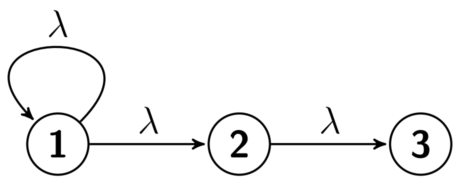


$$
\begin{aligned}
\dot{z}_1 &= (\mu + i\omega)\, z_1 - |z_1|^2\, z_1 - \lambda z_1 \\
\dot{z}_2 &= (\mu + i\omega)\, z_2 - |z_2|^2\, z_2 - \lambda z_1 \\
\dot{z}_3 &= (\mu + i\omega)\, z_3 - |z_3|^2\, z_3 - \lambda z_2 .
\end{aligned} \tag{1}
$$

Figure 1: Representative example of a three-cell feedforward network. Arrows indicate coupling, with coupling strength λ. Each cell represents a dynamical system assumed to be operating near a Hopf bifurcation.

The authors in [24, 25, 26] have found that the coupling causes the amplitudes of oscillations that arise from the onset of the Hopf bifurcation to grow at a larger rate. If μ is the bifurcation parameter, and $\mu = 0$ is the onset of a supercritical Hopf bifurcation, then the third cell undergoes oscillations of amplitude approximately equal to $\mu^{1/6}$, rather than the *expected* amplitude of $\mu^{1/2}$. This phenomenon showcases an accelerated growth rate that has the potential for the design and fabrication of advanced filters in signal processing [31, 32]). An example of a time series, obtained from simulations of Eq. (1), which exhibits this growth phenomenon, is shown in Fig. 2(a). Without self-coupling on the first cell, see Fig. 2(b), the amplification effect is even larger.

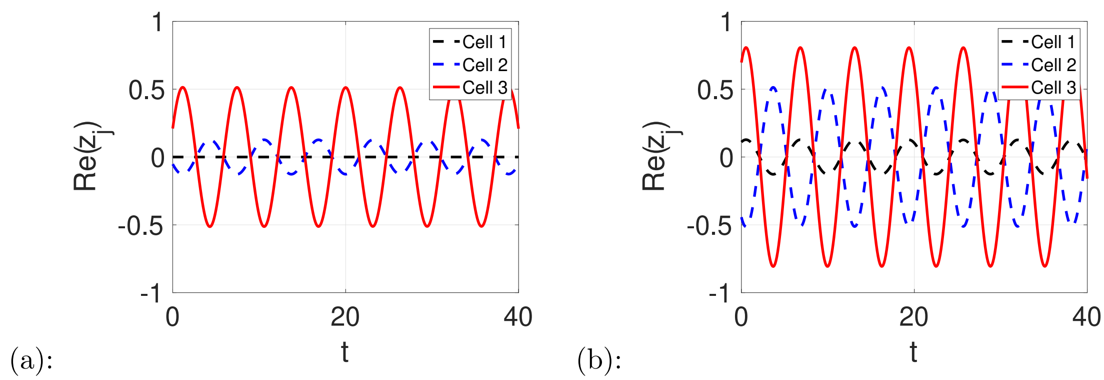


Figure 2: Signal amplification in a feedforward network (a) with self-coupling of the first cell and (b) without self-coupling. Parameters are: $\mu = (0.5)^6$, $\omega = 1$, $\lambda = 1$. (a): With self-coupling. (b): Without self-coupling.

As the feedforward network grows in size, the growth rate of oscillations in the final cell is determined by taking successive cube roots [31]. Thus, in a five-cell In a feedforward network, the growth rate should be proportional to the 54^{th} root of the bifurcation parameter, as shown in [33]. Achieving such large growth rates is exciting indeed, as it can lead to novel mechanisms for signal amplification beyond the simple square-root growth rate. This, of course, means that large amplitude oscillations may arise very soon after the onset of the Hopf bifurcation.

The phenomenon of such large-amplitude oscillations in the third cell can be understood as a type of nonlinear resonance as well as being the result of the combination of the unidirectional coupling and the higher-degree nonlinearities [30]. Related articles proving that anomalous growth rates can occur for equilibria in (unusual) regular networks and for bifurcations at simple eigenvalues can be found in [34, 35].

2.2 Beam Steering

Beam steering involves strategies to manipulate and control the direction of a radiating far-field intensity pattern [11, 12, 13]. In antennas and radar systems, beam steering can be achieved either by switching the antenna elements or by controlling the phase differences between oscillating components, which typically consist of arrays of nonlinear oscillators. Overall, modern methods for beam steering exploit the inherent nonlinearities of individual components and the collective dynamics of oscillator arrays to manipulate phase shifts. None of those methods includes, however, signal amplification.

Recently, the idea of employing a feedforward network, which enables simultaneous beam steering and signal amplification, was introduced in [27, 29]. These works showed that the branches of oscillations in the feedforward network can, under certain conditions, exhibit stable phase-locking patterns, in which the phase difference between consecutive oscillators is constant. Those phase differences are critical to steering a beam towards a desired location. The beam steering can be accomplished as follows.

The total radiation pattern of an antenna array at a point P is given by the equation

$$E(P) = A(\Psi)\, E_0 e^{ikr_0}, \tag{2}$$

where $E_0 e^{ikr_0}$ is the electric field produced by a single patch element, $\Psi = kd \sin\varphi$, in which k is a free-space wave vector, d is the distance between consecutive array elements, and φ is the angle of emission of the beam. The term $A(\Psi)$ is an *Array Factor* defined by

$$A(\Psi) = \frac{\sin\left(\frac{N\Psi}{2}\right)}{N\sin\left(\frac{\Psi}{2}\right)} e^{i(N-1)\Psi/2}. \tag{3}$$

The absolute value of the Array Factor, $|A(\Psi)|$, is symmetric with respect to $\Psi = 0$, and it always attains a maximum at $\Psi = 0$, which corresponds to an angle of emission of $\varphi = 0$. This angle is also known as the *broadside direction*, as it is normal to the plane of the array. A critical observation is that varying Ψ changes the direction of the radiating beam. Thus, the problem of beam steering translates into finding strategies to vary Ψ. One possibility is to employ a feedforward network and tune it to a phase-locking regime. If θ is the constant phase-locking angle among consecutive oscillators, then it has been shown [27, 29] that this angle enters into the array factor as

$$A(\Psi) = \frac{\sin\left(\frac{N(\Psi+\theta)}{2}\right)}{N\sin\left(\frac{\Psi+\theta}{2}\right)} e^{i(N-1)(\Psi+\theta)/2}. \tag{4}$$

By varying the phase-locking angle, we can control the beam direction, thereby steering the beam pattern from angle Ψ to $\Psi + \theta$. Figure 3 illustrates the strategy of beam steering through varying the phase-locking angle, θ, in a feedforward array with 20 nonlinear oscillators.

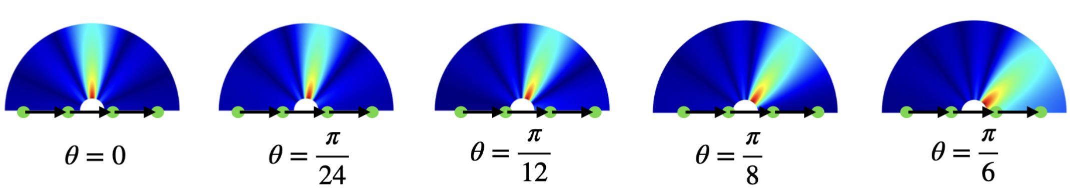


Figure 3: Beam steering can be achieved in a feedforward network by varying the phase-locking angle θ. The sequence of snapshots shows the radiating pattern with 20 nonlinear oscillators.

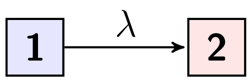
Together, the larger growth rate of oscillation amplitudes and phase-locking characteristics enable innovative solutions and improvements (e.g., simultaneous signal amplification and beam steering) over conventional methods.

3 A two-cell feedforward network of cells with pitchfork bifurcations

In this section, we study the effects of inhomogeneity in the excitation parameter of a feedforward network of cells, each of which admits a supercritical pitchfork bifurcation. We examine a two-cell system and conduct a thorough study of its equilibria via an asymptotic analysis of cubic roots, a numerical investigation, and a singularity analysis.

3.1 A two-cell system: the setup

We consider a simple two-cell system exhibiting steady-state bifurcations:



$$\begin{aligned} \dot{x} &= \mu x - x^3, \\ \dot{y} &= (\mu + \varepsilon) y - y^3 - \lambda x. \end{aligned} \tag{5}$$

Figure 4: A schematic of a two-cell feed-forward network with inhomogeneous cells. Each cell represents a system prone to a pitchfork bifurcation.

The parameter μ represents the input signal: $\mu = 0$ corresponds to the absence of an input signal, while $\mu > 0$ means that there is a signal that we want to detect. The parameter ε represents the inhomogeneity of the cells. It may be positive or negative, due to manufacturing imperfections or intentional design. The parameter λ is the coupling strength.

Varying the parameter μ unfolds an ensemble of qualitatively different bifurcation diagrams depending on the relationship between μ and ε, demonstrating an unexpectedly complex set of stable and unstable equilibria – see Fig. 5.

3.2 Structure and stability of equilibria

We start analyzing system (5) by finding its equilibria and establishing their stability at various combinations of μ and ε. The first cell, whose state variable is x, is a closed system. At $\mu < 0$, its only equilibrium is $x = 0$, while at $\mu > 0$ it has three equilibria: 0 and $\pm\sqrt{\mu}$. Throughout this and the next sections, we assume that $\lambda > 0$, because $\lambda < 0$ is equivalent to switching the sign of x.

The second cell, with state variable y, is influenced by the first cell. If $x = 0$, its equilibria are the roots of the cubic polynomial

$$p_0(y) := (\mu + \varepsilon) y - y^3. \tag{6}$$

If $\mu + \varepsilon < 0$, $p_0(y)$ has a unique root at $y = 0$. If $\mu + \varepsilon > 0$, it has three roots, 0 and $\pm\sqrt{\mu + \varepsilon}$.

If $\mu > 0$ and the first cell is at a nonzero equilibrium, i.e., $x = \pm\sqrt{\mu}$, the equilibria of the second cell are the roots of the cubic polynomials

$$p_+(y) := (\mu + \varepsilon) y - y^3 - \lambda\sqrt{\mu} \quad \text{and} \quad p_-(y) := (\mu + \varepsilon) y - y^3 + \lambda\sqrt{\mu}, \tag{7}$$

respectively. The critical relationship between μ, ε, and λ, at which the polynomials $p_\pm(y)$ have exactly two roots is (see Eq. (A-3)):

$$2(\mu + \varepsilon)^{3/2} = 3\sqrt{3}\lambda\mu^{1/2}. \tag{8}$$