Statement 5): By statement 1), all solutions of system (17) converge to the set of equilibria, which equals the set of critical points of the potential $U(\theta)$. By the definition of $\mathbb{S}^1$-synchronizing graphs, the phase-synchronized equilibrium manifold $\overline{\mathrm{Arc}}_n(0)$ is the only stable equilibrium set, and all others are unstable. Hence, for all initial condition $\theta(0) \in \mathbb{T}^n$, which are not on the stable manifolds of unstable equilibria, the corresponding solution $\theta(t)$ will reach the phase-synchronized equilibrium manifold $\overline{\mathrm{Arc}}_n(0)$. ■

Remark 2: (**Control-theoretic perspective on synchronization)** As established in Theorem 4.3, the set of phase-synchronized solutions $\overline{\mathrm{Arc}}_n(0)$ of the coupled oscillator model (1) is locally stable provided that all natural frequencies are identical. For non-uniform (but sufficiently identical) natural frequencies, phase synchronization is not possible but a certain degree of phase cohesiveness can still be achieved. Hence, the coupled oscillator model (1) can be regarded as an exponentially stable system subject to the disturbance $\omega \in \mathbf{1}_n^\perp$, and classic control-theoretic concepts such as input-to-state stability, practical stability, and ultimate boundedness [139] or their incremental versions [141] can be used to study synchronization. In control-theoretic terminology, synchronization and phase cohesiveness can then also be described as "practical phase synchronization". Compared to prototypical nonlinear control examples, various additional challenges arise in the analysis of the coupled oscillator model (1) due to the bounded and non-monotone sinusoidal coupling and the compact state space $\mathbb{T}^n$ containing numerous equilibria; see the analysis approaches in Section IV and [64], [74], [95].□

C. Phase Balancing

In general, only few results are known about the phase balancing problem. This asymmetry is partially caused by the fact that phase synchrony is required in more applications than phase balancing. Moreover, the phase-synchronized set $\overline{\mathrm{Arc}}_n(0)$ admits a very simple geometric characterization, whereas the phase-balanced set Bal_n has a complicated structure consisting of numerous disjoint subsets. The number of these subsets grows with the number of nodes n in a combinatorial fashion.

Consider the coupled oscillator model (17) with identical natural frequencies. By inverting the direction of time, we get

$$\dot{\theta}_i = \sum\nolimits_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)\,, \qquad i \in \{1, \dots, n\}\,. \quad (20)$$

In the following, we say that the interaction graph $G(\mathcal{V}, \mathcal{E}, A)$ is *circulant* if the adjacency matrix $A = A^T$ is a circulant matrix. Circulant graphs are highly symmetric graphs including complete graphs, bipartite graphs, and ring graphs.[3] For circulant and uniformly weighted graphs, the coupled oscillator model (20) achieves phase balancing. The following theorem summarizes different results, which were originally presented in [28], [29].

Theorem 4.4: **(Phase balancing)** Consider the coupled oscillator model (20) with a connected, uniformly weighted, and circulant graph $G(\mathcal{V}, \mathcal{E}, A)$. The following statements hold:

1) **Local phase balancing:** The phase-balanced set Bal_n is locally asymptotically stable; and
2) **Almost global stability:** If the graph $G(\mathcal{V}, \mathcal{E}, A)$ is complete, then the region of attraction of the stable phase-balanced set Bal_n is almost all of $\mathbb{T}^n$.

The proof of Theorem 4.4 follows a similar reasoning as the proof of Theorem 4.3: convergence is established by potential function arguments and local (in)stability of equilibria by Jacobian arguments. We omit the proof here and refer to [28, Theorem 1] and [29, Theorem 2] for details.

For general connected graphs, the conclusions of Theorem 4.4 are not true. As a remedy to achieve locally stable and globally attractive phase balancing, higher order models need to be considered, see the models proposed in [29], [56].

D. Synchronization in Complete Networks

For a complete coupling graph with uniform weights $a_{ij} = K/n$, where $K > 0$ is the *coupling gain*, the coupled oscillator model (1) reduces to the celebrated Kuramoto model

$$\dot{\theta}_i = \omega_i - \frac{K}{n} \sum\nolimits_{j=1}^{n} \sin(\theta_i - \theta_j)\,, \qquad i \in \{1, \dots, n\}\,. \quad (21)$$

By means of the order parameter $re^{\mathrm{i}\psi} = \frac{1}{n}\sum_{j=1}^{n} e^{\mathrm{i}\theta_j}$, the Kuramoto model (21) can be rewritten in the insightful form

$$\dot{\theta}_i = \omega_i - Kr\sin(\theta_i - \psi)\,, \quad i \in \{1, \dots, n\}\,. \quad (22)$$

Equation (22) gives the intuition that the oscillators synchronize by coupling to a mean field represented by the order parameter $re^{\mathrm{i}\psi}$. Intuitively, for small coupling strength K each oscillator rotates with its natural frequency ω_i, whereas for large coupling strength K all angles $\theta_i(t)$ will be entrained by the mean field $re^{\mathrm{i}\psi}$ and the oscillators synchronize. The threshold from incoherence to synchronization occurs for some critical coupling K_{critical}. This phase transition has been the source of numerous investigations starting with Kuramoto's analysis [5], [6]. Various necessary, sufficient, implicit, and explicit estimates of the critical coupling strength K_{critical} for both the on-set as well as the ultimate stage of synchronization have been proposed [5]–[8], [28], [52], [53], [64], [74], [75], [82]–[87], [95]–[97], [100], [103]–[107], [110], and we refer to [74] for a comprehensive overview.

The mean field approach to the equations (22) can be made mathematically rigorous by a time-scale separation [96] or in the continuum limit as the number of oscillators tends to infinity and the natural frequencies ω are sampled from a distribution function $g : \mathbb{R} \to \mathbb{R}_{\geq 0}$. In the continuum limit and for a symmetric, continuous, and unimodal distribution $g(\omega)$, Kuramoto himself showed in an insightful and ingenuous analysis [5], [6] that the incoherent state (a uniform distribution of the oscillators on the unit circle $\mathbb{S}^1$) supercritically bifurcates for the critical coupling strength

$$K_{\text{critical}} = \frac{2}{\pi g(0)}\,. \quad (23)$$

[3]Further info on circulant graphs and a gallery can be found at `http://mathworld.wolfram.com/CirculantGraph.html`.

In [8], [87], [104], it was found that the bipolar (bimodal double-delta) distribution (respectively the uniform distribution) yield the largest (respectively smallest) threshold K_{critical} over all distributions $g(\omega)$ with bounded support. We refer [7], [8] for further references and to [88], [109]–[111] for recent contributions on the continuum limit model.

In the finite-dimensional case, the necessary synchronization condition (15) gives a lower bound for K_{critical} as

$$K \geq \frac{n}{2(n-1)} \cdot (\omega_{\max} - \omega_{\min})\,. \tag{24}$$

Three recent articles [84]–[86] independently derived a set of implicit consistency equations for the *exact* critical coupling strength K_{critical} for which synchronized solutions exist. Verwoerd and Mason provided the following implicit formulae to compute K_{critical} [86, Theorem 3]:

$$\begin{aligned} &K_{\text{critical}} = nu^* / \sum\nolimits_{i=1}^{n} \sqrt{1-(\Omega_i/u^*)^2}\,, \\ &2\sum\nolimits_{i=1}^{n} \sqrt{1-(\Omega_i/u^*)^2} = \sum\nolimits_{i=1}^{n} 1/\sqrt{1-(\Omega_i/u^*)^2}, \end{aligned} \tag{25}$$

where $\Omega_i = \omega_i - \omega_{\text{sync}}$ and $u^* \in [\|\Omega\|_\infty, 2\|\Omega\|_\infty]$. The implicit formulae (25) can also be extended to bipartite graphs [82]. A local stability analysis is carried out in [84], [85].

From the point of analyzing or designing a sufficiently strong coupling, the exact formulae (25) have three drawbacks. First, they are implicit and thus not suited for performance or robustness estimates in case of additional coupling strength for a given $K > K_{\text{critical}}$. Second, the corresponding region of attraction of a synchronized solution is unknown. Third and finally, the particular natural frequencies ω_i are typically time-varying, uncertain, or even unknown in the applications listed in Section I. In this case, the exact value of K_{critical} needs to be estimated in continuous time, or a conservatively strong coupling $K \gg K_{\text{critical}}$ has to be chosen.

The following theorem states an explicit bound on the critical coupling strength together with performance estimates, convergence rates, and a guaranteed semi-global region of attraction for synchronization. This bound is tight and thus necessary and sufficient when considering arbitrary distributions of the natural frequencies with compact support. The result has been originally presented in [74, Theorem 4.1].

Theorem 4.5: **(Synchronization in the Kuramoto model)** Consider the Kuramoto model (21) with natural frequencies $\omega = (\omega_1, \dots, \omega_n)$ and coupling strength K. The following three statements are equivalent:

(i) the coupling strength K is larger than the maximum non-uniformity among the natural frequencies, that is,

$$K > K_{\text{critical}} \triangleq \omega_{\max} - \omega_{\min}\,; \tag{26}$$

(ii) there exists an arc length $\gamma_{\max} \in\,]\pi/2, \pi]$ such that the Kuramoto model (21) synchronizes exponentially for all possible distributions of the natural frequencies ω_i supported on the compact interval $[\omega_{\min}, \omega_{\max}]$ and for all initial phases $\theta(0) \in \text{Arc}_n(\gamma_{\max})$; and

(iii) there exists an arc length $\gamma_{\min} \in [0, \pi/2[$ such that the Kuramoto model (21) has a locally exponentially stable synchronization manifold in $\overline{\text{Arc}}_n(\gamma_{\min})$ for all possible distributions of the natural frequencies ω_i supported on the compact interval $[\omega_{\min}, \omega_{\max}]$.

If the three equivalent conditions (i), (ii), and (iii) hold, then the ratio K_{critical}/K and the arc lengths $\gamma_{\min} \in [0, \pi/2[$ and $\gamma_{\max} \in\,]\pi/2, \pi]$ are related uniquely via $\sin(\gamma_{\min}) = \sin(\gamma_{\max}) = K_{\text{critical}}/K$, and the following statements hold:

1) **phase cohesiveness:** the set $\overline{\text{Arc}}_n(\gamma) \subseteq \bar{\Delta}_G(\gamma)$ is positively invariant for every $\gamma \in [\gamma_{\min}, \gamma_{\max}]$, and each trajectory starting in $\text{Arc}_n(\gamma_{\max})$ approaches asymptotically $\overline{\text{Arc}}_n(\gamma_{\min})$;
2) **frequency synchronization:** the asymptotic synchronization frequency is the average frequency $\omega_{\text{sync}} = \frac{1}{n}\sum_{i=1}^{n} \omega_i$, and, given phase cohesiveness in $\overline{\text{Arc}}_n(\gamma)$ for some fixed $\gamma < \pi/2$, the exponential synchronization rate is no worse than $\lambda_{\text{K}} = -K\cos(\gamma)$; and
3) **order parameter:** the asymptotic value of the magnitude of the order parameter, denoted by $r_\infty \triangleq \lim_{t\to\infty} \frac{1}{n}|\sum_{j=1}^{n} e^{\mathrm{i}\theta_j(t)}|$, is bounded as

$$1 \geq r_\infty \geq \cos\Bigl(\frac{\gamma_{\min}}{2}\Bigr) = \sqrt{\frac{1+\sqrt{1-(K_{\text{critical}}/K)^2}}{2}}\,.$$

Proof: In the following, we sketch the proof of Theorem 4.5 and refer to [74, Theorem 4.1] for further details.

Implication (i) $\Longrightarrow$ (ii): In a first step, it is shown that the phase cohesive set $\overline{\text{Arc}}_n(\gamma)$ is positively invariant for every $\gamma \in [\gamma_{\min}, \gamma_{\max}]$. By assumption, the angles $\theta_i(t)$ belong to the set $\overline{\text{Arc}}_n(\gamma)$ at time $t = 0$, that is, they are all contained in an arc of length γ. We aim to show that all angles remain in $\overline{\text{Arc}}_n(\gamma)$ for all subsequent times $t > 0$ by means of the contraction Lyapunov function (19). Note that $\overline{\text{Arc}}_n(\gamma)$ is positively invariant if and only if $V(\theta(t))$ does not increase at any time t such that $V(\theta(t)) = \gamma$. The *upper Dini derivative* of $V(\theta(t))$ along trajectories of (21) is given by

$$D^+V(\theta(t)) = \limsup_{h\downarrow 0} \frac{V(\theta(t+h)) - V(\theta(t))}{h}\,.$$

Written out in components and after trigonometric simplifications [74], we obtain that the derivative is bounded as

$$D^+V(\theta(t)) \leq \omega_{\max} - \omega_{\min} - K\sin(\gamma)\,.$$

It follows that the length of the arc formed by the angles is non-increasing in $\overline{\text{Arc}}_n(\gamma)$ if and only if

$$K\sin(\gamma) \geq K_{\text{critical}}\,, \tag{27}$$

where K_{critical} is as stated in equation (26). For $\gamma \in [0, \pi]$ the left-hand side of (27) is a concave function of γ that achieves its maximum at $\gamma^* = \pi/2$. Therefore, there exists an open set of arc lengths $\gamma \in [0, \pi]$ satisfying equation (27) if and only if equation (27) is true with the strict equality sign at $\gamma^* = \pi/2$, which corresponds to condition (26). Additionally, if these two equivalent statements are true, then there exists a unique $\gamma_{\min} \in [0, \pi/2[$ and a $\gamma_{\max} \in\,]\pi/2, \pi]$ that satisfy equation (27) with the equality sign, namely $\sin(\gamma_{\min}) = \sin(\gamma_{\max}) = K_{\text{critical}}/K$. For every $\gamma \in [\gamma_{\min}, \gamma_{\max}]$ it follows that the arc-length $V(\theta(t))$ is non-increasing, and it is strictly decreasing for $\gamma \in\,]\gamma_{\min}, \gamma_{\max}[$. Among other things, this shows that statement (i) implies statement 1). By means of Lemma 3.1, statement 3) then follows from statement 1).

The frequency dynamics of the Kuramoto model (21) can be obtained by differentiating the Kuramoto model (21) as

$$\frac{d}{dt}\,\dot{\theta}_i = \sum\nolimits_{j=1}^{n} \tilde{a}_{ij}(t)\,(\dot{\theta}_j - \dot{\theta}_i)\,, \tag{28}$$

where $\tilde{a}_{ij}(t) = (K/n)\cos(\theta_i(t) - \theta_j(t))$. For $K > K_{\text{critical}}$, we just proved that for every $\theta(0) \in \text{Arc}_n(\gamma_{\max})$ and for all $\gamma \in \,]\gamma_{\min}, \gamma_{\max}[$ there exists a finite time $T \geq 0$ such that $\theta(t) \in \overline{\text{Arc}}_n(\gamma)$ for all $t \geq T$. Consequently, the terms $\tilde{a}_{ij}(t)$ are strictly positive for all $t \geq T$. Notice also that system (28) evolves on the tangent space of $\mathbb{T}^n$, that is, the Euclidean space $\mathbb{R}^n$. Now fix $\gamma \in \,]\gamma_{\min}, \pi/2[$ and let $T \geq 0$ such that $\tilde{a}_{ij}(t) > 0$ for all $t \geq T$. In this case, the frequency dynamics (28) can be analyzed as linear time-varying consensus system. Consider the *disagreement vector* $x = \dot{\theta} - \omega_{\text{sync}}\mathbf{1}_n$ as an error coordinate. By standard consensus arguments [48]–[50], it can be shown that the disagreement vector satisfies $\|x(t)\| \leq \|x(0)\|e^{-\lambda_{\text{K}} t}$ for all $t \geq T$. This proves statement 2) and the implication (i) $\Longrightarrow$ (ii).

Implication (ii) $\Longrightarrow$ (i): To show that condition (26) is also necessary for synchronization, it suffices to construct a counter example for which $K \leq K_{\text{critical}}$ and the oscillators do not achieve exponential synchronization even though all $\omega_i \in [\omega_{\min}, \omega_{\max}]$ and $\theta(0) \in \text{Arc}_n(\gamma)$ for every $\gamma \in \,]\pi/2, \pi]$. A basic instability mechanism under which synchronization breaks down is caused by a bipolar distribution of the natural frequencies. Let the index set $\{1, \dots, n\}$ be partitioned by the two non-empty sets $\mathcal{I}_1$ and $\mathcal{I}_2$. Let $\omega_i = \omega_{\min}$ for $i \in \mathcal{I}_1$ and $\omega_i = \omega_{\max}$ for $i \in \mathcal{I}_2$, and assume that at some time $t \geq 0$ it holds that $\theta_i(t) = -\gamma/2$ for $i \in \mathcal{I}_1$ and $\theta_i(t) = +\gamma/2$ for $i \in \mathcal{I}_2$ and for some $\gamma \in [0, \pi[$. By construction, at time t all oscillators are contained in an arc of length $\gamma \in [0, \pi[$. Assume now that $K < K_{\text{critical}}$ and the oscillators synchronize. It can be shown [74] that the evolution of the arc length $V(\theta(t))$ satisfies the *equality*

$$D^+V(\theta(t)) = \omega_{\max} - \omega_{\min} - K\sin(\gamma)\,. \tag{29}$$

Clearly, for $K < K_{\text{critical}}$ the arc length $V(\theta(t)) = \gamma$ is increasing for any arbitrary $\gamma \in [0, \pi]$. Thus, the phases are not bounded in $\overline{\text{Arc}}_n(\gamma)$. This contradicts the assumption that the oscillators synchronize for $K < K_{\text{critical}}$ from every initial condition $\theta(0) \in \overline{\text{Arc}}_n(\gamma)$. For $K = K_{\text{critical}}$, we know from [84], [85] that phase-locked equilibria have a zero eigenvalue with a two-dimensional Jacobian block, and thus synchronization cannot occur. This instability via a two-dimensional Jordan block is also visible in (29) since $D^+V(\theta(t))$ is increasing for $\theta(t) \in \overline{\text{Arc}}_n(\gamma)$, $\gamma \in \,]\pi/2, \pi]$ until all oscillators change orientation, just as in the example in Subsection III-B. This proves the implication (ii) $\Longrightarrow$ (i).

Equivalence (i),(ii) $\Leftrightarrow$ (iii): The proof relies on Jacobian arguments and will be omitted here, see [74] for details. ∎

Theorem 4.5 places a hard bound on the critical coupling strength K_{critical} for all distributions of ω_i supported on the compact interval $[\omega_{\min}, \omega_{\max}]$. For a particular distribution $g(\omega)$ supported on $[\omega_{\min}, \omega_{\max}]$ the bound (26) is only sufficient and possibly a factor 2 larger than the necessary bound (24). The exact critical coupling lies somewhere in between and can be obtained from the implicit equations (25).

Since the bound (26) on K_{critical} is exact [74] for the worst-case bipolar distribution $\omega_i \in \{\omega_{\min}, \omega_{\max}\}$, Figure 7 reports numerical findings for the other extreme case [87] of a uniform distribution $g(\omega) = 1/2$ supported for $\omega_i \in [-1, 1]$. All

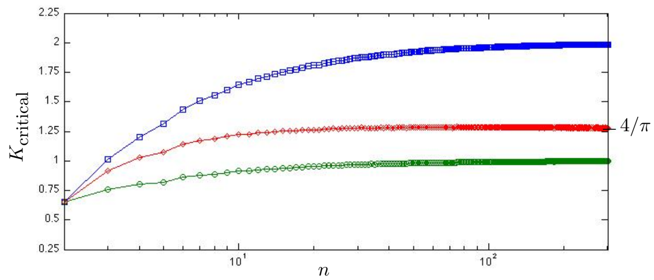

Fig. 7. Statistical analysis of the necessary and explicit bound (24) (◇), the exact and implicit bound (25) (○), and the sufficient, tight, and explicit bound (26) (□) for $n \in [2, 300]$ oscillators in a semi-log plot, where the coupling gains for each n are averaged over 1000 simulations.

three displayed bounds are identical for $n = 2$ oscillators. As n increases, the sufficient bound (26) converges to the width $\omega_{\max} - \omega_{\min} = 2$ of the support of $g(\omega)$, and the necessary bound (24) accordingly to half of the width. The exact bound (25) converges to $4(\omega_{\max} - \omega_{\min})/(2\pi) = 4/\pi$ in agreement with condition (23) predicted for the continuum limit.

Finally, let us mention that Theorem 4.5 can be extended to discontinuously switching and slowly time-varying natural frequencies [74]. For a particular sampling distribution $g(\omega)$, the critical quantity in condition (26), the support $\omega_{\max} - \omega_{\min}$, can be estimated by extreme value statistics, see [89].

E. Synchronization in Sparse Networks

As summarized in Subsection III-D, the quest for sharp and concise synchronization for non-complete coupling graph $G(\mathcal{V}, \mathcal{E}, A)$ is an important and outstanding problem emphasized in every review article on coupled oscillator networks [7], [8], [44]–[46], [74]. The approaches known for phase synchronization in arbitrary graphs or the contraction approach to frequency synchronization (used in the proof of Theorem 4.5) do not generally extend to arbitrary natural frequencies $\omega \in \mathbf{1}_n^{\perp}$ and connected coupling graphs $G(\mathcal{V}, \mathcal{E}, A)$, or do so only under extremely conservative conditions.

One Lyapunov function advocated for classic Kuramoto oscillators (21) is the function $W : \text{Arc}_n(\pi) \to \mathbb{R}$ defined for angles θ in an open semi-circle and given by [52], [53]

$$W(\theta) = \frac{1}{4}\sum\nolimits_{i,j=1}^{n} |\theta_i - \theta_j|^2 = \frac{1}{2}\left\|B_c^T\theta\right\|_2^2\,, \tag{30}$$

where $B_c \in \mathbb{R}^{n \times (n(n-1)/2)}$ is an incidence matrix of the complete graph. As shown in [64, Theorem 4.4], the Lyapunov function (30) generalizes also to the coupled oscillator model (1). Indeed, an even more general model is considered in [64], and a Lyapunov analysis yields the following result.