

Taxicab geometry

Taxicab geometry, considered by Hermann Minkowski in the 19th century, is a form of geometry in which the usual distance function or metric of Euclidean geometry is replaced by a new metric in which the distance between two points is the sum of the absolute differences of their coordinates. The taxicab metric is also known as **rectilinear distance**, **L_1 distance** or **ℓ_1 norm** (see L^p space), **city block distance**, **Manhattan distance**, or **Manhattan length**, with corresponding variations in the name of the geometry.^[1] The latter names allude to the grid layout of most streets on the island of Manhattan, which causes the shortest path a car could take between two intersections in the borough to have length equal to the intersections' distance in taxicab geometry.

Formal description

The taxicab distance, d_1 , between two vectors \mathbf{p}, \mathbf{q} in an n -dimensional real vector space with fixed Cartesian coordinate system, is the sum of the lengths of the projections of the line segment between the points onto the coordinate axes. More formally,

$$d_1(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\|_1 = \sum_{i=1}^n |p_i - q_i|,$$

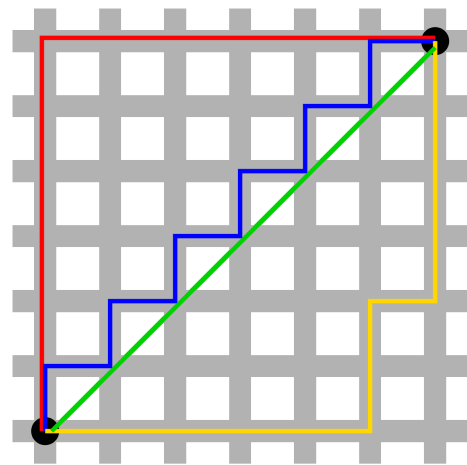
where

$$\mathbf{p} = (p_1, p_2, \dots, p_n) \text{ and } \mathbf{q} = (q_1, q_2, \dots, q_n)$$

are vectors.

For example, in the plane, the taxicab distance between (p_1, p_2) and (q_1, q_2) is $|p_1 - q_1| + |p_2 - q_2|$.

Taxicab distance depends on the rotation of the coordinate system, but does not depend on its reflection about a coordinate axis or its translation. Taxicab geometry satisfies all of Hilbert's axioms (a formalization of Euclidean geometry) except for the side-angle-side axiom, as one can generate two triangles each with two sides and the angle between them the same, and have them not be congruent.



Taxicab geometry versus Euclidean distance: In taxicab geometry all three pictured lines (red, yellow, and blue) have the same length (12) for the same route. In Euclidean geometry, the green line has length $6\sqrt{2} \approx 8.48$, and is the unique shortest path.

A circle is a set of points with a fixed distance, called the *radius*, from a point called the *center*. In taxicab geometry, distance is determined by a different metric than in Euclidean geometry, and the shape of circles changes as well. Taxicab circles are squares with sides oriented at a 45° angle to the coordinate axes. The image to the right shows why this is true, by showing in red the set of all points with a fixed distance from a center, shown in blue. As the size of the city blocks diminishes, the points become more numerous and become a rotated square in a continuous taxicab geometry. While each side would have length $\sqrt{2}r$ using a Euclidean metric, where r is the circle's radius, its length in taxicab geometry is $2r$. Thus, a circle's circumference is $8r$. Thus, the value of a geometric analog to π is 4 in this geometry. The formula for the unit circle in taxicab geometry is $|x| + |y| = 1$ in Cartesian coordinates and

$$r = \frac{1}{|\sin \theta| + |\cos \theta|}$$

in polar coordinates.

A circle of radius r for the Chebyshev distance (L_∞ metric) on a plane is also a square with side length $2r$ parallel to the coordinate axes, so planar Chebyshev distance can be viewed as equivalent by rotation and scaling to planar taxicab distance. However, this equivalence between L_1 and L_∞ metrics does not generalize to higher dimensions.

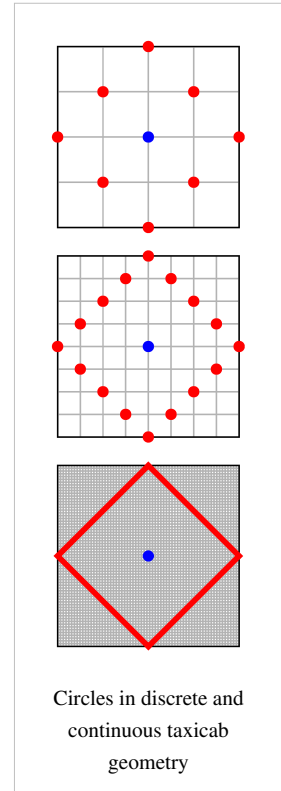
The use of Manhattan distance leads to a strange concept: when the resolution of the Taxicab geometry is made larger, approaching infinity (the size of division of the axis approaches 0), it seems intuitive that the Manhattan distance would approach the Euclidean metric

$$\sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2},$$

but it does not. This is essentially a consequence of being forced to adhere to single-axis movement: when following the Manhattan metric, one cannot move diagonally (in more than one axis simultaneously).

Whenever each pair in a collection of these circles has a nonempty intersection, there exists an intersection point for the whole collection; therefore, the Manhattan distance forms an injective metric space.

A circle of radius 1 (using this distance) is the von Neumann neighborhood of its center.



Measures of distances in chess

In chess, the distance between squares on the chessboard for rooks is measured in Manhattan distance; kings and queens use Chebyshev distance, and bishops use the Manhattan distance (between squares of the same color) on the chessboard rotated 45 degrees, i.e., with its diagonals as coordinate axes. To reach from one square to another, only kings require the number of moves equal to the distance; rooks, queens and bishops require one or two moves (on an empty board, and assuming that the move is possible at all in the bishop's case).

Notes

[1] Manhattan distance (<http://www.nist.gov/dads/HTML/manhattanDistance.html>)

References

- Eugene F. Krause (1987). *Taxicab Geometry*. Dover. ISBN 0-486-25202-7.

External links

- city-block metric (<http://planetmath.org/encyclopedia/CityBlockMetric.html>) on PlanetMath
 - Weisstein, Eric W., "Taxicab Metric (<http://mathworld.wolfram.com/TaxicabMetric.html>)" from MathWorld.
 - Manhattan distance (<http://www.nist.gov/dads/HTML/manhattanDistance.html>). Paul E. Black, Dictionary of Algorithms and Data Structures (<http://www.nist.gov/dads/>), NIST
 - Taxi! (<http://www.ams.org/featurecolumn/archive/taxi.html>) - AMS column about Taxicab geometry
 - TaxicabGeometry.net (<http://www.taxicabgeometry.net>) - a website dedicated to taxicab geometry research and information
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