Numerical Relativity Implementations

Created December 6, 2023; Latest January 13, 2024

RidiculeAmuser

0.1. Conventions

$$\gamma_{\mu\nu} \Leftrightarrow (-,+,+,+)$$

All indices are raised and lowered with $\tilde{\gamma}_{ij}$ and $\tilde{\gamma}^{ij}$ unless specified otherwise.

1. BSSN W-Formalism Equations

Field Magnitudes: $W = \det(\gamma_{ij})^{-\frac{1}{6}},\ \tilde{\gamma}_{ij} = W^2\gamma_{ij} \quad \Leftrightarrow \quad \tilde{\gamma}^{ij} = W^{-2}\gamma^{ij}$

Conjugate Momenta: $K = \gamma^{ij}K_{ij},\ \tilde{A}_{ij} = W^2\left(K_{ij} - \frac{1}{3}\gamma_{ij}K\right)$

Constraint Variables: $\tilde{\Gamma}^i = \tilde{\Gamma}^i{}_{jk}\tilde{\gamma}^{jk}$

Constraint Equations: $\det\left(\tilde{\gamma}_{ij}\right) \stackrel{\text{def}}{=} \boldsymbol{\mathcal{D}} = 1, \quad \tilde{\gamma}_{ij}\tilde{A}^{ij} \stackrel{\text{def}}{=} \boldsymbol{\mathcal{A}} = 0, \quad \tilde{\boldsymbol{\mathcal{G}}}^i \stackrel{\text{def}}{=} \tilde{\Gamma}^i - \tilde{\Gamma}^i{}_{jk}\tilde{\gamma}^{jk} = 0$

Energy Momentum Constraints:

$$\boxed{\begin{aligned} \boldsymbol{\mathcal{H}} &\stackrel{\text{def}}{=} \tilde{\boldsymbol{\mathcal{R}}}_{ij}\tilde{\gamma}^{ij} + \frac{2}{3}K^2 - \tilde{A}^{ij}\tilde{A}_{ij} - 16\pi\rho - 2\Lambda = 0 \\ \boldsymbol{\mathcal{M}}_j &\stackrel{\text{def}}{=} \tilde{D}^i\tilde{A}_{ij} - \frac{2}{3}\partial_j K - 3\frac{\partial^i W}{W}\tilde{A}_{ij} - 8\pi J_j = 0 \end{aligned}}$$

Raw Equations:

$$\boxed{\begin{gathered} \partial_t W = \beta^i\partial_i W + \frac{1}{3}W(\alpha K - \boldsymbol{\mathcal{B}}) \\ \partial_t\tilde{\gamma}_{ij} = \beta^m\partial_m\tilde{\gamma}_{ij} + 2\tilde{\gamma}_{m(i}\partial_{j)}\beta^m - \frac{2}{3}\tilde{\gamma}_{ij}\ \boldsymbol{\mathcal{B}} - 2\alpha\tilde{A}_{ij} \\ \partial_t K = \beta^i\partial_i K - \tilde{\gamma}^{ij}\tilde{\boldsymbol{\mathcal{V}}}_{ij} + \alpha\tilde{A}^{ij}\tilde{A}_{ij} + \frac{1}{3}\alpha K^2 + \alpha[4\pi(S+\rho) - \Lambda] \\ \partial_t\tilde{A}_{ij} = \beta^m\partial_m\tilde{A}_{ij} + 2\tilde{A}_{m(i}\partial_{j)}\beta^m - \frac{2}{3}\tilde{A}_{ij}\ \boldsymbol{\mathcal{B}} + \alpha K\tilde{A}_{ij} - 2\alpha\tilde{A}_{im}\tilde{A}^m{}_j \\ + \left[\alpha\left(\tilde{\boldsymbol{\mathcal{R}}}_{ij} - 8\pi\tilde{S}_{ij}\right) - \tilde{\boldsymbol{\mathcal{V}}}_{ij}\right]^{\text{TF}} \\ \partial_t\tilde{\Gamma}^i = \beta^m\partial_m\tilde{\Gamma}^i - \tilde{\Gamma}^m\partial_m\beta^i + \frac{2}{3}\tilde{\Gamma}^i\ \boldsymbol{\mathcal{B}} + \partial^m\partial_m\beta^i + \frac{1}{3}\partial^i\ \boldsymbol{\mathcal{B}} \\ -2\tilde{A}^{im}\left[3\alpha\frac{\partial_m W}{W} + \partial_m\alpha\right] + 2\alpha\tilde{\Gamma}^i{}_{jk}\tilde{A}^{jk} - \frac{4}{3}\alpha\partial^i K - 16\alpha\pi\tilde{J}^i \end{gathered}}$$

1.1. Auxilliary Variables

$$\tilde{\mathcal{V}}_{ij} = W\Big[W\tilde{D}_i\tilde{D}_j\alpha + 2\partial_{(i}W\partial_{j)}\alpha - \tilde{\gamma}_{ij}\tilde{\gamma}^{ab}\partial_a W\partial_b\alpha\Big]$$

$$\Gamma^i{}_{jk} = \tilde{\Gamma}^i{}_{jk} - W^{-1}\Big(\delta^i{}_k\partial_j W + \delta^i{}_j\partial_k W - \tilde{\gamma}_{jk}\partial^i W\Big)$$

$$\tilde{\mathcal{R}}^W{}_{ij} = \tilde{\gamma}_{ij}\Big[W\tilde{D}^m\tilde{D}_m W - 2\partial_m W\partial^m W\Big] + W\tilde{D}_i\tilde{D}_j W$$

$$\tilde{\mathcal{R}}^C{}_{ij} = W^2\bigg(\tilde{\gamma}_{m(i}\partial_{j)}\tilde{\Gamma}^m + \tilde{\Gamma}^m\tilde{\Gamma}_{(ij)m} + \tilde{\gamma}^{mn}\Big[2\tilde{\Gamma}^k{}_{m(i}\tilde{\Gamma}_{j)kn} + \tilde{\Gamma}^k{}_{im}\tilde{\Gamma}_{kjn}\Big] - \frac{1}{2}\partial^m\partial_m\tilde{\gamma}^{ij}\bigg)$$

$$\tilde{\mathcal{R}}_{ij} = \tilde{\mathcal{R}}^W{}_{ij} + \tilde{\mathcal{R}}^C{}_{ij}, \quad \mathcal{B} = \partial_m\beta^m$$

1.2. Normal Vectors:

$$n_\alpha = (-\alpha, 0, 0, 0) = -\alpha(\mathrm{d}t)_\alpha$$

$$n^\alpha = \frac{1}{\alpha}\Big((\partial_t)_\mu - \beta^\mu\Big) = \left(\frac{1}{\alpha}, -\frac{\beta^x}{\alpha}, -\frac{\beta^y}{\alpha}, -\frac{\beta^z}{\alpha}\right)$$

$$\mathcal{P}^\alpha{}_\mu = \delta^\alpha{}_\mu + n^\alpha n_\mu = \begin{pmatrix} 0 & \beta^x & \beta^y & \beta^z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

1.3. Tensorial Relations

$$\tilde{\Gamma}^{ij}{}_j = -\frac{1}{2}\frac{\partial^i\,\mathcal{D}}{\mathcal{D}} - \partial_j\tilde{\gamma}^{ij} \quad \Rightarrow \quad \tilde{\Gamma}^i = -\partial_j\tilde{\gamma}^{ij}$$

$$\tilde{\Gamma}^j{}_{ij} = \frac{1}{2}\frac{\partial_i\,\mathcal{D}}{\mathcal{D}} \coloneqq 0$$

$$\tilde{\mathcal{R}}^W = 4W\tilde{D}^m\tilde{D}_m W - 6\partial_m W\partial^m W$$

$$\tilde{\mathcal{V}} = W\Big[W\tilde{D}_i\tilde{D}^i\alpha - \partial_i W\partial^i\alpha\Big]$$

$$\tilde{D}_i\mathcal{M}_j = \tilde{D}_i\tilde{D}^k\tilde{A}_{kj} - \frac{2}{3}\tilde{D}_i\tilde{D}_j K - 3\big(\partial^k\ln W\big)\tilde{D}_i\tilde{A}_{kj} - 3\tilde{A}_{kj}\tilde{D}_i\tilde{D}^k\ln W - 8\pi\tilde{D}_i J_j$$

2. Constraint Stabilizing Modifications

$$\partial_t \tilde{\Gamma}^i = \left(\partial_t \tilde{\Gamma}^i\right)^{\mathrm{BSSN}} - \left((1+\kappa_1)\max(\lambda^i, 0) + \frac{2}{3}[\boldsymbol{\mathcal{B}} - 2\alpha K]\right)\tilde{\boldsymbol{\mathcal{G}}}^i$$

$$\lambda^i = \frac{2}{3}[\boldsymbol{\mathcal{B}} - 2\alpha K] - \partial_i \beta^i - \frac{2}{5}\alpha \tilde{A}^i{}_i \quad \text{(No summation)}$$

$$\partial_t K = (\partial_t K)^{\mathrm{BSSN}} - \underbrace{\cancel{s_1 \alpha (\Delta h)^2 \tilde{D}^i \boldsymbol{\mathcal{M}}_i}}_{\text{Shown to interfere with other constraints.}} - \kappa_2 \alpha K\, \boldsymbol{\mathcal{H}}$$

$$\partial_t \tilde{A}_{ij} = \left(\partial_t \tilde{A}_{ij}\right)^{\mathrm{BSSN}} + \kappa_3 \alpha \left[\boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{M}}_i, W, \ldots)^{\mathrm{TF}}\right]_{(ij)} + \kappa_7 \alpha\, \boldsymbol{\mathcal{H}}\, \tilde{A}^{ij}$$

$$\text{Candidate:} \quad \boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{M}}_i, W, \ldots) = \tilde{D}_i \boldsymbol{\mathcal{M}}_j + 3W^{-1}\boldsymbol{\mathcal{M}}_i \partial_j W \qquad \left(\text{Inspired by } \frac{\delta\, \|\boldsymbol{\mathcal{M}}_i\|^2}{\delta \tilde{A}_{ij}}\right)$$

$$\partial_t \tilde{\gamma}_{ij} = \left(\partial_t \tilde{\gamma}_{ij}\right)^{\mathrm{BSSN}} + \kappa_4 \beta_{(i} \boldsymbol{\mathcal{G}}_{j)} - \frac{1}{5}\tilde{\gamma}_{ij} \boldsymbol{\mathcal{G}}^k \beta_k - \kappa_5 \tilde{D}_{(i} \boldsymbol{\mathcal{G}}_{j)}$$

$$\partial_t W = (\partial_t W)^{\mathrm{BSSN}} + \kappa_6 f(\boldsymbol{\mathcal{H}}, \ldots)$$

$$\Rightarrow \partial_t\, \boldsymbol{\mathcal{H}} = (\partial_t\, \boldsymbol{\mathcal{H}})^{\mathrm{BSSN}} + 4\kappa_6 \left[f(\boldsymbol{\mathcal{H}}, \ldots)\tilde{D}^m \tilde{D}_m W + W \tilde{D}_m \tilde{D}^m f(\boldsymbol{\mathcal{H}}, \ldots) - 3\partial_m f(\boldsymbol{\mathcal{H}}, \ldots)\partial^m W\right]$$

$$\text{Candidate:} \qquad f(\boldsymbol{\mathcal{H}}, \ldots) = \boldsymbol{\mathcal{H}}\left[1 - \min\left(2, \frac{\tilde{D}_i \tilde{D}^i W}{W}\right)\right] \qquad \left(\text{Inspired by } \frac{\delta \boldsymbol{\mathcal{H}}^2}{\delta W}\right)$$

Comments on constraint damping:

κ_1 : Essential for stability and physicality.

κ_2 : Reduces constraints but harms physicality and dynamics.

κ_3 : Essential for stability and physicality, unstable at strengths higher than 0.125 with $\Delta h = 1$. Dissipative.

κ_4 : Helpful for stability and constraint damping.

κ_5 : Helpful for stability and constraint damping for $\boldsymbol{\mathcal{G}}^i$. Causes dissipative effects at high strength.

κ_6 : Very helpful to reduce $\boldsymbol{\mathcal{H}}$ constraints. Unstable at high strength.

κ_7 : Helpful for stability and damping $\boldsymbol{\mathcal{H}}$ constraints. Dynamics unknown.

2.1. Singularity Detection

$$\mathcal{S} := \max\left(0, 5\frac{\partial_i \partial^i W}{W}\right) > 1 \quad \text{(Near potential singularity)}$$

2.2. Kreiss Oliger Dissipation

$$\partial_t T^{ijk\dots}{}_{abc\dots} = \left(\partial_t T^{ijk\dots}{}_{abc\dots}\right)^{\mathrm{BSSN}} + \sigma \frac{(\Delta h)^{2n-1}}{4} \left(-\frac{1}{4}\right)^{n-1} \Delta^n \; T^{ijk\dots}{}_{abc\dots}$$

Note: Dissipative and severely harms dynamics at higher strengths. Apply when necessary.

2.3. Boundary Conditions

At outgoing boundary, apply the following with appropriate *1-sided derivatives.*

$$\partial_t T^{ab\dots}{}_{cd\dots} = -\frac{v}{r}\left(r^i \partial_i T^{ab\dots}{}_{cd\dots} + T^{ab\dots}{}_{cd\dots} - \left(T^{ab\dots}{}_{cd\dots}\right)_\infty\right)$$

2.4. Numerical Constraints

$$\tilde{\gamma}_{ij} := \tilde{\gamma}_{(ij)} \; \det(\tilde{\gamma})^{-\frac{1}{3}}, \quad \tilde{A}_{ij} := \tilde{A}_{(ij)} - \frac{1}{3}\gamma^{ij}\tilde{A}_{(ij)}$$

2.5. Gauge Conditions

1+Log Slicing Conditions:

$$\partial_t \alpha = \beta^i \partial_i \alpha - 2\alpha K, \quad [\alpha]_{t=0} = 1$$

$$\partial_t \beta^i = \kappa_8 \tilde{\Gamma}^i - N\beta^i, \quad \left[\beta^i\right]_{t=0} = 0$$

Note: Numerics depend heavily on κ_8 and dynamics are strongly affected.

2.6. Non-zero Vacuum Energy

With a non-zero vacuum energy, the ambient, fields at asymptotic infinity evolve as follows:

$$\partial_t W_\infty = W_\infty \frac{\alpha_\infty K_\infty}{3}$$

$$\partial_t K_\infty = \alpha_\infty \left[\frac{1}{3} {K_\infty}^2 - \Lambda\right]$$

$$\partial_t \left(\tilde{\gamma}_\infty\right)_{ij} = 0, \;\; \partial_t \left(\tilde{A}_\infty\right)_{ij} = 0, \;\; \partial_t \tilde{\Gamma}^i_\infty = 0$$

Using the 1+log slicing gauge condition:

$$\partial_t \alpha_\infty = -2\alpha_\infty K_\infty$$

$$\partial_t \beta^i_\infty = 0$$

3. Numerical Implementation

3.1. Finite Difference Coefficients

Distance From Edge	Index 0	Index 1	Index 2	Index 3	Index 4	Index 5	Index 6
0L	$-\frac{49}{20}$	$\frac{6}{1}$	$-\frac{15}{2}$	$\frac{20}{3}$	$-\frac{15}{4}$	$\frac{6}{5}$	$-\frac{1}{6}$
1L	$-\frac{1}{6}$	$-\frac{77}{60}$	$\frac{5}{2}$	$-\frac{5}{3}$	$\frac{5}{6}$	$-\frac{1}{4}$	$\frac{1}{30}$
2L	$\frac{1}{30}$	$-\frac{2}{5}$	$-\frac{7}{12}$	$\frac{4}{3}$	$-\frac{1}{2}$	$\frac{2}{15}$	$-\frac{1}{60}$
Non-Boundary	$-\frac{1}{60}$	$\frac{3}{20}$	$-\frac{3}{4}$	$\frac{0}{1}$	$\frac{3}{4}$	$-\frac{3}{20}$	$\frac{1}{60}$
2R	$\frac{1}{60}$	$-\frac{2}{15}$	$\frac{1}{2}$	$-\frac{4}{3}$	$\frac{7}{12}$	$\frac{2}{5}$	$-\frac{1}{30}$
1R	$-\frac{1}{30}$	$\frac{1}{4}$	$-\frac{5}{6}$	$\frac{5}{3}$	$-\frac{5}{2}$	$\frac{77}{60}$	$\frac{1}{6}$
0R	$\frac{1}{6}$	$-\frac{6}{5}$	$\frac{15}{4}$	$-\frac{20}{3}$	$\frac{15}{2}$	$-\frac{6}{1}$	$\frac{49}{20}$

3.2. Coordinate Grid

The following coordinate transforms are performed on cartesian axes to convert from domain space to physical space:

$$\mathcal{F}(i) = \frac{r\Delta h}{2}\tanh^{-1}\left(\frac{2i+1}{r} - 1\right)$$

$$\mathcal{F}^{-1}(p) = \frac{r}{2}\tanh\left(\frac{2p}{r\Delta h}\right) - \frac{1}{2}$$

Where (r) is the resolution of the domain on one axis and (Δh) is the base size of a voxel. Given 2 binary black holes with masses 4.83 and positions $\pm 32.51\ \hat{x}$, the following voxel sizes Δh are most appropriate for the resolutions listed below:

Resolution (r)	Voxel Size (Δh)	Relative Qual. (Voxels)	Internal Domain Size
100	≥ 1.792	≤ -6.541	≥ 273.73
125	≥ 1.131	≤ 2.199	≥ 232.39
150	≥ 0.814	≤ 8.968	≥ 212.22
175	≥ 0.630	≤ 14.683	≥ 200.30
200	≥ 0.522	≤ 19.764	≥ 196.87
225	≥ 0.436	≤ 24.425	≥ 190.89
$r \to \infty$	$\geq 82r^{-1}$	$\leq 0.1358r$	N.A.

3.3. Data Formatting and Structures

3.3.1. Data Formats

Data Type	Implementation
`float(N)`	Standard 32 `bit` floating point precision format
`f3Comp`	`float3` format, compressed into 4 `bytes` with shared 5 `bit` exponent.
`f3x3Comp`	`float3x3` format, compressed into 12 `bytes` with 3 `f3Comp` row vectors.
`Sf3x3Comp`	Symmetric `float3x3` format, `f3Comp` for diagonal and off-diagonal.
`Sfloat3x3`	Symmetric uncompressed `float3x3` matrix, with a stride of 24 `bytes`.
`CompressedChristoffel`	Conformal Christoffel Symbols $\tilde{\Gamma}^i{}_{jk}$, stored as `Sf3x3Comp[3]`

Immediate Voxel Data Structure:

Field Variable	Data Type	Stride (`bytes`)
$\tilde{\gamma}_{ij}$	`Sfloat3x3`	24
$\tilde{A}_{ij}$	`Sfloat3x3`	24
$\tilde{\Gamma}^i$	`float3`	12
(β^i, α)	`float4`	16
W	`float`	4
K	`float`	4
Summary:	`Voxel` $\times 3$	$84 \times 3 = 252$

Derivative Data Structure:

Field Variable	Data Type	Stride (`bytes`)
$\partial_k \tilde{\gamma}_{ij}$	`Sf3x3Comp[3]`	24
$\partial_k \tilde{A}_{ij}$	`Sf3x3Comp[3]`	24
$\partial_j \tilde{\Gamma}^i$	`f3x3Comp`	12
$\partial_j \beta^i$	`f3x3Comp`	12
$(\partial_i K, \partial_i \alpha, \partial_i W)$	`f3x3Comp`	12
$\tilde{\Gamma}^i{}_{jk},\ \tilde{\Gamma}_{ijk}$	`CompressedChristoffel`	$24 \times 2 = 48$
Summary:	`CompScalarVectorDerivs`, `CompTensorDerivs`, `CompressedChristoffel`	132

Auxiliary Variables:

Field Variable	Data Type	Stride (`bytes`)
S_{ij}	`Sf3x3Comp`	8
(J_i, ρ)	(`f3Comp`, `float`)	8

3.4. Modifications in Numerical Treatment

For all field variables to be differentiated, for example $\partial_k \tilde{\gamma}_{ij}, \partial_k \tilde{A}_{ij}$, etc:

Add random noise on the order of $\sim$ 0.5% of the derivative magnitude:
The truncation error of the `f3Comp` and related formats are of 0.5% as they store 8 `bit` significands. As such, adding random noise on the order of the truncation error removes bias from the stored values.

Apply Kreiss-Oliger Dissipation with strength on order of $\sigma = 0.25$.
Kreiss Oliger Dissipation (K.O.) at 6^{th} order with a strength of $\sigma = 0.25$ effectively damps away high frequency noise and certain constraint violating modes.
Apply K.O. dissipation over the entire domain, including at the boundaries; Use 1-sided derivatives at said boundaries.

Apply Modified Sommerfeld Boundary Conditions
At the boundaries, apply the sommerfeld boundary conditions as described in (2.3).
The velocity of the sommerfeld radiation v is set to α_∞. However, the lie derivative of all fields are applied $\left(\mathcal{L}_\beta\right)$.

For example:

$$\partial_t \tilde{\gamma}_{ij} = \underbrace{\beta^m \partial_m \tilde{\gamma}_{ij} + 2\tilde{\gamma}_{m(i}\partial_{j)}\beta^m - \frac{2}{3}\tilde{\gamma}_{ij}\ \mathcal{B}}_{\mathcal{L}_\beta} - \frac{\alpha_\infty}{r}\left(r^k \partial_k \tilde{\gamma}_{ij} + \tilde{\gamma}_{ij} - \left(\tilde{\gamma}_{ij}\right)_\infty\right)$$

Exceptions are made to the following fields for non-zero vacuum energies:

$$\partial_t W = \underbrace{\beta^i \partial_i W - \frac{1}{3} W\ \mathcal{B}}_{\mathcal{L}_\beta} + \frac{1}{3} W \alpha_\infty K_\infty - \frac{\alpha_\infty}{r}\left(r^i \partial_i W + W - W_\infty\right)$$

$$\partial_t K = \underbrace{\beta^i \partial_i K}_{\mathcal{L}_\beta} + \frac{1}{3}\alpha_\infty [K_\infty]^2 - \alpha_\infty \Lambda - \frac{\alpha_\infty}{r}\left(r^i \partial_i K + K - K_\infty\right)$$

With the 1+log gauge slicing conditions:

$$\partial_t \alpha = \underbrace{\beta^i \partial_i \alpha}_{\mathcal{L}_\beta} - 2\alpha K_\infty - \frac{\alpha_\infty}{r}\left(r^i \partial_i \alpha + \alpha - \alpha_\infty\right)$$

4. Initial Conditions

4.1. Bowen-York Black Hole Initial Conditions

For a pure black hole initial state with masses m_b, momenta P_b and spin S_b (Indices raised and lowered with δ_{ij}):

$$n_b^i = \frac{r_b^i}{\mathcal{r}_b},\ \ \widehat{K}^{ij} = 3\sum_b \left(\mathcal{r}_b^{-2} \left[P_b^{(i} n_b^{j)} - \frac{1}{2}\left(\delta^{ij} - n_b^i n_b^j\right) (P_b)_k n_b^k \right] + \mathcal{r}_b^{-3} n_b^{(i} \varepsilon^{j)kl} (S_b)_k (n_b)_l \right)$$

$$\xi = \sum_b \frac{m_b}{2\mathcal{r}_b}, \quad \mathcal{H} \Rightarrow\ \nabla^2 u = -\frac{1}{8} \widehat{K}_{ij} \widehat{K}^{ij} (\xi + u + 1)^{-7}$$

$$\lim_{r \to \infty} u = 0, \quad \lim_{r \to \infty} \widehat{K}^{ij} = 0, \quad \lim_{r \to \infty} \xi = 0$$

Field Initialization:

$$K_{ij} = W \widehat{K}_{ij}; \quad W = (\xi + u + 1)^{-2}$$

$$\tilde{\gamma}_{ij} = \delta_{ij},\ \tilde{\Gamma}^i = 0,\ \tilde{A}_{ij} = W^2 K_{ij},\ K = 0$$

$$\alpha = 1,\ \beta^i = 0$$

4.2. Matter Simulations

Energy Momentum Tensor Projections in ADM ($T_{\alpha\beta}\ \Rightarrow\ S_{\alpha\beta};\ J_\alpha;\ \rho$)

$$S_{\alpha\beta} = \mathcal{P}^{\mu}{}_{\alpha} T_{\mu\nu} \mathcal{P}^{\nu}{}_{\beta} \qquad \text{(Spatial Stress Tensor)}$$

$$J_\alpha = -T_{\mu\nu} \mathcal{P}^{\mu}{}_{\alpha} n^\nu \qquad \text{(Momentum Density)}$$

$$\rho = -T_{\mu\nu} n^\mu n^\nu \qquad \text{(Energy Density)}$$

$$\tilde{S}_{ij} = W^2 S_{ij},\ \tilde{J}^i = W^{-2} J^i$$

$$S = S_{\mu\nu} \gamma^{\mu\nu} = S_{ij} \gamma^{ij} = \tilde{S}_{ij} \tilde{\gamma}^{ij}$$

4.3. Elliptic Equation Solver: Multi-grid Relaxation

Method:

1. Values of ξ and $\widehat{K}_{ij}\widehat{K}^{ij}$ are computed with the Bowen-York initial data at full resolution.
2. An initial guess for u is computed:

$$u = \sum_b \frac{\sqrt{|S_b| + P_b^i (P_b)_i + m_b^2}}{2 r_b} - \xi$$

3. ξ, u and $\widehat{K}_{ij}\widehat{K}^{ij}$ are downsampled and averaged to specified resolutions given in the editor.
4. u is solved with ξ and $\widehat{K}_{ij}\widehat{K}^{ij}$ at the lowest resolution with the ***method of relaxation***, while maintaining outgoing boundary conditions.
5. Once convergence has been attained, u is upscaled to the next higher resolution.
6. Repeat steps 4 and 5 until the original resolution has been attained.

Analysis:

The naive ***method of relaxation*** fails to converge in a reasonable amount of time for high resolutions due to its poor ability to correct for lower frequency errors while remaining stable for high frequency modes.

A multi-grid approach speeds up convergence by solving for lower frequency errors with a much larger relaxation factor than otherwise possible, and solving higher frequencies last.

In the Fourier basis (r represents the relaxation factor):

$$\Delta u = \left(\underbrace{\mathcal{F}\left[-\frac{1}{8} \widehat{K}_{ij}\widehat{K}^{ij} (\xi + u + 1)^{-7} \right] (\omega^i)}_{(1)} - \underbrace{\omega_i \omega^i u}_{(2)} \right) r$$

Whose iteration Δu is unstable when the following condition is true.

$$\left| \frac{\partial(\Delta u)}{\partial u} \right| > 2$$

The term (1) does not contributeZ significantly to the condition while (2) explodes for high frequencies. Specifically,

$$\omega_i \omega^i > \frac{2}{r} \quad \Rightarrow \quad \text{(Guaranteed) Instability}$$

By limiting the resolution, one limits ω^i such that r can be made much larger at each step.

5. Diagnostic Equations

The total **ADM mass** of spacetime is:

$$E_{ADM} = \frac{1}{16\pi} \lim_{r\to\infty} \int_{\mathcal{S}} W^{-1} \left[4\partial^l \ln(W) + \left[\tilde{\Gamma}_{\text{Analytic}} \right]^l \right] \mathrm{d}S_l$$

The total **ADM linear momentum** of spacetime is:

$$\left[P_{ADM} \right]_i = \frac{1}{8\pi} \lim_{r\to\infty} \int_{\mathcal{S}} W^{-3} \left[\tilde{A}^j{}_m - \frac{2}{3} \delta^j{}_i K \right] \mathrm{d}S_j$$

The total **ADM angular momentum** of spacetime is:

$$\left[J_{ADM} \right]_i = \frac{1}{8\pi} \varepsilon_{is}{}^m \lim_{r\to\infty} \int_{\mathcal{S}} x^s \left[\tilde{A}^j{}_m - \frac{2}{3} \delta^j{}_m K \right] W^{-3} \,\mathrm{d}S_j$$