Optimal Submarine Cable Path Planning and Trunk-and-Branch Tree Network Topology Design

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***Abstract*— We study the path planning of submarine cable systems with trunk-and-branch tree topology on the surface of the earth. Existing work on path planning represents the earth's surface by triangulated manifolds and takes account of laying cost of the cable including material, labor, alternative protection levels, terrain slope and survivability of the cable. Survivability issues include the risk of future cable break associated with laying the cable through sensitive and risky areas, such as, in particular, earthquake-prone regions. The key novelty of this paper is an examination and solution of the path planning of submarine cable systems with trunk-and-branch tree topology. We formulate the problem as a Steiner minimal tree problem on irregular 2D manifolds in $\mathbb{R}^3$. For a given Steiner topology, we propose a polynomial time computational complexity numerical method based on the dynamic programming principle. If the topology is unknown, a branch and bound algorithm is adopted. Simulations are performed on real-world three-dimensional geographical data.**

***Index Terms*— Optical fiber cables, path optimization, Steiner minimal tree, cost effectiveness, seismic resilience, manifolds.**

I. Introduction

As A critical part of information communications technology infrastructure, submarine cables play a crucial role in the transport of information; they carry over 99% of global voice and data traffic [1]. Operational submarine cable networks, traditionally designed with a 25 years lifespan [2], have reached 1.2 million kilometers [3]. Driven by expected technological advances and growth in Internet services (including cloud services and 5G), data bandwidth demand is expected to double every two years for the foreseeable future; the growth of submarine cables must follow [1]. The construction cost of submarine cables is approximately 24,000 USD per kilometer in today's market. Although this estimate varies depending on various factors, it is clear that a long-haul submarine cable over thousands of kilometers, will cost tens or even hundreds of millions of dollars. Path planning, a procedure of path design for laying the cables, is an important part of constructing a submarine cable system. Commonly, path planning of submarine cables over thousands of kilometers has been done manually, meter by meter, based on the knowledge and experience of experts [4], [5].

One important issue that has to be considered by cable surveyors is how to take earthquake prone-areas into account in the path planning process. In 2006, eight submarine cables were damaged by the Hengchun earthquake, causing a total of 18 cable cuts [6]. This resulted in severe disruptions for several weeks to the Internet services in several Asian regions including China, Hong Kong, Taiwan, and the Philippines [6]. Additionally, human activities such as fishing, anchoring and resource exploration, can incur damage to submarine cables [4].

Submarine cable networks are optical networks where nodes are connected by undersea cables. Submarine cable networks can be designed in a range of topologies and architectures. These include point-to-point, ring, mesh, trunk-and-branch and festoon architectures [7]. See Fig. 1 for several variants of trunk-and-branch architecture [8]. In the present paper we restrict ourselves to trunk-and-branch architectures that have tree topologies. The design choice depends on the connectivity and traffic requirements between the various sites, possibly separated by distances of several thousands of kilometers [9]. Point-to-point and trunk-and-branch tree are the most commonly used architectures (topologies) in submarine cable systems. Note that path planning of submarine cable systems with ring and festoon topologies can be reduced to multiple independent point-to-point cable path planning problems. According to the Submarine Cable Almanac of 2016 [10], of all 258 submarine cable systems in the world, 172 cable systems use point-to-point, ring and festoon topologies, and 76 cable systems use trunk-and-branch tree topologies. In support of our comments on the popularity of trunk-and-branch tree topologies in practice [7], we cite the following examples:

- Asia America Gateway (AAG) cable system that connects South-East Asia with the mainland of the United States.
- African Coast to Europe (ACE) cable system that connects 23 countries.

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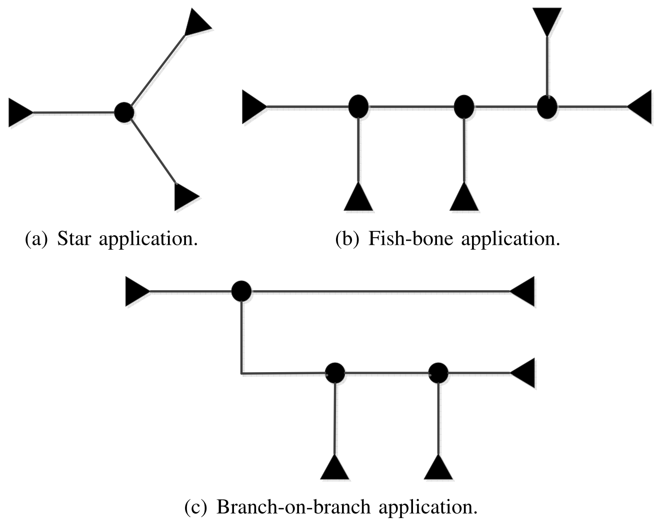

Fig. 1. Variants of trunk-and-branch tree architecture [8]. Triangles represent terminals and circles represent branching units.

- Asia Submarine-cable Express (ASE) cable system that connects Japan, Philippines, Hong Kong, Malaysia, Singapore and Cambodia.

In the present paper, we go a step beyond point-to-point path planning to consider submarine cable path planning for trunk-and-branch tree network topology design on the surface of the earth. A branching unit is a piece of equipment used for divaricating a submarine cable. Such divarication is needed if a cable is required to serve multiple destinations including, for example, a landing site [8]. To take account of large scale real landforms with mountains and valleys, etc., we formulate a new optimal path design problem for submarine cable systems with trunk-and-branch tree topology on the surface of the earth as a *Steiner Minimal Tree* (SMT) problem on an irregular two-dimensional (2D) manifold in three-dimensional (3D) Euclidean space. As in [11]–[14], we model the surface of the earth by triangulated manifolds and use the number of repairs as the risk index for earthquakes. We then use a weighted sum cost to take both the laying cost of the cable system and the number of repairs into account. Note that the existing SMT algorithms for problems in a plane involving homogeneous cost, e.g., the GeoSteiner algorithm [15], cannot be directly applied to a problem in the plane involving non-homogeneous cost, nor to an irregular 2D manifold in 3D Euclidean space. As pointed out in [16], [17], the success of the existing SMT algorithms in the plane is largely attributable to the angle condition for Steiner points in the plane. Namely, any two edges that intersect at a Steiner point meet at an angle of $120°$, see Section IV for details. This condition implies that for any edge in a full Steiner tree in a plane, we can determine its orientation by the orientation of any edge of the tree. This fact no longer holds for the case of a general Steiner tree in higher dimensional Euclidean space ($\mathbb{R}^d, d \geq 3$) [17]. It fails too for a problem in the plane involving non-homogeneous cost, and for an irregular 2D manifold in 3D Euclidean space, since the least cost path connecting any two points is a geodesic, i.e., a curve and not necessarily a straight line. The numerical optimization method of Smith's *branch-and-bound* (B&B) algorithm [17], which is an iterative numerical approach, finds the *relatively minimal tree* (RMT) for a given Steiner topology in $\mathbb{R}^d, d \geq 2$ by interpreting a Steiner tree as a mechanical system that consists of ideal springs. As it relies on metrics and vector operations in Euclidean space, this numerical optimization method also cannot be directly applied to a problem in the plane involving non-homogeneous cost, nor to an irregular 2D manifold in 3D Euclidean space. Methods using a mathematical programming formulation [18], [19], also fail for the same reason. The main technical advances and contributions of this work beyond [11]–[14] are as follows.

1) Based on the earth's surface and the cable repair models, we approach the optimal path design problem for submarine cable systems with trunk-and-branch tree topology on the surface of the earth as an SMT problem on irregular 2D manifold in a 3D Euclidean space. To our best knowledge, this problem has not been considered in the literature.
2) For the first time, we propose a polynomial time complexity numerical method, based on dynamic programming, to find the RMT on an irregular 2D manifold in a 3D Euclidean space for a given topology; this method is of independent interest. Several techniques are suggested to further reduce computational cost. Using this method, the location of the branching units and the geodesics connecting the branching units/terminals with the minimum total cost of the cable network are determined for the given topology.
3) We apply the B&B algorithm of Smith [17] to optimize over Steiner topologies. Applications with 2D and 3D real landforms have demonstrated that the proposed method can achieve optimal submarine cable path planning and trunk-and-branch tree network topology design.

The remainder of the paper is organized as follows. The models of the surface of the earth, laying cost and cable repair are introduced in Section III. There we formulate optimal path design for submarine cable systems with trunk-and-branch tree topology as an SMT problem on an irregular 2D manifold in a 3D Euclidean space. In Section IV, we first use our algorithm for a given Steiner topology and then provide computational complexity analysis and methods of computational cost reduction. The B&B algorithm of Smith [17] is incorporated in Section IV for the case of unknown Steiner topologies. We apply our proposed algorithm to real-world 3D data, and present the corresponding results in Section V. We conclude the paper in Section VI.

II. Related Work

Existing work on avoidance of or path planning through earthquake-prone areas has mainly focused on the case of point-to-point cable path planning. In [20], [21], a resilience-aware cable path design was proposed, but was limited to cables on a 2D plane. In [22], a path analysis based on a raster graphic approach was described, with the aim of obtaining the least cumulative cost path. Optimization, in this case, used Dijkstra's algorithm with objectives to minimize cost and potential cable failure from earthquakes. One weakness of the raster-based approach is that a path is limited to only lateral or diagonal links when moving from one cell to an adjacent cell. In [11], an approach based on the fast marching method (FMM) [23], [24] was used to solve the multi-objective path planning problem on the earth's surface. In addition to earthquake prone areas, a range of constraints, such as anchorage areas and fisheries, areas with wind or underwater turbines and so on, were considered in [11]. The superiority of FMM over Dijkstra's algorithm in terms of solution quality was demonstrated in [25]. In [26], both path planning and choice of shielding levels were optimized along the path. To this end, a multi-objective optimization problem on a graph was formulated, where the two conflicting objectives were the laying cost and a measure of the likelihood of cable failures. This multi-objective optimization problem was solved by a variant of the label setting algorithm. A heuristic label-setting algorithm based on interval partitions was also invoked to reduce runtime; this is especially important for long distance cables. In [12], the same problem was considered and an FMM-based method was presented that is applicable to problems of realistic size. In [14], risk reduction for capsize of the remotely operated cable laying vehicle during cable burying operations in uneven terrains was included in the laying cost model. This instability risk depends on both the direction of the path and the slope of the terrain. A computationally efficient algorithm based on the ordered upwind method was used to solve the problem. In short, the work of [11], [12], [14], [20]–[22], [26] focused only on point-to-point cable path planning.

Among the significant literature on path selection of cable networks, we note that Sebastian *et al.* [27] sought to identify vulnerable points in geographic networks by determining the location of the worst-case failure based on certain network connectivity measures. Saito [28] considered a planar physical cable network for which a node/path replacement strategy was provided. Tran and Saito [29] proposed a method to obtain a set of geographical routes from a set of candidate routes subject to a cost constraint. Then, in [30], they presented a dynamic programming approach to add new links and routes to a given network. Msongaleli *et al.* [31] used Integer Linear Programming to select a path from a set of submarine cables. Zhang *et al.* [32] addressed the problem of selecting cables that require shielding in an existing network. In [13], a summary life-cycle cost model of submarine cables depending on multiple design considerations was proposed to solve the path planning problem connecting a given site to an existing cable network. None of these papers considered path optimization for cable systems with trunk-and-branch tree topology.

Dating back to the Fermat-Torricelli problem in the early 17th century, the SMT problem in Euclidean space is a classical NP-hard geometric combinatorial optimization problem, for which many authors have provided exact solutions. Gilbert and Pollak [33] firstly proposed an algorithm to the general SMT problem in d-dimensional Euclidean space. In their algorithm, the first step is to enumerate all Steiner topologies and then calculate the RMT corresponding to each topology. Since the number of Steiner topologies grows super-exponentially with the number of terminals, the algorithm in [33] can only be applied to very small instances of the SMT problem. Using the fact that an SMT in the plane can be decomposed into smaller so-called *full Steiner trees*, the *GeoSteiner* algorithm [15] solves typical SMT problems with thousands of terminals in Euclidean plane. The GeoSteiner algorithm uses geometrical properties (including the node degree and the angle conditions) to discover the nonexistence or the nonoptimality of a full Steiner tree for a given full topology. The planar *full Steiner topology* (FST) concatenation heuristic of the GeoSteiner algorithm heavily utilize planar geometry, and these do not readily generalize to higher dimensions [19], [34] and irregular 2D manifolds in $\mathbb{R}^3$. Using B&B, Smith [17] proposed an implicit enumeration scheme to generate full Steiner topologies that can be applied to the SMT problem in d-dimensional Euclidean space. Smith's B&B algorithm [17] consists of the tree generation step and the numerical optimization step. In the numerical optimization step, a system of linear equations is obtained from an interpretation of a Steiner tree as a mechanical system; an iterative method is used to find the Steiner points of the Steiner tree for the given Steiner topology. The metric in [17] is Euclidean, so it cannot be directly applied to our case of a 2D manifold $\mathbb{M}$ in $\mathbb{R}^3$. Using a *conic formulation* for the subproblem of locating the Steiner nodes, Fampa *et al.* [19] improved Smith's B&B algorithm. Nonetheless, the Euclidean SMT problem in cases other than the 2D plane is still challenging; see [15], [17], [19], [33] for relevant details. For an overview of the SMT problem in Euclidean space, the reader is referred to [35]. Although much work has been done on the SMT problem in Euclidean space, where the edges of the SMT are straight lines, to the best of our knowledge, no one has considered the challenging SMT problem on an irregular 2D manifold in a 3D Euclidean space where the edges of the SMT are minimum cost geodesics; this problem requires new methodologies.

III. Models and Problem Formulation

Let $\mathbb{D}$ denote a closed and bounded path-connected region on the earth's surface. Given n nodes $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ (called *terminals*) on $\mathbb{D}$ to be connected in a cable network with trunk-and-branch tree topology, we aim to find the connecting network, that may use extra nodes, called *Steiner nodes* (that might be collocated with terminals as discussed later), with minimal total cost (weighted sum of the laying cost and the breakage risk). Such a minimum cost network will be an SMT, to determine which we will have to calculate the number of Steiner nodes to be used, the location of the Steiner nodes, and finally the edges of the tree (geodesics).

In the following, we reuse notation from [11], [12], [14]. We use $\gamma \in \mathbb{D}$ (geodesic) to denote a cable connecting two end nodes A, $B \in \mathbb{D}$. Curves in $\mathbb{D}$ are assumed parameterized according to the *natural parametrization* [36], i.e., parameterizing a curve γ as a function of arc length denoted by s, so that a curve $\gamma : [0, l(\gamma)] \to \mathbb{D}$ is a function from the interval $[0, l(\gamma)]$, taking values in $\mathbb{D}$, where $l(\gamma)$ is the length of the curve γ. This apparently circuitous definition turns out not to be a problem in practice because of the method used to find such curves. We first describe briefly our models of landforms, cable laying cost, and the number of potential required repairs. We then formulate optimal path design for submarine cable systems with trunk-and-branch tree topology as an SMT problem on an irregular 2D manifold in 3D Euclidean space $\mathbb{R}^3$. We again emphasize that the edges of the SMT here are minimal cost geodesics in $\mathbb{D}$ rather than straight lines, significantly adding to the computational complexity of the problem. The overall cost of the SMT, the sum of the costs of all the edges of the SMT, represents the total cost of all the cables in the cable network with trunk-and-branch tree topology.

A. The Earth'S Surface Model

As in [11], [12], [14], to accurately represent the earth's surface, we use a triangulated piecewise-linear 2D manifold $\mathbb{M}$ in $\mathbb{R}^3$ to approximate the region $\mathbb{D}$. Each node $\mathbf{x}$ on $\mathbb{M}$ is represented by 3D coordinates (x, y, z), where $z = \xi(x, y)$ is the altitude of the geographic location (x, y). For details of this representation, the reader is referred to [11].

B. Laying Cost Model

As in [11], for any node $\mathbf{x} = (x, y, z) \in \mathbb{M}, z = \xi(x, y)$, we use $h(\mathbf{x})$ to represent the laying cost per unit length. The laying cost of submarine cables is related to many factors, including the local site attributes (soil type, elevation, etc.), labor, licenses (e.g. right of way), protection level and the direction of the path. For a thorough discussion of the factors affecting cable laying cost, the reader is referred to [13]. For a cable represented by a Lipschitz continuous [37] curve γ connecting two nodes A and B in $\mathbb{M}$, the total laying cost of the cable γ is represented by $\mathbb{H}(\gamma)$. Applying the additive assumption of the laying cost as discussed in [11], $\mathbb{H}(\gamma)$ can be written as

$$\mathbb{H}(\gamma) = \int_0^{l(\gamma)} h(\gamma(s))ds. \tag{1}$$

C. Cable Repair Model

As in [11], the repair rate at location $\mathbf{x} = (x, y, z) \in \mathbb{M}, z = \xi(x, y)$ is defined as $g(\mathbf{x})$. We assume that the repair rate function $g(\mathbf{x})$ is also continuous. Let $\mathbb{G}(\gamma)$ denote the total number of repairs of a cable γ. Again, we assume that $\mathbb{G}(\gamma)$ is additive. That is, $\mathbb{G}(\gamma)$ can be written as [11]

$$\mathbb{G}(\gamma) = \int_0^{l(\gamma)} g(\gamma(s))ds. \tag{2}$$

In [11], [12], [14], [26], ground motion intensities, such as Peak Ground Velocity (PGV), are used to calculate the repair rate g taking into consideration the risk caused by earthquakes. As pointed out in [11], [12], [14], [26], other natural hazards (e.g. landslides, debris flows, volcanoes, storms, hurricanes) that may damage cables can be dealt with in the same way using the laying cost model and cable repair model provided that they are local and additive in nature. For details, the reader is referred to [11], [12], [14], [26].

D. Problem Formulation

To consider both the laying cost and the breakage risk in cable path planning, a weighted sum approach is used. Specifically, for any node $\mathbf{x} = (x, y, z) \in \mathbb{M}, z = \xi(x, y)$, let $f(\mathbf{x}) = \alpha \cdot h(\mathbf{x}) + g(\mathbf{x})$ be a weighted cost at $\mathbf{x}$, where $\alpha \in \mathbb{R}^1_+ \cup \{0\}$ can be regarded as the exchange rate between the laying cost and the risk. Then the total cost of a cable γ is

$$c(\gamma) = \int_0^{l(\gamma)} f(\gamma(s))ds. \tag{3}$$

Based on the models of landforms, laying cost and cable repair and Eq. (3), the problem of path design for submarine cable systems with trunk-and-branch tree topology is restated as follows. Given n terminals $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ on the 2D triangulated manifold $\mathbb{M}$ in $\mathbb{R}^3$ to be connected as a network, find the Steiner tree with minimal weighted cost.

IV. Solutions

The formulated SMT problem on the 2D triangulated manifold $\mathbb{M}$ in $\mathbb{R}^3$ is clearly NP-hard since it generalizes the SMT problem in Euclidean plane. Before proceeding, we first introduce some Steiner tree terminologies following [15], [33].

By the *topology* of a tree, we mean its combinatorial structure; that is, a specification of which pairs of terminals/Steiner nodes are connected by an edge. The topology specifies connections but not the positions of nodes. Note that there are at most $n - 2$ Steiner nodes in the SMT with n terminals. A topology with n terminals and m Steiner nodes $(0 \leq m \leq n - 2)$ is a *Steiner topology* if the degree of each Steiner node is equal to three and the degree of each terminal is three or less. A Steiner topology for n terminals is called an FST if it has exactly $n-2$ Steiner nodes and each terminal has degree one; otherwise, it is called a *nonfull Steiner topology*. The minimum cost tree for a given Steiner topology $\mathcal{T}$ is called an RMT for $\mathcal{T}$, and is unique. A tree with n location-specified terminals in a d-dimensional $(d \geq 2)$ Euclidean space is a *Steiner tree* with the following properties [33], [35].

- There are at most $n - 2$ Steiner nodes in the tree;
- Node degree condition: all the terminals will be of degree one, two, or three and the Steiner nodes are all of degree three;
- Angle condition: any two edges meet at an angle of at least $120°$. The three edges connecting a Steiner node lie in a plane and meet at an angle of $120°$.

A solution of an SMT problem is a Steiner tree with minimal cost. A Steiner tree is always an RMT for its topology, but not vice versa. An RMT is a Steiner tree if all of its angles are $120°$ or more. A Steiner tree with n terminals is a *full Steiner tree* if it has $n-2$ Steiner nodes and each terminal has degree one. Any Steiner tree can be concatenated by smaller full Steiner trees. A Steiner tree is called *degenerate* if the costs of some edges are zero. Accordingly, the nonfull Steiner topologies are degenerate FSTs. For any SMT with a nonfull Steiner topology, we can always find an FST that corresponds to this nonfull Steiner topology such that the RMT for the

FST is degenerate. Therefore, we only need to focus on FST. Denote the Steiner nodes by $\mathbf{x}_{n+1}, \mathbf{x}_{n+2}, \ldots, \mathbf{x}_{2n-2}$.

To solve the formulated SMT problem on a 2D triangulated manifold $\mathbb{M}$ in $\mathbb{R}^3$, we follow the framework of [17]. That is, an implicit enumeration scheme, the B&B algorithm of [17], is adopted to enumerate the Steiner topologies. For a topology to be expanded (i.e., a Steiner topology), a non-iterative algorithm is proposed to find the RMT for the topology. Next in this section, we first introduce our proposed algorithm for finding the RMT for a known topology and a computational complexity analysis of the proposed algorithm, followed by a description of techniques that reduce the computational cost. Lastly, the B&B algorithm of [17] is incorporated.

A. RMT for a Known Topology

In this section, we propose a non-iterative numerical algorithm to optimally find the coordinates of the Steiner nodes for any prespecified Steiner topology $\mathcal{T}$. Note that $\mathcal{T}$ is not necessarily full or nondegenerate. We assume that in $\mathcal{T}$ there exists at least one Steiner node, all terminals are of degree one and $\mathcal{T}$ is connected. If there is no Steiner node, the problem becomes a minimum spanning tree problem (MST), and can be solved by running FMM and Kruskal's MST algorithm [38]. Note that, for n terminals to be connected, the cost of the MST provides an upper bound for the cost of SMT.

In order to more precisely describe the problem, let $N = \{1, 2, \ldots, n\}$ be the set of indices associated with the given terminals $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathbb{M}$ and $S = \{n+1, n+2, \ldots, n+m\}, m \leq n-2$ be the set of the indices associated with additional Steiner nodes $\mathbf{x}_{n+1}, \mathbf{x}_{n+2}, \ldots, \mathbf{x}_{n+m} \in \mathbb{M}$. Recall that the coordinates of all terminals are given but those of Steiner nodes are not. We impose that all terminals are the grid nodes of $\mathbb{M}$. This is a fairly natural assumption; after all we are only given the coordinates of grid nodes of $\mathbb{M}$ in practice. Note that $\mathcal{T}$ describes the adjacency relationship between the nodes: $\mathcal{T}$ describe a graph (more precisely, tree) abstractly. Let $V = N \cup S$ and $E = E_1 \cup E_2$ be the set of all nodes and edges, respectively, i.e., $\mathcal{T} = (V, E)$, where $E_1 = \{(i,j) | i \in N, j \in S\}$ and $E_2 = \{(i,j) | i \in S, j \in S\}$. Fig. 2 shows an example of a Steiner topology with seven terminals and five Steiner nodes.

For a cable γ_{ij} that connects two nodes $\mathbf{x}_i \in \mathbb{M}$ and $\mathbf{x}_j \in \mathbb{M}$, let $c(\mathbf{x}_i, \mathbf{x}_j) = \int_{\mathbf{x}_i}^{\mathbf{x}_j} f(\mathbf{x}(s)) ds$ denote the cumulative weighted cost (hereafter called "cost" for brevity) over the cable path γ_{ij}. Note that the cable is not required to traverse edges of triangles of $\mathbb{M}$. More practical and better solutions can be obtained if we assume that the cable can pass through the interior of triangles of $\mathbb{M}$. Given the graph $\mathcal{T}$ and the coordinates of all terminals, our aims are therefore to find coordinates $\mathbf{X} = \{\mathbf{x}_{n+1}, \mathbf{x}_{n+2}, \ldots, \mathbf{x}_{n+m}\}$, $\mathbf{x}_{n+j} \in \mathbb{M}$, $j = 1, 2, \ldots, m$ and the paths $\Gamma = \{\gamma(e) | e \in E\}$ (i.e., the geodesics corresponding to the edges in E) such that the total cost of the physical cable network corresponding to $\mathcal{T}$ is minimized, that is,

$$\min_{\mathbf{X},\Gamma} \Psi(\mathbf{X}, \Gamma) = \sum_{(i,j)\in E_1} c(\mathbf{x}_i, \mathbf{x}_j) + \sum_{(i,j)\in E_2} c(\mathbf{x}_i, \mathbf{x}_j). \quad (4)$$

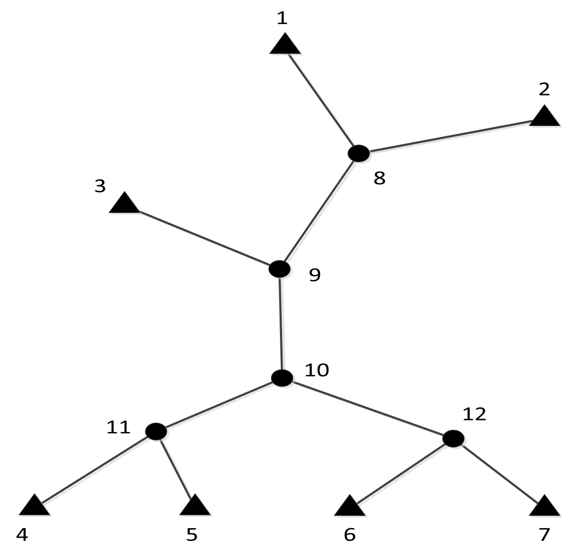

Fig. 2. An example of a full Steiner topology. Triangles represent terminals and circles represent Steiner nodes. Note that the coordinates of terminals are known but those of Steiner nodes are not.

If the coordinates of Steiner nodes were known, then Problem (4) would reduce to a set of least-cost path problems, each of which is to find a least cost path connecting two known nodes on $\mathbb{M}$, and this problem is solved in [11]. Accordingly, one way to solve Problem (4) is to enumerate all the combinations of the coordinates of the Steiner nodes and then apply the method in [11] to find all the paths. This is infeasible in practice since there are large number of nodes in $\mathbb{M}$.

Note that we can rewrite Eq. (4) as

$$\begin{aligned}
&\min_{\mathbf{X},\Gamma} \ \Psi(\mathbf{X}, \Gamma) \\
&\quad = \min_{\mathbf{X}} \left(\sum_{(i,j)\in E_1} \min_{\gamma_{ij}} c(\mathbf{x}_i, \mathbf{x}_j) + \sum_{(i,j)\in E_2} \min_{\gamma_{ij}} c(\mathbf{x}_i, \mathbf{x}_j) \right) \\
&\quad = \min_{\mathbf{X}} \left(\sum_{j\in S} \sum_{\substack{i\in N \\ (i,j)\in E_1}} \min_{\gamma_{ij}} c(\mathbf{x}_i, \mathbf{x}_j) + \sum_{(i,j)\in E_2} \min_{\gamma_{ij}} c(\mathbf{x}_i, \mathbf{x}_j) \right) \\
&\quad = \min_{\mathbf{X}} \left(\sum_{j\in S} \bar{c}(\mathbf{x}_j) + \sum_{(i,j)\in E_2} \min_{\gamma_{ij}} c(\mathbf{x}_i, \mathbf{x}_j) \right), \qquad (5)
\end{aligned}$$

where $\bar{c}(\mathbf{x}_j) = \sum_{i\in N,(i,j)\in E_1} \min_{\gamma_{ij}} c(\mathbf{x}_i, \mathbf{x}_j)$ is the sum of the minimum cost from each terminal that neighbours $\mathbf{x}_j$ to $\mathbf{x}_j$, which can be calculated using FMM with the terminal being the source node. Let $\bar{c}(\mathbf{x}_j) = 0$ if no terminal is adjacent to the Steiner node $\mathbf{x}_j$.

We call subgraph $T = (S, E_2)$, the subtree composed of only Steiner nodes and the edges connect them, the *skeleton tree* of the graph $\mathcal{T}$. The skeleton tree T can be viewed as the tree where the terminals shrink to their connected Steiner nodes. Note that, for now, we only know the topology of the skeleton tree T. We choose an arbitrary node $r \in T$ as the root of the tree T and then assign the edges of T an orientation towards the root. As a result, T becomes a *directed rooted tree* (i.e., anti-arborescence). Fig. 3 shows the skeleton tree corresponding to the Steiner topology in Fig. 2 with Node 9 being the root. Essentially, visiting the nodes in T in order from the leaves to the root gives rise to a *topological order* of nodes of T. That is, a node i in T is not visited until all of its children have been visited. We reorder the Steiner nodes by a topological order, specifically, by traversing T in level-order (i.e., breadth-first) towards the root, so that all children of a node are visited before it is visited. Without loss of generality, we denote the reordered Steiner node sequence as $1, 2, \ldots, m$, where m corresponds to the root of T, and let $\mathbf{x}_{n+i} = \bar{\mathbf{x}}_i$, $i = 1, 2, \ldots, m$. Since altering the order of Steiner nodes $\mathbf{x}_{n+i}, i = 1, 2, \ldots, m$ in the minimization of (5) does not alter the optimum, we rewrite (5) as

$$\min_{\mathbf{X},\Gamma} \Psi(\mathbf{X}, \Gamma) = \min_{\bar{\mathbf{x}}_m, \bar{\mathbf{x}}_{m-1}, \ldots, \bar{\mathbf{x}}_1} \Phi(\bar{\mathbf{x}}_m, \bar{\mathbf{x}}_{m-1}, \ldots, \bar{\mathbf{x}}_1), \quad (6)$$

where

$$\Phi(\bar{\mathbf{x}}_m, \bar{\mathbf{x}}_{m-1}, \ldots, \bar{\mathbf{x}}_1) = \bar{c}(\bar{\mathbf{x}}_m) + \sum_{\substack{(j,m)\in E_2 \\ j\in C(m)}} \min_{\gamma_{jm}} c(\bar{\mathbf{x}}_j, \bar{\mathbf{x}}_m) + \Phi(\bar{\mathbf{x}}_{m-1}, \ldots, \bar{\mathbf{x}}_1). \quad (7)$$

In Eq. (7), $C(m)$ is the set of children of node m. Let $\Phi(\bar{\mathbf{x}}_i) = \bar{c}(\bar{\mathbf{x}}_i)$ for any leaf i. Evidently, if we define

$$\Phi^*(\bar{\mathbf{x}}_m, \bar{\mathbf{x}}_{m-1}, \ldots, \bar{\mathbf{x}}_1) = \min_{\bar{\mathbf{x}}_m, \bar{\mathbf{x}}_{m-1}, \ldots, \bar{\mathbf{x}}_1} \Phi(\bar{\mathbf{x}}_m, \bar{\mathbf{x}}_{m-1}, \ldots, \bar{\mathbf{x}}_1), \quad (8)$$

then

$$\Phi^*(\bar{\mathbf{x}}_m, \bar{\mathbf{x}}_{m-1}, \ldots, \bar{\mathbf{x}}_1) = \min_{\bar{\mathbf{x}}_m} \left(\bar{c}(\bar{\mathbf{x}}_m) + \sum_{\substack{(j,m)\in E_2 \\ j\in C(m)}} \min_{\gamma_{jm}} c(\bar{\mathbf{x}}_j, \bar{\mathbf{x}}_m) + \Phi^*(\bar{\mathbf{x}}_{m-1}, \ldots, \bar{\mathbf{x}}_1) \right). \quad (9)$$

We have now converted Problem (4) to a dynamic programming problem. Eq. (9) is the corresponding Bellman equation. To solve it, we assume that all Steiner nodes are grid nodes in $\mathbb{M}$.

Next, we construct a directed acyclic graph (DAG) $G = (V', E')$ based on T in the following way. Each (Steiner) node $i \in T$ is associated with a subset A_i of V', where A_i are grid nodes of $\mathbb{M}$ and the weight on each node is $\bar{c}_i(\mathbf{x})$. It follows that $V' = \cup_{i\in S} A_i$; that is, V' is composed of m copies of grid nodes of $\mathbb{M}$. For an arc $e = (i, j) \in E_2$, where j is the parent of i, we construct a complete connection (we will later discuss how to avoid unnecessary connections to reduce computational cost) from A_i to A_j for G, that it, there is an arc $\varepsilon = (p, q)$ from every $p \in A_i$ to every $q \in A_j$ in G. The cost on the arc ε is defined as the minimum cost from node $\mathbf{x}_p$ to node $\mathbf{x}_q$, i.e., $w(\varepsilon) = w(p, q) = \min c(\mathbf{x}_p, \mathbf{x}_q)$. Fig. 4 shows an example of the DAG corresponding to the Steiner topology in Fig. 2 and the skeleton tree in Fig. 3.

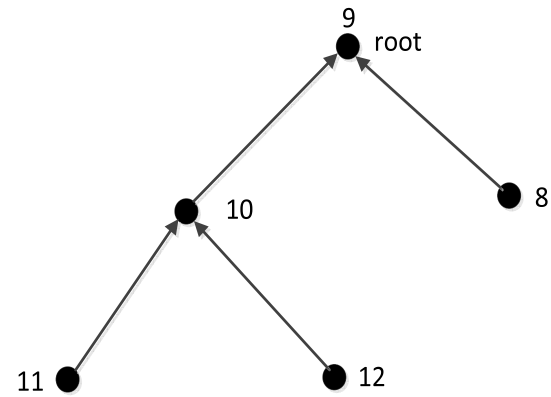

Fig. 3. The skeleton tree corresponding to the Steiner topology in Fig. 2 with node 9 being the root of the tree. One reordered Steiner node sequence is 11, 12, 10, 8, 9.

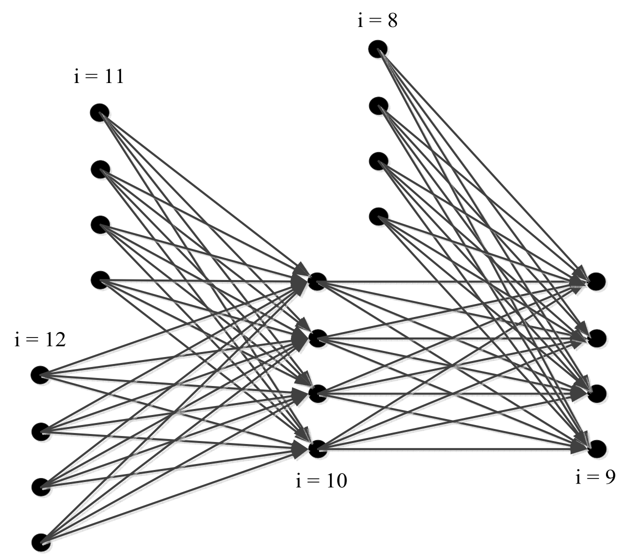

Fig. 4. The DAG $G = (V', E')$ corresponding to the Steiner topology in Fig. 2 and the skeleton tree in Fig. 3.

For a node $\mathbf{x}_p \in A_i$, define ϕ_i^p to be the following minimum cumulative cost (MCC),

$$\phi_p^i = \bar{c}(\mathbf{x}_p) + \sum_{j\in C(i)} \min_{q\in A_j} \left(w(\mathbf{x}_q, \mathbf{x}_p) + \phi_q^j \right), \quad (10)$$

where $C(i)$ is the set of children of Steiner node i. From the Bellman equation (9), and the DAG modeling, we propose an algorithm named DAG-Least-Cost-Tree that finds the tree on DAG with the minimum cost and therefore finds the coordinates of the Steiner nodes. The DAG-Least-Cost-Tree algorithm is listed as Algorithm 1. The statements in Lines 1–12 form the initialization procedure, and implementation of the Bellman equation (9) is in Lines 13–26. Once the iteration arrives at the root, in Line 27 the grid node $\hat{p}$ with MCC ϕ_p^m in A_m is chosen. The node $\hat{p}$ is the physical Steiner node corresponding to the root. The coordinates of the remaining Steiner nodes are derived by tracing back on G starting from $\hat{p}$. Theorem 1 establishes the correctness of DAG-Least-Cost-Tree algorithm.

Algorithm 1 Algorithm for DAG-Least-Cost-Tree

Require: $G = (V', E')$, T with topologically sorted order.
Ensure: Coordinates of Steiner nodes $\bar{\mathbf{x}}_i$, $i = 1, \ldots, m$
1: **for** $i = 1, \ldots, m$ **do**
2: **for** each node $\mathbf{x}_p \in A_i$ **do**
3: **if** i is a leaf in T **then**
4: $\phi_p^i = \bar{c}(\mathbf{x}_p)$; $\pi(\mathbf{x}_p) = \text{NIL}$;
5: **else**
6: **for** each child j of i **do**
7: $\pi(\mathbf{x}_p, j) = \text{NIL}$;
8: **end for**
9: $\phi_p^i = 0$;
10: **end if**
11: **end for**
12: **end for**
13: **for** $i = 1, \ldots, m$, i is not a leaf **do**
14: **for** each node $\mathbf{x}_p \in A_i$ **do**
15: **for** each child j of i **do**
16: $\psi = \infty$;
17: **for** each node $\mathbf{x}_q \in A_j$ **do**
18: **if** $\psi > \phi_q^j + w(\mathbf{x}_q, \mathbf{x}_p)$ **then**
19: $\psi = \phi_q^j + w(\mathbf{x}_q, \mathbf{x}_p)$; $\pi(\mathbf{x}_p, j) = q$;
20: **end if**
21: **end for**
22: $\phi_p^i = \phi_p^i + \psi$;
23: **end for**
24: $\phi_p^i = \phi_p^i + \bar{c}(\mathbf{x}_p)$;
25: **end for**
26: **end for**
27: Let $\hat{p}_m = \arg\min_{\mathbf{x}_p \in A_m} \phi_p^m$;
28: Trace back from $\hat{p}_m$ to leaves via π;
29: Return $\mathbf{x}_{\hat{p}_1}, \ldots, \mathbf{x}_{\hat{p}_m}$.

Theorem 1: Algorithm DAG-Least-Cost-Tree, runs on the weighted DAG $G = (V', E')$, and finds the coordinates of Steiner nodes of topology $\mathcal{T}$ and therefore the least cost tree corresponding to $\mathcal{T}$.

As mentioned above, to calculate $\bar{c}(\mathbf{x}_j)$, we firstly run FMM for each terminal to derive the corresponding level set which gives the MCC from the terminal to each grid node in $\mathbb{M}$. For each $\mathbf{x}_j, j \in S$, $\bar{c}(\mathbf{x}_j)$ is derived by summing all corresponding level sets over the terminals adjacent to Steiner node j. For each pair of grid nodes $\mathbf{x}_i, \mathbf{x}_j \in \mathbb{M}$, we calculate the minimum cost from $\mathbf{x}_i$ to $\mathbf{x}_j$ by running FMM again. Combining with Algorithm 1 above, we summarize the complete procedure for calculating the RMT corresponding to a given Steiner topology $\mathcal{T}$ as below.

1) Run FMM for each terminal node $\mathbf{x}_i, i = 1, 2, \ldots, n$ to obtain the corresponding level set (distance map) d_i;
2) For each Steiner node i, $i \in S$, let $\bar{c}_i(\mathbf{x}_j) = \sum_{i \in N, (i,j) \in E_1} d_j(\mathbf{x}_j)$ for each grid node $\mathbf{x}_j \in \mathbb{M}$ according to the given topology $\mathcal{T}$;
3) For each pair of grid nodes $\mathbf{x}_i, \mathbf{x}_j \in \mathbb{M}$, run FMM for calculating the minimum cost from $\mathbf{x}_i$ to $\mathbf{x}_j$;
4) Based on the Steiner topology $\mathcal{T}$ and the grid nodes of $\mathbb{M}$, construct the DAG $G = (V', E')$;
5) Run DAG-Least-Cost-Tree Algorithm 1 on DAG G and find the minimum cost tree. The nodes on the minimum cost tree $\mathbf{x}_{\hat{p}_1}, \ldots, \mathbf{x}_{\hat{p}_m}$ are the Steiner nodes;
6) Find the geodesics $\Gamma = \{\gamma(e) | e \in E\}$ by the steepest descent method using the corresponding distance map.

Remarks: Unlike for the GeoSteiner algorithm [15] and Smith's B&B algorithm [17], the angle condition is not a limitation for our algorithm above. Although Steiner topologies (including the angle condition) are required for optimality, in practice we may not be able to guarantee a perfect $120°$ between any two edges. In addition, the above algorithm is not limited in application to Steiner topologies. It works for any given trunk-and-branch network topologies with tree structure, and might be of independent interest.

B. Computational Complexity Analysis

For a grid consisting of K nodes on $\mathbb{M}$, the complexity of the steps 1 and 2 of the procedure listed above are $\mathcal{O}(nK \log K)$ and $\mathcal{O}(mK)$, respectively. Step 3 requires the calculation of the cost for each pair of grid nodes in $\mathbb{M}$, and this has complexity $\mathcal{O}(K^2 \log K)$. The DAG G has mK vertices and $(m-1)K^2$ arcs, so finding the least cost tree on G by using a topological order takes $\mathcal{O}(mK + (m-1)K^2)$ operations. The computational complexity of the entire algorithm is, therefore, $\mathcal{O}(K^2(\log K + m - 1))$. One could compute the cost for each pair of grid nodes on the manifold $\mathbb{M}$ before running the algorithm and store the results in a database. However, it will require at least $\mathcal{O}(K^2)$ of memory.

C. Computational Cost Reduction

Various strategies can be used to reduce the computational cost of the proposed method.

- The total cost of an SMT should be less than that of an MST. This provides an upper bound for the cost of an SMT, consequent on finding a reasonable MST. This can be done in two ways [38]: first, calculate the complete graph on the terminals and then run Kruskal's algorithm to find an MST. This has computational complexity of $\mathcal{O}\left(\frac{n(n-1)}{2} K \log K + \frac{n(n-1)}{2} \log(\frac{n(n-1)}{2})\right)$. The second way involves a combination of the FMM and Kruskal's algorithm, for details see [38].
- For each Steiner node i, we calculate $\tilde{d}_i = \min_{\mathbf{x}_j \in \mathbb{M}} \bar{c}(\mathbf{x}_j)$. Line 19 in Algorithm 1 shows that we need to calculate $w(\mathbf{x}_q, \mathbf{x}_p)$, namely, the minimum cost from node $\mathbf{x}_q$ to node $\mathbf{x}_p$ for every pair of grid nodes on $\mathbb{M}$. As mentioned above, we could compute them before running the algorithm and store the results in a database. By Eq. (10), we can reduce the computational cost of Algorithm 1, specifically operations from Line 17 to Line 21, as follows. For a node $\mathbf{x}_p \in A_i$ corresponding to a Steiner node i, we run FMM with an upper bound $b = c_{\text{MST}} - c_p - \sum_{j=i+1}^{m} \tilde{d}_j$. Here c_{MST} is the cost of the MST, and c_p is the cost of node $\mathbf{x}_p$. Running FMM with an upper bound b means that FMM stops once the level set value on the front from the source node exceeds b. Essentially, this operation (of running FMM

with an upper bound) avoids unnecessary arcs in DAG $G = (V', E')$.

D. SMT Without Knowing the Topology

If the topology is unknown, then the SMT has to be found by searching all FSTs. This is more complicated and time-consuming. We adopt the B&B algorithm of [17]. One starts the B&B algorithm with an upper bound on the cost of the SMT on all the n given terminals. This upper bound can be set to $+\infty$, or the cost of found, as above, by using Kruskal's MST algorithm on the complete graph on the terminals [38], or indeed any heuristically discovered tree on the n given terminals. Then one selects three of the n given terminals and designates the unique FST for the three terminals as the root node in the B&B tree. That is, the root node at depth $k = 0$ in the B&B tree corresponds to the unique FST for the three selected terminals. The selection of the three terminals can be made arbitrarily. Given a leaf at depth k in the B&B tree that corresponds to an FST with $k+1$ Steiner nodes, $k+3$ terminals and $2k+3$ edges, branching from this leaf is carried out by adding a new terminal and creating children of this leaf with topologies derived as follows. For a given edge $e = (i, j)$ in the FST, delete it from the FST, and add three new edges (i, s), (s, j), and (s, t), where s is the new Steiner node and t is the new terminal, generating a new FST with one more Steiner node, one more terminal and two more edges. Since there are $2k+3$ edges in the old FST, $2k+3$ new topologies will be created for this leaf. Therefore, the nodes at depth k in the B&B tree enumerate all FSTs that have $k+3$ terminals. It can be shown that the cost of the RMT corresponding to the new FST is always no less than that of the RMT corresponding to the old FST. That is, the above procedure of adding a new terminal cannot decrease the minimal cost of the tree. As a result, if the minimal cost of the RMT for the FST at some node in the B&B tree is no less than the upper bound, all the descendants of this node can be discarded. Otherwise, a new terminal is added to the FST corresponding to the node and one child node is generated for each edge of the FST as above. The proposed approach in Section IV-A is then applied to calculate the RMT for the new FST and the upper bound is updated accordingly. The algorithm proceeds until all FSTs have been considered. The SMT on the n given terminals is then the RMT with the minimum cost.

For $n, n \geq 3$ terminals, there are $F(n) = 1 \cdot 3 \cdot 5 \cdots (2n-5) = \frac{(2n-4)!}{2^{n-2}(n-2)!}$ FSTs [17]. In the worst case, the above-mentioned B&B algorithm has to traverse $\sum_{i=3}^{n} F(i)$ FSTs, which is superexponential. Fortunately, in its execution, the B&B algorithm often quickly finds better upper bound than in the worst case scenario, allowing pruning of more branches that are not optimal and thus accelerating the algorithm. In [17], it was reported that the B&B algorithm was able to solve the SMT problems in $\mathbb{R}^d, d \geq 3$ with 12 terminals. Based on Smith's algorithm and by applying geometric conditions, Laarhoven and Anstreicher [39] solved the SMT problems in $\mathbb{R}^d, d \geq 3$ with 16 terminals. However, it is still infeasible to solve problems for $d \geq 3$ with more than 18 terminals [40]. Note that for a given FST, finding the RMT on irregular 2D manifolds in $\mathbb{R}^3$ is more challenging than in a general Euclidean space. Our future work will include the development of more efficient algorithms for SMT problems on irregular 2D manifolds in $\mathbb{R}^3$ where the topology is unknown *a priori*.

V. Applications

In this section, we illustrate the application of our algorithm to scenarios based on 2D and 3D landforms where the Steiner topologies are not prespecified. We start with a simple case of 2D topography, where we assume uniform cost, to validate our proposed method. Then, we apply the algorithm to a scenario comprising real 3D landforms.

A. Application of the Algorithm to a 2D Landform

In this example, as shown in Fig. 5, we assume that a cable system with trunk-and-branch tree network topology is planned on a planar (2D) region [0.5 km,11.5 km] $\times$ [0.5 km,11.5 km]. There are five terminals, of which coordinates are $\mathbf{x}_1 = (4.0 \text{ km}, 1.0 \text{ km})$, $\mathbf{x}_2 = (8.0 \text{ km}, 2.0 \text{ km})$, $\mathbf{x}_3 = (11.0 \text{ km}, 9.0 \text{ km})$, $\mathbf{x}_4 = (6.0 \text{ km}, 11.0 \text{ km})$ and $\mathbf{x}_5 = (1.0 \text{ km}, 6.0 \text{ km})$, respectively. To apply our numerical method, we discretize both the horizontal and vertical axes into equal bins with sizes 0.02 km. We assume the weighted cost in this region is uniform, i.e., the weighted cost at each node is the same. The (normalized) cost δ in dollars per kilometer is assumed to be 1 in this example. In this case, the coordinates of the three Steiner nodes obtained by our proposed approach are $\mathbf{x}_6 = (4.98 \text{ km}, 2.32 \text{ km})$, $\mathbf{x}_7 = (3.66 \text{ km}, 5.70 \text{ km})$ and $\mathbf{x}_8 = (6.56 \text{ km}, 9.52 \text{ km})$, and the total cost of the cable system is 21.86 dollars. The angles between the edges connecting the Steiner nodes to their adjacent nodes can be calculated using the cosine rule; explicitly, $\angle \mathbf{x}_1\mathbf{x}_6\mathbf{x}_2 = 120.54°$, $\angle \mathbf{x}_1\mathbf{x}_6\mathbf{x}_7 = 122.07°$, $\angle \mathbf{x}_2\mathbf{x}_6\mathbf{x}_7 = 117.38°$, $\angle \mathbf{x}_5\mathbf{x}_7\mathbf{x}_6 = 117.76°$, $\angle \mathbf{x}_5\mathbf{x}_7\mathbf{x}_8 = 120.77°$, $\angle \mathbf{x}_6\mathbf{x}_7\mathbf{x}_8 = 121.46°$, $\angle \mathbf{x}_4\mathbf{x}_8\mathbf{x}_7 = 122.07°$, $\angle \mathbf{x}_3\mathbf{x}_8\mathbf{x}_7 = 120.52°$, $\angle \mathbf{x}_3\mathbf{x}_8\mathbf{x}_4 = 117.41°$. These show that the angle condition is satisfied well beyond any potential numerical error. The exact coordinates of the three Steiner nodes obtained by GeoSteiner algorithm [15] are $\bar{\mathbf{x}}_6 = (5.04 \text{ km}, 2.37 \text{ km})$, $\bar{\mathbf{x}}_7 = (3.66 \text{ km}, 5.66 \text{ km})$ and $\bar{\mathbf{x}}_8 = (6.61 \text{ km}, 9.55 \text{ km})$, and the total cost of the cable system is 21.83 dollars. Note that this 2D uniform laying cost toy example is only used to validate our algorithm, since the GeoSteiner algorithm [15] is able to derive exact solutions more efficiently in this case.

B. Application of the Algorithm to 3D Landforms

In this section, we apply our method to a 3D realistic scenario. The bathymetric data used here is from the Global Multi-Resolution Topography (GMRT) synthesis, a multi-resolution compilation of edited multibeam sonar data collected by scientists and institutions worldwide. The data can be downloaded from the Marine Geoscience Data System at the Lamont-Doherty Earth Observation of Columbia University.

The object region $\mathbb{D}$ is from the northwest corner (36.000° N, 2.000° E) to southeast corner (44.000° N, 9.000° E), which is a subregion of the Mediterranean as

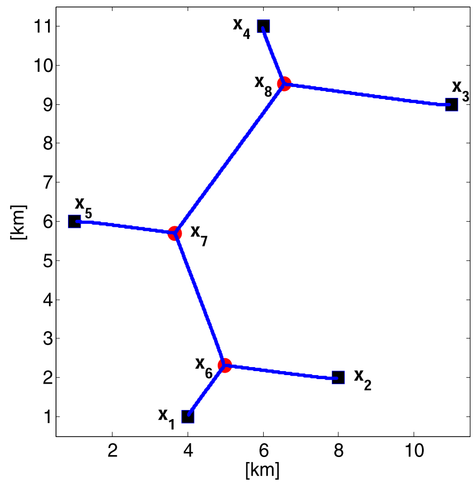

Fig. 5. A 2D example with five terminals. The red solid nodes are Steiner nodes obtained by our method.

Fig. 6. Region $\mathbb{D}$. Source: google earth.

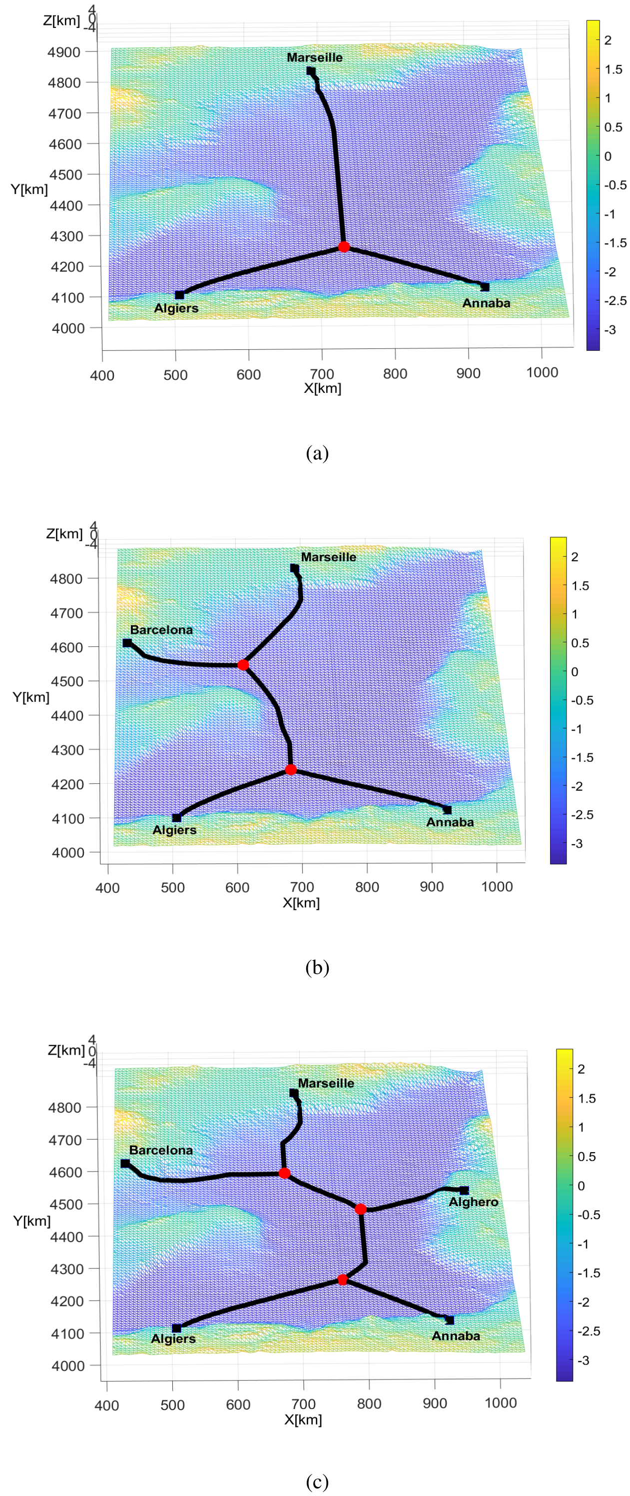

Fig. 7. (a) Minimum cost cable network connecting Marseille, Algiers and Annaba. (b) Minimum cost cable network connecting Marseille, Algiers, Annaba and Barcelona. (c) Minimum cost cable network connecting Marseille, Algiers, Annaba, Barcelona and Alghero. The red solid nodes are Steiner nodes obtained by our method.

shown by Fig. 6. The spatial resolutions are $0.0704°$ and $0.0673°$ in longitude and latitude, respectively. To test our method, we propose, firstly, to lay cables connecting Marseille (France), Algiers (Algeria) and Annaba (Algeria). The locations of these three cities are ($43.297°$ N, $5.359°$ E), ($36.761°$ N, $3.074°$ E), and ($36.928°$ N, $7.760°$ E), respectively. In fact, a 1,300 km submarine cable system with one branching unit, laid in 2005, called Med Cable Network is in service for the three cities [10], but the path data are not publicly available. Then we redesign the cable network by connecting Barcelona ($41.386°$ N, $2.190°$ E, Spain) where two branching units are installed. Lastly, we propose connecting Alghero ($40.557°$ N, $8.312°$ E, Italy) with the other four cities, resulting in a submarine cable network with three branching units.

Before calculating the SMT, a coordinate transformation is applied to convert the geographic data from latitude and longitude coordinates to Universal Transverse Mercator coordinates. Using the landform model described in Section III, 23, 562 faces are created for this region, and the triangulated manifold approximation of the landforms is shown in Fig. 7.

In practice, a range of realistic restrictions have significant impact on the cable path design [13]. For example, according to the recommendations of the International Cable Protection Committee (ICPC) [41], the length of a cable in shallow water should be as short as possible and the cable should always be laid in the sea except for areas near landing sites. These constraints reduce the risk posed by human activities. It is preferable to lay the cable in deeper ocean [41]. We enforce these depth constraints by imposing a modified cost function in units of USD ($), namely,

$$f(\mathbf{x}) = \begin{cases} 20000, & \text{if } z \geq 0 \text{ km}, \\ 20000 - 20000 \times |z| & \text{if } 0 \text{ km} > z \geq -0.2 \text{ km}, \\ \dfrac{6400}{|z| + 0.2}, & \text{otherwise}. \end{cases}$$

The SMTs obtained by our method in the three cases are shown in Fig. 7. In the three-terminal case, a Steiner node is obtained at (4241.39 km, 732.18 km, -2.89 km) as shown by Fig. 7(a) and the total cost is $3, 232, 366 and the total length of the cable system is 1, 102 km. In the four-terminal case, two Steiner nodes are obtained at (4225.19 km, 683.24 km, -2.89 km) and (4530.22 km, 610.88 km, -2.66 km) as shown by Fig. 7(b) and the total cost is $4, 283, 458 and the total length of the cable system is 1, 331 km. In the five-terminal case, three Steiner nodes are obtained at (4242.30 km, 761.95 km, -3.17 km), (4457.84 km, 790.48 km, -2.84 km) and (4568.85 km, 675.22 km, -2.62 km) as shown by Fig. 7(c). The total laying cost of the cable system is $5, 416, 779 and the total length of the cable system is 1, 601 km. Observe that the modified cost function, including a depth term, successfully limits the length of cable in shallow water.

VI. Conclusions

We have addressed the problem of submarine cable path planning and trunk-and-branch tree network topology design on the surface of the earth taking account of laying cost and cable breakage risk. The problem has been formulated as an SMT problem on an irregular 2D manifold in $\mathbb{R}^3$. By applying dynamic programming, we have proposed a polynomial time computational complexity numerical algorithm for a given Steiner topology. To accommodate unknown topologies, the B&B algorithm is incorporated. The proposed method may also be applied to the design of oil and natural gas pipelines, electrical power distribution networks, and other network design problems.

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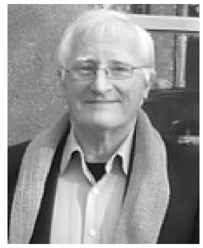

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