

# Numerical Relativity Implementations

Created December 6, 2023; Latest January 10, 2024

RidiculeAmuser

## 0.1. Conventions

$$\gamma_{\mu\nu} \Leftrightarrow (-, +, +, +)$$

All indices are raised and lowered with  $\tilde{\gamma}_{ij}$  and  $\tilde{\gamma}^{ij}$  unless specified otherwise.

## 1. BSSN W-Formalism Equations

Field Magnitudes:  $W = \det(\gamma_{ij})^{-\frac{1}{6}}, \tilde{\gamma}_{ij} = W^2 \gamma_{ij} \Leftrightarrow \tilde{\gamma}^{ij} = W^{-2} \gamma^{ij}$

Conjugate Momenta:  $K = \gamma^{ij} K_{ij}, \tilde{A}_{ij} = W^2 (K_{ij} - \frac{1}{3} \gamma_{ij} K)$

Constraint Variables:  $\tilde{\Gamma}^i = \tilde{\Gamma}^i_{jk} \tilde{\gamma}^{jk}$

Constraint Equations:  $\det(\tilde{\gamma}_{ij}) \stackrel{\text{def}}{=} \mathcal{D} = 1, \tilde{\gamma}_{ij} \tilde{A}^{ij} \stackrel{\text{def}}{=} \mathcal{A} = 0, \tilde{\mathcal{G}}^i \stackrel{\text{def}}{=} \tilde{\Gamma}^i - \tilde{\Gamma}^i_{jk} \tilde{\gamma}^{jk} = 0$

Energy Momentum Constraints:

$$\mathcal{H} \stackrel{\text{def}}{=} \tilde{\mathcal{R}}_{ij} \tilde{\gamma}^{ij} + \frac{2}{3} K^2 - \tilde{A}^{ij} \tilde{A}_{ij} - 16\pi\rho - 2\Lambda = 0$$

$$\mathcal{M}_j \stackrel{\text{def}}{=} \tilde{D}^i \tilde{A}_{ij} - \frac{2}{3} \partial_j K - 3 \frac{\partial^i W}{W} \tilde{A}_{ij} - 8\pi J_j$$

Raw Equations:

$$\partial_t W = \beta^i \partial_i W + \frac{1}{3} W (\alpha K - \mathcal{B})$$

$$\partial_t \tilde{\gamma}_{ij} = \beta^m \partial_m \tilde{\gamma}_{ij} + 2\tilde{\gamma}_{m(i} \partial_{j)} \beta^m - \frac{2}{3} \tilde{\gamma}_{ij} \mathcal{B} - 2\alpha \tilde{A}_{ij}$$

$$\partial_t K = \beta^i \partial_i K - \tilde{\gamma}^{ij} \tilde{\mathcal{V}}_{ij} + \alpha \tilde{A}^{ij} \tilde{A}_{ij} + \frac{1}{3} \alpha K^2 + \alpha [4\pi(S + \rho) - \Lambda]$$

$$\partial_t \tilde{A}_{ij} = \beta^m \partial_m \tilde{A}_{ij} + 2\tilde{A}_{m(i} \partial_{j)} \beta^m - \frac{2}{3} \tilde{A}_{ij} \mathcal{B} + \alpha K \tilde{A}_{ij} - 2\alpha \tilde{A}_{im} \tilde{A}^m_j$$

$$+ [\alpha (\tilde{\mathcal{R}}_{ij} - 8\pi \tilde{S}_{ij}) - \tilde{\mathcal{V}}_{ij}]^{\text{TF}}$$

$$\partial_t \tilde{\Gamma}^i = \beta^m \partial_m \tilde{\Gamma}^i - \tilde{\Gamma}^m \partial_m \beta^i + \frac{2}{3} \tilde{\Gamma}^i \mathcal{B} + \partial^m \partial_m \beta^i + \frac{1}{3} \partial^i \mathcal{B}$$

$$- 2\tilde{A}^{im} \left[ 3\alpha \frac{\partial_m W}{W} + \partial_m \alpha \right] + 2\alpha \tilde{\Gamma}^i_{jk} \tilde{A}^{jk} - \frac{4}{3} \alpha \partial^i K - 16\alpha \pi \tilde{J}^i$$

### 1.1. Auxilliary Variables

$$\begin{aligned}
\tilde{\mathcal{V}}_{ij} &= W \left[ W \tilde{D}_i \tilde{D}_j \alpha + 2 \partial_{(i} W \partial_{j)} \alpha - \tilde{\gamma}_{ij} \tilde{\gamma}^{ab} \partial_a W \partial_b \alpha \right] \\
\Gamma^i_{jk} &= \tilde{\Gamma}^i_{jk} - W^{-1} \left( \delta^i_k \partial_j W + \delta^i_j \partial_k W - \tilde{\gamma}_{jk} \partial^i W \right) \\
\tilde{\mathcal{R}}^W_{ij} &= \tilde{\gamma}_{ij} \left[ W \tilde{D}^m \tilde{D}_m W - 2 \partial_m W \partial^m W \right] + W \tilde{D}_i \tilde{D}_j W \\
\tilde{\mathcal{R}}^C_{ij} &= W^2 \left( \tilde{\gamma}_{m(i} \partial_{j)} \tilde{\Gamma}^m + \tilde{\Gamma}^m \tilde{\Gamma}_{(ij)m} + \tilde{\gamma}^{mn} \left[ 2 \tilde{\Gamma}^k_{m(i} \tilde{\Gamma}_{j)kn} + \tilde{\Gamma}^k_{im} \tilde{\Gamma}_{kjn} \right] - \frac{1}{2} \partial^m \partial_m \tilde{\gamma}^{ij} \right) \\
\tilde{\mathcal{R}}_{ij} &= \tilde{\mathcal{R}}^W_{ij} + \tilde{\mathcal{R}}^C_{ij}, \quad \mathcal{B} = \partial_m \beta^m
\end{aligned}$$

### 1.2. Normal Vectors:

$$\begin{aligned}
n_\alpha &= (-\alpha, 0, 0, 0) = -\alpha (\text{dt})_\alpha \\
n^\alpha &= \frac{1}{\alpha} \left( (\partial_t)_\mu - \beta^\mu \right) = \left( \frac{1}{\alpha}, -\frac{\beta^x}{\alpha}, -\frac{\beta^y}{\alpha}, -\frac{\beta^z}{\alpha} \right) \\
\mathcal{P}^\alpha{}_\mu &= \delta^\alpha{}_\mu + n^\alpha n_\mu = \begin{pmatrix} 0 & \beta^x & \beta^y & \beta^z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

### 1.3. Tensorial Relations

$$\begin{aligned}
\tilde{\Gamma}^{ij}{}_j &= -\frac{1}{2} \frac{\partial^i \mathcal{D}}{\mathcal{D}} - \partial_j \tilde{\gamma}^{ij} \Rightarrow \tilde{\Gamma}^i = -\partial_j \tilde{\gamma}^{ij} \\
\tilde{\Gamma}^j{}_{ij} &= \frac{1}{2} \frac{\partial_i \mathcal{D}}{\mathcal{D}} := 0 \\
\tilde{\mathcal{R}}^W &= 4W \tilde{D}^m \tilde{D}_m W - 6 \partial_m W \partial^m W \\
\tilde{\mathcal{V}} &= W \left[ W \tilde{D}_i \tilde{D}^i \alpha - \partial_i W \partial^i \alpha \right] \\
\tilde{D}_i \mathcal{M}_j &= \tilde{D}_i \tilde{D}^k \tilde{A}_{kj} - \frac{2}{3} \tilde{D}_i \tilde{D}_j K - 3(\partial^k \ln W) \tilde{D}_i \tilde{A}_{kj} - 3 \tilde{A}_{kj} \tilde{D}_i \tilde{D}^k \ln W - 8\pi \tilde{D}_i J_j
\end{aligned}$$

## 2. Constraint Stabilizing Modifications

$$\partial_t \tilde{\Gamma}^i = (\partial_t \tilde{\Gamma}^i)^{\text{BSSN}} - \left( (1 + \kappa_1) \max(\lambda^i, 0) + \frac{2}{3} [\mathcal{B} - 2\alpha K] \right) \tilde{\mathcal{G}}^i$$

$$\lambda^i = \frac{2}{3} [\mathcal{B} - 2\alpha K] - \partial_i \beta^i - \frac{2}{5} \alpha \tilde{A}^i{}_{;i} \quad (\text{No summation})$$

$$\partial_t K = (\partial_t K)^{\text{BSSN}} - \underbrace{s_1 \alpha (\Delta h)^2 \tilde{D}^i \mathcal{M}_i}_{\text{Shown to interfere with other constraints.}} - \kappa_2 \alpha K \mathcal{H}$$

$$\partial_t \tilde{A}_{ij} = (\partial_t \tilde{A}_{ij})^{\text{BSSN}} + \kappa_3 \alpha [\mathcal{F}(\mathcal{M}_i, W, \dots)^{\text{TF}}]_{(ij)} + \kappa_7 \alpha \mathcal{H} \tilde{A}_{ij}$$

$$\text{Candidate: } \mathcal{F}(\mathcal{M}_i, W, \dots) = \tilde{D}_i \mathcal{M}_j + 3W^{-1} \mathcal{M}_i \partial_j W \quad \left( \text{Inspired by } \frac{\delta \|\mathcal{M}_i\|^2}{\delta \tilde{A}_{ij}} \right)$$

$$\partial_t \tilde{\gamma}_{ij} = (\partial_t \tilde{\gamma}_{ij})^{\text{BSSN}} + \kappa_4 \beta_{(i} \mathcal{G}_{j)} - \frac{1}{5} \tilde{\gamma}_{ij} \mathcal{G}^k \beta_k - \kappa_5 \tilde{D}_{(i} \mathcal{G}_{j)}$$

$$\partial_t W = (\partial_t W)^{\text{BSSN}} + \kappa_6 f(\mathcal{H}, \dots)$$

$$\Rightarrow \partial_t \mathcal{H} = (\partial_t \mathcal{H})^{\text{BSSN}} + 4\kappa_7 [f(\mathcal{H}, \dots) \tilde{D}^m \tilde{D}_m W + W \tilde{D}_m \tilde{D}^m f(\mathcal{H}, \dots) - 3\partial_m f(\mathcal{H}, \dots) \partial^m W]$$

$$\text{Candidate: } f(\mathcal{H}, \dots) = W \mathcal{H} - 5 \mathcal{H} \tilde{D}_i \tilde{D}^i W \quad \left( \text{Inspired by } \frac{\delta \mathcal{H}^2}{\delta W} \right)$$

**Comments on constraint damping:**

$\kappa_1$  : Essential for stability and physicality.

$\kappa_2$  : Reduces constraints but harms physicality and dynamics.

$\kappa_3$  : Essential for stability and physicality, unstable at strengths higher than 0.125 with  $\Delta h = 1$ . Dissipative.

$\kappa_4$  : Helpful for stability and constraint damping.

$\kappa_5$  : Helpful for stability and constraint damping for  $\mathcal{G}^i$ . Causes dissipative effects at high strength.

$\kappa_6$  : Very helpful to reduce  $\mathcal{H}$  constraints. Unstable at high strength.

$\kappa_7$  : Helpful for stability and damping  $\mathcal{H}$  constraints. Dynamics unknown.

## 2.1. Singularity Detection

$$\mathcal{S} := \max\left(0, 5\frac{\partial_i \partial^i W}{W}\right) > 1 \quad (\text{Near potential singularity})$$

## 2.2. Kreiss Oliger Dissipation

$$\partial_t T^{ijk\dots}_{abc\dots} = (\partial_t T^{ijk\dots}_{abc\dots})^{\text{BSSN}} + \sigma \frac{(\Delta h)^{2n-1}}{4} \left(-\frac{1}{4}\right)^{n-1} \Delta^n T^{ijk\dots}_{abc\dots}$$

**Note:** Dissipative and severely harms dynamics at higher strengths. Apply when necessary.

## 2.3. Boundary Conditions

At outgoing boundary, apply the following with appropriate *1-sided derivatives*.

$$\partial_t T^{ab\dots}_{cd\dots} = -\frac{v}{r} \left( r^i \partial_i T^{ab\dots}_{cd\dots} + T^{ab\dots}_{cd\dots} - (T^{ab\dots}_{cd\dots})_\infty \right)$$

## 2.4. Numerical Constraints

$$\tilde{\gamma}_{ij} := \tilde{\gamma}_{(ij)} \det(\tilde{\gamma})^{-\frac{1}{3}}, \quad \tilde{A}_{ij} := \tilde{A}_{(ij)} - \frac{1}{3} \gamma^{ij} \tilde{A}_{(ij)}$$

## 2.5. Gauge Conditions

1+Log Slicing Conditions:

$$\partial_t \alpha = \beta^i \partial_i \alpha - 2\alpha K, \quad [\alpha]_{t=0} = 1$$

$$\partial_t \beta^i = \kappa_8 \tilde{\Gamma}^i - N \beta^i, \quad [\beta^i]_{t=0} = 0$$

**Note:** Numerics depend heavily on  $\kappa_8$  and dynamics are strongly affected.

## 2.6. Non-zero Vacuum Energy

With a non-zero vacuum energy, the ambient, fields at asymptotic infinity evolve as follows:

$$\partial_t W_\infty = W_\infty \frac{\alpha_\infty K_\infty}{3}$$

$$\partial_t K_\infty = \alpha_\infty \left[ \frac{1}{3} K_\infty^2 - \Lambda \right]$$

$$\partial_t (\tilde{\gamma}_\infty)_{ij} = 0, \quad \partial_t (\tilde{A}_\infty)_{ij} = 0, \quad \partial_t \tilde{\Gamma}_\infty^i = 0$$

Using the 1+log slicing gauge condition:

$$\partial_t \alpha_\infty = -2\alpha_\infty K_\infty$$

$$\partial_t \beta_\infty^i = 0$$

### 3. Numerical Implementation

#### 3.1. Finite Difference Coefficients

Distance From Edge	Index 0	Index 1	Index 2	Index 3	Index 4	Index 5	Index 6
0L	$-\frac{49}{20}$	$\frac{6}{1}$	$-\frac{15}{2}$	$\frac{20}{3}$	$-\frac{15}{4}$	$\frac{6}{5}$	$-\frac{1}{6}$
1L	$-\frac{1}{6}$	$-\frac{77}{60}$	$\frac{5}{2}$	$-\frac{5}{3}$	$\frac{5}{6}$	$-\frac{1}{4}$	$\frac{1}{30}$
2L	$\frac{1}{30}$	$-\frac{2}{5}$	$-\frac{7}{12}$	$\frac{4}{3}$	$-\frac{1}{2}$	$\frac{2}{15}$	$-\frac{1}{60}$
Non-Boundary	$-\frac{1}{60}$	$\frac{3}{20}$	$-\frac{3}{4}$	$\frac{0}{1}$	$\frac{3}{4}$	$-\frac{3}{20}$	$\frac{1}{60}$
2R	$\frac{1}{60}$	$-\frac{2}{15}$	$\frac{1}{2}$	$-\frac{4}{3}$	$\frac{7}{12}$	$\frac{2}{5}$	$-\frac{1}{30}$
1R	$-\frac{1}{30}$	$\frac{1}{4}$	$-\frac{5}{6}$	$\frac{5}{3}$	$-\frac{5}{2}$	$\frac{77}{60}$	$\frac{1}{6}$
0R	$\frac{1}{6}$	$-\frac{6}{5}$	$\frac{15}{4}$	$-\frac{20}{3}$	$\frac{15}{2}$	$-\frac{6}{1}$	$\frac{49}{20}$

#### 3.2. Coordinate Grid

The following coordinate transforms are performed on cartesian axes to convert from domain space to physical space:

$$\mathcal{F}(i) = \frac{r\Delta h}{2} \tanh^{-1} \left( \frac{2i+1}{r} - 1 \right)$$

$$\mathcal{F}^{-1}(p) = \frac{r}{2} \tanh \left( \frac{2p}{r\Delta h} \right) - \frac{1}{2}$$

Where  $(r)$  is the resolution of the domain on one axis and  $(\Delta h)$  is the base size of a voxel. Given 2 binary black holes with masses 4.83 and positions  $\pm 32.51 \hat{x}$ , the following voxel sizes  $\Delta h$  are most appropriate for the resolutions listed below:

Resolution $(r)$	Voxel Size $(\Delta h)$	Relative Qual. (Voxels)	Internal Domain Size
100	$\geq 1.792$	$\leq -6.541$	$\geq 273.73$
125	$\geq 1.131$	$\leq 2.199$	$\geq 232.39$
150	$\geq 0.814$	$\leq 8.968$	$\geq 212.22$
175	$\geq 0.630$	$\leq 14.683$	$\geq 200.30$
200	$\geq 0.522$	$\leq 19.764$	$\geq 196.87$
225	$\geq 0.436$	$\leq 24.425$	$\geq 190.89$
$r \rightarrow \infty$	$\geq 82r^{-1}$	$\leq 0.1358r$	N.A.

### 3.3. Data Formatting and Structures

#### 3.3.1. Data Formats

Data Type	Implementation
float(N)	Standard 32 bit floating point precision format
f3Comp	float3 format, compressed into 4 bytes with shared 5 bit exponent.
f3x3Comp	float3x3 format, compressed into 12 bytes with 3 f3Comp row vectors.
Sf3x3Comp	Symmetric float3x3 format, f3Comp for diagonal and off-diagonal.
Sfloat3x3	Symmetric uncompressed float3x3 matrix, with a stride of 24 bytes.
CompressedChristoffel	Conformal Christoffel Symbols $\tilde{\Gamma}_{jk}^i$ , stored as Sf3x3Comp[3]

Immediate Voxel Data Structure:

Field Variable	Data Type	Stride (bytes)
$\tilde{\gamma}_{ij}$	Sfloat3x3	24
$\tilde{A}_{ij}$	Sfloat3x3	24
$\tilde{\Gamma}^i$	float3	12
$(\beta^i, \alpha)$	float4	16
$W$	float	4
$K$	float	4
Summary:	Voxel $\times$ 3	$84 \times 3 = 252$

Derivative Data Structure:

Field Variable	Data Type	Stride (bytes)
$\partial_k \tilde{\gamma}_{ij}$	Sf3x3Comp[3]	24
$\partial_k \tilde{A}_{ij}$	Sf3x3Comp[3]	24
$\partial_j \tilde{\Gamma}^i$	f3x3Comp	12
$\partial_j \beta^i$	f3x3Comp	12
$(\partial_i K, \partial_i \alpha, \partial_i W)$	f3x3Comp	12
$\tilde{\Gamma}_{jk}^i, \tilde{\Gamma}_{ijk}$	CompressedChristoffel	$24 \times 2 = 48$
Summary:	CompScalarVectorDerivs, CompTensorDerivs, CompressedChristoffel	132

Auxiliary Variables:

Field Variable	Data Type	Stride (bytes)
$S_{ij}$	Sf3x3Comp	8
$(J_i, \rho)$	(f3Comp, float)	8

### 3.4. Modifications in Numerical Treatment

For all field variables to be differentiated, for example  $\partial_k \tilde{\gamma}_{ij}, \partial_k \tilde{A}_{ij}$ , etc:

**Add random noise on the order of  $\sim 0.5\%$  of the derivative magnitude:**

The truncation error of the `f3Comp` and related formats are of  $0.5\%$  as they store 8 bit significands. As such, adding random noise on the order of the truncation error removes bias from the stored values.

**Apply Kreiss-Oliger Dissipation with strength on order of  $\sigma = 0.25$ .**

Kreiss Oliger Dissipation (K.O.) at  $6^{th}$  order with a strength of  $\sigma = 0.25$  effectively damps away high frequency noise and certain constraint violating modes.

*Apply K.O. dissipation over the entire domain, including at the boundaries; Use 1-sided derivatives at said boundaries.*

**Apply Modified Sommerfeld Boundary Conditions**

At the boundaries, apply the sommerfeld boundary conditions as described in (2.3).

The velocity of the sommerfeld radiation  $v$  is set to  $\alpha_\infty$ . However, the lie derivative of all fields are applied ( $\mathcal{L}_\beta$ ).

For example:

$$\partial_t \tilde{\gamma}_{ij} = \underbrace{\beta^m \partial_m \tilde{\gamma}_{ij} + 2\tilde{\gamma}_{m(i} \partial_{j)} \beta^m - \frac{2}{3} \tilde{\gamma}_{ij} \mathcal{B}}_{\mathcal{L}_\beta} - \frac{\alpha_\infty}{r} \left( r^k \partial_k \tilde{\gamma}_{ij} + \tilde{\gamma}_{ij} - \left( \tilde{\gamma}_{ij} \right)_\infty \right)$$

Exceptions are made to the following fields for non-zero vacuum energies:

$$\partial_t W = \underbrace{\beta^i \partial_i W - \frac{1}{3} W \mathcal{B}}_{\mathcal{L}_\beta} + \frac{1}{3} W \alpha_\infty K_\infty - \frac{\alpha_\infty}{r} (r^i \partial_i W + W - W_\infty)$$

$$\partial_t K = \underbrace{\beta^i \partial_i K}_{\mathcal{L}_\beta} + \frac{1}{3} \alpha_\infty [K_\infty]^2 - \alpha_\infty \Lambda - \frac{\alpha_\infty}{r} (r^i \partial_i K + K - K_\infty)$$

With the 1+log gauge slicing conditions:

$$\partial_t \alpha = \underbrace{\beta^i \partial_i \alpha}_{\mathcal{L}_\beta} - 2\alpha K_\infty - \frac{\alpha_\infty}{r} (r^i \partial_i \alpha + \alpha - \alpha_\infty)$$

## 4. Initial Conditions

### 4.1. Bowen-York Black Hole Initial Conditions

For a pure black hole initial state with masses  $m_b$ , momenta  $P_b$  and spin  $S_b$  (Indices raised and lowered with  $\delta_{ij}$ ):

$$n_b^i = \frac{r_b^i}{r_b}, \quad \widehat{K}^{ij} = 3 \sum_b \left( r_b^{-2} \left[ P_b^{(i} n_b^{j)} - \frac{1}{2} (\delta^{ij} - n_b^i n_b^j) (P_b)_k n_b^k \right] + r_b^{-3} n_b^{(i} \varepsilon^{j)kl} (S_b)_k (n_b)_l \right)$$

$$\xi = \sum_b \frac{m_b}{2r_b}, \quad \mathcal{H} \Rightarrow \nabla^2 u = -\frac{1}{8} \widehat{K}_{ij} \widehat{K}^{ij} (\xi + u)^{-7}$$

$$\lim_{r \rightarrow \infty} u = 1, \quad \lim_{r \rightarrow \infty} \widehat{K}^{ij} = 0, \quad \lim_{r \rightarrow \infty} \xi = 0$$

Field Initialization:

$$K_{ij} = W \widehat{K}_{ij}; \quad W = (\xi + u)^{-2}$$

$$\tilde{\gamma}_{ij} = \delta_{ij}, \quad \tilde{\Gamma}^i = 0, \quad \tilde{A}_{ij} = W^2 K_{ij}, \quad K = 0$$

$$\alpha = 1, \quad \beta^i = 0$$

### 4.2. Matter Simulations

Energy Momentum Tensor Projections in ADM ( $T_{\alpha\beta} \Rightarrow S_{\alpha\beta}; J_\alpha; \rho$ )

$$S_{\alpha\beta} = \mathcal{P}^\mu{}_\alpha T_{\mu\nu} \mathcal{P}^\nu{}_\beta \quad (\text{Spatial Stress Tensor})$$

$$J_\alpha = -T_{\mu\nu} \mathcal{P}^\mu{}_\alpha n^\nu \quad (\text{Momentum Density})$$

$$\rho = -T_{\mu\nu} n^\mu n^\nu \quad (\text{Energy Density})$$

$$\tilde{S}_{ij} = W^2 S_{ij}, \quad \tilde{J}^i = W^{-2} J^i$$

$$S = S_{\mu\nu} \gamma^{\mu\nu} = S_{ij} \gamma^{ij} = \tilde{S}_{ij} \tilde{\gamma}^{ij}$$



## 5. Diagnostic Equations

The total **ADM mass** of spacetime is:

$$E_{ADM} = \frac{1}{16\pi} \lim_{r \rightarrow \infty} \int_{\mathcal{S}} W^{-1} \left[ 4\partial^l \ln(W) + [\tilde{\Gamma}_{\text{Analytic}}]^l \right] dS_l$$

The total **ADM linear momentum** of spacetime is:

$$[P_{ADM}]_i = \frac{1}{8\pi} \lim_{r \rightarrow \infty} \int_{\mathcal{S}} W^{-3} \left[ \tilde{A}^j_m - \frac{2}{3} \delta^j_i K \right] dS_j$$

The total **ADM angular momentum** of spacetime is:

$$[J_{ADM}]_i = \frac{1}{8\pi} \varepsilon_{is}{}^m \lim_{r \rightarrow \infty} \int_{\mathcal{S}} x^s \left[ \tilde{A}^j_m - \frac{2}{3} \delta^j_m K \right] W^{-3} dS_j$$