Índice

1. Teoría de números

1.1. Funciones básicas

1.1.1. Función piso y techo

```
lli piso(lli a, lli b){
  if((a >= 0 && b > 0) || (a < 0 && b < 0)){
    return a / b;
  }else{
    if(a % b == 0) return a / b;
    else return a / b - 1;
  }
}

lli techo(lli a, lli b){
  if((a >= 0 && b > 0) || (a < 0 && b < 0)){
    if(a % b == 0) return a / b;
    else return a / b + 1;
  }else{
    return a / b;
  }
}
```

1.1.2. Exponenciación y multiplicación binaria

```
lli power(lli b, lli e){
  lli ans = 1;
  while(e){
    if(e & 1) ans *= b;
    e >>= 1;
    b *= b;
  }
  return ans;
}

lli multMod(lli a, lli b, lli n){
  lli ans = 0;
  a %= n, b %= n;
  if(abs(b) > abs(a)) swap(a, b);
  if(b < 0){
    a *= -1, b *= -1;
  }
  while(b){
    if(b & 1) ans = (ans + a) % n;
    b >>= 1;
    a = (a + a) % n;
  }
  return ans;
}

uint64_t mul_mod(uint64_t a, uint64_t b, uint64_t m){
  if(a >= m) a %= m;
  if(b >= m) b %= m;
  uint64_t c = (long double)a * b / m;
  int64_t r = (int64_t)(a * b - c * m) % (int64_t)m;
  return r < 0 ? r + m : r;
}
```

1.1.3. Mínimo común múltiplo y máximo común divisor

```
lli gcd(lli a, lli b){
  lli r;
  while(b != 0) r = a % b, a = b, b = r;
  return a;
}

lli lcm(lli a, lli b){
  return b * (a / gcd(a, b));
}

lli gcd(vector<lli> & nums){
  lli ans = 0;
  for(lli & num : nums) ans = gcd(ans, num);
  return ans;
}

lli lcm(vector<lli> & nums){
  lli ans = 1;
  for(lli & num : nums) ans = lcm(ans, num);
  return ans;
}
```

1.1.4. Euclides extendido e inverso modular

```
lli extendedGcd(lli a, lli b, lli & s, lli & t){
  lli q, r0 = a, r1 = b, ri, s0 = 1, s1 = 0, si, t0 = 0, t1 = 1,
  ↪  ti;
  while(r1){
    q = r0 / r1;
    ri = r0 % r1, r0 = r1, r1 = ri;
    si = s0 - s1 * q, s0 = s1, s1 = si;
    ti = t0 - t1 * q, t0 = t1, t1 = ti;
  }
  s = s0, t = t0;
  return r0;
}

lli modularInverse(lli a, lli m){
  lli r0 = a, r1 = m, ri, s0 = 1, s1 = 0, si;
  while(r1){
    si = s0 - s1 * (r0 / r1), s0 = s1, s1 = si;
    ri = r0 % r1, r0 = r1, r1 = ri;
  }
  if(r0 < 0) s0 *= -1;
  if(s0 < 0) s0 += m;
  return s0;
}
```

1.1.5. Todos los inversos módulo p

```
//find all inverses (from 1 to p-1) modulo p
vector<lli> allInverses(lli p){
  vector<lli> ans(p);
  ans[1] = 1;
  for(lli i = 2; i < p; ++i)
    ans[i] = p - (p / i) * ans[p % i] % p;
  return ans;
}
```

1.1.6. Exponenciación binaria modular

```
lli powerMod(lli b, lli e, lli m){
  lli ans = 1;
  b %= m;
  if(e < 0){
    b = modularInverse(b, m);
    e *= -1;
  }
  while(e){
    if(e & 1) ans = (ans * b) % m;
    e >>= 1;
    b = (b * b) % m;
  }
  return ans;
}
```

1.1.7. Teorema chino del residuo

```
pair<lli, lli> chinese(vector<lli> & a, vector<lli> & m){
  lli prod = 1, p, ans = 0;
  for(lli & ni : m) prod *= ni;
  for(int i = 0; i < a.size(); ++i){
    p = prod / m[i];
    ans += (a[i] % m[i]) * modularInverse(p, m[i]) % prod * p %
    ↪  prod;
    while(ans >= prod) ans -= prod; while(ans < 0) ans += prod;
  }
  return {ans, prod};
}
```

1.1.8. Teorema chino del residuo generalizado

```
//generalized chinese remainder theorem
//the modulos doesn't need to be pairwise coprime
pair<lli, lli> crt(const vector<lli> & a, const vector<lli> & m){
  lli a0 = a[0] % m[0], m0 = m[0], a1, m1, s, t, d, M;
  for(int i = 1; i < a.size(); ++i){
    a1 = a[i] % m[i], m1 = m[i];
    d = extendedGcd(m0, m1, s, t);
    if((a0 - a1) % d != 0) return {0, 0}; //error, no solution
    M = m0 * (m1 / d);
    a0 = a0 * t % M * (m1 / d) % M + a1 * s % M * (m0 / d) % M;
    while(a0 >= M) a0 -= M; while(a0 < 0) a0 += M;
    m0 = M;
  }
  while(a0 >= m0) a0 -= m0; while(a0 < 0) a0 += m0;
```

```
  return {a0, m0};
}
```

1.1.9. Coeficiente binomial

```
lli ncr(lli n, lli r){
  if(r < 0 || r > n) return 0;
  r = min(r, n - r);
  lli ans = 1;
  for(lli den = 1, num = n; den <= r; den++, num--)
    ans = ans * num / den;
  return ans;
}
```

1.1.10. Fibonacci

```
//very fast fibonacci
inline void modula(lli & n, lli mod){
  while(n >= mod) n -= mod;
}

lli fibo(lli n, lli mod){
  array<lli, 2> F = {1, 0};
  lli p = 1;
  for(lli v = n; v >>= 1; p <<= 1);
  array<lli, 4> C;
  do{
    int d = (n & p) != 0;
    C[0] = C[3] = 0;
    C[d] = F[0] * F[0] % mod;
    C[d+1] = (F[0] * F[1] << 1) % mod;
    C[d+2] = F[1] * F[1] % mod;
    F[0] = C[0] + C[2] + C[3];
    F[1] = C[1] + C[2] + (C[3] << 1);
    modula(F[0], mod), modula(F[1], mod);
  }while(p >>= 1);
  return F[1];
}
```

1.2. Cribas

1.2.1. Criba de divisores

```
vector<lli> divisorsSum;
vector<vector<int>> divisors;
void divisorsSieve(int n){
  divisorsSum.resize(n + 1, 0);
  divisors.resize(n + 1);
  for(int i = 1; i <= n; ++i){
    for(int j = i; j <= n; j += i){
      divisorsSum[j] += i;
      divisors[j].push_back(i);
    }
  }
}
```

1.2.2. Criba de primos

```
vector<int> primes;
vector<bool> isPrime;
void primesSieve(int n){
  isPrime.resize(n + 1, true);
  isPrime[0] = isPrime[1] = false;
  primes.push_back(2);
  for(int i = 4; i <= n; i += 2) isPrime[i] = false;
  int limit = sqrt(n);
  for(int i = 3; i <= n; i += 2){
    if(isPrime[i]){
      primes.push_back(i);
      if(i <= limit)
        for(int j = i * i; j <= n; j += 2 * i)
          isPrime[j] = false;
    }
  }
}
```

1.2.3. Criba de factor primo más pequeño

```
vector<int> lowestPrime;
void lowestPrimeSieve(int n){
  lowestPrime.resize(n + 1, 1);
```

```
  lowestPrime[0] = lowestPrime[1] = 0;
  for(int i = 2; i <= n; ++i) lowestPrime[i] = (i & 1 ? i : 2);
  int limit = sqrt(n);
  for(int i = 3; i <= limit; i += 2)
    if(lowestPrime[i] == i)
      for(int j = i * i; j <= n; j += 2 * i)
        if(lowestPrime[j] == j) lowestPrime[j] = i;
}
```

1.2.4. Criba de factor primo más grande

```
vector<int> greatestPrime;
void greatestPrimeSieve(int n){
  greatestPrime.resize(n + 1, 1);
  greatestPrime[0] = greatestPrime[1] = 0;
  for(int i = 2; i <= n; ++i) greatestPrime[i] = i;
  for(int i = 2; i <= n; i++)
    if(greatestPrime[i] == i)
      for(int j = i; j <= n; j += i)
        greatestPrime[j] = i;
}
```

1.2.5. Criba de factores primos

```
vector<vector<int>> primeFactors;
void primeFactorsSieve(lli n){
  primeFactors.resize(n + 1);
  for(int i = 0; i < primes.size(); ++i){
    int p = primes[i];
    for(int j = p; j <= n; j += p)
      primeFactors[j].push_back(p);
  }
}
```

1.2.6. Criba de la función φ de Euler

```
vector<int> Phi;
void phiSieve(int n){
  Phi.resize(n + 1);
  for(int i = 1; i <= n; ++i) Phi[i] = i;
  for(int i = 2; i <= n; ++i)
    if(Phi[i] == i)
      for(int j = i; j <= n; j += i)
        Phi[j] -= Phi[j] / i;
}
```

1.2.7. Criba de la función μ

```
vector<int> Mu;
void muSieve(int n){
  Mu.resize(n + 1, -1);
  Mu[0] = 0, Mu[1] = 1;
  for(int i = 2; i <= n; ++i)
    if(Mu[i])
      for(int j = 2*i; j <= n; j += i)
        Mu[j] -= Mu[i];
}
```

1.2.8. Triángulo de Pascal

```
vector<vector<lli>> Ncr;
void ncrSieve(lli n){
  Ncr.resize(n + 1);
  Ncr[0] = {1};
  for(lli i = 1; i <= n; ++i){
    Ncr[i].resize(i + 1);
    Ncr[i][0] = Ncr[i][i] = 1;
    for(lli j = 1; j <= i / 2; j++)
      Ncr[i][i - j] = Ncr[i][j] = Ncr[i - 1][j - 1] + Ncr[i -
      ↪ 1][j];
  }
}
```

1.2.9. Segmented sieve

```
vector<int> segmented_sieve(int limit){
  const int L1D_CACHE_SIZE = 32768;
  int raiz = sqrt(limit);
  int segment_size = max(raiz, L1D_CACHE_SIZE);
  int s = 3, n = 3;
  vector<int> primes(1, 2), tmp, next;
  vector<char> sieve(segment_size);
```

```
  vector<bool> is_prime(raiz + 1, 1);
  for(int i = 2; i * i <= raiz; i++)
    if(is_prime[i])
      for(int j = i * i; j <= raiz; j += i)
        is_prime[j] = 0;
  for(int low = 0; low <= limit; low += segment_size){
    fill(sieve.begin(), sieve.end(), 1);
    int high = min(low + segment_size - 1, limit);
    for(; s * s <= high; s += 2){
      if(is_prime[s]){
        tmp.push_back(s);
        next.push_back(s * s - low);
      }
    }
    for(size_t i = 0; i < tmp.size(); i++){
      int j = next[i];
      for(int k = tmp[i] * 2; j < segment_size; j += k)
        sieve[j] = 0;
      next[i] = j - segment_size;
    }
    for(; n <= high; n += 2)
      if(sieve[n - low])
        primes.push_back(n);
  }
  return primes;
}
```

1.2.10. Criba de primos lineal

```
vector<int> linearPrimeSieve(int n){
  vector<int> primes;
  vector<bool> isPrime(n+1, true);
  for(int i = 2; i <= n; ++i){
    if(isPrime[i])
      primes.push_back(i);
    for(int p : primes){
      int d = i * p;
      if(d > n) break;
      isPrime[d] = false;
      if(i % p == 0) break;
    }
  }
  return primes;
}
```

1.2.11. Criba lineal para funciones multiplicativas

```
//suppose f(n) is a multiplicative function and
//we want to find f(1), f(2), ..., f(n)
//we have f(pq) = f(p)f(q) if gcd(p, q) = 1
//and f(p^a) = g(p, a), where p is prime and a>0
vector<int> generalSieve(int n, function<int(int, int)> g){
  vector<int> f(n+1, 1), cnt(n+1), acum(n+1), primes;
  vector<bool> isPrime(n+1, true);
  for(int i = 2; i <= n; ++i){
    if(isPrime[i]){ //case base: f(p)
      f[i] = g(i, 1);
      primes.push_back(i);
      cnt[i] = 1;
      acum[i] = i;
    }
    for(int p : primes){
      int d = i * p;
      if(d > n) break;
      isPrime[d] = false;
      if(i % p == 0){ //gcd(i, p) != 1
        f[d] = f[i / acum[i]] * g(p, cnt[i] + 1);
        cnt[d] = cnt[i] + 1;
        acum[d] = acum[i] * p;
        break;
      }else{ //gcd(i, p) = 1
        f[d] = f[i] * g(p, 1);
        cnt[d] = 1;
        acum[d] = p;
      }
    }
  }
  return f;
}
```

1.3. Factorización

1.3.1. Factorización de un número

```
vector<pair<lli, int>> factorize(lli n){
  vector<pair<lli, int>> f;
  for(lli p : primes){
    if(p * p > n) break;
    int pot = 0;
    while(n % p == 0){
      pot++;
      n /= p;
    }
    if(pot) f.emplace_back(p, pot);
  }
  if(n > 1) f.emplace_back(n, 1);
  return f;
}
```

1.3.2. Potencia de un primo que divide a un factorial

```
lli potInFactorial(lli n, lli p){
  lli ans = 0, div = n;
  while(div /= p) ans += div;
  return ans;
}
```

1.3.3. Factorización de un factorial

```
vector<pair<lli, lli>> factorizeFactorial(lli n){
  vector<pair<lli, lli>> f;
  for(lli p : primes){
    if(p > n) break;
    f.emplace_back(p, potInFactorial(n, p));
  }
  return f;
}
```

1.3.4. Factorial módulo p

```
//Finds (n!/p^m) mod p^s, where m is the largest power of p
//that divides n!, p must be prime
lli factmod(lli n, lli p, int s){
  lli ans = 1;
  lli ps = power(p, s);
  while(n > 1){
    lli q = n / ps, r = n % ps;
    ans = ans * (q % 2 == 1 && !(p == 2 && s >= 3) ? ps-1 : 1) %
    ↪ ps;
    for(lli i = 2; i <= r; ++i){
      if(i % p == 0) continue;
      ans = ans * i % ps;
    }
    n /= p;
  }
  return ans;
}
```

1.3.5. Factorización usando Pollard-Rho

```
bool isPrimeMillerRabin(lli n){
  if(n < 2) return false;
  if(!(n & 1)) return n == 2;
  lli d = n - 1, s = 0;
  for(; !(d & 1); d >>= 1, ++s);
  for(int a : {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37}){
    if(n == a) return true;
    lli m = powerMod(a, d, n);
    if(m == 1 || m == n - 1) continue;
    int k = 0;
    for(; k < s; ++k){
      m = m * m % n;
      if(m == n - 1) break;
    }
    if(k == s) return false;
  }
  return true;
}

mt19937_64
↪ rng(chrono::steady_clock::now().time_since_epoch().count());
```

```
lli aleatorio(lli a, lli b){
  std::uniform_int_distribution<lli> dist(a, b);
  return dist(rng);
}
lli getFactor(lli n){
  lli a = aleatorio(1, n - 1), b = aleatorio(1, n - 1);
  lli x = 2, y = 2, d = 1;
  while(d == 1){
    x = x * (x + b) % n + a;
    y = y * (y + b) % n + a;
    y = y * (y + b) % n + a;
    d = gcd(abs(x - y), n);
  }
  return d;
}

map<lli, int> fact;
void factorizePollardRho(lli n, bool clean = true){
  if(clean) fact.clear();
  while(n > 1 && !isPrimeMillerRabin(n)){
    lli f = n;
    for(; f == n; f = getFactor(n));
    n /= f;
    factorizePollardRho(f, false);
    for(auto & it : fact){
      while(n % it.first == 0){
        n /= it.first;
        ++it.second;
      }
    }
  }
  if(n > 1) ++fact[n];
}
```

1.4. Funciones aritméticas famosas

1.4.1. Función σ

```
//divisor power sum of n
//if pot=0 we get the number of divisors
//if pot=1 we get the sum of divisors
lli sigma(lli n, lli pot){
  lli ans = 1;
  auto f = factorize(n);
  for(auto & factor : f){
    lli p = factor.first;
    int a = factor.second;
    if(pot){
      lli p_pot = power(p, pot);
      ans *= (power(p_pot, a + 1) - 1) / (p_pot - 1);
    }else{
      ans *= a + 1;
    }
  }
  return ans;
}
```

1.4.2. Función Ω

```
//number of total primes with multiplicity dividing n
int Omega(lli n){
  int ans = 0;
  auto f = factorize(n);
  for(auto & factor : f)
    ans += factor.second;
  return ans;
}
```

1.4.3. Función ω

```
//number of distinct primes dividing n
int omega(lli n){
  int ans = 0;
  auto f = factorize(n);
  for(auto & factor : f)
    ++ans;
  return ans;
}
```

1.4.4. Función φ de Euler

```
//number of coprimes with n less than n
lli phi(lli n){
  lli ans = n;
```

```
  auto f = factorize(n);
  for(auto & factor : f)
    ans -= ans / factor.first;
  return ans;
}
```

1.4.5. Función μ

```
//1 if n is square-free with an even number of prime factors
//-1 if n is square-free with an odd number of prime factors
//0 is n has a square prime factor
int mu(lli n){
  int ans = 1;
  auto f = factorize(n);
  for(auto & factor : f){
    if(factor.second > 1) return 0;
    ans *= -1;
  }
  return ans;
}
```

1.5. Orden multiplicativo, raíces primitivas y raíces de la unidad

1.5.1. Función λ de Carmichael

```
//the smallest positive integer k such that for
//every coprime x with n, x^k=1 mod n
lli carmichaelLambda(lli n){
  lli ans = 1;
  auto f = factorize(n);
  for(auto & factor : f){
    lli p = factor.first;
    int a = factor.second;
    lli tmp = power(p, a);
    tmp -= tmp / p;
    if(a <= 2 || p >= 3) ans = lcm(ans, tmp);
    else ans = lcm(ans, tmp >> 1);
  }
  return ans;
}
```

1.5.2. Orden multiplicativo módulo m

```
// the smallest positive integer k such that x^k = 1 mod m
lli multiplicativeOrder(lli x, lli m){
  if(gcd(x, m) != 1) return 0;
  lli order = phi(m);
  auto f = factorize(order);
  for(auto & factor : f){
    lli p = factor.first;
    int a = factor.second;
    order /= power(p, a);
    lli tmp = powerMod(x, order, m);
    while(tmp != 1){
      tmp = powerMod(tmp, p, m);
      order *= p;
    }
  }
  return order;
}
```

1.5.3. Número de raíces primitivas (generadores) módulo m

```
//number of generators modulo m
lli numberOfGenerators(lli m){
  lli phi_m = phi(m);
  lli lambda_m = carmichaelLambda(m);
  if(phi_m == lambda_m) return phi(phi_m);
  else return 0;
}
```

1.5.4. Test individual de raíz primitiva módulo m

```
//test if order(x, m) = phi(m), i.e., x is a generator for Z/mZ
bool testPrimitiveRoot(lli x, lli m){
  if(gcd(x, m) != 1) return false;
  lli order = phi(m);
  auto f = factorize(order);
  for(auto & factor : f){
    lli p = factor.first;
    if(powerMod(x, order / p, m) == 1) return false;
  }
  return true;
```

```
}
```

1.5.5. Test individual de raíz k-ésima de la unidad módulo m

```
//test if x^k = 1 mod m and k is the smallest for such x, i.e.,
↪ x^(k/p) != 1 for every prime divisor of k
bool testPrimitiveKthRootUnity(lli x, lli k, lli m){
  if(powerMod(x, k, m) != 1) return false;
  auto f = factorize(k);
  for(auto & factor : f){
    lli p = factor.first;
    if(powerMod(x, k / p, m) == 1) return false;
  }
  return true;
}
```

1.5.6. Encontrar la primera raíz primitiva módulo m

```
lli findFirstGenerator(lli m){
  lli order = phi(m);
  if(order != carmichaelLambda(m)) return -1; //just an
  ↪ optimization, not required
  auto f = factorize(order);
  for(lli x = 1; x < m; x++){
    if(gcd(x, m) != 1) continue;
    bool test = true;
    for(auto & factor : f){
      lli p = factor.first;
      if(powerMod(x, order / p, m) == 1){
        test = false;
        break;
      }
    }
    if(test) return x;
  }
  return -1; //not found
}
```

1.5.7. Encontrar la primera raíz k-ésima de la unidad módulo m

```
lli findFirstPrimitiveKthRootUnity(lli k, lli m){
  if(carmichaelLambda(m) % k != 0) return -1; //just an
  ↪ optimization, not required
  auto f = factorize(k);
  for(lli x = 1; x < m; x++){
    if(powerMod(x, k, m) != 1) continue;
    bool test = true;
    for(auto & factor : f){
      lli p = factor.first;
      if(powerMod(x, k / p, m) == 1){
        test = false;
        break;
      }
    }
    if(test) return x;
  }
  return -1; //not found
}
```

1.5.8. Logaritmo discreto

```
// Solves for x in the equation a^x = b mod m
pair<lli, lli> discreteLogarithm(lli a, lli b, lli m){
  lli m1 = m, pw = 1, d, x, y, nonRep = 0;
  for(; (d = gcd(a, m1)) > 1; ++nonRep, m1 /= d, pw = pw * a % m){
    if(pw == b) return {nonRep, 0}; //aperiodic solution found
  }
  d = extendedGcd(pw, m, x, y);
  if(b % d > 0) return {-1, 0}; //solution not found
  b = x * (b / d) % m;
  if(b < 0) b += m;
  lli order = multiplicativeOrder(a, m1);
  lli n = sqrt(order) + 1;
  lli a_n = powerMod(a, n, m1);
  unordered_map<lli, lli> firstHalf;
  pw = a_n;
  for(lli p = 1; p <= n; ++p, pw = pw * a_n % m1){
    firstHalf[pw] = p;
  }
  pw = b % m1;
```

```
  for(lli q = 0; q <= n; ++q, pw = pw * a % m1){
    if(firstHalf.count(pw)) return {nonRep + (n * firstHalf[pw] -
    ↪ q) % order, order}; //periodic solution found
  }
  return {-1, 0}; //solution not found
}
```

1.5.9. Raíz k-ésima discreta

```
// x^k = b mod m, m has at least one generator
vector<lli> discreteRoot(lli k, lli b, lli m){
  if(b % m == 0) return {0};
  lli g = findFirstGenerator(m);
  lli power = powerMod(g, k, m);
  auto y0 = discreteLogarithm(power, b, m);
  if(y0.first == -1) return {};
  lli phi_m = phi(m);
  lli d = gcd(k, phi_m);
  vector<lli> x(d);
  x[0] = powerMod(g, y0.first, m);
  lli inc = powerMod(g, phi_m / d, m);
  for(lli i = 1; i < d; i++)
    x[i] = x[i - 1] * inc % m;
  sort(x.begin(), x.end());
  return x;
}
```

1.5.10. Algoritmo de Tonelli-Shanks para raíces cuadradas módulo p

```
//finds x such that x^2 = a mod p
lli sqrtMod(lli a, lli p){
  a %= p;
  if(a < 0) a += p;
  if(a == 0) return 0;
  assert(powerMod(a, (p - 1) / 2, p) == 1);
  if(p % 4 == 3) return powerMod(a, (p + 1) / 4, p);
  lli s = p - 1;
  int r = 0;
  while((s & 1) == 0) ++r, s >>= 1;
  lli n = 2;
  while(powerMod(n, (p - 1) / 2, p) != p - 1) ++n;
  lli x = powerMod(a, (s + 1) / 2, p);
  lli b = powerMod(a, s, p);
  lli g = powerMod(n, s, p);
  while(true){
    lli t = b;
    int m = 0;
    for(; m < r; ++m){
      if(t == 1) break;
      t = t * t % p;
    }
    if(m == 0) return x;
    lli gs = powerMod(g, 1 << (r - m - 1), p);
    g = gs * gs % p;
    x = x * gs % p;
    b = b * g % p;
    r = m;
  }
}
```

1.6. Particiones

1.6.1. Función P (particiones de un entero positivo)

```
lli mod = 1e9 + 7;

vector<lli> P;

//number of ways to write n as a sum of positive integers
lli partitionsP(int n){
  if(n < 0) return 0;
  if(P[n]) return P[n];
  int pos1 = 1, pos2 = 2, inc1 = 4, inc2 = 5;
  lli ans = 0;
  for(int k = 1; k <= n; k++){
    lli tmp = (n >= pos1 ? P[n - pos1] : 0) + (n >= pos2 ? P[n -
    ↪ pos2] : 0);
    if(k & 1) ans += tmp;
    else ans -= tmp;
    if(n < pos2) break;
    pos1 += inc1, pos2 += inc2;
    inc1 += 3, inc2 += 3;
  }
  ans %= mod;
```

```
  if(ans < 0) ans += mod;
  return ans;
}

void calculateFunctionP(int n){
  P.resize(n + 1);
  P[0] = 1;
  for(int i = 1; i <= n; i++)
    P[i] = partitionsP(i);
}
```

1.6.2. Función Q (particiones de un entero positivo en distintos sumandos)

```
vector<lli> Q;

bool isPerfectSquare(int n){
  int r = sqrt(n);
  return r * r == n;
}

int s(int n){
  int r = 1 + 24 * n;
  if(isPerfectSquare(r)){
    int j;
    r = sqrt(r);
    if((r + 1) % 6 == 0) j = (r + 1) / 6;
    else j = (r - 1) / 6;
    if(j & 1) return -1;
    else return 1;
  }else{
    return 0;
  }
}

//number of ways to write n as a sum of distinct positive integers
//number of ways to write n as a sum of odd positive integers
lli partitionsQ(int n){
  if(n < 0) return 0;
  if(Q[n]) return Q[n];
  int pos = 1, inc = 3;
  lli ans = 0;
  int limit = sqrt(n);
  for(int k = 1; k <= limit; k++){
    if(k & 1) ans += Q[n - pos];
    else ans -= Q[n - pos];
    pos += inc;
    inc += 2;
  }
  ans <<= 1;
  ans += s(n);
  ans %= mod;
  if(ans < 0) ans += mod;
  return ans;
}

void calculateFunctionQ(int n){
  Q.resize(n + 1);
  Q[0] = 1;
  for(int i = 1; i <= n; i++)
    Q[i] = partitionsQ(i);
}
```

1.6.3. Número de factorizaciones ordenadas

```
//number of ordered factorizations of n
lli orderedFactorizations(lli n){
  //skip the factorization if you already know the powers
  auto fact = factorize(n);
  int k = 0, q = 0;
  vector<int> powers(fact.size() + 1);
  for(auto & f : fact){
    powers[k + 1] = f.second;
    q += f.second;
    ++k;
  }
  vector<lli> prod(q + 1, 1);
  //we need Ncr until the max_power+Omega(n) row
  //module if needed
  for(int i = 0; i <= q; i++){
    for(int j = 1; j <= k; j++){
      prod[i] = prod[i] * Ncr[powers[j] + i][powers[j]];
    }
  }
  lli ans = 0;
  for(int j = 1; j <= q; j++){
```

```
    int alt = 1;
    for(int i = 0; i < j; i++){
      ans = ans + alt * Ncr[j][i] * prod[j - i - 1];
      alt *= -1;
    }
  }
  return ans;
}
```

1.6.4. Número de factorizaciones no ordenadas

```
//Number of unordered factorizations of n with
//largest part at most m
//Call unorderedFactorizations(n, n) to get all of them
//Add this to the main to speed up the map:
//mem.reserve(1024); mem.max_load_factor(0.25);
struct HASH{
  size_t operator()(const pair<int,int>&x)const{
    return hash<long long>()(((long long)x.first)^(((long
    ↪ long)x.second)<<32));
  }
};
unordered_map<pair<int, int>, lli, HASH> mem;
lli unorderedFactorizations(int m, int n){
  if(m == 1 && n == 1) return 1;
  if(m == 1) return 0;
  if(n == 1) return 1;
  if(mem.count({m, n})) return mem[{m, n}];
  lli ans = 0;
  int l = sqrt(n);
  for(int i = 1; i <= l; ++i){
    if(n % i == 0){
      int a = i, b = n / i;
      if(a <= m) ans += unorderedFactorizations(a, b);
      if(a != b && b <= m) ans += unorderedFactorizations(b, a);
    }
  }
  return mem[{m, n}] = ans;
}
```

1.7. Otros

1.7.1. Cambio de base

```
string decimalToBaseB(lli n, lli b){
  string ans = "";
  lli d;
  do{
    d = n % b;
    if(0 <= d && d <= 9) ans = (char)(48 + d) + ans;
    else if(10 <= d && d <= 35) ans = (char)(55 + d) + ans;
    n /= b;
  }while(n != 0);
  return ans;
}

lli baseBtoDecimal(const string & n, lli b){
  lli ans = 0;
  for(const char & d : n){
    if(48 <= d && d <= 57) ans = ans * b + (d - 48);
    else if(65 <= d && d <= 90) ans = ans * b + (d - 55);
    else if(97 <= d && d <= 122) ans = ans * b + (d - 87);
  }
  return ans;
}
```

1.7.2. Fracciones continuas

```
//continued fraction of (p+sqrt(n))/q, where p,n,q are positive
↪ integers
//returns a vector of terms and the length of the period,
//the periodic part is taken from the right of the array
pair<vector<lli>, int> ContinuedFraction(lli p, lli n, lli q){
  vector<lli> coef;
  lli r = sqrt(n);
  //Skip this if you know that n is not a perfect square
  if(r * r == n){
    lli num = p + r;
    lli den = q;
    lli residue;
    while(den){
      residue = num % den;
```

```
      coef.push_back(num / den);
      num = den;
      den = residue;
    }
    return {coef, 0};
  }
  if((n - p * p) % q != 0){
    n *= q * q;
    p *= q;
    q *= q;
    r = sqrt(n);
  }
  lli a = (r + p) / q;
  coef.push_back(a);
  int period = 0;
  map<pair<lli, lli>, int> pairs;
  while(true){
    p = a * q - p;
    q = (n - p * p) / q;
    a = (r + p) / q;
    //if p=0 and q=1, we can just ask if q==1 after inserting a
    if(pairs.count({p, q})){
      period -= pairs[{p, q}];
      break;
    }
    coef.push_back(a);
    pairs[{p, q}] = period++;
  }
  return {coef, period};
}
```

1.7.3. Ecuación de Pell

```
//first solution (x, y) to the equation x^2-ny^2=1, n IS NOT a
↪ perfect aquare
pair<lli, lli> PellEquation(lli n){
  vector<lli> cf = ContinuedFraction(0, n, 1).first;
  lli num = 0, den = 1;
  int k = cf.size() - 1;
  for(int i = ((k & 1) ? (2 * k - 1) : (k - 1)); i >= 0; i--){
    lli tmp = den;
    int pos = i % k;
    if(pos == 0 && i != 0) pos = k;
    den = num + cf[pos] * den;
    num = tmp;
  }
  return {den, num};
}
```

1.7.4. Números de Bell

```
//number of ways to partition a set of n elements
//the nth bell number is at Bell[n][0]
vector<vector<int>> Bell;
void bellNumbers(int n){
  Bell.resize(n + 1);
  Bell[0] = {1};
  for(int i = 1; i <= n; ++i){
    Bell[i].resize(i + 1);
    Bell[i][0] = Bell[i - 1][i - 1];
    for(int j = 1; j <= i; ++j)
      Bell[i][j] = Bell[i][j - 1] + Bell[i - 1][j - 1];
  }
}
```

1.7.5. Números de Stirling

```
//s(n, k) represents the number of permutations
//of n elements with k disjoint cycles
vector<vector<lli>> stirling1;
void stirlingNumber1stKind(lli n){
  stirling1.resize(n+1, vector<lli>(n+1));
  stirling1[0][0] = 1;
  for(int i = 1; i <= n; ++i)
    for(int j = 1; j <= i; ++j)
      stirling1[i][j] = (i-1) * stirling1[i-1][j] +
      ↪ stirling1[i-1][j-1];
}

//S(n, k) represents the number of ways to
//partition a set of n object into k non-empty
//distinct subsets
vector<vector<lli>> stirling2;
void stirlingNumber2ndKind(lli n){
  stirling2.resize(n+1, vector<lli>(n+1));
```

```
  stirling2[0][0] = 1;
  for(int i = 1; i <= n; ++i)
    for(int j = 1; j <= i; ++j)
      stirling2[i][j] = j * stirling2[i-1][j] +
      ↪ stirling2[i-1][j-1];
}
```

1.7.6. Números de Euler

```
//euler(n, k) represents the number of permutations
//of 1,...,n with exactly k numbers greater than
//the previous number
vector<vector<lli>> euler;
void eulerianNumbers(lli n){
  euler.resize(n+1, vector<lli>(n+1));
  for(int i = 1; i <= n; ++i){
    euler[i][0] = 1;
    for(int j = 1; j < i; ++j)
      euler[i][j] = (i-j) * euler[i-1][j-1] + (j+1) *
      ↪ euler[i-1][j];
  }
}
```

1.7.7. Prime counting function in sublinear time

```
const lli inv_2 = modularInverse(2, Mod);
const lli inv_6 = modularInverse(6, Mod);
const lli inv_30 = modularInverse(30, Mod);

lli sum(lli n, int k){
  n %= Mod;
  if(k == 0) return n;
  if(k == 1) return n * (n + 1) % Mod * inv_2 % Mod;
  if(k == 2) return n * (n + 1) % Mod * (2*n + 1) % Mod * inv_6 %
  ↪ Mod;
  if(k == 3) return powMod(n * (n + 1) % Mod * inv_2 % Mod, 2,
  ↪ Mod);
  if(k == 4) return n * (n + 1) % Mod * (2*n + 1) % Mod *
  ↪ (3*n*(n+1)%Mod -1) % Mod * inv_30 % Mod;
  return 1;
}
```

```
//finds the sum of the kth powers of the primes
//less than or equal to n (0<=k<=4, add more if you need)
lli SumPrimePi(lli n, int k){
  lli v = sqrt(n), p, temp, q, j, end, i, d;
  vector<lli> lo(v+2), hi(v+2);
  vector<bool> used(v+2);
  for(p = 1; p <= v; p++){
    lo[p] = sum(p, k) - 1;
    hi[p] = sum(n/p, k) - 1;
  }
  for(p = 2; p <= v; p++){
    if(lo[p] == lo[p-1]) continue;
    temp = lo[p-1];
    q = p * p;
    hi[1] -= (hi[p] - temp) * powMod(p, k, Mod) % Mod;
    if(hi[1] < 0) hi[1] += Mod;
    j = 1 + (p & 1);
    end = (v <= n/q) ? v : n/q;
    for(i = p + j; i <= 1 + end; i += j){
      if(used[i]) continue;
      d = i * p;
      if(d <= v)
        hi[i] -= (hi[d] - temp) * powMod(p, k, Mod) % Mod;
      else
        hi[i] -= (lo[n/d] - temp) * powMod(p, k, Mod) % Mod;
      if(hi[i] < 0) hi[i] += Mod;
    }
    if(q <= v)
      for(i = q; i <= end; i += p*j)
        used[i] = true;
    for(i = v; i >= q; i--){
      lo[i] -= (lo[i/p] - temp) * powMod(p, k, Mod) % Mod;
      if(lo[i] < 0) lo[i] += Mod;
    }
  }
  return hi[1] % Mod;
}
```

1.7.8. Suma de la función piso

```
//finds sum(floor(p*i/q), 1<=i<=n)
lli floorsSum(lli p, lli q, lli n){
    lli t = gcd(p, q);
```

```
    p /= t, q /= t;
    lli s = 0, z = 1;
    while(q && n){
        t = p/q;
        s += z*t*n*(n+1)/2;
        p -= q*t;
        t = n/q;
        s += z*p*t*(n+1) - z*t*(p*q*t + p + q - 1)/2;
        n -= q*t;
        t = n*p/q;
        s += z*t*n;
        n = t;
        swap(p, q);
        z = -z;
    }
    return s;
}
```

1.7.9. Periodo de Pisano

```
lli pisano_prime(lli p){
  if(p == 2) return 3;
  if(p == 5) return 20;
  lli order = 0;
  if(p%10 == 1 || p%10 == 9) order = p - 1;
  else order = 2*p + 2;
  auto fact = factorize(order);
  for(auto par : fact){
    lli q; int a;
    tie(q, a) = par;
    order /= power(q, a);
    while(!(fibo(order, p) == 0 && fibo(order+1, p) == 1)){
      order *= q;
    }
  }
  return order;
}

lli pisano(lli mod){
  lli ans = 1;
  auto fact = factorize(mod);
  for(auto par : fact){
    lli p; int a;
    tie(p, a) = par;
    ans = lcm(ans, power(p, a-1) * pisano_prime(p));
  }
  return ans;
}
```

2. Números racionales

2.1. Estructura fraccion

```
struct fraccion{
    ll num, den;
    fraccion(){
        num = 0, den = 1;
    }
    fraccion(ll x, ll y){
        if(y < 0)
            x *= -1, y *=-1;
        ll d = __gcd(abs(x), abs(y));
        num = x/d, den = y/d;
    }
    fraccion(ll v){
        num = v;
        den = 1;
    }
    fraccion operator+(const fraccion& f) const{
        ll d = __gcd(den, f.den);
        return fraccion(num*(f.den/d) + f.num*(den/d), den*(f.den/d));
    }
    fraccion operator-() const{
        return fraccion(-num, den);
    }
    fraccion operator-(const fraccion& f) const{
        return *this + (-f);
    }
    fraccion operator*(const fraccion& f) const{
        return fraccion(num*f.num, den*f.den);
    }
    fraccion operator/(const fraccion& f) const{
        return fraccion(num*f.den, den*f.num);
    }
    fraccion operator+=(const fraccion& f){
        *this = *this + f;
        return *this;
    }
    fraccion operator-=(const fraccion& f){
        *this = *this - f;
        return *this;
    }
    fraccion operator++(int xd){
        *this = *this + 1;
        return *this;
    }
    fraccion operator--(int xd){
        *this = *this - 1;
        return *this;
    }
    fraccion operator*=(const fraccion& f){
        *this = *this * f;
        return *this;
    }
    fraccion operator/=(const fraccion& f){
        *this = *this / f;
        return *this;
    }
    bool operator==(const fraccion& f) const{
        ll d = __gcd(den, f.den);
        return (num*(f.den/d) == (den/d)*f.num);
    }
    bool operator!=(const fraccion& f) const{
        ll d = __gcd(den, f.den);
        return (num*(f.den/d) != (den/d)*f.num);
    }
    bool operator >(const fraccion& f) const{
        ll d = __gcd(den, f.den);
        return (num*(f.den/d) > (den/d)*f.num);
    }
    bool operator <(const fraccion& f) const{
        ll d = __gcd(den, f.den);
        return (num*(f.den/d) < (den/d)*f.num);
    }
    bool operator >=(const fraccion& f) const{
        ll d = __gcd(den, f.den);
        return (num*(f.den/d) >= (den/d)*f.num);
    }
    bool operator <=(const fraccion& f) const{
        ll d = __gcd(den, f.den);
        return (num*(f.den/d) <= (den/d)*f.num);
    }
    fraccion inverso() const{
        return fraccion(den, num);
    }
```

```
    fraccion fabs() const{
        fraccion nueva;
        nueva.num = abs(num);
        nueva.den = den;
        return nueva;
    }
    double value() const{
      return (double)num / (double)den;
    }
    string str() const{
        stringstream ss;
        ss << num;
        if(den != 1) ss << "/" << den;
        return ss.str();
    }
};

ostream &operator<<(ostream &os, const fraccion & f) {
    return os << f.str();
}

istream &operator>>(istream &is, fraccion & f){
    ll num = 0, den = 1;
    string str;
    is >> str;
    size_t pos = str.find("/");
    if(pos == string::npos){
        istringstream(str) >> num;
    }else{
        istringstream(str.substr(0, pos)) >> num;
        istringstream(str.substr(pos + 1)) >> den;
    }
    f = fraccion(num, den);
    return is;
}
```

3. Álgebra lineal

3.1. Estructura matrix

```
template <typename T>
struct matrix{
  vector<vector<T>> A;
  int m, n;

  matrix(int m, int n): m(m), n(n){
    A.resize(m, vector<T>(n, 0));
  }

  vector<T> & operator[] (int i){
    return A[i];
  }

  const vector<T> & operator[] (int i) const{
    return A[i];
  }

  static matrix identity(int n){
    matrix<T> id(n, n);
    for(int i = 0; i < n; i++)
      id[i][i] = 1;
    return id;
  }

  matrix operator+(const matrix & B) const{
    assert(m == B.m && n == B.n); //same dimensions
    matrix<T> C(m, n);
    for(int i = 0; i < m; i++)
      for(int j = 0; j < n; j++)
        C[i][j] = A[i][j] + B[i][j];
    return C;
  }

  matrix operator+=(const matrix & M){
    *this = *this + M;
    return *this;
  }

  matrix operator-() const{
```

```
    matrix<T> C(m, n);
    for(int i = 0; i < m; i++)
      for(int j = 0; j < n; j++)
        C[i][j] = -A[i][j];
    return C;
  }

  matrix operator-(const matrix & B) const{
    return *this + (-B);
  }

  matrix operator-=(const matrix & M){
    *this = *this + (-M);
    return *this;
  }

  matrix operator*(const matrix & B) const{
    assert(n == B.m); //#columns of 1st matrix = #rows of 2nd
    ↪  matrix
    matrix<T> C(m, B.n);
    for(int i = 0; i < m; i++)
      for(int j = 0; j < B.n; j++)
        for(int k = 0; k < n; k++)
          C[i][j] += A[i][k] * B[k][j];
    return C;
  }

  matrix operator*(const T & c) const{
    matrix<T> C(m, n);
    for(int i = 0; i < m; i++)
      for(int j = 0; j < n; j++)
        C[i][j] = A[i][j] * c;
    return C;
  }

  matrix operator*=(const matrix & M){
    *this = *this * M;
    return *this;
  }

  matrix operator*=(const T & c){
    *this = *this * c;
    return *this;
  }

  matrix operator^(lli b) const{
    matrix<T> ans = matrix<T>::identity(n);
    matrix<T> A = *this;
    while(b){
      if(b & 1) ans *= A;
      b >>= 1;
      if(b) A *= A;
    }
    return ans;
  }

  matrix operator^=(lli n){
    *this = *this ^ n;
    return *this;
  }

  bool operator==(const matrix & B) const{
    if(m != B.m || n != B.n) return false;
    for(int i = 0; i < m; i++)
      for(int j = 0; j < n; j++)
        if(A[i][j] != B[i][j]) return false;
    return true;
  }

  bool operator!=(const matrix & B) const{
    return !(*this == B);
  }

  void scaleRow(int k, T c){
    for(int j = 0; j < n; j++)
      A[k][j] *= c;
  }

  void swapRows(int k, int l){
    swap(A[k], A[l]);
  }

  void addRow(int k, int l, T c){
    for(int j = 0; j < n; j++)
      A[k][j] += c * A[l][j];
  }
```

3.2. Transpuesta y traza

```
matrix<T> transpose(){
  matrix<T> tr(n, m);
  for(int i = 0; i < m; i++)
    for(int j = 0; j < n; j++)
      tr[j][i] = A[i][j];
  return tr;
}

T trace(){
  T sum = 0;
  for(int i = 0; i < min(m, n); i++)
    sum += A[i][i];
  return sum;
}
```

3.3. Gauss Jordan

```
//full: true: reduce above and below the diagonal, false: reduce only below
//makeOnes: true: make the elements in the diagonal ones, false: leave the diagonal unchanged
//For every elemental operation that we apply to the matrix,
//we will call to callback(operation, k, l, value).
//operation 1: multiply row "k" by "value"
//operation 2: swap rows "k" and "l"
//operation 3: add "value" times the row "l" to the row "k"
//It returns the rank of the matrix, and modifies it
int gauss_jordan(bool full = true, bool makeOnes = true, function<void(int, int, int, T)>callback = NULL){
  int i = 0, j = 0;
  while(i < m && j < n){
    if(A[i][j] == 0){
      for(int f = i + 1; f < m; f++){
        if(A[f][j] != 0){
          swapRows(i, f);
          if(callback) callback(2, i, f, 0);
          break;
        }
      }
    }
    if(A[i][j] != 0){
      T inv_mult = A[i][j].inverso();
      if(makeOnes && A[i][j] != 1){
        scaleRow(i, inv_mult);
        if(callback) callback(1, i, 0, inv_mult);
      }
      for(int f = (full ? 0 : (i + 1)); f < m; f++){
        if(f != i && A[f][j] != 0){
          T inv_adit = -A[f][j];
          if(!makeOnes) inv_adit *= inv_mult;
          addRow(f, i, inv_adit);
          if(callback) callback(3, f, i, inv_adit);
        }
      }
      i++;
    }
    j++;
  }
  return i;
}

void gaussian_elimination(){
  gauss_jordan(false);
}
```

3.4. Matriz escalonada por filas y reducida por filas

```
matrix<T> reducedRowEchelonForm(){
  matrix<T> asoc = *this;
  asoc.gauss_jordan();
  return asoc;
}

matrix<T> rowEchelonForm(){
  matrix<T> asoc = *this;
  asoc.gaussian_elimination();
  return asoc;
}
```

3.5. Matriz inversa

```
bool invertible(){
  assert(m == n); //this is defined only for square matrices
```

```
    matrix<T> tmp = *this;
    return tmp.gauss_jordan(false) == n;
}

matrix<T> inverse(){
    assert(m == n); //this is defined only for square matrices
    matrix<T> tmp = *this;
    matrix<T> inv = matrix<T>::identity(n);
    auto callback = [&](int op, int a, int b, T e){
        if(op == 1){
            inv.scaleRow(a, e);
        }else if(op == 2){
            inv.swapRows(a, b);
        }else if(op == 3){
            inv.addRow(a, b, e);
        }
    };
    assert(tmp.gauss_jordan(true, true, callback) == n); //check
    ↪ non-invertible
    return inv;
}
```

3.6. Determinante

```
T determinant(){
    assert(m == n); //only square matrices have determinant
    matrix<T> tmp = *this;
    T det = 1;
    auto callback = [&](int op, int a, int b, T e){
        if(op == 1){
            det /= e;
        }else if(op == 2){
            det *= -1;
        }
    };
    if(tmp.gauss_jordan(false, true, callback) != n) det = 0;
    return det;
}
```

3.7. Matriz de cofactores y adjunta

```
matrix<T> minor(int x, int y){
    matrix<T> M(m-1, n-1);
    for(int i = 0; i < m-1; ++i)
        for(int j = 0; j < n-1; ++j)
            M[i][j] = A[i < x ? i : i+1][j < y ? j : j+1];
    return M;
}

T cofactor(int x, int y){
    T ans = minor(x, y).determinant();
    if((x + y) % 2 == 1) ans *= -1;
    return ans;
}

matrix<T> cofactorMatrix(){
    matrix<T> C(m, n);
    for(int i = 0; i < m; i++)
        for(int j = 0; j < n; j++)
            C[i][j] = cofactor(i, j);
    return C;
}

matrix<T> adjugate(){
    if(invertible()) return inverse() * determinant();
    return cofactorMatrix().transpose();
}
```

3.8. Factorización $PA = LU$

```
tuple<matrix<T>, matrix<T>, matrix<T>> PA_LU(){
    matrix<T> U = *this;
    matrix<T> L = matrix<T>::identity(n);
    matrix<T> P = matrix<T>::identity(n);
    auto callback = [&](int op, int a, int b, T e){
        if(op == 2){
            L.swapRows(a, b);
            P.swapRows(a, b);
            L[a][a] = L[b][b] = 1;
            L[a][a + 1] = L[b][b - 1] = 0;
        }else if(op == 3){
            L[a][b] = -e;
```

```
    }
  };
  U.gauss_jordan(false, false, callback);
  return {P, L, U};
}
```

3.9. Polinomio característico

```
vector<T> characteristicPolynomial(){
  matrix<T> M(n, n);
  vector<T> coef(n + 1);
  matrix<T> I = matrix<T>::identity(n);
  coef[n] = 1;
  for(int i = 1; i <= n; i++){
    M = (*this) * M + I * coef[n - i + 1];
    coef[n - i] = -((*this) * M).trace() / i;
  }
  return coef;
}
```

3.10. Gram-Schmidt

```
matrix<T> gram_schmidt(){
  //vectors are rows of the matrix (also in the answer)
  //the answer doesn't have the vectors normalized
  matrix<T> B = (*this) * (*this).transpose();
  matrix<T> ans = *this;
  auto callback = [&](int op, int a, int b, T e){
    if(op == 1){
      ans.scaleRow(a, e);
    }else if(op == 2){
      ans.swapRows(a, b);
    }else if(op == 3){
      ans.addRow(a, b, e);
    }
  };
  B.gauss_jordan(false, false, callback);
  return ans;
}
```

3.11. Recurrencias lineales

```
//Solves a linear homogeneous recurrence relation of degree "deg"
//of the form F(n) = a(d-1)*F(n-1) + a(d-2)*F(n-2) + ... +
↪  a(1)*F(n-(d-1)) + a(0)*F(n-d)
//with initial values F(0), F(1), ..., F(d-1)
//It finds the nth term of the recurrence, F(n)
//The values of a[0,...,d) are in the array P[]
lli solveRecurrence(const vector<lli> & P, const vector<lli> &
↪  init, lli n){
  int deg = P.size();
  vector<lli> ans(deg), R(2*deg);
  ans[0] = 1;
  lli p = 1;
  for(lli v = n; v >>= 1; p <<= 1);
  do{
    int d = (n & p) != 0;
    fill(R.begin(), R.end(), 0);
    for(int i = 0; i < deg; i++)
      for(int j = 0; j < deg; j++)
        (R[i + j + d] += ans[i] * ans[j]) %= mod;
    for(int i = deg-1; i >= 0; i--)
      for(int j = 0; j < deg; j++)
        (R[i + j] += R[i + deg] * P[j]) %= mod;
    copy(R.begin(), R.begin() + deg, ans.begin());
  }while(p >>= 1);
  lli nValue = 0;
  for(int i = 0; i < deg; i++)
    (nValue += ans[i] * init[i]) %= mod;
  return nValue;
}
```

3.12. Berlekamp-Massey

```
//Finds the shortest linear recurrence relation for the
//given init values. Only works for prime modulo.
vector<lli> BerlekampMassey(const vector<lli> & init){
  vector<lli> cur, ls;
  lli ld;
  for(int i = 0, m; i < init.size(); ++i){
    lli eval = 0;
    for(int j = 0; j < cur.size(); ++j)
      eval = (eval + init[i-j-1] * cur[j]) % mod;
```

```
    eval -= init[i];
    if(eval < 0) eval += mod;
    if(eval == 0) continue;
    if(cur.empty()){
      cur.resize(i + 1);
      m = i;
      ld = eval;
    }else{
      lli k = eval * inverse(ld, mod) % mod;
      vector<lli> c(i - m - 1);
      c.push_back(k);
      for(int j = 0; j < ls.size(); ++j)
        c.push_back((mod-ls[j]) * k % mod);
      if(c.size() < cur.size()) c.resize(cur.size());
      for(int j = 0; j < cur.size(); ++j){
        c[j] += cur[j];
        if(c[j] >= mod) c[j] -= mod;
      }
      if(i - m + ls.size() >= cur.size())
        ls = cur, m = i, ld = eval;
      cur = c;
    }
  }
  if(cur.empty()) cur.push_back(0);
  reverse(cur.begin(), cur.end());
  return cur;
}
```

3.13. Simplex

```
/*
Parametric Self-Dual Simplex method
Solve a canonical LP:
  min or max. c x
  s.t. A x <= b
    x >= 0
*/
#include <bits/stdc++.h>
using namespace std;
const double eps = 1e-9, oo = numeric_limits<double>::infinity();

typedef vector<double> vec;
typedef vector<vec> mat;

pair<vec, double> simplexMethodPD(mat &A, vec &b, vec &c, bool
↪ mini = true){
  int n = c.size(), m = b.size();
  mat T(m + 1, vec(n + m + 1));
  vector<int> base(n + m), row(m);

  for(int j = 0; j < m; ++j){
    for(int i = 0; i < n; ++i)
      T[j][i] = A[j][i];
    row[j] = n + j;
    T[j][n + j] = 1;
    base[n + j] = 1;
    T[j][n + m] = b[j];
  }

  for(int i = 0; i < n; ++i)
    T[m][i] = c[i] * (mini ? 1 : -1);

  while(true){
    int p = 0, q = 0;
    for(int i = 0; i < n + m; ++i)
      if(T[m][i] <= T[m][p])
        p = i;

    for(int j = 0; j < m; ++j)
      if(T[j][n + m] <= T[q][n + m])
        q = j;

    double t = min(T[m][p], T[q][n + m]);

    if(t >= -eps){
      vec x(n);
      for(int i = 0; i < m; ++i)
        if(row[i] < n) x[row[i]] = T[i][n + m];
      return {x, T[m][n + m] * (mini ? -1 : 1)}; // optimal
    }

    if(t < T[q][n + m]){
      // tight on c -> primal update
      for(int j = 0; j < m; ++j)
        if(T[j][p] >= eps)
          if(T[j][p] * (T[q][n + m] - t) >= T[q][p] * (T[j][n + m]
          ↪ - t))
```

```
        q = j;

    if(T[q][p] <= eps)
      return {vec(n), oo * (mini ? 1 : -1)}; // primal
      ↪  infeasible
  }else{
    // tight on b -> dual update
    for(int i = 0; i < n + m + 1; ++i)
      T[q][i] = -T[q][i];

    for(int i = 0; i < n + m; ++i)
      if(T[q][i] >= eps)
        if(T[q][i] * (T[m][p] - t) >= T[q][p] * (T[m][i] - t))
          p = i;

    if(T[q][p] <= eps)
      return {vec(n), oo * (mini ? -1 : 1)}; // dual infeasible
  }

  for(int i = 0; i < m + n + 1; ++i)
    if(i != p) T[q][i] /= T[q][p];

  T[q][p] = 1; // pivot(q, p)
  base[p] = 1;
  base[row[q]] = 0;
  row[q] = p;

  for(int j = 0; j < m + 1; ++j){
    if(j != q){
      double alpha = T[j][p];
      for(int i = 0; i < n + m + 1; ++i)
        T[j][i] -= T[q][i] * alpha;
    }
  }
}

  return {vec(n), oo};
}

int main(){
  int m, n;
  bool mini = true;
  cout << "Numero de restricciones: ";
  cin >> m;
  cout << "Numero de incognitas: ";
  cin >> n;
  mat A(m, vec(n));
  vec b(m), c(n);
  for(int i = 0; i < m; ++i){
    cout << "Restriccion #" << (i + 1) << ": ";
    for(int j = 0; j < n; ++j){
      cin >> A[i][j];
    }
    cin >> b[i];
  }
  cout << "[0]Max o [1]Min?: ";
  cin >> mini;
  cout << "Coeficientes de " << (mini ? "min" : "max") << " z: ";
  for(int i = 0; i < n; ++i){
    cin >> c[i];
  }
  cout.precision(6);
  auto ans = simplexMethodPD(A, b, c, mini);
  cout << (mini ? "Min" : "Max") << " z = " << ans.second << ",
  ↪  cuando: \n";
  for(int i = 0; i < ans.first.size(); ++i){
    cout << "x_" << (i + 1) << " = " << ans.first[i] << "\n";
  }
  return 0;
}
```

4. FFT

4.1. Declaraciones previas

```
using lli = long long int;
using comp = complex<double>;
const double PI = acos(-1.0);

int nearestPowerOfTwo(int n){
  int ans = 1;
  while(ans < n) ans <<= 1;
  return ans;
}
```

4.2. FFT con raíces de la unidad complejas

```
void fft(vector<comp> & X, int inv){
  int n = X.size();
  for(int i = 1, j = 0; i < n - 1; ++i){
    for(int k = n >> 1; (j ^= k) < k; k >>= 1);
    if(i < j) swap(X[i], X[j]);
  }
  vector<comp> wp(n>>1);
  for(int k = 1; k < n; k <<= 1){
    for(int j = 0; j < k; ++j)
      wp[j] = polar(1.0, PI * j / k * inv);
    for(int i = 0; i < n; i += k << 1){
      for(int j = 0; j < k; ++j){
        comp t = X[i + j + k] * wp[j];
        X[i + j + k] = X[i + j] - t;
        X[i + j] += t;
      }
    }
  }
  if(inv == -1)
    for(int i = 0; i < n; ++i)
      X[i] /= n;
}
```

4.3. FFT con raíces de la unidad en $\mathbb{Z}_p$ (NTT)

```
int inverse(int a, int n){
  int r0 = a, r1 = n, ri, s0 = 1, s1 = 0, si;
  while(r1){
    si = s0 - s1 * (r0 / r1), s0 = s1, s1 = si;
    ri = r0 % r1, r0 = r1, r1 = ri;
  }
  if(s0 < 0) s0 += n;
  return s0;
}

lli powerMod(lli b, lli e, lli m){
  lli ans = 1;
  e %= m-1;
  if(e < 0) e += m-1;
  while(e){
    if(e & 1) ans = ans * b % m;
    e >>= 1;
    b = b * b % m;
  }
  return ans;
}

template<int prime, int gen>
void ntt(vector<int> & X, int inv){
  int n = X.size();
  for(int i = 1, j = 0; i < n - 1; ++i){
    for(int k = n >> 1; (j ^= k) < k; k >>= 1);
    if(i < j) swap(X[i], X[j]);
  }
  vector<lli> wp(n>>1, 1);
  for(int k = 1; k < n; k <<= 1){
    lli wk = powerMod(gen, inv * (prime - 1) / (k<<1), prime);
    for(int j = 1; j < k; ++j)
      wp[j] = wp[j - 1] * wk % prime;
    for(int i = 0; i < n; i += k << 1){
      for(int j = 0; j < k; ++j){
        int u = X[i + j], v = X[i + j + k] * wp[j] % prime;
        X[i + j] = u + v < prime ? u + v : u + v - prime;
        X[i + j + k] = u - v < 0 ? u - v + prime : u - v;
      }
    }
  }
```

```
  if(inv == -1){
    lli nrev = inverse(n, prime);
    for(int i = 0; i < n; ++i)
      X[i] = X[i] * nrev % prime;
  }
}
```

4.3.1. Valores para escoger el generador y el módulo

Generador (g)	Tamaño máximo del arreglo (n)	Módulo p
3	2^{16}	$1 \times 2^{16} + 1 = 65537$
10	2^{18}	$3 \times 2^{18} + 1 = 786433$
3	2^{19}	$11 \times 2^{19} + 1 = 5767169$
3	$\mathbf{2^{20}}$	$7 \times 2^{20} + 1 = \mathbf{7340033}$
3	2^{21}	$11 \times 2^{21} + 1 = 23068673$
3	2^{22}	$25 \times 2^{22} + 1 = 104857601$
3	2^{22}	$235 \times 2^{22} + 1 = 985661441$
26	2^{23}	$105 \times 2^{23} + 1 = 880803841$
3	$\mathbf{2^{23}}$	$119 \times 2^{23} + 1 = \mathbf{998244353}$
11	2^{24}	$45 \times 2^{24} + 1 = 754974721$
3	2^{25}	$5 \times 2^{25} + 1 = 167772161$
3	2^{26}	$7 \times 2^{26} + 1 = 469762049$
31	2^{27}	$15 \times 2^{27} + 1 = 2013265921$

4.4. Multiplicación de polinomios (convolución lineal)

```
vector<comp> convolution(vector<comp> A, vector<comp> B){
  int sz = A.size() + B.size() - 1;
  int size = nearestPowerOfTwo(sz);
  A.resize(size), B.resize(size);
  fft(A, 1), fft(B, 1);
  for(int i = 0; i < size; i++)
    A[i] *= B[i];
  fft(A, -1);
  A.resize(sz);
  return A;
}
```

```
template<int prime, int gen>
vector<int> convolution(vector<int> A, vector<int> B){
  int sz = A.size() + B.size() - 1;
  int size = nearestPowerOfTwo(sz);
  A.resize(size), B.resize(size);
  ntt<prime, gen>(A, 1), ntt<prime, gen>(B, 1);
  for(int i = 0; i < size; i++)
    A[i] = (lli)A[i] * B[i] % prime;
  ntt<prime, gen>(A, -1);
  A.resize(sz);
  return A;
}

const int p = 7340033, g = 3; //default values for NTT
```

4.5. Aplicaciones

4.5.1. Multiplicación de números enteros grandes

```
string multiplyNumbers(const string & a, const string & b){
  int sgn = 1;
  int pos1 = 0, pos2 = 0;
  while(pos1 < a.size() && (a[pos1] < '1' || a[pos1] > '9')){
    if(a[pos1] == '-') sgn *= -1;
    ++pos1;
  }
  while(pos2 < b.size() && (b[pos2] < '1' || b[pos2] > '9')){
    if(b[pos2] == '-') sgn *= -1;
    ++pos2;
  }
  vector<int> X(a.size() - pos1), Y(b.size() - pos2);
  if(X.empty() || Y.empty()) return "0";
  for(int i = pos1, j = X.size() - 1; i < a.size(); ++i)
    X[j--] = a[i] - '0';
  for(int i = pos2, j = Y.size() - 1; i < b.size(); ++i)
    Y[j--] = b[i] - '0';
  X = convolution<p, g>(X, Y);
  stringstream ss;
  if(sgn == -1) ss << "-";
  int carry = 0;
  for(int i = 0; i < X.size(); ++i){
    X[i] += carry;
```

```
    carry = X[i] / 10;
    X[i] %= 10;
  }
  while(carry){
    X.push_back(carry % 10);
    carry /= 10;
  }
  for(int i = X.size() - 1; i >= 0; --i)
    ss << X[i];
  return ss.str();
}
```

4.5.2. Recíproco de un polinomio

```
vector<int> inversePolynomial(const vector<int> & A){
  vector<int> R(1, inverse(A[0], p));
  //R(x) = 2R(x)-A(x)R(x)^2
  while(R.size() < A.size()){
    int c = 2 * R.size();
    R.resize(c);
    vector<int> TR = R;
    TR.resize(2 * c);
    vector<int> TF(TR.size());
    for(int i = 0; i < c && i < A.size(); ++i)
      TF[i] = A[i];
    ntt<p, g>(TR, 1);
    ntt<p, g>(TF, 1);
    for(int i = 0; i < TR.size(); ++i)
      TR[i] = (lli)TR[i] * TR[i] % p * TF[i] % p;
    ntt<p, g>(TR, -1);
    for(int i = 0; i < c; ++i){
      R[i] = R[i] + R[i] - TR[i];
      if(R[i] < 0) R[i] += p;
      if(R[i] >= p) R[i] -= p;
    }
  }
  R.resize(A.size());
  return R;
}
```

4.5.3. Raíz cuadrada de un polinomio

```
const int inv2 = inverse(2, p);

vector<int> sqrtPolynomial(const vector<int> & A){
  int r0 = 1; //verify that r0^2 = A[0] mod p
  vector<int> R(1, r0);
  //R(x) = R(x)/2 + A(x)/(2R(x))
  while(R.size() < A.size()){
    int c = 2 * R.size();
    R.resize(c);
    vector<int> TF(c);
    for(int i = 0; i < c && i < A.size(); ++i)
      TF[i] = A[i];
    vector<int> IR = inversePolynomial(R);
    TF = convolution<p, g>(TF, IR);
    for(int i = 0; i < c; ++i){
      R[i] = R[i] + TF[i];
      if(R[i] >= p) R[i] -= p;
      R[i] = (lli)R[i] * inv2 % p;
    }
  }
  R.resize(A.size());
  return R;
}
```

4.5.4. Logaritmo y exponencial de un polinomio

```
vector<int> derivative(vector<int> A){
  for(int i = 0; i < A.size(); ++i)
    A[i] = (lli)A[i] * i % p;
  if(!A.empty()) A.erase(A.begin());
  return A;
}

vector<int> integral(vector<int> A){
  for(int i = 0; i < A.size(); ++i)
    A[i] = (lli)A[i] * (inverse(i+1, p)) % p;
  A.insert(A.begin(), 0);
  return A;
}

vector<int> logarithm(vector<int> A){
```

```
  assert(A[0] == 1);
  int n = A.size();
  A = convolution<p, g>(derivative(A), inversePolynomial(A));
  A.resize(n);
  A = integral(A);
  A.resize(n);
  return A;
}

vector<int> exponential(const vector<int> & A){
  assert(A[0] == 0);
  //E(x) = E(x)(1-ln(E(x))+A(x))
  vector<int> E(1, 1);
  while(E.size() < A.size()){
    int c = 2*E.size();
    E.resize(c);
    vector<int> S = logarithm(E);
    for(int i = 0; i < c && i < A.size(); ++i){
      S[i] = A[i] - S[i];
      if(S[i] < 0) S[i] += p;
    }
    S[0] = 1;
    E = convolution<p, g>(E, S);
    E.resize(c);
  }
  E.resize(A.size());
  return E;
}
```

4.5.5. Cociente y residuo de dos polinomios

```
//returns Q(x), where A(x)=B(x)Q(x)+R(x)
vector<int> quotient(vector<int> A, vector<int> B){
  int n = A.size(), m = B.size();
  if(n < m) return vector<int>{0};
  reverse(A.begin(), A.end());
  reverse(B.begin(), B.end());
  A.resize(n-m+1), B.resize(n-m+1);
  A = convolution<p, g>(A, inversePolynomial(B));
  A.resize(n-m+1);
  reverse(A.begin(), A.end());
  return A;
}

//returns R(x), where A(x)=B(x)Q(x)+R(x)
vector<int> remainder(vector<int> A, const vector<int> & B){
  int n = A.size(), m = B.size();
  if(n >= m){
    vector<int> C = convolution<p, g>(quotient(A, B), B);
    A.resize(m-1);
    for(int i = 0; i < m-1; ++i){
      A[i] -= C[i];
      if(A[i] < 0) A[i] += p;
    }
  }
  return A;
}
```

4.5.6. Multievaluación rápida

```
//evaluates all the points in P(x), both the size of P and points
↪ must be the same
vector<int> multiEvaluate(const vector<int> & P, const vector<int>
↪ & points){
  int n = points.size();
  vector<vector<int>> prod(2*n - 1);
  function<void(int, int, int)> pre = [&](int v, int l, int r){
    if(l == r) prod[v] = vector<int>{(p - points[l]) % p, 1};
    else{
      int y = (l + r) / 2;
      int z = v + (y - l + 1) * 2;
      pre(v + 1, l, y);
      pre(z, y + 1, r);
      prod[v] = convolution<p, g>(prod[v + 1], prod[z]);
    }
  };
  pre(0, 0, n - 1);

  function<int(const vector<int>&, int)> eval = [&](const
  ↪ vector<int> & poly, int x0){
    int ans = 0;
    for(int i = (int)poly.size()-1; i >= 0; --i){
      ans = (lli)ans * x0 % p + poly[i];
      if(ans >= p) ans -= p;
    }
    return ans;
```

```
  };

  vector<int> res(n);
  function<void(int, int, int, vector<int>)> evaluate = [&](int v,
  ↪  int l, int r, vector<int> poly){
    poly = remainder(poly, prod[v]);
    if(poly.size() < 400){
      for(int i = l; i <= r; ++i)
        res[i] = eval(poly, points[i]);
    }else{
      if(l == r)
        res[l] = poly[0];
      else{
        int y = (l + r) / 2;
        int z = v + (y - l + 1) * 2;
        evaluate(v + 1, l, y, poly);
        evaluate(z, y + 1, r, poly);
      }
    }
  };
  evaluate(0, 0, n - 1, P);
  return res;
}
```

4.5.7. DFT con tamaño de vector arbitrario (algoritmo de Bluestein)

```
//it evaluates 1, w^2, w^4, ..., w^(2n-2) on the polynomial a(x)
//in this example we do a DFT with arbitrary size
vector<comp> bluestein(vector<comp> A){
  int n = A.size();
  int m = nearestPowerOfTwo(2*n-1);
  comp w = polar(1.0, PI / n), w1 = w, w2 = 1;
  vector<comp> p(m), q(m), b(n);
  for(int k = 0; k < n; ++k, w2 *= w1, w1 *= w*w){
    b[k] = w2;
    p[k] = A[k] * b[k];
    q[k] = (comp)1 / b[k];
    if(k) q[m-k] = q[k];
  }
  fft(p, 1), fft(q, 1);
  for(int i = 0; i < m; i++)
    p[i] *= q[i];
  fft(p, -1);
  for(int k = 0; k < n; ++k)
    A[k] = b[k] * p[k];
  return A;
}
```

4.6. Convolución de dos vectores reales con solo dos FFT's

```
//A and B are real-valued vectors
//just do 2 fft's instead of 3
vector<comp> convolutionTrick(const vector<comp> & A, const
↪  vector<comp> & B){
  int sz = A.size() + B.size() - 1;
  int size = nearestPowerOfTwo(sz);
  vector<comp> C(size);
  comp I(0, 1);
  for(int i = 0; i < A.size() || i < B.size(); ++i){
    if(i < A.size()) C[i] += A[i];
    if(i < B.size()) C[i] += I*B[i];
  }
  fft(C, 1);
  vector<comp> D(size);
  for(int i = 0, j = 0; i < size; ++i){
    j = (size-1) & (size-i);
    D[i] = (conj(C[j]*C[j]) - C[i]*C[i]) * 0.25 * I;
  }
  fft(D, -1);
  D.resize(sz);
  return D;
}
```

4.7. Convolución con módulo arbitrario

```
//convolution with arbitrary modulo using only 4 fft's
vector<int> convolutionMod(const vector<int> & A, const
↪  vector<int> & B, int mod){
  int s = sqrt(mod);
  int sz = A.size() + B.size() - 1;
  int size = nearestPowerOfTwo(sz);
  vector<comp> a(size), b(size);
  for(int i = 0; i < A.size(); ++i)
    a[i] = comp(A[i] % s, A[i] / s);
```

```
  for(int i = 0; i < B.size(); ++i)
    b[i] = comp(B[i] % s, B[i] / s);
  fft(a, 1), fft(b, 1);
  comp I(0, 1);
  vector<comp> c(size), d(size);
  for(int i = 0, j = 0; i < size; ++i){
    j = (size-1) & (size-i);
    comp e = (a[i] + conj(a[j])) * 0.5;
    comp f = (conj(a[j]) - a[i]) * 0.5 * I;
    comp g = (b[i] + conj(b[j])) * 0.5;
    comp h = (conj(b[j]) - b[i]) * 0.5 * I;
    c[i] = e * g + I * (e * h + f * g);
    d[i] = f * h;
  }
  fft(c, -1), fft(d, -1);
  vector<int> D(sz);
  for(int i = 0, j = 0; i < sz; ++i){
    j = (size-1) & (size-i);
    int p0 = (lli)round(real(c[i])) % mod;
    int p1 = (lli)round(imag(c[i])) % mod;
    int p2 = (lli)round(real(d[i])) % mod;
    D[i] = p0 + s*(p1 + (lli)p2*s % mod) % mod;
    if(D[i] >= mod) D[i] -= mod;
    if(D[i] < 0) D[i] += mod;
  }
  return D;
}

//convolution with arbitrary modulo using CRT
//slower but with no precision errors
const int a = 998244353, b = 985661441, c = 754974721;
const lli a_b = inverse(a, b), a_c = inverse(a, c), b_c =
↪ inverse(b, c);
vector<int> convolutionModCRT(const vector<int> & A, const
↪ vector<int> & B, int mod){
  vector<int> P = convolution<a, 3>(A, B);
  vector<int> Q = convolution<b, 3>(A, B);
  vector<int> R = convolution<c, 11>(A, B);
  vector<int> D(P.size());
  for(int i = 0; i < D.size(); ++i){
    int x1 = P[i] % a;
    if(x1 < 0) x1 += a;
    int x2 = a_b * (Q[i] - x1) % b;
    if(x2 < 0) x2 += b;
    int x3 = (a_c * (R[i] - x1) % c - x2) * b_c % c;
    if(x3 < 0) x3 += c;
    D[i] = x1 + a*(x2 + (lli)x3*b % mod) % mod;
    if(D[i] >= mod) D[i] -= mod;
    if(D[i] < 0) D[i] += mod;
  }
  return D;
}
```

4.8. Transformada rápida de Walsh–Hadamard

```
//Fast Walsh-Hadamard transform, works with any modulo p
//op: 0(OR), 1(AND), 2(XOR), A.size() must be power of 2
void fwt(vector<int> & A, int op, int inv){
  int n = A.size();
  for(int k = 1; k < n; k <<= 1)
    for(int i = 0; i < n; i += k << 1)
      for(int j = 0; j < k; ++j){
        int u = A[i + j], v = A[i + j + k];
        int sum = u + v < p ? u + v : u + v - p;
        int rest = u - v < 0 ? u - v + p : u - v;
        if(inv == -1){
          if(op == 0)
            A[i + j + k] = rest ? p - rest : 0;
          else if(op == 1)
            A[i + j] = rest;
          else if(op == 2)
            A[i + j] = sum, A[i + j + k] = rest;
        }else{
          if(op == 0)
            A[i + j + k] = sum;
          else if(op == 1)
            A[i + j] = sum;
          else if(op == 2)
            A[i + j] = sum, A[i + j + k] = rest;
        }
      }
  if(inv == -1 && op == 2){
    lli nrev = inverse(n, p);
    for(int i = 0; i < n; ++i)
      A[i] = A[i] * nrev % p;
  }
}
```

5. Geometría

5.1. Estructura point

```
ld eps = 1e-9, inf = numeric_limits<ld>::max();

bool geq(ld a, ld b){return a-b >= -eps;}     //a >= b
bool leq(ld a, ld b){return b-a >= -eps;}     //a <= b
bool ge(ld a, ld b){return a-b > eps;}        //a > b
bool le(ld a, ld b){return b-a > eps;}        //a < b
bool eq(ld a, ld b){return abs(a-b) <= eps;}  //a == b
bool neq(ld a, ld b){return abs(a-b) > eps;}  //a != b

struct point{
  ld x, y;
  point(): x(0), y(0){}
  point(ld x, ld y): x(x), y(y){}

  point operator+(const point & p) const{return point(x + p.x, y +
  ↪  p.y);}

  point operator-(const point & p) const{return point(x - p.x, y -
  ↪  p.y);}

  point operator*(const ld & k) const{return point(x * k, y * k);}

  point operator/(const ld & k) const{return point(x / k, y / k);}

  point operator+=(const point & p){*this = *this + p; return
  ↪  *this;}

  point operator-=(const point & p){*this = *this - p; return
  ↪  *this;}

  point operator*=(const ld & p){*this = *this * p; return *this;}

  point operator/=(const ld & p){*this = *this / p; return *this;}

  point rotate(const ld angle) const{
    return point(x * cos(angle) - y * sin(angle), x * sin(angle) +
    ↪  y * cos(angle));
  }
  point rotate(const ld angle, const point & p){
    return p + ((*this) - p).rotate(angle);
  }
  point perpendicular() const{
    return point(-y, x);
  }

  ld dot(const point & p) const{
    return x * p.x + y * p.y;
  }
  ld cross(const point & p) const{
    return x * p.y - y * p.x;
  }
  ld norm() const{
    return x * x + y * y;
  }
  long double length() const{
    return sqrtl(x * x + y * y);
  }

  point normalize() const{
    return (*this) / length();
  }

  point projection(const point & p) const{
    return (*this) * p.dot(*this) / dot(*this);
  }
  point normal(const point & p) const{
    return p - projection(p);
  }

  bool operator==(const point & p) const{
    return eq(x, p.x) && eq(y, p.y);
  }
  bool operator!=(const point & p) const{
    return !(*this == p);
  }
  bool operator<(const point & p) const{
    if(eq(x, p.x)) return le(y, p.y);
    return le(x, p.x);
  }
  bool operator>(const point & p) const{
    if(eq(x, p.x)) return ge(y, p.y);
    return ge(x, p.x);
  }
```

```
};

istream &operator>>(istream &is, point & P){
  is >> P.x >> P.y;
  return is;
}

ostream &operator<<(ostream &os, const point & p) {
  return os << "(" << p.x << ", " << p.y << ")";
}

int sgn(ld x){
  if(ge(x, 0)) return 1;
  if(le(x, 0)) return -1;
  return 0;
}
```

5.2. Líneas y segmentos

5.2.1. Verificar si un punto pertenece a una línea o segmento

```
bool pointInLine(const point & a, const point & v, const point &
↪  p){
  //line a+tv, point p
  return eq((p - a).cross(v), 0);
}

bool pointInSegment(point a, point b, const point & p){
  //segment ab, point p
  if(a > b) swap(a, b);
  return pointInLine(a, b - a, p) && !(p < a || p > b);
}
```

5.2.2. Intersección de líneas

```
int intersectLinesInfo(const point & a1, const point & v1, const
↪  point & a2, const point & v2){
  //line a1+tv1
  //line a2+tv2
  ld det = v1.cross(v2);
  if(eq(det, 0)){
    if(eq((a2 - a1).cross(v1), 0)){
      return -1; //infinity points
    }else{
      return 0; //no points
    }
  }else{
    return 1; //single point
  }
}

point intersectLines(const point & a1, const point & v1, const
↪  point & a2, const point & v2){
  //lines a1+tv1, a2+tv2
  //assuming that they intersect
  ld det = v1.cross(v2);
  return a1 + v1 * ((a2 - a1).cross(v2) / det);
}
```

5.2.3. Intersección línea-segmento

```
int intersectLineSegmentInfo(const point & a, const point & v,
↪  const point & c, const point & d){
  //line a+tv, segment cd
  point v2 = d - c;
  ld det = v.cross(v2);
  if(eq(det, 0)){
    if(eq((c - a).cross(v), 0)){
      return -1; //infinity points
    }else{
      return 0; //no point
    }
  }else{
    return sgn(v.cross(c - a)) != sgn(v.cross(d - a)); //1: single
    ↪  point, 0: no point
  }
}
```

5.2.4. Intersección de segmentos

```
int intersectSegmentsInfo(const point & a, const point & b, const
↪  point & c, const point & d){
  //segment ab, segment cd
  point v1 = b - a, v2 = d - c;
```

```
  int t = sgn(v1.cross(c - a)), u = sgn(v1.cross(d - a));
  if(t == u){
    if(t == 0){
      if(pointInSegment(a, b, c) || pointInSegment(a, b, d) ||
      ↪  pointInSegment(c, d, a) || pointInSegment(c, d, b)){
        return -1; //infinity points
      }else{
        return 0; //no point
      }
    }else{
      return 0; //no point
    }
  }else{
    return sgn(v2.cross(a - c)) != sgn(v2.cross(b - c)); //1:
    ↪  single point, 0: no point
  }
}
```

5.2.5. Distancia punto-recta

```
ld distancePointLine(const point & a, const point & v, const point
↪  & p){
  //line: a + tv, point p
  return abs(v.cross(p - a)) / v.length();
}
```

5.3. Círculos

5.3.1. Distancia punto-círculo

```
ld distancePointCircle(const point & p, const point & c, ld r){
  //point p, center c, radius r
  return max((ld)0, (p - c).length() - r);
}
```

5.3.2. Proyección punto exterior a círculo

```
point projectionPointCircle(const point & p, const point & c, ld
↪  r){
  //point p (outside the circle), center c, radius r
  return c + (p - c) / (p - c).length() * r;
}
```

5.3.3. Puntos de tangencia de punto exterior

```
pair<point, point> pointsOfTangency(const point & p, const point &
↪  c, ld r){
  //point p (outside the circle), center c, radius r
  point v = (p - c).normalize() * r;
  ld theta = acos(r / (p - c).length());
  return {c + v.rotate(-theta), c + v.rotate(theta)};
}
```

5.3.4. Intersección línea-círculo y segmento-círculo

```
vector<point> intersectLineCircle(const point & a, const point &
↪  v, const point & c, ld r){
  //line a+tv, center c, radius r
  ld A = v.dot(v);
  ld B = (a - c).dot(v);
  ld C = (a - c).dot(a - c) - r * r;
  ld D = B*B - A*C;
  if(eq(D, 0)) return {a + v * (-B/A)}; //line tangent to circle
  else if(le(D, 0)) return {}; //no intersection
  else{ //two points of intersection (chord)
    D = sqrt(D);
    ld t1 = (-B - D) / A;
    ld t2 = (-B + D) / A;
    return {a + v * t1, a + v * t2};
  }
}

vector<point> intersectSegmentCircle(const point & a, const point
↪  & b, const point & c, ld r){
  vector<point> P = intersectLineCircle(a, b - a, c, r), ans;
  for(const point & p : P){
    if(pointInSegment(a, b, p)) ans.push_back(p);
  }
  return ans;
}
```

5.3.5. Centro y radio a través de tres puntos

```
pair<point, ld> getCircle(const point & m, const point & n, const
↪ point & p){
  //find circle that passes through points p, q, r
  point c = intersectLines((n + m) / 2, (n - m).perpendicular(),
  ↪ (p + n) / 2, (p - n).perpendicular());
  ld r = (c - m).length();
  return {c, r};
}
```

5.3.6. Intersección de círculos

```
vector<point> intersectionCircles(const point & c1, ld r1, const
↪ point & c2, ld r2){
  //circle 1 with center c1 and radius r1
  //circle 2 with center c2 and radius r2
  ld A = 2*r1*(c2.y - c1.y);
  ld B = 2*r1*(c2.x - c1.x);
  ld C = (c1 - c2).dot(c1 - c2) + r1*r1 - r2*r2;
  ld D = A*A + B*B - C*C;
  if(eq(D, 0)) return {c1 + point(B, A) * r1 / C};
  else if(le(D, 0)) return {};
  else{
    D = sqrt(D);
    ld cos1 = (B*C + A*D) / (A*A + B*B);
    ld sin1 = (A*C - B*D) / (A*A + B*B);
    ld cos2 = (B*C - A*D) / (A*A + B*B);
    ld sin2 = (A*C + B*D) / (A*A + B*B);
    return {c1 + point(cos1, sin1) * r1, c1 + point(cos2, sin2) *
    ↪ r1};
  }
}
```

5.3.7. Contención de círculos

```
int circleInsideCircle(const point & c1, ld r1, const point & c2,
↪ ld r2){
  //test if circle 2 is inside circle 1
  //returns "-1" if 2 touches internally 1, "1" if 2 is inside 1,
  ↪ "0" if they overlap
  ld l = r1 - r2 - (c1 - c2).length();
  return (ge(l, 0) ? 1 : (eq(l, 0) ? -1 : 0));
}

int circleOutsideCircle(const point & c1, ld r1, const point & c2,
↪ ld r2){
  //test if circle 2 is outside circle 1
  //returns "-1" if they touch externally, "1" if 2 is outside 1,
  ↪ "0" if they overlap
  ld l = (c1 - c2).length() - (r1 + r2);
  return (ge(l, 0) ? 1 : (eq(l, 0) ? -1 : 0));
}

int pointInCircle(const point & c, ld r, const point & p){
  //test if point p is inside the circle with center c and radius
  ↪ r
  //returns "0" if it's outside, "-1" if it's in the perimeter,
  ↪ "1" if it's inside
  ld l = (p - c).length() - r;
  return (le(l, 0) ? 1 : (eq(l, 0) ? -1 : 0));
}
```

5.3.8. Tangentes

```
vector<vector<point>> commonExteriorTangents(const point & c1, ld
↪ r1, const point & c2, ld r2){
  //returns a vector of segments or a single point
  if(le(r1, r2)) return commonExteriorTangents(c2, r2, c1, r1);
  if(c1 == c2 && eq(r1, r2)) return {};
  int in = circleInsideCircle(c1, r1, c2, r2);
  if(in == 1) return {};
  else if(in == -1) return {{c1 + (c2 - c1).normalize() * r1}};
  else{
    pair<point, point> t;
    if(eq(r1, r2))
      t = {c1 - (c2 - c1).perpendicular(), c1 + (c2 -
      ↪ c1).perpendicular()};
    else
      t = pointsOfTangency(c2, c1, r1 - r2);
    t.first = (t.first - c1).normalize();
    t.second = (t.second - c1).normalize();
    return {{c1 + t.first * r1, c2 + t.first * r2}, {c1 + t.second
    ↪ * r1, c2 + t.second * r2}};
  }
```

```
}

vector<vector<point>> commonInteriorTangents(const point & c1, ld
↪ r1, const point & c2, ld r2){
  if(c1 == c2 && eq(r1, r2)) return {};
  int out = circleOutsideCircle(c1, r1, c2, r2);
  if(out == 0) return {};
  else if(out == -1) return {{c1 + (c2 - c1).normalize() * r1}};
  else{
    auto t = pointsOfTangency(c2, c1, r1 + r2);
    t.first = (t.first - c1).normalize();
    t.second = (t.second - c1).normalize();
    return {{c1 + t.first * r1, c2 - t.first * r2}, {c1 + t.second
    ↪ * r1, c2 - t.second * r2}};
  }
}
```

5.3.9. Intersección polígono-círculo

```
ld signed_angle(const point & a, const point & b){
  return sgn(a.cross(b)) * acosl(a.dot(b) / (a.length() *
  ↪ b.length()));
}

ld intersectPolygonCircle(const vector<point> & P, const point &
↪ c, ld r){
  //Gets the area of the intersection of the polygon with the
  ↪ circle
  int n = P.size();
  ld ans = 0;
  for(int i = 0; i < n; ++i){
    point p = P[i], q = P[(i+1)%n];
    bool p_inside = (pointInCircle(c, r, p) != 0);
    bool q_inside = (pointInCircle(c, r, q) != 0);
    if(p_inside && q_inside){
      ans += (p - c).cross(q - c);
    }else if(p_inside && !q_inside){
      point s1 = intersectSegmentCircle(p, q, c, r)[0];
      point s2 = intersectSegmentCircle(c, q, c, r)[0];
      ans += (p - c).cross(s1 - c) + r*r * signed_angle(s1 - c, s2
      ↪ - c);
    }else if(!p_inside && q_inside){
      point s1 = intersectSegmentCircle(c, p, c, r)[0];
      point s2 = intersectSegmentCircle(p, q, c, r)[0];
      ans += (s2 - c).cross(q - c) + r*r * signed_angle(s1 - c, s2
      ↪ - c);
    }else{
      auto info = intersectSegmentCircle(p, q, c, r);
      if(info.size() <= 1){
        ans += r*r * signed_angle(p - c, q - c);
      }else{
        point s2 = info[0], s3 = info[1];
        point s1 = intersectSegmentCircle(c, p, c, r)[0];
        point s4 = intersectSegmentCircle(c, q, c, r)[0];
        ans += (s2 - c).cross(s3 - c) + r*r * (signed_angle(s1 -
        ↪ c, s2 - c) + signed_angle(s3 - c, s4 - c));
      }
    }
  }
  return abs(ans)/2;
}
```

5.3.10. Smallest enclosing circle

```
pair<point, ld> mec2(vector<point> & S, const point & a, const
↪ point & b, int n){
  ld hi = inf, lo = -hi;
  for(int i = 0; i < n; ++i){
    ld si = (b - a).cross(S[i] - a);
    if(eq(si, 0)) continue;
    point m = getCircle(a, b, S[i]).first;
    ld cr = (b - a).cross(m - a);
    if(le(si, 0)) hi = min(hi, cr);
    else lo = max(lo, cr);
  }
  ld v = (ge(lo, 0) ? lo : le(hi, 0) ? hi : 0);
  point c = (a + b) / 2 + (b - a).perpendicular() * v / (b -
  ↪ a).norm();
  return {c, (a - c).norm()};
}

pair<point, ld> mec(vector<point> & S, const point & a, int n){
  random_shuffle(S.begin(), S.begin() + n);
  point b = S[0], c = (a + b) / 2;
  ld r = (a - c).norm();
  for(int i = 1; i < n; ++i){
```

```
    if(ge((S[i] - c).norm(), r)){
      tie(c, r) = (n == S.size() ? mec(S, S[i], i) : mec2(S, a,
      ↪  S[i], i));
    }
  }
  return {c, r};
}

pair<point, ld> smallestEnclosingCircle(vector<point> S){
  assert(!S.empty());
  auto r = mec(S, S[0], S.size());
  return {r.first, sqrt(r.second)};
}
```

5.4. Polígonos

5.4.1. Perímetro y área de un polígono

```
ld perimeter(vector<point> & P){
  int n = P.size();
  ld ans = 0;
  for(int i = 0; i < n; i++){
    ans += (P[i] - P[(i + 1) % n]).length();
  }
  return ans;
}

ld area(vector<point> & P){
  int n = P.size();
  ld ans = 0;
  for(int i = 0; i < n; i++){
    ans += P[i].cross(P[(i + 1) % n]);
  }
  return abs(ans / 2);
}
```

5.4.2. Envolvente convexa (convex hull) de un polígono

```
vector<point> convexHull(vector<point> P){
  sort(P.begin(), P.end());
  vector<point> L, U;
  for(int i = 0; i < P.size(); i++){
    while(L.size() >= 2 && leq((L[L.size() - 2] -
    ↪  P[i]).cross(L[L.size() - 1] - P[i]), 0)){
      L.pop_back();
    }
    L.push_back(P[i]);
  }
  for(int i = P.size() - 1; i >= 0; i--){
    while(U.size() >= 2 && leq((U[U.size() - 2] -
    ↪  P[i]).cross(U[U.size() - 1] - P[i]), 0)){
      U.pop_back();
    }
    U.push_back(P[i]);
  }
  L.pop_back();
  U.pop_back();
  L.insert(L.end(), U.begin(), U.end());
  return L;
}
```

5.4.3. Verificar si un punto pertenece al perímetro de un polígono

```
bool pointInPerimeter(vector<point> & P, const point & p){
  int n = P.size();
  for(int i = 0; i < n; i++){
    if(pointInSegment(P[i], P[(i + 1) % n], p)){
      return true;
    }
  }
  return false;
}
```

5.4.4. Verificar si un punto pertenece a un polígono

```
int pointInPolygon(vector<point> & P, const point & p){
  if(pointInPerimeter(P, p)){
    return -1; //point in the perimeter
  }
  point bottomLeft = (*min_element(P.begin(), P.end())) -
  ↪  point(M_E, M_PI);
  int n = P.size();
  int rays = 0;
  for(int i = 0; i < n; i++){
```

```
    rays += (intersectSegmentsInfo(p, bottomLeft, P[i], P[(i + 1)
    ↪ % n]) == 1 ? 1 : 0);
  }
  return rays & 1; //0: point outside, 1: point inside
}
```

5.4.5. Verificar si un punto pertenece a un polígono convexo $O(\log n)$

```
//point in convex polygon in log(n)
//first do preprocess: seg=process(P),
//then for each query call pointInConvexPolygon(seg, p - P[0])
vector<point> process(vector<point> & P){
  int n = P.size();
  rotate(P.begin(), min_element(P.begin(), P.end()), P.end());
  vector<point> seg(n - 1);
  for(int i = 0; i < n - 1; ++i)
    seg[i] = P[i + 1] - P[0];
  return seg;
}

bool pointInConvexPolygon(vector<point> & seg, const point & p){
  int n = seg.size();
  if(neq(seg[0].cross(p), 0) && sgn(seg[0].cross(p)) !=
  ↪ sgn(seg[0].cross(seg[n - 1])))
    return false;
  if(neq(seg[n - 1].cross(p), 0) && sgn(seg[n - 1].cross(p)) !=
  ↪ sgn(seg[n - 1].cross(seg[0])))
    return false;
  if(eq(seg[0].cross(p), 0))
    return geq(seg[0].length(), p.length());
  int l = 0, r = n - 1;
  while(r - l > 1){
    int m = l + ((r - l) >> 1);
    if(geq(seg[m].cross(p), 0)) l = m;
    else r = m;
  }
  return eq(abs(seg[l].cross(seg[l + 1])), abs((p -
  ↪ seg[l]).cross(p - seg[l + 1])) + abs(p.cross(seg[l])) +
  ↪ abs(p.cross(seg[l + 1])));
}
```

5.4.6. Cortar un polígono con una recta

```
bool lineCutsPolygon(vector<point> & P, const point & a, const
↪ point & v){
  //line a+tv, polygon P
  int n = P.size();
  for(int i = 0, first = 0; i <= n; ++i){
    int side = sgn(v.cross(P[i%n]-a));
    if(!side) continue;
    if(!first) first = side;
    else if(side != first) return true;
  }
  return false;
}

vector<vector<point>> cutPolygon(vector<point> & P, const point &
↪ a, const point & v){
  //line a+tv, polygon P
  int n = P.size();
  if(!lineCutsPolygon(P, a, v)) return {P};
  int idx = 0;
  vector<vector<point>> ans(2);
  for(int i = 0; i < n; ++i){
    if(intersectLineSegmentInfo(a, v, P[i], P[(i+1)%n])){
      point p = intersectLines(a, v, P[i], P[(i+1)%n] - P[i]);
      if(P[i] == p) continue;
      ans[idx].push_back(P[i]);
      ans[1-idx].push_back(p);
      ans[idx].push_back(p);
      idx = 1-idx;
    }else{
      ans[idx].push_back(P[i]);
    }
  }
  return ans;
}
```

5.4.7. Centroide de un polígono

```
point centroid(vector<point> & P){
  point num;
  ld den = 0;
  int n = P.size();
```

```
  for(int i = 0; i < n; ++i){
    ld cross = P[i].cross(P[(i + 1) % n]);
    num += (P[i] + P[(i + 1) % n]) * cross;
    den += cross;
  }
  return num / (3 * den);
}
```

5.4.8. Pares de puntos antipodales

```
vector<pair<int, int>> antipodalPairs(vector<point> & P){
  vector<pair<int, int>> ans;
  int n = P.size(), k = 1;
  auto f = [&](int u, int v, int w){return
  ↪  abs((P[v%n]-P[u%n]).cross(P[w%n]-P[u%n]));};
  while(ge(f(n-1, 0, k+1), f(n-1, 0, k))) ++k;
  for(int i = 0, j = k; i <= k && j < n; ++i){
    ans.emplace_back(i, j);
    while(j < n-1 && ge(f(i, i+1, j+1), f(i, i+1, j)))
      ans.emplace_back(i, ++j);
  }
  return ans;
}
```

5.4.9. Diámetro y ancho

```
pair<ld, ld> diameterAndWidth(vector<point> & P){
  int n = P.size(), k = 0;
  auto dot = [&](int a, int b){return
  ↪  (P[(a+1)%n]-P[a]).dot(P[(b+1)%n]-P[b]);};
  auto cross = [&](int a, int b){return
  ↪  (P[(a+1)%n]-P[a]).cross(P[(b+1)%n]-P[b]);};
  ld diameter = 0;
  ld width = inf;
  while(ge(dot(0, k), 0)) k = (k+1) % n;
  for(int i = 0; i < n; ++i){
    while(ge(cross(i, k), 0)) k = (k+1) % n;
    //pair: (i, k)
    diameter = max(diameter, (P[k] - P[i]).length());
    width = min(width, distancePointLine(P[i], P[(i+1)%n] - P[i],
    ↪  P[k]));
  }
  return make_pair(diameter, width);
}
```

5.4.10. Smallest enclosing rectangle

```
pair<ld, ld> smallestEnclosingRectangle(vector<point> & P){
  int n = P.size();
  auto dot = [&](int a, int b){return
  ↪  (P[(a+1)%n]-P[a]).dot(P[(b+1)%n]-P[b]);};
  auto cross = [&](int a, int b){return
  ↪  (P[(a+1)%n]-P[a]).cross(P[(b+1)%n]-P[b]);};
  ld perimeter = inf, area = inf;
  for(int i = 0, j = 0, k = 0, m = 0; i < n; ++i){
    while(ge(dot(i, j), 0)) j = (j+1) % n;
    if(!i) k = j;
    while(ge(cross(i, k), 0)) k = (k+1) % n;
    if(!i) m = k;
    while(le(dot(i, m), 0)) m = (m+1) % n;
    //pairs: (i, k) , (j, m)
    point v = P[(i+1)%n] - P[i];
    ld h = distancePointLine(P[i], v, P[k]);
    ld w = distancePointLine(P[j], v.perpendicular(), P[m]);
    perimeter = min(perimeter, 2 * (h + w));
    area = min(area, h * w);
  }
  return make_pair(area, perimeter);
}
```

5.5. Par de puntos más cercanos

```
bool comp1(const point & a, const point & b){
  return le(a.y, b.y);
}
pair<point, point> closestPairOfPoints(vector<point> P){
  sort(P.begin(), P.end(), comp1);
  set<point> S;
  ld ans = inf;
  point p, q;
  int pos = 0;
  for(int i = 0; i < P.size(); ++i){
    while(pos < i && geq(P[i].y - P[pos].y, ans)){
      S.erase(P[pos++]);
```

```
    }
    auto lower = S.lower_bound({P[i].x - ans - eps, -inf});
    auto upper = S.upper_bound({P[i].x + ans + eps, -inf});
    for(auto it = lower; it != upper; ++it){
      ld d = (P[i] - *it).length();
      if(le(d, ans)){
        ans = d;
        p = P[i];
        q = *it;
      }
    }
    S.insert(P[i]);
  }
  return {p, q};
}
```

5.6. Vantage Point Tree (puntos más cercanos a cada punto)

```
struct vantage_point_tree{
  struct node
  {
    point p;
    ld th;
    node *l, *r;
  }*root;

  vector<pair<ld, point>> aux;

  vantage_point_tree(vector<point> &ps){
    for(int i = 0; i < ps.size(); ++i)
      aux.push_back({ 0, ps[i] });
    root = build(0, ps.size());
  }

  node *build(int l, int r){
    if(l == r)
      return 0;
    swap(aux[l], aux[l + rand() % (r - l)]);
    point p = aux[l++].second;
    if(l == r)
      return new node({ p });
    for(int i = l; i < r; ++i)
      aux[i].first = (p - aux[i].second).dot(p - aux[i].second);
    int m = (l + r) / 2;
    nth_element(aux.begin() + l, aux.begin() + m, aux.begin() +
    ↪  r);
    return new node({ p, sqrt(aux[m].first), build(l, m), build(m,
    ↪  r) });
  }

  priority_queue<pair<ld, node*>> que;

  void k_nn(node *t, point p, int k){
    if(!t)
      return;
    ld d = (p - t->p).length();
    if(que.size() < k)
      que.push({ d, t });
    else if(ge(que.top().first, d)){
      que.pop();
      que.push({ d, t });
    }
    if(!t->l && !t->r)
      return;
    if(le(d, t->th)){
      k_nn(t->l, p, k);
      if(leq(t->th - d, que.top().first))
        k_nn(t->r, p, k);
    }else{
      k_nn(t->r, p, k);
      if(leq(d - t->th, que.top().first))
        k_nn(t->l, p, k);
    }
  }

  vector<point> k_nn(point p, int k){
    k_nn(root, p, k);
    vector<point> ans;
    for(; !que.empty(); que.pop())
      ans.push_back(que.top().second->p);
    reverse(ans.begin(), ans.end());
    return ans;
  }
};
```

5.7. Suma Minkowski

```
vector<point> minkowskiSum(vector<point> A, vector<point> B){
  int na = (int)A.size(), nb = (int)B.size();
  if(A.empty() || B.empty()) return {};

  rotate(A.begin(), min_element(A.begin(), A.end()), A.end());
  rotate(B.begin(), min_element(B.begin(), B.end()), B.end());

  int pa = 0, pb = 0;
  vector<point> M;

  while(pa < na && pb < nb){
    M.push_back(A[pa] + B[pb]);
    ld x = (A[(pa + 1) % na] - A[pa]).cross(B[(pb + 1) % nb] -
    ↪  B[pb]);
    if(leq(x, 0)) pb++;
    if(geq(x, 0)) pa++;
  }

  while(pa < na) M.push_back(A[pa++] + B[0]);
  while(pb < nb) M.push_back(B[pb++] + A[0]);

  return M;
}
```

5.8. Triangulación de Delaunay

```
//Delaunay triangulation in O(n log n)
const point inf_pt(inf, inf);

struct QuadEdge{
  point origin;
  QuadEdge* rot = nullptr;
  QuadEdge* onext = nullptr;
  bool used = false;
  QuadEdge* rev() const{return rot->rot;}
  QuadEdge* lnext() const{return rot->rev()->onext->rot;}
  QuadEdge* oprev() const{return rot->onext->rot;}
  point dest() const{return rev()->origin;}
};

QuadEdge* make_edge(const point & from, const point & to){
  QuadEdge* e1 = new QuadEdge;
  QuadEdge* e2 = new QuadEdge;
  QuadEdge* e3 = new QuadEdge;
  QuadEdge* e4 = new QuadEdge;
  e1->origin = from;
  e2->origin = to;
  e3->origin = e4->origin = inf_pt;
  e1->rot = e3;
  e2->rot = e4;
  e3->rot = e2;
  e4->rot = e1;
  e1->onext = e1;
  e2->onext = e2;
  e3->onext = e4;
  e4->onext = e3;
  return e1;
}

void splice(QuadEdge* a, QuadEdge* b){
  swap(a->onext->rot->onext, b->onext->rot->onext);
  swap(a->onext, b->onext);
}

void delete_edge(QuadEdge* e){
  splice(e, e->oprev());
  splice(e->rev(), e->rev()->oprev());
  delete e->rot;
  delete e->rev()->rot;
  delete e;
  delete e->rev();
}

QuadEdge* connect(QuadEdge* a, QuadEdge* b){
  QuadEdge* e = make_edge(a->dest(), b->origin);
  splice(e, a->lnext());
  splice(e->rev(), b);
  return e;
}

bool left_of(const point & p, QuadEdge* e){
  return ge((e->origin - p).cross(e->dest() - p), 0);
}

bool right_of(const point & p, QuadEdge* e){
```

```
  return le((e->origin - p).cross(e->dest() - p), 0);
}

ld det3(ld a1, ld a2, ld a3, ld b1, ld b2, ld b3, ld c1, ld c2, ld
↪  c3) {
  return a1 * (b2 * c3 - c2 * b3) - a2 * (b1 * c3 - c1 * b3) + a3
   ↪  * (b1 * c2 - c1 * b2);
}

bool in_circle(const point & a, const point & b, const point & c,
↪  const point & d) {
  ld det = -det3(b.x, b.y, b.norm(), c.x, c.y, c.norm(), d.x, d.y,
   ↪  d.norm());
  det += det3(a.x, a.y, a.norm(), c.x, c.y, c.norm(), d.x, d.y,
   ↪  d.norm());
  det -= det3(a.x, a.y, a.norm(), b.x, b.y, b.norm(), d.x, d.y,
   ↪  d.norm());
  det += det3(a.x, a.y, a.norm(), b.x, b.y, b.norm(), c.x, c.y,
   ↪  c.norm());
  return ge(det, 0);
}

pair<QuadEdge*, QuadEdge*> build_tr(int l, int r, vector<point> &
↪  P){
  if(r - l + 1 == 2){
    QuadEdge* res = make_edge(P[l], P[r]);
    return make_pair(res, res->rev());
  }
  if(r - l + 1 == 3){
    QuadEdge *a = make_edge(P[l], P[l + 1]), *b = make_edge(P[l +
     ↪  1], P[r]);
    splice(a->rev(), b);
    int sg = sgn((P[l + 1] - P[l]).cross(P[r] - P[l]));
    if(sg == 0)
      return make_pair(a, b->rev());
    QuadEdge* c = connect(b, a);
    if(sg == 1)
      return make_pair(a, b->rev());
    else
      return make_pair(c->rev(), c);
  }
  int mid = (l + r) / 2;
  QuadEdge *ldo, *ldi, *rdo, *rdi;
  tie(ldo, ldi) = build_tr(l, mid, P);
  tie(rdi, rdo) = build_tr(mid + 1, r, P);
  while(true){
    if(left_of(rdi->origin, ldi)){
      ldi = ldi->lnext();
      continue;
    }
    if(right_of(ldi->origin, rdi)){
      rdi = rdi->rev()->onext;
      continue;
    }
    break;
  }
  QuadEdge* basel = connect(rdi->rev(), ldi);
  auto valid = [&basel](QuadEdge* e){return right_of(e->dest(),
  ↪  basel);};
  if(ldi->origin == ldo->origin)
    ldo = basel->rev();
  if(rdi->origin == rdo->origin)
    rdo = basel;
  while(true){
    QuadEdge* lcand = basel->rev()->onext;
    if(valid(lcand)){
      while(in_circle(basel->dest(), basel->origin, lcand->dest(),
       ↪  lcand->onext->dest())){
        QuadEdge* t = lcand->onext;
        delete_edge(lcand);
        lcand = t;
      }
    }
    QuadEdge* rcand = basel->oprev();
    if(valid(rcand)){
      while(in_circle(basel->dest(), basel->origin, rcand->dest(),
       ↪  rcand->oprev()->dest())){
        QuadEdge* t = rcand->oprev();
        delete_edge(rcand);
        rcand = t;
      }
    }
    if(!valid(lcand) && !valid(rcand))
      break;
    if(!valid(lcand) || (valid(rcand) && in_circle(lcand->dest(),
    ↪  lcand->origin, rcand->origin, rcand->dest())))
      basel = connect(rcand, basel->rev());
    else
```

```
      basel = connect(basel->rev(), lcand->rev());
  }
  return make_pair(ldo, rdo);
}

vector<tuple<point, point, point>> delaunay(vector<point> & P){
  sort(P.begin(), P.end());
  auto res = build_tr(0, (int)P.size() - 1, P);
  QuadEdge* e = res.first;
  vector<QuadEdge*> edges = {e};
  while(le((e->dest() - e->onext->dest()).cross(e->origin -
  ↪  e->onext->dest()), 0))
    e = e->onext;
  auto add = [&P, &e, &edges](){
    QuadEdge* curr = e;
    do{
      curr->used = true;
      P.push_back(curr->origin);
      edges.push_back(curr->rev());
      curr = curr->lnext();
    }while(curr != e);
  };
  add();
  P.clear();
  int kek = 0;
  while(kek < (int)edges.size())
    if(!(e = edges[kek++])->used)
      add();
  vector<tuple<point, point, point>> ans;
  for(int i = 0; i < (int)P.size(); i += 3){
    ans.push_back(make_tuple(P[i], P[i + 1], P[i + 2]));
  }
  return ans;
}
```

6. Grafos

6.1. Disjoint Set

```
struct disjointSet{
  int N;
  vector<short int> rank;
  vi parent, count;

  disjointSet(int N): N(N), parent(N), count(N), rank(N){}

  void makeSet(int v){
    count[v] = 1;
    parent[v] = v;
  }

  int findSet(int v){
    if(v == parent[v]) return v;
    return parent[v] = findSet(parent[v]);
  }

  void unionSet(int a, int b){
    a = findSet(a), b = findSet(b);
    if(a == b) return;
    if(rank[a] < rank[b]){
      parent[a] = b;
      count[b] += count[a];
    }else{
      parent[b] = a;
      count[a] += count[b];
      if(rank[a] == rank[b]) ++rank[a];
    }
  }
};
```

6.2. Definiciones

```
struct edge{
  int source, dest, cost;

  edge(): source(0), dest(0), cost(0){}
```

```
  edge(int dest, int cost): dest(dest), cost(cost){}

  edge(int source, int dest, int cost): source(source),
  ↪  dest(dest), cost(cost){}

  bool operator==(const edge & b) const{
    return source == b.source && dest == b.dest && cost == b.cost;
  }
  bool operator<(const edge & b) const{
    return cost < b.cost;
  }
  bool operator>(const edge & b) const{
    return cost > b.cost;
  }
};

struct path{
  int cost = inf;
  deque<int> vertices;
  int size = 1;
  int prev = -1;
};

struct graph{
  vector<vector<edge>> adjList;
  vector<vb> adjMatrix;
  vector<vi> costMatrix;
  vector<edge> edges;
  int V = 0;
  bool dir = false;

  graph(int n, bool dir): V(n), dir(dir), adjList(n), edges(n),
  ↪  adjMatrix(n, vb(n)), costMatrix(n, vi(n)){
    for(int i = 0; i < n; ++i)
      for(int j = 0; j < n; ++j)
        costMatrix[i][j] = (i == j ? 0 : inf);
  }

  void add(int source, int dest, int cost){
    adjList[source].emplace_back(source, dest, cost);
    edges.emplace_back(source, dest, cost);
    adjMatrix[source][dest] = true;
    costMatrix[source][dest] = cost;
    if(!dir){
      adjList[dest].emplace_back(dest, source, cost);
      adjMatrix[dest][source] = true;
      costMatrix[dest][source] = cost;
    }
  }

  void buildPaths(vector<path> & paths){
    for(int i = 0; i < V; i++){
      int u = i;
      for(int j = 0; j < paths[i].size; j++){
        paths[i].vertices.push_front(u);
        u = paths[u].prev;
      }
    }
  }
```

6.3. DFS genérica

```
  void dfs(int u, vi & status, vi & parent){
    status[u] = 1;
    for(edge & current : adjList[u]){
      int v = current.dest;
      if(status[v] == 0){ //not visited
        parent[v] = u;
        dfs(v, status, parent);
      }else if(status[v] == 1){ //explored
        if(v == parent[u]){
          //bidirectional node u<-->v
        }else{
          //back edge u-v
        }
      }else if(status[v] == 2){ //visited
        //forward edge u-v
      }
    }
    status[u] = 2;
  }
```

6.4. Dijkstra

```
  vector<path> dijkstra(int start){
    priority_queue<edge, vector<edge>, greater<edge>> cola;
```

```
  vector<path> paths(V);
  cola.emplace(start, 0);
  paths[start].cost = 0;
  while(!cola.empty()){
    int u = cola.top().dest; cola.pop();
    for(edge & current : adjList[u]){
      int v = current.dest;
      int nuevo = paths[u].cost + current.cost;
      if(nuevo == paths[v].cost && paths[u].size + 1 <
      ↪  paths[v].size){
        paths[v].prev = u;
        paths[v].size = paths[u].size + 1;
      }else if(nuevo < paths[v].cost){
        paths[v].prev = u;
        paths[v].size = paths[u].size + 1;
        cola.emplace(v, nuevo);
        paths[v].cost = nuevo;
      }
    }
  }
  buildPaths(paths);
  return paths;
}
```

6.5. Bellman Ford

```
vector<path> bellmanFord(int start){
  vector<path> paths(V, path());
  vi processed(V);
  vb inQueue(V);
  queue<int> Q;
  paths[start].cost = 0;
  Q.push(start);
  while(!Q.empty()){
    int u = Q.front(); Q.pop(); inQueue[u] = false;
    if(paths[u].cost == inf) continue;
    ++processed[u];
    if(processed[u] == V){
      cout << "Negative cycle\n";
      return {};
    }
    for(edge & current : adjList[u]){
      int v = current.dest;
      int nuevo = paths[u].cost + current.cost;
      if(nuevo == paths[v].cost && paths[u].size + 1 <
      ↪  paths[v].size){
        paths[v].prev = u;
        paths[v].size = paths[u].size + 1;
      }else if(nuevo < paths[v].cost){
        if(!inQueue[v]){
          Q.push(v);
          inQueue[v] = true;
        }
        paths[v].prev = u;
        paths[v].size = paths[u].size + 1;
        paths[v].cost = nuevo;
      }
    }
  }
  buildPaths(paths);
  return paths;
}
```

6.6. Floyd

```
vector<vi> floyd(){
  vector<vi> tmp = costMatrix;
  for(int k = 0; k < V; ++k)
    for(int i = 0; i < V; ++i)
      for(int j = 0; j < V; ++j)
        if(tmp[i][k] != inf && tmp[k][j] != inf)
          tmp[i][j] = min(tmp[i][j], tmp[i][k] + tmp[k][j]);
  return tmp;
}
```

6.7. Cerradura transitiva $O(V^3)$

```
vector<vb> transitiveClosure(){
  vector<vb> tmp = adjMatrix;
  for(int k = 0; k < V; ++k)
    for(int i = 0; i < V; ++i)
      for(int j = 0; j < V; ++j)
        tmp[i][j] = tmp[i][j] || (tmp[i][k] && tmp[k][j]);
  return tmp;
}
```

6.8. Cerradura transitiva $O(V^2)$

```
vector<vb> transitiveClosureDFS(){
  vector<vb> tmp(V, vb(V));
  function<void(int, int)> dfs = [&](int start, int u){
    for(edge & current : adjList[u]){
      int v = current.dest;
      if(!tmp[start][v]){
        tmp[start][v] = true;
        dfs(start, v);
      }
    }
  };
  for(int u = 0; u < V; u++)
    dfs(u, u);
  return tmp;
}
```

6.9. Verificar si el grafo es bipartito

```
bool isBipartite(){
  vi side(V, -1);
  queue<int> q;
  for (int st = 0; st < V; ++st){
    if(side[st] != -1) continue;
    q.push(st);
    side[st] = 0;
    while(!q.empty()){
      int u = q.front();
      q.pop();
      for (edge & current : adjList[u]){
        int v = current.dest;
        if(side[v] == -1) {
          side[v] = side[u] ^ 1;
          q.push(v);
        }else{
          if(side[v] == side[u]) return false;
        }
      }
    }
  }
  return true;
}
```

6.10. Orden topológico

```
vi topologicalSort(){
  int visited = 0;
  vi order, indegree(V);
  for(auto & node : adjList){
    for(edge & current : node){
      int v = current.dest;
      ++indegree[v];
    }
  }
  queue<int> Q;
  for(int i = 0; i < V; ++i){
    if(indegree[i] == 0) Q.push(i);
  }
  while(!Q.empty()){
    int source = Q.front();
    Q.pop();
    order.push_back(source);
    ++visited;
    for(edge & current : adjList[source]){
      int v = current.dest;
      --indegree[v];
      if(indegree[v] == 0) Q.push(v);
    }
  }
  if(visited == V) return order;
  else return {};
}
```

6.11. Detectar ciclos

```
bool hasCycle(){
  vi color(V);
  function<bool(int, int)> dfs = [&](int u, int parent){
    color[u] = 1;
    bool ans = false;
    int ret = 0;
    for(edge & current : adjList[u]){
      int v = current.dest;
      if(color[v] == 0)
        ans |= dfs(v, u);
      else if(color[v] == 1 && (dir || v != parent || ret++))
```

```
      ans = true;
    }
    color[u] = 2;
    return ans;
  };
  for(int u = 0; u < V; ++u)
    if(color[u] == 0 && dfs(u, -1))
      return true;
  return false;
}
```

6.12. Puentes y puntos de articulación

```
pair<vb, vector<edge>> articulationBridges(){
  vi low(V), label(V);
  vb points(V);
  vector<edge> bridges;
  int time = 0;
  function<int(int, int)> dfs = [&](int u, int p){
    label[u] = low[u] = ++time;
    int hijos = 0, ret = 0;
    for(edge & current : adjList[u]){
      int v = current.dest;
      if(v == p && !ret++) continue;
      if(!label[v]){
        ++hijos;
        dfs(v, u);
        if(label[u] <= low[v])
          points[u] = true;
        if(label[u] < low[v])
          bridges.push_back(current);
        low[u] = min(low[u], low[v]);
      }
      low[u] = min(low[u], label[v]);
    }
    return hijos;
  };
  for(int u = 0; u < V; ++u)
    if(!label[u])
      points[u] = dfs(u, -1) > 1;
  return make_pair(points, bridges);
}
```

6.13. Componentes fuertemente conexas

```
vector<vi> scc(){
  vi low(V), label(V);
  int time = 0;
  vector<vi> ans;
  stack<int> S;
  function<void(int)> dfs = [&](int u){
    label[u] = low[u] = ++time;
    S.push(u);
    for(edge & current : adjList[u]){
      int v = current.dest;
      if(!label[v]) dfs(v);
      low[u] = min(low[u], low[v]);
    }
    if(label[u] == low[u]){
      vi comp;
      while(S.top() != u){
        comp.push_back(S.top());
        low[S.top()] = V + 1;
        S.pop();
      }
      comp.push_back(S.top());
      S.pop();
      ans.push_back(comp);
      low[u] = V + 1;
    }
  };
  for(int u = 0; u < V; ++u)
    if(!label[u]) dfs(u);
  return ans;
}
```

6.14. Árbol mínimo de expansión (Kruskal)

```
vector<edge> kruskal(){
  sort(edges.begin(), edges.end());
  vector<edge> MST;
  disjointSet DS(V);
  for(int u = 0; u < V; ++u)
    DS.makeSet(u);
  int i = 0;
```

```
  while(i < edges.size() && MST.size() < V - 1){
    edge current = edges[i++];
    int u = current.source, v = current.dest;
    if(DS.findSet(u) != DS.findSet(v)){
      MST.push_back(current);
      DS.unionSet(u, v);
    }
  }
  return MST;
}
```

6.15. Máximo emparejamiento bipartito

```
bool tryKuhn(int u, vb & used, vi & left, vi & right){
  if(used[u]) return false;
  used[u] = true;
  for(edge & current : adjList[u]){
    int v = current.dest;
    if(right[v] == -1 || tryKuhn(right[v], used, left, right)){
      right[v] = u;
      left[u] = v;
      return true;
    }
  }
  return false;
}

bool augmentingPath(int u, vb & used, vi & left, vi & right){
  used[u] = true;
  for(edge & current : adjList[u]){
    int v = current.dest;
    if(right[v] == -1){
      right[v] = u;
      left[u] = v;
      return true;
    }
  }
  for(edge & current : adjList[u]){
    int v = current.dest;
    if(!used[right[v]] && augmentingPath(right[v], used, left,
    ↪  right)){
      right[v] = u;
      left[u] = v;
      return true;
    }
  }
  return false;
}

//vertices from the left side numbered from 0 to l-1
//vertices from the right side numbered from 0 to r-1
//graph[u] represents the left side
//graph[u][v] represents the right side
//we can use tryKuhn() or augmentingPath()
vector<pair<int, int>> maxMatching(int l, int r){
  vi left(l, -1), right(r, -1);
  vb used(l);
  for(int u = 0; u < l; ++u){
    tryKuhn(u, used, left, right);
    fill(used.begin(), used.end(), false);
  }
  vector<pair<int, int>> ans;
  for(int u = 0; u < r; ++u){
    if(right[u] != -1){
      ans.emplace_back(right[u], u);
    }
  }
  return ans;
}
```

6.16. Circuito euleriano

7. Árboles

7.1. Estructura tree

```
struct tree{
  vi parent, level, weight;
  vector<vi> dists, DP;
  int n, root;

  void dfs(int u, graph & G){
    for(edge & curr : G.adjList[u]){
      int v = curr.dest;
      int w = curr.cost;
      if(v != parent[u]){
        parent[v] = u;
        weight[v] = w;
        level[v] = level[u] + 1;
        dfs(v, G);
      }
    }
  }

  tree(int n, int root): n(n), root(root), parent(n), level(n),
  ↪  weight(n), dists(n, vi(20)), DP(n, vi(20)){
    parent[root] = root;
  }

  tree(graph & G, int root): n(G.V), root(root), parent(G.V),
  ↪  level(G.V), weight(G.V), dists(G.V, vi(20)), DP(G.V,
  ↪  vi(20)){
    parent[root] = root;
    dfs(root, G);
  }

  void pre(){
    for(int u = 0; u < n; u++){
      DP[u][0] = parent[u];
      dists[u][0] = weight[u];
    }
    for(int i = 1; (1 << i) <= n; ++i){
      for(int u = 0; u < n; ++u){
        DP[u][i] = DP[DP[u][i - 1]][i - 1];
        dists[u][i] = dists[u][i - 1] + dists[DP[u][i - 1]][i -
        ↪  1];
      }
    }
  }
```

7.2. k-ésimo ancestro

```
  int ancestor(int p, int k){
    int h = level[p] - k;
    if(h < 0) return -1;
    int lg;
    for(lg = 1; (1 << lg) <= level[p]; ++lg);
    lg--;
    for(int i = lg; i >= 0; --i){
      if(level[p] - (1 << i) >= h){
        p = DP[p][i];
      }
    }
    return p;
  }
```

7.3. LCA

```
  int lca(int p, int q){
    if(level[p] < level[q]) swap(p, q);
    int lg;
    for(lg = 1; (1 << lg) <= level[p]; ++lg);
    lg--;
    for(int i = lg; i >= 0; --i){
      if(level[p] - (1 << i) >= level[q]){
        p = DP[p][i];
      }
    }
    if(p == q) return p;

    for(int i = lg; i >= 0; --i){
      if(DP[p][i] != -1 && DP[p][i] != DP[q][i]){
        p = DP[p][i];
        q = DP[q][i];
      }
    }
```

```
    return parent[p];
  }
```

7.4. Distancia entre dos nodos

```
  int dist(int p, int q){
    if(level[p] < level[q]) swap(p, q);
    int lg;
    for(lg = 1; (1 << lg) <= level[p]; ++lg);
    lg--;
    int sum = 0;
    for(int i = lg; i >= 0; --i){
      if(level[p] - (1 << i) >= level[q]){
        sum += dists[p][i];
        p = DP[p][i];
      }
    }
    if(p == q) return sum;

    for(int i = lg; i >= 0; --i){
      if(DP[p][i] != -1 && DP[p][i] != DP[q][i]){
        sum += dists[p][i] + dists[q][i];
        p = DP[p][i];
        q = DP[q][i];
      }
    }
    sum += dists[p][0] + dists[q][0];
    return sum;
  }
```

7.5. HLD

7.6. Link Cut

8. Flujos

8.1. Estructura flowEdge

```
template<typename T>
struct flowEdge{
  int dest;
  T flow, capacity, cost;
  flowEdge *res;

  flowEdge(): dest(0), flow(0), capacity(0), cost(0), res(NULL){}
  flowEdge(int dest, T flow, T capacity, T cost = 0): dest(dest),
  ↪ flow(flow), capacity(capacity), cost(cost), res(NULL){}

  void addFlow(T flow){
    this->flow += flow;
    this->res->flow -= flow;
  }
};
```

8.2. Estructura flowGraph

```
template<typename T>
struct flowGraph{
  T inf = numeric_limits<T>::max();
  vector<vector<flowEdge<T>*>> adjList;
  vector<int> dist, pos;
  int V;
  flowGraph(int V): V(V), adjList(V), dist(V), pos(V){}
  ~flowGraph(){
    for(int i = 0; i < V; ++i)
      for(int j = 0; j < adjList[i].size(); ++j)
        delete adjList[i][j];
  }
  void addEdge(int u, int v, T capacity, T cost = 0){
    flowEdge<T> *uv = new flowEdge<T>(v, 0, capacity, cost);
    flowEdge<T> *vu = new flowEdge<T>(u, capacity, capacity,
    ↪ -cost);
    uv->res = vu;
    vu->res = uv;
    adjList[u].push_back(uv);
    adjList[v].push_back(vu);
```

```
}
```

8.3. Algoritmo de Edmonds-Karp $O(VE^2)$

```
//Maximun Flow using Edmonds-Karp Algorithm O(VE^2)
T edmondsKarp(int s, int t){
  T maxFlow = 0;
  vector<flowEdge<T>*> parent(V);
  while(true){
    fill(parent.begin(), parent.end(), nullptr);
    queue<int> Q;
    Q.push(s);
    while(!Q.empty() && !parent[t]){
      int u = Q.front(); Q.pop();
      for(flowEdge<T> *v : adjList[u]){
        if(!parent[v->dest] && v->capacity > v->flow){
          parent[v->dest] = v;
          Q.push(v->dest);
        }
      }
    }
    if(!parent[t]) break;
    T f = inf;
    for(int u = t; u != s; u = parent[u]->res->dest)
      f = min(f, parent[u]->capacity - parent[u]->flow);
    for(int u = t; u != s; u = parent[u]->res->dest)
      parent[u]->addFlow(f);
    maxFlow += f;
  }
  return maxFlow;
}
```

8.4. Algoritmo de Dinic $O(V^2E)$

```
//Maximun Flow using Dinic Algorithm O(EV^2)
T blockingFlow(int u, int t, T flow){
  if(u == t) return flow;
  for(int &i = pos[u]; i < adjList[u].size(); ++i){
    flowEdge<T> *v = adjList[u][i];
    if(v->capacity > v->flow && dist[u] + 1 == dist[v->dest]){
      T fv = blockingFlow(v->dest, t, min(flow, v->capacity -
      ↪  v->flow));
      if(fv > 0){
        v->addFlow(fv);
        return fv;
      }
    }
  }
  return 0;
}
T dinic(int s, int t){
  T maxFlow = 0;
  dist[t] = 0;
  while(dist[t] != -1){
    fill(dist.begin(), dist.end(), -1);
    queue<int> Q;
    Q.push(s);
    dist[s] = 0;
    while(!Q.empty()){
      int u = Q.front(); Q.pop();
      for(flowEdge<T> *v : adjList[u]){
        if(dist[v->dest] == -1 && v->flow != v->capacity){
          dist[v->dest] = dist[u] + 1;
          Q.push(v->dest);
        }
      }
    }
    if(dist[t] != -1){
      T f;
      fill(pos.begin(), pos.end(), 0);
      while(f = blockingFlow(s, t, inf))
        maxFlow += f;
    }
  }
  return maxFlow;
}
```

8.5. Flujo máximo de costo mínimo

```
//Max Flow Min Cost
pair<T, T> maxFlowMinCost(int s, int t){
  vector<bool> inQueue(V);
  vector<T> distance(V), cap(V);
  vector<flowEdge<T>*> parent(V);
  T maxFlow = 0, minCost = 0;
```

```
    while(true){
      fill(distance.begin(), distance.end(), inf);
      fill(parent.begin(), parent.end(), nullptr);
      fill(cap.begin(), cap.end(), 0);
      distance[s] = 0;
      cap[s] = inf;
      queue<int> Q;
      Q.push(s);
      while(!Q.empty()){
        int u = Q.front(); Q.pop(); inQueue[u] = 0;
        for(flowEdge<T> *v : adjList[u]){
          if(v->capacity > v->flow && distance[v->dest] >
          ↪  distance[u] + v->cost){
            distance[v->dest] = distance[u] + v->cost;
            parent[v->dest] = v;
            cap[v->dest] = min(cap[u], v->capacity - v->flow);
            if(!inQueue[v->dest]){
              Q.push(v->dest);
              inQueue[v->dest] = true;
            }
          }
        }
      }
      if(!parent[t]) break;
      maxFlow += cap[t];
      minCost += cap[t] * distance[t];
      for(int u = t; u != s; u = parent[u]->res->dest)
        parent[u]->addFlow(cap[t]);
    }
    return {maxFlow, minCost};
  }
```

8.6. Hungariano

```
//Given a m*n cost matrix (m<=n), it finds a maximum cost
↪  assignment.
//The actual assignment is in the vector returned.
//To find the minimum, negate the values.
template<typename T>
pair<T, vector<int>> hungarian(const vector<vector<T>> & a){
  int m = a.size(), n = a[0].size();
  assert(m <= n);
  vector<int> x(m, -1), y(n, -1);
  vector<T> px(m, numeric_limits<T>::min()), py(n, 0);
  for(int u = 0; u < m; ++u)
    for(int v = 0; v < n; ++v)
      px[u] = max(px[u], a[u][v]);
  for(int u = 0, p, q; u < m; ){
    vector<int> s(m + 1, u), t(n, -1);
    for(p = q = 0; p <= q && x[u] < 0; ++p){
      for(int k = s[p], v = 0; v < n && x[u] < 0; ++v){
        if(px[k] + py[v] == a[k][v] && t[v] < 0){
          s[++q] = y[v], t[v] = k;
          if(s[q] < 0)
            for(p = v; p >= 0; v = p)
              y[v] = k = t[v], p = x[k], x[k] = v;
        }
      }
    }
    if(x[u] < 0){
      T delta = numeric_limits<T>::max();
      for(int i = 0; i <= q; ++i)
        for(int v = 0; v < n; ++v)
          if(t[v] < 0)
            delta = min(delta, px[s[i]] + py[v] - a[s[i]][v]);
      for(int i = 0; i <= q; ++i)
        px[s[i]] -= delta;
      for(int v = 0; v < n; ++v)
        py[v] += (t[v] < 0 ? 0 : delta);
    }else{
      ++u;
    }
  }
  T cost = 0;
  for(int u = 0; u < m; ++u)
    cost += a[u][x[u]];
  return {cost, x};
}
```

9. Estructuras de datos

9.1. Segment Tree

9.1.1. Minimalistic: Point updates, range queries

```
template<typename T>
struct SegmentTree{
  int N;
  vector<T> ST;

  //build from an array in O(n)
  SegmentTree(int N, vector<T> & arr): N(N){
    ST.resize(N << 1);
    for(int i = 0; i < N; ++i)
      ST[N + i] = arr[i];
    for(int i = N - 1; i > 0; --i)
      ST[i] = ST[i << 1] + ST[i << 1 | 1];
  }

  //single element update in i
  void update(int i, T value){
    ST[i += N] = value; //update the element accordingly
    while(i >>= 1)
      ST[i] = ST[i << 1] + ST[i << 1 | 1];
  }

  //single element update in [l, r]
  void update(int l, int r, T value){
    l += N, r += N;
    for(int i = l; i <= r; ++i)
      ST[i] = value;
    l >>= 1, r >>= 1;
    while(l >= 1){
      for(int i = r; i >= l; --i)
        ST[i] = ST[i << 1] + ST[i << 1 | 1];
      l >>= 1, r >>= 1;
    }
  }

  //range query, [l, r]
  T query(int l, int r){
    T res = 0;
    for(l += N, r += N; l <= r; l >>= 1, r >>= 1){
      if(l & 1) res += ST[l++];
      if(!(r & 1)) res += ST[r--];
    }
    return res;
  }
};
```

9.1.2. Dynamic: Range updates and range queries

```
template<typename T>
struct SegmentTreeDin{
  SegmentTreeDin *left, *right;
  int l, r;
  T sum, lazy;

  SegmentTreeDin(int start, int end, vector<T> & arr): left(NULL),
  ↪ right(NULL), l(start), r(end), sum(0), lazy(0){
    if(l == r) sum = arr[l];
    else{
      int half = l + ((r - l) >> 1);
      left = new SegmentTreeDin(l, half, arr);
      right = new SegmentTreeDin(half+1, r, arr);
      sum = left->sum + right->sum;
    }
  }

  void propagate(T dif){
    sum += (r - l + 1) * dif;
    if(l != r){
      left->lazy += dif;
      right->lazy += dif;
    }
  }

  T sum_query(int start, int end){
    if(lazy != 0){
      propagate(lazy);
      lazy = 0;
    }
    if(end < l || r < start) return 0;
    if(start <= l && r <= end) return sum;
```

```
    else return left->sum_query(start, end) +
    ↪  right->sum_query(start, end);
  }

  void add_range(int start, int end, T dif){
    if(lazy != 0){
      propagate(lazy);
      lazy = 0;
    }
    if(end < l || r < start) return;
    if(start <= l && r <= end) propagate(dif);
    else{
      left->add_range(start, end, dif);
      right->add_range(start, end, dif);
      sum = left->sum + right->sum;
    }
  }

  void add_pos(int i, T sum){
    add_range(i, i, sum);
  }
};
```

9.1.3. Static: Range updates and range queries

```
template<typename T>
struct SegmentTreeEst{
  int size;
  vector<T> sum, lazy;

  void rec(int pos, int l, int r, vector<T> & arr){
    if(l == r) sum[pos] = arr[l];
    else{
      int half = l + ((r - l) >> 1);
      rec(2*pos+1, l, half, arr);
      rec(2*pos+2, half+1, r, arr);
      sum[pos] = sum[2*pos+1] + sum[2*pos+2];
    }
  }

  SegmentTreeEst(int n, vector<T> & arr): size(n){
    int h = ceil(log2(n));
    sum.resize((1 << (h + 1)) - 1);
    lazy.resize((1 << (h + 1)) - 1);
    rec(0, 0, n - 1, arr);
  }

  void propagate(int pos, int l, int r, T dif){
    sum[pos] += (r - l + 1) * dif;
    if(l != r){
      lazy[2*pos+1] += dif;
      lazy[2*pos+2] += dif;
    }
  }

  T sum_query_rec(int start, int end, int pos, int l, int r){
    if(lazy[pos] != 0){
      propagate(pos, l, r, lazy[pos]);
      lazy[pos] = 0;
    }
    if(end < l || r < start) return 0;
    if(start <= l && r <= end) return sum[pos];
    else{
      int half = l + ((r - l) >> 1);
      return sum_query_rec(start, end, 2*pos+1, l, half) +
      ↪  sum_query_rec(start, end, 2*pos+2, half+1, r);
    }
  }

  T sum_query(int start, int end){
    return sum_query_rec(start, end, 0, 0, size - 1);
  }

  void add_range_rec(int start, int end, int pos, int l, int r, T
  ↪  dif){
    if(lazy[pos] != 0){
      propagate(pos, l, r, lazy[pos]);
      lazy[pos] = 0;
    }
    if(end < l || r < start) return;
    if(start <= l && r <= end) propagate(pos, l, r, dif);
    else{
      int half = l + ((r - l) >> 1);
      add_range_rec(start, end, 2*pos+1, l, half, dif);
      add_range_rec(start, end, 2*pos+2, half+1, r, dif);
      sum[pos] = sum[2*pos+1] + sum[2*pos+2];
    }
```

```
  }

  void add_range(int start, int end, T dif){
    add_range_rec(start, end, 0, 0, size - 1, dif);
  }

  void add_pos(int i, T sum){
    add_range(i, i, sum);
  }
};
```

9.1.4. Persistent: Point updates, range queries

```
template<typename T>
struct StPer{
  StPer *left, *right;
  int l, r;
  T sum;

  StPer(int start, int end): left(NULL), right(NULL), l(start),
  ↪  r(end), sum(0){
    if(l != r){
      int half = l + ((r - l) >> 1);
      left = new StPer(l, half);
      right = new StPer(half+1, r);
    }
  }
  StPer(int start, int end, T val): left(NULL), right(NULL),
  ↪  l(start), r(end), sum(val){}
  StPer(int start, int end, StPer* left, StPer* right):
  ↪  left(left), right(right), l(start), r(end){
    sum = left->sum + right->sum;
  }

  T sum_query(int start, int end){
    if(end < l || r < start) return 0;
    if(start <= l && r <= end) return sum;
    else return left->sum_query(start, end) +
    ↪  right->sum_query(start, end);
  }

  StPer* update(int pos, T val){
    if(l == r) return new StPer(l, r, sum + val);
    int half = l + ((r - l) >> 1);
    if(pos <= half) return new StPer(l, r, left->update(pos, val),
    ↪  right);
    return new StPer(l, r, left, right->update(pos, val));
  }
};
```

9.2. Fenwick Tree

```
template<typename T>
struct FenwickTree{
  int N;
  vector<T> bit;

  //build from array in O(n), indexed in 0
  FenwickTree(int N, vector<T> & arr): N(N){
    bit.resize(N);
    for(int i = 0; i < N; ++i){
      bit[i] += arr[i];
      if((i | (i + 1)) < N)
        bit[i | (i + 1)] += bit[i];
    }
  }

  //single element increment
  void update(int pos, T value){
    while(pos < N){
      bit[pos] += value;
      pos |= pos + 1;
    }
  }

  //range query, [0, r]
  T query(int r){
    T res = 0;
    while(r >= 0){
      res += bit[r];
      r = (r & (r + 1)) - 1;
    }
    return res;
  }

  //range query, [l, r]
```

```
  T query(int l, int r){
    return query(r) - query(l - 1);
  }
};
```

9.3. SQRT Decomposition

```
struct MOquery{
  int l, r, index, S;
  bool operator<(const MOquery & q) const{
    int c_o = l / S, c_q = q.l / S;
    if(c_o == c_q)
      return r < q.r;
    return c_o < c_q;
  }
};

template<typename T>
struct SQRT{
  int N, S;
  vector<T> A, B;

  SQRT(int N): N(N){
    this->S = sqrt(N + .0) + 1;
    A.assign(N, 0);
    B.assign(S, 0);
  }

  void build(vector<T> & arr){
    A = vector<int>(arr.begin(), arr.end());
    for(int i = 0; i < N; ++i) B[i / S] += A[i];
  }

  //single element update
  void update(int pos, T value){
    int k = pos / S;
    A[pos] = value;
    T res = 0;
    for(int i = k * S, end = min(N, (k + 1) * S) - 1; i <= end;
    ↪  ++i) res += A[i];
    B[k] = res;
  }

  //range query, [l, r]
  T query(int l, int r){
    T res = 0;
    int c_l = l / S, c_r = r / S;
    if(c_l == c_r){
      for(int i = l; i <= r; ++i) res += A[i];
    }else{
      for(int i = l, end = (c_l + 1) * S - 1; i <= end; ++i) res
      ↪  += A[i];
      for(int i = c_l + 1; i <= c_r - 1; ++i) res += B[i];
      for(int i = c_r * S; i <= r; ++i) res += A[i];
    }
    return res;
  }

  //range queries offline using MO's algorithm
  vector<T> MO(vector<MOquery> & queries){
    vector<T> ans(queries.size());
    sort(queries.begin(), queries.end());
    T current = 0;
    int prevL = 0, prevR = -1;
    int i, j;
    for(const MOquery & q : queries){
      for(i = prevL, j = min(prevR, q.l - 1); i <= j; ++i){
        //remove from the left
        current -= A[i];
      }
      for(i = prevL - 1; i >= q.l; --i){
        //add to the left
        current += A[i];
      }
      for(i = max(prevR + 1, q.l); i <= q.r; ++i){
        //add to the right
        current += A[i];
      }
      for(i = prevR; i >= q.r + 1; --i){
        //remove from the right
        current -= A[i];
      }
      prevL = q.l, prevR = q.r;
      ans[q.index] = current;
    }
    return ans;
  }
```

```
};
```

9.4. AVL Tree

```
template<typename T>
struct AVLNode{
  AVLNode<T> *left, *right;
  short int height;
  int size;
  T value;

  AVLNode(T value = 0): left(NULL), right(NULL), value(value),
  ↪  height(1), size(1){}

  inline short int balance(){
    return (right ? right->height : 0) - (left ? left->height :
    ↪  0);
  }

  AVLNode *maxLeftChild(){
    AVLNode *ret = this;
    while(ret->left) ret = ret->left;
    return ret;
  }
};

template<typename T>
struct AVLTree{
  AVLNode<T> *root;

  AVLTree(): root(NULL){}

  inline int nodeSize(AVLNode<T> *& pos){return pos ? pos->size:
  ↪  0;}

  inline int nodeHeight(AVLNode<T> *& pos){return pos ?
  ↪  pos->height: 0;}

  inline void update(AVLNode<T> *& pos){
    if(!pos) return;
    pos->height = 1 + max(nodeHeight(pos->left),
    ↪  nodeHeight(pos->right));
    pos->size = 1 + nodeSize(pos->left) + nodeSize(pos->right);
  }

  int size(){return nodeSize(root);}

  void leftRotate(AVLNode<T> *& x){
    AVLNode<T> *y = x->right, *t = y->left;
    y->left = x, x->right = t;
    update(x), update(y);
    x = y;
  }

  void rightRotate(AVLNode<T> *& y){
    AVLNode<T> *x = y->left, *t = x->right;
    x->right = y, y->left = t;
    update(y), update(x);
    y = x;
  }

  void updateBalance(AVLNode<T> *& pos){
    if(!pos) return;
    short int bal = pos->balance();
    if(bal > 1){
      if(pos->right->balance() < 0) rightRotate(pos->right);
      leftRotate(pos);
    }else if(bal < -1){
      if(pos->left->balance() > 0) leftRotate(pos->left);
      rightRotate(pos);
    }
  }

  void insert(AVLNode<T> *&pos, T & value){
    if(pos){
      value < pos->value ? insert(pos->left, value) :
      ↪  insert(pos->right, value);
      update(pos), updateBalance(pos);
    }else{
      pos = new AVLNode<T>(value);
    }
  }

  AVLNode<T> *search(T & value){
    AVLNode<T> *pos = root;
    while(pos){
      if(value == pos->value) break;
```

```
      pos = (value < pos->value ? pos->left : pos->right);
    }
    return pos;
  }

  void erase(AVLNode<T> *&pos, T & value){
    if(!pos) return;
    if(value < pos->value) erase(pos->left, value);
    else if(value > pos->value) erase(pos->right, value);
    else{
      if(!pos->left) pos = pos->right;
      else if(!pos->right) pos = pos->left;
      else{
        pos->value = pos->right->maxLeftChild()->value;
        erase(pos->right, pos->value);
      }
    }
    update(pos), updateBalance(pos);
  }

  void insert(T value){insert(root, value);}

  void erase(T value){erase(root, value);}

  void updateVal(T old, T New){
    if(search(old))
      erase(old), insert(New);
  }

  T kth(int i){
    assert(0 <= i && i < nodeSize(root));
    AVLNode<T> *pos = root;
    while(i != nodeSize(pos->left)){
      if(i < nodeSize(pos->left)){
        pos = pos->left;
      }else{
        i -= nodeSize(pos->left) + 1;
        pos = pos->right;
      }
    }
    return pos->value;
  }

  int lessThan(T & x){
    int ans = 0;
    AVLNode<T> *pos = root;
    while(pos){
      if(x > pos->value){
        ans += nodeSize(pos->left) + 1;
        pos = pos->right;
      }else{
        pos = pos->left;
      }
    }
    return ans;
  }

  int lessThanOrEqual(T & x){
    int ans = 0;
    AVLNode<T> *pos = root;
    while(pos){
      if(x < pos->value){
        pos = pos->left;
      }else{
        ans += nodeSize(pos->left) + 1;
        pos = pos->right;
      }
    }
    return ans;
  }

  int greaterThan(T & x){
    int ans = 0;
    AVLNode<T> *pos = root;
    while(pos){
      if(x < pos->value){
        ans += nodeSize(pos->right) + 1;
        pos = pos->left;
      }else{
        pos = pos->right;
      }
    }
    return ans;
  }

  int greaterThanOrEqual(T & x){
    int ans = 0;
    AVLNode<T> *pos = root;
```

```
    while(pos){
      if(x > pos->value){
        pos = pos->right;
      }else{
        ans += nodeSize(pos->right) + 1;
        pos = pos->left;
      }
    }
    return ans;
  }

  int equalTo(T & x){
    return lessThanOrEqual(x) - lessThan(x);
  }

  void build(AVLNode<T> *& pos, vector<T> & arr, int i, int j){
    if(i > j) return;
    int m = i + ((j - i) >> 1);
    pos = new AVLNode<T>(arr[m]);
    build(pos->left, arr, i, m - 1);
    build(pos->right, arr, m + 1, j);
    update(pos);
  }

  void build(vector<T> & arr){
    build(root, arr, 0, (int)arr.size() - 1);
  }

  void output(AVLNode<T> *pos, vector<T> & arr, int & i){
    if(pos){
      output(pos->left, arr, i);
      arr[++i] = pos->value;
      output(pos->right, arr, i);
    }
  }

  void output(vector<T> & arr){
    int i = -1;
    output(root, arr, i);
  }
};
```

9.5. Treap

```
template<typename T>
struct TreapNode{
  TreapNode<T> *left, *right;
  T value;
  int key, size;

  //fields for queries
  bool rev;
  T sum, add;

  TreapNode(T value = 0): value(value), key(rand()), size(1),
  ↪  left(NULL), right(NULL), sum(value), add(0), rev(false){}
};

template<typename T>
struct Treap{
  TreapNode<T> *root;

  Treap(): root(NULL) {}

  inline int nodeSize(TreapNode<T>* t){return t ? t->size: 0;}

  inline T nodeSum(TreapNode<T>* t){return t ? t->sum : 0;}

  inline void update(TreapNode<T>* &t){
    if(!t) return;
    t->size = 1 + nodeSize(t->left) + nodeSize(t->right);
    t->sum = t->value; //reset node fields
    push(t->left), push(t->right); //push changes to child nodes
    t->sum = t->value + nodeSum(t->left) + nodeSum(t->right);
    ↪  //combine(left,t,t), combine(t,right,t)
  }

  int size(){return nodeSize(root);}

  void merge(TreapNode<T>* &t, TreapNode<T>* t1, TreapNode<T>*
  ↪  t2){
    if(!t1) t = t2;
    else if(!t2) t = t1;
    else if(t1->key > t2->key)
      merge(t1->right, t1->right, t2), t = t1;
    else
```

```
    merge(t2->left, t1, t2->left), t = t2;
  update(t);
}

void split(TreapNode<T>* t, T & x, TreapNode<T>* &t1,
↪  TreapNode<T>* &t2){
  if(!t)
    return void(t1 = t2 = NULL);
  if(x < t->value)
    split(t->left, x, t1, t->left), t2 = t;
  else
    split(t->right, x, t->right, t2), t1 = t;
  update(t);
}

void insert(TreapNode<T>* &t, TreapNode<T>* x){
  if(!t) t = x;
  else if(x->key > t->key)
    split(t, x->value, x->left, x->right), t = x;
  else
    insert(x->value < t->value ? t->left : t->right, x);
  update(t);
}

TreapNode<T>* search(T & x){
  TreapNode<T> *t = root;
  while(t){
    if(x == t->value) break;
    t = (x < t->value ? t->left : t->right);
  }
  return t;
}

void erase(TreapNode<T>* &t, T & x){
  if(!t) return;
  if(t->value == x)
    merge(t, t->left, t->right);
  else
    erase(x < t->value ? t->left : t->right, x);
  update(t);
}

void insert(T & x){insert(root, new TreapNode<T>(x));}

void erase(T & x){erase(root, x);}

void updateVal(T & old, T & New){
  if(search(old))
    erase(old), insert(New);
}

T kth(int i){
  assert(0 <= i && i < nodeSize(root));
  TreapNode<T> *t = root;
  while(i != nodeSize(t->left)){
    if(i < nodeSize(t->left)){
      t = t->left;
    }else{
      i -= nodeSize(t->left) + 1;
      t = t->right;
    }
  }
  return t->value;
}

int lessThan(T & x){
  int ans = 0;
  TreapNode<T> *t = root;
  while(t){
    if(x > t->value){
      ans += nodeSize(t->left) + 1;
      t = t->right;
    }else{
      t = t->left;
    }
  }
  return ans;
}

//OPERATIONS FOR IMPLICIT TREAP
inline void push(TreapNode<T>* t){
  if(!t) return;
  //add in range example
  if(t->add){
    t->value += t->add;
    t->sum += t->add * nodeSize(t);
    if(t->left) t->left->add += t->add;
    if(t->right) t->right->add += t->add;
```

```
    t->add = 0;
  }
  //reverse range example
  if(t->rev){
    swap(t->left, t->right);
    if(t->left) t->left->rev ^= true;
    if(t->right) t->right->rev ^= true;
    t->rev = false;
  }
}

void split2(TreapNode<T>* t, int i, TreapNode<T>* &t1,
↪ TreapNode<T>* &t2){
  if(!t)
    return void(t1 = t2 = NULL);
  push(t);
  int curr = nodeSize(t->left);
  if(i <= curr)
    split2(t->left, i, t1, t->left), t2 = t;
  else
    split2(t->right, i - curr - 1, t->right, t2), t1 = t;
  update(t);
}

inline int aleatorio(){
  return (rand() << 15) + rand();
}

void merge2(TreapNode<T>* &t, TreapNode<T>* t1, TreapNode<T>*
↪ t2){
  push(t1), push(t2);
  if(!t1) t = t2;
  else if(!t2) t = t1;
  else if(aleatorio() % (nodeSize(t1) + nodeSize(t2)) <
  ↪ nodeSize(t1))
    merge2(t1->right, t1->right, t2), t = t1;
  else
    merge2(t2->left, t1, t2->left), t = t2;
  update(t);
}

//insert the element "x" at position "i"
void insert_at(T & x, int i){
  if(i > nodeSize(root)) return;
  TreapNode<T> *t1 = NULL, *t2 = NULL;
  split2(root, i, t1, t2);
  merge2(root, t1, new TreapNode<T>(x));
  merge2(root, root, t2);
}

//delete element at position "i"
void erase_at(int i){
  if(i >= nodeSize(root)) return;
  TreapNode<T> *t1 = NULL, *t2 = NULL, *t3 = NULL;
  split2(root, i, t1, t2);
  split2(t2, 1, t2, t3);
  merge2(root, t1, t3);
}

void update_at(TreapNode<T>* t, T & x, int i){
  push(t);
  assert(0 <= i && i < nodeSize(t));
  int curr = nodeSize(t->left);
  if(i == curr)
    t->value = x;
  else if(i < curr)
    update_at(t->left, x, i);
  else
    update_at(t->right, x, i - curr - 1);
  update(t);
}

T nth(TreapNode<T>* t, int i){
  push(t);
  assert(0 <= i && i < nodeSize(t));
  int curr = nodeSize(t->left);
  if(i == curr)
    return t->value;
  else if(i < curr)
    return nth(t->left, i);
  else
    return nth(t->right, i - curr - 1);
}

//update value of element at position "i" with "x"
void update_at(T & x, int i){update_at(root, x, i);}

//ith element
```

```
  T nth(int i){return nth(root, i);}

  //add "val" in [l, r]
  void add_update(T & val, int l, int r){
    TreapNode<T> *t1 = NULL, *t2 = NULL, *t3 = NULL;
    split2(root, l, t1, t2);
    split2(t2, r - l + 1, t2, t3);
    t2->add += val;
    merge2(root, t1, t2);
    merge2(root, root, t3);
  }

  //reverse [l, r]
  void reverse_update(int l, int r){
    TreapNode<T> *t1 = NULL, *t2 = NULL, *t3 = NULL;
    split2(root, l, t1, t2);
    split2(t2, r - l + 1, t2, t3);
    t2->rev ^= true;
    merge2(root, t1, t2);
    merge2(root, root, t3);
  }

  //rotate [l, r] k times to the right
  void rotate_update(int k, int l, int r){
    TreapNode<T> *t1 = NULL, *t2 = NULL, *t3 = NULL, *t4 = NULL;
    split2(root, l, t1, t2);
    split2(t2, r - l + 1, t2, t3);
    k %= nodeSize(t2);
    split2(t2, nodeSize(t2) - k, t2, t4);
    merge2(root, t1, t4);
    merge2(root, root, t2);
    merge2(root, root, t3);
  }

  //sum query in [l, r]
  T sum_query(int l, int r){
    TreapNode<T> *t1 = NULL, *t2 = NULL, *t3 = NULL;
    split2(root, l, t1, t2);
    split2(t2, r - l + 1, t2, t3);
    T ans = nodeSum(t2);
    merge2(root, t1, t2);
    merge2(root, root, t3);
    return ans;
  }

  void inorder(TreapNode<T>* t){
    if(!t) return;
    push(t);
    inorder(t->left);
    cout << t->value << " ";
    inorder(t->right);
  }

  void inorder(){inorder(root);}
};
```

9.6. Sparse table

9.6.1. Normal

```
template<typename T>
struct SparseTable{
  vector<vector<T>> ST;
  vector<int> logs;
  int K, N;

  SparseTable(vector<T> & arr){
    N = arr.size();
    K = log2(N) + 2;
    ST.assign(K + 1, vector<T>(N));
    logs.assign(N + 1, 0);
    for(int i = 2; i <= N; ++i)
      logs[i] = logs[i >> 1] + 1;
    for(int i = 0; i < N; ++i)
      ST[0][i] = arr[i];
    for(int j = 1; j <= K; ++j)
      for(int i = 0; i + (1 << j) <= N; ++i)
        ST[j][i] = min(ST[j - 1][i], ST[j - 1][i + (1 << (j -
        ↪ 1))]); //put the function accordingly
  }

  T sum(int l, int r){ //non-idempotent functions
    T ans = 0;
    for(int j = K; j >= 0; --j){
      if((1 << j) <= r - l + 1){
        ans += ST[j][l];
        l += 1 << j;
```

```
      }
    }
    return ans;
  }

  T minimal(int l, int r){ //idempotent functions
    int j = logs[r - l + 1];
    return min(ST[j][l], ST[j][r - (1 << j) + 1]);
  }
};
```

9.6.2. Disjoint

```
//build on O(n log n), queries in O(1) for any operation
template<typename T>
struct DisjointSparseTable{
  vector<vector<T>> left, right;
  int K, N;

  DisjointSparseTable(vector<T> & arr){
    N = arr.size();
    K = log2(N) + 2;
    left.assign(K + 1, vector<T>(N));
    right.assign(K + 1, vector<T>(N));
    for(int j = 0; (1 << j) <= N; ++j){
      int mask = (1 << j) - 1;
      T acum = 0; //neutral element of your operation
      for(int i = 0; i < N; ++i){
        acum += arr[i]; //your operation
        left[j][i] = acum;
        if((i & mask) == mask) acum = 0; //neutral element of your
        ↪  operation
      }
      acum = 0; //neutral element of your operation
      for(int i = N-1; i >= 0; --i){
        acum += arr[i]; //your operation
        right[j][i] = acum;
        if((i & mask) == 0) acum = 0; //neutral element of your
        ↪  operation
      }
    }
  }
  T query(int l, int r){
    if(l == r) return left[0][l];
    int i = 31 - __builtin_clz(l^r);
    return left[i][r] + right[i][l]; //your operation
  }
};
```

9.7. Wavelet Tree

```
struct WaveletTree{
  int lo, hi;
  WaveletTree *left, *right;
  vector<int> freq;
  vector<int> pref; //just use this if you want sums

  //queries indexed in base 1, complexity for all queries:
  ↪  O(log(max_element))
  //build from [from, to) with non-negative values in range [x, y]
  //you can use vector iterators or array pointers
  WaveletTree(vector<int>::iterator from, vector<int>::iterator
  ↪  to, int x, int y): lo(x), hi(y){
    if(from >= to) return;
    int m = (lo + hi) / 2;
    auto f = [m](int x){return x <= m;};
    freq.reserve(to - from + 1);
    freq.push_back(0);
    pref.reserve(to - from + 1);
    pref.push_back(0);
    for(auto it = from; it != to; ++it){
      freq.push_back(freq.back() + f(*it));
      pref.push_back(pref.back() + *it);
    }
    if(hi != lo){
      auto pivot = stable_partition(from, to, f);
      left = new WaveletTree(from, pivot, lo, m);
      right = new WaveletTree(pivot, to, m + 1, hi);
    }
  }

  //kth element in [l, r]
  int kth(int l, int r, int k){
    if(l > r) return 0;
    if(lo == hi) return lo;
```

```cpp
    int lb = freq[l - 1], rb = freq[r];
    int inLeft = rb - lb;
    if(k <= inLeft) return left->kth(lb + 1, rb, k);
    else return right->kth(l - lb, r - rb, k - inLeft);
  }

  //number of elements less than or equal to k in [l, r]
  int lessThanOrEqual(int l, int r, int k){
    if(l > r || k < lo) return 0;
    if(hi <= k) return r - l + 1;
    int lb = freq[l - 1], rb = freq[r];
    return left->lessThanOrEqual(lb + 1, rb, k) +
    ↪  right->lessThanOrEqual(l - lb, r - rb, k);
  }

  //number of elements equal to k in [l, r]
  int equalTo(int l, int r, int k){
    if(l > r || k < lo || k > hi) return 0;
    if(lo == hi) return r - l + 1;
    int lb = freq[l - 1], rb = freq[r];
    int m = (lo + hi) / 2;
    if(k <= m) return left->equalTo(lb + 1, rb, k);
    else return right->equalTo(l - lb, r - rb, k);
  }

  //sum of elements less than or equal to k in [l, r]
  int sum(int l, int r, int k){
    if(l > r || k < lo) return 0;
    if(hi <= k) return pref[r] - pref[l - 1];
    int lb = freq[l - 1], rb = freq[r];
    return left->sum(lb + 1, rb, k) + right->sum(l - lb, r - rb,
    ↪  k);
  }
};
```

9.8. Ordered Set C++

```cpp
#include <ext/pb_ds/assoc_container.hpp>
#include <ext/pb_ds/tree_policy.hpp>
using namespace __gnu_pbds;

template<typename T>
using ordered_set = tree<T, null_type, less<T>, rb_tree_tag,
↪  tree_order_statistics_node_update>;

int main(){
  int t, n, m;
  ordered_set<int> conj;
  while(cin >> t && t != -1){
    cin >> n;
    if(t == 0){ //insert
      conj.insert(n);
    }else if(t == 1){ //search
      if(conj.find(n) != conj.end()) cout << "Found\n";
      else cout << "Not found\n";
    }else if(t == 2){ //delete
      conj.erase(n);
    }else if(t == 3){ //update
      cin >> m;
      if(conj.find(n) != conj.end()){
        conj.erase(n);
        conj.insert(n);
      }
    }else if(t == 4){ //lower bound
      cout << conj.order_of_key(n) << "\n";
    }else if(t == 5){ //get nth element
      auto pos = conj.find_by_order(n);
      if(pos != conj.end()) cout << *pos << "\n";
      else cout << "-1\n";
    }
  }
  return 0;
}
```

9.9. Splay Tree

9.10. Red Black Tree

10. Cadenas

10.1. Trie

```
struct Node{
    bool isWord = false;
  map<char, Node*> letters;
};

struct Trie{
  Node* root;

  Trie(){
    root = new Node();
  }

  inline bool exists(Node * actual, const char & c){
    return actual->letters.find(c) != actual->letters.end();
  }

  void InsertWord(const string& word){
    Node* current = root;
    for(auto & c : word){
      if(!exists(current, c))
        current->letters[c] = new Node();
      current = current->letters[c];
    }
    current->isWord = true;
  }

  bool FindWord(const string& word){
    Node* current = root;
    for(auto & c : word){
      if(!exists(current, c))
        return false;
      current = current->letters[c];
    }
    return current->isWord;
  }

  void printRec(Node * actual, string acum){
    if(actual->isWord){
      cout << acum << "\n";
    }
    for(auto & next : actual->letters)
      printRec(next.second, acum + next.first);
  }

  void printWords(const string & prefix){
    Node * actual = root;
    for(auto & c : prefix){
      if(!exists(actual, c)) return;
      actual = actual->letters[c];
    }
    printRec(actual, prefix);
  }
};
```

10.2. KMP

```
struct kmp{
  vector<int> aux;
  string pattern;

  kmp(string pattern){
    this->pattern = pattern;
    aux.resize(pattern.size());
    int i = 1, j = 0;
    while(i < pattern.size()){
      if(pattern[i] == pattern[j])
        aux[i++] = ++j;
      else{
        if(j == 0) aux[i++] = 0;
        else j = aux[j - 1];
      }
    }
  }

  vector<int> search(string & text){
    vector<int> ans;
    int i = 0, j = 0;
    while(i < text.size() && j < pattern.size()){
      if(text[i] == pattern[j]){
        ++i, ++j;
        if(j == pattern.size()){
          ans.push_back(i - j);
```

```
        j = aux[j - 1];
      }
    }else{
      if(j == 0) ++i;
      else j = aux[j - 1];
    }
  }
  return ans;
 }
};
```

10.3. Aho-Corasick

```
const int M = 26;
struct node{
  vector<int> child;
  int p = -1;
  char c = 0;
  int suffixLink = -1, endLink = -1;
  int id = -1;

  node(int p = -1, char c = 0) : p(p), c(c){
    child.resize(M, -1);
  }
};

struct AhoCorasick{
  vector<node> t;
  vector<int> lenghts;
  int wordCount = 0;

  AhoCorasick(){
    t.emplace_back();
  }

  void add(const string & s){
    int u = 0;
    for(char c : s){
      if(t[u].child[c-'a'] == -1){
        t[u].child[c-'a'] = t.size();
        t.emplace_back(u, c);
      }
      u = t[u].child[c-'a'];
    }
    t[u].id = wordCount++;
    lenghts.push_back(s.size());
  }

  void link(int u){
    if(u == 0){
      t[u].suffixLink = 0;
      t[u].endLink = 0;
      return;
    }
    if(t[u].p == 0){
      t[u].suffixLink = 0;
      if(t[u].id != -1) t[u].endLink = u;
      else t[u].endLink = t[t[u].suffixLink].endLink;
      return;
    }
    int v = t[t[u].p].suffixLink;
    char c = t[u].c;
    while(true){
      if(t[v].child[c-'a'] != -1){
        t[u].suffixLink = t[v].child[c-'a'];
        break;
      }
      if(v == 0){
        t[u].suffixLink = 0;
        break;
      }
      v = t[v].suffixLink;
    }
    if(t[u].id != -1) t[u].endLink = u;
    else t[u].endLink = t[t[u].suffixLink].endLink;
  }

  void build(){
    queue<int> Q;
    Q.push(0);
    while(!Q.empty()){
      int u = Q.front(); Q.pop();
      link(u);
      for(int v = 0; v < M; ++v)
        if(t[u].child[v] != -1)
          Q.push(t[u].child[v]);
    }
```

```
  }

  int match(const string & text){
    int u = 0;
    int ans = 0;
    for(int j = 0; j < text.size(); ++j){
      int i = text[j] - 'a';
      while(true){
        if(t[u].child[i] != -1){
          u = t[u].child[i];
          break;
        }
        if(u == 0) break;
        u = t[u].suffixLink;
      }
      int v = u;
      while(true){
        v = t[v].endLink;
        if(v == 0) break;
        ++ans;
        int idx = j + 1 - lenghts[t[v].id];
        cout << "Found word #" << t[v].id << " at position " <<
        ↪  idx << "\n";
        v = t[v].suffixLink;
      }
    }
    return ans;
  }
};
```

10.4. Rabin-Karp

10.5. Suffix Array

10.6. Función Z

11. Varios

11.1. Lectura y escritura de __int128

```
//cout for __int128
ostream &operator<<(ostream &os, const __int128 & value){
  char buffer[64];
  char *pos = end(buffer) - 1;
  *pos = '\0';
  __int128 tmp = value < 0 ? -value : value;
  do{
    --pos;
    *pos = tmp % 10 + '0';
    tmp /= 10;
  }while(tmp != 0);
  if(value < 0){
    --pos;
    *pos = '-';
  }
  return os << pos;
}

//cin for __int128
istream &operator>>(istream &is, __int128 & value){
  char buffer[64];
  is >> buffer;
  char *pos = begin(buffer);
  int sgn = 1;
  value = 0;
  if(*pos == '-'){
    sgn = -1;
    ++pos;
  }else if(*pos == '+'){
    ++pos;
  }
  while(*pos != '\0'){
    value = (value << 3) + (value << 1) + (*pos - '0');
    ++pos;
  }
  value *= sgn;
  return is;
}
```

11.2. Longest Common Subsequence (LCS)

```
int lcs(string & a, string & b){
  int m = a.size(), n = b.size();
  vector<vector<int>> aux(m + 1, vector<int>(n + 1));
  for(int i = 1; i <= m; ++i){
    for(int j = 1; j <= n; ++j){
      if(a[i - 1] == b[j - 1])
        aux[i][j] = 1 + aux[i - 1][j - 1];
      else
        aux[i][j] = max(aux[i - 1][j], aux[i][j - 1]);
    }
  }
  return aux[m][n];
}
```

11.3. Longest Increasing Subsequence (LIS)

```
int lis(vector<int> & arr){
  if(arr.size() == 0) return 0;
  vector<int> aux(arr.size());
  int ans = 1;
  aux[0] = arr[0];
  for(int i = 1; i < arr.size(); ++i){
    if(arr[i] < aux[0])
      aux[0] = arr[i];
    else if(arr[i] > aux[ans - 1])
      aux[ans++] = arr[i];
    else
      aux[lower_bound(aux.begin(), aux.begin() + ans, arr[i]) -
      ↪ aux.begin()] = arr[i];
  }
  return ans;
}
```

11.4. Levenshtein Distance

```
int LevenshteinDistance(string & a, string & b){
  int m = a.size(), n = b.size();
  vector<vector<int>> aux(m + 1, vector<int>(n + 1));
  for(int i = 1; i <= m; ++i)
    aux[i][0] = i;
  for(int j = 1; j <= n; ++j)
    aux[0][j] = j;
  for(int j = 1; j <= n; ++j)
    for(int i = 1; i <= m; ++i)
      aux[i][j] = min({aux[i-1][j] + 1, aux[i][j-1] + 1,
      ↪ aux[i-1][j-1] + (a[i-1] != b[j-1])});
  return aux[m][n];
}
```

11.5. Día de la semana

```
//0:saturday, 1:sunday, ..., 6:friday
int dayOfWeek(int d, int m, lli y){
  if(m == 1 || m == 2){
    m += 12;
    --y;
  }
  int k = y % 100;
  lli j = y / 100;
  return (d + 13*(m+1)/5 + k + k/4 + j/4 + 5*j) % 7;
}
```

11.6. 2SAT

```
struct satisfiability_twosat{
  int n;
  vector<vector<int>> imp;

  satisfiability_twosat(int n) : n(n), imp(2 * n) {}

  void add_edge(int u, int v){imp[u].push_back(v);}

  int neg(int u){return (n << 1) - u - 1;}

  void implication(int u, int v){
    add_edge(u, v);
    add_edge(neg(v), neg(u));
  }

  vector<bool> solve(){
    int size = 2 * n;
    vector<int> S, B, I(size);
```

```
    function<void(int)> dfs = [&](int u){
      B.push_back(I[u] = S.size());
      S.push_back(u);

      for(int v : imp[u])
        if(!I[v]) dfs(v);
        else while (I[v] < B.back()) B.pop_back();

      if(I[u] == B.back())
        for(B.pop_back(), ++size; I[u] < S.size(); S.pop_back())
          I[S.back()] = size;
    };

    for(int u = 0; u < 2 * n; ++u)
      if(!I[u]) dfs(u);

    vector<bool> values(n);

    for(int u = 0; u < n; ++u)
      if(I[u] == I[neg(u)]) return {};
      else values[u] = I[u] < I[neg(u)];

    return values;
  }
};
```

11.7. Código Gray

```
//gray code
int gray(int n){
  return n ^ (n >> 1);
}

//inverse gray code
int inv_gray(int g){
  int n = 0;
  while(g){
    n ^= g;
    g >>= 1;
  }
  return n;
}
```

11.8. Contar número de unos en binario en un rango

```
//count the number of 1's in the i-th bit of all
//representations in binary of numbers in [1,n]
lli count(lli n, int i){
  if(n <= 0) return 0ll;
  lli ans = ((n + 1) >> (i + 1)) << i;
  ans += max(((n + 1) & ((1ll << (i + 1)) - 1)) - (1ll << i),
  ↪  0ll);
  return ans;
}
```

11.9. Números aleatorios en C++11

```
//Random number generation in C++11
mt19937_64
↪  rng(chrono::steady_clock::now().time_since_epoch().count());

//Generate a random integer in [a, b], you can also use long long
↪  int
int aleatorio_int(int a, int b){
  uniform_int_distribution<int> dist(a, b);
  return dist(rng);
}

//Generate a random double in [a, b], you can also use long double
double aleatorio_double(double a, double b){
  uniform_real_distribution<double> dist(a, b);
  return dist(rng);
}
```

12. Fórmulas y notas

12.1. Números de Stirling del primer tipo

$\left[{n \atop k}\right]$ representa el número de permutaciones de n elementos en exactamente k ciclos disjuntos.

$$\begin{aligned}
\left[{0 \atop 0}\right] &= 1 \\
\left[{0 \atop n}\right] &= \left[{n \atop 0}\right] = 0 && , \quad n > 0 \\
\left[{n \atop k}\right] &= (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right] && , \quad k > 0 \\
\sum_{k=0}^{n} \left[{n \atop k}\right] &= n! \\
\sum_{k=0}^{\infty} \left[{n \atop k}\right] x^k &= \prod_{k=0}^{n-1}(x+k)
\end{aligned}$$

12.2. Números de Stirling del segundo tipo

$\left\{{n \atop k}\right\}$ representa el número de formas de particionar un conjunto de n objetos distinguibles en k subconjuntos no vacíos.

$$\begin{aligned}
\left\{{0 \atop 0}\right\} &= 1 \\
\left\{{0 \atop n}\right\} &= \left\{{n \atop 0}\right\} = 0 && , \quad n > 0 \\
\left\{{n \atop k}\right\} &= k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\} && , \quad k > 0 \\
&= \sum_{j=0}^{k} \frac{j^n}{j!} \cdot \frac{(-1)^{k-j}}{(k-j)!}
\end{aligned}$$

12.3. Números de Euler

$\left\langle{n \atop k}\right\rangle$ representa el número de permutaciones de 1 a n en donde exactamente k números son mayores que el número anterior, es decir, cuántas permutaciones tienen k "ascensos".

$$\begin{aligned}
\left\langle{1 \atop 0}\right\rangle &= 1 \\
\left\langle{n \atop k}\right\rangle &= (n-k)\left\langle{n-1 \atop k-1}\right\rangle + (k+1)\left\langle{n-1 \atop k}\right\rangle && , \quad n \geq 2 \\
&= \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k+1-j)^n \\
\sum_{k=0}^{n-1} \left\langle{n \atop k}\right\rangle &= n!
\end{aligned}$$

12.4. Números de Catalan

$$\begin{aligned}
C_0 &= 1 \\
C_n &= \frac{1}{n+1}\binom{2n}{n} = \sum_{j=0}^{n-1} C_j C_{n-1-j} \\
\sum_{n=0}^{\infty} C_n x^n &= \frac{1-\sqrt{1-4x}}{2x}
\end{aligned}$$

12.5. Números de Bell

B_n representa el número de formas de particionar un conjunto de n elementos.

$$\begin{aligned}
B_n &= \sum_{k=0}^{n} \left\{{n \atop k}\right\} = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k \\
\sum_{k=0}^{\infty} \frac{B_n}{n!} x^n &= e^{e^x - 1}
\end{aligned}$$

12.6. Números de Bernoulli

$$B_0{}^+ = 1$$

$$B_n{}^+ = 1 - \sum_{k=0}^{n-1} \binom{n}{k} \frac{B_k{}^+}{n-k+1}$$

$$\sum_{m=0}^{\infty} \frac{B_n{}^+ x^n}{n!} = \frac{x}{1-e^{-x}} = \frac{1}{\frac{1}{1!} - \frac{x}{2!} + \frac{x^2}{3!} - \frac{x^3}{4!} + \cdots}$$

12.7. Fórmula de Faulhaber

$$S_p(n) = \sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{k=0}^{p} \binom{p+1}{k} B_k{}^+ n^{p+1-k}$$

12.8. Función Beta

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2\int_0^{\pi/2} \sin^{2x-1}(\theta)\cos^{2x-1}(\theta)d\theta$$

$$= \int_0^1 t^{x-1}(1-t)^{y-1}dt = \int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}}dt$$

12.9. Función zeta de Riemann

La siguiente fórmula converge rápido para valores pequeños de n ($n \approx 20$):

$$\zeta(s) \approx \frac{1}{d_0(1-2^{1-s})} \sum_{k=1}^{n} \frac{(-1)^{k-1} d_k}{k^s}$$

$$d_k = \sum_{j=k}^{n} \frac{4^j}{n+j} \binom{n+j}{2j}$$

12.10. Funciones generadoras

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k \right) x^n = \frac{1}{1-x} \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n = \frac{1}{(1-x)^k}$$

$$\sum_{n=0}^{\infty} p_n x^n = \frac{1}{\prod_{k=1}^{\infty}(1-x^k)} = \frac{1}{\sum_{n=-\infty}^{\infty} x^{\frac{1}{2}n(3n+1)}}$$

$$\sum_{n=0}^{\infty} n^k x^n = \frac{\sum_{i=0}^{k-1} \left\langle {k \atop i} \right\rangle x^{i+1}}{(1-x)^{k+1}} \quad , \quad k \geq 1$$

Sean $a_1, a_2, \ldots, a_n$ números complejos. Sean $p_m = \sum_{i=1}^{n} a_i^m$ y s_m el m-ésimo polinomio elemental simétrico de $a_1, a_2, \ldots, a_n$. Entonces se cumple que $xS'(x) + P(x)S(x) = 0$, donde $P(x) = \sum_{m=1}^{\infty} p_m x_m$ y $S(x) = \prod_{i=1}^{n}(1-a_i x) = \sum_{m=0}^{n} (-1)^m s_m x^m$.

12.11. Números armónicos

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \approx \ln(n) + \gamma + \frac{1}{2n} - \frac{1}{12n^2}$$

$$\gamma \approx 0.577215664901532860606512$$

12.12. Aproximación de Stirling

$$\ln(n!) \approx n\ln(n) - n + \frac{1}{2}\ln(2\pi n)$$

$$\#\text{ de dígitos de } n! = 1 + \left\lfloor n\log\left(\frac{n}{e}\right) + \frac{1}{2}\log(2\pi n)\right\rfloor \quad (n \geq 30)$$

12.13. Ternas pitagóricas

- Una terna de enteros positivos (a, b, c) es pitagórica si $a^2 + b^2 = c^2$. Además es primitiva si $\gcd(a, b, c) = 1$.
- Generador de ternas primitivas:

$$\begin{aligned} a &= m^2 - n^2 \\ b &= 2mn \\ c &= m^2 + n^2 \end{aligned}$$

  donde $n \geq 1$, $m > n$, $\gcd(m, n) = 1$ y m, n tienen distinta paridad.

- Árbol de ternas pitagóricas primitivas: al multiplicar cualquiera de estas matrices:

$$\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix}, \quad \begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

  por una terna primitiva $\mathbf{v}^{\mathbf{T}}$, obtenemos otra terna primitiva diferente. En particular, si empezamos con $\mathbf{v} = (3, 4, 5)$, podremos generar todas las ternas primitivas.

12.14. Árbol de Stern–Brocot

Todos los racionales positivos se pueden representar como un árbol binario de búsqueda completo infinito con raíz $\frac{1}{1}$.

- Dado un racional $q = [a_0; a_1, a_2, \ldots, a_k]$ donde $a_k \neq 1$, sus hijos serán $[a_0; a_1, a_2, \ldots, a_k + 1]$ y $[a_0; a_1, a_2, \ldots, a_k - 1, 2]$, y su padre será $[a_0; a_1, a_2, \ldots, a_k - 1]$.
- Para hallar el camino de la raíz $\frac{1}{1}$ a un racional q, se usa búsqueda binaria iniciando con $L = \frac{0}{1}$ y $R = \frac{1}{0}$. Para hallar M se supone que $L = \frac{a}{b}$ y $R = \frac{c}{d}$, entonces $M = \frac{a+c}{b+d}$.

12.15. Combinatoria

- Principio de las casillas: al colocar n objetos en k lugares hay al menos $\lceil \frac{n}{k} \rceil$ objetos en un mismo lugar.
- Número de funciones: sean A y B conjuntos con $m = |A|$ y $n = |B|$. Sea $f : A \to B$:
  - Si $m \leq n$, entonces hay $m!\binom{n}{m}$ funciones inyectivas f.
  - Si $m = n$, entonces hay $n!$ funciones biyectivas f.
  - Si $m \geq n$, entonces hay $n!\left\{ {m \atop n} \right\}$ funciones suprayectivas f.
- Barras y estrellas: ¿cuántas soluciones en los enteros no negativos tiene la ecuación $\sum_{i=1}^{k} x_i = n$? Tiene $\binom{n+k-1}{k-1}$ soluciones.
- ¿Cuántas soluciones en los enteros positivos tiene la ecuación $\sum_{i=1}^{k} x_i = n$? Tiene $\binom{n-1}{k-1}$ soluciones.
- Desordenamientos: $a_0 = 1$, $a_1 = 0$, $a_n = (n-1)(a_{n-1} + a_{n-2}) = na_{n-1} + (-1)^n$.
- Sea $f(x)$ una función. Sea $g_n(x) = xg'_{n-1}(x)$ con $g_0(x) = f(x)$. Entonces $g_n(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k f^{(k)}(x)$.
- Supongamos que tenemos $m+1$ puntos: $(0, y_0)$, $(1, y_1)$, …, (m, y_m). Entonces el polinomio $P(x)$ de grado m que pasa por todos ellos es:

$$P(x) = \left[\prod_{i=0}^{m}(x - i)\right](-1)^m \sum_{i=0}^{m} \frac{y_i(-1)^i}{(x-i)i!(m-i)!}$$

- Sea $a_0, a_1, \ldots$ una recurrencia lineal homogénea de grado d dada por $a_n = \sum_{i=1}^{d} b_i a_{n-i}$ para $n \geq d$ con términos iniciales $a_0, a_1, \ldots, a_{d-1}$. Sean $A(x)$ y $B(x)$ las funciones generadoras de las sucesiones a_n y b_n respectivamente, entonces se cumple que $A(x) = \frac{A_0(x)}{1 - B(x)}$, donde $A_0(x) = \sum_{i=0}^{d-1} \left[a_i - \sum_{j=0}^{i-1} a_j b_{i-j} \right] x^i$.
- Si queremos obtener otra recurrencia c_n tal que $c_n = a_{kn}$, las raíces del polinomio característico de c_n se obtienen al elevar todas las raíces del polinomio característico de a_n a la k-ésima potencia; y sus términos iniciales serán $a_0, a_k, \ldots, a_{k(d-1)}$.

12.16. Grafos

- Sea d_n el número de grafos con n vértices etiquetados: $d_n = 2^{\binom{n}{2}}$.
- Sea c_n el número de grafos conexos con n vértices etiquetados. Tenemos la recurrencia: $c_1 = 1$ y $d_n = \sum_{k=1}^{n} \binom{n-1}{k-1} c_k d_{n-k}$. También se cumple, usando funciones generadoras exponenciales, que $C(x) = 1 + \ln(D(x))$.
- Sea t_n el número de torneos fuertemente conexos en n nodos etiquetados. Tenemos la recurrencia $t_1 = 1$ y $d_n = \sum_{k=1}^{n} \binom{n}{k} t_k d_{n-k}$. Usando funciones generadoras exponenciales, tenemos que $T(x) = 1 - \frac{1}{D(x)}$.
- Número de spanning trees en un grafo completo con n vértices etiquetados: n^{n-2}.
- Número de bosques etiquetados con n vértices y k componentes conexas: kn^{n-k-1}.
- Para un grafo no dirigido simple G con n vértices etiquetados de 1 a n, sea $Q = D - A$, donde D es la matriz diagonal de los grados de cada nodo de G y A es la matriz de adyacencia de G. Entonces el número de spanning trees de G es igual a cualquier cofactor de Q.
- Sea G un grafo. Se define al polinomio $P_G(x)$ como el polinomio cromático de G, en donde $P_G(k)$ nos dice cuántas k-coloraciones de los vértices admite G. Ejemplos comunes:
  - Grafo completo de n nodos: $P(x) = x(x-1)(x-2)\ldots(x-(n-1))$
  - Grafo vacío de n nodos: $P(x) = x^n$
  - Árbol de n nodos: $P(x) = x(x-1)^{n-1}$
  - Ciclo de n nodos: $P(x) = (x-1)^n + (-1)^n(x-1)$

12.17. Teoría de números

$$
\begin{aligned}
(f * e)(n) &= f(n) \\
(\varphi * \mathbf{1})(n) &= n \\
(\mu * \mathbf{1})(n) &= e(n) \\
\varphi(n^k) &= n^{k-1}\varphi(n) \\
\sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} k &= \frac{n\varphi(n)}{2} \quad , \quad n \geq 2 \\
\sum_{k=1}^{n} \operatorname{lcm}(k,n) &= \frac{n}{2} + \frac{n}{2}\sum_{d|n} d\varphi(d) = \frac{n}{2} + \frac{n}{2}\prod_{p^a|n} \frac{p^{2a+1}+1}{p+1} \\
\sum_{k=1}^{n} \gcd(k,n) &= \sum_{d|n} d\varphi\left(\frac{n}{d}\right) = \prod_{p^a|n} p^{a-1}(1+(a+1)(p-1))
\end{aligned}
$$

- Lifting the exponent: sea p un primo, x, y enteros y n un entero positivo tal que $p \mid x - y$ pero $p \nmid x$ ni $p \nmid y$. Entonces:
  - Si p es impar: $v_p(x^n - y^n) = v_p(x-y) + v_p(n)$
  - Si $p = 2$ y n es par: $v_p(x^n - y^n) = v_p(x-y) + v_p(n) + v_p(x+y) - 1$

  donde $v_p(n)$ es el exponente de p en la factorización en primos de n.

- Suma de dos cuadrados: sea $\chi_4(n)$ una función multiplicativa igual a 1 si $n \equiv 1 \mod 4$, -1 si $n \equiv 3 \mod 4$ y cero en otro caso. Entonces, el número de soluciones enteras (a, b) de la ecuación $a^2 + b^2 = n$ es $4(\chi_4 * 1)(n) = 4 \sum_{d|n} \chi_4(d)$.

- Teorema de Lucas:

$$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{k_i} \pmod{p}$$

$$m = \sum_{i=0}^{k} m_i p^i \quad , \quad n = \sum_{i=0}^{k} n_i p^i$$

$$0 \le m_i, n_i < p$$

- Sean $a, b, c \in \mathbb{Z}$ con $a \neq 0$ y $b \neq 0$. La ecuación $ax + by = c$ tiene como soluciones:

$$x = \frac{x_0 c - bk}{d}$$

$$y = \frac{y_0 c + ak}{d}$$

para toda $k \in \mathbb{Z}$ si y solo si $d|c$, donde $ax_0 + by_0 = \gcd(a, b) = d$ (Euclides extendido). Si a y b tienen el mismo signo, hay exactamente $\max\left(\left\lfloor \frac{x_0 c}{|b|} \right\rfloor + \left\lfloor \frac{y_0 c}{|a|} \right\rfloor + 1, 0\right)$ soluciones no negativas. Si tienen el signo distinto, hay infinitas soluciones no negativas.

- Dada una función aritmética f con $f(1) \neq 0$, existe otra función aritmética g tal que $(f * g)(n) = e(n)$, dada por:

$$g(1) = \frac{1}{f(1)}$$

$$g(n) = -\frac{1}{f(1)} \sum_{d|n, d<n} f\left(\frac{n}{d}\right) g(d) \quad , \quad n > 1$$

- Sean $h(n) = \sum_{k=1}^{n} f\left(\left\lfloor \frac{n}{k} \right\rfloor\right) g(k)$, $G(n) = \sum_{k=1}^{n} g(k)$ y $m = \lfloor \sqrt{n} \rfloor$, entonces:

$$h(n) = \sum_{k=1}^{\lfloor n/m \rfloor} f\left(\left\lfloor \frac{n}{k} \right\rfloor\right) g(k) + \sum_{k=1}^{m-1} \left(G\left(\left\lfloor \frac{n}{k} \right\rfloor\right) - G\left(\left\lfloor \frac{n}{k+1} \right\rfloor\right)\right) f(k)$$

- Sean $F(n) = \sum_{k=1}^{n} f(k)$, $G(n) = \sum_{k=1}^{n} g(k)$, $h(n) = (f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right)$ y $H(n) = \sum_{k=1}^{n} h(k)$, entonces:

$$H(n) = \sum_{k=1}^{n} f(k) G\left(\left\lfloor \frac{n}{k} \right\rfloor\right)$$

- Sean $\Phi_p(n) = \sum_{k=1}^{n} k^p \varphi(k)$ y $M_p(n) = \sum_{k=1}^{n} k^p \mu(k)$. Aplicando lo anterior, podemos calcular $\Phi_p(n)$ y $M_p(n)$ con complejidad $O(n^{2/3})$ si precalculamos con fuerza bruta los primeros $\lfloor n^{2/3} \rfloor$ valores, y para los demás, usamos las siguientes recurrencias (DP con `map`):

$$\Phi_p(n) = S_{p+1}(n) - \sum_{k=2}^{\lfloor n/m \rfloor} k^p \Phi_p\left(\left\lfloor \frac{n}{k} \right\rfloor\right) - \sum_{k=1}^{m-1} \left(S_p\left(\left\lfloor \frac{n}{k} \right\rfloor\right) - S_p\left(\left\lfloor \frac{n}{k+1} \right\rfloor\right)\right) \Phi_p(k)$$

$$M_p(n) = 1 - \sum_{k=2}^{\lfloor n/m \rfloor} k^p M_p\left(\left\lfloor \frac{n}{k} \right\rfloor\right) - \sum_{k=1}^{m-1} \left(S_p\left(\left\lfloor \frac{n}{k} \right\rfloor\right) - S_p\left(\left\lfloor \frac{n}{k+1} \right\rfloor\right)\right) M_p(k)$$

- En general, si queremos hallar $F(n)$ y existe una función mágica $g(n)$ tal que $G(n)$ y $H(n)$ se puedan calcular en $O(1)$, entonces:

$$F(n) = \frac{1}{g(1)} \left[H(n) - \sum_{k=2}^{\lfloor n/m \rfloor} g(k) F\left(\left\lfloor \frac{n}{k} \right\rfloor\right) - \sum_{k=1}^{m-1} \left(G\left(\left\lfloor \frac{n}{k} \right\rfloor\right) - G\left(\left\lfloor \frac{n}{k+1} \right\rfloor\right)\right) F(k)\right.$$

12.18. Primos

$10^2 + 1$, $10^3 + 9$, $10^4 + 7$, $10^5 + 3$, $10^6 + 3$, $10^7 + 19$, $10^8 + 7$, $10^9 + 7$, $10^{10} + 19$, $10^{11} + 3$, $10^{12} + 39$, $10^{13} + 37$, $10^{14} + 31$, $10^{15} + 37$, $10^{16} + 61$, $10^{17} + 3$, $10^{18} + 3$.

$10^2 - 3$, $10^3 - 3$, $10^4 - 27$, $10^5 - 9$, $10^6 - 17$, $10^7 - 9$, $10^8 - 11$, $10^9 - 63$, $10^{10} - 33$, $10^{11} - 23$, $10^{12} - 11$, $10^{13} - 29$, $10^{14} - 27$, $10^{15} - 11$, $10^{16} - 63$, $10^{17} - 3$, $10^{18} - 11$.

12.19. Números primos de Mersenne

Números primos de la forma $M_p = 2^p - 1$ con p primo. Todos los números perfectos pares son de la forma $2^{p-1}M_p$ y viceversa.

Los primeros 47 valores de p son: 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 756839, 859433, 1257787, 1398269, 2976221, 3021377, 6972593, 13466917, 20996011, 24036583, 25964951, 30402457, 32582657, 37156667, 42643801, 43112609.

12.20. Números primos de Fermat

Números primos de la forma $F_p = 2^{2^p} + 1$, solo se conocen cinco: 3, 5, 17, 257, 65537. Un polígono de n lados es construible si y solo si n es el producto de algunas potencias de dos y distintos primos de Fermat.