*where $\varphi, \psi, f$, and $g$ are suitable $C^k$ scalar functions in $(\mathbb{R}^2, \mathbf{0})$, with $k \geqslant r$, and $R$ is a real-valued function such that $R(\mathbf{x}) \neq 0$ in a full Lebesgue measure open subset of $(\mathbb{R}^2, \mathbf{0})$. If the map $\mathbf{\Phi} = (\varphi, \psi)$ preserves dimension, then the system is $C^r$ integrable in $(\mathbb{R}^2, \mathbf{0})$ provided that $\varphi(\mathbf{0}) = \psi(\mathbf{0}) = 0$ and $f^2(\mathbf{0}) + g^2(\mathbf{0}) \neq 0$.*

The above corollary extends a similar and previous result proved in [3], which essentially corresponds to the above one but fixing $\psi(x, y) = y^p/p$ and where the hypothesis that $\mathbf{\Phi}(x, y) = (\varphi(x, y), y^p/p)$ preserves dimension is replaced by assuming that $\varphi_x(x, y) \not\equiv 0$. Corollary 1 also covers similar results that appear in [4] and in the references of both papers.

**Remark 5.** *The differential system (5) admits a more compact version by using the notation of 1-forms. Notice that the reduced system through $\mathbf{\Phi}$ can be written as $f(u, v)\,\mathrm{d}v - g(u, v)\,\mathrm{d}u = 0$, while system (5) writes as*

$$f(\varphi(x, y), \psi(x, y))\,\mathrm{d}(\psi(x, y)) - f(\varphi(x, y), \psi(x, y))\,\mathrm{d}(\varphi(x, y)) = 0,$$

*where notice that we have omitted the multiplicative factor $R(x, y)$. Indeed, this expression is the pull-back of the initial 1-form:*

$$\mathbf{\Phi}^*(f(u, v)\,\mathrm{d}v - g(u, v)\,\mathrm{d}u) = 0.$$

Since in the planar case, the weak and strong versions of the integrability problem coincide, we will use, for simplicity, the adjective integrable instead of completely integrable. Next we list several integrable examples in the plane. In the first seven examples the functions $\mathbf{\Phi}$ that we will use preserve dimension, which can be proved easily using Lemma 1.

**Example 1.** By using $R = 1$ and the polynomial functions

$$\mathbf{\Phi}(\mathbf{x}) = \left(xy, x + \frac{y^2}{2}\right) \quad \text{and} \quad \mathbf{G}(\mathbf{u}) = \big(u, 1 - u\big)^T,$$

system (3) becomes

$$\begin{aligned} \dot{x} &= x(-1 + xy + y^2), \\ \dot{y} &= y(1 - x - xy), \end{aligned}$$

which is a Kolmogorov planar system. Since $\mathbf{\Phi}(\mathbf{0}) = \mathbf{0}$ and $\mathbf{G}(\mathbf{0}) \neq \mathbf{0}$, this system is $C^\omega$ integrable in $(\mathbb{R}^2, \mathbf{0})$ by Theorem 1. Moreover, since system $\dot{\mathbf{u}} = \mathbf{G}(\mathbf{u})$ has the first integral $I(\mathbf{u}) = ue^{-u-v}$, our Kolmogorov system is $C^\omega$ integrable in whole real plane, with the analytic Darboux first integral $H(\mathbf{x}) = xy \exp\big(-x - xy - y^2/2\big)$. Notice that the reduced system is free of finite singularities, while the Kolmogorov system has two singularities on the plane: a saddle at $\mathbf{0}$ and a center at a point in the first quadrant.

**Example 2.** By taking $R = 1$ and the polynomial functions

$$\mathbf{\Phi}(\mathbf{x}) = \left(\frac{x^2}{2} + \frac{by^2}{2}, \frac{y^2}{2}\right) \quad \text{and} \quad \mathbf{G}(\mathbf{u}) = \big(1, -2u\big)^T,$$

where $b \in \mathbb{R}$ is a parameter, system (3) becomes

$$\begin{aligned}\dot{x} &= y + bx^2y + b^2y^3,\\ \dot{y} &= -x^3 - bxy^2,\end{aligned}$$

which is a planar system with a nilpotent singularity at the origin. Since $\mathbf{\Phi}(\mathbf{0}) = \mathbf{0}$ and $\mathbf{G}(\mathbf{0}) \neq \mathbf{0}$, this system is $C^\omega$ integrable in $(\mathbb{R}^2, \mathbf{0})$ by Theorem 1. Furthermore, since system $\dot{\mathbf{u}} = \mathbf{G}(\mathbf{u})$ has the first integral $I(\mathbf{u}) = v + u^2$, our system is polynomially integrable in whole real plane, with the polynomial first integral $H(\mathbf{x}) = y^2/2 + (x^2/2 + by^2/2)^2$, which has a minimum at the origin. Since the system is hamiltonian, the origin is a center.

**Example 3.** For each $k \in \mathbb{N}$, set $a = \sqrt{2} + k$. The linear differential system $\dot{\mathbf{u}} = \mathbf{G}_k(\mathbf{u})$ given by

$$\begin{aligned}\dot{u} &= (a+3)u + 2av,\\ \dot{v} &= -2au - (4a-3)v,\end{aligned} \tag{6}$$

has a saddle at the origin and the first quadrant is foliated by orbits of a hyperbolic sector of the saddle. By using $R = 1$ and the polynomial function $\mathbf{\Phi}(\mathbf{x}) = \big(x^2, y^2\big)$, system (3) writes as

$$\begin{aligned}\dot{x} &= 2(a+3)x^2y + 4ay^3,\\ \dot{y} &= -4ax^3 - 2(4a-3)xy^2,\end{aligned} \tag{7}$$

which has a unique degenerated singularity at $\mathbf{0}$.

The form of $\mathbf{\Phi}(\mathbf{x})$ implies that the orbits of system (6) contained in the first quadrant come from closed curves around the origin in the $xy$-plane. Hence, system (7) has a degenerated global center at $\mathbf{0}$. Furthermore, system (6) is $C^k$ integrable in $\mathbb{R}^2$, with $C^k$ first integral $I(\mathbf{u}) = (u+2v)(2u+v)^{a-1}$. Since $\mathbf{\Phi}(\mathbf{x})$ is defined on $\mathbb{R}^2$, Remark 2 implies that system (7) is $C^k$ integrable in $\mathbb{R}^2$, with $C^k$ first integral

$$H(\mathbf{x}) = (x^2 + 2y^2)(2x^2 + y^2)^{a-1}.$$

**Example 4.** The reversible planar quadratic differential systems with a center at the origin write as

$$\begin{aligned}\dot{x} &= y + \alpha xy,\\ \dot{y} &= -x + \beta x^2 + \gamma y^2,\end{aligned}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$, and are also called Loud systems. By Poincaré linearizability Theorem it is already known that they have an analytic first integral at $(\mathbb{R}^2, \mathbf{0})$. Let us prove this fact as a consequence of Theorem 1. They can be written in the form (3) with $R = 1$,

$$\mathbf{\Phi}(\mathbf{x}) = \left(x, \frac{y^2}{2}\right) \quad \text{and} \quad \mathbf{G}(\mathbf{u}) = \big(1 + \alpha u, -u + \beta u^2 + 2\gamma v\big)^T.$$

Since $\mathbf{\Phi}(\mathbf{0}) = \mathbf{0}$ and $\mathbf{G}(\mathbf{0}) \neq \mathbf{0}$, the Loud system is locally $C^\omega$ integrable by Theorem 1. Observe that the reduced system is also quadratic. But, if

$\beta(\gamma-\alpha) \neq 0$, then we can get a reduced system that is linear. Indeed, in such a case the Loud system can be written in the form (3) with $R = 1$,

$$\mathbf{\Phi}(\mathbf{x}) = \left(x^2 + \frac{(\gamma-\alpha)\,y^2}{\beta}, (\gamma-\alpha)\,x\right)$$

and

$$\mathbf{G}(\mathbf{u}) = \left(-\frac{\beta\,\gamma\,u}{(\gamma-\alpha)^2} + \frac{(\gamma-\alpha-\beta)\,v}{(\gamma-\alpha)^3}, -\frac{1}{2}\,\frac{\beta\,\alpha\,v}{(\gamma-\alpha)^2} - \frac{1}{2}\,\frac{\beta}{\gamma-\alpha}\right)^T.$$

**Example 5.** The analytic Liénard system

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -xg(x^2) - xyf(x^2),\end{aligned}$$

where $g$ and $f$ are $C^\omega$ functions in $(\mathbb{R}, 0)$ and $g(0) \neq 0$, can be written in the form (3) with $R = 1$,

$$\mathbf{\Phi}(\mathbf{x}) = \left(x^2, y\right) \quad \text{and} \quad \mathbf{G}(\mathbf{u}) = \left(v, -\frac{g(u) + vf(u)}{2}\right)^T.$$

Since $\mathbf{\Phi}(\mathbf{0}) = \mathbf{0}$ and $\mathbf{G}(\mathbf{0}) = (0, -g(0)/2) \neq \mathbf{0}$, this Liénard system is locally $C^\omega$ integrable by Theorem 1.

**Example 6.** The quadratic differential system

$$\begin{aligned}\dot{x} &= x(ax + by + c),\\ \dot{y} &= y\left(\tfrac{ab}{B}x + By - c\right),\end{aligned}$$

where $a, b \in \mathbb{R}$ and $c, B \in \mathbb{R}^* := \mathbb{R}\backslash\{0\}$, is a (4-parametric) subcase of the classical Lotka–Volterra systems. It can be written in the form (3) with $R = 1$,

$$\mathbf{\Phi}(\mathbf{x}) = \left(xy, y - \frac{a}{B}x\right) \quad \text{and} \quad \mathbf{G}(\mathbf{u}) = \left((b+B)u, Bv - c\right)^T.$$

Since $\mathbf{\Phi}(\mathbf{0}) = \mathbf{0}$ and $\mathbf{G}(\mathbf{0}) = (0, -c) \neq \mathbf{0}$, this particular Lotka–Volterra system is locally $C^\omega$ integrable by Theorem 1. Moreover, the reduced system $\dot{\mathbf{u}} = \mathbf{G}(\mathbf{u})$ can be solved easily, whence we get that it has

$$I(\mathbf{u}) = u\left(Bv - c\right)^{-\frac{b+B}{B}}$$

as a $C^\omega$ first integral in $(\mathbb{R}^2, \mathbf{0})$. Thus,

$$H(\mathbf{x}) = xy\left(By - ax - c\right)^{-\frac{b+B}{B}}$$

is a $C^\omega$ first integral in $(\mathbb{R}^2, \mathbf{0})$ of our particular Lotka–Volterra system. Furthermore, if $b = -(p+q)B/q$, with $p, q \in \mathbb{N}$, then the system is globally polynomially integrable, with the polynomial first integral $H_{p,q}(\mathbf{x}) = (xy)^q(By - ax - c)^p$, cf. [18].