Dividing Eq. (8) by $\lambda^{3/2}$ and introducing $\tilde{\mu} := \mu/\lambda$ and $\tilde{\varepsilon}$, we obtain an equation where λ is eliminated: $3(\tilde{\mu}+\tilde{\varepsilon})^{3/2} = 3\sqrt{3}\tilde{\mu}^{1/2}$. This means that the parameter λ in Eq. (8) merely scales μ and ε. In other words, we can measure μ and ε in λ-units. The curve defined by Eq. (8) separates the regions where $p_{\pm}(y)$ have one and three solutions – see Fig. 5.

The stability of the equilibria of Eq. (5) is determined by its Jacobian

$$J = \begin{bmatrix} \mu - 3x^2 & 0 \\ -\lambda & \mu + \varepsilon - 3y^2 \end{bmatrix} \tag{9}$$

evaluated at these equilibria. The Jacobian is lower-triangular due to the feedforward structure of the network; therefore, its eigenvalues are its diagonal entries, $\mu - 3x^2$ and $\mu + \varepsilon - 3y^2$. We consider four cases with a distinct structure of the equilibria. The summarizing phase diagram in the (μ, ε)-plane and representative bifurcation diagrams are displayed in Fig. 5. A detailed description of the structure and stability of the roots of $p_{\pm}(y)$ is given in Appendix A.

3.3 What is the system's response to a jump in the excitation parameter?

In Section 3.2, we found the equilibria of system (5) and determined their stability. The next question, motivated by the sensor design, concerns how the system will respond to a sudden jump in the excitation parameter μ from zero to a small positive value.

To answer this question, we study the basin structure of system (5). At positive μ, the system may have two or four attractors (sinks) of stable node type, depending on the values of μ and ε – see Fig. 6(a). As found in Section 3.2, four sinks exist if $0 < \varepsilon < \lambda$ and $0 < \mu < \mu_1^*(\varepsilon)$, where $\mu_1^*(\varepsilon)$ is the smallest positive root of the equation $2(\mu+\varepsilon)^{3/2} = 3\sqrt{3}\lambda\mu^{1/2}$, which is well-approximated by $\mu_1^*(\varepsilon) \approx \frac{4\varepsilon^3}{27\lambda^2}$ at $\varepsilon < 0.5\lambda$ (see Eq. (A-3)) and Fig. 25(b)). Otherwise, system (5) has two sinks.

If $\mu = 0$, the system resides at $(x = 0, y = 0)$ if $\varepsilon \le 0$, and at $(x = 0, y = \sqrt{\varepsilon})$ or $(x = 0, y = -\sqrt{\varepsilon})$, if $\varepsilon > 0$. These initial conditions lie on the basin boundary at any basin structure for $\mu > 0$, making it uncertain where the system switches due to a sudden jump in μ from zero to a positive value. As μ suddenly jumps to a positive value, x switches to $x = \sqrt{\mu}$ or $x = -\sqrt{\mu}$. To understand where y jumps, we consider four cases.

- **Case $\boldsymbol{\varepsilon < 0}$.** In this case, the initial y is zero, and system (5) has two symmetric sinks. The qualitative behavior of solution branches for y is shown in the bifurcation diagram in the bottom right in Fig. 5. Thus, y will jump from zero to one of two sinks, and the magnitude of the jump will be the same whether x jumps to $\sqrt{\mu}$ or $-\sqrt{\mu}$. The magnitudes of the jump in y versus μ at $\varepsilon = -0.1$ and $\varepsilon = -0.01$ are plotted, respectively, in olive and yellow in Fig. 6(b). Comparing these plots with the red plot of the jump in y at $\varepsilon = 0$, we conclude that negative ε may only weaken the jump in y compared to the one at $\varepsilon = 0$. Hence, it is not beneficial for signal amplification to reduce the excitation coefficient in the second cell. However, if ε happens to be slightly negative due to a manufacturing imprecision, the amplification of the response in y will be almost as strong as it is at $\varepsilon = 0$.
- **Case $\boldsymbol{0 < \varepsilon < \lambda}$, $\boldsymbol{0 < \mu < \mu^*(\varepsilon)}$.** Suppose that $y = \sqrt{\varepsilon}$ before the jump in μ. If the new value of μ satisfies $0 < \mu < \mu_1^*(\varepsilon)$, the system acquires four sinks as shown in Fig. 6(a, left). Depending on whether x switches to $\sqrt{\mu}$ or $-\sqrt{\mu}$, y will jump, respectively, to the positive blue branch or the positive red branch as shown in Fig. 6(b, right). In this case, the response in y is weaker than at $\varepsilon = 0$ – see the light and dark blue plots in Fig. 6(b, left) at $\mu < \mu_1^*(\varepsilon)$. If the initial $y = -\sqrt{\varepsilon}$, the situation is similar.
- **Case $\boldsymbol{0 < \varepsilon < \lambda}$, $\boldsymbol{\mu > \mu^*(\varepsilon)}$.** Suppose that $y = \sqrt{\varepsilon}$ before the jump in μ. Since the system has only two sinks at $\mu > \mu_1^*(\varepsilon)$, it will jump at one of them depending on the sign of x. Contrary to the case where $\varepsilon < 0$, the magnitude of these two possible jumps will be different, as evident from Fig. 6(b). It is impossible to predict which branch this jump will be to, the upper red or the lower blue in Fig. 6(b, right), because its starting point is

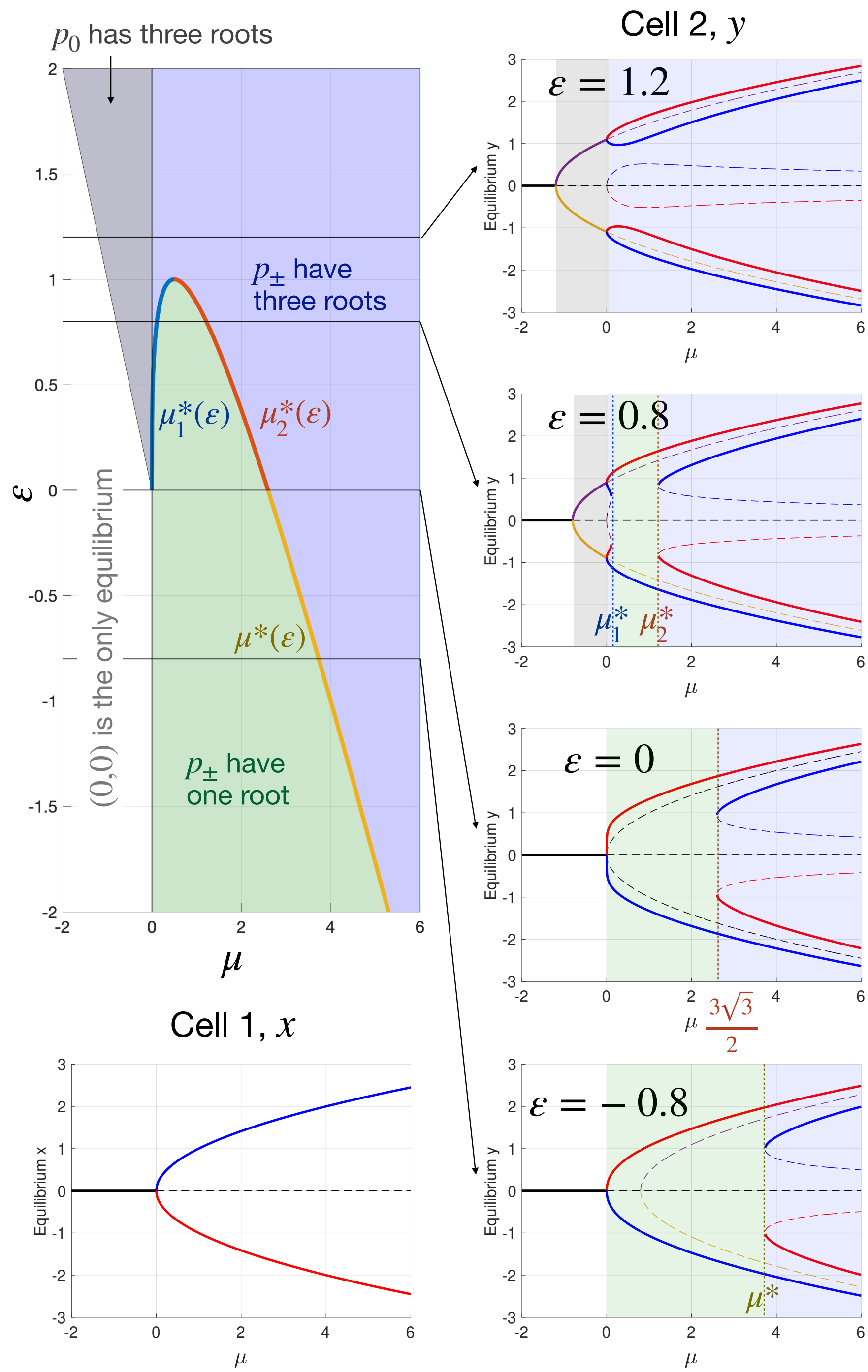

Figure 5: A phase diagram and representative bifurcation diagrams for system (5). Solid and dashed lines represent stable and unstable branches, respectively. The branches of y existing at $x = \sqrt{\mu}$ and $x = -\sqrt{\mu}$ are blue and red, respectively. Black, yellow, and purple branches of y correspond to $x = 0$. The curve separating the one- and three-root regions, plotted in blue, red, and yellow, is defined by Eq. (8).

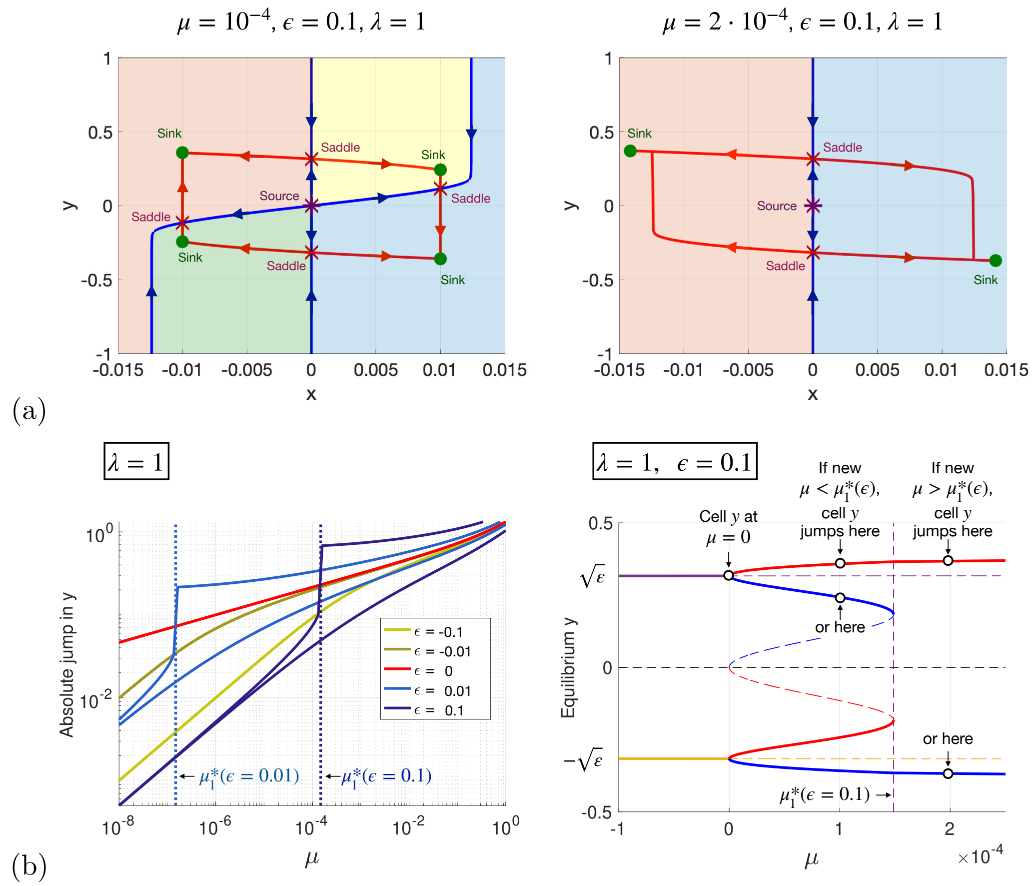

Figure 6: (a): Two possible topologically different basin and heteroclinic structures of system (5). For $0 < \varepsilon < \lambda$, as μ increases from 0, the sinks with yellow and green basins collide with the nearest saddles and disappear. Then these pairs of sink and saddle reappear as μ keeps increasing. (b, left): The absolute value of the jump in y versus μ in log-log scale occurring due to a sudden change in μ from zero to a positive value. If $\varepsilon > 0$, two absolute values of the jump in y are possible at each μ-jump, depending on whether the initial y is $\sqrt{\varepsilon}$ or $-\sqrt{\varepsilon}$ and whether x jumps to $\sqrt{\mu}$ or $-\sqrt{\mu}$. The dotted vertical line corresponds to the critical values of μ at which sinks and saddles collide as described in (a), i.e., a saddle-node bifurcation takes place. (b, right): A zoom-in of the bifurcation diagram at $\varepsilon = 0.1$. Stable equilibrium branches for y are shown with solid lines, while unstable ones with dashed lines. The vertical dashed line is at the critical value of μ at which the saddle-node bifurcation takes place.