II. NATURE OF THE MAIN RESULT

Considering the limit $N \to \infty$, the state of the oscillator system at time t can be described by a continuous distribution function, $f(\omega, \theta, t)$, in frequency ω and phase θ for the problems in Eqs. (1) and (2) or by $f^\sigma(\omega, \theta, t)$ with $\sigma = 1, 2, \ldots, s$ for the problem in Eq. (3), where

$$\int_0^{2\pi} f(\omega, \theta, t) d\theta = g(\omega) \quad \text{or} \quad \int_0^{2\pi} f^\sigma(\omega, \theta, t) d\theta = g^\sigma(\omega) \ ,$$

and $g(\omega)$ and $g^\sigma(\omega)$ are time independent oscillator frequency distributions.

Our main result is as follows. For initial distribution functions $f(\omega, \theta, 0)$ satisfying a certain set of conditions that we will specify later in this paper, we show that

(i) the evolution of $f(\omega, \theta, t)$ from $f(\omega, \theta, 0)$ continues to satisfy the specified conditions,

(ii) for appropriate $g(\omega)$ [or $g^\sigma(\omega)$], the macroscopic system state obeys a finite set of nonlinear ordinary differential equations, which we obtain explicitly.

Concerning (i), we define a distribution function $h(\omega, \theta)$ as a function for which $h \geq 0$ and $\int_0^{2\pi} d\theta \int d\omega h = 1$, and the distribution functions $h(\omega, \theta)$ satisfying our conditions form a manifold M in the space D of all possible distribution functions. What point (i) says is that initial states in $M \subset D$ evolve to other states in M. Thus M is 'invariant' under the dynamics. Concerning point (ii), we use the so-called 'order-parameter' description to define the macroscopic system state. We define the order parameter (or parameters in the case of Eq. (3)) subsequently (Eq. (5)) in terms of an integral over the distribution function f (or f^σ for (3)), where this order-parameter integral globally quantifies the degree to which the entire ensemble of oscillators (or ensembles σ for (3)) behaves in a coherent manner. According to point (ii) the evolution of the order parameters is exactly finite dimensional even though the manifold M and the dynamics of the distribution function f as it evolves in M are infinite dimensional.

The macroscopic dynamics we obtain allows for much simplified investigation of the systems we study. For example, we obtain an exact closed form solution for the nonlinear time evolution of the Kuramoto problem, Eq. (1), for the case of Lorentzian $g(\omega)$. Our formulation will be practically useful if at least some of the macroscopic order-parameter attractors and bifurcations of the full dynamics in the space D are replicated in M. In this regard, we note that numerical solutions of the system (2) for large N have been carried out in Ref. [10], and the resulting macroscopic order-parameter attractors, as well as their bifurcations with variation of system parameters, have been fully mapped out. Comparing these numerical results for the full system (Eq. (2)) with results for the corresponding low dimensional system for the dynamics on M (Eq. (14)), we find that *all* (not just some) of the macroscopic order-parameter attractors and bifurcations of Eq. (2) with Lorentzian $g(\omega)$ are precisely and quantitatively captured by examination of the dynamics on M. These results for the problem given by Eq. (2) suggest that our approach may be useful for other situations such as Eqs. (3). Another notable point is that Ref. [10] also reports numerical simulation results for Eq. (2) with large N for the case of a Gaussian oscillator distribution function, $g(\omega) = (2\pi\Delta^2)^{-1/2}\exp[-(\omega-\omega_0)^2/(2\Delta^2)]$, and the macroscopic order-parameter attractors and bifurcations in this case are found to be the same as those in the Lorentzian case (albeit at different parameter values). Thus, at least for problem (2), phenomena for Lorentzian $g(\omega)$ are not special and should give a useful indication of what can be expected for other unimodal distributions $g(\omega)$.

III. DERIVATION FOR THE EXAMPLE OF THE KURAMOTO PROBLEM

We now support points (i) and (ii) for the case of the Kuramoto problem, Eq. (1). Following that, we will consider other problems, including those associated with Eqs. (2) and (3). Because of its relative simplicity, in this section we use the Kuramoto problem as an example, but we emphasize that our interest is primarily in developing a method that will be useful in less simple cases, such as the problems stated in Eqs. (2) and (3) (see Sec. IV). Following Kuramoto[7, 8], we note that the summation in Eq. (1) can be written as

$$\frac{1}{N}\sum_j \sin[\theta_j - \theta_i] = Im\left\{e^{-i\theta_i}\frac{1}{N}\sum_j e^{i\theta_j}\right\} = Im[re^{-i\theta_i}] \ ,$$

where $r = N^{-1}\sum\exp(i\theta_j)$. Letting $N \to \infty$ in Eq. (1), $f(\omega,\theta,t)$ satisfies the following initial value problem,

$$\partial f/\partial t + \partial/\partial\theta\left\{[\omega + (K/2i)(re^{-i\theta} - r^* e^{i\theta})]f\right\} = 0 \ , \tag{4}$$

$$r = \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\omega f e^{i\theta} \ , \tag{5}$$

where $r(t)$ is the *order parameter*, and Eq. (4) is the continuity equation for the conservation of the number of oscillators. Note that by its definition (5), r satisfies $|r| \leq 1$. Expanding

$f(\omega,\theta,t)$ in a Fourier series in θ, we have

$$f = (g(\omega)/2\pi)\left\{1+[\sum_{n=1}^{\infty} f_n(\omega,t)\exp(in\theta)+c.c.]\right\} ,$$

where *c.c.* stands for complex conjugate. We now consider a restricted class of $f_n(\omega,t)$ such that

$$f_n(\omega,t) = (\alpha(\omega,t))^n ,$$

where $|\alpha(\omega,t)| \leq 1$ to avoid divergence of the series. Substituting this series expansion into Eqs. (4) and (5), we find the remarkable result that this special form of f represents a solution to (4) and (5) if

$$\partial\alpha/\partial t + (K/2)(r\alpha^2 - r^*) + i\omega\alpha = 0 , \tag{6}$$

$$r^* = \int_{-\infty}^{+\infty} d\omega\alpha(\omega,t)g(\omega) . \tag{7}$$

Thus this special initial condition reduces the θ-dependent system, (4), (5) to a problem (6), (7) that is θ-independent. However, we emphasize that Eqs. (6) and (7) still constitute an infinite dimensional dynamical system becuase any initial condition is a *function* of ω, namely $\alpha(\omega,0)$. Performing the summation of the Fourier series using $\sum_{n=1}^{\infty} x^n = x/(1-x)$, we obtain

$$f(\omega,\theta,t) = \frac{g(\omega)}{2\pi}\frac{(1-|\alpha|)(1+|\alpha|)}{(1-|\alpha|)^2+4|\alpha|\sin^2[\frac{1}{2}(\theta-\psi)]} , \tag{8}$$

where $\alpha \equiv |\alpha|e^{-i\psi}$ and ψ real. For $|\alpha| < 1$ we can explicitly verify from Eq. (8) that $f \geq 0$, $\int d\theta f = g(\omega)/2\pi$, and that as $|\alpha| \nearrow 1$ we have $f \to \delta(\theta-\psi)g(\omega)/2\pi$. In order that Eqs. (6) and (7) represent a solution of Eq. (5) for all finite time, we require that, as $\alpha(\omega,t)$ evolves under Eqs. (6) and (7) that $|\alpha(\omega,t)| \leq 1$ continues to be satisfied. This can be shown by substituting $\alpha = |\alpha|e^{-i\psi}$ into Eq. (6), multiplying by $e^{i\psi}$, and taking the real part of the result, thus obtaining

$$\partial|\alpha|/\partial t + (K/2)(|\alpha|^2-1)Re[re^{-i\psi}] = 0 . \tag{9}$$

We see from Eq. (9) that $\partial|\alpha|/\partial t = 0$ at $|\alpha| = 1$. Hence a trajectory of (6), starting with an initial condition satisfying $|\alpha(\omega,0)| < 1$ cannot cross the unit circle in the complex α-plane, and we have $|\alpha(\omega,t)| < 1$ for all finite time, $0 \leq t < +\infty$.

One way to motivate our ansatz, $f_n = \alpha^n$, is to note that the well-known stationary states of the Kuramoto model[7, 8], both the incoherent state ($f = g/2\pi$ corresponding to $\alpha = 0$)