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Effects of Heterogeneity in Two-Cell Feedforward Networks

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Abstract

As the need for higher performance from biological and electronic sensors continues to outpace current technologies, new strategies for designing, developing, and implementing novel sensor systems are emerging. A recently introduced feedforward network-based approach can simultaneously enhance a signal while steering a radiating beam in radio frequency communication systems. Furthermore, the approach is also model-independent, thus making it suitable for other applications. In this work, we aim to understand the effects of inhomogeneities in feedforward arrays, which are inevitable in real-world implementations. We investigate a collection of two-cell feedforward networks composed of pitch-fork cells and Stuart-Landau oscillators and quantify the effects of parameter inhomogeneities using system reduction, analytical and computational bifurcation analyses, and a singularity theory approach. Contrary to common intuition, inhomogeneity in the excitation parameter can be exploited to enhance the network output growth rate. While frequency inhomogeneity in Stuart-Landau networks primarily has an adverse effect on signal amplification, phase locking persists over a surprisingly broad range of inhomogeneity.

Keywords: feedforward networks; signal amplification; Stuart-Landau oscillators; inhomogeneity; disorder; pitch-fork bifurcation; Hopf bifurcation; torus bifurcation.

1 Introduction

1.1 An overview

Signal amplification is a quintessential paradigm in science and engineering, in which an input signal is enhanced to facilitate detection, classification, and subsequent processing. This process allows for the development of highly sensitive sensors. To achieve this highly sought-after effect, various strategies, which depend primarily on the type of signal to be amplified, have been developed across many disciplines. In bioelectrochemistry, fuel cells and electrolyzers, biosensors' response to detect trace numbers of analytes can be enhanced, while reducing noise

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signals, through nanomaterials, enzyme catalysis, and biological reactions [1]. In mass cytometry, thermal-cycling-based DNA enables signal amplification, resulting in higher quantitative measurements of proteins or protein modifications at single-cell resolution [2]. In electronic systems, traditional methods for signal amplification are achieved through transistors. In the lower-frequency stages, for example, many amplifying transistors are integrated on a single substrate, enabling high gain with very high stability and requiring few external components [3]. A novel approach to electric- and magnetic-field sensors exploits the symmetry of networks and infinite-period bifurcations that generate heteroclinic cycles to achieve sensitivities on the order of pico-teslas and femto-amps, respectively [4, 5, 6, 7, 8, 9]. In phototransduction, signal amplification, which involves activation of a relatively small number of G protein-coupled receptors, is achieved through a cascade [10].

Beam steering is another fundamental problem, mainly in engineering, in which the goal is to control the direction of a radiating far-field intensity pattern [11, 12, 13]. Conventional and modern methods involve manipulating the phase shift between oscillating components, which consists of arrays of nonlinear oscillators. This is usually accomplished by leveraging the inherent nonlinearities of individual components and the collective dynamics of the oscillator array. In optics, beam steering can be done by either changing the refractive index of the medium through which the beam is transmitted, or by the use of mirrors, prisms, or rotating diffraction gratings [14]. In acoustics, beam steering is about changing the direction of audio from speakers, and it can be accomplished by changing the magnitude and phase of speakers arranged in an array [15]. Other applications can be found in aerospace communication [16], in light detection and ranging (LiDARs) [17], in laser scanning microscopy [18], in imaging of organs [19], and in antenna and radar systems [20, 12, 21].

Over the past few years, we have been crafting novel strategies to simultaneously address both problems, signal amplification and beam steering. The fundamental principle is the use of feedforward networks as the underlying strategy. Feedforward networks are a specific type of network characterized by a *homogeneous* chain of unidirectionally coupled nodes [22, 23]. The first node or cell may be self-coupled. In both cases, the unidirectional coupling prevents feedback in the system, so that each cell may influence another without being itself affected. This coupling configuration departs from traditional methods, which typically employ bidirectionally coupled nonlinear oscillators. It has been shown that under the right conditions, the feedforward network causes certain bifurcations to exhibit accelerated growth rates [24, 25, 26]. These bifurcations are the source of achieving signal amplification. In follow-up work, we considered the feedforward network as a replacement for the commonly used bidirectionally coupled arrays of nonlinear oscillators. Thus, we first studied the *transmission problem*: generating and steering the radiating patterns that emanate from the sources [27]. A fundamental result of that work was to show the existence of stable phase locking [28] in the collective dynamics. This result is deemed fundamental because the phase-locking angle can be used to steer the beam, while the feedforward dynamics amplify the signal. In subsequent work, we addressed the *reception problem*: understanding the interaction of the feedforward array of nonlinear oscillators with external signals. Studying the reception mode in a feedforward network is more complicated because incident signals introduce time-dependent forcing terms; thus, the model is non-autonomous, whereas in the transmission case it is autonomous. The main contribution from this latter work was to show the regions of parameter space where stable, 1:1 synchronization with the external signals exists [29]. It was also shown that the interaction of the collective pattern of oscillation produced by the feedforward network can lead to narrower main lobes and lower sidelobes than in other commonly used array configurations. It is desirable to have a beamformer system with a narrow mainlobe and low sidelobes, as these characteristics can improve resolution and reduce susceptibility to interference from strong signals.

1.2 A summary of main results

The next phase of work is the actual development and implementation of the technology. But to do that, we must first account for the fact that, in practice, even the most advanced systems are imperfect. That is, they do not necessarily conform to the underlying assumption of homogeneity. In other words, real-life implementation requires us to consider the effects of inhomogeneities or disorder in a system. In the case of beam steering and signal amplification, inhomogeneities may arise naturally through fluctuations in the parameters of each cell, e.g., excitation and frequency parameters, or through variations in the coupling strengths of the cells.

In this work, we abandon the assumption that all nodes are identical and examine the effects of fluctuations on system parameters in a feedforward network. We focus on two-cell arrays of pitchfork and Stuart-Landau cells. Adding inhomogeneities into the model equations introduces unfolding parameters that automatically increase the codimension of the bifurcations. Naturally, higher codimension yields more complex dynamics, which we investigate in this work. We use model reduction and singularity theory to conduct a bifurcation analysis, plot phase and bifurcation diagrams, and obtain a complete quantification of the effects of parameter inhomogeneities.

We show that, contrary to intuition, inhomogeneity in the excitation parameter μ in two-cell pitchfork and Stuart-Landau feedforward networks can result in response amplification beyond that achievable with a homogeneous network.

In addition, feedforward arrays of two Stuart-Landau cells admit inhomogeneities in frequency ω and in the cubic nonlinearity parameter γ. We demonstrate that signal amplification is neutral with respect to the inhomogeneity in γ. By contrast, frequency inhomogeneity has an adverse effect on signal amplification. However, phase locking persists for a surprisingly broad range of frequency gaps. Beyond this range, the system settles on an invariant torus attractor, i.e., exhibits quasiperiodic oscillations.

The results of this work can serve as guiding principles for engineers designing and operating emerging technologies for sensors, beam steering, and signal amplification systems. They also pave the way for the study of large feed-forward arrays with inhomogeneities, planned for future work.

The manuscript is organized as follows. In Section 2, a review of the effect of signal amplification and beam steering through feedforward networks is introduced. Section 3 is devoted to feed-forward arrays of pitch-fork cells. The effects of inhomogeneities in the excitation parameter are studied, with an emphasis on the network's equilibrium states. Section 4 presents a systematic study of the effects of inhomogeneity in frequency and the excitation and cubic nonlinearity parameters in two-cell feed-forward arrays of Stuart-Landau oscillators. The results are discussed in Section 5.

2 Background

In this section, two fundamental applications, which serve as a motivation for the analysis of feedforward networks, *signal amplification* and *beam steering*, are reviewed for completeness purposes.

2.1 Signal amplification via feedforward networks

Feedforward networks are a specific type of network characterized by a homogeneous chain of unidirectionally coupled nodes [30, 22]. The first node may be self-coupled. Under the right conditions, the feedforward network causes certain bifurcations to exhibit accelerated growth rates.

Consider, for example, the three-cell feedforward network shown in Fig. 1 The internal dynamics of each cell are governed by a Hopf bifurcation: