Example 7. The planar Lotka–Volterra system

$$\begin{aligned}\dot{x} &= x(ax+by+c),\\ \dot{y} &= y\left(\tfrac{a(B-b)}{b}x+\tfrac{B}{2}y+\tfrac{Bc}{b}\right),\end{aligned}$$

where $a,c\in\mathbb{R}$, $b,B\in\mathbb{R}^*$ and $bB<0$, can be written in the form (3) with $R=1$,

$$\mathbf{\Phi}(\mathbf{x})=\left(x, cy+axy+\frac{b}{2}y^2\right) \quad \text{and} \quad \mathbf{G}(\mathbf{u})=\left(u, \frac{B}{b}v\right)^T.$$

Thus, $\dot{\mathbf{u}}=\mathbf{G}(\mathbf{u})$ is a reduced differential system for this particular Lotka–Volterra system and it can be solved easily, whence we get that either $I(\mathbf{u})=uv^{-b/B}$ or $I(\mathbf{u})=vu^{-B/b}$ is a C^l first integral in $\mathbb{R}^2$ of the reduced system, with $l=[-b/B]\in\mathbb{N}$ or $l=[-B/b]\in\mathbb{N}$, respectively, where $[r]$ stands for the integer part of the real number r. In addition, if $-b/B$ is a positive rational number p/q, then $I(\mathbf{u})=u^qv^p$ is a polynomial first integral in $\mathbb{R}^2$ of the reduced system.

Since $\mathbf{\Phi}(\mathbf{x})$ is polynomial, $\mathbf{\Phi}(\mathbf{0})=\mathbf{0}$ and system $\dot{\mathbf{u}}=\mathbf{G}(\mathbf{u})$ is polynomially or C^l integrable in $\mathbb{R}^2$, the Lotka–Volterra system is polynomially or C^l integrable in $\mathbb{R}^2$ by Remark 2, with polynomial first integral

$$H(\mathbf{x})=x^q\left(cy+axy+\frac{b}{2}y^2\right)^p,$$

if $-b/B=p/q\in\mathbb{Q}^+$, and C^l first integral either

$$H(\mathbf{x})=x\left(cy+axy+\frac{b}{2}y^2\right)^{-b/B} \quad \text{or} \quad H(\mathbf{x})=\left(cy+axy+\frac{b}{2}y^2\right)x^{-B/b}$$

if $-b/B\in\mathbb{R}^+\setminus\mathbb{Q}^+$ and either $l=[-b/B]\in\mathbb{N}$ or $l=[-B/b]\in\mathbb{N}$, cf. [18].

We finish this section with an example where the function $\mathbf{\Phi}$ does not preserve dimension but the approach developed also allows to prove (complete) integrability.

Example 8. By using $R=1$ and the functions

$$\mathbf{\Phi}(\mathbf{x})=(x,s(x)) \quad \text{and} \quad \mathbf{G}(\mathbf{u})=(1,2u)^T,$$

where $s(x)$ is a C^2 function satisfying that $x^2+s(x)$ is not constant on open intervals, system (3) becomes

$$\begin{aligned}\dot{x} &= 0,\\ \dot{y} &= 2x-s'(x).\end{aligned}$$

Here $\mathbf{\Phi}(\mathbf{0})=\mathbf{0}$ and $\mathbf{G}(\mathbf{0})\neq\mathbf{0}$ but we can not apply Theorem 1 because $\mathbf{\Phi}$ does not preserve dimension. Notice that $\det(D\mathbf{\Phi}(\mathbf{x}))\equiv 0$. Nevertheless, since $I(\mathbf{u})=u^2+v$ is a first integral of the reduced system $\dot{\mathbf{u}}=\mathbf{G}(\mathbf{u})$ and $H(\mathbf{x})=I(\mathbf{\Phi}(\mathbf{x}))=I(x,s(x))=x^2+s(x)$ is a first integral of the system. Clearly, there is a more natural first integral $J(\mathbf{x})=x$.

4. Applications in the space

In this section, we illustrate the application of our main results restricted to the three dimensional real space $\mathbb{R}^3$. As in the 2-dimensional case we state a corollary of Theorem 1 for this case. We introduce the notation $\mathbf{x} = (x, y, z)$, $\mathbf{u} = (u, v, w)$, $\mathbf{\Phi} = (\varphi, \psi, \eta)$ and $\mathbf{G} = (f, g, h)^T$.

Corollary 2. *Consider C^r differential system in the space*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = R \begin{pmatrix} \psi_y\eta_z - \psi_z\eta_y & \varphi_z\eta_y - \varphi_y\eta_z & \varphi_y\psi_z - \varphi_z\psi_y \\ \psi_z\eta_x - \psi_x\eta_z & \varphi_x\eta_z - \varphi_z\eta_x & \varphi_z\psi_x - \varphi_x\psi_z \\ \psi_x\eta_y - \psi_y\eta_x & \varphi_y\eta_x - \varphi_x\eta_y & \varphi_x\psi_y - \varphi_y\psi_x \end{pmatrix} \begin{pmatrix} f(\varphi, \psi, \eta) \\ g(\varphi, \psi, \eta) \\ h(\varphi, \psi, \eta) \end{pmatrix},$$

where $\varphi, \psi, \eta, f, g,$ and h are suitable C^k scalar functions in $(\mathbb{R}^3, \mathbf{0})$, with $k \geqslant r$, and R is a real-valued function such that $R \neq 0$ in a full Lebesgue measure open subset of $(\mathbb{R}^3, \mathbf{0})$ (we have omitted the dependence on (x, y, z) for simplicity). If the map $\mathbf{\Phi} = (\varphi, \psi, \eta)$ preserves dimension, then the system is C^r integrable in $(\mathbb{R}^3, \mathbf{0})$ provided that $\varphi(\mathbf{0}) = \psi(\mathbf{0}) = \eta(\mathbf{0}) = 0$ and $f^2(\mathbf{0}) + g^2(\mathbf{0}) + h^2(\mathbf{0}) \neq 0$.

We continue presenting several examples of application of Theorem 1 when $n = 3$.

Example 9. A Kolmogorov system. Consider

$$R = 1, \quad \mathbf{\Phi}(\mathbf{x}) = \big(xyz, \psi(x, y, z), z\big) \quad \text{and} \quad \mathbf{G}(\mathbf{u}) = \big(0, 0, 1\big)^T,$$

where $\psi(x, y, z)$ is a C^r function in $(\mathbb{R}^3, \mathbf{0})$, with $\psi(\mathbf{0}) = 0$ and such that $\mathbf{\Phi}$ preserves dimension. The system (3) in Theorem 1 becomes

$$\begin{aligned} \dot{x} &= x\big(z\psi_z(x, y, z) - y\psi_y(x, y, z)\big), \\ \dot{y} &= y\big(x\psi_x(x, y, z) - z\psi_z(x, y, z)\big), \\ \dot{z} &= z\big(y\psi_y(x, y, z) - x\psi_x(x, y, z)\big), \end{aligned}$$

which is a Kolmogorov system in $(\mathbb{R}^3, \mathbf{0})$. Since $\mathbf{\Phi}(\mathbf{0}) = \mathbf{0}$ and $\mathbf{G}(\mathbf{0}) \neq \mathbf{0}$, this Kolmogorov system is C^r completely integrable in $(\mathbb{R}^2, \mathbf{0})$ by Theorem 1. Furthermore, if $\psi(x, y, z)$ is a polynomial function, then the system is polynomially completely integrable in $\mathbb{R}^3$, because $\mathbf{\Phi}(\mathbf{x})$ becomes polynomial and the system $\dot{\mathbf{u}} = \mathbf{G}(\mathbf{u})$ is polynomially completely integrable in $\mathbb{R}^3$ with trivial first integrals $I_1(\mathbf{u}) = u$ and $I_2(\mathbf{u}) = v$.

In the next four examples the functions $\mathbf{\Phi}$ that we will use preserve dimension according to Lemma 1.

Example 10. A second Kolmogorov system. Consider the linear differential system $\dot{\mathbf{u}} = \mathbf{G}(\mathbf{u})$ given by

$$\begin{aligned} \dot{u} &= u, \\ \dot{v} &= -v, \\ \dot{w} &= -w. \end{aligned} \tag{8}$$

By taking $R = 1$ and the polynomial function $\mathbf{\Phi}(\mathbf{x}) = \big(yz, xz + x^2, xz - y\big)$, the differential system (3) is

$$\begin{aligned} \dot{x} &= x(xy + 3yz + x^2 z),\\ \dot{y} &= y(2xy + yz - 3x^2 z),\\ \dot{z} &= z(-4xy - 2yz + x^2 z), \end{aligned} \tag{9}$$

which is a Kolmogorov system. System (8) is polynomially completely integrable in $\mathbb{R}^3$, with polynomial first integrals $I_1(\mathbf{u}) = uv$ and $I_2(\mathbf{u}) = uw$. Since $\mathbf{\Phi}(\mathbf{x})$ is polynomial on $\mathbb{R}^3$, Theorem 1 implies that system (9) is polynomially completely integrable in $\mathbb{R}^3$, with functionally independent polynomial first integrals

$$H_1(\mathbf{x}) = yz(xz + x^2) \quad \text{and} \quad H_2(\mathbf{x}) = yz(xz - y).$$

Example 11. A complete integrable Rössler system. The polynomial differential system

$$\begin{aligned} \dot{x} &= -y - z,\\ \dot{y} &= x,\\ \dot{z} &= xz, \end{aligned} \tag{10}$$

is a particular case of the differential system constructed by Rössler [23]. It can be written in the form (3) with the analytic functions $R = 1$,

$$\mathbf{\Phi}(\mathbf{x}) = \Big(\frac{x^2 + y^2}{2} + z, y, z\Big) \quad \text{and} \quad \mathbf{G}(\mathbf{u}) = \big(0, 1, w\big)^T.$$

Since $\Phi(\mathbf{0}) = \mathbf{0}$ and $\mathbf{G}(\mathbf{0}) \neq \mathbf{0}$, system (10) is analytically completely integrable by Theorem 1. Moreover, it is easy to show that system $\dot{\mathbf{u}} = \mathbf{G}(\mathbf{u})$ has the functionally independent first integrals $I_1(\mathbf{u}) = u$ and $I_2(\mathbf{u}) = we^{-v}$. Hence, Rössler system (10) is also Darboux completely integrable, with independent Darboux first integrals

$$H_1(\mathbf{x}) = \frac{x^2 + y^2}{2} + z \qquad \text{and} \qquad H_2(\mathbf{x}) = ze^{-y},$$

cf. [8, 19].

Example 12. A complete integrable Rikitake system. The polynomial differential system

$$\begin{aligned} \dot{x} &= yz,\\ \dot{y} &= xz,\\ \dot{z} &= 1 - xy, \end{aligned} \tag{11}$$

is a particular case of the Rikitake system [22]. It can be written in the form (3) with the analytic functions $R = 1$,

$$\mathbf{\Phi}(\mathbf{x}) = \Big(\frac{x + y}{2}, y - x, x^2 + z^2\Big) \quad \text{and} \quad \mathbf{G}(\mathbf{u}) = \Big(\frac{u}{2}, -\frac{v}{2}, 1\Big)^T.$$

The point $\mathbf{p} = (1, 1, 0)$ is a singular point of the system. Since $\Phi(\mathbf{p}) = (1, 0, 1)$ and $\mathbf{G}(1, 0, 1) \neq \mathbf{0}$, system (11) is analytically completely integrable