

MULTIDIMENSIONAL EXTRAPOLATED GLOBAL PROXIMAL GRADIENT AND APPLICATIONS FOR IMAGE PROCESSING

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Abstract. The proximal gradient method is a generic technique introduced to tackle the non-smoothness in optimization problems, wherein the objective function is expressed as the sum of a differentiable convex part and a non-differentiable regularization term. Such problems with tensor format are of interest in many fields of applied mathematics such as image and video processing. Our goal in this paper is to address the solution of such problems with a more general form of the regularization term. An adapted iterative proximal gradient method is introduced for this purpose. Due to the slowness of the proposed algorithm, we use new tensor extrapolation methods to enhance its convergence. Numerical experiments on color image deblurring are conducted to illustrate the efficiency of our approach.

Key words. Minimization problem, proximal gradient, tensor, extrapolation, convergence acceleration.

1. Introduction. In this paper we are concerned with the minimization problem

$$\min_{\mathcal{X} \in \Omega} \left(\frac{1}{2} \|\mathcal{F}(\mathcal{X}) - \mathcal{B}\|_F^2 + \varphi_\mu(\mathcal{X}) \right), \quad (1.1)$$

where the solution \mathcal{X} and the observation \mathcal{B} are N th-order tensors in $\mathbb{R}^{I_1 \times I_2 \times \dots \times I_N}$, \mathcal{F} is a given linear tensor operator on $\mathbb{R}^{I_1 \times I_2 \times \dots \times I_N}$, and Ω is assumed to be a nonempty closed bounded convex set of the space $\mathbb{R}^{I_1 \times I_2 \times \dots \times I_N}$. In this paper, the space of thenensors that will be considered will be denoted by $\mathbb{T} = \mathbb{R}^{I_1 \times I_2 \times \dots \times I_N}$.

The regularization term consists of the positive real-valued function φ_μ assumed to have the composition form

$$\begin{aligned} \varphi_\mu &= \mu\phi \circ \mathcal{L} : \mathbb{T} \longrightarrow \mathbb{T} \longrightarrow \mathbb{R}_+ \\ \mathcal{X} &\xrightarrow{\mathcal{L}} \mathcal{L}(\mathcal{X}) \xrightarrow{\mu\phi} \mu\phi(\mathcal{L}(\mathcal{X})), \end{aligned} \quad (1.2)$$

where \mathcal{L} is a given linear tensor operator, ϕ is a closed proper lower semicontinuous convex non-differentiable function and μ is a positive regularization parameter. Under these assumptions on φ_μ and Ω , it was proven that there exists a unique solution of the problem (3.2); see [2, 16, 17, 20, 34] for more details.

The convex optimization problem (1.1) provides a general framework that will cover a wide range of high-order regularization problems. Specific choices of ϕ and \mathcal{L} lead to various types of regularisation, including l_1 -regularization propblems [14, 19, 21], l_1 - l_2 -regularization process [28, 31], and total variation regularization problems [1, 5, 33, 35] which aim is preserving sharp edges (discontinuities) in data while removing noise and unwanted fine-scale detail. Section 7 is dedicated to exploring specific instances of these cases.

The problem (1.1) represents a constrained multidimensional regularization problem that has many applications in different areas, such as artificial neural network and machine learning [9, 24], image and video processing [5, 10, 30, 38]. Our goal in this paper is to develop a general gradient-like method to handle the non-differentiability of the regularization term in problem (1.1). Our minimization approach is twofold. Firstly, computing the proximal mapping of the regularization term, we look for a unconstrained minimizer of the problem (1.1), that is a minimisation of (1.1)

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on the whole space \mathbb{T} . This minimizer will be determined by computing the proximal mapping of the regularisation term. Secondly, after compute the unconstrained minimizer, we project it over the convex set Ω by applying the Tseng's splitting algorithm [2].

The framework of the proximal gradient algorithms, to which this work belongs, has been thoroughly investigated under various contexts with an emphasis on establishing conditions under which these algorithms converge. The advantage of the proximal type methods is in their simplicity. However, they have also been recognized as slow methods. The numerical experiments conducted in this paper provide further grounds to that claim by showing that, under some choices of the operator \mathcal{F} and function φ_μ , the sequences produced by the proposed algorithm shares a very slow rate of convergence. In order to overcome this drawback, we propose two accelerated versions by incorporating very recent tensor extrapolation methods introduced in [7, 8, 18]

The rest of this work is structured as follows. In Section 2, we provide a review of standard definitions and some useful properties. Moving on to Section 3, we introduce a general multidimensional double proximal gradient method using the tensorial form of the objective function in the minimization problem (1.1). Section 4 is devoted to discuss some special cases of the problem (1.1). In Section 5, we suggest enhancing the proposed algorithm's speed by incorporating efficient tensorial extrapolation techniques. Section 6 is dedicated to applying the proposed algorithm to color image completion, and inpainting. Finally, in Section 7, we present our conclusion.

2. Preliminaries and notation. Tensor algebra [15, 25, 26] was developed to handle higher-dimensional representation and analysis. Maintaining data or operations in their inherent multi-dimensional format enhances the capacity of systems to accumulate and retain extensive amounts of diverse data, thereby ensuring the accuracy of modeling.

Now, let us introduce some tensor algebra tools needed in this work. For this, we adopt notations used in [25]. Lowercase letters is used for presenting vectors, e.g., x , and uppercase letters are for matrices, e.g., X , while calligraphic letters is devoted for tensors, e.g., \mathcal{X} . We denote by \mathbb{T} the space $\mathbb{R}^{I_1 \times I_2 \times \dots \times I_N}$ and $\mathbb{T}^N := \mathbb{T} \times \mathbb{T} \times \dots \times \mathbb{T}$. Let $\mathcal{X} \in \mathbb{T}$ be an N th-order tensor of size $I_1 \times \dots \times I_N$, which entries are denoted by $\mathcal{X}_{i_1, i_2, \dots, i_N}$ or $\mathcal{X}(i_1, i_2, \dots, i_N)$. Zero tensor is denoted by \mathcal{O} (all its entries equal to zero). We denote by $\mathbf{X}_{(n)} \in \mathbb{R}^{I_n \times (I_1 I_2 \cdots I_{n-1} I_{n+1} \cdots I_N)}$ the n -mode matrix of the tensor \mathcal{X} and we denote by $X_n = \mathcal{X}(:, :, \dots, :, n)$ the frontal slices of \mathcal{X} obtained by fixing the last index at n , $1 \leq n \leq I_N$.

DEFINITION 2.1. Let $k \geq 1$ be an integer. The k -norm and the infinity norm of the tensor \mathcal{X} are defined by

$$\begin{cases} \|\mathcal{X}\|_k = \left(\sum_{i_1=1}^{I_1} \sum_{i_2=1}^{I_2} \cdots \sum_{i_N=1}^{I_N} |\mathcal{X}(i_1, i_2, \dots, i_N)|^k \right)^{1/k} \\ \|\mathcal{X}\|_\infty = \max_{\substack{1 \leq i_j \leq I_j \\ 1 \leq j \leq N}} |\mathcal{X}(i_1, i_2, \dots, i_N)|. \end{cases} \quad (2.1)$$

DEFINITION 2.2. Let $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^{I_1 \times I_2 \times \dots \times I_N}$. The inner product of \mathcal{X} and \mathcal{Y} is defined by

$$\langle \mathcal{X} | \mathcal{Y} \rangle = \sum_{i_1=1}^{I_1} \sum_{i_2=1}^{I_2} \cdots \sum_{i_N=1}^{I_N} \mathcal{X}(i_1, i_2, \dots, i_N) \mathcal{Y}(i_1, i_2, \dots, i_N).$$

It follows immediately that the associated norm is the Frobenius norm denoted as $\|\mathcal{X}\|_F$.

DEFINITION 2.3. [27] The tensor nuclear norm $\|\mathcal{X}\|_*$ of the tensor \mathcal{X} is defined as the sum of the singular values of the n -mode matricization $\mathbf{X}_{(n)}$ of the tensor \mathcal{X} and given by

$$\|\mathcal{X}\|_* = \sum_{n=1}^N \|\mathbf{X}_{(n)}\|_*,$$

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where $\|\mathbf{X}_{(\mathbf{n})}\|_* = \sum_{i=1}^{I_n} \sigma_i(\mathbf{X}_{(\mathbf{n})})$ with $\sigma_i(\mathbf{X}_{(\mathbf{n})})$ stands for the i th singular value of the matrix $\mathbf{X}_{(\mathbf{n})}$.

Now, Let us recall some optimization notions that will be used later. For more details, see [32]. Let $g : \mathbb{T} \rightarrow \mathbb{R} \cup \{\infty\}$ be a closed proper convex function .

DEFINITION 2.4. (*convex conjugate function*) The function g^* defined by

$$g^* : \mathbb{T} \rightarrow \mathbb{R}$$

$$\mathcal{X} \mapsto g^*(\mathcal{X}) = \sup_{\mathcal{Y}} (\langle \mathcal{Y} | \mathcal{X} \rangle - g(\mathcal{Y}))$$

is called the the convex conjugate function of the function g .

PROPOSITION 2.5. [32] Let $\mathcal{Y} \in \mathbb{T}$, then there exists a unique minimizer \mathcal{X} for the problem

$$\min_{\mathcal{X}} \left(g(\mathcal{X}) + \frac{1}{2} \|\mathcal{X} - \mathcal{Y}\|^2 \right).$$

DEFINITION 2.6. (*proximal operator*) The operator prox_g defined by

$$\text{prox}_g : \mathbb{T} \rightarrow \mathbb{R}$$

$$\mathcal{Y} \mapsto \text{prox}_g(\mathcal{Y}) = \underset{\mathcal{X}}{\operatorname{argmin}} \left(g(\mathcal{X}) + \frac{1}{2} \|\mathcal{X} - \mathcal{Y}\|^2 \right),$$

is called the proximal operator of g .

PROPOSITION 2.7. [32] Let $\alpha > 0$, then

$$\text{prox}_{\alpha g}(\mathcal{U}) = \underset{\mathcal{X}}{\operatorname{argmin}} \left(g(\mathcal{X}) + \frac{1}{2\alpha} \|\mathcal{X} - \mathcal{U}\|^2 \right).$$

The operator $\text{prox}_{\alpha g}$ is called the proximal operator of g with the parameter α .

3. Global tensorial double proximal gradient method . The purpose of this paper is the resolution of the generalized constrained tensorial minimization problem (1.1). Let us set

$$f : \mathbb{T} \rightarrow \mathbb{R}_+$$

$$\mathcal{X} \mapsto f(\mathcal{X}) = \frac{1}{2} \|\mathcal{F}(\mathcal{X}) - \mathcal{B}\|_F^2,$$
(3.1)

where \mathcal{F} and \mathcal{B} are those in (1.1). Then, the problem (1.1) is expressed as

$$\min_{\mathcal{X} \in \Omega} (f(\mathcal{X}) + \varphi_\mu(\mathcal{X})).$$
(3.2)

It is clear that the function f is differentiable, and its gradient is given by

$$\nabla f(\mathcal{X}) = \mathcal{F}^T(\mathcal{F}(\mathcal{X}) - \mathcal{B}).$$

The non-differentiability of ϕ renders the function φ_μ non-differentiable, increasing the complexity of solving the problem.

3.1. Global tensorial double proximal gradient algorithm. In this subsection, we present a noteworthy extension of the gradient descent method tailored for addressing the general tensorial convex minimization problem (1.1). The literature has seen diverse adaptations of the gradient descent technique to tackle various minimization problems, including nonlinear minimization problems [16], fractional optimization problems [4], among others. The proximal gradient

method serves as a generalized version of the gradient descent method, particularly adept at handling non-differentiability in the cost function; see [2, 3, 20, 32].

First, we consider the minimization problem

$$\min_{\mathcal{X} \in \mathbb{T}} (f(\mathcal{X}) + \varphi_\mu(\mathcal{X})). \quad (3.3)$$

Assume that, at the iteration k , we have formed an iterate tensor \mathcal{X}_k approximating the solution to the unconstrained minimization problem (3.3). The quadratic approximation of f at the iterated tensor \mathcal{X}_k , with $\alpha_k > 0$, is expressed as follows

$$\Phi_k(\mathcal{X}) = f(\mathcal{X}_k) + \langle \mathcal{X} - \mathcal{X}_k | \nabla f(\mathcal{X}_k) \rangle + \frac{1}{2\alpha_k} \|\mathcal{X} - \mathcal{X}_k\|_F^2,$$

where the step size $(\alpha_k)_k$ is a positive constant that depends, as will be shown later, on the Lipschitz constant of ∇f . Then, at each step k , we can approximate the problem (3.3) by replacing f by its approximate Φ_k . We have then, the following minimization problem

$$\min_{\mathcal{X}} \left(\varphi_\mu(\mathcal{X}) + f(\mathcal{X}_k) + \langle \mathcal{X} - \mathcal{X}_k | \nabla f(\mathcal{X}_k) \rangle + \frac{1}{2\alpha_k} \|\mathcal{X} - \mathcal{X}_k\|_F^2 \right),$$

we add the constant quantity $\|\alpha_k \nabla f(\mathcal{X}_k)\|_F^2 - f(\mathcal{X}_k)$ to the cost function, since that does not change the solution, we obtain

$$\min_{\mathcal{X}} \left(\varphi_\mu(\mathcal{X}) + \|\alpha_k \nabla f(\mathcal{X}_k)\|_F^2 + \langle \mathcal{X} - \mathcal{X}_k | \nabla f(\mathcal{X}_k) \rangle + \frac{1}{2\alpha_k} \|\mathcal{X} - \mathcal{X}_k\|_F^2 \right),$$

leading to

$$\min_{\mathcal{X}} \left(\varphi_\mu(\mathcal{X}) + \frac{1}{2\alpha_k} \|\mathcal{X} - \mathcal{X}_k + \alpha_k \nabla f(\mathcal{X}_k)\|_F^2 \right), \quad (3.4)$$

which has a unique minimizer \mathcal{Z}_k defined by

$$\mathcal{Z}_k = \text{prox}_{\alpha_k \varphi_\mu} (\mathcal{X}_k - \alpha_k \nabla f(\mathcal{X}_k)), \quad \forall k \in \mathbb{N}, \quad (3.5)$$

where the operator $\text{prox}_{\alpha_k \varphi_\mu}$ stands for the proximal operator of φ_μ with the parameter α_k . Then, as we can see from the above relation, two crucial elements are necessary to compute \mathcal{Z}_k . The first element is the constant $(\alpha_k)_k$ which will be discussed in Subsection 3.3. The second element is the operator $\text{prox}_{\alpha_k \varphi_\mu}$ which is given in the following proposition.

PROPOSITION 3.1. *For all $\mathcal{Y} \in \mathbb{T}$ and $\alpha > 0$, the proximal operator of $\alpha \varphi_\mu$ is given by*

$$\mathcal{Z} = \text{prox}_{\alpha \varphi_\mu} (\mathcal{Y}) = \mathcal{Y} + \alpha \mathcal{L}^T(\mathcal{P}_*), \quad (3.6)$$

where \mathcal{L}^T is the transpose of \mathcal{L} and \mathcal{P}_* is an optimal solution of

$$\min_{\mathcal{P}} \left(\phi_\mu^*(-\mathcal{P}) + \left\langle \frac{\alpha}{2} \mathcal{L}(\mathcal{L}^T(\mathcal{P})) + \mathcal{L}(\mathcal{Y}) | \mathcal{P} \right\rangle \right), \quad (3.7)$$

with ϕ_μ^* being the conjugate function of $\phi_\mu = \mu \phi$.

Proof. Let $\mathcal{Y} \in \mathbb{T}$ and $\alpha > 0$, then $\text{prox}_\alpha \varphi_\mu(\mathcal{Y})$ is the unique solution of the problem

$$\min_{\mathcal{U}} \left(\varphi_\mu(\mathcal{U}) + \frac{1}{2\alpha} \|\mathcal{U} - \mathcal{Y}\|_F^2 \right).$$

Using (1.2) and the fact that $\varphi_\mu(\mathcal{U}) = \phi_\mu(\mathcal{L}(\mathcal{U}))$, the above constrained minimization problem can be reformulated as the following constrained one

$$\min_{\mathcal{U}, \mathcal{V}} \left(\phi_\mu(\mathcal{V}) + \frac{1}{2\alpha} \|\mathcal{U} - \mathcal{Y}\|_F^2 \right) \text{ subject to } \mathcal{V} = \mathcal{L}(\mathcal{U}). \quad (3.8)$$

The associated Lagrangian operator \mathbf{L} is provided as

$$\mathbf{L}(\mathcal{U}, \mathcal{V}, \mathcal{P}) = \phi_\mu(\mathcal{V}) + \frac{1}{2\alpha} \|\mathcal{U} - \mathcal{Y}\|_F^2 + \langle \mathcal{P} | \mathcal{V} - \mathcal{L}(\mathcal{U}) \rangle, \quad \mathcal{P} \in \mathbb{T}.$$

Consequently, as demonstrated in [37], the solution of the problem (3.8) corresponds to the saddle point of \mathbf{L} , representing the solution to the Lagrangian primal problem

$$\min_{\mathcal{U}, \mathcal{V}} \max_{\mathcal{P}} \mathbf{L}(\mathcal{U}, \mathcal{V}, \mathcal{P}).$$

Due to the separability of the Lagrangian with respect to \mathcal{U} and \mathcal{V} , we can interchange the min-max order according to the well known min-max theorem [17, 20]. Consequently, the Lagrangian dual problem can be expressed as

$$\max_{\mathcal{P}} \left[\underbrace{\min_{\mathcal{U}} \left(\frac{1}{2\alpha} \|\mathcal{U} - \mathcal{Y}\|_F^2 - \langle \mathcal{P} | \mathcal{L}(\mathcal{U}) \rangle \right)}_{h(\mathcal{U})} + \min_{\mathcal{V}} (\phi_\mu(\mathcal{V}) + \langle \mathcal{P} | \mathcal{V} \rangle) \right]. \quad (3.9)$$

On one side, due to the convexity and differentiability of the cost function h of the problem in \mathcal{U} , its minimizer denoted by \mathcal{U}_* is exactly the root of its gradient function. A simple calculation gives $\mathcal{U}_* = \mathcal{Y} + \alpha \mathcal{L}^T(\mathcal{P})$ with the corresponding optimal value equal to

$$h(\mathcal{U}_*) = \frac{1}{2\alpha} \|\mathcal{U}_* - \mathcal{Y}\|_F^2 - \langle \mathcal{P} | \mathcal{L}(\mathcal{U}_*) \rangle = - \left\langle \frac{\alpha}{2} \mathcal{L}(\mathcal{L}^T(\mathcal{P})) + \mathcal{L}(\mathcal{Y}) | \mathcal{P} \right\rangle.$$

On the other side, the right minimization problem in \mathcal{V} is a direct application of the convex conjugate function of ϕ_μ . In fact,

$$\min_{\mathcal{V}} (\phi_\mu(\mathcal{V}) + \langle \mathcal{P} | \mathcal{V} \rangle) = - \max_{\mathcal{V}} (\langle -\mathcal{P} | \mathcal{V} \rangle - \phi_\mu(\mathcal{V})) \stackrel{def}{=} -\phi_\mu^*(-\mathcal{P}).$$

Substituting in (3.9), we obtain the following subsequent dual problem

$$\max_{\mathcal{P}} \left[-\phi_\mu^*(-\mathcal{P}) - \left\langle \frac{\alpha}{2} \mathcal{L}(\mathcal{L}^T(\mathcal{P})) + \mathcal{L}(\mathcal{Y}) | \mathcal{P} \right\rangle \right],$$

which is equivalent to the minimization problem (3.7). \square

We have shown that $\text{prox}_{\alpha\varphi_\mu}(\mathcal{Y}) = \mathcal{Y} + \alpha \mathcal{L}^T(\mathcal{P}_*)$, where \mathcal{P}_* is a minimizer of the problem (3.7), then, at each step k , to compute the proximal operator of φ_μ , we need to solve the minimization problem (3.7). That is, at the step k , if we set

$$\mathcal{Y}_k := \mathcal{X}_k - \alpha_k \nabla f(\mathcal{X}_k), \quad (3.10)$$

then, we have to solve the problem

$$\min_{\mathcal{P}} \left(\phi_\mu^*(-\mathcal{P}) + \left\langle \frac{\alpha_k}{2} \mathcal{L}(\mathcal{L}^T(\mathcal{P})) + \mathcal{L}(\mathcal{Y}_k) | \mathcal{P} \right\rangle \right). \quad (3.11)$$

For the sake of clarity and simplicity, let us consider the operators \mathcal{J} and \mathcal{N}_k defined by

$$\begin{aligned}\mathcal{J} : \quad \mathbb{T} &\longrightarrow \mathbb{R}_+ \\ \mathcal{P} &\longrightarrow \mathcal{J}(\mathcal{P}) = \phi_\mu^*(-\mathcal{P}), \\ \mathcal{N}_k : \quad \mathbb{T} &\longrightarrow \mathbb{R} \\ \mathcal{P} &\longrightarrow \mathcal{N}_k(\mathcal{P}) = \left\langle \frac{\alpha_k}{2} \mathcal{L}(\mathcal{L}^T(\mathcal{P})) + \mathcal{L}(\mathcal{Y}_k) \mid \mathcal{P} \right\rangle.\end{aligned}$$

The problem (3.11) is then formulated as

$$\min_{\mathcal{P}} (\mathcal{N}_k(\mathcal{P}) + \mathcal{J}(\mathcal{P})). \quad (3.12)$$

Since \mathcal{N}_k is a closed proper convex differentiable function, and the functional \mathcal{J} is closed proper and convex, the problem (3.12) shares the same structure and conditions as the problem (3.3), where \mathcal{N}_k and \mathcal{J} play roles analogous to f and φ_μ respectively. Consequently, we use again the same approach (proximal gradient) to address (3.12). Therefore, we can approximate the solution of (3.12) via the sequence $(\mathcal{P}_l)_l$ defined as

$$\forall l \in \mathbb{N}, \quad \mathcal{P}_{l+1} = \text{prox}_{\beta_l \mathcal{J}}(\mathcal{P}_l - \beta_l \nabla \mathcal{N}_k(\mathcal{P}_l)), \quad (3.13)$$

with $\beta_l > 0$ being a step size parameter that depends on the Lipschitz constant of the functional \mathcal{N}_k . Three crucial elements are then necessary to compute the above sequence. The first one is the parameter $(\alpha_k)_k$ which will be discussed in Subsection 3.3. The second one is the gradient of the differentiable function \mathcal{N}_k , and finally the operator $\text{prox}_{\beta_l \mathcal{J}}$. For the gradient of \mathcal{N}_k , it is immediate to see that

$$\forall k, \quad \nabla \mathcal{N}_k(\mathcal{P}) = \alpha_k \mathcal{L}(\mathcal{L}^T(\mathcal{P})) + \mathcal{L}(\mathcal{Y}_k). \quad (3.14)$$

For the proximal operator $\text{prox}_{\beta_l \mathcal{J}}$, let us first recall the following proposition relating the proximal mapping of any proper closed convex function g by their conjugates.

PROPOSITION 3.2. [29]

$$\text{prox}_g(x) + \text{prox}_{g^*}(x) = x, \quad \forall x. \quad (3.15)$$

PROPOSITION 3.3.

For all $\mathcal{P} \in \mathbb{T}$, the proximal mapping of $\beta_l \mathcal{J}$ can be expressed as

$$\text{prox}_{\beta_l \mathcal{J}}(\mathcal{P}) = \mathcal{P} + \text{prox}_{\beta_l \phi_\mu}(-\mathcal{P}), \quad \forall l \in \mathbb{N}.$$

Proof. For any $\mathcal{P} \in \mathbb{T}$, we have

$$\begin{aligned}\text{prox}_{\beta_l \mathcal{J}}(\mathcal{P}) &= \underset{\mathcal{W}}{\operatorname{argmin}} \left(\mathcal{J}(\mathcal{W}) + \frac{1}{2\beta_l} \|\mathcal{W} - \mathcal{P}\|_F^2 \right), \\ &= \underset{\mathcal{W}}{\operatorname{argmin}} \left(\phi_\mu^*(-\mathcal{W}) + \frac{1}{2\beta_l} \|\mathcal{W} - \mathcal{P}\|_F^2 \right), \\ &= -\underset{\mathcal{V}}{\operatorname{argmin}} \left(\phi_\mu^*(\mathcal{V}) + \frac{1}{2\beta_l} \|\mathcal{V} + \mathcal{P}\|_F^2 \right), \\ &= -\text{prox}_{\beta_l \phi_\mu^*}(-\mathcal{P}), \\ &= -\mathcal{P} - \text{prox}_{\beta_l \phi_\mu}(-\mathcal{P}) \quad (\text{according to (3.15)}).\end{aligned}$$

□

Furthermore, since $\phi_\mu = \mu\phi$, the proximal operator $\text{prox}_{\beta_l\phi_\mu}$ is equal to the proximal operator of the function $(\beta_l\mu)\phi$. That is $\text{prox}_{\beta_l\phi_\mu} = \text{prox}_{(\beta_l\mu)\phi}$ which is a direct result of the proximal operator of the function ϕ .

Suppose now that the proximal operator of the function ϕ is known. At iteration k , the steps of the algorithm computing the approximate solution \mathcal{Z}_k in (3.5) of the problem (3.4) can be summarized as illustrated in Algorithm 1.

Algorithm 1 Approximate minimizer \mathcal{Z}_k .

Require: $l_k \in \mathbb{N}$, $\mathcal{X}_k, \mu, \alpha_k, \beta_1, \dots, \beta_{l_k}$

- 1: Compute the tensors \mathcal{Y}_k via (3.10).
 - 2: Compute the operator $\mathcal{L}(\mathcal{N}_k)$ via (3.14).
 - 3: **for** $l = 1 : l_k$ **do**
 - 4: $\mathcal{R}_l = \mathcal{P}_l - \beta_l \mathcal{L}(\mathcal{N}_k)(\mathcal{P}_l)$,
 - 5: $\mathcal{P}_{l+1} = \mathcal{R}_l + \text{prox}_{\beta_l\mu\phi}(-\mathcal{R}_l)$,
 - 6: **end for**
 - 7: Compute \mathcal{Z}_k by $\mathcal{Z}_k = \mathcal{Y}_k + \alpha_k \mathcal{L}^T(\mathcal{P}_{l_k+1})$.
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Notice that the step size scalar $(\alpha_k)_k$ depends on the Lipschitz constant $L(\nabla f)$ while the scalars β_l , $l = 1, \dots, l_k$ are related to $L(\nabla \mathcal{N}_k)$. We will discuss their update process later.

3.2. Approximate unconstrained solution via Tseng's splitting algorithm. The question now is how about the solution of the constrained problem (3.2)? In other words, how can we use \mathcal{Z}_k solution of the problem (4.5) for getting a solution of the problem (3.2) under the nonempty closed and convex constraint Ω . The Tseng's splitting algorithm proposed in [2] is a useful and common tool that provides a response to our question. From the solution \mathcal{Z}_k , this algorithm uses, at step k , the projection orthogonal onto the set Ω to produce an approximate solution \mathcal{X}_{k+1} of (3.2) as follows

$$\begin{cases} \mathcal{Y}_k & \text{and } \mathcal{Z}_k \text{ provided by Algorithm 1 ,} \\ \mathcal{Q}_k = \mathcal{Z}_k - \alpha_k \nabla f(\mathcal{Z}_k), \\ \mathcal{X}_{k+1} = \Pi_\Omega(\mathcal{Z}_k - \mathcal{Y}_k + \mathcal{Q}_k). \end{cases} \quad (3.16)$$

where Π_Ω stands for the orthogonal projection onto Ω .

3.3. Selection of parameters α_k and β_l . Selecting the step size parameters α_k and β_l is regarded as a standard condition that guarantees the convergence of the sequence $(\mathcal{X}_k)_k$ to the minimizer of the problem (3.2). In [2, 16], the authors provide sufficient conditions for this purpose. It is required that the positive values α_k and β_l should be less than the inverses of Lipschitz constants of the operators ∇f and $\nabla \mathcal{N}_k$, denoted by $L(\nabla f)$ and $L(\nabla \mathcal{N}_k)$, respectively. That is, $0 < \alpha_k < \frac{1}{L(\nabla f)}$ and $0 < \beta_l < \frac{1}{L(\nabla \mathcal{N}_k)}$.

For any pair $(\mathcal{X}, \mathcal{Y})$ in $\mathbb{T} \times \mathbb{T}$, we have

$$\begin{aligned} \|\nabla f(\mathcal{X}) - \nabla f(\mathcal{Y})\|_F &= 2 \|\mathcal{F}^T(\mathcal{F}(\mathcal{X})) - \mathcal{F}^T(\mathcal{F}(\mathcal{Y}))\|_F, \\ &= 2 \|\mathcal{F}^T(\mathcal{F}(\mathcal{X} - \mathcal{Y}))\|_F, \\ &\leq 2 \|\mathcal{F}^T \circ \mathcal{F}\| \cdot \|\mathcal{X} - \mathcal{Y}\|_F, \end{aligned}$$

where \circ refer to the composition operation. Then, we can choose the quantity $2 \|\mathcal{F}^T \circ \mathcal{F}\|$ as a Lipschitz constant of the operator ∇f .

As a result, the step size α_k can be chosen as a stable value $\alpha_k \in \left(0, \frac{1}{2\|\mathcal{F}^T \circ \mathcal{F}\|}\right)$. In the scenario involving the operator $\nabla \mathcal{N}_k$ for a given step k , the Lipschitz constant $L(\nabla \mathcal{N}_k)$ is unknown. In such cases, the step sizes (β_l) can be determined using a line search method [32]. This entails applying the proximal gradient method with a straightforward backtracking step size rule, as follows

$$\forall l \in \mathbb{N}, \quad \beta_l = \rho \beta_{l-1}. \quad (3.17)$$

The main steps for the computation of the approximation \mathcal{X}_{k+1} of the problem (3.2) is given in the following algorithm.

Algorithm 2 Global Tensorial Proximal Gradient (GTPG)

Require: $\bar{l} \in \mathbb{N}$, μ , \mathcal{X}_1 , \mathcal{P}_1 , ∇f , prox_{ϕ} , β_0, ρ and $\alpha \in (0, \frac{1}{2\|\mathcal{F}^T \circ \mathcal{F}\|})$

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1: for  $k = 1$ ... until convergence do
2:   Compute the tensor  $\mathcal{Y}_k$  via (3.10)
3:   Compute the operator  $\mathcal{L}(\mathcal{N}_k)$  via (3.14)
4:   for  $l = 1 : \bar{l}$  do
5:     Update the parameter  $\beta_l$  via (3.17)
6:      $\mathcal{R}_l = \mathcal{P}_l - \beta_l \mathcal{L}(\mathcal{N}_k(\mathcal{P}_l))$ 
7:      $\mathcal{P}_{l+1} = \mathcal{R}_l + \text{prox}_{\beta_l \mu \phi}(-\mathcal{R}_l)$ ,
8:   end for
9:   Compute  $\mathcal{Z}_k$  via:  $\mathcal{Z}_k = \mathcal{Y}_k + \alpha \mathcal{L}^T(\mathcal{P}_{l+1})$ 
10:   $\mathcal{Q}_k = \mathcal{Z}_k - \alpha \nabla f(\mathcal{Z}_k)$ 
11:   $\mathcal{X}_{k+1} = \Pi_{\Omega}(\mathcal{Z}_k - \mathcal{Y}_k + \mathcal{Q}_k)$ 
12: end for

```

The following theorem gives a convergence result on the constructed approximate sequence $(\mathcal{X}_k)_k$ produced by Algorithm 2.

THEOREM 3.4. *Let Ω be a closed subset of \mathbb{T} . Then, the sequence $(\mathcal{X}_k)_k$ generated by Algorithm (2) converges to the unique solution, denoted by \mathcal{X}_* , of the problem (3.2) (or equivalently (1.1)). That is, $\lim_{k \rightarrow \infty} \|\mathcal{X}_k - \mathcal{X}_*\|_F = 0$.*

Proof. The functions f and φ_μ are proper lower semicontinuous and convex, with f being Gateau differentiable and uniformly convex on \mathbb{T} and since the set Ω is a closed convex nonempty subset of \mathbb{T} , the strong convergence of the sequence $(\mathcal{X}_k)_k$ follows immediately from the general result given in [2]. \square

4. Special cases.

4.1. Case 1. $\mathcal{L} = id_{\mathbb{T}}$ and $\phi = \|\cdot\|_1$. The problem (1.1) becomes

$$\min_{\mathcal{X} \in \Omega} \left(\frac{1}{2} \|\mathcal{F}(\mathcal{X}) - \mathcal{B}\|_F^2 + \|\mathcal{X}\|_1 \right), \quad (4.1)$$

where $\|\mathcal{X}\|_1 = \sum_{i_1=1}^{I_1} \sum_{i_2=1}^{I_2} \dots \sum_{i_N=1}^{I_N} |\mathcal{X}(i_1, i_2, \dots, i_N)|$. This problem (4.1) is a direct extension, to tensor case, of the popular (vector) l_1 -regularization problems [14, 19, 21]. Furthermore, the tensorial proximal operator $\text{prox}_{\beta_l \mu \phi} := \text{prox}_{\beta_l \mu} \|\cdot\|_1$ can be seen as a direct extension of the soft thresholding operator [32]. Then, similarly to the vector case, $\text{prox}_{\gamma} \|\cdot\|_1$ with any $\gamma > 0$, can be computed via the orthogonal projection on the $\|\cdot\|_\infty$ -unit ball; see [32] for more details. This leads to

$$\left(\text{prox}_{\gamma \|\cdot\|_1}(\mathcal{P})\right)_{i_1, \dots, i_N} = \begin{cases} \mathcal{P}_{i_1, \dots, i_N} - \gamma, & \mathcal{P}_{i_1, \dots, i_N} \geq \gamma \\ 0, & |\mathcal{P}_{i_1, \dots, i_N}| < \gamma \\ \mathcal{P}_{i_1, \dots, i_N} + \gamma, & \mathcal{P}_{i_1, \dots, i_N} \leq -\gamma \end{cases}$$

Under these choices, the function \mathcal{N}_k , and hence its gradient are given as

$$\mathcal{N}_k(\mathcal{P}) = \left\langle \frac{\alpha_k}{2} \mathcal{P} + \mathcal{Y}_k \mid \mathcal{P} \right\rangle \quad \text{and} \quad \nabla \mathcal{N}_k(\mathcal{P}) = \frac{\alpha_k}{2} \mathcal{P} + \mathcal{Y}_k.$$

The algorithm 2 is then recovered as a natural extension to tensor case of the well known *iterative shrinkage-thresholding algorithms* (ISTA) devoted to solve vector l_1 -regularization problems, see [14, 19, 21]. This leads to the following new algorithm named Tensorial Iterative Shrinkage-Thresholding Algorithm (TISTA).

Algorithm 3 Tensorial Iterative Shrinkage-Thresholding Algorithm (TISTA)

Require: $\bar{l} \in \mathbb{N}$, μ , \mathcal{X}_1 , \mathcal{P}_1 , ∇f , prox_ϕ , β_0 , ρ and $\alpha \in (0, \frac{1}{2\|\mathcal{F}^T \circ \mathcal{F}\|})$,

- 1: **for** $k = 1$... until convergence **do**
- 2: $\mathcal{Y}_k = \mathcal{X}_k - \alpha \nabla f(\mathcal{X}_k)$,
- 3: **for** $l = 1 : \bar{l}$ **do**
- 4: Update the parameter β_l via (3.17): $\beta_l = \rho \beta_{l-1}$,
- 5: $\mathcal{R}_l = \mathcal{P}_l - \frac{\alpha \beta_l}{2} \mathcal{P}_l + \mathcal{Y}_k$,
- 6: $\mathcal{P}_{l+1} = \mathcal{R}_l + \text{prox}_{\beta_l \mu \|\cdot\|_1}(\mathcal{R}_l)$.
- 7: **end for**
- 8: Compute \mathcal{Z}_k via: $\mathcal{Z}_k = \mathcal{Y}_k + \alpha \mathcal{P}_{l+1}$,
- 9: $\mathcal{Q}_k = \mathcal{Z}_k - \alpha \nabla f(\mathcal{Z}_k)$,
- 10: $\mathcal{X}_{k+1} = \Pi_\Omega(\mathcal{Z}_k - \mathcal{Y}_k + \mathcal{Q}_k)$.
- 11: **end for**

Assume that the function f in equation (3.1) is such that

$$\begin{aligned} f : \mathbb{T} &\longrightarrow \mathbb{R}_+ \\ \mathcal{X} &\longrightarrow f(\mathcal{X}) = \frac{1}{2} \|\mathcal{A} *_N \mathcal{X} - \mathcal{B}\|_F^2, \end{aligned} \tag{4.2}$$

where $\mathcal{A} \in \mathbb{T}^2$ is an $2N$ th-order tensor of size $I_1 \times I_2 \times \dots \times I_N \times I_1 \times I_2 \times \dots \times I_N$ and $*_N$ stands for the Einstein product operation [18], where $\mathcal{A} *_N \mathcal{X}$ is the $I_1 \times \dots \times I_N$ tensor which entries are given by

$$(\mathcal{A} *_N \mathcal{X})_{i_1, \dots, i_N} = \sum_{j_1 \dots j_N} \mathcal{A}_{i_1, \dots, i_N j_1 \dots j_N} \mathcal{X}_{j_1 \dots j_N}. \tag{4.3}$$

Through a simple calculation, we get $\nabla f(\mathcal{X}) = \mathcal{A} *_N \mathcal{X} - \mathcal{B}$. Substituting in Algorithm 3, we obtain a new algorithm the called Einstein Tensorial Iterative Shrinkage-Thresholding Algorithm (EISTA) and is summarized as follows.

$$\min_{\mathcal{X} \in \Omega} \left(\frac{1}{2} \|\mathcal{A} *_N \mathcal{X} - \mathcal{B}\|_F^2 + \|\mathcal{X}\|_1 \right). \tag{4.4}$$

Algorithm 4 EISTA (Einstein product based ISTA)

Require: $\bar{l} \in \mathbb{N}$, $\mu, \mathcal{A}, \mathcal{B}, \mathcal{X}_1, \mathcal{P}_1$, prox_{ϕ} , β_0, ρ and $\alpha \in (0, \frac{1}{2\|\mathcal{A} *_{\mathcal{N}} \mathcal{A}\|})$.

- 1: **for** $k = 1 \dots$ until convergence **do**
- 2: $\mathcal{Y}_k = \mathcal{X}_k - \alpha(\mathcal{A} *_{\mathcal{N}} \mathcal{X}_k - \mathcal{B})$.
- 3: **for** $l = 1 : \bar{l}$ **do**
- 4: Update the parameter β_l via (3.17): $\beta_l = \rho\beta_{l-1}$
- 5: $\mathcal{R}_l = \mathcal{P}_l - \frac{\alpha\beta_l}{2}\mathcal{P}_l + \mathcal{Y}_k$,
- 6: $\mathcal{P}_{l+1} = \mathcal{R}_l + \text{prox}_{\beta_l\mu\|\cdot\|_1}(\mathcal{R}_l)$.
- 7: **end for**
- 8: Compute \mathcal{Z}_k via: $\mathcal{Z}_k = \mathcal{Y}_k + \alpha_k \mathcal{P}_{l_k+1}$,
- 9: $\mathcal{Q}_k = \mathcal{Z}_k - \alpha_k(\mathcal{A} *_{\mathcal{N}} \mathcal{Z}_k - \mathcal{B})$,
- 10: $\mathcal{X}_{k+1} = \Pi_{\Omega}(\mathcal{Z}_k - \mathcal{Y}_k + \mathcal{Q}_k)$.
- 11: **end for**

4.2. Case 2 $\mathcal{L} = \nabla$ and $\phi = \|\cdot\|_1$. The problem (1.1) becomes

$$\min_{\mathcal{X} \in \Omega} \left(\frac{1}{2} \|\mathcal{F}(\mathcal{X}) - \mathcal{B}\|_F^2 + \mu \|\nabla(\mathcal{X})\|_1 \right), \quad (4.5)$$

where the gradient operator $\nabla(\mathcal{X})$ of the N -order tensor $\mathcal{X} \in \mathbb{T}$ is defined as a column block tensor in \mathbb{T}^N consisting of the partial derivatives $(\nabla_{(n)} \mathcal{X})_n$, i.e., $\nabla \mathcal{X} = (\nabla_{(1)} \mathcal{X}, \dots, \nabla_{(N)} \mathcal{X})$, such that, for $n = 1, \dots, N$, the block tensor $\nabla_{(n)} \mathcal{X}$ is given by

$$(\nabla_{(n)} \mathcal{X})_{i_1, \dots, i_N} = \begin{cases} \mathcal{X}_{i_1, \dots, i_n+1, \dots, i_N} - \mathcal{X}_{i_1, \dots, i_n, \dots, i_N} & \text{if } i_n < I_n \\ 0 & \text{if } i_n = I_n. \end{cases}$$

The problem (4.5) was considered in [5], resulting the algorithm named TDPG and given as follows.

Algorithm 5 [[5]] Tensorial Double Proximal Gradient (TDPG)

Require: $\bar{l} \in \mathbb{N}$, $\mu, \mathcal{X}_1, \mathcal{P}_1, \nabla f, \text{prox}_{\phi}, \beta_0, \rho$ and $\alpha \in (0, \frac{1}{2\|\mathcal{F}^T \circ \mathcal{F}\|})$.

- 1: **for** $k = 1 \dots$ until convergence **do**
- 2: Compute the tensors \mathcal{Y}_k via (3.10)
- 3: Compute the operator $\nabla \mathcal{N}_k$ via (3.14)
- 4: **for** $l = 1 : \bar{l}$ **do**
- 5: Update the parameter β_l via (3.17)
- 6: $\mathcal{R}_l = \mathcal{P}_l - \beta_l \nabla \mathcal{N}_k(\mathcal{P}_l)$
- 7: $\mathcal{P}_{l+1} = \mathcal{R}_l + \text{prox}_{\beta_l\mu\phi}(-\mathcal{R}_l)$,
- 8: **end for**
- 9: Compute \mathcal{Z}_k via: $\mathcal{Z}_k = \mathcal{Y}_k + \alpha \nabla^T(\mathcal{P}_{l_k+1})$
- 10: $\mathcal{Q}_k = \mathcal{Z}_k - \alpha \nabla f(\mathcal{Z}_k)$
- 11: $\mathcal{X}_{k+1} = \Pi_{\Omega}(\mathcal{Z}_k - \mathcal{Y}_k + \mathcal{Q}_k)$
- 12: **end for**

4.3. Case 3. $\mathcal{L} = \nabla$ and $\varphi_{\mu} = \mu \|\mathcal{L}(\cdot)\|_1 + \|\cdot\|_{*}$. Considering these choices, the obtained problem is as follows

$$\min_{\mathcal{X} \in \mathbb{T}} \left(\frac{1}{2} \|\mathcal{F}(\mathcal{X}) - \mathcal{B}\|_F^2 + \mu \|\nabla(\mathcal{X})\|_1 + \|\mathcal{X}\|_{*} \right), \quad (4.6)$$

where $\|\cdot\|_*$ is the tensor nuclear norm given in Definition 2.3. If, in addition, we impose that $\|\mathcal{X}\|_* < \epsilon$ for some $\epsilon > 0$ then, the unconstrained problem (4.6) can be reformulated as the following constrained problem.

$$\min_{\mathcal{X} \in \Omega} \left(\frac{1}{2} \|\mathcal{F}(\mathcal{X}) - \mathcal{B}\|_F^2 + \mu \|\nabla(\mathcal{X})\|_1 \right) \quad \text{s.t. } \Omega = \{\mathcal{X} \in \mathbb{T}, \|\mathcal{X}\|_* < \epsilon\}. \quad (4.7)$$

The above problem has many applications such as in data completion problem and image processing. For instance, let us denote $\mathcal{M} \in \mathbb{T}$ our incomplete data (for example, image with missing pixels). Let E be the set of the indices of the observed (available) elements, i.e. $E = \{(i_1, i_2, \dots, i_N), \mathcal{M}_{i_1, i_2, \dots, i_N} \text{ is observed}\}$. Consider the projection operator $\mathbf{P}_E : \mathbb{T} \rightarrow \mathbb{T}$ given as

$$(\mathbf{P}_E(\mathcal{X}))_{i_1, i_2, \dots, i_N} = \begin{cases} \mathcal{X}_{i_1, i_2, \dots, i_N} & \text{if } (i_1, i_2, \dots, i_N) \in E \\ 0 & \text{otherwise.} \end{cases} \quad (4.8)$$

By taking $\mathcal{F} = \mathbf{P}_E$ and $\mathcal{B} = \mathbf{P}_E(\mathcal{M})$ in (4.7), our proposed algorithm can be applied to fill the missing elements of the data \mathcal{M} . This will be illustrated in the numerical section.

5. Convergence acceleration using Extrapolation . Extrapolation methods [6, 7, 8, 12, 13, 18, 22, 23, 36] are very useful techniques for accelerating the convergence of slowly sequences as those generated by our proposed algorithm (1). In this section, we adopt two recent Extrapolation methods, the Global Tensor Topological Extrapolation Transformation (GT-TET) and the High-Order Singular Values Decomposition based on Minimal Extrapolation Method (HOSVD-MPE).

5.1. GT-TET. Topological Extrapolation Algorithm (TEA) is one of the most popular extrapolation method to enhance convergence of slowly vector sequences. It is well-regarded to its theoretical clarity and numerical efficiency, particularly, when employed to address nonlinear problems, as exemplified by our primary problem (1.1). It was proposed for the first time by Brezenski [11] for vector sequences. Recently, in [18], tensor version of TEA, namely GT-TET, was introduced to address multidimensional sequences.

Let $(\mathcal{S}_n)_n$ be a convergent sequence of tensors, and $m \in \mathbb{N}^*$. Given $2m+1$ terms $\mathcal{S}_n, \mathcal{S}_{n+1}, \dots, \mathcal{S}_{n+2m}$, the GT-TET method provides an approximation $\mathcal{S}_{n,m}^{TET}$ to the limit as

$$\mathcal{S}_{n,m}^{TET} = \mathcal{S}_n + \sum_{j=1}^m c_j^{(n)} \Delta \mathcal{S}_{n+j-1}, \quad (5.1)$$

where differences $\Delta \mathcal{S}_{n+j-1} = \mathcal{S}_{n+j} - \mathcal{S}_{n+j-1}$, and the weights $c_1^{(n)}, c_2^{(n)}, \dots, c_m^{(n)}$ are determined via the solution of the least-squares problem:

$$c^{(n,m)} = \arg \min_{x \in \mathbb{R}^m} \|H^{(n,m)}x - b^{(n,m)}\|, \quad (5.2)$$

where $c^{(n,m)} = [c_1^{(n)}, c_2^{(n)}, \dots, c_m^{(n)}]^T \in \mathbb{R}^m$ and the matrix $H^{(n,m)} \in \mathbb{R}^{m \times m}$ and the vector $b^{(n,m)} \in \mathbb{R}^m$ are given as

$$H^{(n,m)} = \begin{bmatrix} \langle \mathcal{Y}, \Delta^2 \mathcal{S}_n \rangle & \dots & \langle \mathcal{Y}, \Delta^2 \mathcal{S}_{n+m-1} \rangle \\ \langle \mathcal{Y}, \Delta^2 \mathcal{S}_{n+1} \rangle & \dots & \langle \mathcal{Y}, \Delta^2 \mathcal{S}_{n+m} \rangle \\ \vdots & \dots & \vdots \\ \langle \mathcal{Y}, \Delta^2 \mathcal{S}_{n+m-1} \rangle & \dots & \langle \mathcal{Y}, \Delta^2 \mathcal{S}_{n+2m-2} \rangle \end{bmatrix}, b^{(n,m)} = \begin{bmatrix} \langle \mathcal{Y}, \Delta \mathcal{S}_n \rangle \\ \langle \mathcal{Y}, \Delta \mathcal{S}_{n+1} \rangle \\ \vdots \\ \langle \mathcal{Y}, \Delta \mathcal{S}_{n+m-1} \rangle \end{bmatrix}$$

where $\mathcal{Y} \in \mathbb{T}$ is some chosen tensor, and the square deference $\Delta^2 \mathcal{S}_{n+j-1} = \Delta \mathcal{S}_{n+j} - \Delta \mathcal{S}_{n+j-1} = \mathcal{S}_{n+j+1} - 2\mathcal{S}_{n+j} + \mathcal{S}_{n+j-1}$.

Using the standard QR decomposition of the matrix $H^{(n,m)}$, that is $H^{(n,m)} = QR$ with Q orthogonal and R upper triangular, the solution of the problem (5.2) (assuming full rank of $H^{(n,m)}$) is obtained via

$$c^{(n,m)} = R^{-1}y \quad \text{such that} \quad y = Q^T b^{(n,m)}.$$

We summarized the steps of GT-TET in the following algorithm.

Algorithm 6 GT-TET

Require: $\mathcal{S}_n, \mathcal{X}_{n+1}, \dots, \mathcal{S}_{n+2m}$,

Ensure: $\mathcal{T}_{n,m}^{TET}$.

- 1: Compute the differences $\Delta \mathcal{S}_{n+j-1}, j = 1, \dots, 2m - 1$.
 - 2: Compute the square differences $\Delta^2 \mathcal{X}_{n+j-1}, j = 1, \dots, 2m - 2$.
 - 3: Construct the matrix $H^{(n,m)}$ and vector $b^{(n,m)}$.
 - 4: Compute the QR of $H^{(n,m)}$: $H^{(n,m)} = QR$.
 - 5: $y = Q^T b^{(n,m)}$,
 - 6: $c^{(n,m)} = R^{-1}y$.
 - 7: Compute the approximation $\mathcal{T}_{n,m}^{TET}$ as given in (5.1).
-

5.2. HOSVD-MPE. The HOSVD-MPE is a polynomial extrapolation method recently proposed in [7] to accelerate the convergence of tensor sequences. It provides an approximation $\mathcal{X}_{n,m}^{H-M}$ to the limit as

$$\mathcal{X}_{n,m}^{H-M} = \sum_{j=1}^m c_j^{(n)} \mathcal{X}_{n+j-1}, \quad (5.3)$$

the weights $c_1^{(n,m)}, c_2^{(n,m)}, \dots, c_m^{(n,m)}$ are determined by

$$c_j^{(n)} = \frac{\delta_j^{(n)}}{\sum_{i=1}^m \delta_i^{(n)}} \quad \text{for } j = 1, \dots, m. \quad (5.4)$$

where $\delta^{(n,m)} = (\delta_1^{(n)}, \delta_2^{(n)}, \dots, \delta_m^{(n)})^T$ is the solution of the constrained least-squares tensor problem

$$\min_{\substack{x \in \mathbb{R}^m \\ \|x\|_2=1}} \| \mathbb{D}_m^{(n)} \bar{\times}_m x \|_F, \quad (5.5)$$

with $\mathbb{D}_m^{(n)} = [\Delta \mathcal{X}_n, \Delta \mathcal{X}_{n+1}, \dots, \Delta \mathcal{X}_{n+m-1}] \in \mathbb{T}^m$ such that the i^{th} frontal slice, obtained by fixing the last index at i , is given by $[\mathbb{D}_m^{(n)}]_{:, \dots, :, i} = \Delta \mathcal{X}_{n+i-1}$, $1 \leq i \leq m$. In [7], we proposed the following algorithm to compute the solution the desired approximation $\mathcal{X}_{n,m}^{H-M}$ of the problem (5.3).

Algorithm 7 HOSVD-MPE

Require: $\mathcal{X}_n, \mathcal{X}_{n+1}, \dots, \mathcal{X}_{n+m}$.**Ensure:** $\mathcal{X}_{n,m}^{H-M}$

- 1: Compute the tensors $\Delta \mathcal{X}_n, \Delta \mathcal{X}_{n+1}, \dots, \Delta \mathcal{X}_{n+m-1}$
 - 2: Form the tensor $\mathbb{D}_m^{(n)} = [\Delta \mathcal{X}_n, \Delta \mathcal{X}_{n+1}, \dots, \Delta \mathcal{X}_{n+m-1}]$.
 - 3: Compute the m -unfolding matrix of $\mathbb{D}_m^{(n)}$: $N = (\mathbb{D}_m^{(n)})_{(m)}$.
 - 4: Compute the square matrix $M := NN^T \in \mathbb{R}^{m \times m}$.
 - 5: Compute the Eigen Value Decomposition (EVD) of M : $M = V\Sigma V^T$.
 - 6: Determine $\delta^{(n,m)} = Ve_m$ with $e_m = (0, 0, \dots, 1)^T \in \mathbb{R}^m$.
 - 7: Determine $c_1^{(n)}, c_2^{(n)}, \dots, c_m^{(n)}$ via (5.4).
 - 8: Set $\mathcal{X}_{n,m}^{H-M} = \sum_{j=1}^m c_j^{(n)} \mathcal{X}_{n+j-1}$.
-

5.3. Accelerated version of GTPG algorithm. There are several schemes to apply extrapolation methods to a given sequence. In this work we adopt the restarting mode for integrating the GT-TET and HOSVD-MPE methods to Algorithm 2 which is the main algorithm of this work. Algorithms 8 and Algorithm 9 describe the accelerated versions of GTPG using HOSVD-MPE and GT-TET, respectively. We denote by $(\mathcal{T}_k)_k$ the extrapolated sequence produced by these two Algorithms 8 and 9.

Algorithm 8 Accelerated GTPG using HOSVD-MPE (GTPG-HM)

Require: $m \in \mathbb{N}^*$, $\mathcal{T}_1 = \mathcal{X}_1$ and ϵ (stop threshold).**Ensure:** \mathcal{T}_k .

- 1: $k = 2$,
 - 2: Starting from \mathcal{X}_1 , compute $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_m$ via Algorithm 2 (GTPG),
 - 3: Compute $\mathcal{X}_{1,m}^{H-M}$ by HOSVD-MPE algorithm 7,
 - 4: Set $\mathcal{T}_k = \mathcal{X}_{1,m}^{H-M}$,
 - 5: **if** $\frac{\|\mathcal{T}_k - \mathcal{T}_{k-1}\|}{\|\mathcal{T}_{k-1}\|} < \epsilon$ **then**
 - 6: Stop
 - 7: **else**
 - 8: Set $\mathcal{X}_1 = \mathcal{T}_k$ and $k = k + 1$ and go to the second step.
 - 9: **end if**
-

Algorithm 9 Accelerated GTPG using GT-TET (GTPG-TET)

Require: $m \in \mathbb{N}^*$, $\mathcal{T}_1 = \mathcal{X}_1$ and ϵ (stop threshold).**Ensure:** \mathcal{T}_k .

- 1: $k = 2$,
 - 2: Starting from \mathcal{X}_1 , compute $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_{2m+1}$ via Algorithm 2 (GTPG),
 - 3: Compute $\mathcal{X}_{1,m}^{TET}$ by GT-TET algorithm 6,
 - 4: Set $\mathcal{T}_k = \mathcal{X}_{1,m}^{TET}$,
 - 5: **if** $\frac{\|\mathcal{T}_k - \mathcal{T}_{k-1}\|}{\|\mathcal{T}_{k-1}\|} < \epsilon$ **then**
 - 6: Stop
 - 7: **else**
 - 8: Set $\mathcal{X}_1 = \mathcal{T}_k$ and $k = k + 1$ and go to the second step.
 - 9: **end if**
-

6. Numerical experiment. In this section, we illustrated the efficiency of the proposed extrapolated algorithm TISTA-TET, TISTA-HM, TDGP-TET and TDGP-HM (see Table 6.1). As explained in Section 4, some image deblurring problems, such as image completion can be modeled within the context of our primary problem (1.1) and then the applicability of our proposed method to this kind of problems. The objective of image completion problems is to reconstruct missing regions within an observed image by using content-aware information extracted from undisturbed regions, based on the available pixels. Let $\mathcal{X}_{orig} \in \mathbb{R}^{I_1 \times I_2 \times 3}$ denote our original color image, and we uniformly/randomly mask off a portion of its entries that we regard as missing values. Let $\mathcal{B} \in \mathbb{R}^{I_1 \times I_2 \times 3}$ the obtained incomplete image and let be E the set of the indices of the available elements, i.e. $E = \{(i_1, i_2, \dots, i_3), \mathcal{B}_{i_1, i_2, i_3} \text{ is observed}\}$. We test the performance of our algorithms to recover the missing entries via the solution of the following problems

$$\min_{\mathcal{X} \in \Omega} \left(\frac{1}{2} \|\mathbf{P}_E(\mathcal{X}) - \mathcal{B}\|_F^2 + \|\mathcal{X}\|_1 \right), \quad \text{using the lgorithm TISTA 3}$$

$$\min_{\mathcal{X} \in \Omega} \left(\frac{1}{2} \|\mathbf{P}_E(\mathcal{X}) - \mathcal{B}\|_F^2 + \mu \|\nabla(\mathcal{X})\|_1 \right), \quad \text{using the algorithm TDGP 5},$$

where $\Omega = \{\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times 3}, \text{ such that } \|\mathcal{X}\|_* < \epsilon\}$ and \mathbf{P}_E is the projection operator given in (4.8). To assess the efficiency of our algorithm, we measure its performance using the peak signal-to-noise ratio (*PSNR*) using the standard *psnr* function from Matlab, and the relative error *Re* defined as $Re = \frac{\|\mathcal{X}_{prx} - \mathcal{X}_{orig}\|}{\|\mathcal{X}_{orig}\|}$ where \mathcal{X}_{prx} is the approximate solution. The iterations will be stopped when $\frac{\|\mathcal{X}_{k+1} - \mathcal{X}_k\|}{\|\mathcal{X}_k\|} < 10^{-3}$. All computations were conducted using MATLAB R2023a on an 12th Gen Intel(R) Core(TM) i7-1255U 1.70 GHz computer with 16 GB of RAM. We tested and compared the algorithms listed in the following table

Table 6.1: Algorithms

Acronym	meaning
TISTA	basic algorithm TISTA 3
TISTA-TET	extrapolated TISTA using topological extrapolation GT-TET
TISTA-HM	extrapolated TISTA using HOSVD-MPE extrapolation
TDGP	basic algorithm TDGP 5
TDGP-TET	extrapolated TDGP using topological extrapolation GT-TET
TDGP-HM	extrapolated TISTA using HOSVD-MPE extrapolation

In the tow subsequent parts, we will test the performance of our algorithms both for the randomly uncompleted images and uniformly uncompleted ones. Let us first investigate the efficiency of our proposed methods in the case of randomly missing pixels.

6.1. Random missing pixels. In this part, we use the $250 \times 250 \times 3$ color images 'football.png', 'Peppers.bmp' and 'flamingos.jpg', and we randomly mask off about 55% of their entries as shown in Figure 6.1. The corresponding results are shown in Figure 6.2. In one hand, we see that TISTA and TDGP algorithms provide images with nearly the same clarity. In the other side, we can see clearly that the extrapolated versions of TISTA and TDGP provide clearer images. Figures 6.3 and 6.4 illustrate, the relative error and the PSNR curves produced by the six algorithms. We can see clearly that the extrapolated algorithms TISTA-TET, TISTA-HM and TDGP-TET, TDGP-HM converge faster than the basic ones TISTA and TDGP. The curves in Figure

6.5, represent the convergence rates of the four extrapolated methods TISTA-TET, TISTA-HM and TDPG-TET, TDPG-HM . The curves depict the efficiency of the four methods, highlighting the superiority of TISTA-TET and TISTA-HM in comparison to the TDGPG-TET and TDGPG-HM algorithms.

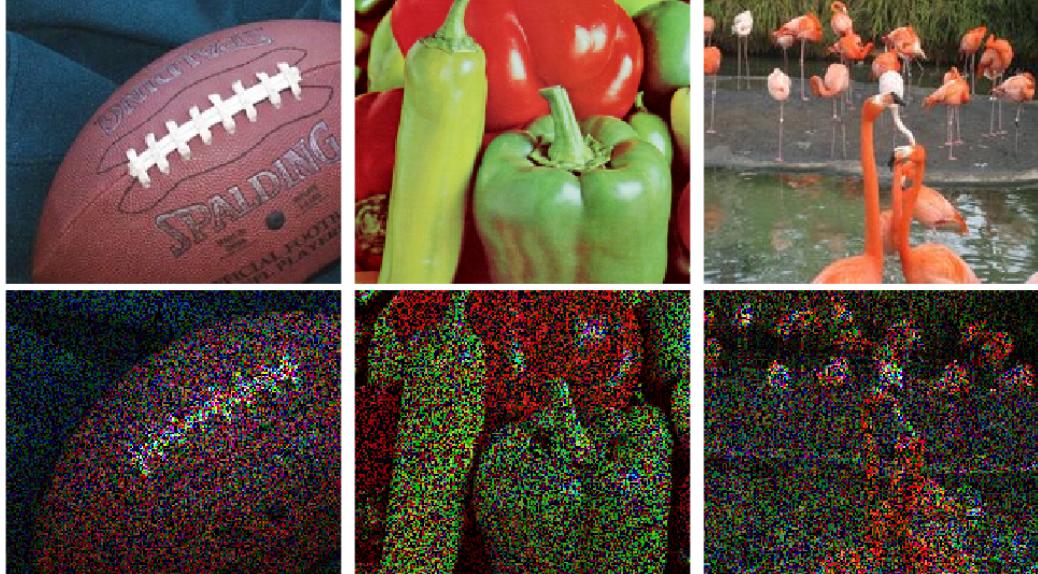


Fig. 6.1: The original images (1st row), the (randomly) 55% incompletely images (2nd row),

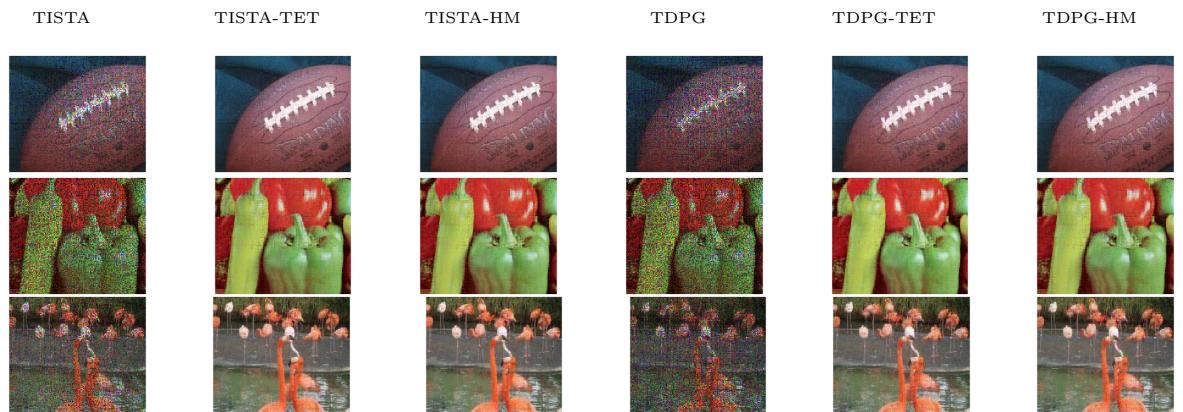


Fig. 6.2: The recovered images by: TISTA (1th column), TISTA-TET (2th column), TISTA-HM (3th column), TDGP (4th column), TDGP-TET (5th column) and b TDGP-HM (6th column).

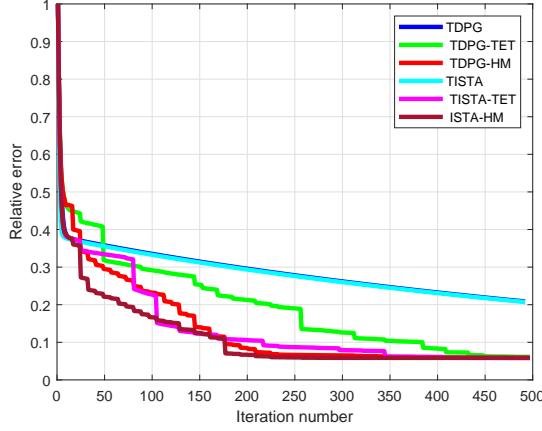


Fig. 6.3: The curves of relative error.

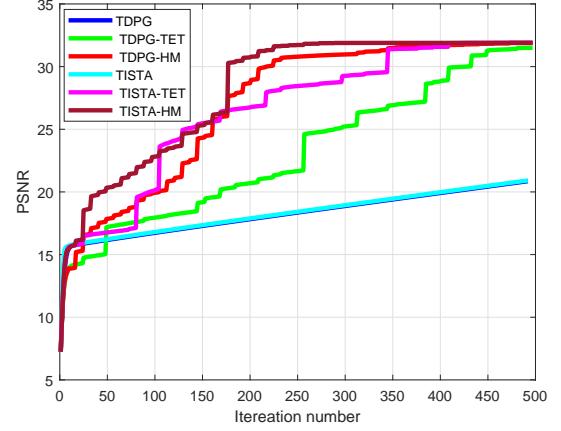


Fig. 6.4: The curves of PSNR.

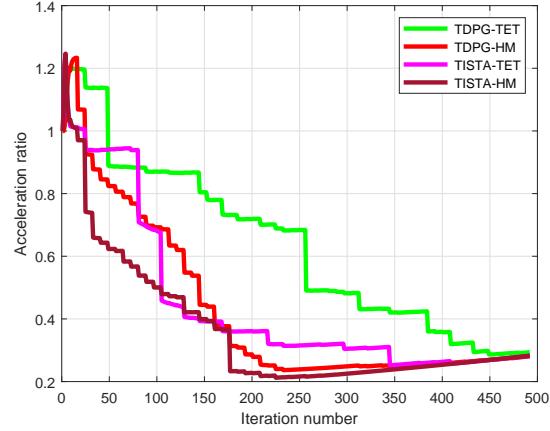


Fig. 6.5: The curves of the acceleration rate.

6.2. Nonrandom missing pixels. In this subsection, we choose the $250 \times 250 \times 3$ color images 'boats.png', 'hand.jpg' and 'soccer.jpg', and we added to these images different masks as shown in Figure 6.6. The recovered images are shown in Figure 6.7. As in the random case, we remark that TISTA and TDPG have nearly the same performance, while their extrapolated versions provide more clear images.

The relative error and the PSNR curves of the six algorithms are shown in Figures 7.1 and 7.2, respectively. The curves behaviours show the applicability and effectiveness of the extrapolated algorithms TISTA-TET, TISTA-HM, TDPG-TET and TDPG-HM.

Figure 7.3 illustrates the convergence rate curves of all extrapolated methods. We can see clearly the fast convergence of TISTA-TET and TDPG-TET as compared to TISTA-HM and TDPG-HM. This shows the advantage of the extrapolated method using the topological epsilon algorithm for nonrandom incompleteness problems.



Fig. 6.6: The original images (1st row) and the uniformly incomplete images (2nd row).

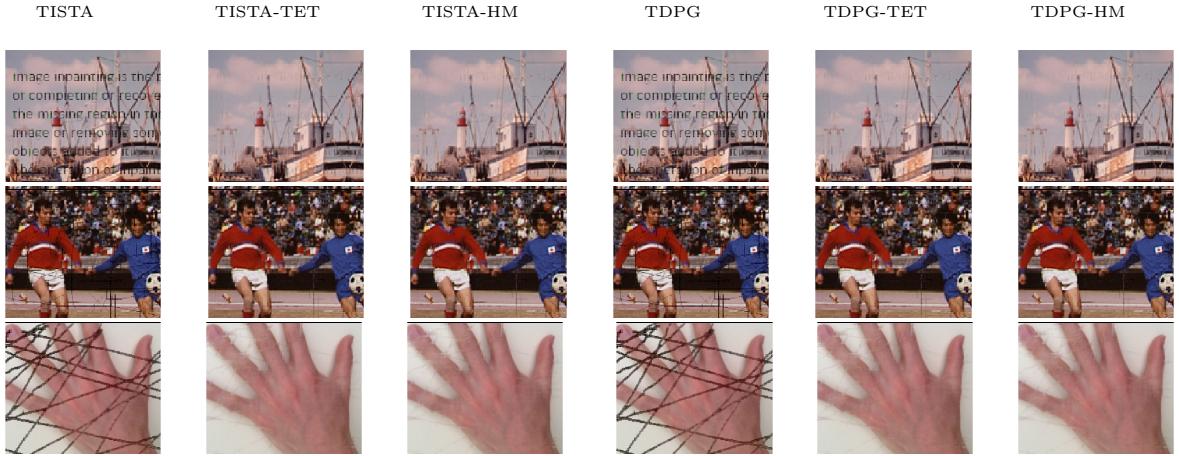


Fig. 6.7: The original images (1st column), the corrupted images (2nd column), the recovered images by TDPG (3rd column), by TDPG-TET (4th column) and by TDPG-HM (5th column).

7. Conclusion. In this work, by embracing a proximal-based approach, we introduced a solution for a constrained multidimensional minimization problem with a general non-smooth regularization term. First, we introduce an approximation of the unconstrained problem (in the whole space). An approximated solution of the constrained problem is obtained by projecting, via Tseng's Algorithm, the unconstrained solution into the closed set Ω . Through numerical experiments on image completion, we have illustrated the efficiency of the adopted approach. Due to the slowness of the proposed algorithms, we have enhancing their convergence rate by incorporating some recent

Table 6.2: Comparison of algorithms for different incompleteness levels 30%, 50% and 80%.

Images	%	Algorithms	PSNR	Relative error	CPU-time (s)
football.png	30%	TDPG/TISTA	34.69/34.69	4.25e-02/4.24e-02	12.93/12.14
		TDPG-TET/TISTA-TET	36.59/36.67	3.45e-02/ 3.42e-02	6.63 /5.72
		TDPG-HM/TISTA-HM	36.52/36.54	3.48e-02/3.47e-02	3.56/4.86
	50%	TDPG/TISTA	28.21/28.21	9.08e-02/9.08e-02	19.83/18.59
		TDPG-TET/TISTA-TET	29.48/29.51	7.81e-02/7.79e-02	10.94/7.05
		TDPG-HM/TISTA-HM	29.52/29.54	7.78e-02/7.76e-02	10.09/7.88
	80%	TDPG/TISTA	19.10/19.12	2.83e-01/2.82e-01	35.10/39.32
		TDPG-TET/TISTA-TET	22.76/22.82	1.71e-01/ 1.70e-01	20.36/14.23
		TDPG-HM/TISTA-HM	22.72/22.69	1.72e-01/ 1.73e-01	24.75/11.89
barbara.bmp	30%	TDPG/TISTA	36.11/36.11	3.38e-02/3.38e-02	12.77/12.78
		TDPG-TET/TISTA-TET	37.83 /87.87	2.75e-02/2.74e-02	6.82/4.4
		TDPG-HM/TISTA-HM	37.77/37.78	2.77e-02/ 2.77e-02	10.12/5.41
	50%	TDPG/TISTA	29.92/29.92	7.00e-02/7.00e-02	32.08/37.08
		TDPG-TET/TISTA-TET	31.39/31.61	5.82e-02/5.67e-02	28.39/33.69
		TDPG-HM/TISTA-HM	31.40/31.40	5.82e-02/ 5.82e-02	19.44/10.86
	80%	TDPG/TISTA	21.44/21.44	1.92e-01/1.92e-01	45.54/48.67
		TDPG-TET/TISTA-TET	23.17/23.07	1.54e-01/1.56e-01	35.96/27.69
		TDPG-HM/TISTA-HM	23.21/23.20	1.53e-01/1.53e-01	34.94/23.32
flamingos.jpg	30%	TDPG/TISTA	34.93/34.92	4.51e-02/4.51e-02	12.82/13.93
		TDPG-TET/TISTA-TET	36.61/36.85	3.63e-02/3.54e-02	6.98/7.88
		TDPG-HM/TISTA-HM	36.60/36.60	3.64e-02/3.64e-02	11.95/6.84
	50%	TDPG/TISTA	28.29/28.29	9.92e-02/ 9.91e-02	22.45/25.30
		TDPG-TET/TISTA-TET	29.66/29.60	8.28e-02/8.35e-02	10.62/9.26
		TDPG-HM/TISTA-HM	29.62/29.63	8.28e-02/8.33e-02	10.15/9.49
	80%	TDPG/TISTA	20.49/20.48	2.57e-01/2.56e-01	43.35 /45.01
		TDPG-TET/TISTA-TET	21.80/21.96	2.20e-01/2.16e-01	24.01/16.95
		TDPG-HM/TISTA-HM	21.97/21.96	2.15e-01/2.14e-01	22.49/1854

tensor extrapolation methods. The suggest numerical tests illustrate the significant role of these extrapolation methods in speeding up the convergence of the proposed algorithms.

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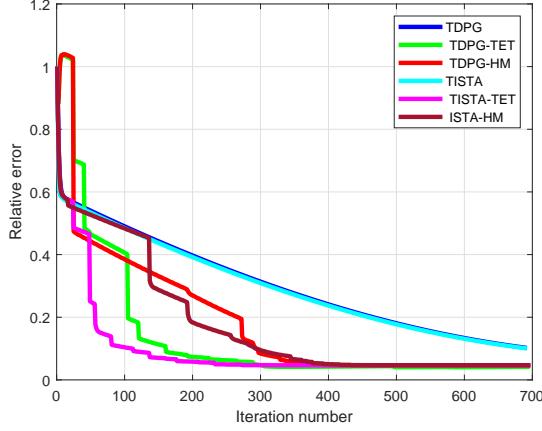


Fig. 7.1: The curves of relative error.

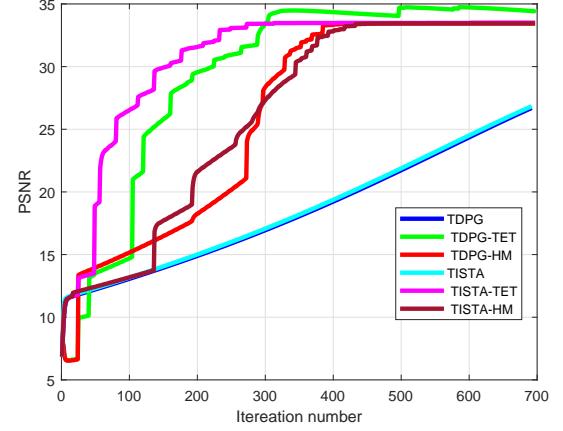


Fig. 7.2: The curves of PSNR.

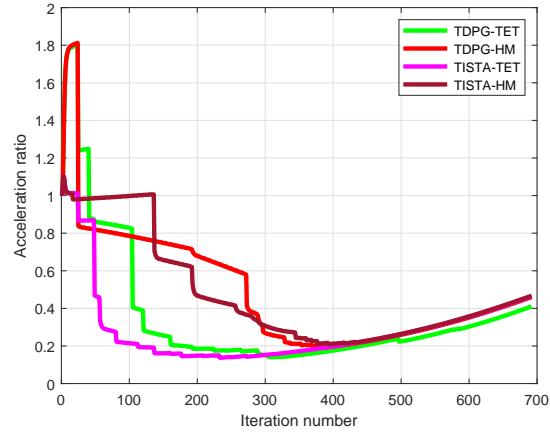


Fig. 7.3: The curves of the acceleration rate.

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