Asymmetry-Induced Synchronization in Oscillator Networks

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A scenario has recently been reported in which in order to stabilize complete synchronization of an oscillator network—a symmetric state—the symmetry of the system itself has to be broken by making the oscillators nonidentical. But how often does such behavior—which we term asymmetry-induced synchronization (*AISync*)—occur in oscillator networks? Here we present the first general scheme for constructing *AISync* systems and demonstrate that this behavior is the norm rather than the exception in a wide class of physical systems that can be seen as multilayer networks. Since a symmetric network in complete synchrony is the basic building block of cluster synchronization in more general networks, *AISync* should be common also in facilitating cluster synchronization by breaking the symmetry of the cluster subnetworks.

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I. INTRODUCTION

A common assumption in the field of network dynamics is that homogeneity in the local dynamics [1, 2] and interaction network [3–5]—or in the combination of both [6, 7]—can facilitate complete synchronization. It has been recently shown, however, that structural heterogeneity in networks of identical oscillators [8] or oscillator heterogeneity in structurally symmetric networks [9] can stabilize otherwise unstable synchronous states, thus effectively breaking the symmetry of a system to stabilize a symmetric state. These scenarios, which we refer to as *asymmetry-induced synchronization* (*AISync*), can be interpreted as the converse of symmetry breaking, and hence as a converse of chimera states [10, 11]. Perhaps the most striking and the strongest form of *AISync* is the one in which oscillators coupled in a symmetric network (i.e., each oscillator plays exactly the same structural role) can converge to identical dynamics only when they themselves are nonidentical; this has been demonstrated, however, exclusively for rotationally symmetric networks and one type of periodic oscillators [9]. Whether such *AISync* behavior can be shown to be common among systems with other symmetric network structures and oscillator dynamics, including experimentally testable ones, has been an open question.

In this article we introduce and analyze a broad class of *AISync* systems that can have general symmetric network structure with multiple link types and general oscillator dynamics (which can be chaotic, periodic, continuous-time, discrete-time, etc.). This in particular includes physical systems previously used in network synchronization experiments, thus providing a recipe for future empirical studies. For this class, we demonstrate that *AISync* is indeed common and provide a full characterization of those networks that support *AISync* behavior, showing that the fraction of such networks is significant over a range of network sizes and link densities.

II. DEFINITION OF *AISync*

To formulate a precise definition of *AISync*, we consider networks of N (not necessarily identical) oscillators coupled through K different types of interactions. The network dynamics is described by

$$\dot{\boldsymbol{X}}_i = \boldsymbol{F}_i(\boldsymbol{X}_i) + \sum_{\alpha=1}^{K} \sum_{\substack{i'=1 \\ i'\neq i}}^{N} A_{ii'}^{(\alpha)} \boldsymbol{H}^{(\alpha)}(\boldsymbol{X}_i, \boldsymbol{X}_{i'}), \qquad (1)$$

where $\boldsymbol{X}_i = \boldsymbol{X}_i(t)$ is the M-dimensional state vector of node i, the function $\boldsymbol{F}_i$ governs the intrinsic dynamics of node i, the adjacency matrix $A^{(\alpha)} = (A_{ii'}^{(\alpha)})$ represents the topology of interactions through links of type α, and $\boldsymbol{H}^{(\alpha)}$ is the coupling function associated with the link type α. A completely synchronous state of the network is defined by $\boldsymbol{X}_1(t) = \boldsymbol{X}_2(t) = \cdots = \boldsymbol{X}_N(t)$.

To isolate the effect of breaking the homogeneity of oscillators, we consider adjacency matrices $A^{(\alpha)}$ that together represent a *symmetric network*, defined as a network in which every node can be mapped to any other node by some permutation of nodes without changing any $A^{(\alpha)}$. Thus, the set of links of any given type must couple every node identically (see Fig. 1(a) for an example). The rationale for using symmetric network structures here is to ensure that any stabilization of complete synchronization by oscillator heterogeneity is due to the reduced system symmetry (as required for *AISync*) and not due to having network heterogeneity and oscillator heterogeneity compensating each other, which may not break the system symmetry.

When restricted to undirected networks with a single link type, our definition of symmetric networks yields the class of vertex-transitive graphs from graph theory [12]. This rich class encompasses Cayley graphs (defined as a network of relations between elements of a finite group; Appendix A) and circulant graphs (defined as a network whose nodes can be arranged in a ring so that the network is invariant under rotations), which have previously been used to study chimera states [13]. Enumerating

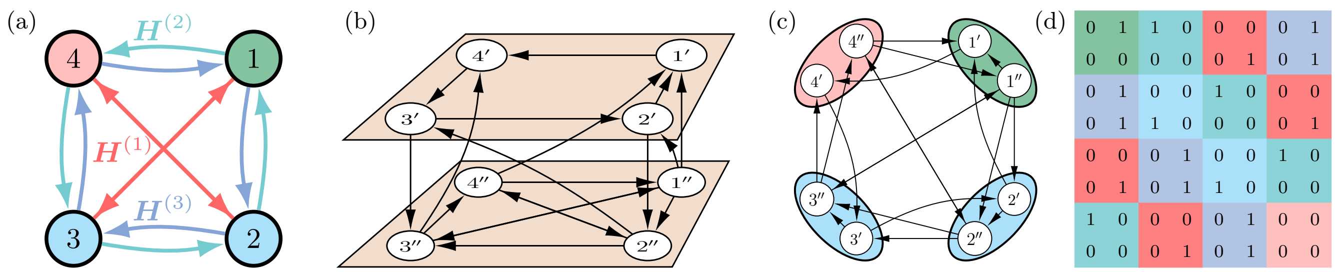

FIG. 1. Multilayer construction of *AISync* networks. (a) Example of a symmetric network of $N = 4$ heterogeneous oscillators and $K = 3$ types of (directed) links with associated coupling functions $\boldsymbol{H}^{(1)}$, $\boldsymbol{H}^{(2)}$ and $\boldsymbol{H}^{(3)}$. (b) One of many possible multilayer networks corresponding to the network in (a), with $L = 2$ layers and $n = LN = 8$ identical subnodes. Subnodes are labeled with node indices, with prime and double prime indicating layer 1 and 2, respectively. (c) Flattened, monolayer representation of the multilayer network in (b). (d) Block structure of the adjacency matrix $\widetilde{A}$ for the monolayer network in (c). Colors indicate different types of nodes ($F^{(i)}$, diagonal blocks) and links ($\widetilde{A}^{(ii')}$, off-diagonal blocks).

all vertex-transitive graphs of a given size N becomes challenging as N grows and has so far been completed only for $N < 32$ [14]. The symmetric networks we consider here generalizes vertex-transitive graphs to the even richer class of networks that can be directed and include multiple link types.

Given a symmetric network structure, the system in Eq. (1) exhibits *AISync* if it satisfies the following two conditions: (C1) there are no asymptotically stable synchronous states for any *homogeneous system* (i.e., with $\boldsymbol{F}_1 = \cdots = \boldsymbol{F}_N$), and (C2) there is an asymptotically *heterogeneous system* (i.e., with $\boldsymbol{F}_i \neq \boldsymbol{F}_{i'}$, for some $i \neq i'$) for which a stable synchronous state exists. A challenge in establishing *AISync* is that the form of Eq. (1) does not guarantee the existence of a completely synchronous state. Another challenge concerns the stability analysis of such a state, since Eq. (1) is beyond the framework normally used in the master stability function (MSF) approach and its generalizations currently available [2, 15–17]: oscillators can be nonidentical (different $\boldsymbol{F}_i$), and the network can host $K > 1$ types of directed interactions.

III. MULTILAYER SYSTEMS CONSIDERED

To overcome these challenges, below we propose a *multilayer construction* that defines a large, general subclass of systems within the class given by Eq. (1). We show that any system in this subclass is guaranteed to have a synchronous state, and the stability of that state can be analyzed by applying the MSF framework to the flattened representation of the system. The MSF approach decouples the oscillator dynamics from the network structure, which enables us to draw conclusions about *AISync* for general oscillator dynamics.

In our multilayer system, each node is composed of L identical *subnodes*, belonging to L different layers and connected by a set of *internal sublinks*. The pattern of these internal sublinks is thus part of the node's properties and determines the heterogeneity across nodes. For a pair of connected nodes, the type of the connecting link is determined by the pattern of *external sublinks* between the subnodes of these two nodes. This construction yields a multilayer network [18–22] of subnodes and sublinks with L layers; see Fig. 1(b) for an $L = 2$ example. Note that in general there is more than one possible multilayer network for a given symmetric network. Networks with such layered structure have been used extensively as realistic models of various natural and man-made systems. The class of systems just defined is broader than most classes of systems used in previous studies of synchronization on multilayer networks [23, 24], since the links between two different layers are not constrained to be one-to-one. The underlying hierarchical organization, in which each node decomposes into interacting subnodes, is shared by many physical systems, such as the multiprocessor nodes in modern supercomputers.

Coupling the dynamics of subnodes diffusively in this network, the multilayer system can be described at the subnode level as

$$\dot{\boldsymbol{x}}_\ell^{(i)} = \boldsymbol{f}(\boldsymbol{x}_\ell^{(i)}) + \sum_{i'=1}^{N} \sum_{\ell'=1}^{L} \widetilde{A}_{\ell\ell'}^{(ii')} \left[\boldsymbol{h}(\boldsymbol{x}_{\ell'}^{(i')}) - \boldsymbol{h}(\boldsymbol{x}_\ell^{(i)})\right], \quad (2)$$

where $\boldsymbol{x}_\ell^{(i)} = \boldsymbol{x}_\ell^{(i)}(t)$ is the m-dimensional state vector for subnode ℓ (i.e., in layer ℓ) of node i, the function $\boldsymbol{f}$ determines the dynamics of every isolated subnode, and $\boldsymbol{h}$ is the coupling function common to all sublinks. Here, for all links of a given type between different nodes, the corresponding coupling matrix $\widetilde{A}^{(ii')} := (\widetilde{A}_{\ell\ell'}^{(ii')})$, $i \neq i'$, is the same and encodes the subnode connection pattern for that link type. In contrast, the subnode connection pattern within each node i is encoded in the matrix $F^{(i)} := (\widetilde{A}_{\ell\ell'}^{(ii)})$. Since the subnode-to-subnode interactions are diffusive, the synchronous state given by $\boldsymbol{x}_\ell^{(i)}(t) = \boldsymbol{s}(t)$, $\forall i, \ell$ with $\dot{\boldsymbol{s}} = \boldsymbol{f}(\boldsymbol{s})$ is guaranteed to exist. Note that the diffusive coupling among subnodes do not necessarily imply that the node-to-node interactions are diffusive, as intralayer synchronization of the form $\boldsymbol{x}_\ell^{(i)} = \boldsymbol{s}_\ell$ among subnodes is also valid as a state of complete synchronization among all nodes. The interactions among nodes do not vanish in this case due to the existence of external sublink connections among different layers. To summarize, Eq. (2) describes a general class of multilayer models of symmetric networks that admit a state corresponding to complete synchronization, $\boldsymbol{X}_i(t) = \boldsymbol{S}(t)$, $\forall i$, when written in the form of Eq. (1) (see Appendix B for details).

IV. ESTABLISHING *AISync*

To facilitate the stability analysis required to establish *AISync*, we flatten the multilayer network representation into a single layer (see Fig. 1(c) for an example). We use $\widetilde{A} = (\widetilde{A}_{jj'})$ to denote the adjacency matrix that encodes the structure of the resulting monolayer network (see Fig. 1(d) for an example). This matrix has a block structure in which the matrices $F^{(i)}$ appearing on the diagonal blocks characterize node properties, while $\widetilde{A}^{(ii')}$ appearing on the off-diagonal blocks reflect the link types. Since subnodes and sublinks are identical, we can directly apply the MSF analysis [25] to the monolayer network and obtain the stability function $\psi(\lambda)$ (see Appendix C for details). The maximum transverse Lyapunov exponent (MTLE) of the synchronous state is then computed as $\Psi := \max_{2\le j\le n} \psi(\lambda_j)$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the corresponding Laplacian matrix $\widetilde{L} := (\widetilde{L}_{jj'})$, defined as $\widetilde{L}_{jj'} := \delta_{jj'} \sum_{k=1}^{n} \widetilde{A}_{jk} - \widetilde{A}_{jj'}$, where $\delta_{jj'}$ is the Kronecker delta function. Here, λ_1 is the identically zero eigenvalue, which is excluded in the definition of Ψ for corresponding to a mode of perturbation that does not affect synchronization. Thus, the synchronous state is asymptotically stable if $\Psi < 0$ and unstable if $\Psi > 0$.

To establish *AISync* for our multilayer system, we first verify that all homogeneous systems have $\Psi > 0$ (i.e., synchronous state $\boldsymbol{x}_\ell^{(i)} = \boldsymbol{s}$, $\forall i, \ell$, is unstable), and check numerically that all other synchronous states $\boldsymbol{x}_\ell^{(i)} = \boldsymbol{s}_\ell$, $\forall i, \ell$, are also unstable. This establishes condition (C1). We then find a heterogeneous system with $\Psi < 0$, which establishes condition (C2). This procedure is detailed in Appendix D.

In the case of linear $\boldsymbol{f}$ and $\boldsymbol{h}$, which is widely used to study consensus dynamics and encompasses a variety of nontrivial stability regions [26], the problem of verifying *AISync* is fully solvable. To see this, we first note that in this case the stability function $\psi(\lambda)$ determines the (common) stability of *all* completely synchronous states of the form $\boldsymbol{x}_\ell^{(i)} = \boldsymbol{s}_\ell$, $\forall i, \ell$, where the subnode states $\boldsymbol{s}_\ell$ can in general be different for different ℓ. Next, for a given (homogeneous or heterogeneous) system, we sort its Laplacian eigenvalues into two groups: $\lambda_1, \ldots, \lambda_{j^*}$, corresponding only to those perturbations parallel to the synchronization manifold, and $\lambda_{j^*+1}, \ldots, \lambda_n$, corresponding to perturbations that are transverse to the manifold and thus destroy synchronization. The stability (of all completely synchronous states) is then determined by $\Psi' := \max_{j^*<j\le n} \psi(\lambda_j)$, noting that both j^* and λ_j generally depend on the network structure. This leads to the following solution for the *AISync* conditions: $\Psi' \ge 0$ for all homogeneous systems and $\Psi' < 0$ for some heterogeneous system (where we include $\Psi' = 0$ in the first condition because $\Psi' = 0$ for linear system would exclude asymptotically stable synchronization).

V. EXAMPLES OF *AISync*

A. Consensus dynamics

Here we establish *AISync* for the system with the symmetric network structure shown in Fig. 2, in which the subnodes follow the consensus dynamics used in Ref. [26]:

$$\dot{\boldsymbol{x}}_i = D\boldsymbol{f}\,\boldsymbol{x}_i - \sum_j \widetilde{L}_{ij}\, D\boldsymbol{h}\,\boldsymbol{x}_j, \tag{3}$$

where

$$D\boldsymbol{f} = \begin{pmatrix} -2 & 2 & -1 & 2 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -3 & 4 \\ 0 & 0 & -1 & 1 \end{pmatrix} \quad D\boldsymbol{h} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}. \tag{4}$$

This leads to the stability region $\psi(\lambda) < 0$ shown in Fig. 2(c), defined by

$$x(x+3) - y^2 - (2x+3)^2 y^2 > 0, \tag{5}$$

where x and y denote the real and imaginary parts of λ, respectively.

For $L = 2$ there are only two possible homogeneous systems, associated with the two possible directions of the internal sublink in each node. The homogeneous system in Fig. 2(a) has Laplacian eigenvalues $\lambda_1 = 0$, $\lambda_2 = 2$, $\lambda_{3,4} \approx 0.5 \pm 0.866i$, and $\lambda_{5,6} \approx 1.5 \pm 0.866i$, where λ_1 and λ_2 correspond to the perturbations parallel to the synchronization manifold and $\lambda_3, \ldots, \lambda_6$ correspond to those in the transverse directions (i.e., $j^* = 2$). Since $\psi(\lambda_j) > 0$ for $j = 3, 4, 5, 6$ [i.e., all these λ_j's fall outside the stability region defined by Eq. (5), as indicated by the red squares in Fig. 2(c)], we have $\Psi' = \max_{2<j\le 6} \psi(\lambda_j) > 0$. The other homogeneous system is not synchronizable since all the single-prime subnodes have no incoming sublink. In contrast, for the heterogeneous system in Fig. 2(b), the Laplacian eigenvalues are $\lambda_1 = 0$, $\lambda_j = 1$ for $1 < j \le 5$, and $\lambda_6 = 2$ (i.e., $j^* = 1$ in this case). As shown by the blue dots in Fig. 2(c), we have $\Psi' = \max_{1<j\le 6} \psi(\lambda_j) < 0$ for this heterogeneous system. We thus see that $\Psi' \ge 0$ (i.e., the synchronous state is not asymptotically stable) for both homogeneous systems and $\Psi' < 0$ (i.e., the synchronous state is asymptotically stable) for a heterogeneous system, establishing *AISync*: the agents can reach consensus only when some of them are different from the others.