1. Cosine similarity measures the similarity between two non-zero vectors using the dot product. It is defined as

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|}.$$

A result of -1 indicates the two vectors are exactly opposite, 0 indicates they are orthogonal, and 1 indicates they are the same.

(a) Write a function in Python that calculates the cosine similarity using only the dot product and norm functions.

(b) In Python, use your function to compute the cosine similarity of the following set of vectors.

$$1 \left\{ \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \right\}, 2 \left\{ \begin{bmatrix} 5 \\ 6 \\ -5 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -5 \end{bmatrix} \right\}, \text{ and } 3 \left\{ \begin{bmatrix} -3 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ -4 \\ 7 \end{bmatrix} \right\}$$

(c) Compute the cosine similarity by hand to compare the results.

1) $\dfrac{-6-3+4}{\sqrt{26} \cdot \sqrt{14}} = \dfrac{-5}{\sqrt{364}} = -0.26207$

2) $\dfrac{30+12+25}{\sqrt{86} \cdot \sqrt{65}} = \dfrac{67}{\sqrt{5590}} = 0.896125\ldots$

3) $\dfrac{-21-4+49}{\sqrt{59} \cdot \sqrt{114}} = \dfrac{24}{\sqrt{6726}} = 0.29263$

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e:\Download\A10>python problem1.py
The answer is:  -0.2620712091804796
The answer is:  0.8961256300885553
The answer is:  0.2926394084397884
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2. Two common loss functions in machine learning are the L1 loss, or Manhattan distance, and L2 loss, known as Euclidean distance. That is,

$$\mathcal{L}_1(\mathbf{u}, \mathbf{v}) = \Sigma_i |u_i - v_i| \text{ and } \mathcal{L}_2(\mathbf{u}, \mathbf{v}) = \Sigma_i (u_i - v_i)^2$$

The L2 loss is particularly sensitive to outliers, since the result is squared.

(a) Create a Python script that generates a plot of the L1 distance versus L2 distance. That is, the y-axis represents the output of each function and the x-axis is the vector u - v

(b) Create a function in Python that, given two vectors, computes the L1 and L2 loss. The loss should be printed out.

(c) To test the above function, randomly generate two 4×1 vectors and pass them as input to your function.

NOT ATTEMPTED

3. Create a function in Python that calculates the projection of one vector onto another using the definition

(a) Calculate the projection of $\mathbf{u} = \begin{bmatrix} 5 \\ 1 \\ 4 \\ -2 \end{bmatrix}$ onto $\mathbf{v} = \begin{bmatrix} 0 \\ 5 \\ 4 \\ 5 \end{bmatrix}$ by hand.

(b) Calculate the same projection using the function you wrote in Python.

a)

$$\frac{1}{6}\begin{bmatrix} 0 \\ 5 \\ 4 \\ 5 \end{bmatrix}$$

$$\frac{0 + 5 + 16 - 10}{0+25+16+25} = \frac{11}{66} = \frac{1}{6} \qquad \begin{bmatrix} 0 \\ 5/6 \\ 2/3 \\ 5/6 \end{bmatrix}$$

4. Find the closest point in the subspace spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ to the point $\mathbf{x}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 7 \\ 3 \\ 4 \end{bmatrix}, \text{ and } \mathbf{x} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$$

$$\frac{12+6+15}{9+36+25} \begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix} + \frac{28-3+12}{49+9+16} \begin{bmatrix} 7 \\ 3 \\ 4 \end{bmatrix}$$

$$\frac{33}{70} \begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix} + \frac{37}{74} \begin{bmatrix} 7 \\ 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} \frac{172}{35} \\ -\frac{93}{70} \\ \frac{61}{14} \end{bmatrix}$$

5. Find the best approximation to $\mathbf{z}$ by vectors of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, where

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix}, \text{ and } \mathbf{z} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

$$\frac{4-3}{4+1+9} \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix} + \frac{10-8+2}{25+4+16+4} \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix}$$

$$\frac{3.5}{49} \cancel{\frac{1}{14}} \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix} + \frac{4}{49} \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix}$$

$$\frac{1}{49}\left(\begin{bmatrix} 7 \\ 0 \\ -3.5 \\ -10.5 \end{bmatrix} + \begin{bmatrix} 20 \\ -8 \\ 16 \\ 8 \end{bmatrix}\right) = \frac{1}{49} \begin{bmatrix} 27 \\ -8 \\ 12.5 \\ -2.5 \end{bmatrix} = \begin{bmatrix} 27/49 \\ -8/49 \\ 25/98 \\ -5/98 \end{bmatrix}$$

6. Verify that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set, and then find the orthogonal projection of $\mathbf{y}$ onto Span$\{\mathbf{u}_1, \mathbf{u}_2\}$, where

$$\mathbf{u}_1 = \begin{bmatrix}3\\4\\0\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}-4\\3\\0\end{bmatrix}, \text{ and } \mathbf{y} = \begin{bmatrix}4\\3\\-2\end{bmatrix}$$

$3\cdot -4 + 3\cdot 4 + 0 = 0$

orthogonal

$$\frac{(4\cdot 3)+3\cdot 4+0}{3^2+4^2} U_1 + \frac{(-4\cdot 4)+(3\cdot 3)+0}{4^2+3^2} U_2$$

$$\frac{24}{25}\begin{bmatrix}3\\4\\0\end{bmatrix} - \frac{5}{25}\begin{bmatrix}-4\\3\\0\end{bmatrix}$$

$$\frac{1}{25}\begin{bmatrix}72\\96\\0\end{bmatrix} - \frac{1}{25}\begin{bmatrix}-20\\15\\0\end{bmatrix}$$

$$\frac{1}{25}\begin{bmatrix}92\\81\\0\end{bmatrix}$$

$$\begin{bmatrix}92/25\\81/25\\0\end{bmatrix}$$

7. Let $\mathbf{y} = \begin{bmatrix}7\\9\end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix}\frac{1}{\sqrt{10}}\\-\frac{3}{\sqrt{10}}\end{bmatrix}$, and $W = \text{Span}\{\mathbf{u}_1\}$.

(a) Let U be the 2×1 matrix whose only column is $\mathbf{u}_1$. Compute U^TU and UU^T.

(b) Compute $\text{proj}_W\mathbf{y}$ and $(UU^T)\mathbf{y}$.

a) $$\begin{bmatrix}\frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{10}}\\-\frac{3}{\sqrt{10}}\end{bmatrix} = \frac{1}{10}+\frac{9}{10} \qquad U^TU = 1$$

$$\begin{bmatrix}\frac{1}{\sqrt{10}}\\-\frac{3}{\sqrt{10}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}}\end{bmatrix} = UU^T = \begin{bmatrix}\frac{1}{10} & -\frac{3}{10}\\ -\frac{3}{10} & \frac{9}{10}\end{bmatrix}$$

b) $$\frac{\begin{bmatrix}7\\9\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{10}}\\-\frac{3}{\sqrt{10}}\end{bmatrix}}{\begin{bmatrix}\frac{1}{\sqrt{10}}\\-\frac{3}{\sqrt{10}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{10}}\\-\frac{3}{\sqrt{10}}\end{bmatrix}}\begin{bmatrix}\frac{1}{\sqrt{10}}\\-\frac{3}{\sqrt{10}}\end{bmatrix} = \frac{\frac{7}{\sqrt{10}} - \frac{27}{\sqrt{10}}}{\frac{1}{10}+\frac{9}{10}}\begin{bmatrix}\frac{1}{\sqrt{10}}\\-\frac{3}{\sqrt{10}}\end{bmatrix} = \frac{-20}{\sqrt{10}}\begin{bmatrix}\frac{1}{\sqrt{10}}\\-\frac{3}{\sqrt{10}}\end{bmatrix} = \begin{bmatrix}-2\\6\end{bmatrix} = \text{proj}_W\mathbf{y}$$

$$\begin{bmatrix}\frac{1}{10} & -\frac{3}{10}\\ -\frac{3}{10} & \frac{9}{10}\end{bmatrix}\begin{bmatrix}7\\9\end{bmatrix} = (UU^T)y = \begin{bmatrix}-2\\6\end{bmatrix}$$