~ **points**



b is the outcome.

(g) Recall the following claim we proved in class.

Claim: Suppose f satisfies strategyproofness, ontoness, and anonymity, then

$$f(P) = \text{median}(p_1, \ldots, p_n, y_1, \ldots, y_{n-1}),$$

where $p_i, i = 1, \ldots, n$ are player peaks and $y_j, j = 1, \ldots, n-1$ are phantom peaks.

In view of the arguments in the previous parts of this question, how would you extend the result to the following? (Fill in the content for the question mark)

Proposed claim: Suppose f satisfies strategyproofness, ontoness, anonymity, and Condorcet consistency then $f(P) = ?$ **1 point**

$f(P) = \text{median}(p_1, p_2, \ldots, p_n)$.

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Problem 3 (10 points). Denote by $K = |A|$ the number of alternatives in A. For each alternative $a \in A$, denote by $j_a(P_i)$ the ranking of a in the strict preference relation P_i (for example, $j_a(P_i) = 1$ when alternative a is the most-preferred alternative according to P_i).

Define a proposed social welfare function F as follows. For each alternative a, compute the sum $s_a = \sum_{i \in N} j_a(P_i)$. We say that alternative a is (weakly) preferred to alternative b in $F(P)$ if and only if $s_a \leqslant s_b$.

(a) Prove that F defines a valid social welfare function, i.e., $F(P)$ induces a weak preference order across all alternatives for every P. Write the arguments in not more than 3 sentences. **2 points**

We need to show that $F(P)$ is complete (every pair of alternatives is comparable), reflexive ($a\, F(P)\, a$), and transitive (if $a F(P) b$ and $b F(P) c \Rightarrow a F(P) c$). Note that $F(P)$ is assigning some score for every position in the votes of the voters and adding them up. Since the final comparison is wrt the scores that are integers, the properties of completeness, reflexivity, and transitivity comes directly as a consequence of the same properties in the set of integers. Hence this is a valid SWF.

(b) Is F **dictatorial**? First answer this in **yes/no**. If yes, explain (in not more than 3 sentences) why. If no, give a counterexample with 3 players and 3 alternatives. **(1 + 1) points**

No.

$$P = \begin{array}{ccc} a & b & b \\ b & a & a \\ c & c & c \\ \hline 1 & 2 & 3 \end{array}$$

agents: 1, 2, 3

Let WLOG 1 is the dictator but $b\, F(P)\, a$. The example can be replicated accordingly irrespective of which agent is the dictator.

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(c) Does F satisfy **unanimity**? First answer this in **yes/no**. If yes, explain (in not more than 3 sentences) why. If no, give a counterexample with 3 players and 3 alternatives. **(1 + 1) points**

Yes. If $a \, P_i \, b \;\; \forall i \in N$ then $j_a(P_i) < j_b(P_i) \;\; \forall i \in N$

$$\Rightarrow \sum_{i \in N} j_a(P_i) = s_a < s_b = \sum_{i \in N} j_b(P_i)$$

Hence, $a \, F(P) \, b$.

(d) Does F satisfy **independence of irrelevant alternatives**? First answer this in **yes/no**. If yes, explain (in not more than 3 sentences) why. If no, give a counterexample with 3 players and ~~3~~ 4 alternatives. **(1 + 1) points**

No.

1	c	c	c
2	d	d	d
3	a	a	b
4	b	b	a

$= P$

$P' =$

c	c	b
d	d	c
a	a	d
b	b	a

$s_a = 3 + 3 + 4 = 10$

$s_b = 4 + 4 + 3 = 11$

$\Rightarrow a \, F(P) \, b$

$s_a = 10$

$s_b = 4 + 4 + 1 = 9$

$\Rightarrow b \, F(P') \, a$.

(e) Does F satisfy **anonymity**? First answer this in **yes/no**. If yes, explain (in not more than 3 sentences) why. If no, give a counterexample with 3 players and 3 alternatives. **(1 + 1) points**

Yes. F only considers the number of agents that place a candidate in a specific position and does not consider the agents identity. Hence even if the identity of the voters are permuted, the outcome of F does not change.

END OF QUESTION PAPER. GOOD LUCK!