A NEW FAMILY OF INTEGRABLE DIFFERENTIAL SYSTEMS IN ARBITRARY DIMENSION

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Abstract. We present a wide class of differential systems in any dimension that are either integrable or complete integrable. In particular, our result enlarges a known family of planar integrable systems. We give an extensive list of examples that illustrates the applicability of our approach. For instance, in the plane this list includes some Liénard, Lotka–Volterra and quadratic systems; in the space, some Kolmogorov, Rikitake and Rössler systems. Examples of complete integrable systems in higher dimensions are also provided.

1. Introduction and main result

This work deals with C^r real autonomous differential systems, *i.e.*, systems of ordinary differential equations

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \qquad \mathbf{x} \in \Omega \subset \mathbb{R}^n, \tag{1}$$

where the dot denotes derivative with respect to an independent real variable t, Ω is an open subset of $\mathbb{R}^n$ and $\mathbf{F}$ is a C^r function in Ω, with $r \in \mathbb{N} \cup \{\infty, \omega\}$, where as usual $\mathbb{N}$ denotes the set of positive integer numbers. Recall that C^ω stands for analytic functions.

A C^k *first integral* in Ω of system (1) is a locally non-constant C^k function $H: \Omega \longrightarrow \mathbb{R}$, with $k \in \mathbb{N} \cup \{\infty, \omega\}$, that satisfies the equation

$$\dot{H}(\mathbf{x}) = \nabla H(\mathbf{x})\, \mathbf{F}(\mathbf{x}) = 0$$

on the whole Ω. This means that H is constant throughout the orbits of the system. In other words, the orbits of the system live in the level hypersurfaces of the first integral H, which implies that we can reduce in one dimension the study of (1).

Recall that the C^k functions $H_1(\mathbf{x}), \dots, H_m(\mathbf{x})$ are *functionally independent in* Ω if their gradients, $\nabla H_1(\mathbf{x}), \dots, \nabla H_m(\mathbf{x})$, are linearly independent in a full Lebesgue measure open subset of Ω. Hence, if system (1) has m functionally independent C^k first integrals in Ω, with $1 \leqslant m < n$, then the

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orbits of the system live in the intersection set of the level hypersurfaces of these first integrals, which generically is a $(n-m)$-dimensional subset of Ω. If $m = n-1$, then the intersection set is in general one-dimensional, and so it is formed by orbits of the system. Thus, if system (1) has $n-1$ functionally independent first integrals, then all orbits of the system are determined by the first integrals except perhaps in a zero Lebesgue measure subset of Ω. Therefore, it is said that system (1) is C^k *completely integrable in* Ω if it has $n-1$ functionally independent C^k first integrals in Ω.

The *weak* or *strong* versions of the integrability problem consist in finding conditions on $\mathbf{F}(\mathbf{x})$ under which system (1) has one first integral or $n-1$ functionally independent first integrals, respectively. Both versions also ask about the type of such first integrals: are they C^k functions, with $k \in \mathbb{N} \cup \{\infty, \omega\}$, polynomial, rational, Darboux, or Liouville? In this sense, differential system (1) is called *polynomially integrable in* Ω if it has a polynomial first integral in Ω and it is called *polynomially completely integrable in* Ω if it has $n-1$ functionally independent polynomial first integrals in Ω. Similarly, we can define rational, Darboux, or Liouville (completely) integrable in Ω.

The integrability of differential systems (1) is a very active research issue in the field of ordinary differential equations. See for instance [1, 2, 15] and references there in, or next discussion. There are different tools and methods to study the integrability of differential systems; for example, Lie symmetries [21], Noether symmetries [24], the Darboux theory of integrability [9, 10, 16], Lax pairs [12, 14, 16], singularity analysis [12], and integrating factors as well as inverse integrating factors [11]. However, in general, the problem of determining whether a concrete differential system (1) is integrable remains unsolved.

In this work, we will focus in C^r real autonomous differential systems of the form

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \qquad \mathbf{x} \in (\mathbb{R}^n, \mathbf{0}), \tag{2}$$

where $(\mathbb{R}^n, \mathbf{0})$ is a small enough open neighborhood of the origin $\mathbf{0}$ of $\mathbb{R}^n$. We are mainly interested in the local strong integrability problem, that is, to find a particular structure on $\mathbf{F}(\mathbf{x})$ under which system (2) is (completely) integrable in $(\mathbb{R}^n, \mathbf{0})$. We shall assume that $\mathbf{0}$ is a singular point of the differential system, that is, $\mathbf{F}(\mathbf{0}) = \mathbf{0}$, because otherwise the classical flow box theorem solves the problem.

Recall that $\operatorname{adj}(\mathbf{A})$ denotes the adjoint of a square matrix $\mathbf{A}$ and it is the transpose of the cofactor matrix of $\mathbf{A}$. If A is of size n, then the identity $\mathbf{A}\operatorname{adj}(\mathbf{A}) = \det(\mathbf{A})\mathbf{I}_n$ holds, where $\mathbf{I}_n$ is the identity matrix of size n. Moreover, in particular, when $\mathbf{A}$ is invertible $\mathbf{A}^{-1} = \operatorname{adj}(\mathbf{A})/\det(\mathbf{A})$.

Before giving our main result we explain the simple idea that leads to its statement. It is well-known that if two differential systems, both defined in $(\mathbb{R}^n, \mathbf{0})$,

$$\dot{\mathbf{x}} = \mathbf{K}(\mathbf{x}) \quad \text{and} \quad \dot{\mathbf{u}} = \mathbf{G}(\mathbf{u}),$$

are conjugated via a C^k diffeomorphism $\mathbf{u} = \mathbf{\Phi}(\mathbf{x})$, $\mathbf{x} = \mathbf{\Phi}^{-1}(\mathbf{u})$, such that $\mathbf{\Phi}(\mathbf{0}) = \mathbf{0}$, then they share all their dynamical properties. In particular, they have the same number of functionally independent first integrals. Let us specify the relation between both differential systems:

$$\begin{aligned}\mathbf{K}(\mathbf{x}) &= D\mathbf{\Phi}^{-1}(\mathbf{\Phi}(\mathbf{x}))\mathbf{G}(\mathbf{\Phi}(\mathbf{x})) = \big(D\mathbf{\Phi}(\mathbf{x})\big)^{-1}\mathbf{G}(\mathbf{\Phi}(\mathbf{x}))\\ &= \frac{\operatorname{adj}(D\mathbf{\Phi}(\mathbf{x})))}{\det(D\mathbf{\Phi}(\mathbf{x}))}\mathbf{G}(\mathbf{\Phi}(\mathbf{x})).\end{aligned}$$

Moreover, if in the differential system $\dot{\mathbf{x}} = \mathbf{K}(\mathbf{x})$ we change the time t by a new one, s, such that $\mathrm{d}t/\mathrm{d}s = R(\mathbf{x})\det(D\mathbf{\Phi}(\mathbf{x}))$, for some real-valued function R that is defined in an open subset $\mathcal{O}$ of $(\mathbb{R}^n, \mathbf{0})$ and it is nonzero on open sets, then we get the new differential system

$$\mathbf{x}' = \frac{\mathrm{d}t}{\mathrm{d}s}\dot{\mathbf{x}} = R(\mathbf{x})\operatorname{adj}(D\mathbf{\Phi}(\mathbf{x})))\mathbf{G}(\mathbf{\Phi}(\mathbf{x})),$$

which is equivalent to $\dot{\mathbf{u}} = \mathbf{G}(\mathbf{u})$ in $\mathcal{O}\backslash\{R(\mathbf{x}) = 0\}$. It is also well-known that this new differential system has the same number of functionally independent first integrals that $\dot{\mathbf{u}} = \mathbf{G}(\mathbf{u})$.

Although the derivation of the last differential system uses that $\mathbf{\Phi}$ is invertible, its structure remains well-defined even when $\mathbf{\Phi}$ is not invertible. This observation serves as the basis for conjecturing that completely integrable differential systems can be constructed by using non-invertible functions $\mathbf{\Phi}$. This is precisely what our main result will establish.

We also must introduce a key concept in our approach. Let $\mathbf{\Phi}\colon (\mathbb{R}^n, \mathbf{0}) \longrightarrow \mathbb{R}^n$ be a C^k map, with $k \in \mathbb{N} \cup \{\infty, \omega\}$. We say that $\mathbf{\Phi}$ *preserves dimension* if the image $\mathbf{\Phi}(U)$ of any open set $U \subset (\mathbb{R}^n, \mathbf{0})$ contains open sets. Of course, diffeomorphisms preserve dimension, but other non-injective or non-exhaustive maps, like for instance the 2-fold map $\mathbf{\Phi}(x, y) = (x^2, y)$, do as well.

Theorem 1. *Consider the C^r differential system, with $r \in \mathbb{N} \cup \{\infty, \omega\}$,*

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) := R(\mathbf{x})\operatorname{adj}(D\mathbf{\Phi}(\mathbf{x}))\,\mathbf{G}(\mathbf{\Phi}(\mathbf{x})), \qquad \mathbf{x} \in (\mathbb{R}^n, \mathbf{0}), \tag{3}$$

with $\mathbf{F}(\mathbf{0}) = \mathbf{0}$, where $\mathbf{\Phi}, \mathbf{G}\colon (\mathbb{R}^n, \mathbf{0}) \longrightarrow \mathbb{R}^n$ are suitable C^k functions, with $k \geq r$, $\mathbf{\Phi}(\mathbf{0}) = \mathbf{0}$, and $R(\mathbf{x})$ is a real-valued function such that $R(\mathbf{x}) \neq 0$ in a full Lebesgue measure open subset of $(\mathbb{R}^n, \mathbf{0})$. Assume also that $\mathbf{\Phi}$ preserves dimension. Then:

(i) If $I\colon (\mathbb{R}^n, \mathbf{0}) \longrightarrow \mathbb{R}$ is a C^k first integral of $\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x})$, then the function $H(\mathbf{x}) := (I \circ \mathbf{\Phi})(\mathbf{x})$ is a C^k first integral of system (3).

(ii) If $I_j\colon (\mathbb{R}^n, \mathbf{0}) \longrightarrow \mathbb{R}$, $j = 1, \ldots, m \leq n-1$, are m functionally independent C^k first integrals of $\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x})$, then the functions $H_j(\mathbf{x}) := (I_j \circ \mathbf{\Phi})(\mathbf{x})$, $j = 1, \ldots, m$, are functionally independent C^k first integrals of system (3).

(iii) If $\mathbf{G}(\mathbf{0}) \neq \mathbf{0}$, then system (3) is C^k completely integrable in $(\mathbb{R}^n, \mathbf{0})$.