well as the set of admissible initial conditions. For $\kappa \gg 1$, practical phase synchronization is achieved for all angles in an open semi-circle. More general coupled oscillator networks display the same phenomenology, but the threshold from incoherence to synchrony is generally unknown.

C. Synchronization Metrics

The notion of phase cohesiveness can be understood as a performance measure for synchronization and phase synchronization is simply the extreme case of phase cohesiveness with $\lim_{t\to\infty}\theta(t) \in \bar{\Delta}_G(0) = \overline{\mathrm{Arc}}_n(0)$. An alternative performance measure is the magnitude of the so-called *order parameter* introduced by Kuramoto [5], [6]:

$$re^{\mathrm{i}\psi} = \frac{1}{n}\sum\nolimits_{j=1}^{n} e^{\mathrm{i}\theta_j}\,.$$

The order parameter is the centroid of all oscillators represented as points on the unit circle in $\mathbb{C}^1$. The magnitude r of the order parameter is a synchronization measure: if all oscillators are phase-synchronized, then $r = 1$, and if all oscillators are spaced equally on the unit circle, then $r = 0$. The latter case is characterized in Subsection III-E. For a complete graph, the magnitude r of the order parameter serves as an *average* performance index for synchronization, and phase cohesiveness can be understood as a *worst-case* performance index. Extensions of the order parameter tailored to non-complete graphs have been proposed in [19], [52], [56].

For a complete graph and for γ sufficiently small, the set $\bar{\Delta}_G(\gamma)$ reduces to $\overline{\mathrm{Arc}}_n(\gamma)$, the arc of length γ containing all oscillators. The order parameter is contained within the convex hull of this arc since it is the centroid of all oscillators, see Figure 6. In this case, the magnitude r of the order parameter can be related to the arc length γ.

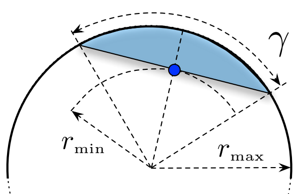

Fig. 6. Schematic illustration of an arc of length $\gamma \in [0, \pi]$, its convex hull (shaded), and the value ● of the corresponding order parameter $re^{\mathrm{i}\psi}$ with minimum magnitude $r_{\mathrm{min}} = \cos(\gamma/2)$ and maximum magnitude $r_{\mathrm{max}} = 1$.

Lemma 3.1: **(Shortest arc length and order parameter, [74, Lemma 2.1])** Given an angle array $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{T}^n$ with $n \geq 2$, let $r(\theta) = \frac{1}{n}|\sum_{j=1}^{n} e^{\mathrm{i}\theta_j}|$ be the magnitude of the order parameter, and let $\gamma(\theta)$ be the length of the shortest arc containing all angles, that is, $\theta \in \overline{\mathrm{Arc}}_n(\gamma(\theta))$. The following statements hold:

1) if $\gamma(\theta) \in [0, \pi]$, then $r(\theta) \in [\cos(\gamma(\theta)/2), 1]$; and
2) if $\theta \in \overline{\mathrm{Arc}}_n(\pi)$, then $\gamma(\theta) \in [2\arccos(r(\theta)), \pi]$.

D. Synchronization Conditions

The coupled oscillator dynamics (1) feature (i) the synchronizing coupling described by the graph $G(\mathcal{V}, \mathcal{E}, A)$ and (ii) the de-synchronizing effect of the non-uniform natural frequencies ω. Loosely speaking, synchronization occurs when the coupling dominates the non-uniformity. Various conditions have been proposed in the synchronization and power systems literature to quantify this trade-off.

The coupling is typically quantified by the algebraic connectivity $\lambda_2(L)$ [44], [45], [52], [64], [127], [128] or the weighted nodal degree $\mathrm{deg}_i \triangleq \sum_{j=1}^{n} a_{ij}$ [64], [97], [117], [129], [130], and the non-uniformity is quantified by either absolute norms $\|\omega\|_p$ or incremental norms $\|B^T\omega\|_p$, where typically $p \in \{2, \infty\}$. Sometimes, these conditions can be evaluated only numerically since they are state-dependent [127], [129] or arise from a non-trivial linearization process, such as the Master stability function formalism [44], [45], [131]. In general, concise and accurate results are known only for specific topologies such as complete graphs [74], linear chains [108], and bipartite graphs [82] with uniform weights.

For arbitrary coupling topologies only sufficient conditions are known [52], [64], [127], [129] as well as numerical investigations for random networks [89], [98], [99], [128], [132]. Simulation studies indicate that these conditions are conservative estimates on the threshold from incoherence to synchrony. Literally, every review article on synchronization draws attention to the problem of finding sharp synchronization conditions [7], [8], [44]–[46], [74], [114].

E. Phase Balancing and Splay State

In certain applications in neuroscience [11]–[13], deep-brain stimulation [26], [27], and vehicle coordination [19], [28]–[31], one is not interested in the coherent behavior with synchronized (or nearly synchronized) phases, but rather in the phenomenon of synchronized frequencies and desychronized phases.

Whereas the phase-synchronized state is characterized by the order parameter r achieving its maximal (unit) magnitude, we say that a solution $\theta : \mathbb{R}_{\geq 0} \to \mathbb{T}^n$ to the coupled oscillator model (1) achieves *phase balancing* if all phases $\theta_i(t)$ converge to $\mathrm{Bal}_n = \{\theta \in \mathbb{T}^n : r(\theta) = |\frac{1}{n}\sum_{j=1}^{n} e^{\mathrm{i}\theta_j}| = 0\}$ as $t \to \infty$, that is, the oscillators are distributed over the unit circle $\mathbb{S}^1$, such that their centroid $re^{\mathrm{i}\psi}$ vanishes. We refer to [28] for a geometric characterization of the balanced state.

One balanced state of particular interest in neuroscience applications [11]–[13], [26], [27] is the so-called splay state $\{\theta \in \mathbb{T}^n : \theta_i = i \cdot 2\pi/n + \varphi \pmod{2\pi}, \varphi \in \mathbb{S}^1, i \in \{1, \dots, n\}\} \subseteq \mathrm{Bal}_n$ corresponding to phases uniformly distributed around the unit circle $\mathbb{S}^1$ with distances $2\pi/n$. Other highly symmetric balanced states consist of multiple clusters of collocated phases, where the clusters themselves are arranged in splay state, see [28], [29].

IV. Analysis of Synchronization

In this section we present several analysis approaches to synchronization in the coupled oscillator model (1). We begin with a few basic ideas to provide important intuition as well as the analytic basis for further analysis.

A. Some Simple Yet Important Insights

The potential energy $U : \mathbb{T}^n \to \mathbb{R}$ of the elastic spring network in Figure 1 is, up to an additive constant, given by

$$U(\theta) = \sum\nolimits_{\{i,j\}\in\mathcal{E}} a_{ij}\big(1 - \cos(\theta_i - \theta_j)\big)\,. \tag{12}$$

By means of the potential energy, the coupled oscillator model (1) can reformulated as the forced gradient system

$$\dot{\theta}_i = \omega_i - \nabla_i U(\theta)\,, \qquad i \in \{1,\dots,n\}\,, \tag{13}$$

where $\nabla_i U(\theta) = \frac{\partial}{\partial\theta_i}U(\theta)$ denotes the partial derivative. It can be easily verified that the phase-synchronized state $\theta_i = \theta_j$ for all $\{i,j\}\in\mathcal{E}$ is a local minimum of the potential energy (12). The gradient formulation (13) clearly emphasizes the competition between the synchronization-enforcing coupling through the potential $U(\theta)$ and the synchronization-inhibiting heterogeneous natural frequencies ω_i.

We next note that ω has to satisfy certain bounds, relative to the weighted nodal degree, in order for a synchronized solution to exist.

Lemma 4.1: **(Necessary sync conditions)** Consider the coupled oscillator model (1) with graph $G(\mathcal{V},\mathcal{E},A)$, frequencies $\omega \in \mathbf{1}_n^{\perp}$, and nodal degree $\deg_i = \sum_{j=1}^n a_{ij}$ for each oscillator $i \in \{1,\dots,n\}$. If there exists a synchronized solution $\theta \in \bar{\Delta}_G(\gamma)$ for some $\gamma \in [0,\pi/2]$, then the following conditions hold:

1) **Absolute bound:** For each node $i \in \{1,\dots,n\}$,

$$\deg_i \sin(\gamma) \geq |\omega_i|\,; \tag{14}$$

2) **Incremental bound:** For all distinct $i,j \in \{1,\dots,n\}$,

$$(\deg_i + \deg_j)\sin(\gamma) \geq |\omega_i - \omega_j|\,. \tag{15}$$

Proof: Since $\omega \in \mathbf{1}_n^{\perp}$, the synchronization frequency ω_{sync} is zero, and phase and frequency synchronized solutions are equilibrium solutions determined by the equations

$$\omega_i = \sum\nolimits_{j=1}^{n} a_{ij}\sin(\theta_i - \theta_j)\,, \qquad i \in \{1,\dots,n\}\,. \tag{16}$$

Since $\sin(\theta_i - \theta_j) \in [-\sin(\gamma), +\sin(\gamma)]$ for $\theta \in \bar{\Delta}_G(\gamma)$, the equilibrium equations (16) have no solution if condition (14) is not satisfied. Since $\omega \in \mathbf{1}_n^{\perp}$, an incremental bound on ω seems to be more appropriate than an absolute bound. The subtraction of the ith and jth equation (16) yields

$$\omega_i - \omega_j = \sum\nolimits_{k=1}^{n} \left(a_{ik}\sin(\theta_i - \theta_k) - a_{jk}\sin(\theta_j - \theta_k)\right)\,.$$

Again, since the coupling is bounded, the above equation has no solution in $\bar{\Delta}_G(\gamma)$ if condition (15) is not satisfied. ■

The following result is fundamental for various approaches to phase and frequency synchronization. To the best of the authors' knowledge this result has been first established in [133], and it has been reproved numerous times.

Lemma 4.2: **(Stable synchronization in $\Delta_G(\pi/2)$)** Consider the coupled oscillator model (1) with a connected graph $G(\mathcal{V},\mathcal{E},A)$ and frequencies $\omega \in \mathbf{1}_n^{\perp}$. The following statements hold:

1) **Jacobian:** The Jacobian $J(\theta)$ of the coupled oscillator model (1) evaluated at $\theta \in \mathbb{T}^n$ is given by

$$J(\theta) = -B\,\mathrm{diag}(\{a_{ij}\cos(\theta_i - \theta_j)\}_{\{i,j\}\in\mathcal{E}})B^T\,;$$

2) **Local stability and uniqueness:** If there exists an equilibrium $\theta^* \in \Delta_G(\pi/2)$, then
   (i) $-J(\theta^*)$ is a Laplacian matrix;
   (ii) the equilibrium manifold $[\theta^*] \in \Delta_G(\pi/2)$ is locally exponentially stable; and
   (iii) this equilibrium manifold is unique in $\bar{\Delta}_G(\pi/2)$.

Proof: Since $\frac{\partial}{\partial\theta_i}\big(\omega_i - \sum_{j=1}^n a_{ij}\sin(\theta_i - \theta_j)\big) = -\sum_{j=1}^n a_{ij}\cos(\theta_i - \theta_j)$ and $\frac{\partial}{\partial\theta_j}\big(\omega_i - \sum_{j=1}^n a_{ij}\sin(\theta_i - \theta_j)\big) = a_{ij}\cos(\theta_i - \theta_j)$, we obtain that the Jacobian is equal to minus the Laplacian matrix of the connected graph $G(\mathcal{V},\mathcal{E},\tilde{A})$ with the (possibly negative) weights $\tilde{a}_{ij} = a_{ij}\cos(\theta_i - \theta_j)$. Equivalently, in compact notation $J(\theta) = -B\,\mathrm{diag}(\{a_{ij}\cos(\theta_i - \theta_j)\}_{\{i,j\}\in\mathcal{E}})B^T$. This completes the proof of statement 1).

The Jacobian $J(\theta)$ evaluated for an equilibrium $\theta^* \in \Delta_G(\pi/2)$ is minus the Laplacian matrix of the graph $G(\mathcal{V},\mathcal{E},\tilde{A})$ with strictly positive weights $\tilde{a}_{ij} = a_{ij}\cos(\theta_i^* - \theta_j^*) > 0$ for every $\{i,j\}\in\mathcal{E}$. Hence, $J(\theta^*)$ is negative semidefinite with the nullspace $\mathbf{1}_n$ arising from the rotational symmetry, see Figure 4. Consequently, the equilibrium point $\theta^* \in \Delta_G(\pi/2)$ is locally (transversally) exponentially stable, or equivalently, the corresponding equilibrium manifold $[\theta^*] \in \Delta_G(\pi/2)$ is locally exponentially stable.

The uniqueness statement follows since the right-hand side of the coupled oscillator model (1) is a one-to-one function (modulo rotational symmetry) for $\theta \in \bar{\Delta}_G(\pi/2)$, see [134, Corollary 1]. This completes the proof of statement 2). ■

By Lemma 4.2, any equilibrium in $\Delta_G(\pi/2)$ is stable which supports the notion of phase cohesiveness as a performance metric. Since the Jacobian $J(\theta)$ is the negative Hessian of the potential $U(\theta)$ defined in (12), Lemma 4.2 also implies that any equilibrium in $\Delta_G(\pi/2)$ is a local minimizer of $U(\theta)$. Of particular interest are so-called $\mathbb{S}^1$-*synchronizing graphs* for which all critical points of (12) are hyperbolic, the phase-synchronized state is the global minimum of $U(\theta)$, and all other critical points are local maxima or saddle points. The class of $\mathbb{S}^1$-synchronizing graphs includes, among others, complete graphs and acyclic graphs [100]–[103].

These basic insights motivated various characterizations and explorations of the critical points and the curvature of the potential $U(\theta)$ in the literature on synchronization [52], [64], [74], [89], [93], [100], [100]–[103], [103] as well as on power systems [61], [116], [127], [129], [133]–[137].

B. Phase Synchronization

If all natural frequencies are identical, $\omega_i \equiv \omega$ for all $i \in \{1,\dots,n\}$, then a transformation of the coupled oscillator model (1) to a rotating frame with frequency ω leads to

$$\dot{\theta}_i = -\sum\nolimits_{j=1}^{n} a_{ij}\sin(\theta_i - \theta_j)\,, \qquad i \in \{1,\dots,n\}\,. \tag{17}$$

The analysis of the coupled oscillator model (17) is particularly simple and local phase synchronization can be concluded by various analysis methods. A sample of different analysis schemes (by far not complete) includes the contraction property [54], [64], [92], [100], [138], quadratic Lyapunov functions [52], [64], linearization [81], [103], or order parameter and potential function arguments [28], [56], [80].

The following theorem on phase synchronization summarizes a collection of results originally presented in [28], [54], [56], [74], [100], [103], and it can be easily proved given the insights developed in Subsection IV-A.

Theorem 4.3: **(Phase synchronization)** Consider the coupled oscillator model (1) with a connected graph $G(\mathcal{V}, \mathcal{E}, A)$ and with frequency $\omega \in \mathbb{R}^n$ (not necessarily zero mean). The following statements are equivalent:

(i) **Stable phase sync:** there exists a locally exponentially stable phase-synchronized solution $\theta \in \overline{\mathrm{Arc}}_n(0)$ (or a synchronization manifold $[\theta] \in \bar{\Delta}_G(0)$); and
(ii) **Uniformity:** there exists a constant $\omega \in \mathbb{R}$ such that $\omega_i = \omega$ for all $i \in \{1, \dots, n\}$.

If the two equivalent cases (i) and (ii) are true, the following statements hold:

1) **Global convergence:** For all initial angles $\theta(0) \in \mathbb{T}^n$ all frequencies $\dot{\theta}_i(t)$ converge to ω and all phases $\theta_i(t) - \omega t \pmod{2\pi}$ converge to the critical points $\{\theta \in \mathbb{T}^n : \nabla U(\theta) = \mathbf{0}_n\}$;
2) **Semi-global stability:** The region of attraction of the phase-synchronized solution $\theta \in \overline{\mathrm{Arc}}_n(0)$ contains the open semi-circle $\mathrm{Arc}_n(\pi)$, and each arc $\overline{\mathrm{Arc}}_n(\gamma)$ is positively invariant for every arc length $\gamma < \pi$;
3) **Explicit phase:** For initial angles in an open semi-circle $\theta(0) \in \mathrm{Arc}_n(\pi)$, the asymptotic synchronization phase is given by[2] $\theta(t) = \sum_{i=1}^n \theta_i(0)/n + \omega t \pmod{2\pi}$;
4) **Convergence rate:** For every initial angle $\theta(0) \in \overline{\mathrm{Arc}}_n(\gamma)$ with $\gamma < \pi$, the exponential convergence rate to phase synchronization is no worse than $\lambda_{\text{ps}} = -\lambda_2(L)\,\mathrm{sinc}(\gamma)$; and
5) **Almost global stability:** If the graph $G(\mathcal{V}, \mathcal{E}, A)$ is $\mathbb{S}^1$-synchronizing, the region of attraction of the phase-synchronized solution $\theta \in \overline{\mathrm{Arc}}_n(0)$ is almost all of $\mathbb{T}^n$.

Proof: Implication (i) $\implies$ (ii): By assumption, there exist constants $\theta_{\text{sync}} \in \mathbb{S}^1$ and $\omega_{\text{sync}} \in \mathbb{R}$ such that $\theta_i(t) = \theta_{\text{sync}} + \omega_{\text{sync}} t \pmod{2\pi}$. In the phase-synchronized case, the dynamics (1) then read as $\omega_{\text{sync}} = \omega_i$ for all $i \in \{1, \dots, n\}$. Hence, a necessary condition for the existence of phase synchronization is that all ω_i are identical.

Implication (ii) $\implies$ (i): Consider the model (1) written in a rotating frame with frequency ω as in (17). Note that the set of phase-synchronized solutions $\bar{\Delta}_G(0)$ is an equilibrium manifold. By Lemma 4.2, we conclude that $\bar{\Delta}_G(0)$ is locally exponentially stable. This concludes the proof of (i) $\Leftrightarrow$ (ii).

[2]This "average" of angles (points on $\mathbb{S}^1$) is well-defined in an open semi-circle. If the parametrization of θ has no discontinuity inside the arc containing all angles, then the average can be obtained by the usual formula.

Statement 1): Note that (17) can be written as the gradient flow $\dot{\theta} = -\nabla U(\theta)$, and the corresponding potential function $U(\theta)$ is non-increasing along trajectories. Since the sublevel sets of $U(\theta)$ are compact and the vector field $\nabla U(\theta)$ is smooth, the invariance principle [139, Theorem 4.4] asserts that every solution converges to set of equilibria of (17).

Statements 2): The coupled oscillator model (17) can be re-written as the consensus-type system

$$\dot{\theta}_i = -\sum\nolimits_{j=1}^n b_{ij}(\theta) \cdot (\theta_i - \theta_j)\,, \qquad i \in \{1, \dots, n\}\,, \tag{18}$$

where the weights $b_{ij}(\theta) = a_{ij}\,\mathrm{sinc}(\theta_i - \theta_j)$ depend explicitly on the system state. Notice that for $\theta \in \overline{\mathrm{Arc}}_n(\gamma)$ and $\gamma < \pi$ the weights $b_{ij}(\theta)$ are upper and lower bounded as $b_{ij}(\theta) \in [a_{ij}\,\mathrm{sinc}(\gamma), a_{ij}]$ Assume that the initial angles $\theta_i(0)$ belong to the set $\overline{\mathrm{Arc}}_n(\gamma)$, that is, they are all contained in an arc of length $\gamma \in [0, \pi[$. In this case, a natural Lyapunov function to establish phase synchronization can be obtained from the *contraction property*, which aims at showing that the convex hull containing all oscillators is decreasing, see [54], [64], [92], [100], [140] and the review [138, Section 2].

Recall the geodesic distance between two angles on $\mathbb{S}^1$ and define the continuous function $V : \mathbb{T}^n \to [0, \pi]$ by

$$V(\psi) = \max\{|\psi_i - \psi_j| \mid i, j \in \{1, \dots, n\}\}. \tag{19}$$

Notice that, if all angles are contained in an arc at time t, then the arc length $V(\theta(t)) = \max_{i,j \in \{1,\dots,n\}} |\theta_i(t) - \theta_j(t)|$ is a Lyapunov function candidate for phase synchronization. Indeed, it can be shown that $V(\theta(t))$ decreases along trajectories of (18) for $\theta(0) \in \overline{\mathrm{Arc}}_n(\gamma)$ and for all $\gamma < \pi$. The analysis is complicated by the following fact: the function $V(\theta(t))$ is continuous but not necessarily differentiable when the maximum geodesic distance (that is, the right-hand side of (19)), is attained by more than one pair of oscillators. We omit the explicit calculations here and refer to [54], [64], [74], [83], [92] for a detailed analysis.

Statement 3): By statement 2), the set $\mathrm{Arc}_n(\pi)$ is positively invariant, and for $\theta(0) \in \mathrm{Arc}_n(\pi)$ the average $\sum_{i=1}^n \theta_i(t)/n$ is well defined for $t \geq 0$. A summation over all equations of the model (17) yields $\sum_{i=1}^n \dot{\theta}_i(t) = 0$, or equivalently, $\sum_{i=1}^n \theta_i(t)$ is constant for all $t \geq 0$. In particular, for $t = 0$ we have that $\sum_{i=1}^n \theta_i(t) = \sum_{i=1}^n \theta_i(0)$ and for a phase-synchronized solution we have that $\sum_{i=1}^n \theta_{\text{sync}} = \sum_{i=1}^n \theta_i(0)$. Hence, the explicit synchronization phase is given by $\sum_{i=1}^n \theta_i(0)/n$. In the original coordinates (non-rotating frame) the synchronization phase is given by $\sum_{i=1}^n \theta_i(0)/n + \omega t \pmod{2\pi}$.

Statement 4): Given the invariance of the set $\overline{\mathrm{Arc}}_n(\gamma)$ for any $\gamma < \pi$, the system (18) can be analyzed as a linear time-varying consensus system with initial condition $\theta(0) \in \overline{\mathrm{Arc}}_n(\gamma)$, and bounded time-varying weights $b_{ij}(\theta(t)) \in [a_{ij}\,\mathrm{sinc}(\gamma), a_{ij}]$ for all $t \geq 0$. The worst-case convergence rate λ_{ps} can then be obtained by a standard symmetric consensus analysis, see [52], [53], [64], [74]. For instance, it can be shown that the deviation of the angles $\theta(t)$ from their average, $\|\theta(t) - (\sum_{i=1}^n \theta(t)/n)\mathbf{1}_n\|_2^2$ (the *disagreement function*) decays exponentially with rate λ_{ps}.