Course: 3CS110ME24

Course: Federated Learning

Unit 1
Introduction

Probability of Events

We are now ready to define what is meant by probability.

Specifically, we talk about "the **probability of an event**" as the relative frequency with which that event can be expected to occur.

The probability of an event may be obtained in three different ways:

(1) *empirically*,

(2) *theoretically*, and

(3) *subjectively*.

Calculating probabilities: How likely is it?

- If each outcome of an experiment is *equally likely*, the **probability** of an event is the fraction of favorable outcomes.

$$\text{Probability of an event} = \frac{\text{Number of favorable outcomes}}{\text{Total number of possible equally likely outcomes}}$$

- A probability of an event is the fraction of favorable outcomes.

$$\text{Probability} = \frac{\text{Favorable outcomes}}{\text{Total outcomes}}$$

- A Probability must be between 0 and 1.
- The probability of an event is 0 $\Leftrightarrow$ the event can never occur.
- The probability of an event is 1 $\Leftrightarrow$ the event will always occur.

Calculating probabilities: How likely is it?

When you toss a fair coin, heads and tails are **equally likely** outcomes. Favorable outcomes are outcomes in a specified event. For equally likely outcomes, the **<u>theoretical probability</u>** of an event is the ratio of the number of favorable outcomes to the total number of outcomes.

Calculating probabilities: How likely is it?

- **Example:** Suppose I flip two identical coins. What is the probability that I get two heads?
- **Solution:** There are four equally likely outcomes.

Nickel	Dime
H	H
H	T
T	H
T	T

$$P(\mathrm{HH}) = \frac{\text{Favorable outcomes}}{\text{Total outcomes}} = \frac{1}{4}$$

Calculating probabilities: How likely is it?

Example 1A: Finding Theoretical Probability

Each letter of the word PROBABLE is written on a separate card. The cards are placed face down and mixed up. What is the probability that a randomly selected card has a consonant?

There are 8 possible outcomes and 5 favorable outcomes.

$$P(\text{consonant}) = \frac{5}{8} = 62.5\%$$

Calculating probabilities: How likely is it?

Example 1B: Finding Theoretical Probability

Two number cubes are rolled. What is the probability that the difference between the two numbers is 4?

1 1	1 2	1 3	1 4	1 5	1 6
2 1	2 2	2 3	2 4	2 5	2 6
3 1	3 2	3 3	3 4	3 5	3 6
4 1	4 2	4 3	4 4	4 5	4 6
5 1	5 2	5 3	5 4	5 5	5 6
6 1	6 2	6 3	6 4	6 5	6 6

Calculating probabilities: How likely is it?

Example 1B: Finding Theoretical Probability

Two number cubes are rolled. What is the probability that the difference between the two numbers is 4?

1 1	1 2	1 3	1 4	1 5	1 6
2 1	2 2	2 3	2 4	2 5	2 6
3 1	3 2	3 3	3 4	3 5	3 6
4 1	4 2	4 3	4 4	4 5	4 6
5 1	5 2	5 3	5 4	5 5	5 6
6 1	6 2	6 3	6 4	6 5	6 6

There are 36 possible outcomes.

$$P(\text{difference is 4}) = \frac{\text{number of outcomes with a difference of 4}}{36}$$

$$P(\text{difference is 4}) = \frac{4}{36} = \frac{1}{9}$$

4 outcomes with a difference of 4: (1, 5), (2, 6), (5, 1), and (6, 2)

Calculating probabilities: How likely is it?

Example: Suppose I have a 50-50 chance of getting through a certain traffic light without having to stop. I go through this light on my way to work and again on my way home.

To work	To home
Stop	Stop
Stop	Don't stop
Don't stop	Stop
Don't stop	Don't stop

1. What is the probability of having to stop at this light at least once on a workday?
2. What is the probability of not having to stop at all?

Calculating probabilities: How likely is it?

- **Solution:**

1. 50-50 chance: the probability of stopping at the light is ½ and the probability of not stopping is ½

$$\frac{\text{Favorable outcomes}}{\text{Total outcomes}} = \frac{3}{4}$$

2. One of the possible outcomes (Don't stop-Don't stop) corresponds to not having to stop at all:

$$\frac{\text{Favorable outcomes}}{\text{Total outcomes}} = \frac{1}{4}$$

Calculating probabilities: How likely is it?

$$\text{Probability} \text{of event } \textbf{not} \text{ occurring} \\ = 1 - \text{Probablity of event occurring}$$

- **Example:** There are several sections of English offered. There are some English teachers I like and some I don't. I enroll in a section of English without knowing the teacher. A friend of mine has calculated that the probability that I get a teacher I like is:

$$\text{P(Teacher I like)} = \frac{7}{17}$$

What is the probability that I will get a teacher that I don't like?

- **Solution:** $$\text{P(Teacher I don't like)} = 1 - \text{P(Teacher I like)} \\ = 1 - \frac{7}{17} = \frac{10}{17}$$

Calculating probabilities: How likely is it?

- **Example:** Suppose we toss a pair of standard six-sided dice.

1. What is the probability that a we get a 7?
2. What is the probability that we get any sum but 7?

- **Solution:**

Red die	Green die
1	6
2	5
3	4
4	3
5	2
6	1

1. Probability of a 7

$$= \frac{6}{36} = \frac{1}{6} = 0.17 = 17\%$$

2. Probability of event **not** getting a 7

$$= 1 - \frac{1}{6} = \frac{5}{6} = 0.83 = 83\%$$

Calculating probabilities: How likely is it?

Two integers from 1 to 10 are randomly selected. The same number may be chosen twice. What is the probability that both numbers are less than 9?

Calculating probabilities: How likely is it?

Two integers from 1 to 10 are randomly selected. The same number may be chosen twice. What is the probability that both numbers are less than 9?

$P(\text{number} < 9) = 1 - P(\text{number} \geq 9)$ *Use the complement.*

$$P(\text{number} < 9) = 1 - \frac{2}{10} = \frac{8}{10}$$

The probability that both numbers are less than 9, is

$$\frac{8}{10} \bullet \frac{8}{10} = \frac{64}{100} = \frac{16}{25}, \text{ or } 64\%.$$

Calculating probabilities: How likely is it?

- The **disjunction** is the event that either A or B occurs. The probability of this disjunction:

$$\boxed{P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)}$$

- **Example:** Suppose a librarian has a cart with 10 paperback algebra books, 15 paperback biology books, 21 hardbound algebra books, and 39 hardbound biology books. What is the probability that a book selected at random from this cart is an algebra book or a paperback book?

Calculating probabilities: How likely is it?

- **Solution:**

Let A be an algebra book and B be a paperback book.

Three probabilities: $\mathrm{P(A)}$, $\mathrm{P(B)}$, and $\mathrm{P(A\ and\ B)}$.

Altogether, there are $10+15+21+39=85$ books.

$$\mathrm{P(A)} = \frac{31}{85}, \quad \mathrm{P(B)} = \frac{25}{85}, \quad \mathrm{P(A\ and\ B)} = \frac{10}{85}.$$

$$\mathrm{P(A\ or\ B)} = \mathrm{P(A)} + \mathrm{P(B)} - \mathrm{P(A\ and\ B)}$$

$$= \frac{31}{85} + \frac{25}{85} - \frac{10}{85} = \frac{46}{85} = 0.54 = 54\%$$

Calculating probabilities: How likely is it?

- **Example:** The surface area of Earth is approximately 197 million square miles. North America covers approximately 9.37 million square miles, and South America covers approximately 6.88 million square miles. A meteor falls from the sky and strikes Earth.

  What is the probability that it strikes North or South America?

- **Solution:** The total area covered by North and South America is $9.37+6.88=16.25$ million square miles.

$$\text{Fraction of the surface area of Earth} = \frac{16.25}{197} = 0.082 = 8.2\%.$$

Calculating probabilities: How likely is it?

You can estimate the probability of an event by using data, or by **experiment**. For example, if a doctor states that an operation "has an 80% probability of success," 80% is an estimate of probability based on similar case histories.

Each repetition of an experiment is a **trial**. The sample space of an experiment is the set of all possible outcomes. The **experimental probability** of an event is the ratio of the number of times that the event occurs, the *frequency*, to the number of trials.

Probability of Events

Empirical probability, also known as **experimental probability**, refers to a probability that is based on **historical data**.

In other words, empirical probability illustrates the **likelihood of an event** occurring based on historical data.

- The **empirical probability** of an event is a probability obtained by experimental evidence.

$$\text{Probability} = \frac{\text{Favorable outcomes}}{\text{Total number of outcomes in the experiment}}$$

Empirical (Observed) Probability $P'(A)$

Calculating probabilities: How likely is it?

- **Example:** Suppose the city posted workers at the intersection, and over a five-week period it counted 16,652 vehicles passing through the intersection, of which 1432 ran a red light. Use these numbers to calculate an empirical probability that cars passing through the intersection will run a red light.
- **Solution:** 1432 out of 16,652 ran the red light.

$$\text{An empirical probability} = \frac{1432}{16{,}652} = 0.09 = 9\%$$

Calculating probabilities: How likely is it?

- A **conditional probability** is the probability that one event occurs given that another has occurred.
- **Example:** The accompanying table of data is adapted from a study of a test for TB among patients diagnosed with extra pulmonary TB .

	Has TB	Does not have TB
Test positive	446	15
Test negative	216	323

Calculate the conditional probability that a person tests positive given that the person has TB.

- **Solution:** $446 + 216 = 662$ people who have TB.

$$\text{P(Positive test given TB is present)} = \frac{\text{True positives}}{\text{All who have TB}} = \frac{446}{662}$$

$$= 0.674 = 67.4\%$$

On February 1, 2003, the space shuttle Columbia exploded. This was the second disaster in 113 space missions for NASA. On the basis of this information, what is the probability that a future mission is successfully completed?

Joint Probability

- The term joint probability refers to a statistical measure that calculates the likelihood of two events occurring together and at the same point in time.
- Put simply, a joint probability is the probability of event Y occurring at the same time that event X occurs.
- In order for joint probability to work, both events must be independent of one another, which means they aren't conditional or don't rely on each other.
- Joint probabilities can be visualized using Venn diagrams.

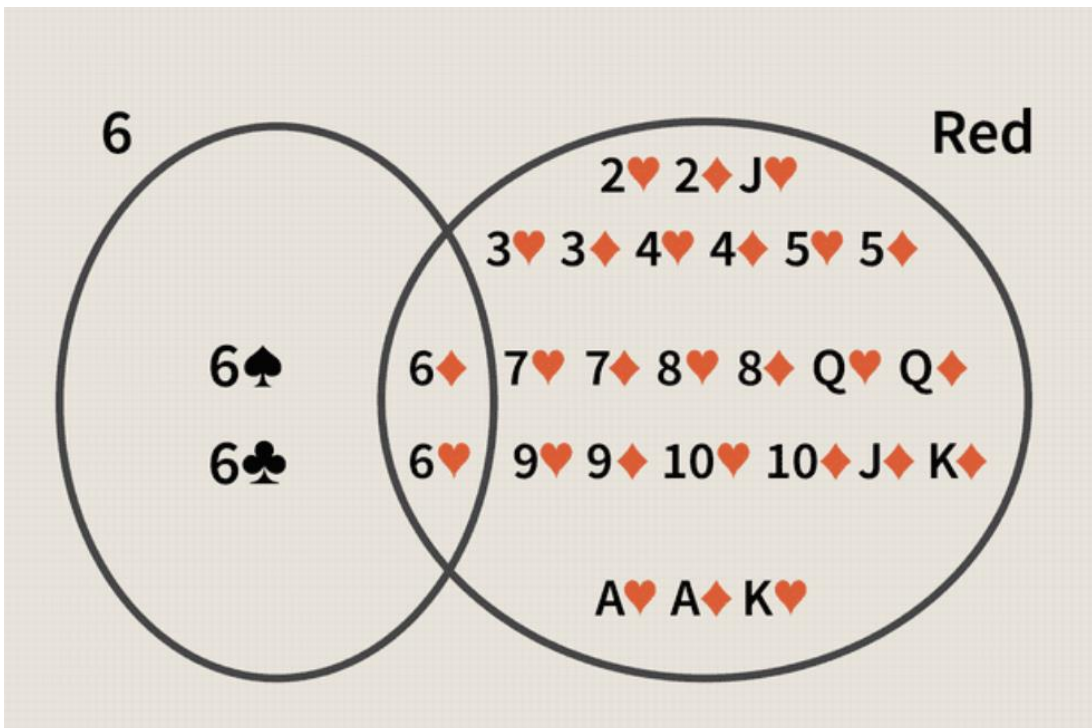
Joint Probability

- So the joint probability of picking a card that is both red and 6 from a deck is P(6 ∩ red) = 2/52 = 1/26
- Since a deck of cards has two red sixes—the *six of hearts and the six of diamonds*.
- Because the events red and 6 are independent, you can also use the following formula to calculate the joint probability:

$$P(6 \cap red) = P(6) \times P(red) = 4/52 \times 26/52 = 1/26$$

Joint Probability

Therefore, the joint probability is also called the intersection of two or more events. A Venn diagram is perhaps the best visual tool to explain an intersection:



How Are Empirical and Theoretical Probabilities Related?

How Are Empirical and Theoretical Probabilities Related?

Consider the rolling of one die and define event A as the occurrence of a “1.”

An ordinary die has six equally likely sides, so the theoretical probability of event A is $P(\text{A}) = \frac{1}{6}$. What does this mean?

Do you expect to see one “1” in each trial of six rolls? Explain. If not, what results do you expect? If we were to roll the die several times and keep track of the proportion of the time event A occurs, we would observe an empirical probability for event A.

How Are Empirical and Theoretical Probabilities Related?

What value would you expect to observe for $P'(A)$? Explain. How are the two probabilities $P(A)$ and $P'(A)$ related? Explain.

Demonstration—law Of Large Numbers

To gain some insight into this relationship, let's perform an experiment. The experiment will consist of 20 trials.

Each trial of the experiment will consist of rolling a die six times and recording the number of times the "1" occurs. Perform 20 trials.

How Are Empirical and Theoretical Probabilities Related?

Each row of Table 4.1 shows the results of one trial; we conduct 20 trials, so there are 20 rows.

Trial	Column 1: Number of 1s Observed	Column 2: Relative Frequency	Column 3: Cumulative Relative Frequency	Trial	Column 1: Number of 1s Observed	Column 2: Relative Frequency	Column 3: Cumulative Relative Frequency
1	1	1/6	1/6 = 0.17	11	1	1/6	10/66 = 0.15
2	2	2/6	3/12 = 0.25	12	0	0/6	10/72 = 0.14
3	0	0/6	3/18 = 0.17	13	2	2/6	12/78 = 0.15
4	1	1/6	4/24 = 0.17	14	1	1/6	13/84 = 0.15
5	0	0/6	4/30 = 0.13	15	1	1/6	14/90 = 0.16
6	1	1/6	5/36 = 0.14	16	3	3/6	17/96 = 0.18
7	2	2/6	7/42 = 0.17	17	0	0/6	17/102 = 0.17
8	2	2/6	9/48 = 0.19	18	1	1/6	18/108 = 0.17
9	0	0/6	9/54 = 0.17	19	0	0/6	18/114 = 0.16
10	0	0/6	9/60 = 0.15	20	1	1/6	19/120 = 0.16

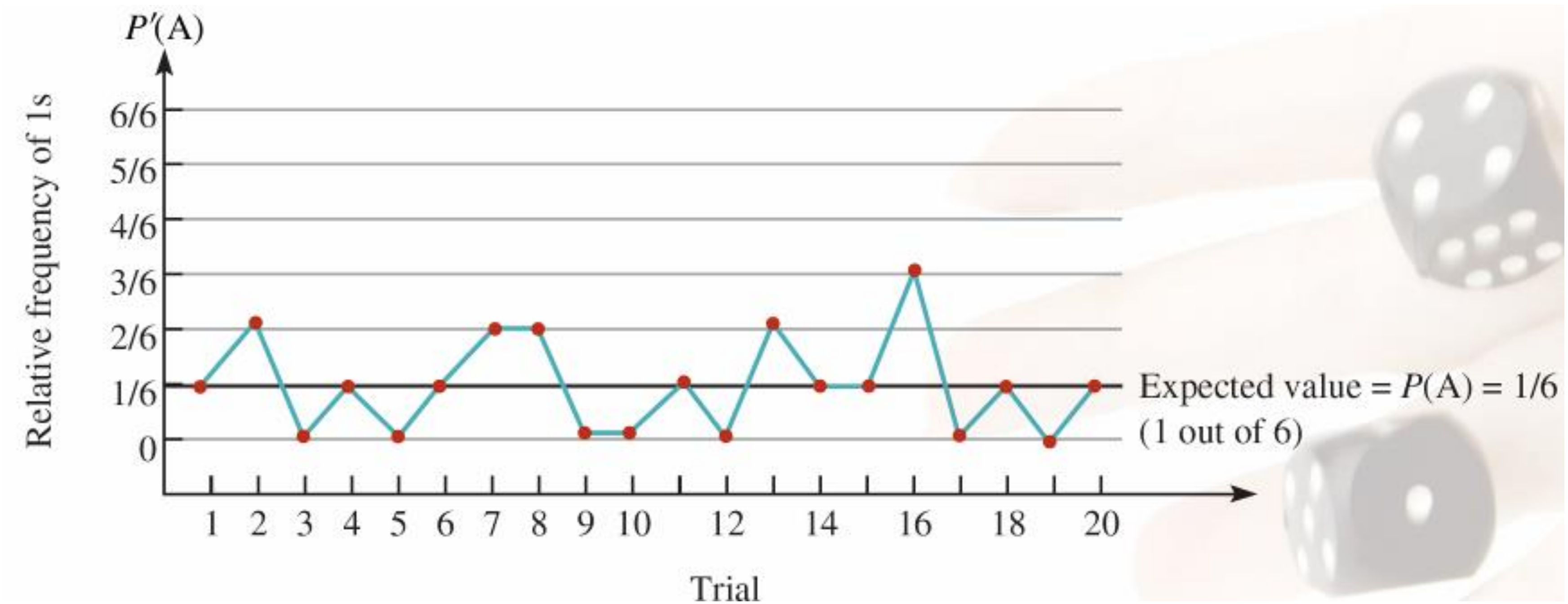
Experimental Results of Rolling a Die Six Times in Each Trial

Table 4.1

How Are Empirical and Theoretical Probabilities Related?

Column 1 lists the number of 1s observed in each trial (set of six rolls); column 2 lists the observed relative frequency for each trial; and column 3 lists the cumulative relative frequency as each trial was completed.

Figure 4.3a shows the fluctuation (above and below) of the observed probability, $P'(A)$ (Table 4.1, column 2), about the theoretical probability, $P(A) = \frac{1}{6}$.

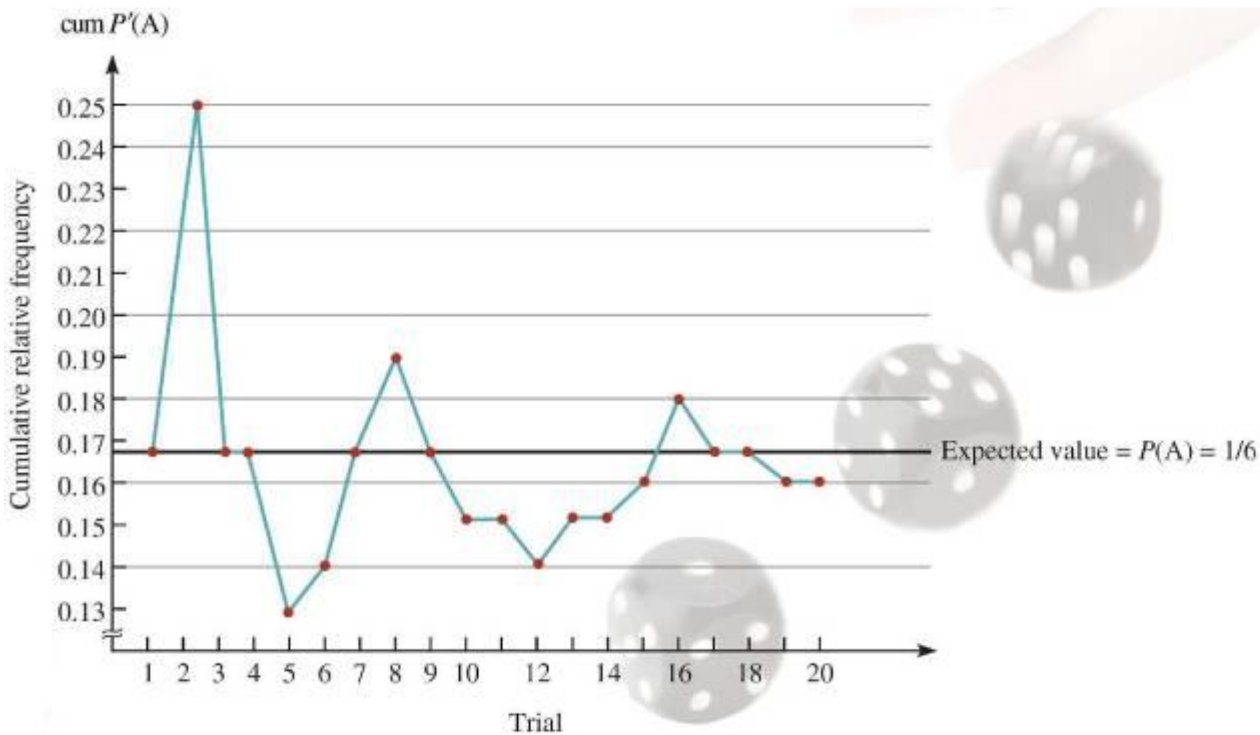


Fluctuations Found in the Die-Tossing Experiment

Figure 4.3 - (a) Relative Frequency

How Are Empirical and Theoretical Probabilities Related?

Whereas Figure 4.3b shows the fluctuation of the cumulative relative frequency (Table 4.1, column 3) and how it becomes more stable.



Fluctuations Found in the Die-Tossing Experiment

Figure 4.3 - (b) Cumulative Relative Frequency

How Are Empirical and Theoretical Probabilities Related?

In fact, the cumulative relative frequency becomes relatively close to the theoretical or expected probability $\frac{1}{6}$, or $0.166\overline{6}$ = 0.167.

A cumulative graph such as that shown in Figure 4.3b demonstrates the idea of a **long-term average.**

Long term average is often referred to as the **law of large numbers,** which states that as the number of times an experiment is repeated increases, the ratio of the number of successful occurrences to the number of trials will tend to approach the theoretical probability of the outcome for an individual trial.

How Are Empirical and Theoretical Probabilities Related?

The law of large numbers is telling us that the larger the number of experimental trials, n, the closer the empirical probability, $P'(\text{A})$, is expected to be to the true or theoretical probability, $P(\text{A})$.

This concept has many applications. The preceding die-tossing experiment is an example in which we can easily compare actual results against what we expected to happen; it gave us a chance to verify the claim of the law of large numbers.

Sometimes we live with the results obtained from large sets of data when the theoretical expectation is unknown.

How Are Empirical and Theoretical Probabilities Related?

One such example occurs in the life insurance industry. The key to establishing proper life insurance rates is using the probability that those insured will live 1, 2, or 3 years, and so forth, from the time they purchase their policies.

These probabilities are derived from actual life and death statistics and hence are empirical probabilities.

They are published by the government and are extremely important to the life insurance industry.

How Are Empirical and Theoretical Probabilities Related?

Law of large number

As the number of times an experiment is repeated increases, the ratio of the number of successful occurrences to the number of trials will tend to approach the theoretical probability of the outcome for an individual trial.

Comparison of Probability and Statistics

Comparison of Probability and Statistics

Probability and **statistics** are two separate but related fields of mathematics. It has been said that “probability is the vehicle of statistics.” That is, if it were not for the laws of probability, the theory of statistics would not be possible.

Let’s illustrate the relationship and the difference between these two branches of mathematics by looking at two sets of poker chips.

On one hand, we know that the probability set contains twenty green, twenty red, and twenty blue poker chips.

Comparison of Probability and Statistics

Probability tries to answer questions such as, “If one chip is randomly drawn from this set, what is the chance that it will be blue?” On the other hand, in the statistics set, we don’t know what the combination of chips is.

We draw a sample and, based on the findings in the sample, make conjectures about what we believe to be in the set.

Note the difference: Probability asks you about the chance that something specific, such as drawing a blue chip, will happen when you know the possibilities (that is, you know the population).

Comparison of Probability and Statistics

Statistics, in contrast, asks you to draw a sample, describe the sample (descriptive statistics), and then make inferences about the population based on the information found in the sample (inferential statistics).

Bayes' Theorem

- **Bayes' theorem** describes the probability of occurrence of an event related to any condition.
- It is also considered for the case of conditional probability.
- Bayes theorem is also known as the formula for the probability of "causes".

Bayes' Theorem

- For example: if we have to calculate the probability of taking a blue ball from the second bag out of three different bags of balls, where each bag contains three different colour balls viz. red, blue, black.
- In this case, the probability of occurrence of an event is calculated depending on other conditions is known as conditional probability.

Bayes' Theorem

- The general statement of Bayes' theorem is "**The conditional probability of an event A, given the occurrence of another event B, is equal to the product of the event of B, given A and the probability of A divided by the probability of event B.**" i.e.

$$P(A|B) = P(B|A)P(A) / P(B)$$

where,

- **P(A)** and **P(B)** are the probabilities of events A and B
- **P(A|B)** is the probability of event A when event B happens
- **P(B|A)** is the probability of event B when A happens

Problem 1 : BT

- Three persons A, B and C have applied for a job in a private company. The chance of their selections is in the ratio 1 : 2 : 4. The probabilities that A, B and C can introduce changes to improve the profits of the company are 0.8, 0.5 and 0.3, respectively. If the change does not take place, find the probability that it is due to the appointment of C.

Solution 1: BT

Let E_1: person A get selected

E_2: person B get selected

E_3: person C get selected

A: Changes introduced but profit not happened

Now, $P(E_1) = 1/(1+2+4) = 1/7$

$P(E_2) = 2/7$ and $P(E_3) = 4/7$

$P(A|E_1)$ = P(Profit not happened by the changes introduces by A) = 1 – P(Profit happened by the changes introduces by A) = 1 – 0.8 = 0.2

$P(A|E_2)$ = P(Profit not happened by the changes introduces by B) = 1 – P(Profit happened by the changes introduces by B) = 1 – 0.5 = 0.5

$P(A|E_3)$ = P(Profit not happened by the changes introduces by C) = 1 – P(Profit happened by the changes introduces by C) = 1 – 0.3 = 0.7

Solution 1: BT

We have to find the probability of not happening profit due to selection of C

$$P(E_3|A) = \frac{P(A|E_3)P(E_3)}{P(A|E_1)P(E_1)+P(A|E_2)P(E_2)+P(A|E_3)P(E_3)}$$

$$P(E_3|A) = \frac{0.7\times\frac{4}{7}}{0.2\times\frac{1}{7}+0.5\times\frac{2}{7}+0.7\times\frac{4}{7}}$$

= 7/10.

∴ the required probability is 0.7.

Problem 2: BT

In a neighbourhood, 90% children were falling sick due flu and 10% due to measles and no other disease. The probability of observing rashes for measles is 0.95 and for flu is 0.08. If a child develops rashes, find the child's probability of having flu.

Solution 2: BT

F: children with flu,

M: children with measles,

R: children showing the symptom of rash

$P(F) = 90\% = 0.9$,

$P(M) = 10\% = 0.1$

$P(R|F) = 0.08$,

$P(R|M) = 0.95$

$$P(F|R) = \frac{P(R|F)P(F)}{P(R|M)P(M)+P(R|F)P(F)}$$

$$P(F|R) = \frac{0.08\times0.9}{0.95\times0.1+0.08\times0.9}$$

$$= 0.072/(0.095 + 0.072) = 0.072/0.167 \approx 0.43$$

$$\Rightarrow P(F|R) = 0.43$$