and the partially synchronized state with $|r| = const. > 0$, both conform to $f_n = \alpha^n$. Thus one view of the ansatz is that it specifies a family of distribution functions that connect these two states in a natural way.

To proceed further, we now introduce another restriction on our assumed form of f: we require that $\alpha(\omega, t)$ be analytically continuable from real ω into the complex ω-plane, that this continuation has no singularities in the lower half ω-plane, and that $|\alpha(\omega, t)| \to 0$ as $Im(\omega) \to -\infty$. If these conditions are satisfied for the initial condition, $\alpha(\omega, 0)$, then they are also satisfied for $\alpha(\omega, t)$ for $\infty > t > 0$. To see that this is so, we first note that for large negative $\omega_i = Im(\omega)$, Eq. (6) is approximately $\partial\alpha/\partial t = -|\omega_i|\alpha$, and thus $\alpha(\omega, t) \to 0$ as $\omega_i \to -\infty$ will continue to be satisfied if $\alpha(\omega, 0) \to 0$ as $\omega_i \to -\infty$. Next we note from[13] that $\alpha(\omega, t)$ is analytic in any region of the complex ω-plane for which $\alpha(\omega, 0)$ is analytic provided that the solution $\alpha(\omega, t)$ to Eq. (6) exists. To establish existence for $0 \leq t < +\infty$ it suffices to show that the solution to Eq. (6) cannot become infinite at a finite value of t. This can be ruled out by noting that our derivation of (9) with ω now complex carries through except that there is now an addition term $-\omega_i|\alpha|$ on the left hand side of the equation. Thus at $|\alpha| = 1$ we have $\partial|\alpha|/\partial t = \omega_i|\alpha| < 0$, and we conclude that, if $|\alpha(\omega, 0)| < 1$ everywhere in the lower half complex ω-plane, then $|\alpha(\omega, t)| < 1$ for all finite time $0 \leq t < +\infty$ everywhere in the lower half complex ω-plane.

Regarding the initial condition $\alpha(\omega, 0)$, we note that, if $|\alpha(\omega, 0)| \leq 1$ for ω real, if the continuation $\alpha(\omega, 0)$ is analytic everywhere in the lower half ω-plane, and if the continuation satisfies $|\alpha(\omega, 0)| \to 0$ as $\omega_i \to -\infty$, then the continuation satisfies $|\alpha(\omega, 0)| < 1$ everywhere in the lower half complex ω-plane[14]. Examples of possible initial conditions are $k \exp(-i\omega c)$ with $Re(c) > 0$ and $|k| \leq 1$, $k/(\omega - d)$ with $|k| \leq Im(d)$, and $\int_0^\infty k(c) \exp(-i\omega c) dc$ with $\int_0^\infty |k(c)| dc \leq 1$.

We can now specify the invariant manifold M on which our dynamics takes place. It is the space of functions of the real variables (ω, θ) of the form given by Eq. (8) where $|\alpha(\omega, t)| \leq 1$ for real ω; $\alpha(\omega, t)$ can be analytically continued from the real ω-axis into the lower half ω-plane; and, when continued into the lower half ω-plane, $\alpha(\omega, t)$ has no singularities there and approaches zero as $\omega_i \to -\infty$.

Now taking $g(\omega)$ to be Lorentzian

$$g(\omega) = g_L(\omega) \equiv (\Delta/\pi)[(\omega - \omega_0)^2 + \Delta^2]^{-1} ,$$

we can do the ω integral in Eq. (7) by closing the contour by a large semicircle in the lower half ω-plane. Writing $g_L(\omega) = (2\pi i)^{-1}[(\omega - \omega_0 - i\Delta)^{-1} - (\omega - \omega_0 + i\Delta)^{-1}]$, we see that the integral is given by the residue of the pole at $\omega = \omega_0 - i\Delta$. By a change of variables $(\theta, \omega) \to (\theta - \omega_0 t, (\omega - \omega_0)/\Delta)$, we can, without loss of generality set $\omega_0 = 0$, $\Delta = 1$. Thus we obtain $r(t) = \alpha^*(-i, t)$. Putting this result into Eq. (6) and setting $\omega = -i$, we obtain the nonlinear evolution of the order parameter $r = \rho e^{-i\phi}$ ($\rho \geq 0$ and ϕ real):

$$d\rho/dt + \left(1 - \frac{1}{2}K\right)\rho + \frac{1}{2}K\rho^3 = 0\ , \tag{10}$$

and $d\phi/dt = 0$. Thus the dynamics is described by the single real nonlinear, first order, ordinary differential equation, Eq. (10). The solution of Eq. (10) is,

$$\frac{\rho(t)}{R} = \left|1 + \left[\left(\frac{R}{\rho(0)}\right)^2 - 1\right] e^{(1-\frac{1}{2}K)t}\right|^{-1/2}\ , \tag{11}$$

where $R = |1 - (2/K)|^{1/2}$. We see that for $K < K_c = 2$, the order parameter goes to zero exponentially with increasing time, while for $K > 2$ it exponentially asymptotes to the finite value $[1 - (2/K)]^{1/2}$, in agreement with the known time-asymptotic results for the case $g = g_L$ (e.g., see Ref.[8]). Plots of the nonlinear evolution of $\rho(t)$ are shown in Fig. 1. Linearization of Eq. (10) yields an exponential damping rate of $[1-(K/2)]$ for perturbations

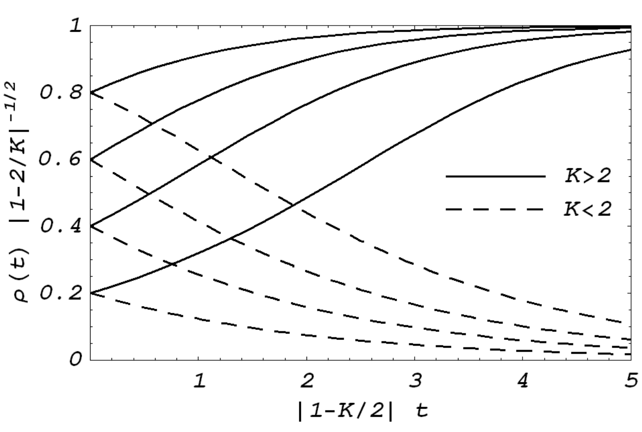

FIG. 1: The order parameter $\rho = |r|$ versus time.

around $\rho = 0$ for $K < 2$, which becomes unstable for $K > K_c = 2$, at which point the stable nonlinear equilibrium at $\rho = \sqrt{1 - (2/K)}$ comes into existence. For $K > K_c$ linearization of (10) around the equilibrium $\rho = \sqrt{1 - (2/K)}$ yields a corresponding perturbation damping rate $[(K/2) - 1]$. For $g = g_L$ the latter damping rate can also be obtained from the recent stability analyses of solutions of Eqs. (4) and (5) [10, 15]. We emphasize that our solution for $r(t)$ obeys two uncoupled first order real ordinary differential equations (Eq. (10) and $d\phi/dt = 0$), while the problem for $\alpha(\omega, t)$ (Eqs. (6) and (7)) is an infinite dimensional dynamical system (i.e., to obtain $\alpha(\omega, t)$ we need to specify an initial *function* of ω, $\alpha(\omega, 0)$). This is further reflected by the fact that linearization of Eqs. (6) and (7) about their equilibria yields a problem with a continuous spectrum of neutral modes[15, 16]. Thus the microscopic dynamics in M of the distribution function is infinite dimensional, while the macroscopic dynamics of the order parameter is low dimensional.

IV. GENERALIZATIONS

a. Other distributions $g(\omega)$

So far we have restricted our discussion to the case of the Lorentzian $g_L(\omega)$. We now consider

$$g(\omega) = g_4(\omega) \equiv (\sqrt{2}/\pi)(\omega^4 + 1)^{-1} \ ,$$

which decreases with increasing ω as ω^{-4}, in contrast to $g_L(\omega)$ which decreases as ω^{-2}. The distribution $g_4(\omega)$ has four poles at $\omega = (\pm 1 \pm i)/\sqrt{2}$. Proceeding as before, we apply the residue method to the integral (7) to obtain

$$r(t) = \frac{1}{2}[(1+i)r_1(t) + (1-i)r_2(t)] \ ,$$

where

$$r_{1,2} = \alpha^*((\mp 1 - i)/\sqrt{2}, t)$$

and $r_{1,2}(t)$ obey the two coupled nonlinear ordinary differential equations,

$$dr_{1,2}/dt + (K/2)[r^* r_{1,2}^2 - r] + [(1 \mp i)/\sqrt{2}]r_{1,2} = 0 \ . \tag{12}$$

Thus we obtain a system of two first order complex nonlinear differential equations. Indeed, the above considerations can be applied to any $g(\omega)$ that is a rational function of ω (i.e., $g(\omega) = P_1(\omega)/P_2(\omega)$ where $P_1(\omega)$ and $P_2(\omega)$ are polynomials in ω). The requirement that $g(\omega)$ be normalizable ($\int g(\omega) d\omega = 1$) and real puts restrictions on the possible $P_{1,2}(\omega)$; e.g., $P_2(\omega)$ must have even degree, $2m$, and all its roots must come in complex conjugate pairs (it cannot have a root on the real ω axis). Such a $g(\omega)$ has m poles in the lower half ω-plane, and application of our method yields m complex, first order ordinary differential equations