Constructing Composite Stellar Profiles

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April 6, 2015

1 Introduction

If you have a set of profiles for number of stars, with each profile covering the same range of radii, deriving a composite profile is simple; simply add together all the profiles, without any clever weights (i.e. add together all the photons from all the stars). If some of the stars have missing points on their profiles, as is the case if some are saturated, things are not so straightforward, but a simple procedure is derived in the next section.

In determining stellar power-law wings such a procedure is required; you need to know the PSF's core shape (and magnitude), while only for saturated stars is the wing shape well determined, and reasonably insensitive to the sky level.

2 Theory

Let us assume that we have a set of N radial profiles of stars, each of which has n points; write the i^{th} point on the j^{th} profile as P_i^j. If all the profiles are identical except for a scale factor α^j, we have

$$P_i^j = \alpha^j f_i + \epsilon_{ij}$$

with $\langle \epsilon_{ij} \rangle = 0$ and $\langle \epsilon_{ij}^2 \rangle = \sigma_{ij}^2$; we assume that the ϵ_{ij} for different i and j are independent.

Taking a logarithm on each side, this becomes [1]

$$\begin{aligned}\ln P_i^j &= \ln \alpha^j + \ln f_i + \ln\left(1 + \frac{\epsilon_{ij}}{\alpha^j f_i}\right) \\ &\sim \ln \alpha^j + \ln f_i + \frac{\epsilon_{ij}}{\alpha^j f_i}\end{aligned}$$

If we write the error term as ϵ_{ij}/P_i^j this is a linear problem, which can be written in canonical form $\mathbf{y} = M\boldsymbol{\theta} + \boldsymbol{\epsilon}$, where $\mathbf{y}$ represents the vector of the P_i^js, M a matrix of coefficients, $\boldsymbol{\theta}$ a vector of $\ln\alpha$ and $\ln f$, and $\boldsymbol{\epsilon}$ a vector of errors. Specifically,

$$\begin{pmatrix} \ln P_1^1 \\ \ln P_2^1 \\ \vdots \\ \ln P_1^2 \\ \ln P_2^2 \\ \vdots \\ \ln P_{n-1}^N \\ \ln P_n^N \end{pmatrix} = \left(\begin{array}{cccc|cccc} 1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline 1 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 1 \end{array}\right) \left(\begin{array}{c} \ln f_1 \\ \ln f_2 \\ \vdots \\ \ln f_n \\ \hline \ln \alpha_1 \\ \ln \alpha_2 \\ \vdots \\ \ln \alpha_N \end{array}\right) + \begin{pmatrix} \epsilon_{11}/P_1^1 \\ \epsilon_{21}/P_2^1 \\ \vdots \\ \epsilon_{12}/P_1^2 \\ \epsilon_{22}/P_2^2 \\ \vdots \\ \epsilon_{n-1,N}/P_{n-1}^N \\ \epsilon_{nN}/P_n^N \end{pmatrix}$$

Let us write $W_{ij} \equiv \left(P_i^j/\sigma_{ij}\right)^2$, so that $\boldsymbol{\epsilon}$'s covariance matrix V is simply

$$V = \begin{pmatrix} 1/W_{11} & 0 & 0 & \dots \\ 0 & 1/W_{21} & 0 & \dots \\ 0 & 0 & 1/W_{31} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The minimum-least-squares estimate for the $\ln f_i$ and $\ln \alpha^j$ is given by

[1] We have of course introduced a small bias by this procedure. If x has mean $\langle x \rangle \equiv \mu$ and variance σ^2, then $\langle \ln x \rangle \approx \ln \mu - \sigma^2/(2\mu^2)$

$$\left(M^TV^{-1}M\right)^{-1}M^TV^{-1}\mathbf{y} =$$

$$\left(\begin{array}{cccc|cccc}
\sum_j W_{1j} & 0 & \dots & 0 & W_{11} & W_{12} & \dots & W_{1N} \\
0 & \sum_j W_{2j} & \dots & 0 & W_{21} & W_{22} & \dots & W_{2N} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & \sum_j W_{nj} & W_{n1} & W_{n2} & \dots & W_{nN} \\
\hline
W_{11} & W_{21} & \dots & W_{n1} & \sum_i W_{i1} & 0 & \dots & 0 \\
W_{12} & W_{22} & \dots & W_{n2} & 0 & \sum_i W_{i2} & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
W_{1N} & W_{2N} & \dots & W_{nN} & 0 & 0 & \dots & \sum_i W_{iN}
\end{array}\right)^{-1}
\left(\begin{array}{c}
\sum_j W_{1j}\ln P_1^j \\
\sum_j W_{2j}\ln P_2^j \\
\vdots \\
\sum_j W_{nj}\ln P_n^j \\
\hline
\sum_i W_{i1}\ln P_i^1 \\
\sum_i W_{i2}\ln P_i^2 \\
\vdots \\
\sum_i W_{iN}\ln P_i^N
\end{array}\right)$$

This matrix is singular, as one of the α^js simply sets the scale for the composite profile f_i; I have chosen to solve this problem by setting $f_1 = 1$ and deleting the first row of M and consequently the first row and column from $M^TV^{-1}M$ (equivalently, you could set $\alpha_0 = 1$ and eliminate *its* row and column — but this would mean that all the uncertainty in scaling the first profile to the composite appears in the composite's error bars).

If a point in the profile is missing, you can simply set the corresponding W_{ij} to zero; providing that each pair of neighbouring points is represented in at least one of the input profiles, and that each profile contains at least two usable points, the matrix should not be singular.

The covariance matrix for the $\ln f_i$ and $\ln\alpha^j$ is given by $\left(M^TV^{-1}M\right)^{-1}$

3 Example from Photo

A simulated r' frame was run through photo's correct frames and bright object finder, resulting in a list of 11 objects with more than 100000 counts, or about 16th magnitude. Their profiles are shown in the figure, along with the derived composite profile. Note that this composite is *not* a Gaussian-plus-powerlaw fit to the data, it is simply the f_i derived as in the previous section, scaled by the α^j.

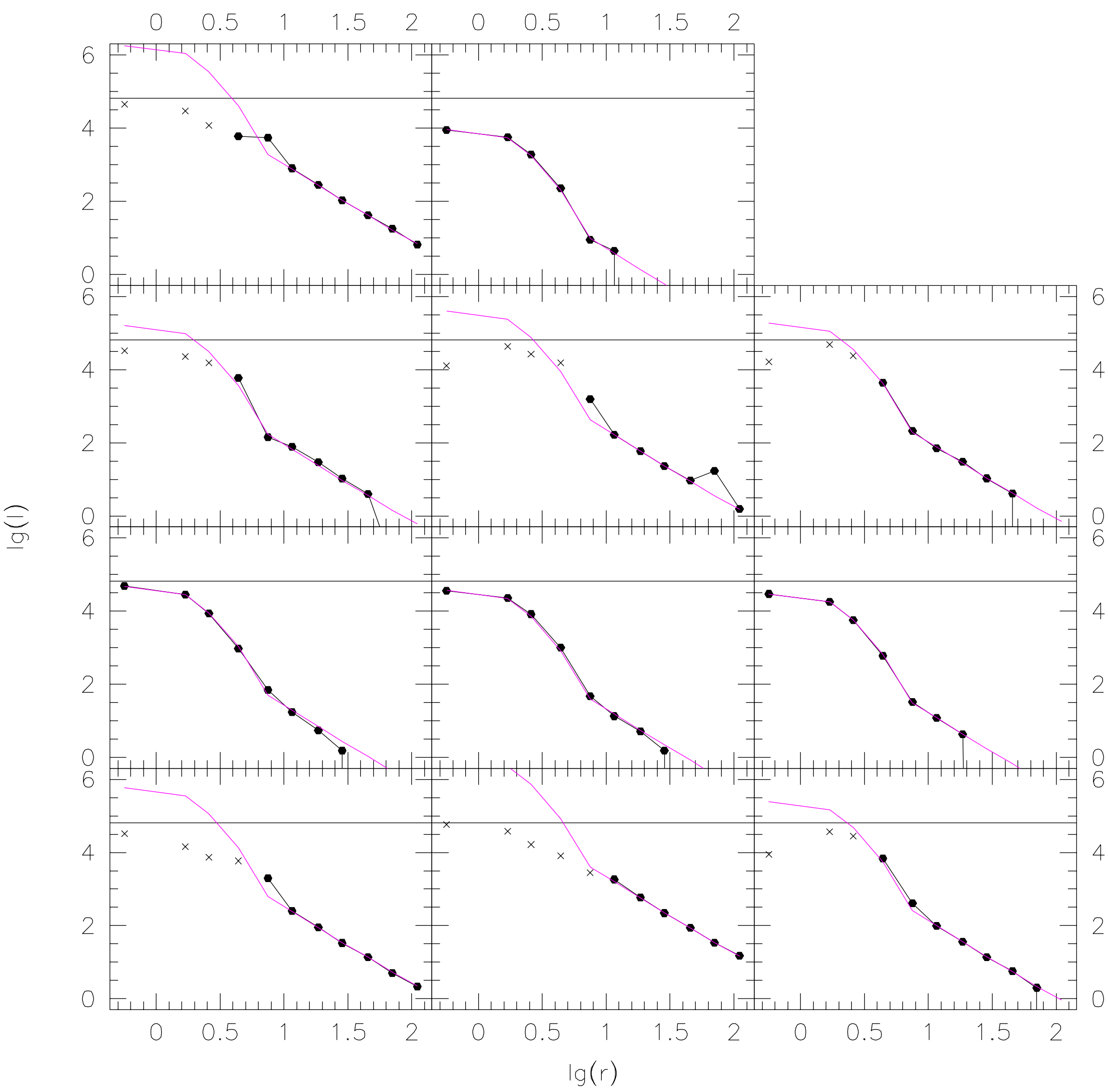

Figure 1: The radial profiles of 11 stars from a simulated photo frame. For each star, the solid points are those used to derive the composite, while the crosses mark points that may have been contaminated by saturation. The magenta line indicates the derived composite profile scaled to the data; the horizontal black line is the saturation level of the chip.