by Remark 3 and Theorem 1. Moreover, it is easy to show that system $\dot{\mathbf{u}} = \mathbf{G}(\mathbf{u})$ has the functionally independent first integrals $I_1(\mathbf{u}) = uv$ and $I_2(\mathbf{u}) = v^2 e^w$. Hence, Rikitake system (11) is also Darboux completely integrable, with independent Darboux first integrals

$$H_1(\mathbf{x}) = \frac{y^2 - x^2}{2} \qquad \text{and} \qquad H_2(\mathbf{x}) = (y - x)^2\, e^{x^2+z^2},$$

cf. [17, 20].

Example 13. A complete integrable Lotka–Volterra system. The polynomial differential system

$$\begin{aligned}\dot{x} &= x(z - cy),\\ \dot{y} &= y(x - az),\\ \dot{z} &= z\Big(y - \frac{x}{ac}\Big),\end{aligned} \tag{12}$$

with $a, c \in \mathbb{R}^+$, is part of the so-called (a, b, c) Lotka–Volterra systems [13].

If $c \geqslant 1$, then system (12) can be written in the form (3), with the C^1 functions $R(\mathbf{x}) = (1/c)z^{1-c}$,

$$\mathbf{\Phi}(\mathbf{x}) = \big(acz + cy + x, y, xz^c\big), \quad \text{and} \quad \mathbf{G}(\mathbf{u}) = \Big(0, v, -\frac{w}{a}\Big)^T.$$

If $c < 1$, then system (12) can be written in the form (3), with the C^1 functions $R(\mathbf{x}) = x^{1-1/c}$,

$$\mathbf{\Phi}(\mathbf{x}) = \big(acz + cy + x, y, x^{1/c}z\big), \quad \text{and} \quad \mathbf{G}(\mathbf{u}) = \Big(0, v, -\frac{w}{ac}\Big)^T.$$

In both cases, $c \geqslant 1$ and $c < 1$, the reduced differential system can be solved easily, whence we get that $I_1(\mathbf{u}) = u$ is a polynomial first integral of $\dot{\mathbf{u}} = \mathbf{G}(\mathbf{u})$. Moreover, on one hand, for case $c \geqslant 1$ we obtain that if $a \geqslant 1$, then $I_2(\mathbf{u}) = vw^a$ is a C^β first integral in $\mathbb{R}^3$ of the reduced system, with $\beta = [a] \in \mathbb{N}$, and if $a < 1$, then $I_2(\mathbf{u}) = v^{1/a}w$ is a C^β first integral in $\mathbb{R}^3$ of the reduced system, with $\beta = [1/a] \in \mathbb{N}$. On the other hand, for case $c < 1$ it follows that if $a \geqslant 1$, then $I_2(\mathbf{u}) = v^{1/c}w^a$ is a C^β first integral in $\mathbb{R}^3$ of the reduced system, with $\beta = \min\{[a], [1/c]\} \in \mathbb{N}$, and if $a < 1$, then $I_2(\mathbf{u}) = v^{1/(ac)}w$ is a C^β first integral in $\mathbb{R}^3$ of the reduced system, with $\beta = [1/(ac)] \in \mathbb{N}$.

Since $\mathbf{\Phi}(\mathbf{0}) = \mathbf{0}$, $\mathbf{\Phi}(\mathbf{x})$ is a polynomial or C^α function, with $\alpha = [c] \in \mathbb{N}$ if $c \geqslant 1$ and $\alpha = [1/c] \in \mathbb{N}$ if $c < 1$, and system $\dot{\mathbf{u}} = \mathbf{G}(\mathbf{u})$ is polynomially or C^β completely integrable in $\mathbb{R}^3$, the Lotka–Volterra system is polynomially or C^k completely integrable in $\mathbb{R}^3$ by Remark 2, with $k \geqslant 1$. More precisely, the Lotka–Volterra system always has the polynomial first integral

$$H_1(\mathbf{x}) = x + cy + acz.$$

If $a = p/q \in \mathbb{Q}^+$ and $c = p_1/q_1 \in \mathbb{Q}^+$, then the Lotka–Volterra system is polynomially completely integrable in $\mathbb{R}^3$, with the additional independent polynomial first integral

$$H_2(\mathbf{x}) = x^{pq_1} y^{qq_1} z^{pp_1}.$$

If $c \notin \mathbb{Q}^+$ and $c > 1$, then the Lotka–Volterra system is C^k completely integrable in $\mathbb{R}^3$, with the additional functionally independent C^k first integral either

$$H_2(\mathbf{x}) = x^a y z^{ac} \quad \text{or} \quad H_2(\mathbf{x}) = x y^{1/a} z^c,$$

where $k \geqslant \min\big\{[a], [ac]\big\}$ if $a > 1$ or $k \geqslant \min\big\{[1/a], [c]\big\}$ if $a < 1$.

If $c \notin \mathbb{Q}^+$ and $c < 1$, then the Lotka–Volterra system is C^k completely integrable in $\mathbb{R}^3$, with the additional functionally independent C^k first integral either

$$H_2(\mathbf{x}) = x^{a/c} y^{1/c} z^a \quad \text{or} \quad H_2(\mathbf{x}) = x^{1/c} y^{1/(ac)} z,$$

where $k \geqslant \min\big\{[1/c], [a]\big\}$ if $a > 1$ or $k \geqslant \min\big\{[1/c], [1/(ac)]\big\}$ if $a < 1$, cf. [6, 13].

We conclude this section with an example where the function $\mathbf{\Phi}$ does not preserve dimension but the approach developed also allows to prove complete integrability.

Example 14. By using $R = 1$ and the polynomial functions

$$\mathbf{\Phi}(\mathbf{x}) = \big(xz - y^2, yz - y^2, xz - yz\big) \quad \text{and} \quad \mathbf{G}(\mathbf{u}) = \big(0, 0, 1\big)^T$$

system (3) becomes

$$\begin{aligned} \dot{x} &= 2xy - xz - 2y^2, \\ \dot{y} &= -yz, \\ \dot{z} &= -2yz + z^2. \end{aligned}$$

Here $\mathbf{\Phi}(\mathbf{0}) = \mathbf{0}$ and $\mathbf{G}(\mathbf{0}) \neq \mathbf{0}$ but we can not apply Theorem 1 because $\mathbf{\Phi} = (\varphi, \psi, \eta)$ does not preserve dimension. In fact, its components satisfy $\varphi - \psi - \eta = 0$. Notice that $\det(D\mathbf{\Phi}(\mathbf{x})) \equiv 0$. Nevertheless the system is polynomially completely integrable in $(\mathbb{R}^3, \mathbf{0})$. The reason is that $\mathbf{\Phi}(\mathbf{x})$ is polynomial and its reduced system is polynomially completely integrable, with first integrals $I_1(\mathbf{u}) = u$ and $I_2(\mathbf{u}) = v$. Hence their transformed first integrals $H_1(\mathbf{x}) = xz - y^2$ and $H_2(\mathbf{x}) = yz - y^2$ are also functionally independent.

5. Applications in higher dimensions

In this section, we illustrate the application of our main result in higher dimensions. In particular, an example for arbitrary dimension is presented. Again the functions $\mathbf{\Phi}$ in next examples preserve dimension according to Lemma 1.

Example 15. Some 4D complete integrable Kolmogorov systems. For each $l \in \mathbb{N}$, consider the polynomial Kolmogorov system

$$\begin{aligned} \dot{x}_1 &= x_1(x_2^l - x_4^l), \\ \dot{x}_2 &= x_2(x_3^l - x_1^l), \\ \dot{x}_3 &= x_3(x_4^l - x_2^l), \\ \dot{x}_4 &= x_4(x_1^l - x_3^l), \end{aligned} \tag{13}$$

which for $l = 1$ is a Lotka–Volterra system. This differential system can be written in the form (3) with the analytic functions $R = 1$,

$$\mathbf{\Phi}(\mathbf{x}) = \big(x_1, x_1x_3, x_2x_4, x_1^l + x_2^l + x_3^l + x_4^l\big)$$

and $\mathbf{G}(\mathbf{u}) = (1/l, 0, 0, 0)^T$. Since $\Phi(\mathbf{0}) = \mathbf{0}$ and $\mathbf{G}(\mathbf{0}) \neq \mathbf{0}$, system (13) is analytically completely integrable by Theorem 1. Moreover, the differential system $\dot{\mathbf{u}} = \mathbf{G}(\mathbf{u})$ is polynomially completely integrable, with polynomial first integrals $I_i(\mathbf{u}) = u_{i+1}$, for $i = 1, 2, 3$. Hence, system (13) is polynomially completely integrable, with independent polynomial first integrals

$$H_1(\mathbf{x}) = x_1x_3, \quad H_2(\mathbf{x}) = x_2x_4 \quad \text{and} \quad H_3(\mathbf{x}) = \sum_{i=1}^{4} x_i^l.$$

Example 16. Some 4D complete integrable nilpotent systems. Consider the polynomial differential systems

$$\begin{aligned} \dot{x}_1 &= P_1\big(x_2 + A_1(x_1)\big), \\ \dot{x}_2 &= P_2\big(x_3 + \tfrac{1}{d_2}A_2(x_1)\big) - A_1'(x_1)P_1\big(x_2 + A_1(x_1)\big), \\ \dot{x}_3 &= P_3\big(x_4 + \tfrac{1}{d_3}A_3(x_1)\big) - \tfrac{A_2'(x_1)}{d_2}P_1\big(x_2 + A_1(x_1)\big), \\ \dot{x}_4 &= -\tfrac{A_3'(x_1)}{d_3}P_1\big(x_2 + A_1(x_1)\big), \end{aligned} \tag{14}$$

where $P_i \in \mathbb{R}[s]$ is a monic polynomial of degree d_i, with $d_2d_3 \neq 0$, $P_1^2(\mathbf{0}) + P_2^2(\mathbf{0}) + P_3^2(\mathbf{0}) \neq 0$, and $A_i(x_1) = a_i + b_ix_1$, $a_i, b_i \in \mathbb{R}$, for $i = 1, 2, 3$. This system is a *nilpotent differential system* since its right-hand side is a nilpotent map [7], that is, the linearization matrix of the differential system at each point of $\mathbb{R}^4$ is nilpotent.

System (14) can be written in the form (3) with the analytic functions

$$\mathbf{\Phi}(\mathbf{x}) = \Big(x_1, x_2 + A_1(x_1), x_3 + \frac{1}{d_2}A_2(x_1), x_4 + \frac{1}{d_3}A_3(x_1)\Big)$$

and

$$\mathbf{G}(\mathbf{u}) = \Big(P_1(u_2), P_2(u_3), P_3(u_4), 0\Big)^T.$$

Since $\mathbf{\Phi}(\mathbf{0}) = \mathbf{0}$, $\mathbf{G}(\mathbf{0}) = (P_1(\mathbf{0}), P_2(\mathbf{0}), P_3(\mathbf{0}), 0) \neq \mathbf{0}$, and the reduced system $\dot{\mathbf{u}} = \mathbf{G}(\mathbf{u})$ is polynomially completely integrable, the differential system (14) is polynomially completely integrable in $\mathbb{R}^4$ by Theorem 1.