1. Find an orthogonal basis for the given set of vectors using the Gram-Schmidt process.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ -3 \end{bmatrix} \right\}$$

$$V_1 = X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\frac{V_1}{\|V_1\|} = \frac{1}{\sqrt{3}} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1\sqrt{3} \end{bmatrix}$$

$$V_2 = X_2 - \frac{\langle X_2, V_1 \rangle}{\|V_1\|^2} V_1$$

$$\begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix} - \frac{\langle (-3,2,0), (1,1,1) \rangle}{|(1,1,1)|^2} (1,1,1)$$

$$\begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix} - \frac{-3-2+0}{3 \cdot 1^2} (1,1,1)$$

$$\begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix} - \frac{-5}{3} (1,1,1)$$

$$\begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 5/3 \\ \downarrow \end{bmatrix}$$

$$\begin{bmatrix} -4/3 \\ -1/3 \\ 5/3 \end{bmatrix}$$

$$\frac{V_2}{\|V_2\|} = \frac{1}{\sqrt{14/3}} \begin{bmatrix} -4/3 \\ -1/3 \\ 5/3 \end{bmatrix} = \begin{bmatrix} -\frac{2\sqrt{42}}{21} \\ -\frac{\sqrt{42}}{42} \\ \frac{5\sqrt{42}}{42} \end{bmatrix}$$

$$V_3 = X_3 - \frac{\langle X_3, V_1 \rangle}{\|V_1\|^2} V_1 - \frac{\langle X_3, V_2 \rangle}{\|V_2\|^2} V_2$$

$$V_3 = \begin{bmatrix} -3 \\ 4 \\ -3 \end{bmatrix} - \frac{-2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -4/3 \\ -1/3 \\ 5/3 \end{bmatrix}$$

$$V_3 = \begin{bmatrix} -3 \\ 9/2 \\ -3/2 \end{bmatrix}$$

$$\frac{V_3}{\|V_3\|} = \frac{21}{\sqrt{14}} \begin{bmatrix} -3 \\ 9/2 \\ -3/2 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{14}}{7} \\ \frac{3\sqrt{14}}{14} \\ -\frac{\sqrt{14}}{14} \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{3} & -2\sqrt{42}/21 & -\sqrt{14}/7 \\ 1/\sqrt{3} & -\sqrt{42}/42 & 3\sqrt{14}/14 \\ 1/\sqrt{3} & 5\sqrt{42}/42 & -\sqrt{14}/14 \end{bmatrix}$$

2. Orthogonalize the given set of vectors using the Gram-Schmidt process.

$$\left\{ \begin{bmatrix} 4 \\ 5 \\ -1 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 1 \\ -3 \end{bmatrix} \right\}$$

$$V_1 = \begin{bmatrix} 4 \\ 5 \\ -1 \\ -4 \end{bmatrix}$$

$$\frac{V_1}{\|V_1\|} = \frac{1}{\sqrt{58}} \begin{bmatrix} 4 \\ 5 \\ -1 \\ -4 \end{bmatrix} = \begin{bmatrix} 2\sqrt{58}/29 \\ 5\sqrt{58}/58 \\ -\sqrt{58}/58 \\ -2\sqrt{58}/29 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} -3 \\ -1 \\ 1 \\ -3 \end{bmatrix} - \frac{-6}{\sqrt{58}} \begin{bmatrix} 4 \\ 5 \\ -1 \\ -4 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} -75/29 \\ -14/29 \\ 26/29 \\ -99/29 \end{bmatrix}$$

$$\frac{V_2}{\|V_2\|} = \frac{29}{\sqrt{16298}} \begin{bmatrix} -75/29 \\ -14/29 \\ 26/29 \\ -99/29 \end{bmatrix} = \begin{bmatrix} -75\sqrt{16298}/16298 \\ -7\sqrt{16298}/8149 \\ 13\sqrt{16298}/8149 \\ -99\sqrt{16298}/16298 \end{bmatrix}$$

$$\begin{bmatrix} \frac{2\sqrt{58}}{29} & -\frac{75\sqrt{16298}}{16298} \\ \frac{5\sqrt{58}}{58} & -\frac{7\sqrt{16298}}{8149} \\ -\frac{\sqrt{58}}{58} & \frac{13\sqrt{16298}}{8149} \\ -\frac{2\sqrt{58}}{29} & -\frac{99\sqrt{16298}}{16298} \end{bmatrix}$$

3. Compute the QR factorization of the given matrix A using `scipy.linalg.qr`. Verify R by hand using the Q matrix that was computed. Save your script as `problem3.py`.

$$A = \begin{bmatrix} 1 & 0 & 4 \\ -2 & 3 & -2 \\ -2 & 0 & 6 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/3 & 0.298 & 0.894 \\ 2/3 & -0.745 & 0 \\ 2/3 & 0.596 & 0.447 \end{bmatrix}$$

$$Q^T = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ -0.298 & -0.745 & 0.596 \\ 0.894 & 0 & 0.447 \end{bmatrix}$$

$$Q^T A = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ -0.298 & -0.745 & 0.596 \\ 0.894 & 0 & 0.447 \end{bmatrix} \begin{bmatrix} 1 & 0 & 4 \\ -2 & 3 & -2 \\ -2 & 0 & 6 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 2 & 4/3 \\ 0 & -2.235 & 3.874 \\ 0 & 0 & 6.258 \end{bmatrix} =$$

```
Value of r:
 [[-3.          2.          1.33333333]
 [ 0.         -2.23606798  3.87585116]
 [ 0.          0.          6.26099034]]
```

4. Find the least squares solution given A and $\mathbf{b}$.

$A^T A x = A^T b$

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{24} \begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} \frac{11}{24} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{1}{8} \end{bmatrix}$$

$$A^T B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

$$\begin{bmatrix} 11/24 & -\frac{1}{8} \\ -1/8 & 1/8 \end{bmatrix} \begin{bmatrix} 6 \\ 14 \end{bmatrix} =$$

$$\boxed{\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

5. Using the dataset `dataset1.txt`, available through Canvas, find the least squares solution using `np.linalg.lstsq`. You can load the data using `np.loadtxt`. After finding the least squares solution, plot the data and the solution using `matplotlib`.

Code in zip, plot not correctly implemented

6. Using the dataset `dataset2.txt`, available through Canvas, find the least squares solution using `np.linalg.lstsq`. You can load the data using `np.loadtxt`. After finding the least squares solution, plot the data and the solution using `matplotlib`.

Code in Zip, plot not correctly implemented.

7. Find a basis for the eigenspace corresponding to each listed eigenvalue.

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \lambda = 1, 2, 3$$

$\lambda = 1$

$$\begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \xrightarrow{RR} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda = 2$

$$\begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \xrightarrow{RR} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$\lambda = 3$

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

8. Use `np.linalg.eig` to calculate the eigenvalues and eigenvectors of the given matrix. Using `matplotlib`, plot the standard basis vectors, the vectors defined by the columns of A, and the calculated eigenvectors. Save your script as `problem8.py`.

Code in Zip

$$A = \begin{bmatrix} 1 & -2 \\ -4 & 1 \end{bmatrix}$$