oscillations. This feature can be used to create new mechanisms for weak-signal detection and amplification, as well as for new engineering applications, such as beam steering. While related mathematical models may assume identical cells, or oscillators, in practice, we must account for the fact that even the most sophisticated circuit components have imperfections. These imperfections can be due to manufacturing processes that introduce, for example, slight fluctuations or inhomogeneities in the internal operating parameters of each unit, or to variations in the oscillators' coupling circuitry. On the other hand, it is worth checking whether inhomogeneities may have a positive effect and should therefore be introduced by design. Motivated by these considerations, we have studied the effects of inhomogeneities on the collective response of two-cell feedforward networks.

We have investigated the effects of parameter inhomogeneities in two-cell feedforward networks consisting of (*i*) pitchfork cells and (*ii*) Stuart-Landau cells. We used analysis of cubic roots, numerical simulations, and singularity theory for pitchfork cells, and model reduction, numerical simulations, and singularity theory for Stuart-Landau cells. Singularity theory is a powerful tool for identifying loci of bifurcation and hysteresis, where the number of equilibrium points changes. The other analytical and numerical tools we employed were instrumental in finding equilibria and periodic solutions and in assessing their stability.

The effects of fluctuations in the excitation parameters in pitchfork cells on the number of equilibrium points and transitions between them to amplified values were studied. Our analysis of the basins of attraction reveals that macroscopic jumps in equilibrium points are possible under certain conditions. In practice, it can be very difficult to control the initial conditions of electronic components, so the analysis has led to a new configuration that guarantees the network response will always transition to the highest amplification rate. This new configuration can prove useful for amplifying DC (Direct Current) signals, for instance.

The combined effects of inhomogeneities in the excitation, frequency, and cubic nonlinearity parameters of Stuart-Landau cells were also investigated. A phase-space analysis revealed complex behavior, e.g., tori bifurcations into quasi-periodic oscillations. The phase-locked attractor persists through a surprisingly broad range of inhomogeneities in the natural frequency. While this inhomogeneity reduces the second cell's amplification factor relative to the first, the effect is small for small inhomogeneities. Inhomogeneity in the excitation parameter may amplify the response of the second cell, but when combined with other inhomogeneities, may lead to instability of the phase-locked solution. The effect of small inhomogeneity in the cubic nonlinearity is primarily neutral: it makes the graph of the amplitude of the second cell versus the inhomogeneity in frequency slanted, without changing its height.

The work conducted in this manuscript is not exhaustive. Indeed, there are many other issues worth investigating. Chief among them is the interplay between external signals and the feedforward response. In beam steering, for instance, a feedforward network can be used to transmit (while amplified) signals or to receive incoming radio-frequency signals. In the homogeneous case, 1:1 synchronization with the external signal has been demonstrated, but in the inhomogeneous case, the effects of the interaction remain unknown. Another important issue is the effects of inhomogeneous coupling and disorder on the delay of a signal traveling down the chain of oscillators in the feedforward array. And, of course, inhomogeneities in larger arrays can lead to much more complicated dynamics worth investigating. These and many other related issues remain open and constitute future work.

Code availability

Our codes for generating figures are available in the GitHub repository
https://github.com/mar1akc/TwoCellBifurcatingNetworks.

ACKNOWLEDGEMENTS

Appendix

A   Equilibria of the feedforward network of pitchfork cells

In this appendix, we will elaborate on the dependence of the roots of the cubic polynomials $p_\pm$ on parameters ε, μ, and λ.

Let $\mu > 0$ and $\mu + \varepsilon > 0$. Let $x = \pm\sqrt{\mu}$, i.e., the first cell is in one of its stable equilibria. The equilibria of the second cell y are the roots of the cubic polynomials

$$p_+(y) := (\mu + \varepsilon)y - y^3 - \lambda\sqrt{\mu} \quad \text{and} \quad p_-(y) := (\mu + \varepsilon)y - y^3 + \lambda\sqrt{\mu}, \tag{A-1}$$

respectively. These polynomials tend to $\pm\infty$ as $y \to \mp\infty$. They are nonmonotone if $\mu + \varepsilon > 0$ and strictly decreasing otherwise. If $\mu + \varepsilon > 0$, their minimum and maximum are located at $y_{\min} = -\sqrt{\frac{\mu+\varepsilon}{3}}$ and $y_{\max} = \sqrt{\frac{\mu+\varepsilon}{3}}$, respectively, as sketched in Fig. 25(a). Their values at these extrema are

$$
\begin{aligned}
p_+(y_{\min}) &= -\frac{2}{3\sqrt{3}}(\mu + \varepsilon)^{3/2} - \lambda\sqrt{\mu}, \quad & p_+(y_{\max}) &= \frac{2}{3\sqrt{3}}(\mu + \varepsilon)^{3/2} - \lambda\sqrt{\mu}, \\
p_-(y_{\min}) &= -\frac{2}{3\sqrt{3}}(\mu + \varepsilon)^{3/2} + \lambda\sqrt{\mu}, \quad & p_-(y_{\max}) &= \frac{2}{3\sqrt{3}}(\mu + \varepsilon)^{3/2} + \lambda\sqrt{\mu}.
\end{aligned}
\tag{A-2}
$$

Therefore, if $x = \sqrt{\mu}$, y has one or three equilibria, depending on whether $p_+(y_{\max})$ is less or greater than zero. If $x = -\sqrt{\mu}$, y has one or three equilibria depending on whether $p_-(y_{\max})$ is greater or less than zero. In both cases, the steady-state bifurcation in y occurs at the parameter values satisfying

$$2(\mu + \varepsilon)^{3/2} = 3\sqrt{3}\lambda\mu^{1/2}. \tag{A-3}$$

The left-hand side of Eq. (A-3) is convex in μ while the right-hand side is concave in μ. Hence, Eq. (A-3) has at most two roots. If $\varepsilon < 0$, it has one root on its domain $\mu \geq -\varepsilon > 0$. If $\varepsilon \geq 0$, it transitions from having two roots to having no roots at the value of ε where Eq. (A-3) holds and the derivatives of its right- and left-hand side are equal. The latter condition is equivalent to
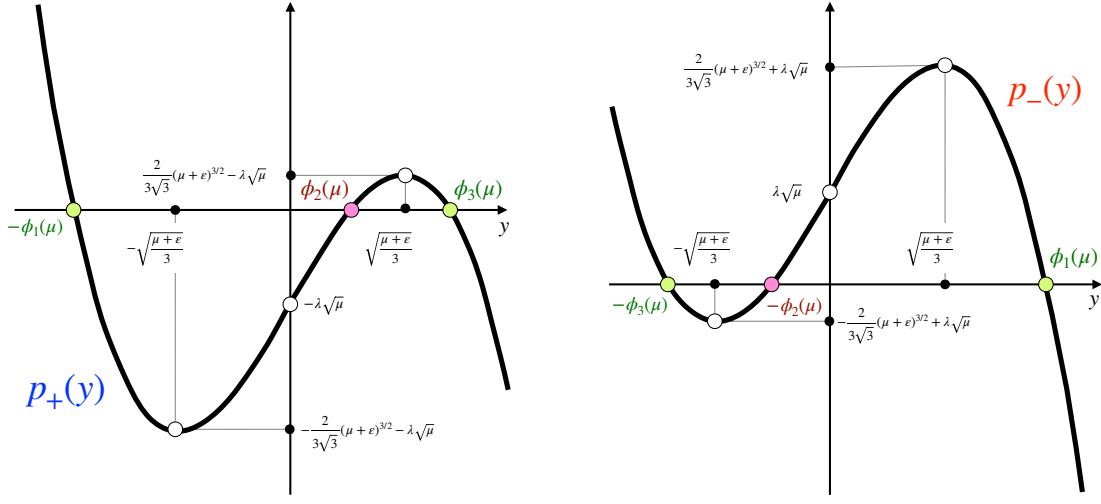
$$(\mu + \varepsilon)^{1/2} = \frac{\lambda\sqrt{3}}{2\mu^{1/2}}. \tag{A-4}$$

Substituting Eq. (A-4) to Eq. (A-3) and taking into account that $\lambda > 0$, we find that if Eq. (A-3) has only one root, it must be $\mu = \frac{1}{2}\lambda$. Then Eq. (A-4) yields the critical value of ε:
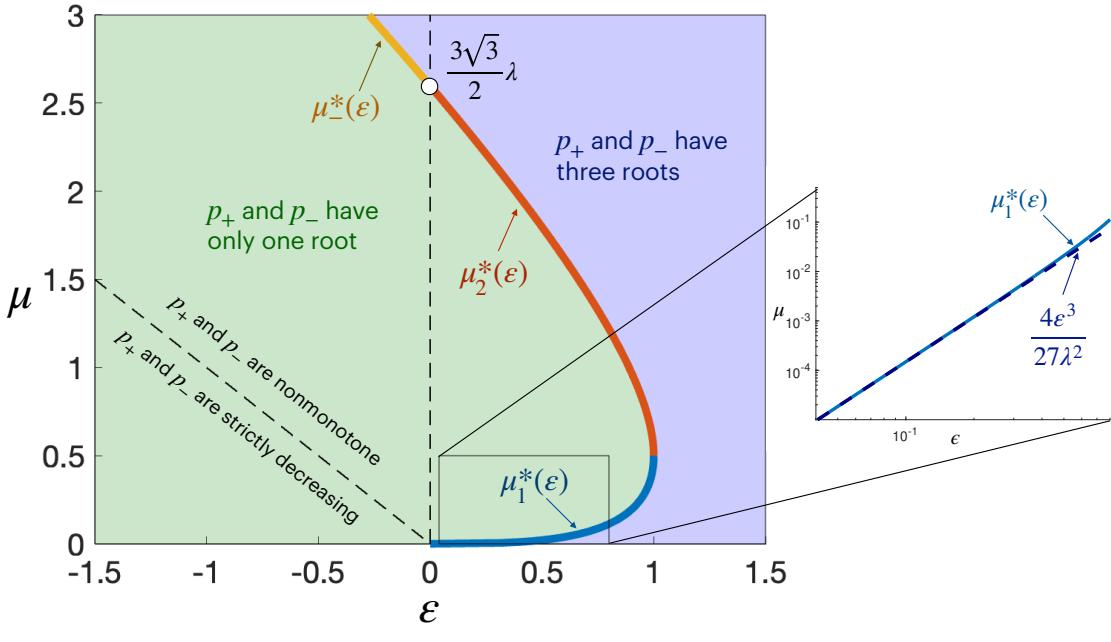
$$\varepsilon = \lambda. \tag{A-5}$$

Therefore, if $\varepsilon < \lambda$, Eq. (A-3) has two roots, if $\varepsilon > \lambda$, it has no roots, and it has a unique root $\mu = \frac{1}{2}\lambda$ at $\varepsilon = \lambda$.

Now we describe the structure and stability of the roots of polynomials $p_+(y)$ and $p_-(y)$. The stability type is defined by the eigenroots of the Jacobian in Eq. (9): $\mu - 3x^2$ and $\mu + \varepsilon - 3y^2$. We assume that $\lambda > 0$. A complete phase diagram of real roots of p_+ and p_- is shown in Figs. 5 and 25(b).

(a)

(b)

Figure 25: (a): The cubic polynomials $p_+(y)$ and $p_-(y)$ defined in (A-1). (b): A phase diagram of real roots of polynomials p_+ and p_- in the (ε, μ)-plane at $\lambda = 1$. The curve consisting of blue, red, and yellow pieces, corresponding, respectively, to $\mu_1^*(\varepsilon)$, $\mu_2^*(\varepsilon)$, and $\mu^*(\varepsilon)$, is defined by Eq. (A-3).

33