Supplementary Notes

Supplementary Note 1: Analysis of the stability landscape around $\boldsymbol{\beta}_=$

Our detailed analysis of the stability landscape around the point $\boldsymbol{\beta}_= = (\beta_=, \ldots, \beta_=)$, $\beta_= = 2\sqrt{\alpha_2}$, is divided into three parts. We first derive a formula for the maximum Lyapunov exponent $\lambda^{\max}$ in terms of the coefficients of the characteristic polynomial in a neighborhood of $\boldsymbol{\beta}_=$, which allows us to express a necessary condition for $\lambda^{\max}$ to decrease along a given path (Sec. 1). We then derive equations that relate these coefficients to $\boldsymbol{\beta}$ (Sec. 2). Finally, we apply the Implicit Function Theorem (IFT) to these equations and show that, for a generic system, any descending path must be tangent to a system-specific hyperplane in the $\boldsymbol{\beta}$-space (Sec. 3). In that section, we also derive an explicit formula for $\lambda^{\max}$ in terms of the parameterization of the path when the path is transverse to the hyperplane. Two example cases, $n = 2$ and $n = 3$, are presented for illustration in Sec. 4.

1 Maximum Lyapunov exponent $\boldsymbol{\lambda}^{\max}$ on arbitrary path through $\boldsymbol{\beta}_=$

In the main text, we introduced the $2n \times 2n$ Jacobian matrix $\mathbf{J}$ that characterizes the linearized dynamics of the generators. Recall that

$$\mathbf{J} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{P} & -\mathbf{B} \end{pmatrix}, \tag{S1}$$

where $\mathbf{O}$ and $\mathbf{I}$ denote the null and the identity matrix of size n, respectively, the $n \times n$ matrix $\mathbf{P} = (P_{ik})$ is given by Eq. (3) in the main text, and $\mathbf{B}$ is the diagonal matrix of elements β_i. We define

$$\overline{\mathbf{J}} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{P}/\beta_=^2 & -\mathbf{B}/\beta_= \end{pmatrix}, \tag{S2}$$

assuming $\beta_= > 0$ (or, equivalently, $\alpha_2 > 0$). Note that ν is an eigenvalue of $\overline{\mathbf{J}}$ if and only if $\beta_=\nu$ is an eigenvalue of $\mathbf{J}$, since the characteristic polynomial of $\overline{\mathbf{J}}$ can be written as

$$\begin{aligned} \det(\overline{\mathbf{J}} - \nu\mathbf{I}) &= \det(\nu^2\mathbf{I} + \nu\mathbf{B}/\beta_= + \mathbf{P}/\beta_=^2) \\ &= \beta_=^{-2n} \det(\beta_=^2\nu^2\mathbf{I} + \beta_=\nu\mathbf{B} + \mathbf{P}) \\ &= \beta_=^{-2n} \det(\mathbf{J} - \beta_=\nu\mathbf{I}). \end{aligned} \tag{S3}$$

Consider a (possibly curved) path $\boldsymbol{\gamma}$ passing through the point $\boldsymbol{\beta}_=$ in the space of all $\boldsymbol{\beta}$, parametrized by ε through a differentiable vector function $\boldsymbol{\beta} = \boldsymbol{\gamma}(\varepsilon)$ satisfying $\boldsymbol{\gamma}(0) = \boldsymbol{\beta}_=$. We denote the parametrized eigenvalues of $\overline{\mathbf{J}}$ as $\nu_{j\pm} = \nu_{j\pm}(\varepsilon)$, $j = 1, \ldots, n$. The eigenvalues of $\mathbf{P}$ are denoted by $\alpha_1, \ldots, \alpha_n$, among which we have $\alpha_1 = 0$, and we assume that they are all real, distinct, and indexed so that $0 < \alpha_2 < \cdots < \alpha_n$. The assumption that these eigenvalues are distinct will be crucial in the arguments that follow. When $\varepsilon = 0$, we have $\mathbf{B} = \beta_= \mathbf{I}$, so

$$\det(\overline{\mathbf{J}} - \nu\mathbf{I}) = \beta_=^{-2n} \det[\beta_=^2(\nu^2 + \nu)\mathbf{I} + \mathbf{P}], \tag{S4}$$

and this implies that $\beta_=^2(\nu^2 + \nu) = \alpha_j$, or equivalently $\nu = \bigl(-1 \pm \sqrt{1 - \alpha_j/\alpha_2}\bigr)/2$, whenever ν is an eigenvalue of $\overline{\mathbf{J}}$. We thus index $\nu_{j_\pm}(\varepsilon)$ so that

$$\nu_{j_\pm}(\varepsilon) \xrightarrow{\varepsilon \to 0} \nu_{j_\pm}(0) = \frac{-1 \pm \sqrt{1 - \alpha_j/\alpha_2}}{2}, \quad j = 1, \ldots, n. \tag{S5}$$

Note that $\nu_{1_+}(\varepsilon) \to \nu_{1_+}(0) = 0$ and $\nu_{1_-}(\varepsilon) \to \nu_{1_-}(0) = -1$ are not relevant for determining the stability of synchronous states since these eigenvalues are associated with perturbation modes that do not affect synchronization. Thus, the Lyapunov exponent $\lambda^{\max}$ determining the stability is the largest real component among the remaining eigenvalues:

$$\lambda^{\max}(\varepsilon) = \beta_= \cdot \max_{2 \le j \le n} \max\bigl\{Re(\nu_{j_+}(\varepsilon)), Re(\nu_{j_-}(\varepsilon))\bigr\}. \tag{S6}$$

For $\varepsilon = 0$, which corresponds to the point $\boldsymbol{\beta}_=$, it follows from the formula for $\nu_{j_\pm}(0)$ in Eq. (S5) and $\alpha_2 < \alpha_j$, $\forall j \ge 3$, that

$$\lambda^{\max}(0) = \beta_= \cdot Re(\nu_{2_+}(0)) = -\frac{\beta_=}{2} = -\sqrt{\alpha_2} = \lambda_=^{\max}. \tag{S7}$$

Since eigenvalues are continuous functions of the matrix elements and $\boldsymbol{\gamma}$ is a continuous function, $\nu_{j_\pm}(\varepsilon)$ changes with ε continuously. Thus, for $\varepsilon \neq 0$, the eigenvalue $\nu_{2_+}(\varepsilon)$ determines the maximum in Eq. (S6) for sufficiently small ε, and hence we have

$$\lambda^{\max}(\varepsilon) = \beta_= \cdot Re(\nu_{2_+}(\varepsilon)) = \beta_= \cdot Re\left(-\frac{c_2}{2} + \frac{1}{2}\sqrt{c_2^2 - 4d_2}\right), \tag{S8}$$

where we factored the characteristic polynomial of $\overline{\mathbf{J}}$ into quadratic factors as

$$\det(\overline{\mathbf{J}} - \nu\mathbf{I}) = \bigl(\nu^2 + c_1\nu\bigr)\bigl(\nu^2 + c_2\nu + d_2\bigr)\cdots\bigl(\nu^2 + c_n\nu + d_n\bigr), \tag{S9}$$

(noting that $d_1 = 0$ always holds because $\nu = 0$ is an eigenvalue of $\overline{J}$ associated with $\alpha_1 = 0$). Equations (S8) and (S9) imply that $\lambda^{\max}$ can be viewed as a function of c_2 and d_2 and thus defines a landscape over the (c_2, d_2)-plane, while the path $\boldsymbol{\gamma}(\varepsilon)$ in the $\boldsymbol{\beta}$-space corresponds a path on the (c_2, d_2)-plane defined by the functions $c_2 = c_2(\varepsilon)$ and $d_2 = d_2(\varepsilon)$. Defining $f(c_2, d_2) \equiv Re\bigl(-c_2 + \sqrt{c_2^2 - 4d_2}\bigr)/2$, we see that the condition for $\lambda^{\max}$ to be decreasing along the path $\boldsymbol{\gamma}(\varepsilon)$ starting at $\boldsymbol{\beta}_=$ in the $\boldsymbol{\beta}$-space is equivalent to the condition that the corresponding path $(c_2(\varepsilon), d_2(\varepsilon))$ starting at $(c_2, d_2) = (1, 1/4)$ immediately enters the following triangular region of the (c_2, d_2)-plane:

$$\bigl\{(c_2, d_2) :\; f(c_2, d_2) < -1/2\bigr\} = \bigl\{(c_2, d_2) :\; 2(d_2 - 1/4) > c_2 - 1 > 0\bigr\}. \tag{S10}$$

The latter condition implies that the derivatives of $c_2(\varepsilon)$ and $d_2(\varepsilon)$ at $\varepsilon = 0$ satisfy $2d_2'(0) \geq c_2'(0)$. We may assume $c_2'(0) \geq 0$ without loss of generality. (If not, we simply need to make the change of variable, $\varepsilon \to -\varepsilon$.) In Sec. 3, we show that, for a generic choice of the matrix $\mathbf{P}$, we have $d_2'(0) = 0$. This implies that, if $\lambda^{\max}$ decreases along the path, we have $c_2'(0) = 0$. In general, if $c_2'(0) = d_2'(0) = 0$, the nonlinearity of $c_2(\varepsilon)$ and $d_2(\varepsilon)$ determines whether $\lambda^{\max}$ decreases along the path in either direction. We further show that the condition $c_2'(0) = 0$ translates to the path $\boldsymbol{\gamma}$ being tangent to a specific hyperplane, which we denote by L, at the point $\boldsymbol{\beta}_=$.

2 Equations relating coefficients c_i and d_i to parameter ε

We now derive equations that define the coefficients c_i and d_i implicitly as functions $c_i = c_i(\varepsilon)$ and $d_i = d_i(\varepsilon)$. We first rewrite the characteristic polynomial of $\overline{\mathbf{J}}$ as

$$\det(\overline{\mathbf{J}} - \nu\mathbf{I}) = \det(\nu^2\mathbf{I} + \nu\mathbf{B}/\beta_= + \mathbf{P}/\beta_=^2) = \det(\nu^2\mathbf{I} + \nu\overline{\mathbf{B}} + \mathbf{D}), \tag{S11}$$

where we used a similarity transformation $\mathbf{Q}^{-1}(\mathbf{P}/\beta_=^2)\mathbf{Q} = \mathbf{D}$ and $\mathbf{B}/\beta_=$ as $\mathbf{Q}^{-1}(\mathbf{B}/\beta_=)\mathbf{Q} = \overline{\mathbf{B}}$ (based on the diagonalization of the matrix $\mathbf{P}$), and $\mathbf{D}$ is the diagonal matrix with diagonal elements $\overline{\alpha}_i \equiv \alpha_i/\beta_=^2 = \frac{1}{4}\alpha_i/\alpha_2$. Along the path $\boldsymbol{\gamma}(\varepsilon) = (\gamma_1(\varepsilon), \ldots, \gamma_n(\varepsilon))$, the components of the matrix $\overline{\mathbf{B}}$ can be expressed as

$$\overline{B}_{ij}(\varepsilon) = \sum_{\ell=1}^{n} u_{i\ell} v_{j\ell} \gamma_\ell(\varepsilon)/\beta_=, \tag{S12}$$

where $u_{i\ell}$ and $v_{i\ell}$ are the ℓth component of the left and right eigenvectors of $\mathbf{P}$ associated with the eigenvalue α_i, respectively. We now use the definition of the determinant,

$$\det(\mathbf{A}) \equiv \sum_{\sigma} \mathrm{sign}(\sigma) \prod_{i=1}^{n} A_{i\sigma(i)} = \sum_{\sigma} \mathrm{sign}(\sigma) \cdot A_{1\sigma(1)} \cdots A_{n\sigma(n)}, \tag{S13}$$

where the summation $\sum_\sigma$ is taken over all possible permutations σ of indices, and $\mathrm{sign}(\sigma)$ is the sign of permutation σ (i.e., $\mathrm{sign}(\sigma) = 1$ when σ is an even permutation and $\mathrm{sign}(\sigma) = -1$ when σ is an odd permutation). With this definition, we can write the coefficient of the ν^k term (for $k = 1, \ldots, 2n-1$; no constant term corresponding to $k = 0$, since $\nu = 0$ is an eigenvalue) of the polynomial in Eq. (S11) as

$$\sum_{\sigma} \mathrm{sign}(\sigma) \sum_{\{k_i\}} \prod_{i=1}^{n} E_{i\sigma(i)}^{(k_i)} \cdot \chi\big(\textstyle\sum_i k_i = k\big), \tag{S14}$$

where the summation $\sum_{\{k_i\}}$ is taken over all possible combinations of $k_i = 0, 1, 2$, $i = 1, \ldots, n$. The function χ is an indicator function defined by $\chi\big(\sum_i k_i = k\big) = 1$ if $\sum_i k_i = k$ and