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# A Appendix

## A.1 Definition of Ideal Budget Range

**Ideal Budget Range.** Based on the observation of token elasticity, a token cost bottom range exists during searching for the optimal budget. In this range, the token costs approach the token cost lowest bound. Before or after the range, the token cost will increase. We define such a bottom range as "ideal budget range". It's worth noting that the budget continuously degrades during the search. Only the token cost rebounds. That's why we refer to this observation as token elasticity. To summarize, ideal budget range is an range that minimizes actual token consumption. Let $\boldsymbol{\beta} = \{\beta_1, \beta_2, ..., \beta_N\}$ denote all possible budgets that can maintain answer correctness. A rolling window $W \in \boldsymbol{\beta}$ is applied iteratively over $\boldsymbol{\beta}$. Let $k$ represent the range size, which is adaptively determined during our evaluation as $\frac{N}{3}$, where $N$ is the total number of possible budgets. A budget range is defined as:

$$W_k(i) = \{\boldsymbol{\beta}_j \mid i \leq j \leq i + k - 1\},$$
$$1 \leq i \leq |\boldsymbol{\beta}| - k + 1$$

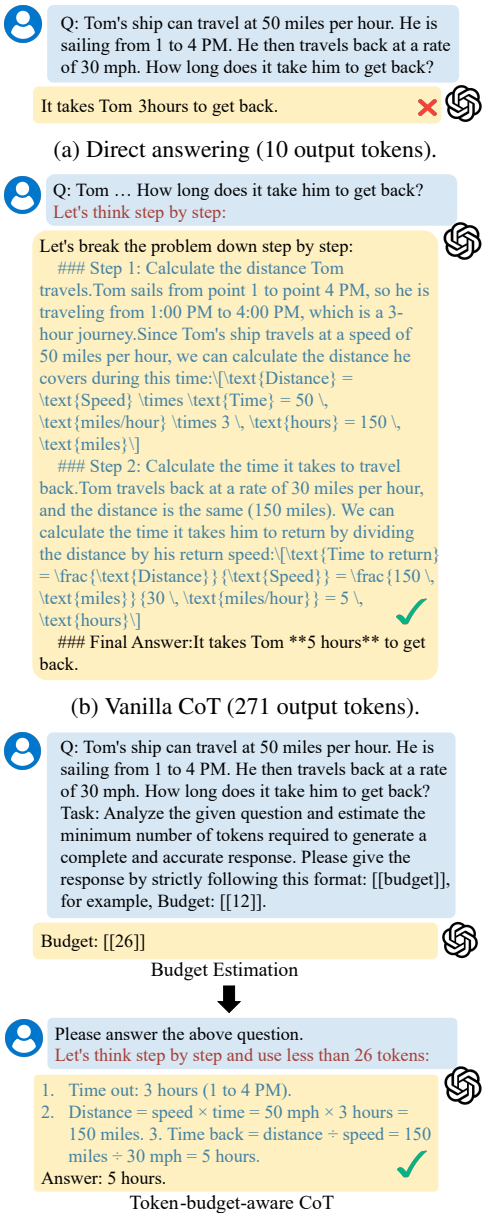


Figure 7: An intuitive example to illustrate the workflow of TALE-EP on GPT-4o-mini (OpenAI, 2024a).

The ideal budget range $W^*$ is defined as:

$$W_k^* = \arg\min_i \left( \sum_{\beta_j \in W_k(i)} \mathbb{T}(\beta_j) \right), \quad (1)$$

where $\mathbb{T}$ denote the actual token consumption for a given budget $\beta \in \boldsymbol{\beta}$. We aim to estimate a budget located in the ideal budget ranges without any search process. In that case, TALE obtains the ideal budget within acceptable sacrifice.

## A.2 Effectiveness of Budget Estimation.

In this RQ, we evaluate the effectiveness of the budget estimation performance. An ideal estimated budget should be located around the optimal