exponentially in time for $K > K_c$ ($K < K_c$) (later shown rigorously and quantitatively in Ref. [16]). Crawford[22], using center manifold theory, obtains (Eq. (108) of Ref. [22]) an equation of the form $d\rho/dt = a(K - K_c)\rho + b\rho^3 + O(\rho^5)$ for K near K_c. Another work of interest is that of Watanabe and Strogatz[23] who consider the case where all oscillators have the same frequency for both finite and infinite N. By use of a nonlinear transformation of the phase variables $\theta_i(t)$, these authors show that the dynamics reduces to a solution of three coupled first order ordinary differential equations. Thus, while macroscopic behavior of order-parameter dynamics has been previously addressed for the standard Kuramoto problem, it has, until now, never been demonstrated fully (e.g., without the restriction of [22] to small amplitude, or the restriction of [23] to identical frequencies). Our paper does this and also demonstrates that our technique can be usefully applied to a host of other important related problems.

Our work also suggests other future lines of study. For example, can any rigorous results be obtained relevant to whether our macroscopic order-parameter attractors obtained by considering f in the manifold M have general validity[24]? Are there interesting qualitative differences between the behavior for Lorentzian $g(\omega)$ as compared to other monotonic symmetric oscillator distribution functions $g(\omega)$? What other systems, in addition to those discussed in Sec. IV, can our method be applied to?

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