Notice that if $\mathbf{u} = \mathbf{\Phi}(\mathbf{x})$ preserves dimension and system $\dot{\mathbf{u}} = \mathbf{G}(\mathbf{u})$ is completely integrable in $(\mathbb{R}^n, \mathbf{0})$, then Theorem 1 ensures that system (3) is also completely integrable in $(\mathbb{R}^n, \mathbf{0})$. In such a case, the phase portrait of system (3) can be obtained from the phase portrait of $\dot{\mathbf{u}} = \mathbf{G}(\mathbf{u})$. As we will see, in many cases we will take $R = 1$. It is clear that this function simply corresponds to a reparameterization of the trajectories of the system and it has no relevance when people is interested on integrability matters. For brevity, under the hypothesis of Theorem 1, we call the system $\dot{\mathbf{u}} = \mathbf{G}(\mathbf{u})$ *a reduced differential system for system* (3) *through* $\mathbf{\Phi}$.

To the best of our knowledge, our theorem extends known results for $n = 2$ (see Section 3) and provides a new mechanism in the search of (complete) integrable differential systems in any dimension.

Next remark presents two examples that show the importance of the hypothesis on $\mathbf{\Phi}$ in Theorem 1.

Remark 1. *(i) Consider* $\mathbf{x} = (x, y)$, $\mathbf{u} = (u, v)$, $R = 1$, $\mathbf{G}(\mathbf{u}) = (1, 2u)^T$ *and* $\mathbf{\Phi}(\mathbf{x}) = (x, -x^2 + k)$, *for some constant* $k \in \mathbb{R}$. *Then* $I(\mathbf{u}) = u^2 + v$ *is a first integral of* $\dot{\mathbf{u}} = \mathbf{G}(\mathbf{u})$ *but* $H(\mathbf{x}) = I(\mathbf{\Phi}(\mathbf{x})) \equiv k$ *it is not a first integral of* $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$. *Obviously,* $\mathbf{\Phi}$ *does not preserve dimension.*

(ii) Consider $\mathbf{x} = (x, y, z)$, $\mathbf{u} = (u, v, w)$, $R = 1$, $\mathbf{G}(\mathbf{u}) = (0, v, -w)^T$ *and* $\mathbf{\Phi}(\mathbf{x}) = (yz, y, z)$. *Then* $I_1(\mathbf{u}) = u$ *and* $I_2(\mathbf{u}) = uv$ *are functionally independent first integrals of* $\dot{\mathbf{u}} = \mathbf{G}(\mathbf{u})$ *and* $H_1(\mathbf{x}) = I_1(\mathbf{\Phi}(\mathbf{x})) = yz$ *and* $H_2(\mathbf{x}) = I_2(\mathbf{\Phi}(\mathbf{x})) = yz$ *are also first integrals of* $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, *but they are not functionally independent because both coincide. Again,* $\mathbf{\Phi}$ *does not preserve dimension.*

Nevertheless, even when $\mathbf{\Phi}$ does not preserve dimension, the approach of the theorem can sometimes work to provide complete integrable systems. For instance, when $m = n - 1$ first integrals I_j are explicit and functionally independent it may happen that the corresponding H_j are also functionally independent. This situation is illustrated in Example 8 for $n = 2$ (see Section 3) and in Example 14 for $n = 3$ (see Section 4). In fact, in these examples the determinant of $D\mathbf{\Phi}(\mathbf{x})$ vanishes identically.

In general, Theorem 1 has two immediate applications: it allows to construct complete integrable differential systems having singularities from differential systems without singularities and it gives a method to obtain systems in any dimension with a singularity at the origin and with local (complete) integrability.

We also notice that when in system (3) we consider $\mathbf{F} = \mathbf{G}$ and $\mathbf{\Phi}$ an involution, then it is said that the system is *orbitally* $\mathbf{\Phi}$*-symmetric* and in such a case the system presents strong symmetries that make its study simpler. See [5] and references there in.

The paper is organized as follows. Section 2 provides the proof of Theorem 1 as well as some comments on it. In Section 3, we compare the family of integrable planar systems derived from Theorem 1 with previous results on the integrability of planar analytic differential systems. Additionally, we present several examples of the application of Theorem 1 in the real plane, including families of Liénard, Lotka–Volterra, and quadratic differential systems. Section 4 focuses on examples of the application of Theorem 1 in the three dimensional setting, which include differential systems of Kolmogorov, Rikitake and Rössler type. Finally, in Section 5 we present some applications in higher dimensions.

2. Proof of Theorem 1

In this section we will prove Theorem 1 and state some remarks around natural extensions.

Proof of Theorem 1. (i) Let $I\colon (\mathbb{R}^n, \mathbf{0}) \longrightarrow \mathbb{R}$ be a C^k first integral of $\dot{\mathbf{u}} = \mathbf{G}(\mathbf{u})$, that is,

$$\nabla I(\mathbf{u})\, \mathbf{G}(\mathbf{u}) = 0, \quad \text{for all} \quad \mathbf{u} \in (\mathbb{R}^n, \mathbf{0}). \tag{4}$$

Let us prove that the C^k function $H(\mathbf{x}) := (I \circ \mathbf{\Phi})(\mathbf{x})$ it is a first integral of system (3),

$$\begin{aligned} \dot{H}(\mathbf{x}) &= \nabla H(\mathbf{x})\, R(\mathbf{x})\, \operatorname{adj}(D\mathbf{\Phi}(\mathbf{x}))\, \mathbf{G}(\mathbf{\Phi}(\mathbf{x})) \\ &= \nabla I(\mathbf{\Phi}(\mathbf{x}))\, D\mathbf{\Phi}(\mathbf{x})\, R(\mathbf{x})\, \operatorname{adj}(D\mathbf{\Phi}(\mathbf{x}))\, \mathbf{G}(\mathbf{\Phi}(\mathbf{x})) \\ &= \nabla I(\mathbf{\Phi}(\mathbf{x}))\, R(\mathbf{x})\, \det(D\mathbf{\Phi}(\mathbf{x}))\, \mathbf{I}_n\, \mathbf{G}(\mathbf{\Phi}(\mathbf{x})) \\ &= R(\mathbf{x})\, \det(D\mathbf{\Phi}(\mathbf{x}))\, \nabla I(\mathbf{\Phi}(\mathbf{x}))\, \mathbf{G}(\mathbf{\Phi}(\mathbf{x})) = 0, \quad \text{for} \quad \mathbf{x} \in (\mathbb{R}^n, \mathbf{0}), \end{aligned}$$

where we have used that $\nabla I(\mathbf{\Phi}(\mathbf{x}))\, \mathbf{G}(\mathbf{\Phi}(\mathbf{x})) = 0$, which follows by replacing $\mathbf{u} = \mathbf{\Phi}(\mathbf{x})$ in equation (4). Hence it only remains to prove that H is not constant on open sets. We already know that this is the case for the function I, because it is a first integral of $\dot{\mathbf{u}} = \mathbf{G}(\mathbf{u})$. Then, since $\mathbf{\Phi}$ preserves dimension this property is translated to H, as we wanted to prove.

(ii) Assume now that $\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x})$ has $m \leq n-1$ functionally independent C^k first integrals $I_1(\mathbf{u}), \ldots, I_m(\mathbf{u})$ of the system defined in $(\mathbb{R}^n, \mathbf{0})$. Hence, for each $j \in \{1, \ldots, m\}$ we have

$$\nabla I_j(\mathbf{u})\, \mathbf{G}(\mathbf{u}) = 0, \quad \text{for} \quad \mathbf{u} \in (\mathbb{R}^n, \mathbf{0})$$

and, moreover, $\nabla I_1(\mathbf{u}), \ldots, \nabla I_m(\mathbf{u})$ are linearly independent for almost any point $\mathbf{u} \in (\mathbb{R}^n, \mathbf{0})$.

We now define, for each $j \in \{1, \ldots, m\}$, the C^k function $H_j(\mathbf{x}) := (I_j \circ \mathbf{\Phi})(\mathbf{x})$. By item (i) each H_j is a first integral of system (3).

We will prove by contradiction that all $H_1(\mathbf{x}), \ldots, H_m(\mathbf{x})$ are also functionally independent. Assume that the C^k first integrals $H_1(\mathbf{x}), \ldots, H_m(\mathbf{x})$ are functionally dependent in $(\mathbb{R}^n, \mathbf{0})$, that is, there exists an open subset $U \subset (\mathbb{R}^n, \mathbf{0})$ such that for each $\mathbf{x} \in U$ there are real constants $\lambda_1(\mathbf{x}), \ldots, \lambda_m(\mathbf{x})$ not all of them zero, such that

$$\lambda_1(\mathbf{x}) \nabla H_1(\mathbf{x}) + \cdots + \lambda_m(\mathbf{x}) \nabla H_m(\mathbf{x}) = \mathbf{0}.$$

Hence,

$$\big(\lambda_1(\mathbf{x}) \nabla I_1(\mathbf{\Phi}(\mathbf{x})) + \cdots + \lambda_m(\mathbf{x}) \nabla I_m(\mathbf{\Phi}(\mathbf{x}))\big)\, D\mathbf{\Phi}(\mathbf{x}) = \mathbf{0}.$$

Since $\mathbf{\Phi}$ preserves dimension, we can assume, without lost of generality, that $\det(D\mathbf{\Phi}(\mathbf{x})) \neq 0$ for each $\mathbf{x} \in U$. Therefore,

$$\lambda_1(\mathbf{x})\nabla I_1(\mathbf{u}) + \cdots + \lambda_m(\mathbf{x})\nabla I_m(\mathbf{u}) = 0,$$

where we have chosen $\mathbf{u} = \mathbf{\Phi}(\mathbf{x})$ in a non empty open subset of $\mathbf{\Phi}(U)$. The last equality contradicts the linear independence of $\nabla I_1(\mathbf{u}), \ldots, \nabla I_m(\mathbf{u})$ in a full Lebesgue measure open subset of $(\mathbb{R}^n, \mathbf{0})$.

(iii) When $\mathbf{G}(\mathbf{0}) \neq \mathbf{0}$, the flow box theorem guarantees that the C^k differential system $\dot{\mathbf{u}} = \mathbf{G}(\mathbf{u})$ is C^k completely integrable in $(\mathbb{R}^n, \mathbf{0})$, that is, there are $n-1$ functionally independent C^k first integrals $I_1(\mathbf{u}), \ldots, I_{n-1}(\mathbf{u})$ of the system defined in $(\mathbb{R}^n, \mathbf{0})$. Hence the result follows by item (ii). □

Next results are straightforward consequences of Theorem 1.

Remark 2. *If differential system $\dot{\mathbf{u}} = \mathbf{G}(\mathbf{u})$ is C^β completely integrable in $(\mathbb{R}^n, \mathbf{0})$ and it is a reduced differential system for system (3) through a C^α function $\mathbf{\Phi}\colon (\mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^n, \mathbf{0})$, that preserves dimension, then system (3) is C^k completely integrable in $(\mathbb{R}^n, \mathbf{0})$, with $k \geqslant \min\{\alpha, \beta\}$.*

Remark 3. We have considered differential systems in $(\mathbb{R}^n, \mathbf{0})$ only for simplicity. Of course, we can consider differential systems in $(\mathbb{R}^n, \mathbf{p})$, with $\mathbf{p} \in \mathbb{R}^n$ a singular point. For these systems we can use C^k functions $\mathbf{\Phi}, \mathbf{G}\colon (\mathbb{R}^n, \mathbf{p}) \longrightarrow (\mathbb{R}^n, \mathbf{q})$. Therefore, we have analogous results to Theorem 1 if these functions satisfy that $\mathbf{\Phi}(\mathbf{p}) = \mathbf{q}$, $\mathbf{G}(\mathbf{q}) \neq \mathbf{0}$ and the other hypotheses of the theorem. Moreover, Remark 2 also holds.

Remark 4. Under the hypotheses of Theorem 1, if $\dot{\mathbf{u}} = \mathbf{G}(\mathbf{u})$ is completely integrable in $\mathbb{R}^n$ and $\mathbf{\Phi}$ is defined on whole $\mathbb{R}^n$, then system (3) is completely integrable in $\mathbb{R}^n$, that is *globally completely integrable*.

In order to guarantee that a C^k map $\mathbf{\Phi}$ preserves dimension and therefore the application of Theorem 1, we state the following result.

Lemma 1. *Let $\mathbf{\Phi}\colon (\mathbb{R}^n, \mathbf{0}) \longrightarrow \mathbb{R}^n$ be a C^k map, with $k \in \mathbb{N} \cup \{\infty, \omega\}$. If $\det(D\mathbf{\Phi}(\mathbf{x})) \neq 0$ in a full Lebesgue measure open subset of $(\mathbb{R}^n, \mathbf{0})$, then $\mathbf{\Phi}$ preserves dimension.*

Proof. Follows from the Inverse Function Theorem. □

3. Applications in the plane

In this section, we illustrate the application of our main result restricted to the real plane $\mathbb{R}^2$. We start by giving a corollary of Theorem 1 which is a version of that result when $n = 2$. We introduce the notations $\mathbf{x} = (x, y)$, $\mathbf{u} = (u, v)$, $\mathbf{\Phi} = (\varphi, \psi)$ and $\mathbf{G} = (f, g)^T$.

Corollary 1. *Consider C^r planar differential system*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = R(x,y) \begin{pmatrix} \psi_y(x,y) & -\varphi_y(x,y) \\ -\psi_x(x,y) & \varphi_x(x,y) \end{pmatrix} \begin{pmatrix} f(\varphi(x,y), \psi(x,y)) \\ g(\varphi(x,y), \psi(x,y)) \end{pmatrix}, \quad (5)$$