Pareto Optimal Solution: Let $x^* \in X$,If there is not $x \neq x^* \in X$, s.t. $f_i(x) \leq f_i(x^*), (i = 1, 2, \cdots m)$ and at least one $i_0, s.t. f_{i_0}(x) < f_{i_0}(x^*)$, it is called a Par eto optimal solution of the problem (CMOP).

Pareto Front: Let X^* be the full optimal solution of the problem, then $F(X^*) = \{F(x) \mid x \in X^*\}$ is called as the Pareto Front of the multi-objective optimization problem.

KKT condition: If $x^* \in X \subset R^n$, $\lambda_i^* > 0, i \in I$, $w_i^* \geq 0$ and at least one $w_i^* > 0$, and the following conditions:

$$\sum_{i=1}^{m} w_i^* \nabla g_i(x^*) + \sum_{i=1}^{a} \lambda_i^* \nabla g_i(x^*) = 0$$
$$g_i(x^*) \leq 0, \lambda_i^* > 0, \forall i \in I$$
$$\lambda_i^* g_i(x^*) = 0, \forall i \in I$$

$x^* \in X \subset R^n$ is called the KKT point of the original multi-objective optimization problem (CMOP).

Pareto critical point:

If x^* is the KKT point of the problem:

$$\min \sum_{i=1}^{m} w_i f_i(x)$$
$$s.t. g_i(x) \leq 0 \ , i \in I$$

The KKT point of, where $w_i \geq 0$, and at least one $w_i > 0$. Then x^* is called the Pareto critical point of the original multi-objective constraint optimization problem (CMOP).

1.3 MOSQP Algorithm

In 2015, Fliege and Vaz [1] using the idea of single target SQP method, developed the SQP method to solve the constraint multi-objective optimization problem, the algorithm is divided into two stages, by solving the secondary planning subproblem obtain search direction, it uses the L1 accurate penalty function as a value function line search step, to obtain approximate Pareto frontier with constraint multi-objective optimization problem. The algorithmic framework is provided as follows:

Algorithm 2:

Step1 Initialize the iteration point set $X_0 = \{x_1, x_2, \cdots x_N\}, x_i \in R^n$;

Step2 Spread out K times for each iteration point in $X_j = \{x_1^j, x_2^j, \cdots x_N^j\}, x_i^j \in R^n$.

Let $T=\varnothing$,

Consider each objective function to construct the corresponding single-objective optimization problem:

$$\begin{aligned}&(\mathrm{P}_i)\quad \min\ f_i(x), i\in M=\{1,\cdots,m\}\\ &s.t.\qquad g_i(x)\le 0, i\in I=\{1,2,\cdots a\}\end{aligned}$$

Solving for the quadratic programming sub-problem

$$\begin{aligned}\mathbf{QP}(x^j,B^j)\quad &\min \frac{1}{2}(d^j)^T B^j d^j+\nabla f_i(x^j)^T d^j\\ &s.t.\ \nabla g_i(x^j)^T d+g_i(x^j)\le 0 \quad i\in I\end{aligned}$$

And a linear search by the Armijo step rule：

$P(x^j+\alpha^j d^j,\pi)\le P(x^j,\pi)+\sigma\alpha^j\nabla P(x^j,\pi)d^j$, where $\sigma\in(0,1)$ 。

Let $T=T\bigcup\{x^j+\alpha^j d^j\}$

Let $X^{j+1}=\text{non-dominate}\{X^j\bigcup T\}, j=j+1$

If $j=K$, stop , let $X=X^K$.

Step3 Iteratively optimization for each point set in $X=\{x_1^0,x_2^0,\cdots,x_{M^*}^0\}$.

Let $S=\varnothing$：

Obtaining the iteration direction by solving a quadratic programming subproblem with information about all the objective functions：

$$\begin{aligned}\mathbf{QP}(x^j,B^j)\quad &\min \frac{1}{2}(d^j)^T B^j d^j+\sum_{i=1}^{m}\nabla f_i(x^j)^T d^j\\ &s.t.\ \nabla g_i(x^j)^T d^j+g_i(x^j)\le 0 \quad i\in I\end{aligned}$$

And a linear search by the Armijo step rule：

$P(x^j+\alpha^j d^j,\pi)\le P(x^j,\pi)+\sigma\alpha^j\nabla P(x^j,\pi)d^j$, where $\sigma\in(0,1)$ 。

Let $x^{j+1}=x^j+\alpha^j d^j$, $j=j+1$, if $\|d^j\|=0$, Stop。

$S=S\bigcup x_l^j, l=\{1,2,\cdots,M^*\}$, Iterate over the next iteration point.

The non-dominant point in S is retained as the final Pareto optimal solution.

In Spread stage, this paper proposes a feasible spread mode to ensure that the diffusion stage is only carried out in the feasible domain, so that the initial iteration point set of the Pareto optimization solution stage is all in the feasible domain, and we cancel the non-dominant solution at each diffusion, which is because some non-dominant solutions still retain the possibility of declining towards a certain function.

2 A smooth approximation of the low-order penalty functions

2.1 Smooth function

Consider the L1-type penalty function

$$P(x,\pi)=f(x)+\pi\sum_{i=1}^{l}g_i^+(x)$$

Introducing a function $h:R\to R$

$$h(t)=\begin{cases}0 & t<0\\ t & t\geq 0\end{cases}$$

Then

$$g_i^+(x)=h(g_i(x))$$

According to the function $h(t)$, then

$$h^k(t)=\begin{cases}0 & t<0\\ t^k & t\geq 0\end{cases}$$

When $k\in(0,1]$, it cannot be able Thus increasing the difficulty for considering a low-order penalty function as a value function. Consider the following smoothness approximation function:

$$h_\varepsilon^k(t)=\begin{cases}0 & t<-\varepsilon\\ \varepsilon^{k(1-b)}b^{-k}(t+\varepsilon)^{kb} & -\varepsilon\leq t<0\\ (t+\varepsilon b^{-1})^k & t\geq 0\end{cases}$$

Where $\varepsilon>0$, $b>0$.

2.2 Properties of the smooth functions

Theorem 2.2.1 For any $0<k<+\infty$, $\varepsilon>0$, $b>0$, the following properties to hold true:

(1) $h_\varepsilon^k(t)$ is continuous in R;

(2) $\forall t\in R, h_\varepsilon^m(t)\geq h^m(t)$;

(3) $\forall t\in R, \lim_{\varepsilon\to 0}h_\varepsilon^m(t)=h^m(t)$;

(4)If $0<\varepsilon_1<\varepsilon_2$, then $\forall t\in R, h_{\varepsilon_1}^k(t)\leq h_{\varepsilon_2}^k(t)$。

Proof: