Example 17. A n-dimensional family of Kolmogorov complete integrable systems. Consider the polynomial functions $R = 1$,

$$\mathbf{\Phi}(\mathbf{x}) = \Big(x_1, x_1x_2, x_2x_3, \dots, x_{n-1}x_n\Big)$$

and the C^r function

$$\mathbf{G}(\mathbf{u}) = \big(G_1(\mathbf{u}), 1 + G_2(\mathbf{u}), \dots, 1 + G_{n-1}(\mathbf{u}), 0\big),$$

with $\mathbf{u} \in (\mathbb{R}^n, \mathbf{0})$ and $G_i(\mathbf{0}) = 0$ for $i \geqslant 2$.

From the identity $\mathbf{A}\,\mathrm{adj}(\mathbf{A}) = \det(\mathbf{A})\mathbf{I}_n$ and by defining

$$P_{ij} := (-1)^{i+j} \prod_{\mu=1}^{j-2} x_\mu \prod_{\nu=j+1}^{n-1} x_\nu,$$

with $\prod_{\mu=1}^{j-2} x_\mu \equiv 1$ if $j \leqslant 2$ and $\prod_{\nu=j+1}^{n-1} x_\nu \equiv 1$ if $j \geqslant n-1$, it follows that

$$\mathrm{adj}(D\mathbf{\Phi}(\mathbf{x})) = \begin{pmatrix} x_1P_{11} & 0 & \cdots & 0 \\ x_2P_{21} & x_2P_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_nP_{n1} & x_nP_{n2} & \cdots & P_{nn} \end{pmatrix}.$$

Hence, system (3) becomes

$$\begin{aligned} \dot{x}_1 &= x_1P_{11}G_1(\mathbf{\Phi}(\mathbf{x})), \\ \dot{x}_2 &= x_2\Big(P_{21}G_1(\mathbf{\Phi}(\mathbf{x})) + P_{22}\big(1 + G_1(\mathbf{\Phi}(\mathbf{x}))\big)\Big), \\ &\ \vdots \\ \dot{x}_n &= x_n\Big(P_{n1}G_1(\mathbf{\Phi}(\mathbf{x})) + \textstyle\sum_{j=2}^{n-1} P_{nj}\big(1 + G_j(\mathbf{\Phi}(\mathbf{x}))\big)\Big), \end{aligned}$$

which is a C^r Kolmogorov system in $(\mathbb{R}^n, \mathbf{0})$. Since $\Phi(\mathbf{0}) = \mathbf{0}$ and $\mathbf{G}(\mathbf{0}) \neq \mathbf{0}$, this system is C^r completely integrable in $(\mathbb{R}^n, \mathbf{0})$ by Theorem 1.

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