For $n+1 \le s \le 2n-1$, there is no nonzero term since there is no i for which $k_i = 0$ (see Eq. (S24)), and thus

$$G_{2n-1,s} = 0, \quad n+1 \le s \le 2n-1. \tag{S30}$$

Now we can use Eqs. (S25)–(S30) to show that any pair of columns of $\mathbf{G}$ are linearly independent. For $1 \le s \le n$ and $n+1 \le s' \le 2n-1$, the last two components ($k = 2n-2$ and $k = 2n-1$) of the sth and s'th column vectors form the two-dimensional vectors $(n-1, 1)^T$ and $(1, 0)^T$, respectively, which are linearly independent. This implies that the full $(2n-1)$-dimensional vectors in the sth and s'th columns of $\mathbf{G}$ are also linearly independent. For $1 \le s < s' \le n$, since the last components ($k = 2n-1$) of the sth and s'th column vectors are both equal to one, it suffices to show that the second component ($k = 2$) is different in order to establish that they are linearly independent. From Eq. (S26), we have

$$\begin{aligned} G_{2s} - G_{2s'} &= \sum_{t \ne s} \prod_{i \ne t,s} \overline{\alpha}_i - \sum_{t \ne s'} \prod_{i \ne t,s'} \overline{\alpha}_i \\ &= \Big(\sum_{t \ne s,s'} \prod_{i \ne t,s} \overline{\alpha}_i + \prod_{i \ne s',s} \overline{\alpha}_i \Big) - \Big(\sum_{t \ne s,s'} \prod_{i \ne t,s'} \overline{\alpha}_i + \prod_{i \ne s,s'} \overline{\alpha}_i \Big) \\ &= \sum_{t \ne s,s'} \Big(\prod_{i \ne t,s,s'} \overline{\alpha}_i \Big) (\overline{\alpha}_{s'} - \overline{\alpha}_s) > 0, \end{aligned} \tag{S31}$$

since $\overline{\alpha}_{s'} > \overline{\alpha}_s$ and $\overline{\alpha}_i > 0$, $\forall i$, and hence all terms in the summation are positive. Thus, we have $G_{2s} \ne G_{2s'}$, implying that the sth and s'th column vectors are linearly independent. For $n+1 \le s < s' \le 2n-1$, the argument is similar to the case of $1 \le s < s' \le n$; it suffices to show that the first component ($k = 1$) is different, and Eq. (S25) yields

$$\begin{aligned} G_{1s} - G_{1s'} &= \sum_{t \ne \hat{s}} \prod_{i \ne t,\hat{s}} \overline{\alpha}_i - \sum_{t \ne \hat{s}'} \prod_{i \ne t,\hat{s}'} \overline{\alpha}_i \\ &= \sum_{t \ne \hat{s},\hat{s}'} \Big(\prod_{i \ne t,\hat{s},\hat{s}'} \overline{\alpha}_i \Big) (\overline{\alpha}_{\hat{s}'} - \overline{\alpha}_{\hat{s}}) > 0, \end{aligned} \tag{S32}$$

where we have defined $\hat{s}' \equiv s' - n + 1$ (and recall that $\hat{s} = s - n + 1$). Combining all of the above, we have that the sth and s'th column vectors of $\mathbf{G}$ are linearly independent for all pairs of distinct s and s', which establishes that $\mathbf{G}$ is non-singular.

The IFT can now be applied to Eq. (S18) to conclude that the functions $c_i(\varepsilon)$ and $d_i(\varepsilon)$ are continuously differentiable. Furthermore, their derivatives satisfy the set of equations obtained by substituting these functions into Eq. (S18) and differentiating both sides with respect to ε:

$$\frac{d}{d\varepsilon} F_k(c_1(\varepsilon), \ldots, c_n(\varepsilon), d_1(\varepsilon), \ldots, d_n(\varepsilon), \varepsilon) = 0. \tag{S33}$$

Using Eq. (S20), this can be written as

$$\sum_{\{k_i\}} \chi\left(\sum_i k_i = k\right) \cdot \sum_{\ell=1}^{n} \left[\left(\prod_{i \neq \ell} a_i^{(k_i)}(\varepsilon) \right) \cdot \frac{d a_\ell^{(k_\ell)}(\varepsilon)}{d\varepsilon} - \right.$$
$$\left. - \sum_{\sigma} \mathrm{sign}(\sigma) \left(\prod_{i \neq \ell} E_{i\sigma(i)}^{(k_i)}(\varepsilon) \right) \cdot \frac{d E_{\ell\sigma(\ell)}^{(k_\ell)}(\varepsilon)}{d\varepsilon} \right] = 0, \quad \text{(S34)}$$

where we have now written the dependence of $a_i^{(k_i)}$ on ε explicitly. Setting $\varepsilon = 0$ and using Eq. (S21), we obtain

$$\sum_{\ell=1}^{n} \left(\sum_{\{k_i\}} \prod_{i \neq \ell} a_i^{(k_i)}(0) \cdot \chi\left(\sum_i k_i = k\right) \right) \cdot \left[\frac{d a_\ell^{(k_\ell)}(0)}{d\varepsilon} - \frac{d E_{\ell\ell}^{(k_\ell)}(0)}{d\varepsilon} \right] = 0. \quad \text{(S35)}$$

Noting that $k_\ell = 0, 1, 2$ and that the derivatives are both zero if $k_\ell = 2$ (see Eqs. (S15) and (S17)), we can rearrange the summation in Eq. (S35) to write

$$\sum_{s=1}^{n} \left(\sum_{\{k_i\}} \prod_{i \neq s} a_i^{(k_i)}(0) \cdot \chi\big(s, k, \{k_i\}\big) \right) \cdot [x_s'(0) - y_s'(0)]$$
$$+ \sum_{s=n+1}^{2n-1} \left(\sum_{\{k_i\}} \prod_{i \neq \hat{s}} a_i^{(k_i)}(0) \cdot \chi\big(s, k, \{k_i\}\big) \right) \cdot [x_s'(0) - y_s'(0)] = 0, \quad \text{(S36)}$$

where we recall that $x_s(\varepsilon) = c_s(\varepsilon) = a_s^{(1)}(\varepsilon)$ for $s = 1, \ldots, n$ and $x_s(\varepsilon) = d_{s-n+1}(\varepsilon) = a_s^{(0)}(\varepsilon)$ for $s = n+1, \ldots, 2n-1$, and we use the notations $y_s(\varepsilon) = \overline{B}_{ss}(\varepsilon)$ for $s = 1, \ldots, n$ and $y_s(\varepsilon) = \overline{\alpha}_{s-n+1}$ for $s = n+1, \ldots, 2n-1$. From Eq. (S23), we see that Eq. (S36) is equivalent to

$$\sum_{s=1}^{2n-1} G_{ks}[x_s'(0) - y_s'(0)] = 0, \quad \text{(S37)}$$

which can be put in vector form as $\mathbf{G}(\mathbf{x} - \mathbf{y}) = \mathbf{0}$ using the notations $\mathbf{x} = (x_1'(0), \ldots, x_{2n-1}'(0))^T$ and $\mathbf{y} = (y_1'(0), \ldots, y_{2n-1}'(0))^T$. Thus, since $\mathbf{G}$ is non-singular, we have $\mathbf{x} = \mathbf{y}$, and hence $x_i'(0) = y_i'(0)$. It then follows that

$$c_i'(0) = x_i'(0) = y_i'(0) = \frac{d\overline{B}_{ii}(0)}{d\varepsilon} = \frac{d}{d\varepsilon} \left(\sum_{\ell=1}^{n} u_{i\ell} v_{i\ell} \gamma_\ell(\varepsilon)/\beta_= \right) \Bigg|_{\varepsilon=0} = \sum_{\ell=1}^{n} u_{i\ell} v_{i\ell} \gamma_\ell'(0)/\beta_= \quad \text{(S38)}$$

for $i = 1, \ldots, n$, where we used the expression for $\overline{B}_{ij}$ from Eq. (S12). It also follows that

$$d_i'(0) = x_{i+n-1}'(0) = y_{i+n-1}'(0) = \frac{d}{d\varepsilon} \left(\overline{\alpha}_i \right) \Big|_{\varepsilon=0} = 0 \quad \text{(S39)}$$

for $i = 1, \ldots, n$, since $\overline{\alpha}_i$ is a constant that does not depend on ε. In particular, we have

$$c_2'(0) = \sum_{\ell=1}^{n} u_{2\ell} v_{2\ell} \gamma_\ell'(0)/\beta_= \quad \text{and} \quad d_2'(0) = 0. \tag{S40}$$

Now, recall the argument at the end of Sec. 1 that $d_2'(0) = 0$ implies $c_2'(0) = 0$ if $\lambda^{\max}$ decreases along the path $\boldsymbol{\gamma}(\varepsilon)$ in the $\boldsymbol{\beta}$-space. In view of that argument, we see from Eq. (S40) that, if the eigenvalues of $\mathbf{P}$ are all distinct, the vector $(\gamma_1'(0), \ldots, \gamma_n'(0))^T$ is parallel to the hyperplane L whenever $\lambda^{\max}$ decreases along the path, where L is uniquely defined by the equation

$$\sum_{i=1}^{n} u_{2i} v_{2i} (\beta_i - \beta_=) = 0. \tag{S41}$$

In other words, any descending path on the $\lambda^{\max}$-landscape must be tangent to the hyperplane L at $\boldsymbol{\beta}_=$.

If the path $\boldsymbol{\gamma}(\varepsilon)$ is not tangent to L (and hence $c_2'(0) \neq 0$), the expansion of $c_2(\varepsilon)$ and $d_2(\varepsilon)$ around $\varepsilon = 0$, which reads

$$\begin{aligned} c_2(\varepsilon) &= 1 + c_2'(0)\varepsilon + O(\varepsilon^2), \\ d_2(\varepsilon) &= \frac{1}{4} + O(\varepsilon^2), \end{aligned} \tag{S42}$$

can be substituted into Eq. (S8) to obtain the following approximation for $\lambda^{\max}(\varepsilon)$:

$$\lambda^{\max}(\varepsilon) = \begin{cases} \lambda_=^{\max} - \beta_= c_2'(0)\varepsilon/2 + O(\varepsilon^2), & \varepsilon \leq 0, \\ \lambda_=^{\max} + \beta_= \sqrt{c_2'(0)\varepsilon/2} + O(\varepsilon), & \varepsilon > 0. \end{cases} \tag{S43}$$

(Recall we are assuming $c_2'(0) \geq 0$.) This establishes that the point $\boldsymbol{\beta}_=$ is a local minimizer of $\lambda^{\max}$ along any path that transversally intersects with the hyperplane L at $\boldsymbol{\beta}_=$.

4 Explicit calculation of matrix G for small n

Here, we derive an expression for the matrix $\mathbf{G}$ in Eq. (S22) for $n = 2$ and for $n = 3$. As in Sec. 1–3 above, we assume $\beta_= > 0$, or equivalently, $\alpha_2 > 0$.

Example 1: For $n = 2$, Eq. (S11) becomes

$$\begin{aligned} \det(\overline{\mathbf{J}} - \nu\mathbf{I}) = \det(\nu^2\mathbf{I} + \nu\overline{\mathbf{B}} + \mathbf{D}) &= \det\begin{pmatrix} \nu^2 + \overline{B}_{11}\nu & \overline{B}_{12}\nu \\ \overline{B}_{21}\nu & \nu^2 + \overline{B}_{22}\nu + 1/4 \end{pmatrix} \\ &= (\nu^2 + \overline{B}_{11}\nu)(\nu^2 + \overline{B}_{22}\nu + 1/4) - \overline{B}_{12}\overline{B}_{21}\nu^2 \\ &= \nu^4 + (\overline{B}_{11} + \overline{B}_{22})\nu^3 + (\overline{B}_{11}\overline{B}_{22} - \overline{B}_{12}\overline{B}_{21} + 1/4)\nu^2 + \overline{B}_{11}\nu/4, \\ &= \nu^4 + \frac{\beta_1 + \beta_2}{\beta_=}\nu^3 + \left(\frac{\beta_1\beta_2}{\beta_=^2} + \frac{1}{4}\right)\nu^2 + \overline{B}_{11}\nu/4, \end{aligned}$$