while Eq. (S9) becomes

$$\begin{aligned}\det(\overline{\mathbf{J}} - \nu\mathbf{I}) &= (\nu^2 + c_1\nu)(\nu^2 + c_2\nu + d_2) \\ &= \nu^4 + (c_1 + c_2)\nu^3 + (c_1c_2 + d_2)\nu^2 + c_1d_2\nu.\end{aligned} \tag{S44}$$

Thus, by comparing coefficients for the same powers of ν, we have

$$\begin{aligned}F_1(c_1, c_2, d_2, \varepsilon) &= c_1d_2 - \overline{B}_{11}/4, \\ F_2(c_1, c_2, d_2, \varepsilon) &= c_1c_2 + d_2 - \left(\frac{\beta_1\beta_2}{\beta_=^2} + \frac{1}{4}\right), \\ F_3(c_1, c_2, d_2, \varepsilon) &= c_1 + c_2 - \frac{\beta_1 + \beta_2}{\beta_=},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial F_1}{\partial c_1} &= d_2, & \frac{\partial F_1}{\partial c_2} &= 0, & \frac{\partial F_1}{\partial d_2} &= c_1, \\ \frac{\partial F_2}{\partial c_1} &= c_2, & \frac{\partial F_2}{\partial c_2} &= c_1, & \frac{\partial F_2}{\partial d_2} &= 1, \\ \frac{\partial F_3}{\partial c_1} &= 1, & \frac{\partial F_3}{\partial c_2} &= 1, & \frac{\partial F_3}{\partial d_2} &= 0,\end{aligned}$$

$$\frac{\partial \mathbf{F}}{\partial(c, d)} = \begin{pmatrix} d_2 & 0 & c_1 \\ c_2 & c_1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \mathbf{G} = \frac{\partial \mathbf{F}}{\partial(c, d)}\bigg|_{\varepsilon=0} = \begin{pmatrix} 1/4 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Therefore, $\mathbf{G}$ is non-singular (regardless of the value of $\alpha_2 > 0$).

Example 2: For $n = 3$, it is convenient to first write out the terms in the first sum of the first line in Eq. (S20) and differentiate them with respect to c_i and d_i (since the second sum does not contain c_i or d_i). The terms from the first sum are

$$F_1 = c_1d_2d_3 - \cdots, \tag{S45}$$

$$F_2 = d_2d_3 + c_1c_2d_3 + c_1d_2c_3 - \cdots, \tag{S46}$$

$$F_3 = c_1c_2c_3 + c_1d_3 + c_1d_2 + c_2d_3 + d_2c_3 - \cdots, \tag{S47}$$

$$F_4 = d_2 + d_3 + c_1c_2 + c_2c_3 + c_3c_1 - \cdots, \tag{S48}$$

$$F_5 = c_1 + c_2 + c_3 - \cdots. \tag{S49}$$

Using this, we see that the matrix of partial derivatives is

$$\frac{\partial \mathbf{F}}{\partial(c, d)} = \begin{pmatrix} d_2d_3 & 0 & 0 & c_1d_3 & c_1d_2 \\ c_2d_3 + d_2c_3 & c_1d_3 & c_1d_2 & d_3 + c_1c_3 & d_2 + c_1c_2 \\ c_2c_3 + d_2 + d_3 & c_1c_3 + d_3 & c_1c_2 + d_2 & c_1 + c_3 & c_1 + c_2 \\ c_2 + c_3 & c_3 + c_1 & c_1 + c_2 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}, \tag{S50}$$

from which we obtain the matrix $\mathbf{G}$ by setting $c_1 = c_2 = c_3 = 1$, $d_2 = \overline{\alpha}_2 = 1/4$, and $d_3 = \overline{\alpha}_3$:

$$\mathbf{G} = \begin{pmatrix} \overline{\alpha}_3/4 & 0 & 0 & \overline{\alpha}_3 & 1/4 \\ 1/4 + \overline{\alpha}_3 & \overline{\alpha}_3 & 1/4 & 1 + \overline{\alpha}_3 & 5/4 \\ 5/4 + \overline{\alpha}_3 & 1 + \overline{\alpha}_3 & 5/4 & 2 & 2 \\ 2 & 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}. \tag{S51}$$

From this, we obtain $\det(\mathbf{G}) = -\overline{\alpha}_3(1 - 4\overline{\alpha}_3)^2/64 = -\alpha_3(\alpha_2 - \alpha_3)^2/(256\alpha_2^3)$ and see that $\mathbf{G}$ is non-singular if and only if $0 < \alpha_2 < \alpha_3$. This condition is indeed satisfied by the 3-generator system considered in the main text ($0 < \alpha_2 \approx 75.5 < \alpha_3 \approx 178.5$).