Theorem 4.6: **(Frequency synchronization I)** Consider the coupled oscillator model (1) with a connected graph $G(\mathcal{V}, \mathcal{E}, A)$ and $\omega \in \mathbf{1}_n^{\perp}$. Assume that the algebraic connectivity is larger than a critical value, that is,

$$\lambda_2(L) > \lambda_{\text{critical}} \triangleq \left\| B_c^T \omega \right\|_2 , \tag{31}$$

where $B_c \in \mathbb{R}^{n \times n(n-1)/2}$ is the incidence matrix of the complete graph. Accordingly, define $\gamma_{\max} \in \,]\pi/2, \pi]$ and $\gamma_{\min} \in [0, \pi/2[$ as unique solutions to $(\pi/2) \cdot \operatorname{sinc}(\gamma_{\max}) = \sin(\gamma_{\min}) = \lambda_{\text{critical}}/\lambda_2(L)$. The following statements hold:

1) **phase cohesiveness:** the set $\left\{\theta \in \operatorname{Arc}_n(\pi) : \|B_c^T\theta\|_2 \leq \gamma\right\} \subseteq \bar{\Delta}_G(\gamma)$ is positively invariant for every $\gamma \in [\gamma_{\min}, \gamma_{\max}]$, and each trajectory starting in the set $\left\{\theta \in \operatorname{Arc}_n(\pi) : \left\|B_c^T\theta\right\|_2 < \gamma_{\max}\right\}$ asymptotically reaches the set $\left\{\theta \in \operatorname{Arc}_n(\pi) : \|B_c^T\theta\|_2 \leq \gamma_{\min}\right\}$; and
2) **frequency synchronization:** for every $\theta(0) \in \operatorname{Arc}_n(\pi)$ with $\left\|B_c^T\theta(0)\right\|_2 < \gamma_{\max}$ the frequencies $\dot{\theta}_i(t)$ synchronize exponentially to the average frequency $\omega_{\text{sync}} = \frac{1}{n}\sum_{i=1}^{n}\omega_i$, and, given phase cohesiveness in $\bar{\Delta}_G(\gamma)$ for some fixed $\gamma < \pi/2$, the exponential synchronization rate is no worse than $\lambda_{\text{fe}} = -\lambda_2(L)\cos(\gamma)$.

The proof of Theorem 4.6 follows a similar ultimate-boundedness strategy as the proof of Theorem 4.5 by using the Lyapunov function (30). It can be found in Appendix B.

For classic Kuramoto oscillators (21), condition (31) reduces to $K > \left\|B_c^T\omega\right\|_2$. Clearly, the condition $K > \left\|B_c^T\omega\right\|_2$ is more conservative than the bound (26) which reads as $K > \|B_c^T\omega\|_\infty = \omega_{\max} - \omega_{\min}$. One reason for this conservatism is that the analysis leading to condition (31) requires *all* phase distances $|\theta_i - \theta_j|$ to be bounded, whereas according to Lemma 4.2 only *pairwise* phase distances $|\theta_i - \theta_j|$, $\{i, j\} \in \mathcal{E}$, need to be bounded for stable synchronization. The following result exploits these weaker assumptions and states a sharper (but only local) synchronization condition.

Theorem 4.7: **(Frequency synchronization II)** Consider the coupled oscillator model (1) with a connected graph $G(\mathcal{V}, \mathcal{E}, A)$ and $\omega \in \mathbf{1}_n^{\perp}$. There exists a locally exponentially stable equilibrium manifold $[\theta] \in \Delta_G(\pi/2)$ if

$$\lambda_2(L) > \left\|B^T\omega\right\|_2 . \tag{32}$$

Moreover, if condition (32) holds, then $[\theta]$ is phase cohesive in $\{\theta \in \mathbb{T}^n : \|B^T\theta\|_2 \leq \gamma_{\min}\} \subseteq \bar{\Delta}_G(\gamma_{\min})$, where $\gamma_{\min} \in [0, \pi/2[$ satisfies $\sin(\gamma_{\min}) = \|B^T\omega\|_2/\lambda_2(L)$.

The strategy to prove Theorem 4.7 is inspired by the ingenuous analysis in [52, Section IIV.B]. It relies on the insight gained from Lemma 4.2 that any synchronization manifold $[\theta] \in \Delta_G(\pi/2)$ is locally stable, and it formulates the existence of such a synchronization manifold as a fixed point problem. Here, we follow the basic proof strategy in [52], but we provide a more accurate result together with a self-contained proof which is reported in Appendix C.

V. Conclusions and Open Research Directions

In this paper we introduced the reader to the coupled oscillator model (1), we reviewed several applications, we discussed different synchronization notions, and we presented different analysis approaches to phase synchronization, phase balancing, and frequency synchronization.

Despite the vast literature, the countless applications, and the numerous theoretic results on the synchronization properties of model (1), many interesting and important problems are still open. In the following, we summarize limitations of the existing analysis approaches and present a few worthwhile directions for future research.

First, in many applications the coupling between the oscillators is not purely sinusoidal. For instance, phase delays in neuroscience [13], time delays in sensor networks [37], or transfer conductances in power networks [63] lead to a "shifted coupling" of the form $\sin(\theta_i - \theta_j - \varphi_{ij})$, where $\varphi_{ij} \in [-\pi/2, \pi/2]$. In this case and also for other "skewed" or "symmetry-breaking" coupling functions, many of the presented analysis schemes either fail or lead to overly conservative results. Another interesting class of oscillator networks are systems of pulse-coupled oscillators featuring hybrid dynamics: impulsive coupling at discrete time instants and uncoupled continuous dynamics otherwise. This class of oscillator networks displays a very interesting phenomenology. For instance, the behavior of identical oscillators coupled in a complete graph strongly depends on the curvature of the uncoupled dynamics [142]. Most of the results known for continuously-coupled oscillators still need to be extended to pulse-coupled oscillators with hybrid dynamics.

Second, in many applications [12], [24], [34], [63], [67] the coupled oscillator dynamics are not given by a simple first-order phase model of the form (1). Rather, the dynamics are of higher order, or sometimes there is no readily available phase variable to describe the limit cycle attracting the coupled dynamics. The analysis of oscillator networks with more general oscillator dynamics is largely unexplored. Whereas advances have been made for the simple case of phase synchronization of linear or passive oscillator networks, the case of frequency synchronization of non-identical oscillators with higher-order dynamics is not well-studied.

Third, despite the vast scientific interest the quest for sharp, concise, and closed-form synchronization conditions for arbitrary complex graphs has been so far in vain [7], [8], [44]–[46], [74]. As suggested by Lemma 4.1, Lemma 4.2, Theorem 4.5, and the proof of Theorem 4.7, the proper metric for the synchronization problem is the incremental ∞-norm $\|B^T\theta\|_\infty = \max_{\{i,j\}\in\mathcal{E}} |\theta_i - \theta_j|$. In the authors' opinion, a Banach space analysis of the coupled oscillator model (1) with the incremental ∞-norm will most likely deliver the sharpest possible conditions. However, such an analysis is very challenging for arbitrary natural frequencies $\omega \in \mathbf{1}_n^{\perp}$ and connected and weighted coupling graphs $G(\mathcal{V}, \mathcal{E}, A)$. Recent work [114] by the authors puts forth a novel algebraic condition for synchronization with a rigorous analysis for specific classes of graphs and with (only) a statistical validation for generic weighted graphs.

Fourth and finally, a few interesting and open theoretical challenges include the following. First, most of the presented analysis approaches and conditions do not extend to time-varying or directed coupling graphs $G(\mathcal{V}, \mathcal{E}, A)$, and alternative methods need to be developed. Second, most known estimates on the region of attraction of a synchronized solution are conservative. The semi-circle estimates given in Theorem 4.3 and Theorem 4.5 rely on convexity of $\mathrm{Arc}_n(\pi)$ and are overly conservative. We refer to [63], [112] for a set of interesting results and conjectures on the region of attraction. Third, the presented analysis approaches are restricted to synchronized equilibria inside the set $\Delta_G(\pi/2)$. Other interesting equilibrium configurations outside $\Delta_G(\pi/2)$ include splay state equilibria or frequency-synchronized equilibria with phases spread over an entire semi-circle.

We sincerely hope that this tutorial article stimulates further exciting research on synchronization in coupled oscillators, both on the theoretical side as well as in the countless applications.

APPENDIX

A. Modeling of the spring-interconnected particles

Consider the spring network in Figure 1 consisting of a group of particles constrained to rotate around a circle of unit radius. For simplicity, we assume that the particles are allowed to move freely on the circle and exchange their order without collisions. Each particle is characterized by its phase angle $\theta_i \in \mathbb{S}^1$ and frequency $\dot{\theta}_i \in \mathbb{R}$, and its inertial and damping coefficients are $M_i > 0$ and $D_i > 0$.

The external forces and torques acting on each particle are (i) a viscous damping force $D_i\dot{\theta}_i$ opposing the direction of motion, (ii) a non-conservative force $\omega_i \in \mathbb{R}$ along the direction of motion depicting a preferred natural rotation frequency, and (iii) an elastic restoring torque between interacting particles i and j coupled by an ideal elastic spring with stiffness $a_{ij} > 0$ and zero rest length.

To compute the elastic torque between the particles, we parametrize the position of each particle i by the unit vector $p_i = [\cos(\theta_i)\,,\,\sin(\theta_i)]^T \in \mathbb{S}^1 \subset \mathbb{R}^2$. The elastic Hookean energy stored in the springs is the function $E : \mathbb{T}^n \to \mathbb{R}$ given up to an additive constant by

$$\begin{aligned}E(\theta) &= \sum\nolimits_{\{i,j\}\in\mathcal{E}} \frac{a_{ij}}{2}\|p_i - p_j\|_2^2 \\ &= \sum\nolimits_{\{i,j\}\in\mathcal{E}} a_{ij}\big(1 - \cos(\theta_i)\cos(\theta_j) - \sin(\theta_i)\sin(\theta_j)\big) \\ &= \sum\nolimits_{\{i,j\}\in\mathcal{E}} a_{ij}\big(1 - \cos(\theta_i - \theta_j)\big)\,,\end{aligned}$$

where we employed the trigonometric identity $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ in the last equality. Hence, we obtain the restoring torque acting on particle i as

$$T_i(\theta) = -\frac{\partial}{\partial\theta_i}E(\theta) = -\sum\nolimits_{\{i,j\}\in\mathcal{E}} a_{ij}\sin(\theta_i - \theta_j)\,.$$

Therefore, the network of spring-interconnected particles depicted in Figure 1 obeys the dynamics

$$M_i\ddot{\theta}_i + D_i\dot{\theta}_i = \omega_i - \sum\nolimits_{\{i,j\}\in\mathcal{E}} a_{ij}\sin(\theta_i - \theta_j)\,, \quad i \in \{1,\dots,n\}\,. \tag{33}$$

The coupled oscillator model (1) is then obtained as the kinematic variant or the overdamped limit of the spring network (33) with zero inertia $M_i = 0$ and unit damping $D_i = 1$ for all oscillators $i \in \{1,\dots,n\}$.

B. Proof of Theorem 4.6

Assume that $\theta(0) \in \overline{\mathrm{Arc}}_n(\rho)$ for $\rho \in [0,\pi[$. Recall that the angular differences are well defined for θ in the open semi-circle $\mathrm{Arc}_n(\pi)$, and define the vector of phase differences $\delta \triangleq B_c^T\theta = (\theta_2 - \theta_1, \dots) \in [-\pi,+\pi]^{n(n-1)/2}$. By taking the derivative $d/dt\,\delta(t)$ the phase differences satisfy

$$\begin{aligned}\dot{\delta} &= B_c^T\omega - B_c^T B\,\mathrm{diag}(\{a_{ij}\}_{\{i,j\}\in\mathcal{E}})\,\mathbf{sin}(B^T\theta) \\ &= B_c^T\omega - B_c^T B_c\,\mathrm{diag}(\{a_{ij}\}_{i,j\in\{1,\dots,n\},i<j})\,\mathbf{sin}(\delta),\end{aligned} \tag{34}$$

where $\mathbf{sin}(x) = (\sin(x_1),\dots,\sin(x_n))$ for a vector $x \in \mathbb{R}^n$. Notice that for $\theta(0) \in \mathrm{Arc}_n(\pi)$ the δ-dynamics (34) are well-defined for an open interval of time. In the following, we will show that the set $\{\delta \in \mathbb{R}^n : \|\delta\|_2 < \gamma_{\max}\}$ is positively invariant under condition (31). As a consequence, the set $\{\delta \in \mathbb{R}^n : \|\delta\|_\infty < \gamma_{\max} \le \pi\}$ is positively invariant as well, and the δ-coordinates are well defined for all $t \ge 0$.

The Lyapunov function (30) reads in δ-coordinates as $W(\delta) = \frac{1}{2}\|\delta\|^2$, and its derivative along trajectories of (34) is

$$\begin{aligned}\dot{W}(\delta) &= \delta^T B_c^T\omega - \delta^T B_c^T B_c\,\mathrm{diag}(\{a_{ij}\}_{i<j})\,\mathbf{sin}(\delta) \\ &= \delta^T B_c^T\omega - n\,\delta^T\,\mathrm{diag}(\{a_{ij}\}_{i<j})\,\mathbf{sin}(\delta)\,,\end{aligned} \tag{35}$$

where the second equality follows from the identity

$$\delta^T B_c^T B_c = \theta^T B_c B_c^T B_c = \theta^T(nI_n - \mathbf{1}_{n\times n})B_c = n\theta^T B_c = n\delta.$$

For $\|\delta_2\| \le \rho$, $\rho \in [0,\pi[$, consider the following inequalities

$$\begin{aligned}&n\,\delta^T\,\mathrm{diag}(\{a_{ij}\}_{i<j})\,\mathbf{sin}(\delta) \\ &\quad= n\,(B_c^T\theta)^T\,\mathrm{diag}(\{a_{ij}\,\mathrm{sinc}(\theta_i - \theta_j)\}_{i<j})(B_c^T\theta) \\ &\quad\ge n\,\mathrm{sinc}(\rho)\,(B_c^T\theta)^T\,\mathrm{diag}(\{a_{ij}\}_{i<j})(B_c^T\theta) \\ &\quad\ge \lambda_2(L)\,\mathrm{sinc}(\rho)\|B_c^T\theta\|_2^2 = \lambda_2(L)\,\mathrm{sinc}(\rho)\|\delta\|_2^2\,,\end{aligned}$$

where the last inequality follows from [64, Lemma 4.7]. Hence, the derivative (35) simplifies further to

$$\dot{W}(\delta) \le \delta^T B_c^T\omega - \lambda_2(L)\,\mathrm{sinc}(\rho)\|\delta\|_2^2. \tag{36}$$

In the following we regard $B_c^T\omega$ as external disturbance affecting the otherwise stable δ-dynamics (34) and apply ultimate boundedness arguments [139]. Note that the right-hand side of (36) is strictly negative for

$$\|\delta\|_2 > \mu_c \triangleq \frac{\|B_c^T\omega\|_2}{\lambda_2(L)\,\mathrm{sinc}(\rho)} = \frac{\lambda_{\text{critical}}}{\lambda_2(L)\,\mathrm{sinc}(\rho)}\,.$$

Pick $\epsilon \in]0,1[$. If $\rho \ge \|\delta\|_2 \ge \mu_c/\epsilon$, then the right-hand side of (36) is upper-bounded by

$$\dot{W}(\delta) \le -\big(1-\epsilon\big)\cdot\lambda_2(L)\,\mathrm{sinc}(\rho)W(\delta)\,.$$

In the following, choose μ such that $\rho > \mu > \mu_c$ and let $\epsilon = \mu_c/\mu \in]0,1[$. By standard ultimate boundedness arguments [139, Theorem 4.18], for $\|\delta(0)\|_2 \le \rho$, there is $T \ge 0$ such that $\|\delta(t)\|_2$ is exponentially decaying for $t \in [0,T]$ and $\|\delta(t)\|_2 \le \mu$ for all $t \ge T$. For the choice $\mu = \gamma$ with $\gamma \in [0,\pi/2[$, the condition $\mu > \mu_c$ reduces to

$$\gamma\,\mathrm{sinc}(\rho) > \lambda_{\text{critical}}/\lambda_2(L)\,. \tag{37}$$

Now, we perform a final analysis of the bound (37). The left-hand side of (37) is an increasing function of γ and a decreasing function of ρ. Therefore, there exists some (ρ, γ) in the convex set $\Lambda \triangleq \{(\rho,\gamma) : \rho \in [0,\pi[\,,\ \gamma \in [0,\pi/2[\,,\ \rho > \gamma\}$ satisfying equation (37) if and only if the inequality (37) is true at $\rho = \gamma = \pi/2$, where the left-hand side of (37) achieves its supremum in Λ. The latter condition is equivalent to inequality (31). Additionally, if these two equivalent statements are true, then there is an open set of points in Λ satisfying (37), which is bounded by the unique curve that satisfies inequality (37) with the equality sign, namely $f(\rho,\gamma) = 0$, where $f : \Lambda \to \mathbb{R}$, $f(\rho,\gamma) = \gamma\,\mathrm{sinc}(\rho) - \lambda_{\text{critical}}/\lambda_2(L)$. Consequently, for every $(\rho,\gamma) \in \{(\rho,\gamma) \in \Lambda : f(\rho,\gamma) > 0\}$, it follows for $\|\delta(0)\|_2 \leq \rho$ that there is $T \geq 0$ such that $\|\delta(t)\|_2 \leq \gamma$ for all $t \geq T$. The supremum value for ρ is given by $\rho_{\max} \in \,]\pi/2, \pi]$ solving the equation $f(\rho_{\max}, \pi/2) = 0$ and the infimum value of γ by $\gamma_{\min} \in [0, \pi/2[$ solving the equation $f(\gamma_{\min}, \gamma_{\min}) = 0$.

This proves statement 1) (where we replaced $\rho_{\max}$ by $\gamma_{\max}$) and shows that there is $T \geq 0$ such that $\|B_c^T\theta(t)\|_\infty \leq \|B_c^T\theta(t)\|_2 \leq \gamma_{\min} < \pi/2$ for all $t \geq T$. Thus, $\theta(t) \in \bar{\Delta}_G(\gamma_{\min})$ for $t \geq T$, and frequency synchronization can be established analogously to the proof of Theorem 4.5.

C. Proof of Theorem 4.7

According to Lemma 4.2, there exists a locally exponentially stable synchronization manifold $[\theta] \in \bar{\Delta}_G(\gamma)$, $\gamma \in [0, \pi/2[$, if and only if there is an equilibrium $\theta \in \bar{\Delta}_G(\gamma)$. The equilibrium equations (16) can be rewritten as

$$\omega = L(B^T\theta)\theta\,, \tag{38}$$

where $L(B^T\theta) = B\,\mathrm{diag}(\{a_{ij}\,\mathrm{sinc}(\theta_i - \theta_j)\}_{\{i,j\}\in\mathcal{E}})B^T$ is the Laplacian matrix associated with the graph $G(\mathcal{V},\mathcal{E},\tilde{A})$ with nonnegative edge weights $\tilde{a}_{ij} = a_{ij}\,\mathrm{sinc}(\theta_i - \theta_j)$ for $\theta \in \bar{\Delta}_G(\gamma)$. Since for any weighted Laplacian matrix L, we have that $L \cdot L^\dagger = L^\dagger \cdot L = I_n - (1/n)\mathbf{1}_{n\times n}$ (follows from the singular value decomposition [117]), a multiplication of equation (38) from the left by $B^T L(B^T\theta)^\dagger$ yields

$$B^T L(B^T\theta)^\dagger\omega = B^T\theta\,. \tag{39}$$

Note that the left-hand side of equation (39) is a continuous[4] function for $\theta \in \bar{\Delta}_G(\gamma)$. Consider the formal substitution $x = B^T\theta$, the compact and convex set $\mathcal{S}_\infty(\gamma) = \{x \in B^T\mathbb{R}^n : \|x\|_\infty \leq \gamma\}$, and the continuous map $f : \mathcal{S}_\infty(\gamma) \to \mathbb{R}$ given by $f(x) = B^T L(x)^\dagger\omega$. Then equation (39) reads as the fixed-point equation $f(x) = x$, and we can invoke *Brouwers's Fixed Point Theorem* which states that every continuous map from a compact and convex set to itself has a fixed point, see for instance [143, Section 7, Corollary 8].

Since the analysis of the map f in the ∞-norm is very hard in the general case, we resort to a 2-norm analysis and restrict ourselves to the set $\mathcal{S}_2(\gamma) = \{x \in B^T\mathbb{R}^n : \|x\|_2 \leq \gamma\} \subseteq \mathcal{S}_\infty(\gamma)$. By Brouwer's Fixed Point Theorem, there exists a solution $x \in \mathcal{S}_2(\gamma)$ to the equations $x = f(x)$ if and only if $\|f(x)\|_2 \leq \gamma$ for all $x \in \mathcal{S}_2(\gamma)$, or equivalently if and only if

$$\max_{x\in\mathcal{S}_2(\gamma)} \left\|B^T L(x)^\dagger\omega\right\|_2 \leq \gamma\,. \tag{40}$$

In the following we show that (32) is a sufficient condition for inequality (40).

First, we establish some identities. For a Laplacian matrix L, we obtain $L^\dagger = V\,\mathrm{diag}(0, \{1/\lambda_i(L)\}_{i=2,\dots,n})V^T$, where $\lambda_1(L) = 0$ and $\lambda_i(L) > 0$, $i \in \{2,\dots,n\}$, are the eigenvalues of L and $V \in \mathbb{R}^{n\times n}$ is an associated orthonormal matrix of eigenvectors. It follows that $V\,\mathrm{diag}\,(0,1,\dots,1)\,V^T = I_n - (1/n)\mathbf{1}_{n\times n}$, and since $\omega \perp \mathbf{1}_n$, there exists $\alpha \in \mathbb{R}^{|\mathcal{E}|}$ (not necessarily unique), such that $\omega = B\alpha$. By means of these identities, the left-hand side of (40) can be simplified and upper-bounded for all $x \in \mathcal{S}_2(\gamma)$:

$$\begin{aligned}
&\left\|B^T L(x)^\dagger\omega\right\|_2 = \left\|B^T L(x)^\dagger B\alpha\right\|_2 = \\
&\left\|B^T V(x)\,\mathrm{diag}\left(0, \frac{1}{\lambda_2(L(x))}, \dots, \frac{1}{\lambda_n(L(x))}\right) V^T(x) B\alpha\right\|_2 \\
&\leq \frac{1}{\lambda_2(L(x))} \cdot \left\|B^T V(x)\,\mathrm{diag}\,(0,1,\dots,1)\,V^T(x)B\alpha\right\|_2 \\
&\quad = (1/\lambda_2(L(x))) \cdot \left\|B^T\omega\right\|_2\,.
\end{aligned} \tag{41}$$

Thus, a sufficient condition for inequality (40) to be true can be derived as follows:

$$\begin{aligned}
&\max_{x\in\mathcal{S}_2(\gamma)} \left\|B^T L(x)^\dagger\omega\right\|_2 \leq \left\|B^T\omega\right\|_2 \max_{x\in\mathcal{S}_2(\gamma)} \big(1/\lambda_2\big(L(x)\big)\big) \\
&\quad \leq \left\|B^T\omega\right\|_2 \max_{x\in\{x\in\mathbb{R}^{|\mathcal{E}|}:\, \|x\|_\infty\leq\gamma\}} \big(1/\lambda_2\big(L(x)\big)\big) \\
&\quad = \left\|B^T\omega\right\|_2 / \left(\lambda_2(L)\cdot\mathrm{sinc}(\gamma)\right) \overset{!}{\leq} \gamma\,,
\end{aligned}$$

where we used identity (41), we enlarged the domain $\mathcal{S}_2(\gamma)$ to $\{x \in \mathbb{R}^{|\mathcal{E}|} : \|x\|_\infty \leq \gamma\}$, and we used the fact $\lambda_2(L(x)) \geq \lambda_2(L)\cdot\mathrm{sinc}(\gamma)$ for $\|x\|_\infty \leq \gamma$. In summary, we conclude that there is a locally exponentially stable synchronization manifold $[\theta] \in \{\theta \in \mathbb{T}^n : \|B^T\theta\|_2 \leq \gamma\} \subseteq \bar{\Delta}_G(\gamma)$ if

$$\lambda_2(L)\sin(\gamma) \geq \|B^T\omega\|_2\,. \tag{42}$$

Since the left-hand side of (42) is a concave function of $\gamma \in [0, \pi/2[$, there exists an open set of $\gamma \in [0, \pi/2[$ satisfying equation (42) if and only if equation (42) is true with the strict equality sign at $\gamma^* = \pi/2$, which corresponds to condition (32). Additionally, if these two equivalent statements are true, then there exists a unique $\gamma_{\min} \in [0, \pi/2[$ that satisfies equation (27) with the equality sign, namely $\sin(\gamma_{\min}) = \|B^T\omega\|_2/\lambda_2(L)$. This concludes the proof.

[4] The continuity can be established when re-writing equations (38) and (39) in the quotient space $\mathbf{1}_n^\perp$, where $L(B^T\theta)$ is nonsingular, and using the fact that the inverse of a matrix is a continuous function of its elements.

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