II. SETUP

In this section, we first review the EMS models that exhibit a tachyonic instability in KN black holes. We then present the numerical method used to construct scalarized KN black hole solutions within these models.

A. Einstein-Maxwell-scalar Models

In the EMS models, a scalar field is non-minimally coupled to electromagnetism, potentially inducing a tachyonic instability in KN black holes. The action for this system is

$$S=\frac{1}{16\pi}\int d^4x\sqrt{-g}\left[R-2\partial_\mu\phi\partial^\mu\phi-f\left(\phi\right)F^{\mu\nu}F_{\mu\nu}\right], \tag{1}$$

where R is the Ricci scalar, ϕ is the scalar field, and $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ denotes the electromagnetic field tensor. Varying the action (1) yields the equations of motion for the metric $g_{\mu\nu}$, scalar field ϕ and electromagnetic field A_μ:

$$\begin{aligned} R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}&=2T_{\mu\nu},\\ \Box\phi-\frac{1}{4}\frac{df\left(\phi\right)}{d\phi}F^{\mu\nu}F_{\mu\nu}&=0,\\ \partial_\mu\left[\sqrt{-g}f\left(\phi\right)F^{\mu\nu}\right]&=0, \end{aligned} \tag{2}$$

with the energy-momentum tensor

$$T_{\mu\nu}=\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}g_{\mu\nu}\left(\partial\phi\right)^2+f\left(\phi\right)\left(F_{\mu\rho}F_\nu{}^\rho-\frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right). \tag{3}$$

To allow for spontaneous scalarization of KN black holes, the KN black hole solution with $\phi=0$ must satisfy the equations of motion (2). This requirement imposes a condition on the coupling function $f\left(\phi\right)$, specifically $f'\left(0\right)=df\left(\phi\right)/d\phi|_{\phi=0}=0$. Without loss of generality, we set $f\left(0\right)=1$. Consequently, $f\left(\phi\right)$ can be expanded around $\phi=0$ as

$$f\left(\phi\right)=1+\alpha\phi^2+\mathcal{O}\left(\phi^3\right), \tag{4}$$

where α is a dimensionless coupling constant governing the scalar-electromagnetic interaction strength.

In the EMS models, KN black holes exhibit stability against metric and vector perturbations, similar to the Einstein-Maxwell theory [56]. However, they may develop a tachyonic instability when subjected to scalar perturbation $\delta\phi$, leading to the formation of scalarized black holes.

Linearizing the scalar field equation in the KN black hole background yields

$$\left(\Box-\mu_{\rm eff}^{2}\right)\delta\phi=0, \tag{5}$$

where the effective mass squared is given by $\mu_{\rm eff}^2=\alpha F^{\mu\nu}F_{\mu\nu}$. In the Boyer-Linquist coordinates, for a KN black hole with ADM mass M, angular momentum J and electric charge Q, the effective mass squared is given by

$$\mu_{\rm eff}^{2}=-\frac{\alpha Q^{2}\left(r^{4}-6a^{2}r^{2}\cos^{2}\theta+a^{4}\cos^{4}\theta\right)}{\left(r^{2}+a^{2}\cos^{2}\theta\right)^{4}}, \tag{6}$$

where $a=J/M$ is the ratio of angular momentum to mass. A tachyonic instability could arise when $\mu_{\rm eff}^2<0$, potentially driving the system away from KN black hole solutions. For $\alpha>0$, regions where $\mu_{\rm eff}^2<0$ consistently appear outside the event horizon in KN black holes, although these regions shrink as the black hole spin increases [57]. Conversely, when $\alpha<0$, the $\mu_{\rm eff}^2<0$ regions emerge only when the black hole spin is sufficiently large [55].

However, the condition $\mu_{\rm eff}^2<0$ is only a necessary condition for the appearance of tachyonic instability. To overcome dissipation through the event horizon and spatial infinity, $\mu_{\rm eff}^2$ must be sufficiently negative to induce this instability. In other words, only KN black holes for which $\mu_{\rm eff}^2$ falls below certain threshold values can develop a tachyonic instability. At these thresholds, the tachyonic instability triggers the formation of stationary scalar clouds—regular bound-state solutions to Eq. (5)—that exist outside KN black holes. These scalar clouds mark bifurcation points in the parameter space and signal the onset of scalarized KN black holes. The existence domains for scalar clouds at both fundamental and excited states have been identified for $\alpha>0$ [57] and $\alpha<0$ [55], respectively. It is important to note that the existence domains of these scalar clouds are independent of the specific form of the coupling function $f(\phi)$, provided it satisfies the series expansion given in Eq. (4).

B. Rotating Black Hole Solutions

To construct scalarized KN black hole solutions, we employ a generic ansatz for stationary, axisymmetric and asymptotically-flat black hole solutions [31, 39, 58, 59]:

$$\begin{aligned} ds^{2}&=-e^{2F_{0}}Ndt^{2}+e^{2F_{1}}\left(\frac{dr^{2}}{N}+r^{2}d\theta^{2}\right)+e^{2F_{2}}r^{2}\sin^{2}\theta\left(d\varphi^{2}-\frac{W}{r^{2}}dt\right)^{2},\\ A_{\mu}dx^{\mu}&=\left(A_{t}-A_{\varphi}\frac{W}{r^{2}}\sin\theta\right)dt+A_{\varphi}\sin\theta d\varphi\text{ and }\phi=\phi\left(r,\theta\right). \end{aligned} \tag{7}$$

Here $N \equiv 1 - r_H/r$, where r_H is the black hole horizon radius. The seven functions F_0, F_1, F_2, W, A_t, A_φ and ϕ depend only on the coordinates r and θ.

In the stationary spacetime, two Killing vectors ∂_t and ∂_φ are present. Their linear combination $\xi = \partial_t + \Omega_H \partial\varphi$, where Ω_H is the angular velocity of the black hole horizon, is null and orthogonal to the horizon. The surface gravity κ is then defined by $\kappa^2 = -\left(\nabla_\mu \xi_\nu\right)\left(\nabla^\mu \xi^\nu\right)/2$ and related to the Hawking temperature T_H as [58]

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{4\pi r_H} e^{F_0(r_H,\theta) - F_1(r_H,\theta)}. \tag{8}$$

In the EMS models, the black hole entropy is expressed as $S = A_H/4$, where the area of the horizon A_H is given by

$$A_H = 2\pi r_H^2 \int_0^\pi d\theta \sin\theta e^{F_1(r_H,\theta) + F_2(r_H,\theta)}. \tag{9}$$

Various physical quantities, including the black hole mass M, charge Q, angular momentum J, electrostatic potential Φ and horizon angular velocity Ω_H, can be extracted by analyzing the asymptotic behavior of the metric and gauge field functions near the horizon and at spatial infinity [58, 59]:

$$\begin{aligned} &A_t|_{r=r_H} \sim 0, \quad W|_{r=r_H} \sim r_H^2 \Omega_H, \\ &A_t|_{r=\infty} \sim \Phi - \frac{Q}{r}, \quad W|_{r=\infty} \sim \frac{2J}{r}, \quad e^{2F_0} N\big|_{r=\infty} \sim 1 - \frac{2M}{r}. \end{aligned} \tag{10}$$

These quantities further satisfy the Smarr relation [38, 58, 60]:

$$M = 2T_H S + 2\Omega_H J + \Phi Q, \tag{11}$$

which allows us to assess the accuracy of our numerical solutions.

To obtain scalarized black hole solutions, we numerically solve the coupled partial differential equations derived by substituting the ansatz in Eq. (7) into the equations of motion (2). For numerical implementation, we compactify the radial coordinate r via the transformation

$$x = \frac{\sqrt{r^2 - r_H^2} - r_H}{\sqrt{r^2 - r_H^2} + r_H}, \tag{12}$$

which maps the event horizon $r = r_H$ and spatial infinity $r = \infty$ to $x = -1$ and $x = 1$. Using this compactified coordinate x, the power series expansions near the horizon yield the following boundary conditions at $x = -1$:

$$\partial_x F_0 = \partial_x F_1 = \partial_x F_2 = \partial_x \phi = \partial_x A_\varphi = A_t = W - \Omega_H = 0. \tag{13}$$