for m complex order parameters. For instance, for the example, $g(\omega) = g_4(\omega)$, above, there are two poles in $Im(\omega) < 0$, namely, $\omega = (\pm 1 - i)/\sqrt{2}$, and these two poles result in the two order parameters r_1 and r_2.

b. External driving

We now consider the Kuramoto problem with an external drive, Eq. (2). Again taking the $N \to \infty$ limit for the number of oscillators, we obtain

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \theta}\left\{ f \left[(\omega - \Omega) + \frac{1}{2i}(Kr + \Lambda)e^{-i\theta} - \frac{1}{2i}(Kr + \Lambda)^* e^{i\theta} \right] \right\} = 0 \ , \tag{13}$$

with $r(t)$ given by Eq. (5). In writing Eq. (13), we have utilized a change of variables $\theta \to \theta + \Omega t$ to remove the $e^{i\Omega t}$ time dependence that would otherwise appear multiplying the Λ terms. Again assuming that $g(\omega)$ is a Lorentzian with unit width $\Delta = 1$ peaked at $\omega = \omega_0$, and proceeding as before, we obtain the following equation for $r(t)$,

$$dr/dt + \frac{1}{2}\{(Kr + \Lambda)^* r^2 - (Kr + \Lambda)\} + [1 + i(\Omega - \omega_0)]r = 0 \ . \tag{14}$$

Equilibria are obtained by setting $dr/dt = 0$ in Eq. (14). Depending on parameters (K, Ω, Λ), there are either one or three such equilibria[10]. Also, depending on parameters, there may be an attracting limit cycle. Whether the equilibria are attractors for Eq. (14) depends on their stability which can be assessed by linearization around the equilibria. The equilibria obtained for Eq. (14) and their stability are the same as obtained by the analysis of the full system (13) as performed in Ref.[10]. Furthermore, the bifurcations and stability of the limit cycle are the same as numerically found in Ref.[10]. Thus, for this problem, it appears that the important observable macroscopic dynamics is contained entirely within the invariant manifold M.

c. Communities of oscillators

Turning now to the problem of coupled communities of Kuramoto systems given by Eq. (3), we introduce different Lorentzians for each community,

$$g^\sigma(\omega) = \pi^{-1}[(\omega - \omega_\sigma)^2 + \Delta_\sigma^2]^{-1} \ ,$$

and proceed as before. We obtain a coupled system of equations for the order parameter associated with each community σ,

$$dr_\sigma/dt + (-i\omega_\sigma + \Delta_\sigma)r_\sigma + \frac{1}{2}\sum_{\sigma'=1}^{s} K_{\sigma\sigma'}[r_{\sigma'}^* r_\sigma^2 - r_{\sigma'}] = 0 \ , \tag{15}$$

where $\sigma = 1, 2, \dots, s$. Thus we obtain s complex coupled differential equations where s is the number of communities. We conjecture that, for s large enough [e.g., $s \geq 2$ or 3] and appropriate parameter values, there may be chaotic attracting solutions for Eq. (15). It would be particularly interesting to see whether such solutions in M are also attractors for the macroscopic order-parameter behavior of the full system (3), e.g., by comparing numerical solutions of Eqs. (3) and (15).

d. Time-delayed coupling

In applications time delay in the coupling between dynamical units in a network is often present. For example, the propagation speed of signals between units is finite (e.g., along axons in a neural network), and there may also be an inherent response time of a unit to information that it receives. Thus time delay has been extensively studied in the context of networks of coupled systems, and in particular for the case of coupled phase oscillators[17, 18, 19]. It has been found for such systems that time delay can substantially modify the dynamics, leading to a much richer variety of behaviors. In the context of Eqs. (1)–(3), for example, the response of oscillator i at time t to input from oscillator j is now related to the state θ_j of oscillator j at time $(t - \tau_{ji})$ where τ_{ji} is the time delay for this interaction. Assuming that all the delay times are the same, $\tau_{ji} = \tau$, independent of i and j, the quantities $\theta_j(t)$ appearing in the summations in Eqs. (1)–(3) must now be replaced by $\theta_j(t-\tau)$. Again such a generalization can be straightforwardly incorporated into our method. For example, for the external drive problem (Eq. (2) and Sec. IVb) we have in place of Eq. (2),

$$\frac{d\theta_i(t)}{dt} = \omega_i + \frac{K}{N}\sum_{i=1}^{N} \sin[\theta_j(t-\tau) - \theta_i(t)] + \Lambda \sin[\Omega t - \theta_i(t)] \ . \tag{16}$$

Going to a rotating frame, $\theta_i'(t) = \theta_i(t) - \Omega t$, $\omega' = \omega - \Omega$, Eq. (16) becomes

$$\frac{d\theta_i'(t)}{dt} = \omega_i' + \frac{K}{N}\sum_{i=1}^{N} \sin[\theta_j'(t-\tau) - \theta_i'(t) - \Omega\tau] - \Lambda \sin\theta_i'(t) \ . \tag{17}$$

The summation in Eq. (17) is

$$\frac{K}{N} Im\left\{ e^{-i[\theta_i'(t)+\Omega\tau]} \sum_{j=1}^{N} e^{i\theta_j'(t-\tau)} \right\} = K Im\left\{ e^{-i[\theta_i'(t)+\Omega\tau]} r(t-\tau) \right\} \ . \tag{18}$$

Thus, to include delay, it suffices to replace the term $[Kr(t) + \Lambda]$ in Eqs. (13) and (14) by $[Ke^{-i\Omega\tau} r(t-\tau) + \Lambda]$. E.g., making this substitution in Eq. (14) and setting $\Lambda = 0$,

$\Omega = \omega_0$ yields the following first order delay-differential equation for the order-parameter of the standard Kuramoto model with coupling delay,

$$\frac{dr(t)}{dt} - \frac{K}{2}\left[e^{-i\omega_0\tau} r(t-\tau) - e^{i\omega_0\tau} r^*(t-\tau)(r(t))^2\right] + r(t) = 0 \ , \tag{19}$$

which returns Eq. (10) for $\tau \to 0$. We note that our reduced descriptions with delay (e.g., Eq. (19)) are (in contrast to Eqs. (10), (14) and (15)) now infinite dimensional dynamical systems. For small $|r|$, linearizing Eq. (19) about the incoherent state ($r = 0$), and setting $r \sim e^{st}$ yields a dispersion relation for s,

$$s + 1 = (K/2)\exp[-(s + i\omega_0)\tau] \ , \tag{20}$$

in agreement with Ref. [18]. In addition, steady synchronized states can be found (as in Ref. [19]) by setting $r(t) = r_0 e^{i\eta t}$ in Eq. (19) and solving the result,

$$i\eta - \frac{K}{2}\left[e^{-i(\omega_0+\eta)\tau} - r_0^2 e^{i(\omega_0+\eta)\tau}\right] + 1 = 0 \ , \tag{21}$$

for the real constants η and r_0. Furthermore, through linearization of Eq. (19) about $r = r_0 e^{i\eta t}$, our formulation can be used to study the previously unaddressed problem of assessing the stability of the steady synchronized states, Eq. (21).

e. The Millennium bridge problem, Ref.[20]

Another example is that of the observed oscillation of London's Millennium Bridge induced by the pacing phase entrainment of pedestrians walking across the bridge as modeled by Eqs. (52) and (53) of the paper by B. Eckhardt et al.[20]. In that case, assuming a Lorentzian distribution of natural pacing frequencies for the pedestrians, one can use the method given in our paper to obtain an ordinary differential equation for the mechanical response of the bridge coupled to another ordinary differential equation for the order parameter describing the collective state of the pedestrians.

V. DISCUSSION AND CONCLUSION

Low dimensional descriptions of the classical Kuramoto problem (Eq. (1)) have been previously attempted. An early such attempt was made by Kuramoto and Nishikawa[21] who used a heuristic approach resulting in an integral equation for $r(t)$. On the basis of their work they predict that for small $|r(0)|$ the order-parameter $r(t)$ initially grows (decays)