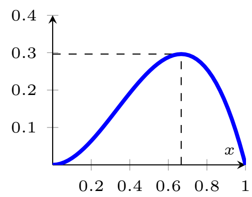


Figure 28: The graph of $y = 2x^2(1-x)$, where $x \equiv |v|^2$.

corresponds to a local minimum of $\tilde{\sigma}^2$. Thus, if $\tilde{\mu} < \frac{3\sqrt{3}}{2\sqrt{2}}$, the maximum of $\tilde{\sigma}^2$ is achieved at the end of the interval $|v|^2 = \frac{1}{2}$. If $\tilde{\mu} < \frac{3\sqrt{3}}{2\sqrt{2}}$, the maximum of $\tilde{\sigma}^2$ may be achieved at the larger root of (B-19) or $x = \frac{1}{2}$. The value of $\tilde{\sigma}^2$ at $x = \frac{1}{2}$ is larger than that at the larger root of Eq. (B-19) for $\tilde{\mu} \in (\frac{3\sqrt{3}}{2\sqrt{2}}, \frac{4\sqrt{2}}{3})$, and the other way around for $\tilde{\mu} > \frac{4\sqrt{2}}{3}$ – see Fig. 11. Noting that $\tilde{\mu} > \frac{4\sqrt{2}}{3}$ corresponds to $x \equiv |v|^2 = \frac{3}{4}$, we conclude that the region spanned by ellipses (B-16) is bounded by the union of

- the parametric curve in Eq. (B-13), where the expression for $\tilde{\mu}$ comes from Eq. (B-19) and the expression for $\tilde{\sigma}$ is obtained by plugging in Eq. (B-19) into Eq. (B-16), and
- the ellipse $|v|^2 = \frac{1}{2}$.

Now it remains to check which part of the region spanned by ellipses (B-16) with $|v|^2 \geq \frac{1}{2}$ satisfies stability condition (B-17). Since we are considering the case $|v|^2 \in (\frac{1}{2}, 1)$, $1 - |v|^2 > 0$ and $1 - 3|v|^2 < 0$. Hence, we rewrite Eq. (B-17) as

$$\tilde{\sigma}^2 > \tilde{\mu}^2(1 - |v|^2)(3|v|^2 - 1). \tag{B-20}$$

Plugging Eq. (B-20) into Eq. (B-16) and dividing the result by $\tilde{\mu}^2$, we obtain

$$2|v|^4(1 - |v|^2) < \frac{1}{\tilde{\mu}^2}, \quad |v|^2 \in (\tfrac{1}{2}, 1), \tag{B-21}$$

cf. Eq. (B-19). Hence, we have the following situation:

- If $\tilde{\mu} < \frac{3\sqrt{3}}{2\sqrt{2}}$, Eq. (B-21) holds for all $|v|^2 \in (\frac{1}{2}, 1)$. Hence, the intersection of the region inside the ellipse (B-16) with $|v|^2 = \frac{1}{2}$ and the strips $0 < \tilde{\mu} < \frac{3\sqrt{3}}{2\sqrt{2}}$, $\tilde{\sigma}^2 > 1$, has an asymptotically stable periodic solution.
- If $\tilde{\mu} > \frac{3\sqrt{3}}{2\sqrt{2}}$, inequality (B-21) holds for $\frac{1}{2} < |v|^2 < x_1$ and $x_2 < |v|^2 < 1$, where $x_1 < x_2$ are the two positive roots of the cubic equation $2x^2(1-x) = \tilde{\mu}^{-2}$. The analysis of this cubic equation and Fig. 28 show that $x_1 \in (0, \frac{2}{3})$ and $x_2 \in (\frac{2}{3}, 1)$.
  The interval $\frac{1}{2} < |v|^2 < x_1$ defines the region between the ellipse Eq. (B-16) with $|v|^2 = \frac{1}{2}$, i.e., $\frac{\tilde{\mu}^2}{8} + \frac{\tilde{\sigma}^2}{2} = 1$, and the contour

$$|v|^2 \in \left(\frac{1}{2}, \frac{2}{3}\right), \quad \tilde{\sigma} = \sqrt{\frac{3|v|^2 - 1}{2|v|^4}}, \quad \tilde{\mu} = \frac{1}{|v|^2\sqrt{2(1 - |v|^2)}}. \tag{B-22}$$

  Note that $|v|^2 = \frac{1}{2}$ defines the point $(\tilde{\sigma} = 1, \tilde{\mu} = 2)$. The interval $x_2 < |v|^2 < 1$ defines the region between the line $|v|^2 = 1$ and the contour

$$|v|^2 \in \left(\frac{2}{3}, 1\right), \quad \tilde{\sigma} = \sqrt{\frac{3|v|^2 - 1}{2|v|^4}}, \quad \tilde{\mu} = \frac{1}{|v|^2\sqrt{2(1 - |v|^2)}}. \tag{B-23}$$

  The union of these two regions is bounded by the curves in Eqs. (B-22) and (B-23) and the line $\tilde{\sigma}^2 = 1$. Finally, as we have shown above, the ellipse (B-16) with $|v|^2 = \frac{1}{2}$, $\frac{\tilde{\mu}^2}{8} + \frac{\tilde{\sigma}^2}{2} = 1$, intersects with the curve (B-23) at $|v|^2 = \frac{3}{4}$, $\tilde{\sigma} = \frac{\sqrt{10}}{3}$, $\tilde{\mu} = \frac{4\sqrt{2}}{3}$.

Thus, an asymptotically stable equilibrium of ODE (22) exists in the union of these two regions and the region with $\tilde{\mu} \le \frac{3\sqrt{3}}{2\sqrt{2}}$, i.e., in the area shaded yellow, pink, and green in Fig. 11. This completes the proof of Statement (3).

- To identify the region where three periodic solutions exist, we return to the consideration of the solutions to Eq. (B-10). We focus on the case $\tilde{\sigma} \ge 0$, as the sought region is symmetric with respect to the $\tilde{\mu}$-axis. For convenience, we denote $|v|^2$ by $x$. We seek a set of critical values of $(\tilde{\sigma}, \tilde{\mu})$, where this equation has exactly two solutions. In this case, the graph of $y = x$ must be tangent to the graph of $y = [\tilde{\mu}^2(1-x)^2 + \tilde{\sigma}^2]^{-1}$ at one of these solutions. Hence, we must solve the system

$$\begin{aligned} x &= \frac{1}{\tilde{\mu}^2(1-x)^2 + \tilde{\sigma}^2}, \\ 1 &= \frac{2\tilde{\mu}^2(1-x)}{(\tilde{\mu}^2(1-x)^2 + \tilde{\sigma}^2)^2}. \end{aligned} \tag{B-24}$$

Substituting the first equation into the second one, we obtain the familiar equation $2\tilde{\mu}^2 x^2(1-x) = 1$. It has a solution if $\tilde{\mu} \ge \frac{3\sqrt{3}}{2\sqrt{2}}$. Substituting $\tilde{\mu}^2 = [2x^2(1-x)]^{-1}]$ to the first equation in Eq. (B-24), we find

$$\tilde{\sigma}^2 = \frac{3x-1}{2x^2}. \tag{B-25}$$

Since $\tilde{\sigma} \ge 0$, we find $x \ge \frac{1}{3}$. The graph of $y = x$ is below the graph of the Witch of Agnesi between two additional solutions to Eq. (B-24) in the interval $x \in (0,1)$. This implies that the region in the $(\tilde{\sigma}, \tilde{\mu})$-space where three solutions exist lies between the curve (B-23) and the curve

$$|v|^2 \in \left(\frac{1}{3}, \frac{2}{3}\right), \quad \tilde{\sigma} = \sqrt{\frac{3|v|^2 - 1}{2|v|^4}}, \quad \tilde{\mu} = \frac{1}{|v|^2\sqrt{2(1-|v|^2)}}. \tag{B-26}$$

Now it remains to check the stability of these three solutions. The proof of Statement (3) reveals that asymptotically stable equilibria lie on ellipses of the form (B-16) with

(a) $|v|^2 \ge 1$, or

(b) $\frac{1}{2} < |v|^2 \le \frac{2}{3}$ and with $\tilde{\mu}$ increasing until they touch the $\det J = 0$ curve in Eq. (B-26) from above, or

(c) $\frac{2}{3} < |v|^2 < 1$ and with $\tilde{\mu}$ increasing until they touch the curve $\det J = 0$ in Eq. (B-23) from below.

Ellipse family (a) does not intersect with families (b) and (c) and foliates the strip $|\tilde{\sigma}| \le 1$. Ellipse family (b) foliates the region under the ellipse $\frac{\tilde{\mu}^2}{8} + \frac{\tilde{\sigma}^2}{2} = 1$ with $|v|^2 = \frac{1}{2}$ and above curve (B-26). Ellipse family (c) foliates the region between the the ellipse with $|v|^2 = \frac{2}{3}$, the line $\tilde{\sigma} = 1$, and curve (B-23). Families (b) and (c) overlap over the region shaded dirty pink in Fig. 11. The tip of the bistability region corresponds to $|v|^2 = \frac{2}{3}$, $\tilde{\sigma} = \frac{3}{2\sqrt{2}}$, $\tilde{\mu} = \frac{3\sqrt{3}}{2\sqrt{2}}$. Curves (B-26) and (B-23) envelope families (b) and (c). This completes the proof of Statement (4).

□

# C Calculations for Section 4.4

Eq. (61c) yields

$$x = \frac{2(\mu + \varepsilon + \gamma\sigma)}{3(1+\gamma^2)}. \tag{C-27}$$

Plugging Eq. (C-27) into Eq. (61b) gives:

$$\frac{4}{3}\frac{(\mu+\varepsilon+\gamma\sigma)^2}{1+\gamma^2} = (\mu+\varepsilon)^2+\sigma^2. \tag{C-28}$$

Plugging Eq. (C-27) into Eq. (61a) and using Eq. (C-28) results in

$$\frac{8}{27}\frac{(\mu+\varepsilon+\gamma\sigma)^3}{(1+\gamma^2)^2} = \lambda^2\mu. \tag{C-29}$$

System (C-27), (C-28), (C-29) can be solved for $\varepsilon$, $\sigma$, and $|u|^2 \equiv x$ in terms of $\lambda$, $\mu$, and $\gamma$ as follows. We express $\mu+\varepsilon+\gamma\sigma$ from Eq. (C-29) as

$$\mu+\varepsilon+\gamma\sigma = \frac{3}{2}\left(\lambda^2\mu(1+\gamma^2)^2\right)^{1/3} =: C(\lambda). \tag{C-30}$$

Then we denote $\mu+\varepsilon$ by $X$, express $\gamma\sigma$ from Eq. (C-30) as $\gamma\sigma = C(\lambda)-X$, multiply Eq. (C-28) by $\gamma^2$, and obtain a quadratic equation for $X$:

$$X^2(1+\gamma^2) - 2XC(\lambda) + C^2(\lambda)\left(1-\frac{4}{3}\frac{\gamma^2}{1+\gamma^2}\right) = 0. \tag{C-31}$$

Its solution is

$$\mu+\varepsilon \equiv X = C(\lambda)\frac{1\pm\frac{1}{\sqrt{3}}\gamma}{1+\gamma^2} \equiv \frac{3}{2}\frac{1\pm\frac{1}{\sqrt{3}}\gamma}{(1+\gamma^2)^{1/3}}\lambda^{2/3}\mu^{1/3}. \tag{C-32}$$

Then, we get

$$\varepsilon = \frac{3}{2}\frac{1\pm\frac{1}{\sqrt{3}}\gamma}{(1+\gamma^2)^{1/3}}\lambda^{2/3}\mu^{1/3}-\mu, \quad \sigma = \frac{3}{2}\frac{\gamma\mp\frac{1}{\sqrt{3}}}{(1+\gamma^2)^{1/3}}\lambda^{2/3}\mu^{1/3}, \quad |u|^2 = \lambda^{2/3}\mu^{1/3}(1+\gamma^2)^{-1/3}. \tag{C-33}$$

# D Additional equations for system (46)

The ODE system for the real and complex parts of the new variable $u = u_R + iu_I$ defined as $z_2 = ue^{i\omega t}$, where $z_2$ is the state variable of the second state of system (46) is

$$\begin{aligned} u_R &= \left(\mu+\varepsilon-|u|^2\right)u_R - \left(\sigma-\gamma|u|^2\right)u_I - \lambda\sqrt{\mu}, \\ u_I &= \left(\mu+\varepsilon-|u|^2\right)u_I + \left(\sigma-\gamma|u|^2\right)u_R. \end{aligned} \tag{D-34}$$

The Jacobian matrix of Eq. (D-34) is

$$J = \begin{bmatrix} (\mu+\varepsilon-|u|^2)-2u_R^2+2\gamma u_R u_I & -2u_R u_I-(\sigma-\gamma|u|^2)+2\gamma u_I^2 \\ (\sigma-\gamma|u|^2)-2\gamma u_R^2-2u_R u_I & (\mu+\varepsilon-|u|^2)-2u_I^2-2\gamma u_I u_R \end{bmatrix}. \tag{D-35}$$

Its determinant and trace are:

$$\det J = \left(\mu+\varepsilon-|u|^2\right)\left(\mu+\varepsilon-3|u|^2\right)+\left(\sigma-\gamma|u|^2\right)\left(\sigma-3\gamma|u|^2\right), \tag{D-36}$$
$$\operatorname{tr} J = 2\left(\mu+\varepsilon-2|u|^2\right). \tag{D-37}$$

An equilibrium of ODE (D-34) is asymptotically stable if $\det J > 0$ and $\operatorname{tr} J < 0$.