$\chi\big(\sum_i k_i = k\big) = 0$ otherwise. The matrices $E_{ij}^{(k)}$ are defined for $k = 0, 1, 2$ as

$$E_{ij}^{(0)} = D_{ij} = \overline{\alpha}_i \delta_{ij}, \quad E_{ij}^{(1)} = \overline{B}_{ij}(\varepsilon), \quad E_{ij}^{(2)} = \delta_{ij}, \tag{S15}$$

corresponding to the matrices $\mathbf{D}$, $\overline{\mathbf{B}}$, and $\mathbf{I}$ in Eq. (S11), respectively (δ_{ij} is the Kronecker delta function). Equations (S14) and (S15) provide an expression for the coefficients of $\det(\overline{\mathbf{J}} - \nu \mathbf{I})$ in terms of $\boldsymbol{\beta}$, and thus of ε, for a given system and its matrix $\mathbf{P}$.

We now derive a different expression for $\det(\overline{\mathbf{J}} - \nu \mathbf{I})$, this time in terms of the characteristic polynomial coefficients c_i and d_i, using Eq. (S9). Ignoring the dependence of c_i and d_i on ε for the moment and regarding them as independent variables, the coefficient of the ν^k term can be written as

$$\sum_{\{k_i\}} \prod_{i=1}^{n} a_i^{(k_i)} \cdot \chi\big(\textstyle\sum_i k_i = k\big), \tag{S16}$$

where the summation is the same as in Eq. (S14) and

$$a_i^{(0)} = d_i, \quad a_i^{(1)} = c_i, \quad a_i^{(2)} = 1. \tag{S17}$$

Setting Eqs. (S14) and (S16) equal to each other, we obtain a set of nonlinear equations that must be satisfied by the variables $c_1, \ldots, c_n$, $d_1, \ldots, d_n$, and ε:

$$F_k(c_1, \ldots, c_n, d_1, \ldots, d_n, \varepsilon) = 0, \quad k = 1, 2, \ldots, 2n-1, \tag{S18}$$

or, in vector form,

$$\mathbf{F}(c_1, \ldots, c_n, d_1, \ldots, d_n, \varepsilon) = \mathbf{0}, \tag{S19}$$

where we have defined $\mathbf{F} \equiv (F_1, \ldots, F_{2n-1})^T$, and the functions F_k are given by

$$\begin{aligned} F_k &= \sum_{\{k_i\}} \prod_{i=1}^{n} a_i^{(k_i)} \cdot \chi\big(\textstyle\sum_i k_i = k\big) - \sum_{\sigma} \mathrm{sign}(\sigma) \sum_{\{k_i\}} \prod_{i=1}^{n} E_{i\sigma(i)}^{(k_i)} \cdot \chi\big(\textstyle\sum_i k_i = k\big) \\ &= \sum_{\{k_i\}} \chi\big(\textstyle\sum_i k_i = k\big) \cdot \left[\prod_{i=1}^{n} a_i^{(k_i)} - \sum_{\sigma} \mathrm{sign}(\sigma) \prod_{i=1}^{n} E_{i\sigma(i)}^{(k_i)} \right]. \end{aligned} \tag{S20}$$

This is the equation that implicitly defines the functions $c_i = c_i(\varepsilon)$ and $d_i = d_i(\varepsilon)$. Note that, when $c_i = 1$, $d_i = \overline{\alpha}_i$, and $\varepsilon = 0$ (corresponding to the point $\boldsymbol{\beta}_{=}$), we have $E_{ij}^{(1)} = \overline{B}_{ij}(0) = \delta_{ij}$, and hence

$$a_i^{(0)} = d_i(0) = \overline{\alpha}_i, \quad a_i^{(1)} = c_i(0) = 1, \quad a_i^{(2)} = 1, \quad E_{ij}^{(k)} = a_i^{(k)} \delta_{ij}, \tag{S21}$$

implying

$$\begin{aligned} F_k(1, \ldots, 1, \overline{\alpha}_1, \ldots, \overline{\alpha}_n, 0) &= \sum_{\{k_i\}} \chi\big(\textstyle\sum_i k_i = k\big) \cdot \left[\prod_{i=1}^{n} a_i^{(k_i)} - \sum_{\sigma} \mathrm{sign}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}^{(k_i)} \delta_{i\sigma(i)} \right] \\ &= \sum_{\{k_i\}} \chi\big(\textstyle\sum_i k_i = k\big) \cdot \left[\prod_{i=1}^{n} a_i^{(k_i)} - \prod_{i=1}^{n} a_i^{(k_i)} \right] = 0, \end{aligned}$$

i.e., Eq. (S18) is satisfied. The functions $c_i(\varepsilon)$ and $d_i(\varepsilon)$ defined through Eq. (S20) thus satisfy $c_i(0) = 1$ and $d_i(0) = \overline{\alpha}_i$.

3 Characterizing descending paths on $\lambda^{\max}$-landscape

Here, we will apply the IFT to show that the functions $c_i(\varepsilon)$ and $d_i(\varepsilon)$ are continuously differentiable for small ε (and thus define a smooth curve in the neighborhood of the point $(1, \overline{\alpha}_i)$ in the (c_i, d_i)-plane), and we determine their first derivatives. The condition under which we can apply the IFT to Eq. (S18) at the point $(c_1, \ldots, c_n, d_1, \ldots, d_n, \varepsilon) = (1, \ldots, 1, \overline{\alpha}_1, \ldots, \overline{\alpha}_n, 0)$ is that the $(2n-1) \times (2n-1)$ matrix

$$\mathbf{G} \equiv \begin{pmatrix} \frac{\partial F_1}{\partial c_1} & \cdots & \frac{\partial F_1}{\partial c_n} & \frac{\partial F_1}{\partial d_2} & \cdots & \frac{\partial F_1}{\partial d_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial F_{2n-1}}{\partial c_1} & \cdots & \frac{\partial F_{2n-1}}{\partial c_n} & \frac{\partial F_{2n-1}}{\partial d_2} & \cdots & \frac{\partial F_{2n-1}}{\partial d_n} \end{pmatrix}, \tag{S22}$$

is non-singular, where the elements of $\mathbf{G}$ are all evaluated at that point. We note that d_1 is excluded from the set of variables here because $d_1 = 0$ always holds. We also note that $\mathbf{G}$ (and whether it is singular or not) is completely determined by $\overline{\alpha}_2, \ldots, \overline{\alpha}_n$, and hence by the matrix $\mathbf{P}$ (see examples in Sec. 4). For notational convenience, define $x_s = c_s$ for $s = 1, \ldots, n$ and $x_s = d_{s-n+1}$ for $s = n+1, \ldots, 2n-1$. Differentiating Eq. (S20), we find an expression for the (k, s)-element of $\mathbf{G}$:

$$G_{ks} = \frac{\partial F_k}{\partial x_s}(1, \ldots, 1, \overline{\alpha}_1, \ldots, \overline{\alpha}_n, 0) = \begin{cases} \displaystyle\sum_{\{k_i\}} \prod_{i \neq s} a_i^{(k_i)} \cdot \chi\big(s, k, \{k_i\}\big), & \text{if } s = 1, \ldots, n, \\ \displaystyle\sum_{\{k_i\}} \prod_{i \neq \hat{s}} a_i^{(k_i)} \cdot \chi\big(s, k, \{k_i\}\big), & \text{if } s = n+1, \ldots, 2n-1, \end{cases} \tag{S23}$$

where the summation is defined as in Eq. (S14); we denote $\hat{s} \equiv s - n + 1$; the values of $a_i^{(k_i)}$ are given by Eq. (S21); and we have defined

$$\chi\big(s, k, \{k_i\}\big) \equiv \begin{cases} 1 & \text{if } \sum_i k_i = k,\ k_s = 1, \text{ and } s = 1, \ldots, n, \\ 1 & \text{if } \sum_i k_i = k,\ k_{\hat{s}} = 0, \text{ and } s = n+1, \ldots, 2n-1, \\ 0 & \text{otherwise.} \end{cases} \tag{S24}$$

Since the eigenvalues of $\mathbf{P}$ are assumed to be distinct, we have $\frac{1}{4} = \overline{\alpha}_2 < \cdots < \overline{\alpha}_n$. We have numerically verified that this condition holds for all networks considered in the main text. We first seek to show that the matrix $\mathbf{G}$ is non-singular by proving that the sth and s'th columns of

$\mathbf{G}$ are linearly independent for all distinct pairs s and s'. To do this, we first note that Eq. (S23) simplifies in the special cases $k = 1$, $k = 2$, $k = 2n-2$, and $k = 2n-1$, as follows.

For $k = 1$, to satisfy $\sum_i k_i = k$, we must have $k_i = 1$ for exactly one value of i and $k_i = 0$ for all the others (recall that each k_i is either 0, 1, or 2). For the case $1 \leq s \leq n$, we can derive a simplified formula, but it is not needed below, so we will skip that case here. For the case $n+1 \leq s \leq 2n-1$, to have $\chi\big(s, k, \{k_i\}\big) = 1$ in Eq. (S24) we must have $k_t = 1$ for some $t \neq \hat{s}$ and $k_i = 0$ for all $i \neq t$ (including $i = \hat{s}$). Since there are $n-1$ possibilities for t, there are that many nonzero terms in the summation in Eq. (S23), which reduces to

$$G_{1s} = \sum_{t\neq\hat{s}} \prod_{i\neq\hat{s}} a_i^{(k_i)} = \sum_{t\neq\hat{s}} \prod_{i\neq t,\hat{s}} \overline{\alpha}_i, \quad n+1 \leq s \leq 2n-1. \tag{S25}$$

For $k = 2$, the condition $\sum_i k_i = k$ implies that we either have (a) $k_i = 2$ for exactly one value of i and $k_i = 0$ for all the others, or (b) $k_i = 1$ for two different values of i, and $k_i = 0$ for all the others. For $1 \leq s \leq n$, we have $\chi\big(s, k, \{k_i\}\big) = 0$ for the terms in Eq. (S23) corresponding to case (a), according to Eq. (S24). For nonzero terms in Eq. (S23) corresponding to case (b), we have $k_s = 1$, $k_t = 1$ with some $t \neq s$, and $k_i = 0$ for all $i \neq s, t$. Putting the two cases together, we obtain

$$G_{2s} = \sum_{t\neq s} \prod_{i\neq s} a_i^{(k_i)} = \sum_{t\neq s} \prod_{i\neq t,s} \overline{\alpha}_i, \quad 1 \leq s \leq n. \tag{S26}$$

A simplified expression for G_{2s} for $n+1 \leq s \leq 2n-1$ can also be obtained, but it is not needed for our purpose.

For $k = 2n-2$, satisfying $\sum_i k_i = k$ requires that either (a) $k_i = 0$ for exactly one value of i and $k_i = 2$ for all the others, or (b) $k_i = 1$ for two different values of i, and $k_i = 2$ for all the others. For $1 \leq s \leq n$, similarly to the case of $k = 2$, there is no nonzero term in Eq. (S23) corresponding to case (a), and for the nonzero terms in Eq. (S23) corresponding to case (b), we have $k_s = 1$, $k_t = 1$ with some $t \neq s$, and $k_i = 2$ for all $i \neq s, t$. Putting these two cases together, we obtain

$$G_{2n-2,s} = \sum_{t\neq s} \prod_{i\neq s} a_i^{(k_i)} = \sum_{t\neq s} 1 = n-1, \quad 1 \leq s \leq n. \tag{S27}$$

For $n+1 \leq s \leq 2n-1$, case (b) does not correspond to any nonzero term in Eq. (S23) because $k_i \neq 0$ for all i. Case (a), on the other hand, yields exactly one term with $k_{\hat{s}} = 0$, leading to

$$G_{2n-2,s} = \prod_{i\neq\hat{s}} a_i^{(k_i)} = 1, \quad n+1 \leq s \leq 2n-1. \tag{S28}$$

For $k = 2n-1$, the only way to satisfy $\sum_i k_i = k$ is to have just one $k_i = 1$ and all the other $k_i = 2$. For $1 \leq s \leq n$, we must have $k_s = 1$ for a nonzero term, so only one term survives in Eq. (S23), and hence

$$G_{2n-1,s} = \prod_{i\neq s} a_i^{(k_i)} = 1, \quad 1 \leq s \leq n. \tag{S29}$$