Low Dimensional Behavior of Large Systems of Globally Coupled Oscillators

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Abstract

It is shown that, in the infinite size limit, certain systems of globally coupled phase oscillators display low dimensional dynamics. In particular, we derive an explicit finite set of nonlinear ordinary differential equations for the macroscopic evolution of the systems considered. For example, an exact, closed form solution for the nonlinear time evolution of the Kuramoto problem with a Lorentzian oscillator frequency distribution function is obtained. Low dimensional behavior is also demonstrated for several prototypical extensions of the Kuramoto model, and time-delayed coupling is also considered.

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Because synchronous behavior in large groups consisting of many coupled oscillators has been widely observed in many situations, the behavior of such systems has long been of interest. Since the problem is difficult to solve in general, much work has been done on the simple paradigmatic case of globally coupled phase oscillators. Even in this simple context, however, much remains unclear, particularly when considering situations in which a large oscillator population interacts with external dynamical systems, or when there are communities of interacting oscillators with different community and connection characteristics, etc. In this paper we consider an approach that allows the study of the time evolving dynamical behavior of these types of systems by an exact reduction to a small number of ordinary differential equations. This reduction is achieved by considering a restricted class of states. In spite of this restriction, for at least one significant example [see preceding article], consideration of our derived ordinary differential equations appears to yield dynamics in precise agreement with results obtained from considerations not imposing this restriction. Thus we believe that our results may be useful in many other contexts.

I. INTRODUCTION

Understanding the generic behavior of systems consisting of large numbers of coupled oscillators is of great interest because such systems occur in a wide variety of significant applications[1]. Examples are the synchronous flashing of groups of fireflies, coordination of oscillatory neurons governing circadian rhythms in animals[2], entrainment in coupled oscillatory chemically reacting cells[3], Josephson junction circuits[4], neutrino oscillations[5], bubbly fluids[6], etc. A key contribution in this area was the introduction of the following model by Kuramoto[7],

$$d\theta_i(t)/dt = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j(t) - \theta_i(t)) \ , \tag{1}$$

where the state of oscillator i is given by its phase $\theta_i(t)$, $(i = 1, 2, \ldots, N)$, ω_i is the natural frequency of oscillator i, and the coupling constant K specifies the strength of the influence of one oscillator on another. It has been shown[7, 8] that in the $N \to \infty$ limit there is a continuous phase transition such that, for K below a critical value ($K < K_c$), no coherent behavior of the system occurs (i.e., there is no global correlation between the oscillator phases), while above the critical coupling strength ($K > K_c$), the system displays global cooperative behavior (i.e., partial or complete synchronization of the phases).

Among other problems related to (1) that we shall also consider are the case where there is a sinusoidal periodic external drive of strength Λ added to the righthand side of (1) (see Refs.[9] and [10]),

$$d\theta_i/dt = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) + \Lambda\sin(\Omega t - \theta_i) \ , \tag{2}$$

and the case where there are several communities of different kinds of oscillators where the evolution of the phases $\theta_i^\sigma(t)$ of oscillators in community σ is given by (see Refs.[11, 12])

$$d\theta_i^\sigma/dt = \omega_i^\sigma + \sum_{\sigma'=1}^{s}\frac{K_{\sigma\sigma'}}{N_{\sigma'}}\sum_{j=1}^{N_{\sigma'}}\sin(\theta_j^{\sigma'} - \theta_i^\sigma) \ . \tag{3}$$

Here $\sigma = 1, 2, \ldots, s$, N_σ is the number of oscillators of type σ, and $K_{\sigma\sigma'}$ is the strength of the coupling from oscillators in community σ' to oscillators in community σ. For all three cases (Eqs. (1), (2), (3)), we are interested in the limit $N \to \infty$. We will also consider such problems with time delayed coupling (e.g., $\theta_j(t) \to \theta_j(t-\tau)$ in Eqs. (1)–(3)).

The problem stated in Eq. (2) was first considered by Sakaguchi[9]. It can, for example, be motivated as a model of circadian rhythm[2]. Circadian rhythm in mammals is governed by the suprachiasmatic nucleus that is located in the brain and consists of a large population of oscillatory neurons. These neurons presumably couple with each other and are also influenced (though the optic nerve) by the daily variation of sunlight (modeled by the term in (2) involving Λ). In [10], we found numerical and analytical evidence that the bifurcations and macroscopic dynamics of (2) with large N appeared to be similar to what might be expected for the dynamics of a two dimensional dynamical system. This observation was the motivation for the present paper.

The problem stated in Eq. (3) has been previously considered in Refs.[11] and [12] where the linear stability of the incoherent state was investigated along with numerical solutions for the nonlinear evolution.